MULTI-AGENT SYSTEM FOR TARGET TRACKING ON A SPHERE AND ITS ASYMPTOTIC BEHAVIOR

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Abstract. We propose a second-order multi-agent system for target tracking on a sphere. The model contains a centripetal force, a bonding force, a velocity alignment operator to the target, and cooperative control between flocking agents. We propose an appropriate regularized rotation operator instead of Rodrigues' rotation operator to derive the velocity alignment operator for target tracking. By the regularized rotation operator, we can decompose the phase of agents into translational and structural parts. By analyzing the translational part of this reference frame decomposition, we can obtain rendezvous results to the given target. If the multi-agent system can obtain the target's position, velocity, and acceleration vectors, then the complete rendezvous occurs. Even in the absence of the target's acceleration information, if the coefficients are sufficiently large enough, then the practical rendezvous occurs.

1. Introduction

Target tracking refers to designing a dynamical system that agents follow given maneuvering target agents using the information of the targets, such as position, velocity, and acceleration. The target tracking problem is applied in various fields, such as mobile sensor networks, virtual reality, and surveillance systems using unmanned aerial vehicles (UAVs) [18, 22, 24]. Most of the relevant literature focuses on the uncertainty of target motions. From a technical point of view, we can divide the models for this field into measurement models, target motion models, and filtering models. The measurement model deals with target information in a sensor coordinate containing additive noise such as image sensors and radar sensor networks [2, 3, 20]. The target motion model is a coupled dynamical system for target tracking. The filtering model is based on the particle filter method and stochastic frameworks estimating the target state such as nonlinear filtering [13, 15] and adaptive filtering [14, 16].

Depending on the structure of the system, we also divide the models for target tracking into two types of systems: single integrator model and double integrator model. For the single integrator model, one can control the velocity of the agents directly. For example, in [10], the authors proposed a tracking algorithm for a slowly moving target using the target's position and bearing angle. Many researchers assume agents can obtain only the target's position and bearing angle for targets maneuvering underwater. From the engineering point of view, it is a reasonable assumption. For the double integrator model, one can control the acceleration of agents. After Olfati-Saber's seminal work [11], researches for the dynamic tracking system using the double integrator model have been extensively conducted. For this kind of model, the tracking agents can have the position and velocity information of the target. Moreover, to avoid collisions between agents or make a formation flight of the agents, a flocking algorithm and cooperative control are frequently used.

The domain or manifolds of agents are also one of the main topics in this field [1, 18] such as the surveillance system for the restricted area or target tracking system on the whole planet. Our goal is to provide a robust navigational feedback system for the target tracking problem on a sphere. Let

γ-agent be a given target governed by the following system:

$$\begin{aligned} \dot{q}_\gamma &= p_\gamma, \\ \dot{p}_\gamma &= -\frac{\|p_\gamma\|^2}{\|q_\gamma\|^2} q_\gamma + U_\gamma(t), \end{aligned} \tag{1.1}$$

where $q_\gamma \in \mathbb{S}^2$, $p_\gamma \in T_{q_\gamma}\mathbb{S}^2$, and U_γ are the position, velocity, and control law of the target agent (γ-agent) on sphere, respectively. To conserve the modulus of $q_\gamma(t) \in \mathbb{S}^2$, we additionally assume that the following condition holds for all $t \geq 0$.

$$q_\gamma(t) \perp U_\gamma(t).$$

Therefore, the control law $U_\gamma(t)$ has the following form: for some $u_\gamma(t) \in \mathbb{R}^3$,

$$U_\gamma(t) = \|q_\gamma(t)\|^2 u_\gamma(t) - \langle u_\gamma(t), q_\gamma(t)\rangle q_\gamma(t).$$

For simplicity, we assume that $u_\gamma(t)$ is continuous.

For a given γ-agent, we propose a novel multi-agent system for the target tracking on a spherical space:

$$\begin{aligned} \dot{q}_i(t) &= p_i(t), \\ \dot{p}_i(t) &= -\frac{\|p_i\|^2}{\|q_i\|^2} q_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|q_i\|^2 q_j - \langle q_i, q_j\rangle q_i) \\ &\quad + c_q(\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma\rangle q_i) + c_p(P_{q_\gamma \to q_i}(p_\gamma) - p_i) + U_i, \end{aligned} \tag{1.2}$$

where $q_i \in \mathbb{S}^2$ and $p_i \in T_{q_i}\mathbb{S}^2$ are the position and velocity of the ith agent, respectively. The first term on the right-hand side of the second equation is the centripetal force term to conserve the modulus of q_i. The second term

$$\sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|q_i\|^2 q_j - \langle q_i, q_j\rangle q_i)$$

is the cooperative control term between agents where the inter-particle force parameter is given by

$$\sigma_{ij} = \sigma(\|x_i - x_j\|^2).$$

The next two terms, $c_q(\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma\rangle q_i)$ and $c_p(P_{q_\gamma \to q_i}(p_\gamma) - p_i)$, are the bonding force and a velocity alignment term between the target and the ith agent, respectively, where $c_q > 0$ and $c_p > 0$ are target tracking coefficients for the position and velocity, respectively. The last term U_i is an extra control law based on the target's information, which will be determined later in (1.5) and (1.6) for each purpose.

Throughout this paper, we assume the initial data satisfies the following admissible conditions on $\mathbb{S}^2$:

$$\|q_i(0)\| = 1, \quad \langle p_i(0), q_i(0)\rangle = 0, \qquad \text{for all } i \in \{1, \ldots, N\}. \tag{1.3}$$

Definition 1.1. *For a given target $(q_\gamma(t), p_\gamma(t))$, let $\{(q_i(t), p_i(t))\}_{i=1}^N$ be the solution to* (1.2)*. We define the two kinds of rendezvouses.*

(1) *An asymptotic complete rendezvous occurs between the agents and the given target, if*

$$\lim_{t\to\infty} \max_{1\le i\le N} \|q_i(t) - q_\gamma(t)\| = 0.$$

(2) *An asymptotic practical rendezvous occurs between the agents and the given target, if*

$$\lim_{c_q, c_p\to\infty} \lim_{t\to\infty} \max_{1\le i\le N} \|q_i(t) - q_\gamma(t)\| = 0.$$

In what follows, we will show that our model contains many robust properties, including the complete rendezvous. Even in the absence of the target acceleration information, the practical rendezvous occurs when the coefficients are large enough. In particular, we obtain a sharp estimate of the distance between the target and agents. There are many other papers on the dynamics on $\mathbb{S}^2$ as well as $\mathbb{R}^n$, but our asymptotic analysis including exponential convergence and practical rendezvous is new on the target tracking problem, to the best of our knowledge.

The derivation of our model is motivated by the decomposition property of flocking dynamics on a flat space. On a flat space, from momentum conservation, the dynamics is represented by the composition of frame reference dynamics and local alignment dynamics as in [11]. In contrast to previous results in $\mathbb{R}^n$, it is hard to expect such a decomposition for the flocking model on $\mathbb{S}^2$. See Sections 2 and 3 for details. In particular, in our previous papers [6, 7, 8], we used Rodrigues' rotation operator $R_{\cdot\to\cdot}$ to derive a flocking system on a sphere since Rodrigues' rotation operator $R_{\cdot\to\cdot}$ is the most natural flocking operator. However, its composition is complex so that it is difficult to analyze. Moreover, it contains an unavoidable singularity at antipodal points due to its geometric characteristics. From this singularity, even though agents are located on $\mathbb{S}^2$, the vanishing point on the communication rate is necessary [6]. Due to this difficulty, the target tracking problem on $\mathbb{S}^2$ has not been well understood.

We remove the singular term from the natural rotation operator $R_{\cdot\to\cdot}$ to obtain *a rotation operator in two dimensions*:

$$P_{z_1\to z_2} := \langle z_1, z_2\rangle I + z_2 z_1^T - z_1 z_2^T, \quad \text{for } z_1 \text{ and } z_2 \text{ in a unit sphere.} \tag{1.4}$$

See also Appendix A for the motivation of the non-singularity rotation operator P and its properties. We will prove that its dynamics consists of the composition of the rigid motion part on $\mathbb{S}^2$ and the local alignment part. Using this property, we derive an $\mathbb{S}^2$-version of the reference frame decomposition in Proposition 3.2 and provide a sufficient condition to obtain a target tracking estimate between multiple agents $\{(q_i(t), p_i(t))\}_{i=1}^N$ and the given target $(q_\gamma(t), p_\gamma(t))$. Moreover, by the regularity of the operator P, we can obtain the following global existence result.

Theorem 1. *Assume that for a continuous function u_γ, a given target $(q_\gamma(t), p_\gamma(t))$ satisfies* (1.1)*. If the initial data $\{(q_i(0), p_i(0))\}_{i=1}^N$ satisfies* (1.3) *and U_i is Lipschitz continuous with respect to $\{(q_i, p_i)\}_{i=1}^N$ with $\langle U_i, q_i\rangle = 0$, then there exists a unique global-in-time solution $\{(q_i(t), p_i(t))\}_{i=1}^N$ to system* (1.2) *and $\{q_i(t)\}_{i=1}^N$ are located on $\mathbb{S}^2$ for all time $t > 0$.*

As in $\mathbb{R}^d$, we notice that the velocity alignment operator between the target and the agents plays an important role in target tracking. In particular, the bonding force between the target and the agents, $c_q(\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma\rangle q_i)$, alone is not enough to track a target on $\mathbb{S}^2$. The velocity alignment operator $c_p(P_{q_\gamma\to q_i}(p_\gamma) - p_i)$ is crucial for the target tracking algorithm. See the simulations in Section 5. In the next two theorems, we present a quantitative analysis of the velocity alignment operator with two different U_i's;

$$U_i = 2\langle w_\gamma, q_i\rangle (q_i \times p_i) + \dot{w}_\gamma(t) \times q_i \tag{1.5}$$

or

$$U_i = 0, \tag{1.6}$$

where w_γ is the angular velocity of the target given by

$$w_\gamma = q_\gamma \times p_\gamma. \tag{1.7}$$

From Theorem 2, if the agents can obtain the exact target information containing acceleration, then the agents can accurately track the target, and the position differences between the target and the agents decay exponentially fast.

Theorem 2. *Let $(q_\gamma(t), p_\gamma(t))$ be a given target satisfying (1.1) with a continuous target control u_γ and $\{q_i(t), p_i(t)\}_{i=1}^N$ be the solution to (1.2) satisfying (1.3). We assume that $\sigma_{ij} = \sigma$ is a positive constant and*

$$U_i = 2\langle w_\gamma, q_i\rangle (q_i \times p_i) + \dot{w}_\gamma(t) \times q_i,$$

where w_γ is the angular velocity defined in (1.7).

If $c_q > \sigma > 0$ or

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^N \|p_i(0) - w_\gamma(0) \times q_i(0)\|^2 \\ &\quad + \frac{\sigma}{2N^2}\sum_{i,j=1}^N \|q_i(0) - q_j(0)\|^2 + \frac{c_q}{N}\sum_{i=1}^N \|q_\gamma(0) - q_i(0)\|^2 < \sigma\left(1 + \frac{c_q}{\sigma}\right)^2,\end{aligned}$$

then the asymptotic complete rendezvous occurs and its convergence rate is exponential, i.e., there are positive constants $\mathcal{C}$, $\mathcal{D}$ such that

$$\|q_i(t) - q_\gamma(t)\|, \ \|p_i(t) - p_\gamma(t)\| \le \mathcal{C}e^{-\mathcal{D}t}.$$

Remark 1.1.
(1) *If the above sufficient condition in Theorem 2 does not hold, then we can find a steady-state solution. This means that the sufficient condition is almost optimal to lead the convergence result in Theorem 2. See Section 5.*
(2) *The author in [11] does not deal with the estimate of the distance between the target and agents. Our model is inspired by [11], but the target tracking estimate and practical rendezvous are novel.*
(3) *The derivation of U_i in the above theorem is technical, but from the frame decomposition in Proposition 3.2, it is a very natural choice to obtain the complete rendezvous.*

The former one in (1.5) corresponds to the case with the target acceleration, while it is unknown in the latter case (1.6). These choices with the different amounts of the target information induce the different accuracies of the target tracking. Since the target information obtained by the agents through observation is usually incomplete, there have been many studies to overcome this incompleteness. For example, many researchers proposed target tracking systems including restricted target information [10, 19], communication-induced delays [12, 17], and additive noise from measurement [9, 23]. The result in Theorem 3 below means that the large coefficients of the system allow the agents to get close enough to the target as needed without acceleration information of the target. In other words, the practical rendezvous occurs.

Theorem 3. *For $(q_\gamma(t), p_\gamma(t))$ satisfying (1.1) with a continuous target control u_γ, let $\{q_i(t), p_i(t)\}_{i=1}^N$ be the solution to (1.2) subject to the initial data satisfying (1.3) and*

$$U_i = 0.$$

Assume that $\sigma_{ij} = \sigma$ is a positive constant and the angular velocity of the target and its time derivative are bounded

$$\|w_\gamma\|, \|\dot{w}_\gamma\| < C_\gamma.$$

If $\|p_i(0) - p_\gamma(0)\| \neq 2$ for all $i \in \{1, \dots, N\}$, then the asymptotic practical rendezvous occurs and

$$\|q_i(t) - q_\gamma(t)\| \le \mathcal{C}e^{-\frac{\mathcal{D}}{4}t} + \frac{\mathcal{C}}{\mathcal{D}},$$

where $\mathcal{C}$ is a positive constant depending on the initial data, σ, and C_γ. The constant $\mathcal{D}$ is given by

$$\mathcal{D} := \begin{cases} c_p - \sqrt{-4c_q + c_p^2}, & \text{if} \quad c_p^2 \ge -4c_q, \\ c_p, & \text{if} \quad c_p^2 < -4c_q. \end{cases}$$

There are technical issues in the proofs of Theorems 2 and 3. We can obtain the complete rendezvous result in Theorem 2 through Lasalle's invariance principle with an energy functional. However, Lasalle's invariance principle does not give a convergence rate. An appropriate Lyapunov functional will be used to obtain the exponential convergence result. In particular, in this case, we derive a closed differential inequality by using six functionals including information on the distance between the target and agents and the distance between agents. The practical rendezvous in Theorem 3 has a more subtle issue. It is necessary to control the distance between the target and agents through the size of the coefficients. However, it is impossible if the coefficients appear in the nonlinear higher-order terms except for the linear terms. If we use a standard functional, the coefficients necessarily occur in the nonlinear terms due to the geometrical characteristics of $\mathbb{S}^2$. This problem will be solved by using new functionals inspired by hyperbolic geometry.

The rest of the paper is organized as follows. In Section 2, we present the global-in-time existence and uniqueness of the solution to (1.2) and target tracking results for $\mathbb{R}^3$. Section 3 is devoted to a reference frame decomposition for the main system. From this decomposition, the solution to the main system is represented by the composition of operators for the translational part and the structural part. Next, we reduce the system for the structural part to a linearized system in Section 4. Using this, we prove the complete and practical rendezvouses of Theorems 2 and 3 in Section 5. In Section 6, we verify our analytic results using numerical simulations. Section 7 is devoted to the summary of our results.

2. Preliminary: Global well-posedness and Motivations

2.1. **The global existence and uniqueness.** In this section, we provide the proof of Theorem 1: there is a unique global-in-time solution to (1.2) and this solution is located on the sphere when the initial data satisfies the admissible conditions in (1.3).

For the local existence and uniqueness, we use the same argument in [6, 7]. For given C^1 functions q_γ, p_γ, and $w_\gamma = q_\gamma \times p_\gamma$, we consider the following system of ODEs:

$$
\begin{aligned}
&\dot{q}_i(t) = p_i(t),\\
&\dot{p}_i(t) = -\frac{\|p_i\|^2}{\|q_i\|^2} q_i + \sum_{j=1}^{N} \frac{\sigma(\|x_i - x_j\|^2)}{N} (\|q_i\|^2 q_j - \langle q_i, q_j\rangle q_i)\\
&\qquad\quad + c_q(\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma\rangle q_i) + c_p(\langle q_\gamma, q_i\rangle p_\gamma - \langle q_i, p_\gamma\rangle q_\gamma - p_i) + U_i.
\end{aligned}
\tag{2.1}
$$

Here, we will choose $U_i = 2\langle w_\gamma, q_i\rangle (q_i \times p_i) + \dot{w}_\gamma(t) \times q_i$ for the complete rendezvous and $U_i = 0$ for the practical rendezvous.

We assume that the initial data $\{(q_i(0), p_i(0))\}_{i=1}^N$ satisfies the admissible condition in (1.3). Then the right-hand side of (2.1) is Lipschitz continuous with respect to $\{(q_i, p_i)\}_{i=1}^N$ in a small neighborhood of $\{(q_i(0), p_i(0))\}_{i=1}^N$ in $\mathbb{R}^{6N}$. By the Picard-Lindelöf Theorem, there is the maximum time interval $[0, T_M)$ in which a solution of (2.1) exists and it is unique.

We next follow the same argument in [6, 7]. On the maximum time interval $[0, T_M)$, we take the inner product between the second equation of (2.1) and x_i to obtain that

$$
\langle \dot{p}_i, q_i\rangle = -\|p_i\|^2 - c_p\langle p_i, q_i\rangle. \tag{2.2}
$$

By (2.2) and the first equation of (2.1), we obtain that

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{N}|\langle p_i, q_i\rangle|^2 &= 2\sum_{i=1}^{N}(\langle \dot{p}_i, q_i\rangle + \langle p_i, \dot{q}_i\rangle)\langle p_i, q_i\rangle\\ &= 2\sum_{i=1}^{N}(\langle \dot{p}_i, q_i\rangle + \|p_i\|^2)\ \langle p_i, q_i\rangle\\ &= -2c_p\sum_{i=1}^{N}|\langle p_i, q_i\rangle|^2.\end{aligned}$$

Note that the initial data satisfies $\sum_{i=1}^{N}|\langle v_i(0), x_i(0)\rangle|^2 = 0$. Therefore, the Gronwall inequality implies that

$$\sum_{i=1}^{N}|\langle v_i(t), x_i(t)\rangle| \equiv 0, \quad \text{for } t > 0,$$

and this implies that

$$\langle v_i(t), x_i(t)\rangle \equiv 0.$$

We take the inner product between $\dot{q}_i$ and q_i. By the first equation of (2.1),

$$\frac{d}{dt}\|q_i\|^2 = 2\langle \dot{q}_i, q_i\rangle = 2\langle p_i, q_i\rangle = 0.$$

Since initial conditions satisfy $\|x_i(0)\| = 1$ and $\langle v_i(0), x_i(0)\rangle = 0$ for all $i \in \{1, \ldots, N\}$, we have

$$\|x_i(t)\| \equiv 1, \quad \text{for } t > 0,\ i \in \{1, \ldots N\}.$$

In conclusion, we can apply the extensibility of solutions in [21, Corollary 2.2] to obtain that

$$T_M = \infty.$$

Moreover, we can easily check that $\{(q_i(t), p_i(t))\}_{i=1}^{N}$ is the unique solution to (1.2) by a standard argument. Therefore, we can obtain the following proposition.

Proposition 2.1. *Let* $\{(q_i(t), p_i(t))\}_{i=1}^{N}$ *be a solution to* (1.2) *with* (1.3). *Then for all* $i \in \{1, \ldots, N\}$ *and* $t > 0$*,*

$$\langle q_i(t), p_i(t)\rangle = 0 \quad \textit{and} \quad \|q_i(t)\| = 1.$$

2.2. **Target tracking problem in $\mathbb{R}^3$.** In this section, we estimate the distance between the target and agents for the following model in $\mathbb{R}^3$:

$$\begin{aligned}\dot{q}_i &= p_i,\\ \dot{p}_i &= \sum_{j=1}^{N}\frac{\psi_{ij}}{N}(p_j - p_i) + \sum_{j=1}^{N}\frac{\sigma_{ij}}{N}(q_j - q_i) + c_q(q_\gamma - q_i) + c_p(p_\gamma - p_i) + u_i,\end{aligned}$$

where $q_i \in \mathbb{R}^3$ and $p_i \in \mathbb{R}^3$ are the position and velocity of the ith agent, respectively. Here, q_γ, p_γ, and u_γ are the position, velocity, and acceleration of a given target (γ-agent) satisfying

$$\begin{aligned}\dot{q}_\gamma &= p_\gamma,\\ \dot{p}_\gamma &= u_\gamma.\end{aligned}$$

A new input parameter u_i will be determined later. Depending on the information of the target, we choose two different u_i's and analyze the corresponding asymptotic behaviors. The argument is straightforward, and thus the reader familiar with target tracking problems in $\mathbb{R}^3$ may skip this section.

If $u_i = 0$, then the above model corresponds to the one in Olfati-Saber's seminal paper [11]. As studied in [11], the system of equations can be decomposed as two second-order systems for the structural dynamics and translational dynamics. For simplicity, we assume that $\psi_{ij} = 0$ and $\sigma_{ij} = \sigma_{ji}$ for all indices i and j in $\{1, \ldots, N\}$. We note that the effect of the flocking term $\sum_{j=1}^{N} \frac{\psi_{ij}}{N}(p_j - p_i)$ is negligible, when $\max_{1\le i,j\le N} \|p_j - p_i\| \ll 1$. See the numerical simulations in Figures 6 and 7.

Let

$$q_c = \frac{1}{N}\sum_{i=1}^{N} q_i, \quad p_c = \frac{1}{N}\sum_{i=1}^{N} p_i,$$

and

$$x_i = q_i - q_c, \quad v_i = p_i - p_c. \tag{2.3}$$

Then, the above dynamics can be decomposed into the translational dynamics (2.4) and the structural dynamics (2.5):

$$\begin{aligned} \dot{q}_c &= p_c, \\ \dot{p}_c &= c_q(q_\gamma - q_c) + c_p(p_\gamma - p_c) + u_i, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \dot{x}_i &= x_i, \\ \dot{v}_i &= \sum_{j=1}^{N} \frac{\sigma_{ij}}{N}(x_j - x_i) - c_q x_i - c_p v_i. \end{aligned} \tag{2.5}$$

The structural dynamics part in (2.5) has been analyzed in [11].

We focus on the translational dynamics part in (2.4) for two different cases of u_i. We first suppose that all of the position p_γ, velocity q_γ, and acceleration u_γ of the target are given. In this case, it is natural to choose $u_i := u_\gamma$. Let

$$q_d = q_c - q_\gamma, \quad p_d = p_c - p_\gamma.$$

Then the translational dynamics in (2.4) can be rewritten as

$$\begin{aligned} \dot{q}_d &= p_d, \\ \dot{p}_d &= -c_q q_d - c_p p_d. \end{aligned}$$

This is a simple linear system of ODEs and it has the following solution;

$$\begin{aligned} q_d(t) = \frac{1}{2\sqrt{c_p^2 - 4c_q}} \Big[&- c_p q_d(0) e^{\frac{1}{2}t\left(-\sqrt{c_p^2-4c_q}-c_p\right)} + q_d(0)\sqrt{c_p^2 - 4c_q}\, e^{\frac{1}{2}t\left(-\sqrt{c_p^2-4c_q}-c_p\right)} \\ &+ c_p q_d(0) e^{\frac{1}{2}t\left(\sqrt{c_p^2-4c_q}-c_p\right)} + q_d(0)\sqrt{c_p^2 - 4c_q}\, e^{\frac{1}{2}t\left(\sqrt{c_p^2-4c_q}-c_p\right)} \\ &- 2p_d(0) e^{\frac{1}{2}t\left(-\sqrt{c_p^2-4c_q}-c_p\right)} + 2p_d(0) e^{\frac{1}{2}t\left(\sqrt{c_p^2-4c_q}-c_p\right)} \Big]. \end{aligned}$$

Therefore, we can easily check that q_d and p_d converge to zero exponentially. This means that the complete rendezvous with an exponential decay rate occurs for any positive c_q and c_p.

If we only know the position and velocity of the target, we cannot expect a complete rendezvous. On the other hand, we can control the maximum position difference between the target and agents if the tracking coefficients for the target are sufficiently large. We refer to [4, 5] for related issues.

For $u_i = 0$, the translational dynamics is given by

$$\begin{aligned} \dot{q}_d &= p_d, \\ \dot{p}_d &= -c_q q_d - c_p p_d - u_\gamma. \end{aligned}$$

As we mentioned above, we cannot expect the complete rendezvous for this case. Alternatively, to obtain the practical rendezvous estimate, we additionally assume that the acceleration of the target is bounded:

$$\limsup \|u_\gamma\| \le C_\gamma, \tag{2.6}$$

for some $C_\gamma > 0$. Then we define auxiliary variables as follows.

$$X_d^1 = \langle q_d, q_d\rangle, \quad X_d^2 = \langle q_d, p_d\rangle, \quad X_d^3 = \langle p_d, p_d\rangle.$$

By the system of the translational dynamics, we can obtain

$$\begin{aligned}
\dot{X}_d^1 &= 2X_d^2,\\
\dot{X}_d^2 &= X_d^3 - c_q X_d^1 - c_p X_d^2 - \langle q_d, u_\gamma\rangle,\\
\dot{X}_d^3 &= -2c_q X_d^2 - 2c_p X_d^3 - 2\langle p_d, u_\gamma\rangle.
\end{aligned}$$

We rewrite the above system of equations as the following inhomogeneous linear system of ODEs:

$$\dot{X}_d = A_d X_d + F_d,$$

where $X_d = (X_d^1, X_d^2, X_d^3)^T$ and $F_d = (0, -\langle q_d, u_\gamma\rangle, -2\langle p_d, u_\gamma\rangle)^T$, and the coefficient matrix is given by

$$M_d = \begin{bmatrix} 0 & 2 & 0 \\ -c_q & -c_p & 1 \\ 0 & -2c_q & -2c_p \end{bmatrix}.$$

Note that M_d has the following eigenvalues.

$$\left\{-c_p,\ -c_p - \sqrt{c_p^2 - 4c_q},\ -c_p + \sqrt{c_p^2 - 4c_q}\right\}.$$

Let $D_d < 0$ be the greatest real part in the above eigenvalues and let

$$-\mu_d = D_d.$$

Then, we have

$$\begin{aligned}
\frac{d}{dt}\|X_d\|^2 &= 2\langle X_d, M_d X_d\rangle + 2\langle X_d, F_d\rangle\\
&\le -2\mu_d\|X_d\|^2 + 2\|X_d\|\|F_d\|,
\end{aligned}$$

this implies that

$$\frac{d}{dt}\|X_d\| \le -\mu_d\|X_d\| + \|F_d\|.$$

From elementary calculations, it follows that for any $\epsilon > 0$,

$$\begin{aligned}
\|F_d\| &\le \|q_d\|\|u_\gamma\| + 2\|p_d\|\|u_\gamma\|\\
&\le \frac{\epsilon\|q_d\|^2}{2} + \frac{1}{2\epsilon}\|u_\gamma\|^2 + \frac{\epsilon\|p_d\|^2}{2} + \frac{2}{\epsilon}\|u_\gamma\|^2\\
&\le \epsilon\|X_d\| + \frac{5}{2\epsilon}\|u_\gamma\|^2.
\end{aligned}$$

We choose $\epsilon = \mu_d/2$ and use the Gronwall inequality and (2.6) to obtain that

$$\begin{aligned}
\|X_d\| &\le e^{-(\mu_d-\epsilon)t}\|X_d(0)\| + \frac{5}{2\epsilon}e^{(\mu_d-\epsilon)t}\int_0^t \|u_\gamma(s)\|^2 e^{(\mu_d-\epsilon)s}ds\\
&\le e^{-(\mu_d-\epsilon)t}\|X_d(0)\| + C_\gamma^2\frac{5}{2\epsilon}e^{-(\mu_d-\epsilon)t}\frac{e^{(\mu_d-\epsilon)t}-1}{\mu_d-\epsilon}.
\end{aligned}$$

This implies that

$$\limsup \|X_d\| \leq \frac{10C_\gamma^2}{\mu_d^2}.$$

Thus, if we choose a sufficiently large tracking coefficients $c_q, c_p > 0$, then we obtain that

$$\limsup_{t\to\infty} \|q_i(t) - q_\gamma(t)\|, \ \limsup_{t\to\infty} \|p_i(t) - p_\gamma(t)\| \ll 1.$$

3. Generalized rotation operator on sphere and reference frame decomposition

In this section, we decompose our model (1.2) on $\mathbb{S}^2$ into structural dynamics and translational dynamics. Due to the complexity of (1.2), the decomposition of agents' positions into a sum of two vectors as the model in $\mathbb{R}^3$ is not suitable for our case. Instead, we observe that a rigid body motion on $\mathbb{S}^2$ can be used as a reference frame. Choosing an appropriate rigid body motion, our model can be represented as the composition of a rigid body motion and local alignment dynamics. The rigid body motion can be derived based on the angular velocity tensor $W_\gamma(t)$ of the γ-agent and a generalized rotation operator S_γ along the given target described below. Recall the given γ-agent trajectory on $\mathbb{S}^2$:

$$\dot{q}_\gamma = p_\gamma,$$

where $q_\gamma \in \mathbb{S}^2$ and $p_\gamma \in T_x\mathbb{S}^2$ are the position and velocity of the given γ-agent, respectively.

Let

$$w_\gamma = q_\gamma \times p_\gamma.$$

By elementary calculation, we have $q_\gamma \times w_\gamma = -p_\gamma$ and

$$\dot{q}_\gamma = w_\gamma \times q_\gamma.$$

For the angular velocity vector $w_\gamma = (w_\gamma^1, w_\gamma^2, w_\gamma^3)^T$, we define the angular velocity tensor $W_\gamma(t)$ of the γ-agent by

$$W_\gamma^t = \begin{bmatrix} 0 & -w_\gamma^3(t) & w_\gamma^2(t) \\ w_\gamma^3(t) & 0 & -w_\gamma^1(t) \\ -w_\gamma^2(t) & w_\gamma^1(t) & 0 \end{bmatrix}.$$

From the above notation, the equation for the γ-agent is written by

$$\dot{q}_\gamma = p_\gamma = W_\gamma^t q_\gamma.$$

Now, we consider the following system of ODEs:

$$\dot{x}(t) = W_\gamma^t x(t). \tag{3.1}$$

We can define the corresponding solution operator $S_\gamma(x_0, t) = S_\gamma^t x_0 : \mathbb{S}^2 \times [0, \infty) \mapsto \mathbb{S}^2$ such that

$$S_\gamma^t x_0 = x(t; x_0), \tag{3.2}$$

where $x(t; x_0)$ is the solution to (3.1) subject to

$$x(0; x_0) = x_0 \in \mathbb{S}^2. \tag{3.3}$$

One can easily check that S_γ^t is a rigid body motion on $\mathbb{S}^2$.

Lemma 3.1. *Let $x_\gamma(t) \in \mathbb{S}^2$ be the position of a γ-agent which is a C^2 function with respect to $t \geq 0$. For the given γ-agent, the solution operator S_γ^t defined above is represented by a matrix and the matrix product. Moreover, for any $x, y \in \mathbb{R}^3$,*

$$\|x\|^2 = \|S_\gamma^t x\|^2, \quad \langle x, y\rangle = \langle S_\gamma^t x, S_\gamma^t y\rangle.$$

Proof. Let $x_\gamma(t)$ be a given C^2 function with $\|x_\gamma(t)\| = 1$. We define the solution operator S_γ^t by (3.1)-(3.3). Take any two vectors x_1^0 and x_2^0 on $\mathbb{S}^2$. Let

$$x_1(t) = S_\gamma^t x_1^0, \quad x_2(t) = S_\gamma^t x_2^0.$$

Equivalently,

$$\dot{x}_1(t) = W_\gamma^t x_1(t), \quad \dot{x}_2(t) = W_\gamma^t x_2(t),$$

subject to

$$x_1(0) = x_1^0, \quad x_2(0) = x_2^0.$$

Then we have

$$\dot{x}_1(t) - \dot{x}_2(t) = W_\gamma^t (x_1(t) - x_2(t)).$$

This implies that

$$\frac{1}{2}\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 = \langle x_1(t) - x_2(t), W_\gamma^t(x_1(t) - x_2(t))\rangle.$$

We note that W_γ is a skew symmetric matrix and this implies that

$$\begin{aligned}\langle x_1(t) - x_2(t), W_\gamma^t(x_1(t) - x_2(t))\rangle &= \langle W_\gamma^T(t)(x_1(t) - x_2(t)), x_1(t) - x_2(t)\rangle \\ &= -\langle W_\gamma(t)(x_1(t) - x_2(t)), x_1(t) - x_2(t)\rangle \\ &= -\langle x_1(t) - x_2(t), W_\gamma(t)(x_1(t) - x_2(t))\rangle.\end{aligned}$$

Therefore, we can obtain that

$$\langle x_1(t) - x_2(t), W_\gamma^t(x_1(t) - x_2(t))\rangle = 0$$

and

$$\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 = 0.$$

Since we choose x_1^0 and x_2^0 arbitrary, $S_\gamma^t : \mathbb{S}^2 \mapsto \mathbb{S}^2$ is a rigid body motion of $\mathbb{S}^2$. This implies that S_γ^t is represented by a matrix and the matrix product. Moreover, the following holds.

$$\|x\|^2 = \|S_\gamma^t x\|^2, \quad \langle x, y\rangle = \langle S_\gamma^t x, S_\gamma^t y\rangle,$$

for any $x, y \in \mathbb{R}^3$. □

In $\mathbb{R}^3$, the agent's position can be decomposed into a sum of two vectors as described in (2.3)-(2.5). Similarly, the agent's position on $\mathbb{S}^2$ is expressed as the composition of the translational operator S_γ^t and the structural vector x_i:

$$q_i(t) = S_\gamma^t x_i(t). \tag{3.4}$$

Notice that $x_\gamma(t) := q_\gamma(0)$ is a time-independent fixed point on $\mathbb{S}^2$ and satisfies

$$q_\gamma(t) = S_\gamma^t x_\gamma(t). \tag{3.5}$$

In the proposition below, we derive a second-order system of x_i in the moving frame.

Proposition 3.2. *Let $(q_\gamma(t), p_\gamma(t))$ be a given γ-agent satisfying*

$$\dot{q}_\gamma = p_\gamma,$$

where $q_\gamma \in \mathbb{S}^2$ and $p_\gamma \in T_x\mathbb{S}^2$. Let S_γ^t be the solution operator defined by (3.1)-(3.3). *If* (3.4) *and* (3.5) *hold, then the followings are equivalent.*

(1) $\{(x_i(t), v_i(t))\}_{i=1}^N$ *satisfies the following structural system of ODEs:*

$$\begin{aligned}
\dot{x}_i &= v_i, \\
\dot{v}_i &= -\frac{\|v_i\|^2}{\|x_i\|^2}x_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|x_i\|^2 x_j - \langle x_i, x_j\rangle x_i) \\
&\qquad\qquad + c_q(\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma\rangle x_i) - c_p v_i + A_i,
\end{aligned} \tag{3.6}$$

subject to initial data $x_i(0) \in \mathbb{S}^2$, $v_i(0) \in T_{x_i(0)}\mathbb{S}^2$ *for all* $i \in \{1, \dots, N\}$.

(2) $\{(q_i(t), p_i(t))\}_{i=1}^N$ *is the solution to main system* (1.2) *subject to* (1.3) *with*

$$U_i = 2\langle w_\gamma, q_i\rangle(q_i \times p_i) + \dot{w}_\gamma(t) \times q_i + S_\gamma^t A_i. \tag{3.7}$$

Proof. For any $x_0 \in \mathbb{S}^2$, we consider $x(t) = S_\gamma^t x_0$. Then

$$\dot{S}_\gamma^t x_0 = \frac{d}{dt}(S_\gamma^t x_0) = \dot{x}(t) = W_\gamma^t x(t) = W_\gamma^t S_\gamma^t x_0. \tag{3.8}$$

Since x_0 is arbitrary and S_γ^t is a 3×3 matrix by Lemma 3.1, we have

$$\dot{S}_\gamma^t = W_\gamma^t S_\gamma^t. \tag{3.9}$$

We note that for any $x \in \mathbb{R}^3$,

$$W_\gamma^t x = w_\gamma \times x. \tag{3.10}$$

We first prove that if $\{(x_i(t), v_i(t))\}_{i=1}^N$ satisfies (3.6), then $\{(q_i(t), \dot{q}_i(t))\}_{i=1}^N$ is the solution to the main system with (3.7), where $q_i(t) = S_\gamma^t x_i(t)$. By the definition,

$$\frac{d}{dt}q_i = \dot{S}_\gamma^t x_i + S_\gamma^t \dot{x}_i.$$

Motivated by the above, we naturally define the corresponding velocity as follows.

$$p_i = \dot{S}_\gamma^t x_i + S_\gamma^t \dot{x}_i. \tag{3.11}$$

Thus, we have

$$\frac{d}{dt}p_i = \ddot{S}_\gamma^t x_i + 2\dot{S}_\gamma^t \dot{x}_i + S_\gamma^t \ddot{x}_i.$$

By (3.6) and Lemma 3.1,

$$\begin{aligned}
S_\gamma^t \ddot{x}_i = &- \frac{\|v_i\|^2}{\|x_i\|^2} S_\gamma^t x_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}\left[\|x_i\|^2 S_\gamma^t x_j - \langle x_i, x_j\rangle S_\gamma^t x_i\right] \\
&+ c_q\left[\|x_i\|^2 S_\gamma^t x_\gamma - \langle x_i, x_\gamma\rangle S_\gamma^t x_i\right] - c_p S_\gamma^t v_i + S_\gamma^t A_i.
\end{aligned} \tag{3.12}$$

From the property of S_γ^t in Lemma 3.1, it follows that

$$\|x_i\|^2 = \|S_\gamma^t x_i\|^2, \quad \langle x_i, x_j\rangle = \langle S_\gamma^t x_i, S_\gamma^t x_j\rangle.$$

As [6, 7, 8], we can easily prove that

$$x_i(t) \in \mathbb{S}^2, \quad v_i(t) \in T_{x_i(t)}\mathbb{S}^2, \quad \text{for all} \quad t \geq 0,\ i \in \{1, \dots, N\}. \tag{3.13}$$

By this modulus conservation and (3.12),

$$\begin{aligned}
S_\gamma^t \ddot{x}_i = &- \|v_i\|^2 q_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}\left[\|q_i\|^2 q_j - \langle q_i, q_j\rangle q_i\right] \\
&+ c_q\left[\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma\rangle q_i\right] + c_p\left[W_\gamma^t q_i - p_i\right] + S_\gamma^t A_i.
\end{aligned} \tag{3.14}$$

Here, we used (3.9) and (3.11) to obtain

$$-S^t_\gamma v_i = W^t_\gamma q_i - p_i. \tag{3.15}$$

By (3.8), (3.9), (3.15) and the definition of q_i and p_i,

$$\begin{aligned}\ddot{S}^t_\gamma x_i + 2\dot{S}^t_\gamma \dot{x}_i &= \dot{W}^t_\gamma S^t_\gamma x_i + W^t_\gamma \dot{S}^t_\gamma x_i + 2\dot{S}^t_\gamma \dot{x}_i\\ &= \dot{W}^t_\gamma q_i + W^t_\gamma W^t_\gamma q_i + 2W^t_\gamma S^t_\gamma \dot{x}_i\\ &= \dot{W}^t_\gamma q_i + W^t_\gamma W^t_\gamma q_i + 2W^t_\gamma (p_i - W^t_\gamma) q_i)\\ &= \dot{W}^t_\gamma q_i - W^t_\gamma W^t_\gamma q_i + 2W^t_\gamma p_i.\end{aligned} \tag{3.16}$$

Clearly, by the skew symmetric property of W_γ,

$$\begin{aligned}\|p_i\|^2 &= \|W^t_\gamma S^t_\gamma x_i\|^2 + 2\langle W^t_\gamma S^t_\gamma x_i, S^t_\gamma \dot{x}_i\rangle + \|S^t_\gamma \dot{x}_i\|^2\\ &= \|W^t_\gamma q_i\|^2 + 2\langle W^t_\gamma q_i, p_i - W_\gamma q_i\rangle + \|v_i\|^2\\ &= \langle q_i, W^t_\gamma W^t_\gamma q_i\rangle - 2\langle q_i, W^t_\gamma p_i\rangle + \|v_i\|^2.\end{aligned}$$

This implies that

$$-\|v_i\|^2 = -\|p_i\|^2 + \langle q_i, W^t_\gamma W^t_\gamma q_i\rangle - 2\langle q_i, W^t_\gamma p_i\rangle. \tag{3.17}$$

By (3.16) and (3.17), we have

$$\begin{aligned}\ddot{S}^t_\gamma x_i + 2\dot{S}^t_\gamma \dot{x}_i - \|v_i\|^2 q_i = -\|p_i\|^2 q_i &+ \langle q_i, W^t_\gamma W^t_\gamma q_i\rangle q_i - W^t_\gamma W^t_\gamma q_i\\ &- 2\langle q_i, W^t_\gamma p_i\rangle q_i + 2W^t_\gamma p_i + \dot{W}^t_\gamma q_i.\end{aligned} \tag{3.18}$$

Thus, by (3.14) and (3.18),

$$\begin{aligned}\dot{p} &= \ddot{S}^t_\gamma x_i + 2\dot{S}^t_\gamma \dot{x}_i + S^t_\gamma \ddot{x}_i\\ &= -\|p_i\|^2 q_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|q_i\|^2 q_j - \langle q_i, q_j\rangle q_i) + c_q(\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma\rangle q_i)\\ &\quad + c_p(W^t_\gamma q_i - p_i) + \langle q_i, W^t_\gamma W^t_\gamma q_i\rangle q_i - W^t_\gamma W^t_\gamma q_i\\ &\quad - 2\langle q_i, W^t_\gamma p_i\rangle q_i + 2W^t_\gamma p_i + \dot{W}^t_\gamma q_i + S^t_\gamma A_i.\end{aligned} \tag{3.19}$$

We note that for any $x \in \mathbb{R}^3$,

$$W^t_\gamma x = w_\gamma \times x. \tag{3.20}$$

From (3.19)-(3.20) and the modulus conservation property of S^t_γ with $x_i(t) \in \mathbb{S}^2$, it follows that

$$\begin{aligned}\dot{p} = -\frac{\|p_i\|^2}{\|q_i\|^2} q_i &+ \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|q_i\|^2 q_j - \langle q_i, q_j\rangle q_i) + c_q(\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma\rangle q_i)\\ &+ c_p(w_\gamma \times q_i - p_i) + 2\langle w_\gamma, q_i\rangle (q_i \times p_i) + \dot{w}_\gamma \times q_i + S^t_\gamma A_i.\end{aligned}$$

Now, if we choose A_i such as

$$2\langle w_\gamma, q_i\rangle(q_i \times p_i) + \dot{w}_\gamma(t) \times q_i + S^t_\gamma A_i = 0,$$

then our model corresponds to $u_i = 0$ case in the flat space case, and if we choose $A_i = 0$ then our model corresponds to $u_i = u_\gamma$ case in the flat space case. From the uniqueness of the solution to the main system, we obtain the desired result.

We next prove that if $\{(q_i(t), p_i(t))\}_{i=1}^N$ is the solution to the main system with (3.7), then $\{(x_i(t), \dot{x}_i(t))\}_{i=1}^N$ satisfies (3.6), where $x_i(t) = S^{-1}_\gamma(t) q_i(t)$. By the first equation of (1.2), we have

$$p_i = \dot{q}_i = \dot{S}^t_\gamma x_i + S^t_\gamma \dot{x}_i = W^t_\gamma S^t_\gamma x_i + S^t_\gamma \dot{x}_i. \tag{3.21}$$

This implies that

$$\begin{aligned}\ddot{q}_i &= \ddot{S}^t_\gamma x_i + 2\dot{S}^t_\gamma \dot{x}_i + S^t_\gamma \ddot{x}_i \\ &= \dot{W}^t_\gamma S^t_\gamma x_i + W^t_\gamma W^t_\gamma S^t_\gamma x_i + 2W^t_\gamma S^t_\gamma \dot{x}_i + S^t_\gamma \ddot{x}_i.\end{aligned} \tag{3.22}$$

By (3.21), we have

$$\begin{aligned}\|p_i\|^2 &= \|W^t_\gamma S^t_\gamma x_i\|^2 + 2\langle W^t_\gamma S^t_\gamma x_i, S^t_\gamma \dot{x}_i\rangle + \|S^t_\gamma \dot{x}_i\|^2 \\ &= \|W^t_\gamma q_i\|^2 + 2\langle W^t_\gamma q_i, p_i - W_\gamma q_i\rangle + \|v_i\|^2 \\ &= \langle q_i, W^t_\gamma W^t_\gamma q_i\rangle - 2\langle q_i, W^t_\gamma p_i\rangle + \|v_i\|^2.\end{aligned} \tag{3.23}$$

The second equation in (1.2) and $q_i(t) \in \mathbb{S}^2$ imply that

$$\begin{aligned}\ddot{q}_i = &-\|p_i\|^2 q_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|q_i\|^2 q_j - \langle q_i, q_j\rangle q_i) + c_q(\|q_i\|^2 q_\gamma - \langle q_i, q_\gamma\rangle q_i) \\ &+ c_p(P_{q_\gamma \to q_i}(p_\gamma) - p_i) + U_i, \\ = &-\|p_i\|^2 S^t_\gamma x_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|S^t_\gamma x_i\|^2 S^t_\gamma x_j - \langle S^t_\gamma x_i, S^t_\gamma x_j\rangle S^t_\gamma x_i) \\ &+ c_q(\|S^t_\gamma x_i\|^2 S^t_\gamma x_\gamma - \langle S^t_\gamma x_i, S^t_\gamma x_\gamma\rangle S^t_\gamma x_i) + c_p(P_{q_\gamma \to q_i}(p_\gamma) - p_i) + U_i.\end{aligned}$$

From the property of S^t_γ in Lemma 3.1, it follows that

$$\begin{aligned}\ddot{q}_i = &-\|p_i\|^2 S^t_\gamma x_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|x_i\|^2 S^t_\gamma x_j - \langle x_i, x_j\rangle S^t_\gamma x_i) \\ &+ c_q\left(\|x_i\|^2 S^t_\gamma x_\gamma - \langle x_i, x_\gamma\rangle S^t_\gamma x_i\right) + c_p(P_{q_\gamma \to q_i}(p_\gamma) - p_i) + U_i.\end{aligned} \tag{3.24}$$

By (3.22)–(3.24),

$$\begin{aligned}S^t_\gamma \ddot{x}_i = &-\left[\dot{W}^t_\gamma S^t_\gamma x_i + W^t_\gamma W^t_\gamma S^t_\gamma x_i + 2W^t_\gamma S^t_\gamma \dot{x}_i\right] \\ &- \|p_i\|^2 S^t_\gamma x_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|x_i\|^2 S^t_\gamma x_j - \langle x_i, x_j\rangle S^t_\gamma x_i) \\ &+ c_q\left(\|x_i\|^2 S^t_\gamma x_\gamma - \langle x_i, x_\gamma\rangle S^t_\gamma x_i\right) + c_p(P_{q_\gamma \to q_i}(p_\gamma) - p_i) + U_i \\ = &-\left[\dot{W}^t_\gamma S^t_\gamma x_i + W^t_\gamma W^t_\gamma S^t_\gamma x_i + 2W^t_\gamma S^t_\gamma \dot{x}_i\right] \\ &- \left(\langle q_i, W^t_\gamma W^t_\gamma q_i\rangle - 2\langle q_i, W^t_\gamma p_i\rangle + \|v_i\|^2\right) S^t_\gamma x_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|x_i\|^2 S^t_\gamma x_j - \langle x_i, x_j\rangle S^t_\gamma x_i) \\ &+ c_q\Big(\|x_i\|^2 S^t_\gamma x_\gamma - \langle x_i, x_\gamma\rangle S^t_\gamma x_i\Big) + c_p(P_{q_\gamma \to q_i}(p_\gamma) - p_i) + U_i.\end{aligned}$$

Note that

$$\begin{aligned}&-\left[\dot{W}^t_\gamma S^t_\gamma x_i + W^t_\gamma W^t_\gamma S^t_\gamma x_i + 2W^t_\gamma S^t_\gamma \dot{x}_i\right] - \Big(\langle q_i, W^t_\gamma W^t_\gamma q_i\rangle - 2\langle q_i, W^t_\gamma p_i\rangle\Big) S^t_\gamma x_i \\ &\qquad = -\left[\dot{W}^t_\gamma q_i + W^t_\gamma W^t_\gamma q_i + 2W^t_\gamma (p_i - W_\gamma q_i)\right] - \Big(\langle q_i, W^t_\gamma W^t_\gamma q_i\rangle - 2\langle q_i, W^t_\gamma p_i\rangle\Big) q_i \\ &\qquad = -\langle q_i, W^t_\gamma W^t_\gamma q_i\rangle q_i + W^t_\gamma W^t_\gamma q_i + 2\langle q_i, W^t_\gamma p_i\rangle q_i - 2W^t_\gamma p_i - \dot{W}^t_\gamma q_i \\ &\qquad = -2\langle w_\gamma, q_i\rangle (q_i \times p_i) - \dot{w}_\gamma \times q_i.\end{aligned}$$

Therefore, by the property of S_γ and the above two equalities, we obtain that $\{(x_i(t), v_i(t))\}_{i=1}^N$ satisfies (3.6) with (3.7).

□

4. Reduction to a linearized system with a negative definite coefficient matrix

In this section, we derive a linearized system from the structural system in (3.6). We define auxiliary variables motivated by the flat case in Section 2 and we extract leading order terms using $\|q_i(t)\| = 1$ and $\langle q_i(t), p_i(t)\rangle = 0$ for all $t \geq 0$ and $i \in \{1, \dots, N\}$. In the system with respect to auxiliary variables, leading order terms form an inhomogeneous linear system of ODEs with a negative definite coefficient matrix.

We consider the following system of ODEs with $\sigma_{ij} = \sigma > 0$ and $c_q, c_p > 0$.

$$\begin{aligned}
&\dot{x}_i = v_i, \\
&\dot{v}_i = -\frac{\|v_i\|^2}{\|x_i\|^2} x_i + \sum_{j=1}^N \frac{\sigma}{N}(\|x_i\|^2 x_j - \langle x_i, x_j\rangle x_i) \\
&\qquad\qquad + c_q(\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma\rangle x_i) - c_p v_i + A_i.
\end{aligned} \tag{4.1}$$

For consistency, we additionally assume that for all $t \geq 0$,

$$\langle A_i(t), x_i(t)\rangle = 0, \quad \text{for all } i \in \{1, \dots, N\},$$

and the initial data satisfies

$$\|x_i(0)\| = 1 \quad \text{and} \quad \langle v_i(0), x_i(0)\rangle = 0, \quad \text{for all } i \in \{1, \dots, N\}.$$

We now define the auxiliary variables as follows.

$$X_\gamma^1 = \frac{1}{N}\sum_{i=1}^N \|x_i - x_\gamma\|^2, \quad X_\gamma^2 = \frac{1}{N}\sum_{i=1}^N \langle x_i - x_\gamma, v_i\rangle, \quad X_\gamma^3 = \frac{1}{N}\sum_{i=1}^N \langle v_i, v_i\rangle,$$

and

$$X^1 = \frac{1}{N^2}\sum_{i,k=1}^N \langle x_i - x_k, x_i - x_k\rangle, \quad X^2 = \frac{1}{N^2}\sum_{i,k=1}^N \langle v_i - v_k, x_i - x_k\rangle,$$
$$X^3 = \frac{1}{N^2}\sum_{i,k=1}^N \langle v_i - v_k, v_i - v_k\rangle.$$

We also define the corresponding inhomogeneous terms as follows.

$$\begin{aligned}
&F_\gamma^1 = 0, \\
&F_\gamma^2 = -\frac{1}{N}\sum_{i=1}^N \frac{\|v_i\|^2}{2}\|x_i - x_\gamma\|^2 + \frac{\sigma}{4N^2}\sum_{i,j=1}^N \|x_i - x_j\|^2 \|x_i - x_\gamma\|^2 \\
&\qquad + \frac{c_q}{4N}\sum_{i=1}^N \|x_i - x_\gamma\|^4 + \frac{1}{N}\sum_{i=1}^N \langle x_i - x_\gamma, A_i\rangle, \\
&F_\gamma^3 = \frac{2}{N}\sum_{i=1}^N \langle v_i, A_i\rangle,
\end{aligned}$$

and

$$
\begin{aligned}
F^1 &= 0,\\
F^2 &= -\frac{1}{N^2}\sum_{i,k=1}^{N}\frac{\|v_i\|^2+\|v_k\|^2}{2}\|x_i-x_k\|^2+\frac{\sigma}{2N^3}\sum_{i,j,k=1}^{N}\|x_i-x_j\|^2\|x_i-x_k\|^2\\
&\quad+\frac{c_q}{2N^2}\sum_{i,k=1}^{N}\|x_\gamma-x_i\|^2\|x_i-x_k\|^2+\frac{1}{N^2}\sum_{i,k=1}^{N}\langle A_i-A_k,x_i-x_k\rangle,\\
F^3 &= \frac{2}{N^2}\sum_{i,k=1}^{N}\big(\|v_i\|^2\langle x_i,v_k\rangle+\|v_k\|^2\langle x_x,v_i\rangle\big)+\frac{2\sigma}{N^3}\sum_{i,j,k=1}^{N}\|x_i-x_j\|^2\langle x_i,v_k\rangle\\
&\quad+\frac{c_q}{N^2}\sum_{i,k=1}^{N}\|x_\gamma-x_i\|^2\langle x_i,v_k\rangle+\frac{2}{N^2}\sum_{i,k=1}^{N}\langle A_i-A_k,v_i-v_k\rangle.
\end{aligned}
$$

Let

$$
X=(X^1_\gamma,X^2_\gamma,X^3_\gamma,X^1,X^2,X^3)^T,\quad F=(F^1_\gamma,F^2_\gamma,F^3_\gamma,F^1,F^2,F^3)^T. \tag{4.2}
$$

Proposition 4.1. *For the auxiliary variable X and the inhomogeneous term F, the following holds.*

$$
\dot{X}=MX+F,
$$

where the coefficient matrix M is given by

$$
M=\begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0\\
-c_q & -c_p & 1 & -\sigma/2 & 0 & 0\\
0 & -2c_q & -2c_p & 0 & \sigma & 0\\
0 & 0 & 0 & 0 & 2 & 0\\
0 & 0 & 0 & -(c_q+\sigma) & -c_p & 1\\
0 & 0 & 0 & 0 & -2(c_q+\sigma) & -2c_p
\end{bmatrix}.
$$

Proof. Clearly,

$$
\frac{d}{dt}X^1_\gamma=2X^2_\gamma.
$$

For X^2_γ, we have

$$
\begin{aligned}
\frac{d}{dt}X^2_\gamma &= X^3_\gamma+\frac{1}{N}\sum_{i=1}^{N}\langle x_i-x_\gamma,\dot{v}_i\rangle\\
&= X^3_\gamma+\frac{1}{N}\sum_{i=1}^{N}\Big\langle x_i-x_\gamma,\ -\|v_i\|^2x_i+\sum_{j=1}^{N}\frac{\sigma}{N}(\|x_i\|^2x_j-\langle x_i,x_j\rangle x_i)\\
&\qquad\qquad+c_q(\|x_i\|^2x_\gamma-\langle x_i,x_\gamma\rangle x_i)-c_pv_i+A_i\Big\rangle\\
&= X^3_\gamma-\frac{1}{N}\sum_{i=1}^{N}\frac{\|v_i\|^2}{2}\|x_i-x_\gamma\|^2+\frac{\sigma}{N^2}\sum_{i,j=1}^{N}\langle x_i-x_\gamma,x_j-\langle x_i,x_j\rangle x_i\rangle\\
&\quad+\frac{c_q}{N}\sum_{i=1}^{N}\langle x_i-x_\gamma,x_\gamma-\langle x_i,x_\gamma\rangle x_i\rangle-\frac{c_p}{N}\sum_{i=1}^{N}\langle x_i-x_\gamma,v_i\rangle+\frac{1}{N}\sum_{i=1}^{N}\langle x_i-x_\gamma,A_i\rangle.
\end{aligned}
$$

Note that by $x_i \in \mathbb{S}^2$ and changing the indices,

$$\begin{aligned}\sum_{i,j=1}^{N}\langle x_i - x_\gamma, x_j - \langle x_i, x_j\rangle x_i\rangle &= -\sum_{i,j=1}^{N}\langle x_\gamma, x_j - \langle x_i, x_j\rangle x_i\rangle \\ &= -\sum_{i,j=1}^{N}\langle x_\gamma, x_i - \langle x_i, x_j\rangle x_i\rangle \\ &= -\sum_{i,j=1}^{N}\frac{\|x_i - x_j\|^2}{2}\langle x_\gamma, x_i\rangle \\ &= -\frac{1}{2}\sum_{i,j=1}^{N}\|x_i - x_j\|^2 + \frac{1}{4}\sum_{i,j=1}^{N}\|x_i - x_j\|^2\|x_i - x_\gamma\|^2.\end{aligned} \tag{4.3}$$

By (4.3), we have

$$\begin{aligned}\frac{d}{dt}X_\gamma^2 = X_\gamma^3 - \frac{1}{N}\sum_{i=1}^{N}\frac{\|v_i\|^2}{2}\|x_i - x_\gamma\|^2 - \frac{\sigma}{2}X^1 + \frac{\sigma}{4N^2}\sum_{i,j=1}^{N}\|x_i - x_j\|^2\|x_i - x_\gamma\|^2 \\ - \frac{c_q}{N}\sum_{i=1}^{N}\|x_i - x_\gamma\|^2 + \frac{c_q}{4N}\sum_{i=1}^{N}\|x_i - x_\gamma\|^4 - c_p X_\gamma^2 + \frac{1}{N}\sum_{i=1}^{N}\langle x_i - x_\gamma, A_i\rangle \\ = -c_q X_\gamma^1 - c_p X_\gamma^2 + X_\gamma^3 - \frac{\sigma}{2}X^1 + F_\gamma^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}X_\gamma^3 &= \frac{1}{N}\sum_{i=1}^{N}\langle v_i, \dot{v}_i\rangle \\ &= \frac{1}{N}\sum_{i=1}^{N}\Big\langle v_i,\ -\|v_i\|^2 x_i + \sum_{j=1}^{N}\frac{\sigma}{N}(\|x_i\|^2 x_j - \langle x_i, x_j\rangle x_i) \\ &\qquad + c_q(\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma\rangle x_i) - c_p v_i + A_i\Big\rangle \\ &= \frac{\sigma}{N^2}\sum_{i,j=1}^{N}\langle v_i, x_j\rangle - \frac{1}{N}\sum_{i=1}^{N}c_q\langle v_i, x_i - x_\gamma\rangle - \frac{1}{N}\sum_{i=1}^{N}c_p\langle v_i, v_i\rangle + \frac{1}{N}\sum_{i=1}^{N}\langle v_i, A_i\rangle.\end{aligned}$$

Thus, we have

$$\frac{d}{dt}X_\gamma^3 = -2c_q X_\gamma^2 - 2c_p X_\gamma^3 - \sigma X^2 + F_\gamma^3. \tag{4.4}$$

For X^1,

$$\frac{d}{dt}X^1 = 2X^2.$$

Similar to the previous cases, we use the second equation in (4.1) to obtain

$$
\begin{aligned}
\frac{d}{dt}X^2 &= X^3 + \frac{1}{N^2}\sum_{i,k=1}^{N} \langle \dot{v}_i - \dot{v}_k, x_i - x_k \rangle \\
&= X^3 + \frac{1}{N^2}\sum_{i,k=1}^{N} \Big\langle - \|v_i\|^2 x_i + \|v_k\|^2 x_k + \sum_{j=1}^{N} \frac{\sigma}{N}[-\langle x_i, x_j\rangle x_i + \langle x_k, x_j\rangle x_k] \\
&\qquad\qquad + c_q\,[-\langle x_i, x_\gamma\rangle x_i + \langle x_k,\ x_\gamma\rangle x_k] - c_p v_i + c_p v_k, x_i - x_k \Big\rangle \\
&\quad + \frac{1}{N^2}\sum_{i,k=1}^{N} \langle A_i - A_k, x_i - x_k\rangle.
\end{aligned}
$$

By $x_i \in \mathbb{S}^2$, we have

$$
\begin{aligned}
\frac{d}{dt}X^2 = X^3 &- \frac{1}{N^2}\sum_{i,k=1}^{N} \frac{\|v_i\|^2 + \|v_k\|^2}{2}\|x_i - x_k\|^2 \\
&- \sigma X^1 + \frac{\sigma}{4N^3}\sum_{i,j,k=1}^{N} \|x_i - x_j\|^2\|x_i - x_k\|^2 + \frac{\sigma}{4N^3}\sum_{i,j,k=1}^{N} \|x_k - x_j\|^2\|x_i - x_k\|^2 \\
&- c_q X^1 + \frac{c_q}{4N^2}\sum_{i,k=1}^{N} \|x_\gamma - x_i\|^2\|x_i - x_k\|^2 + \frac{c_q}{4N^2}\sum_{i,k=1}^{N} \|x_\gamma - x_k\|^2\|x_i - x_k\|^2 \\
&- c_p X^2 + \frac{1}{N^2}\sum_{i,k=1}^{N} \langle A_i - A_k, x_i - x_k\rangle.
\end{aligned}
$$

Changing the indices implies that

$$
\frac{d}{dt}X^2 = -\sigma X^1 - c_q X^1 - c_p X^2 + X^3 + F^2.
$$

Finally, for X^3, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}X^3 = \frac{1}{N^2}\sum_{i,k=1}^{N}\Big\langle &-\|v_i\|^2x_i+\|v_k\|^2x_k+\sum_{j=1}^{N}\frac{\sigma}{N}(-\langle x_i,x_j\rangle x_i+\langle x_k,x_j\rangle x_k)\\ &+c_q\left[-\langle x_i,x_\gamma\rangle x_i+\langle x_k,x_\gamma\rangle x_k\right]-c_pv_i+c_pv_k,\ v_i-v_k\Big\rangle\\ &+\frac{1}{N^2}\sum_{i,k=1}^{N}\langle A_i-A_k,v_i-v_k\rangle\\ =&\frac{1}{N^2}\sum_{i,k=1}^{N}\left(\|v_i\|^2\langle x_i,v_k\rangle+\|v_k\|^2\langle x_x,v_i\rangle\right)\\ &-\sigma X^2+\sum_{i,j,k=1}^{N}\frac{\sigma}{2N^3}\|x_i-x_j\|^2\langle x_i,v_k\rangle+\sum_{i,j,k=1}^{N}\frac{\sigma}{2N^3}\|x_k-x_j\|^2\langle x_k,v_i\rangle\\ &-c_qX^2+\frac{c_q}{4N^2}\sum_{i,k=1}^{N}\|x_\gamma-x_i\|^2\langle x_i,v_k\rangle+\frac{c_q}{4N^2}\sum_{i,k=1}^{N}\|x_\gamma-x_k\|^2\langle x_k,v_i\rangle-c_pX^3\\ &+\frac{1}{N^2}\sum_{i,k=1}^{N}\langle A_i-A_k,v_i-v_k\rangle.\end{aligned}$$

Thus, we conclude that

$$\frac{d}{dt}X^3=-2\sigma X^2-2c_qX^2-2c_pX^3+F^3.$$

□

Note that the eigenvalues of the 6×6 coefficient matrix M have the only negative real part. The above result will be used for the complete rendezvous case.

Remark 4.2. *In* [8]*, we use l^∞-framework to obtain a uniform decay estimate which is independent of N. However, due to X^2 term on the right-hand side of* (4.4)*, we cannot use this l^∞-framework. We obtain only the convergence result depending on N by using the 6×6 system with l^2-framework.*

For the practical rendezvous result, we use a different framework, weighted l^∞-framework. To obtain l^∞-estimate, we define the following functionals:

$$X_i^1=\frac{4\|x_i-x_\gamma\|^2}{4-\|x_i-x_\gamma\|^2},\quad X_i^2=\frac{16\langle x_i-x_\gamma,v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2},\quad X_i^3=\frac{16\langle v_i,v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2},\tag{4.5}$$

and

$$\begin{aligned}&F_i^1=0,\\ &F_i^2=-\frac{\|v_i\|^2}{2}\frac{16\|x_i-x_\gamma\|^2}{\left(4-\|x_i-x_\gamma\|^2\right)^2}+\frac{16\sigma}{N\left(4-\|x_i-x_\gamma\|^2\right)^2}\sum_{j=1}^{N}\langle x_i-x_\gamma,x_j-\langle x_i,x_j\rangle x_i\rangle\\ &\qquad+\frac{16\langle x_i-x_\gamma,A_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2}+\frac{64\langle x_i-x_\gamma,v_i\rangle^2}{\left(4-\|x_i-x_\gamma\|^2\right)^3}\\ &F_i^3=\frac{32\sigma}{N\left(4-\|x_i-x_\gamma\|^2\right)^2}\sum_{j=1}^{N}\langle v_i,x_j\rangle+\frac{32\langle v_i,A_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2}+\frac{64\langle v_i,v_i\rangle\langle x_i-x_\gamma,v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^3}.\end{aligned}\tag{4.6}$$

We note that due to the geometric structure of $\mathbb{S}^2$, the quartic terms with the coefficient c_q in F_γ^2 and F^2 appear. Thus, the standard functional $X(t)$ in the previous argument and Section 2 does not work for this practical rendezvous case. For the complete rendezvous case, we will use the energy functional method and Lasalle's invariance principle to control the quartic terms. However, for the practical rendezvous case, we cannot use the same methodology since the system is not autonomous. Thus, if an extra term with the coefficient c_q appears in F, then it is hard to obtain the desired result. Alternatively, using the functionals in (4.5), we can remove the quartic term with the coefficient c_q as in (4.6).

By the same argument in Proposition 4.3, we have

$$\frac{d}{dt}X_i^1 = 2X_i^2.$$

Using the second equation for the structural system, we obtain the following for X_γ^2.

$$\begin{aligned}
\frac{d}{dt}X_i^2 &= X_i^3 + \frac{16\langle x_i - x_\gamma, \dot{v}_i\rangle}{(4-\|x_i-x_\gamma\|^2)^2} + \frac{64\langle x_i - x_\gamma, v_i\rangle^2}{(4-\|x_i-x_\gamma\|^2)^3} \\
&= X_i^3 + 16\Bigg\langle x_i - x_\gamma,\ -\|v_i\|^2 x_i + \sum_{j=1}^N \frac{\sigma}{N}(\|x_i\|^2 x_j - \langle x_i, x_j\rangle x_i) \\
&\qquad\qquad + c_q(\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma\rangle x_i) - c_p v_i + A_i \Bigg\rangle \Big/ \left(4 - \|x_i - x_\gamma\|^2\right)^2 \\
&\quad + \frac{64\langle x_i - x_\gamma, v_i\rangle^2}{(4-\|x_i-x_\gamma\|^2)^3} \\
&= X_i^3 - \frac{\|v_i\|^2}{2}\frac{16\|x_i-x_\gamma\|^2}{(4-\|x_i-x_\gamma\|^2)^2} + \frac{16\sigma}{N\,(4-\|x_i-x_\gamma\|^2)^2}\sum_{j=1}^N \langle x_i - x_\gamma, x_j - \langle x_i, x_j\rangle x_i\rangle \\
&\quad + \frac{16c_q}{(4-\|x_i-x_\gamma\|^2)^2}\langle x_i - x_\gamma, x_\gamma - \langle x_i, x_\gamma\rangle x_i\rangle - \frac{16c_p}{(4-\|x_i-x_\gamma\|^2)^2}\langle x_i - x_\gamma, v_i\rangle \\
&\quad + \frac{16\langle x_i - x_\gamma, A_i\rangle}{(4-\|x_i-x_\gamma\|^2)^2} + \frac{64\langle x_i - x_\gamma, v_i\rangle^2}{(4-\|x_i-x_\gamma\|^2)^3}.
\end{aligned}$$

Note that

$$\begin{aligned}
\langle x_i - x_\gamma, x_\gamma - \langle x_i, x_\gamma\rangle x_i\rangle &= \langle x_i - x_\gamma, x_\gamma\rangle - \langle x_i, x_\gamma\rangle\langle x_i - x_\gamma, x_i\rangle \\
&= -\|x_i - x_\gamma\|^2 - \langle x_i - x_\gamma, x_\gamma\rangle\langle x_i - x_\gamma, x_i\rangle \\
&= -\|x_i - x_\gamma\|^2 + \frac{\|x_i - x_\gamma\|^4}{4}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{d}{dt}X_i^2 = X_i^3 &- \frac{\|v_i\|^2}{2}\frac{16\|x_i-x_\gamma\|^2}{(4-\|x_i-x_\gamma\|^2)^2} + \frac{16\sigma}{N\,(4-\|x_i-x_\gamma\|^2)^2}\sum_{j=1}^N \langle x_i - x_\gamma, x_j - \langle x_i, x_j\rangle x_i\rangle \\
&- c_q X_i^1 - c_p X_i^2 + \frac{16\langle x_i - x_\gamma, A_i\rangle}{(4-\|x_i-x_\gamma\|^2)^2} + \frac{64\langle x_i - x_\gamma, v_i\rangle^2}{(4-\|x_i-x_\gamma\|^2)^3}.
\end{aligned}$$

For X_i^3, we have

$$
\begin{aligned}
\frac{d}{dt}X_\gamma^3 &= \frac{32\langle v_i, \dot v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{64\langle v_i, v_i\rangle\langle x_i-x_\gamma, v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^3} \\
&= 32\Big\langle v_i,\ -\|v_i\|^2 x_i + \sum_{j=1}^N \frac{\sigma}{N}(\|x_i\|^2 x_j - \langle x_i, x_j\rangle x_i) \\
&\qquad\qquad + c_q(\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma\rangle x_i) - c_p v_i + A_i\Big\rangle / \left(4-\|x_i-x_\gamma\|^2\right)^2 + \frac{64\langle v_i, v_i\rangle\langle x_i-x_\gamma, v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^3} \\
&= \frac{32\sigma}{N\left(4-\|x_i-x_\gamma\|^2\right)^2}\sum_{j=1}^N \langle v_i, x_j\rangle - 32c_q\frac{\langle v_i, x_i-x_\gamma\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2} - 32c_p\frac{\langle v_i, v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2} \\
&\quad + \frac{32\langle v_i, A_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{64\langle v_i, v_i\rangle\langle x_i-x_\gamma, v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^3} \\
&= \frac{32\sigma}{N\left(4-\|x_i-x_\gamma\|^2\right)^2}\sum_{j=1}^N \langle v_i, x_j\rangle - 2c_q X_i^2 - 2c_p X_i^3 + \frac{32\langle v_i, A_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{64\langle v_i, v_i\rangle\langle x_i-x_\gamma, v_i\rangle}{\left(4-\|x_i-x_\gamma\|^2\right)^3}.
\end{aligned}
$$

In conclusion, we have

$$
\begin{aligned}
\frac{d}{dt}X_i^1 &= 2X_i^2 + F_i^1, \\
\frac{d}{dt}X_i^2 &= -c_q X_i^1 - c_p X_i^2 + X_i^3 + F_i^2 \\
\frac{d}{dt}X_i^3 &= -2c_q X_i^2 - 2c_p X_i^3 + F_i^3.
\end{aligned}
$$

Therefore, we have proved the following proposition.

Proposition 4.3. *Let*

$$X_i = (X_i^1, X_i^2, X_i^3)^T, \quad F_i = (F_i^1, F_i^2, F_i^3)^T,$$

where X_i^k, F_i^k, $k = 1, 2, 3$ are functionals defined in (4.5) *and* (4.6).

Then the following holds.

$$\dot X_i = M_\infty X_i + F_i,$$

where the coefficient matrix M_∞ is given by

$$M_\infty = \begin{bmatrix} 0 & 2 & 0 \\ -c_q & -c_p & 1 \\ 0 & -2c_q & -2c_p \end{bmatrix}.$$

5. Asymptotic analysis on the target tracking models: complete and practical rendezvouses

In this section, we provide the proofs of Theorems 2 and 3 in Section 1. Let (q_γ, p_γ) be the phase of the target. We assume that the target satisfies (1.1) for some continuous $u_\gamma(t) \in \mathbb{R}^3$. For the given target $(q_\gamma(t), p_\gamma(t))$, let $\{(q_i(t), p_i(t))\}_{i=1}^N$ be the solution to (1.2). By the argument in Section 4, we have the following equivalent system for $x_i(t) = S_\gamma^{-1}(t)p_i(t)$.

$$
\begin{aligned}
\dot x_i &= v_i, \\
\dot v_i &= -\frac{\|v_i\|^2}{\|x_i\|^2}x_i + \sum_{j=1}^N \frac{\sigma_{ij}}{N}(\|x_i\|^2 x_j - \langle x_i, x_j\rangle x_i) + c_q(\|x_i\|^2 x_\gamma - \langle x_i, x_\gamma\rangle x_i) - c_p v_i + A_i,
\end{aligned}
\tag{5.1}
$$

where S^t_γ is the solution operator defined by (3.1)-(3.3). For the angular velocity $w_\gamma = q_\gamma \times p_\gamma$, A_i is the extra control law given by

$$A_i = S_\gamma^{-1}(t)U_i - 2\langle w_\gamma, q_i\rangle S_\gamma^{-1}(t)[q_i \times p_i] - S_\gamma^{-1}(t)[\dot w_\gamma(t) \times q_i].$$

5.1. **Complete rendezvouses.** We assume that $\sigma_{ij} = \sigma > 0$ and $A_i = 0$, i.e.,

$$U_i = 2\langle w_\gamma, q_i\rangle q_i \times p_i - \dot w_\gamma(t) \times q_i.$$

We first use an energy functional method to obtain the convergence result in Theorem 2 without convergence rate. We now define an energy functional $\mathcal{E} = \mathcal{E}(\{(x_i, v_i)\}_{i=1}^N)$ as follows.

$$\mathcal{E} = \mathcal{E}_k + \mathcal{E}_c,$$

where $\mathcal{E}_k$ is the kinetic energy given by

$$\mathcal{E}_k = \frac{1}{2N}\sum_{i=1}^N \|v_i\|^2,$$

and $\mathcal{E}_c$ is the configuration energy given by

$$\mathcal{E}_c = \frac{\sigma}{4N^2}\sum_{i,j=1}^N \|x_i - x_j\|^2 + \frac{c_q}{2N}\sum_{i=1}^N \|x_\gamma - x_i\|^2.$$

This energy functional has a dissipation property. To obtain this, we take the inner product between v_i and $\dot v_i$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_i\|^2 &= -\frac{\|v_i\|^2}{\|x_i\|^2}\langle x_i, v_i\rangle + \sum_{j=1}^N \frac{\sigma}{N}(\|x_i\|^2\langle x_j, v_i\rangle - \langle x_i, x_j\rangle\langle x_i, v_i\rangle)\\ &\quad + c_q(\|x_i\|^2\langle x_\gamma, v_i\rangle - \langle x_i, x_\gamma\rangle\langle x_i, v_i\rangle) - c_p\langle v_i, v_i\rangle.\end{aligned}$$

Using the orthogonality $\langle x_i, v_i\rangle = 0$ and $\|x_i\| = 1$ in (3.13), we have

$$\frac{1}{2}\frac{d}{dt}\|v_i\|^2 = \sum_{j=1}^N \frac{\sigma}{N}\langle x_j, v_i\rangle + c_q\langle x_\gamma, v_i\rangle - c_p\|v_i\|^2.$$

Therefore,

$$\frac{d}{dt}\mathcal{E}_k = \sum_{i,j=1}^N \frac{\sigma}{N^2}\langle x_j, v_i\rangle + \frac{c_q}{N}\sum_{i=1}^N\langle x_\gamma, v_i\rangle - \frac{c_p}{N}\sum_{i=1}^N \|v_i\|^2.$$

Similarly,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_c &= \frac{\sigma}{2N^2}\sum_{i,j=1}^N \langle x_i - x_j, v_i - v_j\rangle - \frac{c_q}{N}\sum_{i=1}^N\langle x_\gamma, v_i\rangle\\ &= -\frac{\sigma}{2N^2}\sum_{i,j=1}^N (\langle x_i, v_j\rangle + \langle x_j, v_i\rangle) - \frac{c_q}{N}\sum_{i=1}^N\langle x_\gamma, v_i\rangle\\ &= -\frac{\sigma}{N^2}\sum_{i,j=1}^N \langle x_j, v_i\rangle - \frac{c_q}{N}\sum_{i=1}^N\langle x_\gamma, v_i\rangle.\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}(\mathcal{E}_k + \mathcal{E}_c) = -\frac{c_q}{N}\sum_{i=1}^N \|v_i\|^2 = -2c_q\mathcal{E}_k.$$

We notice that (5.1) is autonomous, since x_γ is a constant vector. Moreover, the energy functional $\mathcal{E}$ is zero if and only if

$$v_i = 0 \text{ for all } i \in \{1, \dots, N\}.$$

We can easily prove that the union of the following two sets is the maximum invariant set of $\mathcal{E}$.

$$\left\{\{(x_i,v_i)\}_{i=1}^N : v_i = 0, \quad x_i = x_\gamma \text{ for all } i \in \{1,\dots,N\}\right\}$$

and

$$\left\{\{(x_i,v_i)\}_{i=1}^N : v_i = 0, \quad \frac{\sigma}{N}\sum_{j=1}^N x_j + c_q x_\gamma = 0 \text{ for all } i \in \{1,\dots,N\}\right\}.$$

If we assume that $c_q > \sigma$ or $\mathcal{E}(0) < \frac{\sigma}{2}\left(1+\frac{c_q}{\sigma}\right)^2$, then $\frac{\sigma}{N}\sum_{j=1}^N x_j + c_q x_\gamma \neq 0$. Thus, by Lasalle's invariance principle,

$$\|v_i(t)\| \to 0 \quad \text{and} \quad x_i(t) \to x_\gamma$$

as $t \to \infty$. Therefore, we have proved the following proposition.

Proposition 5.1. *If $c_q > \sigma$ or $\mathcal{E}(0) < \frac{\sigma}{2}\left(1+\frac{c_q}{\sigma}\right)^2$, then*

$$v_i(t) \to 0$$

and

$$x_i(t) \to x_\gamma(t)$$

as $t \to \infty$ for any initial data satisfying $x_i(0) \neq -x_\gamma(0)$ for all $i \in \{1,\dots,N\}$.

Next we consider exponential decay estimates for $\|x_i - x_\gamma\|$ and $\|v_i\|$. For notational simplicity, we define the following two functionals.

$$\mathcal{D}_x(t) = \max_{1\le i\le N} \|x_i(t) - x_\gamma(t)\|^2$$

and

$$\mathcal{D}_v(t) = \max_{1\le i\le N} \|v_i(t)\|^2.$$

Proposition 5.2. *Assume that $A_i = 0$. Then for the functional F defined in (4.2), the following estimate holds*

$$\|F\| \le 8(\sigma + c_q)[\mathcal{D}_x(t) + \mathcal{D}_v(t)]X_\gamma^1.$$

Proof. By elementary calculation, we have

$$\begin{aligned}
|F_\gamma^1| &= 0,\\
|F_\gamma^2| &\le \left(\frac{\mathcal{D}_v(t)}{2} + \frac{\sigma\mathcal{D}_x(t)}{4} + \frac{c_q\mathcal{D}_x(t)}{4}\right)X_\gamma^1,\\
|F_\gamma^3| &= 0,
\end{aligned}$$

and

$$
\begin{aligned}
F^1 &= 0,\\
F^2 &= -\frac{1}{N^2}\sum_{i,k=1}^{N}\frac{\|v_i\|^2+\|v_k\|^2}{2}\|x_i-x_k\|^2+\frac{\sigma}{2N^3}\sum_{i,j,k=1}^{N}\|x_i-x_j\|^2\|x_i-x_k\|^2\\
&\quad+\frac{c_q}{2N^2}\sum_{i,k=1}^{N}\|x_\gamma-x_i\|^2\|x_i-x_k\|^2+\frac{1}{N^2}\sum_{i,k=1}^{N}\langle A_i-A_k,x_i-x_k\rangle,\\
F^3 &= \frac{2}{N^2}\sum_{i,k=1}^{N}\big(\|v_i\|^2\langle x_i,v_k\rangle+\|v_k\|^2\langle x_x,v_i\rangle\big)+\frac{2\sigma}{N^3}\sum_{i,j,k=1}^{N}\|x_i-x_j\|^2\langle x_i,v_k\rangle\\
&\quad+\frac{c_q}{N^2}\sum_{i,k=1}^{N}\|x_\gamma-x_i\|^2\langle x_i,v_k\rangle+\frac{2}{N^2}\sum_{i,k=1}^{N}\langle A_i-A_k,v_i-v_k\rangle.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\|x_i-x_k\|^2 &\le \|x_i-x_\gamma+x_\gamma-x_k\|^2\\
&\le 2\|x_i-x_\gamma\|^2+2\|x_\gamma-x_k\|^2\\
&\le 4\mathcal{D}_x(t),
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
|\langle x_i,v_k\rangle| &= |\langle x_i-x_k,v_k\rangle|\\
&\le |\langle x_i-x_\gamma,v_k\rangle|+|\langle x_\gamma-x_k,v_k\rangle|\\
&\le \mathcal{D}_x(t)+\mathcal{D}_v(t).
\end{aligned}
\tag{5.3}
$$

By (5.2) and (5.3), we have

$$
|F^2| \le \frac{\mathcal{D}_v(t)}{N^2}\sum_{i,k=1}^{N}\|x_i-x_k\|^2+\frac{2\sigma\mathcal{D}_x(t)}{N^2}\sum_{i,k=1}^{N}\|x_i-x_k\|^2+\frac{c_q\mathcal{D}_x(t)}{2N^2}\sum_{i,k=1}^{N}\|x_i-x_k\|^2,
$$

and

$$
\begin{aligned}
|F^3| &\le 4(\mathcal{D}_x(t)+\mathcal{D}_v(t))\frac{1}{N}\sum_{i=1}^{N}\|v_i\|^2+\frac{2\sigma(\mathcal{D}_x(t)+\mathcal{D}_v(t))}{N^2}\sum_{i,j=1}^{N}\|x_i-x_j\|^2\\
&\quad+\frac{c_q(\mathcal{D}_x(t)+\mathcal{D}_v(t))}{N}\sum_{i=1}^{N}\|x_\gamma-x_i\|^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{1}{N^2}\sum_{i,k=1}^{N}\|x_i-x_k\|^2 &= \frac{1}{N^2}\sum_{i,k=1}^{N}\|x_i-x_\gamma+x_\gamma-x_k\|^2\\
&\le 4X_\gamma^1.
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
&|F^1| = 0,\\
&|F^2| \le \left(4\mathcal{D}_v(t)+8\sigma\mathcal{D}_x(t)+2c_q\mathcal{D}_x(t)\right)X_\gamma^1,\\
&|F^3| \le (8\sigma+c_q)\left(\mathcal{D}_x(t)+\mathcal{D}_v(t)\right)X_\gamma^1+4(\mathcal{D}_x(t)+\mathcal{D}_v(t))X_\gamma^3.
\end{aligned}
$$

The above implies the result in this lemma.

□

We are ready to prove Theorem 2. We first check that the coefficient matrix M has the following six eigenvalues.

$$\left\{-c_p,\ -c_p,\ -c_p \pm \sqrt{-4c_q + c_p^2},\ -c_p \pm \sqrt{-4c_q + c_p^2 - 4\sigma}\right\}.$$

Thus, their real parts are all negative. Let D be the greatest real part of the above eigenvalues and we define

$$\mu := -D > 0.$$

Then by Proposition 5.1, for any $\epsilon > 0$, there is $t_0 > 0$ such that if $t > t_0$, then

$$0 \le \mathcal{D}_x(t) + \mathcal{D}_v(t) < \frac{\epsilon}{4\,(1 + 2\sigma + 2c_q)}.$$

From Proposition 4.1 and 5.2, it follows that

$$X(t) = e^{A(t-t_0)}X(t_0) + \int_{t_0}^{t} e^{A(t-s)}F(s)ds.$$

This implies that

$$\begin{aligned}\|X(t)\| &\le e^{-\mu(t-t_0)}\|X(t_0)\| + \int_{t_0}^{t} e^{-\mu(t-s)}\|F(s)\|ds \\ &\le e^{-\mu(t-t_0)}\|X(t_0)\| + \epsilon\int_{t_0}^{t} e^{-\mu(t-s)}\|X(s)\|ds.\end{aligned}$$

Therefore, by the Gronwall inequality, if $t > t_0$, then

$$\|X(t)\| \le \|X(t_0)\|e^{-(\mu-\epsilon)(t-t_0)}.$$

5.2. **Practical rendezvouses.** In this part, we consider the target tracking problem without acceleration information of the target. We assume that $\sigma_{ij} = \sigma > 0$ and target speed and acceleration are bounded:

$$\|w_\gamma(t)\|, \|\dot{w}_\gamma(t)\| < C_\gamma^w, \quad t \ge 0,$$

where $C_\gamma^w > 0$ is a positive constant. We assume that $U_i = 0$. We first check that the coefficient matrix M_∞ in Proposition 4.3 has the following eigenvalues.

$$\left\{-c_p,\ -c_p \pm \sqrt{-4c_q + c_p^2}\right\}.$$

Thus, their real parts are all negative. Let D_∞ be the greatest real part of the above eigenvalues and we define

$$\mu_\infty := -D_\infty > 0.$$

Let

$$X_\infty = \max_{1\le i\le N} \|X_i\|.$$

By Proposition 4.3, for any fixed $t > 0$, there is an index $i_t \in \{1, \dots, N\}$ such that

$$\|X_{i_t}\| = X_\infty$$

and

$$\begin{aligned}\frac{d}{dt}X_\infty^2 &= \frac{d}{dt}\|X_{i_t}\|^2 \\ &= \langle X_{i_t}, M_\infty X_{i_t}\rangle + \langle X_{i_t}, F_{i_t}\rangle \\ &\le -\mu_\infty\|X_{i_t}\|^2 + \|X_{i_t}\|\|F_{i_t}\| \\ &= -\mu_\infty X_\infty^2 + X_\infty\|F_{i_t}\|.\end{aligned}$$

By direct calculation,

$$
\begin{aligned}
&|F_i^1| = 0,\\
&|F_i^2| \le \frac{\|v_i\|^2}{2}\frac{16\|x_i-x_\gamma\|^2}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{16\sigma}{N\left(4-\|x_i-x_\gamma\|^2\right)^2}\sum_{j=1}^{N}|\langle x_i-x_\gamma, x_j-\langle x_i,x_j\rangle x_i\rangle|\\
&\qquad + \frac{16|\langle x_i-x_\gamma, A_i\rangle|}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{64\langle x_i-x_\gamma, v_i\rangle^2}{\left(4-\|x_i-x_\gamma\|^2\right)^3}\\
&|F_i^3| \le \frac{32\sigma}{N\left(4-\|x_i-x_\gamma\|^2\right)^2}\sum_{j=1}^{N}|\langle v_i, x_j\rangle| + \frac{32|\langle v_i, A_i\rangle|}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{64\langle v_i,v_i\rangle|\langle x_i-x_\gamma, v_i\rangle|}{\left(4-\|x_i-x_\gamma\|^2\right)^3}.
\end{aligned}
$$

We note that

$$
\begin{aligned}
\langle x_i-x_\gamma, x_j-\langle x_i,x_j\rangle x_i\rangle &= \langle x_i-x_\gamma, x_j-x_\gamma\rangle + \langle x_i-x_\gamma, x_\gamma\rangle - \langle x_i-x_\gamma, \langle x_i,x_j\rangle x_i\rangle\\
&= \langle x_i-x_\gamma, x_j-x_\gamma\rangle - \frac{1}{2}\|x_i-x_\gamma\|^2 - \frac{\langle x_i,x_j\rangle}{2}\|x_i-x_\gamma\|^2.
\end{aligned}
$$

This implies that

$$
|\langle x_i-x_\gamma, x_j-\langle x_i,x_j\rangle x_i\rangle| \le 2\max_{1\le i\le N}\|x_i-x_\gamma\|^2.
$$

Similarly,

$$
|\langle v_i,x_j\rangle| = |\langle v_i, x_j-x_i\rangle| \le |\langle v_i, x_j-x_\gamma\rangle| + |\langle v_i, x_\gamma-x_i\rangle| \le \|v_i\|^2 + \max_{1\le i\le N}\|x_i-x_\gamma\|^2,
$$

$$
\langle x_i-x_\gamma, v_i\rangle^2 \le 4\|v_i\|^2.
$$

Thus,

$$
\begin{aligned}
&|F_i^1| = 0,\\
&|F_i^2| \le 2X_\infty + \frac{32\sigma\max_{1\le i\le N}\|x_i-x_\gamma\|^2}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{16|\langle x_i-x_\gamma, A_i\rangle|}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{256\|v_i\|^2}{\left(4-\|x_i-x_\gamma\|^2\right)^3},\\
&|F_i^3| \le 2\sigma X_\infty + \frac{32\sigma\max_{1\le i\le N}\|x_i-x_\gamma\|^2}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{32|\langle v_i, A_i\rangle|}{\left(4-\|x_i-x_\gamma\|^2\right)^2} + \frac{256\|v_i\|^3}{\left(4-\|x_i-x_\gamma\|^2\right)^3}.
\end{aligned}
$$

By elementary calculation, we have

$$
|\langle x_i-x_\gamma, A_i\rangle| \le \|x_i-x_\gamma\|^2 + \frac{\|A_i\|^2}{4}.
$$

Note that

$$
\|A_i\|^2 \le 6\|w_\gamma\|^2\|S_\gamma^{-1}(t)p_i\|^2 + 3\|\dot{w}_\gamma\|^2.
$$

Since $p_i(t) = W_\gamma^t S_\gamma^t x_i(t) + S_\gamma^t \dot{x}_i(t)$,

$$
\|S_\gamma^{-1}(t)q_i\|^2 \le 2\|w_\gamma\|^2 + 2\|v_i\|^2
$$

and

$$
\|A_i\|^2 \le 12(C_\gamma^w)^2\|v_i\|^2 + 12(C_\gamma^w)^4 + 3(C_\gamma^w)^2.
$$

Therefore, we have

$$
|\langle x_i-x_\gamma, A_i\rangle| \le \|x_i-x_\gamma\|^2 + 3(C_\gamma^w)^2\|v_i\|^2 + 3(C_\gamma^w)^4 + \frac{3(C_\gamma^w)^2}{4}. \tag{5.4}
$$

Similarly, we have

$$
|\langle v_i, A_i\rangle| \le \|v_i\|^2 + 3(C_\gamma^w)^2\|v_i\|^2 + 3(C_\gamma^w)^4 + \frac{3(C_\gamma^w)^2}{4}. \tag{5.5}
$$

By (5.4)-(5.5) and the above argument, if $\max_{1\le i\le N}\|x_i - x_\gamma\| < \frac{2\sqrt{C_1-1}}{\sqrt{C_1}} < 2$, then

$$
\begin{aligned}
&|F_i^1| = 0,\\
&|F_i^2| \le (2+2\sigma C_1 + 5C_1 + 3(C_\gamma^w)^2)X_\infty + 3(C_\gamma^w)^4 C_1^2 + \frac{3(C_\gamma^w)^2 C_1^2}{4},\\
&|F_i^3| \le \left(2+2\sigma+2\sigma C_1 + 6(C_\gamma^w)^2\right)X_\infty + 6(C_\gamma^w)^4 C_1^2 + \frac{6(C_\gamma^w)^2 C_1^2}{4} + 4X_\infty^{3/2}.
\end{aligned}
$$

We conclude that

$$\|F_i\| \le \left(4+2\sigma+4\sigma C_1+5C_1+9(C_\gamma^w)^2\right)X_\infty + 9(C_\gamma^w)^4 C_1^2 + \frac{9(C_\gamma^w)^2 C_1^2}{4} + 4X_\infty^{3/2}.$$

Therefore, we obtain that

$$\dot{X}_\infty \le -\mu_\infty X_\infty + \left(4+2\sigma+4\sigma C_1+5C_1+9(C_\gamma^w)^2\right)X_\infty + 9(C_\gamma^w)^4 C_1^2 + \frac{9(C_\gamma^w)^2 C_1^2}{4} + 4X_\infty^{3/2}. \tag{5.6}$$

We choose c_q and c_p sufficiently large and take

$$X_\infty(0) = \frac{\sqrt{C_1-1}}{\sqrt{C_1}}.$$

Let $T>0$ be a maximal number such that on $t\in[0,T)$,

$$\max_{1\le i\le N}\|x_i(t)-x_\gamma(t)\| < 2X_\infty(0), \quad t\in[0,T). \tag{5.7}$$

By the initial condition and the continuity of the solution, there is a positive number $T>0$ satisfying (5.7). We claim that if c_q and c_p are sufficiently large, then $T=\infty$. We note that for a given initial data, σ, C_1, C_γ^w are fixed constants. Therefore, on $t\in[0,T)$,

$$\dot{X}_\infty \le -\mu_\infty X_\infty + CX_\infty + C. \tag{5.8}$$

(5.8) implies

$$\dot{X}_\infty \le -\frac{\mu}{2}X_\infty + C, \tag{5.9}$$

if c_q and c_p are sufficiently large. Therefore, by the Gronwall inequality and (5.9),

$$X_\infty(t) \le e^{-\frac{\mu}{2}t}X_\infty(0) + e^{-\frac{\mu}{2}t}\frac{2Ce^{\frac{\mu}{2}t}-2C}{\mu} = e^{-\frac{\mu}{2}t}\left(X_\infty(0)-\frac{2C}{\mu}\right) + \frac{2C}{\mu}.$$

If c_q and c_p are sufficiently large, then μ is sufficiently large and $X_\infty \le X_\infty(0)$. These imply that on $t\in[0,T)$,

$$\max_{1\le i\le N}\|x_i(t)-x_\gamma(t)\| \le X_\infty \le X_\infty(0) < 2X_\infty(0).$$

By the continuity of the solution, we obtain that

$$T=\infty.$$

Finally, by the above, we obtain the following practical rendezvous estimate.

$$X_\infty(t) \le e^{-\frac{\mu}{2}t}\left(X_\infty(0)-\frac{2C}{\mu}\right) + \frac{2C}{\mu}.$$

Thus, we complete the proof of Theorem 3.

6. Simulation results

In this section, we present several numerical simulations for the target tracking problem on the unit sphere and the flat space to verify the asymptotic complete rendezvous and practical rendezvous. We use the fourth-order Runge-Kutta method. We consider six α-agents $\{(q_i, p_i)\}_{i=1}^6$ chasing one target (q_γ, p_γ). We assume that the control law for the target (q_γ, p_γ) is given by

$$u_\gamma(t) = a(\cos t, \sin t, 1),$$

where a is a constant. Throughout this section, we assume that the inter-particle bonding force parameter is given by

$$\sigma = 1.$$

With the extra control law for agents

$$U_i = 2\langle w_\gamma, q_i\rangle(q_i \times p_i) + \dot{w}_\gamma(t) \times q_i,$$

the initial positions and velocities for the agents are randomly chosen in

$$(q_i(0), p_i(0)) \in T\mathbb{S}^2 \cap [-1,1]^3 \times [-1,1]^3$$

as follows:

$$\begin{aligned}
q_1(0) &= (\ \ 0.8132, \ \ 0.4989, -0.2993), & q_2(0) &= (\ \ 0.7198, \ \ 0.4908, \ \ 0.4908),\\
q_3(0) &= (-0.6758, -0.6991, \ \ 0.2330), & q_4(0) &= (-0.7878, \ \ 0.5627, -0.2501),\\
q_5(0) &= (-0.5440, -0.7504, \ \ 0.3752), & q_6(0) &= (-0.8599, -0.3608, \ \ 0.3608),
\end{aligned}$$

and

$$\begin{aligned}
p_1(0) &= (\ \ 0.1028, -0.1884, -0.0347), & p_2(0) &= (-0.1168, \ \ 0.5118, -0.3405),\\
p_3(0) &= (-0.0821, \ \ 0.0857, \ \ 0.0191), & p_4(0) &= (-0.1454, -0.1506, \ \ 0.1189),\\
p_5(0) &= (\ \ 0.2220, -0.1040, \ \ 0.1137), & p_6(0) &= (-0.0003, \ \ 0.3768, \ \ 0.3759).
\end{aligned}$$

The initial data for the target is

$$q_\gamma(0) = (-0.6451, 0.6605, -0.3840) \quad \text{and} \quad p_\gamma(0) = (0.1761, 0.3646, 0.3311).$$

Note that all the initial positions and velocities satisfy the admissible conditions in (1.3). Since $\omega_\gamma = q_\gamma \times p_\gamma$, we can check that

$$\begin{aligned}
U_i &= 2\langle \omega_\gamma, q_i\rangle(q_i \times p_i) + \dot{\omega}_\gamma(t) \times q_i\\
&= 2\langle \omega_\gamma, q_i\rangle(q_i \times p_i) + (\dot{q}_\gamma \times p_\gamma + q_\gamma \times \dot{p}_\gamma) \times q_i\\
&= 2\langle \omega_\gamma, q_i\rangle(q_i \times p_i) + \left(q_\gamma \times \left[-\frac{\|p_\gamma\|^2}{\|q_\gamma\|^2} q_\gamma + \|q_\gamma\|^2 u_\gamma - \langle u_\gamma, q_\gamma\rangle q_\gamma\right]\right) \times q_i\\
&= 2\langle \omega_\gamma, q_i\rangle(q_i \times p_i) + (q_\gamma \times \|q_\gamma\|^2 u_\gamma) \times q_i.
\end{aligned} \tag{6.1}$$

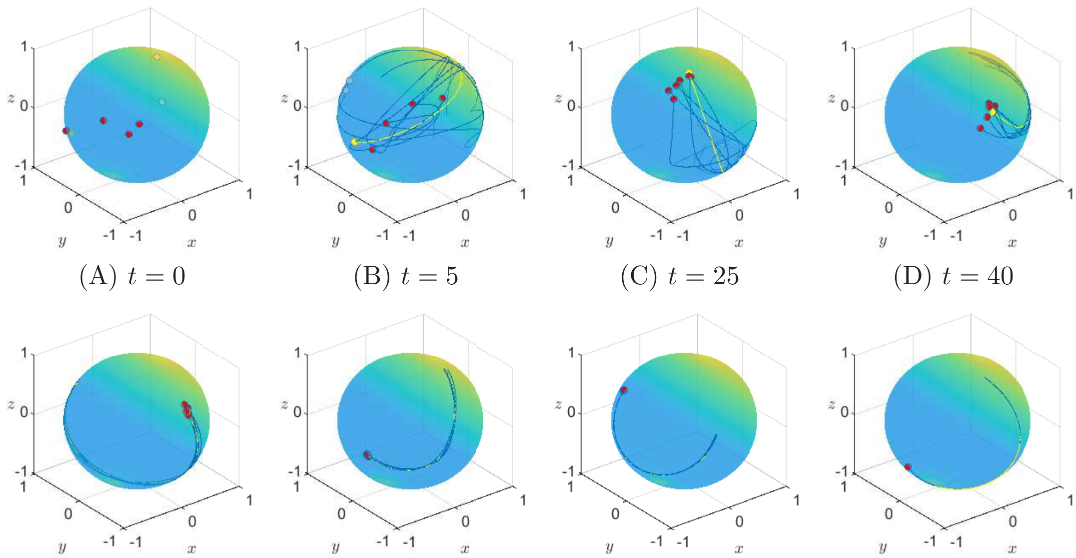
We fix

$$\sigma = 1, \ c_q = 5, \ c_p = 0.1 \quad \text{and} \quad a = 0.5.$$

For this case, the time evolution of (1.2) is given in Figure 1. The red points and blue lines stand for the position $q_i(t)$ at $t = t_0$ and trajectories for the time interval $[t_0 - 3, t_0]$, respectively. The yellow one is for the target agent $q_\gamma(t)$. In addition, we can check that the asymptotic complete rendezvous occurs as we proved in Theorem 2. See Figure 2. Here, the exponential function is $2e^{(-c_p+0.05)(t-40)}$.

For the zero extra control law, i.e. $U_i = 0$, we fix the parameters such that

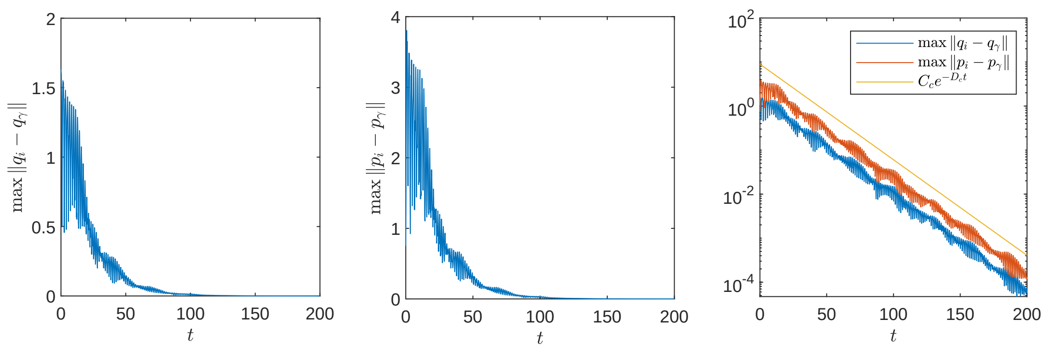
$$\sigma = 1, \ c_q = 4, \ c_p = 4, \ a = 0.5.$$



(A) $t = 0$ (B) $t = 5$ (C) $t = 25$ (D) $t = 40$

(E) $t = 55$ (F) $t = 70$ (G) $t = 100$ (H) $t = 200$

FIGURE 1. The time evolution of (1.2) with extra control law (6.1)



(A) $\max_{1 \le i \le 6} \|q_i(t) - q_\gamma\|$ (B) $\max_{1 \le i \le 6} \|p_i(t) - p_\gamma\|$ (C) Logplot of (A) and (B)

FIGURE 2. The asymptotic complete rendezvous

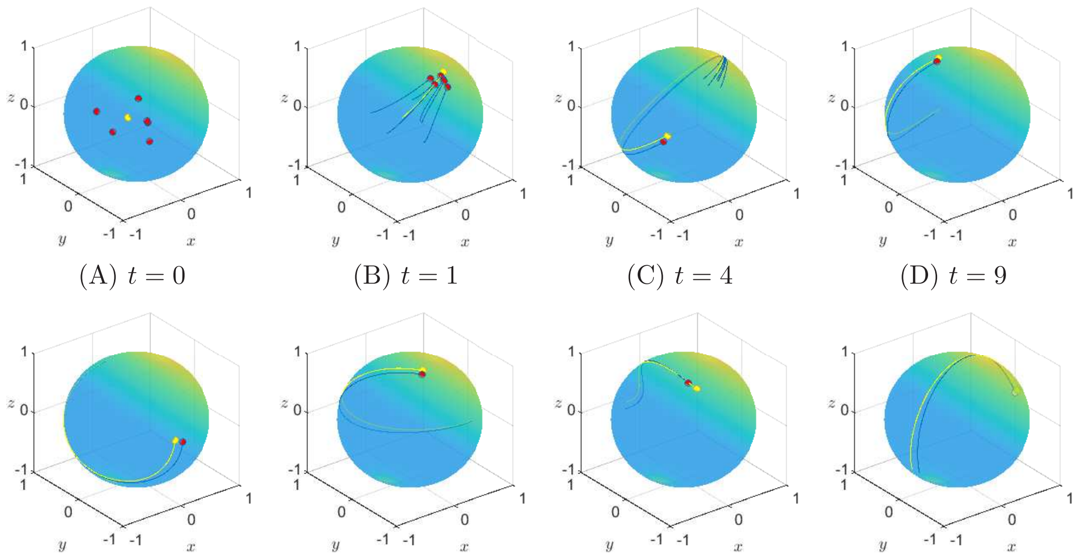
The initial data of agents are randomly chosen but near the target as follows:

$$
\begin{aligned}
&q_1(0) = (-0.8147, -0.5366, \quad 0.2193), && q_2(0) = (-0.4575, -0.8843, \quad 0.0922),\\
&q_3(0) = (-0.4335, -0.8173, \quad 0.3794), && q_4(0) = (-0.8645, -0.2373, \quad 0.4429),\\
&q_5(0) = (-0.4420, -0.7998, \quad 0.4060), && q_6(0) = (-0.4312, -0.6004, \quad 0.6734),
\end{aligned}
$$

and

$$
\begin{aligned}
&p_1(0) = (0.0228, -0.0750, -0.0987), && p_2(0) = (0.2519, -0.1263, \quad 0.0383),\\
&p_3(0) = (0.0200, \quad 0.0169, \quad 0.0594), && p_4(0) = (0.0388, -0.1447, -0.0017),\\
&p_5(0) = (0.0365, \quad 0.1109, \quad 0.2583), && p_6(0) = (0.0081, \quad 0.0050, \quad 0.0097).
\end{aligned}
$$

The initial data for the target is given by



(A) $t = 0$ (B) $t = 1$ (C) $t = 4$ (D) $t = 9$

(E) $t = 35$ (F) $t = 90$ (G) $t = 150$ (H) $t = 200$

FIGURE 3. The time evolution of (1.2) with control law

$$q_\gamma(0) = (-0.6324, -0.6324, 0.4472) \quad \text{and} \quad p_\gamma(0) = (0.4712, -0.1742, 0.4199).$$

Figure 3 shows the time evolution of (1.2) without extra control law.

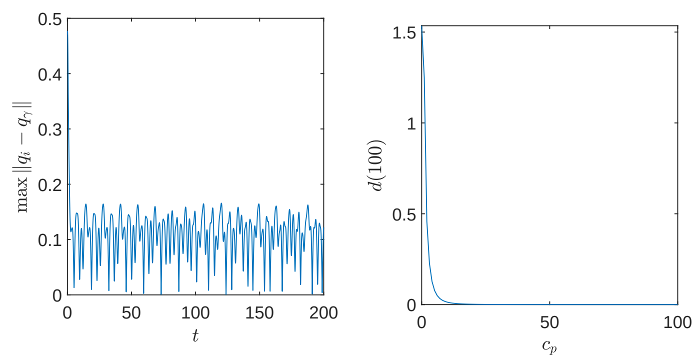
We can see that the maximum distance

$$\max_{1 \le i \le 6} \|q_i(t) - q_\gamma(t)\|$$

between agents and the target is bounded by $2/\sqrt{c_p}$. See Figure 4(A). Let

$$d(t) = \max_{1 \le i \le 6} \|q_i(t) - q_\gamma(t)\|.$$

Figure 4(B) displays $d(t)$ at $t = 100$ with respect to c_p. As c_p increases, the maximum distance between agents and target decreases. Therefore, we observe that the asymptotic practical rendezvous occurs.



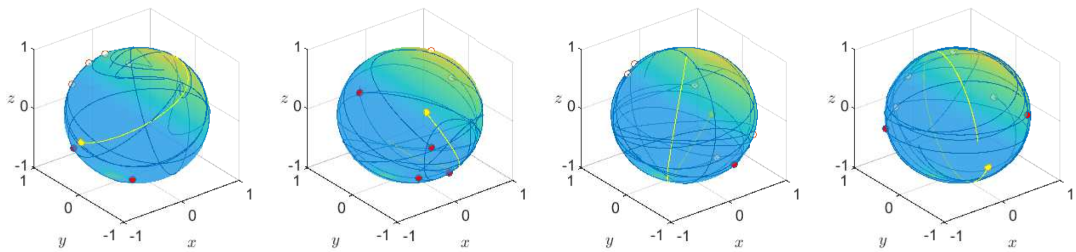
(A) $\max_{1 \le i \le 6} \|q_i(t) - q_\gamma(t)\|$ (B) $d(t)$ at $t = 100$

FIGURE 4. The asymptotic practical rendezvous

With the extra control law, we observed the asymptotic complete rendezvous in Figure 1 and Figure 2. However, if we choose the parameter c_p as zero, then the agents are not able to track the target. See Figure 5. Here, other parameters and initial data are the same as the case in Figure 1. In the absence of the velocity alignment term, the agents easily escape the sphere due to the accumulation of errors. To overcome this, as in [8], we add the following feedback term f_i^0 on the second equation of (1.2).

$$f_i^0 = -k_0\Big(q_i - \frac{q_i}{\|q_i\|}\Big),$$

where $k_0 = 10^4$. From this, we conclude that the velocity alignment operator is crucial in this target tracking algorithm.



(A) $t = 5$ (B) $t = 55$ (C) $t = 100$ (D) $t = 200$

FIGURE 5. The time evolution of (1.2) with extra control law (6.1) and $c_p = 0$

As we mentioned in Subsection 2.2, the flocking term is negligible for the target tracking problem (1.2). With the same parameters of Figure 1 and Figure 3, the numerical results of (1.2) including the rotational flocking term

$$\sum_{j=1}^{N} \frac{\psi_{ij}}{N}(R_{q_j \to q_i}(p_j) - p_i),$$

where $\psi_{ij} = 1$ is given in Figure 6. It is confirmed that the flocking term does not affect the results. See also Figure 7.

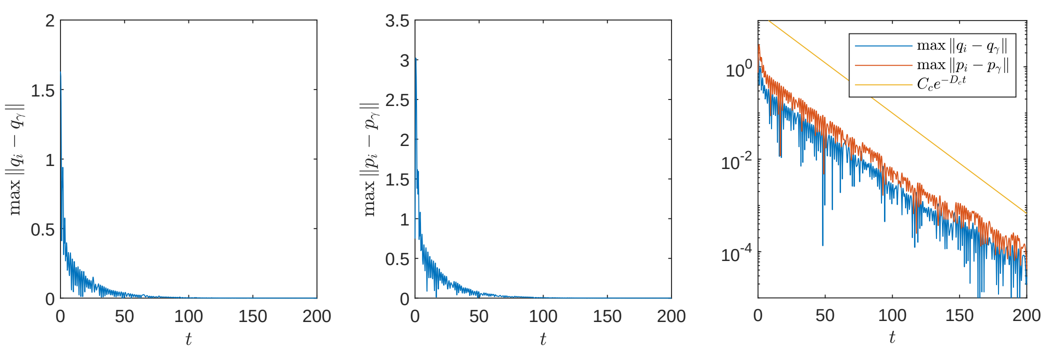


FIGURE 6. The numerical results with flocking term and the same parameters with Figure 2

Finally, we compare the target tracking problems on a sphere and flat space numerically. To compare the two cases, we impose the periodic boundary for the flat space and fix parameters such as $\sigma = 1$, $c_q = 5$, and $c_p = 0.1$. Let

$$u_\gamma = (a\cos t, a\sin t, a),$$

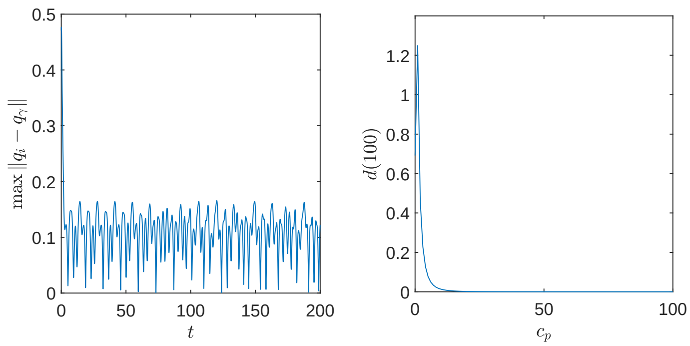
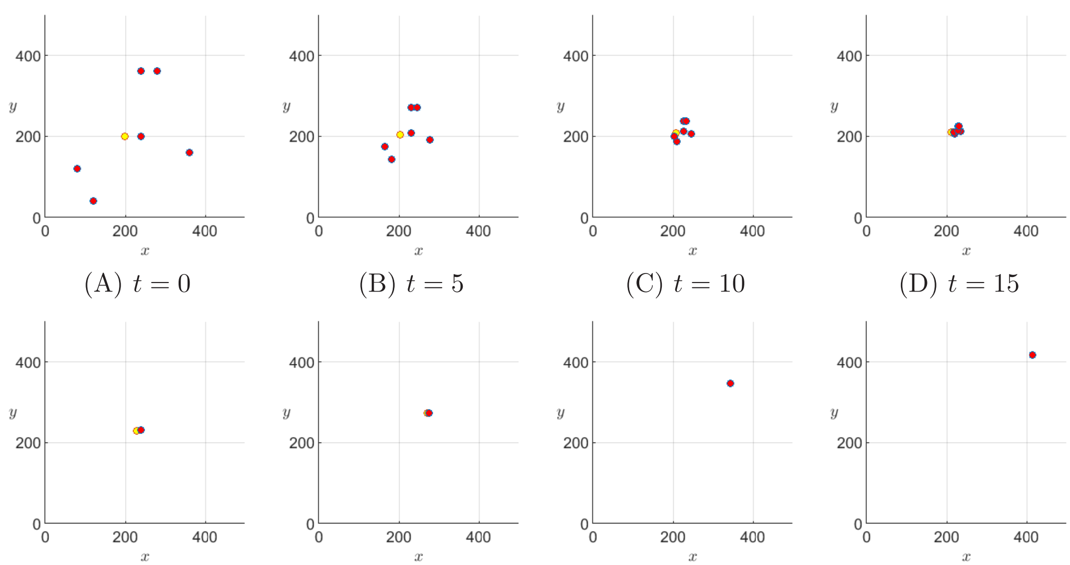


FIGURE 7. The numerical results with flocking term and the same parameters with Figure 4

where $a = 0.5$ and $u_i = u_\gamma$. Then we can observe that the complete rendezvous occurs. See Figure 8. If $u_i = 0$, then we observe the practical rendezvous. See Figure 9.



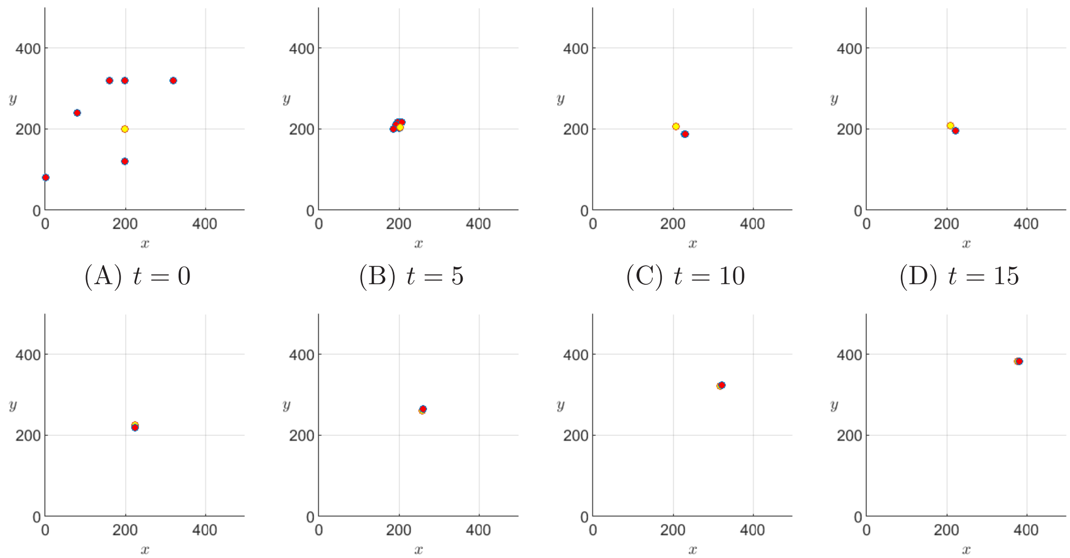
(A) $t = 0$ (B) $t = 5$ (C) $t = 10$ (D) $t = 15$

(E) $t = 40$ (F) $t = 100$ (G) $t = 200$ (H) $t = 300$

FIGURE 8. The snapshops of complete rendezvous on flat space

7. Conclusion

In this paper, we proposed a novel model for target tracking on spherical geometry. With the target's position, velocity, and acceleration, if the initial energy of agents is small or the bonding force between the target and each agent is larger than the one between agents, the complete rendezvous occurs. When only the information of position and velocity is known and the target's angular velocity and its time derivative are bounded, the practical rendezvous is obtained for relatively large intra-bonding forces. The target tracking problems on $\mathbb{S}^2$ with time delay, white noises from the observation, and measurement are also interesting topics. These issues will be discussed in our future researches.



(A) $t = 0$ (B) $t = 5$ (C) $t = 10$ (D) $t = 15$

(E) $t = 40$ (F) $t = 100$ (G) $t = 200$ (H) $t = 300$

FIGURE 9. The snapshops of practical rendezvous on flat space

APPENDIX A. PROPERTIES OF THE ADMISSIBLE ROTATION OPERATOR

In this part, we consider admissible rotation operators on a sphere and their properties. The rotation operator appears naturally for defining the flocking on a sphere [6]. Let $R_{\cdot\to\cdot}$ be Rodrigues' rotation operator given by

$$R_{x_k\to x_i}(v_k) = R(x_k, x_i)\cdot v_k$$

and for $x_k \neq x_i$,

$$R(x_k, x_i) := \langle x_k, x_i\rangle I + x_i x_k^T - x_k x_i^T + (1 - \langle x_k, x_i\rangle)\left(\frac{x_k \times x_i}{|x_k \times x_i|}\right)\left(\frac{x_k \times x_i}{|x_k \times x_i|}\right)^T.$$

Here, x_k, x_i and v_j are three dimensional column vectors. The rotation operator $R_{\cdot\to\cdot}$ has many good properties we desired or needed to be physically established and we can construct a flocking model by replacing the velocity difference term $v_i - v_j$ in the flat space to $R_{x_j\to x_i}v_j(t) - v_i(t)$. See [6] for the details. However, there are some inconvenient points due to the presence of singularity on $R_{\cdot\to\cdot}$. Therefore, we can naturally ask whether such alternatives can be found.

The idea to find the alternative is as follows. First, classify the properties that the rotation operators must satisfy, and find all the operators that satisfy the properties. Next, we will choose one of those operators that meets our needs. Our option will be the simplest of the possible operators. This form has various advantages. It is convenient to calculate, and it shares most of the good properties of the rotation operator $R_{\cdot\to\cdot}$ previously defined. By removing the singularity, we easily show the global-in-time existence and uniqueness of the new model in (1.2). See [6] for the existence and uniqueness of the model with $R_{\cdot\to\cdot}$.

To construct a unit sphere model with the Newtonian equation, we need a modification of $v_j - v_i$ terms, which is the first motivation of the operators $R_{x_j\to x_i}$ in [6]. As we compute the velocity difference between v_i and v_j at the point x_i, we should transform v_j into a tangential vector of the sphere at x_i. We note that the typical ansatz for the flocking motion on a sphere is circle motions. In order to include circle motions along one great circle, the operator should coincide with a rotation operator in two dimensions, a (x_i, x_j)-plane. In other words, an admissible rotation operator M from

z_1 to z_2 can be a 3×3 matrix such that

$$Mz_1 = z_2, \qquad Mz_2 = 2\langle z_1, z_2\rangle z_2 - z_1, \tag{A.1a}$$

$$\langle Mv, z_2\rangle = 0 \text{ for any } z_1, z_2 \in \mathcal{D} \text{ and } v \in T_{z_1}\mathcal{D}. \tag{A.1b}$$

In the next proposition, we can prove that the admissible choices in (A.1) for the rotation operator are equivalent to the following set.

$$\mathcal{A}_{z_1\to z_2} := \left\{P_{z_1\to z_2} + a(z_1 \times z_2)(z_1 \times z_2)^T + b(z_1 - \langle z_1, z_2\rangle z_2)(z_1 \times z_2)^T : a, b \in \mathbb{R}\right\}, \tag{A.2}$$

where $P_{z_1\to z_2}$ is the operator defined in (1.4).

Proposition A.1. *Suppose that unit vectors z_1 and z_2 are linearly independent. Then, a 3×3 matrix M satisfies* (A.1) *if and only if $M \in \mathcal{A}_{z_1\to z_2}$.*

Proof. As two vectors z_1 and z_2 are perpendicular to $z_1 \times z_2$, operator $P_{z_1\to z_2}$ satisfies (A.1) from the direct computation. Note that $\langle z_1 \times z_2, z_i\rangle = 0$ for $i = 1, 2$. From this motivation, we naturally define

$$M := P_{z_1\to z_2} + a(z_1 \times z_2)(z_1 \times z_2)^T + b(z_1 - \langle z_1, z_2\rangle z_2)(z_1 \times z_2)^T \tag{A.3}$$

for any $a, b \in \mathbb{R}$. Then M satisfies (A.1a). Also, as z_2 is perpendicular to both $z_1 \times z_2$ and $(z_1 - \langle z_1, z_2\rangle z_2)$, we conclude (A.1b).

Conversely, choose any 3×3 matrix M' satisfying (A.1). As z_1 and z_2 are linearly independent, $\{z_2,\ z_1 - \langle z_1, z_2\rangle z_2,\ z_1 \times z_2\}$ are a basis of $\mathbb{R}^3$. Therefore, there are $a, b, c \in \mathbb{R}$ such that

$$M' \frac{z_1 \times z_2}{\|z_1 \times z_2\|^2} = a(z_1 \times z_2) + b(z_1 - \langle z_1, z_2\rangle z_2) + cz_2. \tag{A.4}$$

From (A.1b) and $z_1 \times z_2 \in T_{z_1}\mathcal{D}$, it follows that $c = 0$. Therefore, we conclude that

$$Mz_1 \times z_2 = M'z_1 \times z_2$$

for M given in (A.3). On the other hand, (A.1a) show that

$$M(z_2) = M'(z_2) \quad \text{ and } \quad M(z_1 - \langle z_1, z_2\rangle z_2) = M'(z_1 - \langle z_1, z_2\rangle z_2). \tag{A.5}$$

From (A.4) and (A.5), we obtain that $M = M'$. ☐

The set $\mathcal{A}_{z_1\to z_2}$ includes the rotation operators $R_{z_1\to z_2}$ and $P_{z_1\to z_2}$ given in [6] and (1.4), respectively. Here, if we take the following values in (A.3):

$$a = \frac{1 - \langle z_1, z_2\rangle}{\|z_1 \times z_2\|^2} \quad \text{and} \quad b = 0,$$

then the matrix coincides with $R_{z_1\to z_2}$, which preserves the modulus of each vectors. See Lemma 2.3 in [6]. Among several choices in the admissible set in (A.2), $P_{z_1\to z_2}$ can be regarded as the simplest choice such that $a = b = 0$ in (A.2). Moreover, there is no singularity compared to the previous rotation operator $R_{\cdot\to\cdot}$. In addition to this simplicity, the rotation operator $P_{z_1\to z_2}$ also share the following desired transport properties.

Lemma A.2. *For $z_1, z_2 \in \mathcal{D}$, $P_{z_1\to z_2}$ given in* (1.4) *satisfies* (A.1)*. Furthermore, we have*

$$P^T_{z_1\to z_2} = P_{z_2\to z_1} \tag{A.6}$$

and

$$P^T_{z_1\to z_2} P_{z_1\to z_2}(z_1) = z_1, \quad P^T_{z_1\to z_2} P_{z_1\to z_2}(z_2) = z_2.$$

Proof. As two vectors z_1 and z_2 are perpendicular to $z_1 \times z_2$, the properties in (A.1) follow from the direct computation. Also, since the transpose is the linear operator, we have

$$P^T_{z_1\to z_2} = \langle z_1, z_2\rangle I - z_2 z_1^T + z_1 z_2^T,$$

and we conclude (A.6). From (A.1) and (A.6), it holds that

$$P^T_{z_1\to z_2} P_{z_1\to z_2}(z_1) = P^T_{z_1\to z_2}(z_2) = z_1$$

and

$$P^T_{z_1\to z_2} P_{z_1\to z_2}(z_2) = P^T_{z_1\to z_2}(2\langle z_1, z_2\rangle z_2 - z_1) = 2\langle z_1, z_2\rangle z_1 - (2\langle z_1, z_2\rangle z_1 - z_2) = z_2.$$

□

While the two operators $R_{z_1\to z_2}$ and $P_{z_1\to z_2}$ coincide on the (z_1, z_2)-plane from Lemma A.2, the following lemma gives us one difference between the two operators. We can show that $P_{\cdot\to\cdot}$ gives a map between two tangent spaces although the operator is not a bijection if $\langle z_1, z_2\rangle = 0$.

Lemma A.3. *$P_{z_1\to z_2}|_{T_{z_1}\mathcal{D}}$ is a map from $T_{z_1}\mathcal{D}$ to $T_{z_2}\mathcal{D}$. Furthermore, if $\langle z_1, z_2\rangle \neq 0$, then $P_{z_1\to z_2}|_{T_{z_1}\mathcal{D}}$ is a bijection from $T_{z_1}\mathcal{D}$ to $T_{z_2}\mathcal{D}$.*

Proof. As $\mathcal{D}$ is a unit sphere, $v \in T_y\mathcal{D}$ if and only if $\langle v, y\rangle = 0$ for any $y \in \mathbb{R}^3$. Thus, we have

$$\langle v, z_1\rangle = 0 \quad \text{for any vector } v \in T_{z_1}\mathcal{D}. \tag{A.7}$$

From (A.1a) and (A.6), it holds that for any $v \in \mathbb{R}^3$,

$$(P_{z_1\to z_2} v)\cdot z_2 = v^T P^T_{z_1\to z_2} z_2 = v^T P_{z_1\to z_2} r z_2 = v^T z_1 = \langle v, z_1\rangle. \tag{A.8}$$

By (A.7) and (A.8), we conclude that

$$(P_{z_1\to z_2} v)\cdot z_2 = 0 \text{ and thus } P_{z_1\to z_2} v \in T_{z_2}\mathcal{D} \text{ for any vector } v \in T_{z_1}\mathcal{D}.$$

We now assume that $\langle z_1, z_2\rangle \neq 0$ and show that $P_{z_1\to z_2}|_{T_{z_1}\mathcal{D}}$ is bijective between two tangent spaces. First, if $z_1 = z_2$ or $z_1 = -z_2$, we get $P_{z_1\to z_2} = I$ and $P_{z_1\to z_2} = -I$. If not, z_1 and z_2 are linearly independent. From the assumption, $P_{z_1\to z_2}(z_1 \times z_2) = \langle z_1, z_2\rangle (z_1 \times z_2)$ is a nonzero vector. Combining this with (A.1a), we conclude that $P_{z_1\to z_2}|_{T_{z_1}\mathcal{D}}$ is surjective in $T_{z_2}\mathcal{D}$ and thus the determinant of $P_{z_1\to z_2}$ is nonzero. As the inverse function of $P_{z_1\to z_2}$ exists, we conclude that this lemma holds. □

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