

INITIAL PERTURBATION OF THE MEAN CURVATURE FLOW FOR ASYMPTOTICAL CONICAL LIMIT SHRINKER

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ABSTRACT. This is the second paper in the series to study the initial perturbation of mean curvature flow. We study the initial perturbation of mean curvature flow, whose first singularity is modeled by an asymptotic conical shrinker. The noncompactness of the limiting shrinker creates essential difficulties. We introduce the Feynman-Kac formula to get precise asymptotic behaviour of the linearized rescaled mean curvature equation along an orbit. We also develop the invariant cone method to the non-compact setting for the local dynamics near the shrinker. As a consequence, we prove that after a generic initial perturbation, the perturbed rescaled mean curvature flow avoids the conical singularity.

1. INTRODUCTION

In this paper, we extend the idea in our previous paper [SX] to study the initial perturbation of MCF whose first singularity is modeled by an asymptotically conical self-shrinker.

A *mean curvature flow* (MCF) is a family of closed embedded hypersurfaces $\{\mathbf{M}_t\}$ in \mathbb{R}^{n+1} satisfying the equation $\partial_t x = -H\mathbf{n}$. Here x is the position vector, H is the mean curvature, which is minus the trace of the second fundamental forms, and \mathbf{n} is the outward unit normal vector field over M_t . It is known that an MCF always develops singularities within finite time, so the analysis of singularity blowup becomes a central topic when studying MCF. After a spacetime rescaling, an MCF can be turned to a *rescaled mean curvature flow* (RMCF), satisfying the equation

$$(1.1) \quad \partial_t x = - \left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right) \mathbf{n},$$

where $\left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right)$ is called the rescaled mean curvature. MCF and RMCF are related as follows: if $\{\mathbf{M}_\tau\}_{\tau \in [-1, 0]}$ is an MCF, then

$$(1.2) \quad M_t = e^{t/2} \mathbf{M}_{-e^{-t}}$$

is its corresponding RMCF for $t \in [0, \infty)$. A static hypersurface under RMCF is called a *self-shrinker*, which satisfies the equation $H - \frac{\langle x, \mathbf{n} \rangle}{2} = 0$. RMCF was first introduced by Huisken in [H] to study the singularity of MCF. We say a singularity of MCF is modeled by a self-shrinker Σ , if the corresponding RMCF converges to Σ smoothly as time goes to

infinity. If Σ is non-compact, then the convergence is in the sense of C_{loc}^∞ , namely on any compact subset the convergence is smooth.

Colding-Minicozzi introduced ideas from dynamical systems to study MCF (c.f. [CM1, CM2, CM3, CM4] etc). Their dynamical approach views the RMCF (1.1) as the negative gradient flow of the F -functional

$$F(M_t) = \int_{M_t} e^{-\frac{|x|^2}{4}} d\mu$$

and shrinkers as the critical points of F . So it is natural to anticipate that generic RMCF avoids those shrinkers that are saddles in the second variation, modulo translations and dilations. In [CMP], Colding-Minicozzi-Pedersen proposed the conjecture that one can perturb the initial data of an MCF so that the MCF will only encounter singularities modeled by generic self-shrinkers (c.f. [CMP, Conjecture 8.2]). Since Euclidean rigid motion does not change the MCF essentially, in [CM1] Colding-Minicozzi also introduced the entropy

$$\lambda(M) = \sup_{x \in \mathbb{R}^{n+1}, t \in (0, \infty)} F(t^{-1}(M - x))$$

to modulo the translations and dilations. In [SX], we made progress to Colding-Minicozzi's dynamical program. We studied the initial perturbation of a MCF whose first singularity is modeled by a closed embedded self-shrinker.

In this paper, we make further progress to Colding-Minicozzi's program. We study the initial perturbation of MCF whose first singularity is unique and modeled by an asymptotic conical self-shrinker. A self-shrinker Σ is called *asymptotically conical* if after blowing down it converges to a cone. More precisely, $\tau^{-1}\Sigma$ converges to a cone Γ smoothly on any compact subset of $\mathbb{R}^{n+1} \setminus \{0\}$ as $\tau \rightarrow \infty$. We make the following standing assumption throughout the paper if not otherwise mentioned.

(\star) Let $(\mathbf{M}_\tau)_{\tau \in [-1, 0]}$ be an MCF with an unique first-time singularity at the spacetime point $(0, 0)$, and let $(M_t)_{t \in [0, \infty)}$ be the corresponding RMCF with $M_t \rightarrow \Sigma$ in the C_{loc}^∞ sense as $t \rightarrow \infty$, where Σ is an asymptotically conical self-shrinker.

Our main theorem is as follows.

Theorem 1.1. *Assume (\star). Then there exist $\delta_0 > 0$ and an open dense subset \mathcal{S} of $\{u \in C^{2, \alpha}(M_0) \mid \|u\|_{C^{2, \alpha}} = 1\}$, such that for any $0 < \delta < \delta_0$ and any $u_0 \in \mathcal{S}$, there exists $\varepsilon_0 := \varepsilon_0(u_0)$, such that for all $0 < \varepsilon < \varepsilon_0$, there exists $T > 0$, such that the RMCF $\{\widetilde{M}_t\}$ starting from $\widetilde{M}_0 := \{x + \varepsilon u_0(x)\mathbf{n}(x) \mid x \in M_0\}$ satisfies*

$$F(\widetilde{M}_T) < \lambda(\Sigma) - \delta^{2.5}.$$

Moreover, there exists $R = R(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, such that

- (1) $F(\widetilde{M}_T \setminus B_R) < \delta^3$,
- (2) $\widetilde{M}_T \cap B_R$ can be written as the graph of a function $\bar{u}(T) : \Sigma \cap B_R \rightarrow \mathbb{R}$ with $\|\bar{u}(T)\|_{C^{2, \alpha}} = \delta$,
- (3) $F(\mathcal{R}\widetilde{M}_T) < \lambda(\Sigma) - \delta^{2.5}$ for any translation and dilation \mathcal{R} of scale δ .

By Huisken's monotonicity formula, we immediately obtain the following corollary.

Corollary 1.2. *In the setting of Theorem 1.1, there exists a spacetime neighbourhood of $(0, 0)$ with size ε , such that $\{\widetilde{M}_t\}$ has no singularity modeled by Σ in this neighbourhood.*

In other words, after an initial generic perturbation, any singularity (if there is one) in a spacetime neighbourhood of the original singularity can not be modeled by an asymptotic conical self-shrinker Σ . Moreover, the neighborhood is larger than that created by translations and dilations of the same order of magnitude of the initial perturbation.

In [SX], we study the problem of initial perturbation for MCF with compact singularities and obtained a stronger conclusion that after a generic initial perturbation, the perturbed MCF will never generate a singularity modeled by the original limit closed self-shrinker. Here we can only avoid a conical self-shrinker in a spacetime neighbourhood. The difference between these two results illustrates the nature of non-compactness. A similar issue appears in the study of higher multiplicity singularities, see [Su].

As an application, we can study the behaviour of the perturbed RMCF near the original singularity. Our first application shows that after appropriate rescalings, the perturbed MCF will converge to an ancient solution.

Theorem 1.3. *Assume (\star) . Suppose u_0 is a generic smooth function on M_0 with unit $C^{2,\alpha}$ -norm and $\epsilon_i \rightarrow 0$ is a sequence of positive number. Suppose $\{\widetilde{M}_t^i\}$ is the RMCF starting from $M_0 + \epsilon_i u_0 \mathbf{n}$. Then there exists a sequence $\{T_i\}_{i=1}^\infty$, $T_i \rightarrow \infty$ as $i \rightarrow \infty$, such that $\{\widetilde{M}_{t+T_i}\}$ smoothly converging to an ancient RMCF $\{N_t\}_{t \in (-\infty, 0)}$ on any compact spacetime subset, and N_t is not the static flow Σ .*

In [SX], we have proved similar existence results of ancient solutions when the limit shrinker is compact. In [CCMS1], Chodosh-Choi-Mantoulidis-Schulze proved the existence of ancient solutions coming out from an asymptotic conical self-shrinker with dimensional assumption $n \leq 6$ due to the requirement in geometric measure theory. [CCMS1] studied the problem of initial perturbation using the geometric measure theory method, while here we use purely PDE and dynamical system method. Our approach has the disadvantage of being unable to handle multiple singularities, while it also enjoys some merits such as being free of dimension or low entropy assumptions and of allowing generic perturbations, not necessarily one-sided.

In the following, we discuss related backgrounds and give literature reviews.

1.1. Generic MCF. It is known that a closed MCF in \mathbb{R}^{n+1} must generate finite-time singularities, and the singularities are modeled by self-shrinkers (c.f. [H, I, Wh]). The self-shrinkers are minimal surfaces in the Gaussian metric space, and there are many constructions of self-shrinkers (see [KKM], [Ngu] etc). It seems impossible to classify all embedded self-shrinkers even in \mathbb{R}^3 . Therefore, it is very complicated to understand the singular behaviour of an MCF.

Generic MCF is proposed to overcome this issue. The concept of generic MCF was firstly proposed by Huisken in [H] (c.f. [AIC] for similar ideas in the study of MCF in \mathbb{R}^3). Colding-Minicozzi in [CM1] formulated the notion of stability and classified the generic shrinkers that

are spheres \mathbb{S}^n and cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$, $k = 0, 1, \dots, n-1$. The general principle of *survival of the stable* thus hints that a generic MCF avoids all singularities that are non spherical and non cylindrical. The idea of [CM1] is to study the linearization of (1.1) at a shrinker Σ . The linearized equation has the form $\partial_t u = L_\Sigma u$ where $L_\Sigma = \Delta_\Sigma - \frac{1}{2}\langle x, \nabla_\Sigma \cdot \rangle + (|A|^2 + \frac{1}{2})$ is self-adjoint with respect to the Gaussian weighted inner product $\langle u, v \rangle = \int_\Sigma u(x)v(x)e^{-\frac{|x|^2}{4}} d\mu$. It is known that the mean curvature H of Σ is the eigenfunction of L_Σ with eigenvalue 1, i.e. $L_\Sigma H = H$ (we remark that we use different sign convention than [CM1] for the definition of eigenvalues for the purpose of studying dynamics). Moreover, from elliptic operator theory it is known that the leading eigenfunction does not change sign. Shrinkers with positive H are classified by Huisken and Colding-Minicozzi to be spheres and cylinders. So for a non spherical and non cylindrical shrinker, the leading eigenfunction ϕ_1 cannot be H hence the leading eigenvalue λ_1 has to be larger than 1. The idea of [CM1] is then to perturb Σ in the direction of ϕ_1 , which can decrease the entropy strictly.

Note that the perturbations in [CM1] are constructed on the shrinker Σ hence are local in nature. To avoid the shrinker by perturbing the initial condition, we have to control the perturbed RMCF all the way up to leaving a neighborhood of the shrinker. Our main theorem is a consequence of the following key estimate on the global dynamics, which shows that a generic initial perturbation realizes the local perturbation in the ϕ_1 direction used by [CM1].

Theorem 1.4. *In the setting of Theorem 1.1, we have*

$$|\langle \bar{u}(T), \phi_1 \rangle_{W^{1,2}}| \geq (1 - o_\delta(1)) \|\bar{u}(T)\|_{W^{1,2}},$$

where ϕ_1 is the first eigenfunction of the linearized operator on Σ with L^2 -norm 1.

In other words, \widetilde{M}_t drifts to the most unstable direction on Σ .

The proofs of the above stated-theorems are given in Section 4.

1.2. Two dynamical problems. From our work in the compact case [SX], we see that one of the main ingredients is to study the asymptotic behavior of positive solutions to the linearized RMCF equation (also called variational equation)

$$(1.3) \quad \partial_t u = L_{M_t} u,$$

where (M_t) is an RMCF converging to a limit shrinker Σ as $t \rightarrow \infty$ in the C_{loc}^∞ sense. This equation can be considered as the Jacobi equation along an RMCF which governs how nearby orbits diverge. Our dynamical approach to the problem of initial perturbations consists of the study of the following two dynamical problems:

- (1) The asymptotic behavior of the solution to (1.3). In particular, we want to find initial condition $u(0) : M_0 \rightarrow \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|u(t)\|_{L^2(M_t)} = \lambda_1(\Sigma), \quad \frac{u(t)}{\|u(t)\|_{L^2(M_t)}} \rightarrow \phi_1.$$

One easily recognize in this case that $\lambda_1(\Sigma)$ is the leading Lyapunov exponent.

(2) The local (nonlinear) dynamics of the RMCF near the shrinker.

In [SX], we studied the case when Σ is a compact shrinker. The first dynamics problem is addressed by a Harnack estimate given by the Li-Yau estimate. For the second dynamical problem, we write the RMCF equation as $\partial_t u = L_\Sigma u + Q(u, \nabla u, \nabla^2 u)$ in a neighborhood of Σ where each manifold corresponds to the graph of a function u . This problem can be approached by the invariant manifold theory in hyperbolic dynamics, so the dynamics of RMCF in a neighborhood of the shrinker is approximately that of the linear equation $\partial_t u = L_\Sigma u$.

When Σ is noncompact, serious issues arise in both problems. For the first problem, we do not have a uniform Harnack estimate, and the Li-Yau estimate gets worse as time gets longer. In this paper, we introduce a Feynman-Kac representation of the solutions to (1.3), which enables us to prove the following theorem, hence address the first problem (see Section 3.2).

Theorem 1.5. *Let $\{M_t\}_{t \in [0, \infty)}$ be an RMCF with $M_t \rightarrow \Sigma$ as $t \rightarrow \infty$ in the C_{loc}^∞ sense, where Σ is a shrinker that either is compact or satisfies*

- (1) $\limsup_t \lambda_1(M_t) \leq \lambda_1(\Sigma)$;
- (2) *there exists constant $D > 0$ such that $\lambda_1(M_t) - \lambda_2(M_t) \geq D$,*

as $t \rightarrow \infty$ where $\lambda_1(M_t)$ (resp. $\lambda_1(\Sigma)$) is the leading eigenvalue of L_{M_t} (resp. L_Σ) and $\lambda_2(M_t)$ is the second. Let v^ be the solution to the initial value problem (1.3) with initial condition $v_0^* > 0$.*

Then we have

- (1) $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|v^*(t)\|_{L^2(t)} = \lambda_1(\Sigma)$, *the leading eigenvalue of L_Σ ;*
- (2) *Let $\phi_1(t)$ be the first eigenfunction of L_{M_t} on M_t with L^2 -norm 1. There exist constants $1 > c > 0, C > 1$ and a sequence of times $t_i \rightarrow \infty$ such that*

$$\|v^*\|_{Q(M_{t_i})} \leq C \|v^*\|_{L^2(M_{t_i})}, \text{ and } \frac{|\langle v^*(t_i), \phi_1(t_i) \rangle_{L^2(M_{t_i})}|}{\|v^*(t_i)\|_{Q(M_{t_i})}} > c.$$

Here the $L^2(M_t)$ norm is the Gaussian weighted L^2 norm for functions on M_t and $Q(M_t)$ is a norm equivalent to the weighted $W^{1,2}$ -norm on M_t .

We shall apply Theorem 1.5 to control the perturbed RMCF \widetilde{M}_t over a long time T so that both M_t and \widetilde{M}_t are δ -close to Σ in the $C^{2,\alpha}$ norm over a large domain. See the red curve in Figure 1. Item (2) of Theorem 1.5 gives that the difference of the two manifolds \widetilde{M}_T and M_T has a nontrivial projection to the ϕ_1 direction. Here comes the second problem. We wish to approximate the local dynamics near Σ by the linear equation $\partial_t u = L_\Sigma u$ and show that the ϕ_1 -component dominates all other Fourier modes when the perturbed flow leaves a δ -neighborhood of Σ . See the blue curve in Figure 1. The main difficulty is that the flow $\{M_t\}$ cannot be written as a global graph over Σ for any time t so that the linear approximation of the local dynamics can only be done by restricting to a compact domain, which makes the system nonautonomous, since the information outside the compact domain

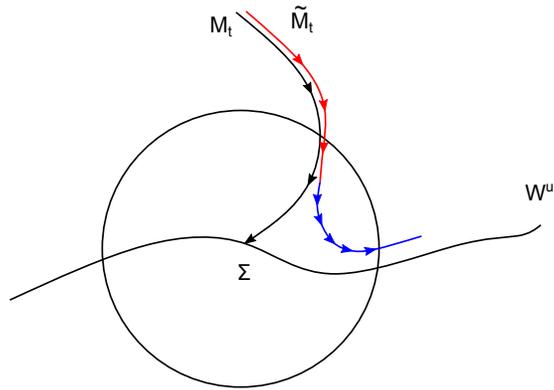


FIGURE 1. Dynamics of the perturbed RMCF \widetilde{M}_t

is discarded. Moreover, the time span for the blue curve in Figure 1, even though is only finite, is rather long depending on the smallness of the initial perturbation.

To overcome this difficulty, one important ingredient in the proof is the pseudo-locality property of MCF. Pseudo-locality is a special property for the nonlinear geometric heat equation, first discovered by Perelman in the setting of Ricci flow, then by Ecker-Huisken in [EH], and later studied by [INS] in the setting of MCF. Roughly speaking, pseudo-locality says that if the MCF is graphical in a small neighbourhood, then it keeps being graphical in a short time. Using the correspondence (1.2) between MCF and RMCF we get that the region close to the singularity in the MCF will be expanded to infinity at an exponential rate in the RMCF. Thus pseudolocality enables us to get control of the dynamics of RMCF over an exponentially growing domain where we can approximate the RMCF using the linear equation $\partial_t u = L_\Sigma u$ over a sufficiently long time. This part will be elaborated in Section 4, where we prove Theorem 1.4.

1.3. Feynman-Kac formula. When the limit self-shrinker Σ is compact, M_t will be very close to Σ when t is sufficiently large. In particular, the geometry of M_t will be uniformly close to Σ , and we can identify the function space of M_t with the function space of Σ . In [SX], this fact is crucial, and it allows us to use a Li-Yau type Harnack inequality, to show that the positive solutions to (1.3) satisfy the estimates in the conclusion (2) of Theorem 1.5.

In the setting of noncompact shrinkers, Li-Yau estimate does not meet our purpose. The new tool we introduce to prove Theorem 1.5 is a Feynman-Kac formula in the setting of RMCF. The Feynman-Kac formula views the equation (1.3) from a dynamical and probabilistic perspective. Indeed, if we consider the dynamical system $\partial_t u = \mathcal{L}_\Sigma u := (\Delta_\Sigma - \frac{1}{2}\langle x, \nabla_\Sigma \cdot \rangle)u$

on a shrinker Σ . The fact (following from the self-adjointness of \mathcal{L}_Σ)

$$\frac{d}{dt} \int_\Sigma u(t) e^{-\frac{|x|^2}{4}} d\mu = \int_\Sigma \mathcal{L}_\Sigma u(t) e^{-\frac{|x|^2}{4}} d\mu = - \int_\Sigma \nabla 1 \cdot \nabla u(t) e^{-\frac{|x|^2}{4}} d\mu = 0$$

can be interpreted as that the dynamical system has $e^{-\frac{|x|^2}{4}} d\mu$ as invariant measure. The situation is then rather similar to the well-known Ornstein-Uhlenbeck process in \mathbb{R}^n . The stochastic differential equation $dX = -Xdt + dW$, where W is the Brownian motion, has the Ornstein-Uhlenbeck operator $\mathcal{L} = \Delta - \langle x, \nabla \cdot \rangle$ as the generator and Gaussian as the invariant measure. Feynman-Kac formula gives a representation of the solution to a linear heat equation in the presence of a potential term, for example, the equation $\partial_t u = \Delta u + Vu$ on \mathbb{R}^n , with initial condition $u(x, 0) = f(x)$ as

$$u(x, t) = \int_\Omega f(\omega(0)) \exp\left(\int_0^t V(\omega(s)) ds\right) d\nu_{x,t}(\omega).$$

where $\nu_{x,t}$ is a probability measure on the “space of all paths” Ω ending at x at time t . We establish a Feynman-Kac type representation of solutions to (1.3) in Section 2. The path integral feature is useful for localizing the linearized equation (1.3) to a neighborhood of the shrinker and to a bounded domain, and enables us to establish the correct exponential growth of the solution in Theorem 1.5. Details are presented in Section 3.

1.4. Asymptotically conical self-shrinkers. Our analysis in this paper in principle should apply to general self-shrinkers and even singularities of other flows. However, we choose to study the singularity modeled by asymptotically conical self-shrinkers for the sake of concreteness and simplicity.

Firstly, asymptotically conical self-shrinkers form an important class of self-shrinkers. In fact, it is not known that whether other types of singularities really exist. Particularly in \mathbb{R}^3 , L. Wang [Wa2] shows that all noncompact shrinkers are only those with finitely many cylindrical or conical ends, and the asymptotics are smooth. Ilmanen [I] conjectured that in \mathbb{R}^3 , any asymptotic cylindrical self-shrinker is actually a standard cylinder. If this conjecture is true, then the self-shrinkers in \mathbb{R}^3 can be classified into three classes: compact, cylinder, and asymptotically conical. The cylinder is known to be generic. Therefore, together with [SX], we know how to perturb away all types of first nongeneric singularities of MCF in \mathbb{R}^3 by generic initial perturbations.

Secondly, Chodosh-Schulze [CS] proved that the tangent flow of an asymptotically conical self-shrinker is unique. Therefore, over any compact region, the RMCF can be written as a graph over the limit shrinker for a sufficiently large time.

Thirdly, asymptotically conical self-shrinkers have some nice properties themselves. For example, in [BW] Bernstein-Wang analyzed the spectrum and eigenfunctions on an asymptotically conical self-shrinker, and prove certain nice bounds. In Section 6, we proved that if the RMCF converging to an asymptotically conical self-shrinker and it models the unique singularity, some geometric quantities converge.

Examples of asymptotically conical self-shrinkers are firstly constructed by Angenent-Ilmanen-Chopp in [AIC] using numerical methods, and later Nguyen [Ngu] and Kapouleas-Kleene-Møller [KKM] constructed examples theoretically. The theory of asymptotically conical self-shrinkers is interesting and has been attracted mathematicians. We refer the readers to [BW] for further detailed discussion on asymptotically conical self-shrinkers.

1.5. Convergence of eigenvalues and eigenfunctions. To apply Theorem 1.5, we have to verify the assumptions on the convergence of eigenvalues. The problem of spectral flow, i.e. how eigenvalues and eigenfunctions of parameter-dependent elliptic operators depend on the parameter, is important in many applications and has been studied widely in literature (c.f. [A, HM, U, Z] etc). However, our setting is rather special since a family of compact manifolds M_t converges to a noncompact one in the C_{loc}^∞ sense and the L^2 -norm is defined with a Gaussian weight.

In Section 6, we prove the following convergence result assuming (\star) , which may have an independent interest.

Theorem 1.6. *Assume (\star) , then we have as $t \rightarrow \infty$*

- (1) $\lambda_1(M_t) \rightarrow \lambda_1(\Sigma)$;
- (2) *there is a constant $D > 0$ such that $\lambda_1(M_t) - \lambda_2(M_t) > D$.*

Section 6 also contains further information on the convergence of eigenfunctions, etc.

1.6. Organization of paper. The paper is organized as follows. In Section 2, we establish the Feynman-Kac formula in the setting of RMCF. In Section 3, we study the asymptotic behavior of the solution to the variational equation using Feynman-Kac. In Section 4, we study the dynamics in a neighborhood of the shrinker, hence completes the proof of the main theorems stated above. In Section 5, we give some graphical estimates for the RMCF close to the shrinker using pseudolocality and Ecker-Huisken, etc. In Section 6, we study the convergence of the leading eigenvalue and eigenfunctions for the L -operator on M_t as $t \rightarrow \infty$. Finally, we have four appendices containing some technical ingredients. In Appendix A, we give estimates for functions on M_t pulled back to Σ on a compact set that we call transplantations. In Appendix B, we introduce polar-spherical transplantation adapted to conical shrinkers and give estimates for the pullback of the L -operator. In Appendix C, we give the proof of Proposition 4.6. In Appendix D, we prove the existence of heat kernel of L on conical shrinker.

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2. FEYNMAN-KAC FORMULA

In this section, we derive a Feynman-Kac formula adapted to the variational equation $u_t = L_{M_t}u$ over the RMCF $\{M_t\}_{t \in [0, \infty)}$. Throughout this section, u, v are functions defined on the RMCF M_t .

2.1. Heat kernels. In this section we study the heat kernel of the equation

$$\partial_t u = L_{M_t}u = \mathcal{L}_{M_t}u + V(x, t)u,$$

where $\{M_t\}_{t \in [0, \infty)}$ is the RMCF with limit $M_t \rightarrow \Sigma$ in C_{loc}^∞ as $t \rightarrow \infty$, V is a smooth potential which is always chosen as 0 or $|A|^2 + \frac{1}{2}$ in the paper and \mathcal{L}_{M_t} is the drifted Laplacian, defined by $\mathcal{L}_{M_t}u = \Delta_{M_t}u - \frac{1}{2}\langle x, \nabla u \rangle$.

Definition 2.1. *The heat kernel of the above equation is a function of the form $\mathcal{H}(x, t; y, s)$, where $x \in M_t$ and $y \in M_s$ and we always assume $s < t$, satisfying*

- (1) $\partial_t \mathcal{H} = \mathcal{L}_{x, t} \mathcal{H} + V \mathcal{H}$,
- (2) $\lim_{t \searrow s} \mathcal{H}(\cdot, t, y, s) e^{-\frac{|y|^2}{4}} = \delta_y$.

With the heat kernel, we can express the solution to the initial value problem

$$(2.1) \quad \begin{cases} \partial_t u = L_{M_t}u \\ u(\cdot, s) = f(\cdot) \end{cases}$$

as

$$(2.2) \quad u(x, t) = \int_{M_s} \mathcal{H}(x, t; y, s) f(y) e^{-|y|^2/4} d\mu_s(y).$$

Of particular importance for us is the following cocycle property.

Theorem 2.2 (cocycle property). *Let \mathcal{H} be the heat kernel for the equation $\partial_t u = \mathcal{L}u + Vu$, then for all $x \in M_r, y \in M_s, z \in M_t$ with $r < s < t$, we have*

$$(2.3) \quad \int_{M_s} \mathcal{H}(z, t; y, s) \mathcal{H}(y, s; x, r) e^{-|y|^2/4} d\mu_s = \mathcal{H}(z, t; x, r).$$

The proof is to take $s \rightarrow 0$ and apply item (2) of Definition 2.1. We refer the readers to [CC, Chapter 26] for the existence of heat kernel and related properties.

2.2. The Trotter product formula for evolving manifolds. The next ingredient is the Trotter product formula. The classical Trotter product formula is as follows: let A and B be two self-adjoint operators bounded from below on a Hilbert space \mathbb{H} and suppose that $A + B$ is self-adjoint on $\mathcal{D}(A) \cap \mathcal{D}(B)$, where $\mathcal{D}(\bullet)$ is the domain of \bullet , then $e^{-t(A+B)} = \lim_n (e^{-\frac{t}{n}A} e^{-\frac{t}{n}B})^n$ in the strong operator norm on \mathbb{H} .

To adapt this formula to our setting of evolving manifolds, we need a non-autonomous Trotter formula, which was developed in [V, VWZ]. Let $\mathcal{T}(t, s)$ be the fundamental solution to the variational equation (1.3), i.e. for all $u(s) \in C^\infty(M_s)$, the function $\mathcal{T}(t, s)u(s) :=$

$u(t) : M_t \rightarrow \mathbb{R}$ solves (1.3) with initial condition $u(s)$. The fundamental solution is related to the heat kernel by

$$u(t) = \mathcal{T}(t, s)u(s) = \int_{M_s} \mathcal{H}(x, t; y, s)u(y, s)e^{-\frac{|y|^2}{4}} d\mu_s(y).$$

So the fundamental solution $\mathcal{T}(t, s)$ can be extended to a linear operator from $L^2(M_s)$ to $L^2(M_t)$. Similarly, we let $\bar{\mathcal{T}}(t, s)$ be the fundamental solution to the equation $\partial_t u = \mathcal{L}_{M_t}u$. Then we get the following result by applying the non-autonomous Trotter product formula in [V] in our setting.

Proposition 2.3. *Let $V(\cdot, \tau) : M_\tau \rightarrow \mathbb{R}$, $0 \leq s \leq \tau \leq t < \infty$ be a smooth function that is also smooth in τ and let $\mathcal{T}(t, s)$ and $\bar{\mathcal{T}}(t, s)$ be as above. Denote $t_k = k(t - s)/n + s$. Then we have*

$$\mathcal{T}(t, s) = \lim_n \prod_{k=0}^{n-1} \left(\bar{\mathcal{T}}(t_{k+1}, t_k) e^{\frac{t-s}{n} V(\cdot, t_k)} \right)$$

in the strong operator topology as linear operators from $L^2(M_s)$ to $L^2(M_t)$.

2.3. Feynman-Kac formula. We next prove the Feynman-Kac formula in the setting of RMCF.

Theorem 2.4. *Let $V(\cdot, \tau) : M_\tau \rightarrow \mathbb{R}$, $0 \leq s \leq \tau \leq t < \infty$ be a smooth function that is also smooth in τ . Then there exists a positive measure $\nu_{x,t}$ on the infinite product space $\Omega := \prod_{0 \leq \tau \leq t} M_\tau$ such that the solution of the Cauchy problem (2.1) is represented as follows in the L^2 -norm*

$$u(x, t) = \int_{\Omega} f(\omega(0)) \exp \left(\int_0^t V(\omega(s), s) ds \right) d\nu_{x,t}(\omega).$$

Proof. We follow the argument in Chapter X of [RS], which avoids using probabilistic language. We introduce the space $\Omega := \prod_{0 \leq \tau \leq t} M_\tau$ of all paths $\{\omega_\tau\}$ along the RMCF such that $\omega_\tau \in M_\tau$ endowed with the product topology. For fixed $\mathbf{t} := (t_1, \dots, t_m)$ with $t_1 < t_2 < \dots < t_m$, we introduce a subspace $\Omega(\mathbf{t})$ of Ω defined as $\Omega(\mathbf{t}) = \prod_{i=1}^m M_{t_i}$. Let $F : \Omega(\mathbf{t}) \rightarrow \mathbb{R}$ be a continuous function on $\Omega(\mathbf{t})$ and $\omega \in \Omega$ be a path along (M_t) . The restriction map $\varphi : \Omega \rightarrow \mathbb{R}$ is defined as $\varphi(\omega) := F(\omega(t_1), \dots, \omega(t_m))$.

We denote by $C_{\text{fin}}(\Omega)$ the set of all such functions on Ω with all possible choices of time slices \mathbf{t} and introduce a linear functional $\mathbb{L}_{x_{m+1}, t_{m+1}}$ for each given point x_{m+1} on the t_{m+1} -slice $M_{t_{m+1}}$

$$\begin{aligned} \mathbb{L}_{x_{m+1}, t_{m+1}}(\varphi) &= \int_{M_{t_m}} \cdots \int_{M_{t_1}} F(x_1, \dots, x_m) \bar{\mathcal{H}}(x_{m+1}, t_{m+1}; x_m, t_m) \cdots \bar{\mathcal{H}}(x_2, t_2; x_1, t_1) \\ &\quad e^{-\frac{1}{4}(|x_0|^2 + \cdots + |x_m|^2)} d\mu_{x_1} \cdots d\mu_{x_m}, \end{aligned}$$

where $\bar{\mathcal{H}}$ is the heat kernel for the heat equation $\partial_t u = \mathcal{L}_{M_t}u$ defined as in Section 2.1 with $V \equiv 0$.

The linear functional \mathbb{L} is well-defined on C_{fin} . Indeed, let \mathbf{t}' be a finite superset of \mathbf{t} , then $F : \Omega(\mathbf{t}) \rightarrow \mathbb{R}$ can be considered as a function $F' : \Omega(\mathbf{t}') \rightarrow \mathbb{R}$ which agrees with F on the time slices in \mathbf{t} and constant on the times slices in $\mathbf{t}' \setminus \mathbf{t}$. Then the cocycle property of the heat kernel enables us to integrate out the variables on the slices in $\mathbf{t}' \setminus \mathbf{t}$.

Then we obtain a bounded positive linear functional $\mathbb{L}_{x_{m+1}, t_{m+1}}$ on $C_{\text{fin}}(\Omega)$, and by Stone-Weierstrass theorem there exists a unique extension to $C(\Omega)$. Then by Riesz representation, we obtain a unique Borel measure $\nu_{x_{m+1}, t_{m+1}}$ such that for all $\varphi \in C(\Omega)$

$$\mathbb{L}_{x_{m+1}, t_{m+1}}(\varphi) = \int_{\Omega} \varphi(\omega) d\nu_{x_{m+1}, t_{m+1}}(\omega).$$

This gives us a representation of solutions of the Cauchy problem (2.1) as follows: for each $x \in M_t$, we have

$$\mathbb{L}_{x,t}f = \int_{\Omega} f(\omega(0)) d\nu_{x,t}(\omega) = \int_{M_0} f(y) \bar{\mathcal{H}}(x, t; y, 0) e^{-\frac{|y|^2}{4}} d\mu_y.$$

Finally, the Trotter product formula shows that (denoting $t_k = tk/n$, $x_n = x$, $t_n = t$)

$$\begin{aligned} u(x, t) &= \lim_n \prod_{k=0}^{n-1} \left(U(t_{k+1}, t_k) e^{\frac{t}{n} V(x_k, t_k)} \right) f \\ &= \lim_n \int_{M_{t_{n-1}}} \cdots \int_{M_0} e^{\frac{t}{n} \sum_k V(x_k, t_k)} \bar{\mathcal{H}}(x, t; x_{n-1}, t_{n-1}) \cdots \bar{\mathcal{H}}(x_1, t_1; x_0, 0) f(x_0) \\ &\quad e^{-\frac{1}{4}(|x_0|^2 + \cdots + |x_{n-1}|^2)} d\mu_{x_0} \cdots d\mu_{x_{n-1}} \\ &= \int_{\Omega} e^{\int_0^t V(\omega(s), s) ds} f(\omega(0)) d\nu_{x,t}(\omega). \end{aligned}$$

This gives the Feynman-Kac formula. □

Remark 2.5. *The measure ν is called Wiener measure.*

2.4. The localization. One notable difficulty in the study of singularities modeled by non-compact self-shrinkers in MCF theory is that in general the manifold M_t cannot be written as a global graph over the limiting shrinker Σ no matter how large t is, where $\{M_t\}_{t \in [0, \infty)}$ is the RMCF with $M_t \rightarrow \Sigma$ in C_{loc}^{∞} as $t \rightarrow \infty$. Therefore it is natural to consider the localized Dirichlet boundary value problem. Let B_R be a big open ball in \mathbb{R}^{n+1} . When t is sufficiently large, we can write part of M_t as a normal graph over $B_R \cap \Sigma$. We denote by M_t^R this part of M_t and by $L_{M_t}^R$ the restriction of L_{M_t} -operator to M_t^R , and introduce the evolutionary Dirichlet boundary value problem

$$(2.4) \quad \begin{cases} \partial_t u &= L_{M_t}^R u \\ u(t, \cdot)|_{\partial M_t^R} &= 0 \end{cases}.$$

For problem (2.4), we can also introduce the heat kernel as in Section 2.1, which also has the cocycle property (c.f. Lemma 26.12 of [CC]). Let us denote by \mathcal{H}^R its heat kernel, and

by $\bar{\mathcal{H}}^R$ the heat kernel for the Dirichlet boundary value problem with $V \equiv 0$. Then we can repeat the argument of Theorem 2.4 to obtain a Wiener measure ν^R that is supported on $\Omega^R := \prod_{0 \leq \tau \leq t} M_\tau^R$ such that the solution to (2.4) can be represented as

$$[\mathcal{T}^R(t, s)u(s, \cdot)](x) = \int_{\Omega} u(s, \omega(s)) \exp\left(\int_s^t V(\omega(\tau), \tau) d\tau\right) d\nu_{x,t}^R(\omega),$$

where we use $\mathcal{T}^R(t, s)$ to denote the fundamental solution to (2.4).

One remarkable property of the Feynman-Kac representation is that we can compare the solution of (2.4) to that of the original Cauchy problem without cutoff. Using parabolic maximum principle, we have pointwisely $\bar{\mathcal{H}}(x, t; y, s) \geq \bar{\mathcal{H}}^R(x, t; y, s)$. Thus we get the following proposition by comparing the argument of Theorem 2.4.

Proposition 2.6. *Let ν be the Wiener measure constructed in Theorem 2.4 and $\mathcal{T}^R(t, s)$ be as above. Then we have pointwise*

$$\int_{\Omega} u(s, \omega(s)) \exp\left(\int_s^t V(\omega(\tau), \tau) d\tau\right) d\nu_{x,t}(\omega) \geq [\mathcal{T}^R(t, s)u(s, \cdot)](x).$$

We remark that using the theory of local operator in [G] and [R], it is possible to show that

$$\int_{\Omega} u(s, \omega(s)) \exp\left(\int_s^t V(\omega(\tau), \tau) d\tau\right) \mathbf{1}_{\{\tau_R(\omega) > t\}} d\nu_{x,t}(\omega) = [\mathcal{T}^R(t, s)u(s, \cdot)](x)$$

where the stopping time $\tau_R(\omega) := \inf\{t \geq 0 \mid \omega(t) \notin M_t^R\}$. However, we avoid that complications since the statement in Proposition 2.6 is enough for our purpose.

3. ASYMPTOTIC BEHAVIOR OF THE SOLUTION TO THE VARIATIONAL EQUATION

In this section, we use the Feynman-Kac formula to study the asymptotic behavior of positive solutions to the variational equation $\partial_t v^* = L_{M_t} v^*$. We shall give the proof of Theorem 1.5 in Section 3.2. We introduce the L^2 -norm and Q -norm on M_t as follows.

Definition 3.1. (1) *The $L^2(M_t)$ -norm is defined by $\|u\|_{L^2(M_t)} = \left(\int_{M_t} |u(x)|^2 e^{-|x|^2/4} d\mu\right)^{1/2}$*

for a function $u : M_t \rightarrow \mathbb{R}$;

(2) *the Q -norm is defined by*

$$\|u\|_{Q(M_t)} = \left(\int_{M_t} \left(|\nabla u(x)|^2 + \Lambda u(x)^2 - (|A|^2 + \frac{1}{2})u(x)^2\right) e^{-|x|^2/4} d\mu\right)^{1/2}.$$

Here we pick $\Lambda > \sup_t \lambda_1(t)$ where $\lambda_1(t)$ is the first eigenvalue of L_{M_t} on M_t . In the case we study later, $\lambda_1(t)$ is uniformly bounded from above, so we can always pick such a Λ .

(3) *We abbreviate $L^2(t)$ for $L^2(M_t)$, and $Q(t)$ for $Q(M_t)$.*

(4) *We also introduce $L^2(\Sigma)$ and $Q(\Sigma)$ similarly and abbreviate as L^2 and Q respectively.*

The conclusion item (2) of Theorem 1.5 is a corollary of item (1) and the energy estimate, and the upper bound estimate $\lim \frac{1}{t} \log \|v^*(t)\|_{L^2(t)} \leq \lambda_1(\Sigma)$ is also an easy corollary of the energy estimate. The main difficulty is to establish the lower bound $\lim \frac{1}{t} \log \|v^*(t)\|_{L^2(t)} \geq \lambda_1(\Sigma)$, for which we use Feynman-Kac to localize to a bounded region and apply the result in the next subsection.

3.1. The cone-preservation property. In this section, we study the asymptotic dynamics of the evolutionary Dirichlet boundary value problem (2.4) with some large R fixed. For each $R > 0$, there exists $T = T(R)$ such that for all $t > T$, we can write M_t^R as a normal graph of a function m_t over $\Sigma^R := \Sigma \cap B_R$, i.e.

$$M_t^R = \text{Graph}\{x + m_t(x)\mathbf{n}(x), \quad x \in \Sigma^R\}.$$

This provides a diffeomorphism $\varphi_t : \Sigma^R \rightarrow M_t^R$, via $x \mapsto x + m_t(x)\mathbf{n}(x)$, that converges to identity in the C^2 norm as $t \rightarrow \infty$.

A function $f : M_t^R \rightarrow \mathbb{R}$ is pulled back by φ_t to a function $f^* = f \circ \varphi_t : \Sigma^R \rightarrow \mathbb{R}$. This is called ‘‘transplantation’’ in [SX]. From the equation $\partial_t v^* = L_{M_t}^R v^*$, we obtain the equation for $v^* := v^* \circ \varphi_t$ as

$$(3.1) \quad \partial_t v^* = L_{\Sigma^R}^R v^* + P(v^*, t),$$

where $L_{\Sigma^R}^R$ is the restriction of L_{Σ} to Σ^R , and $P(v^*, t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly for all $\|v^*\|_{C^2} \leq 1$. Because v^* satisfies a linear equation, we have $P(\lambda v^*, t) = \lambda P(v^*, t)$. Moreover, we have the following estimate for P

$$(3.2) \quad |P(v^*, t)| \leq C \|m\|_{C^2} (|\nabla_{\Sigma^R}^2 v^*| + |\nabla_{\Sigma} v^*| + |v^*|).$$

We next show the $L^2(M_t^R)$ and $L^2(\Sigma^R)$ norms are equivalent under the transplantation. Given $v^* \in L^2(M_t^R)$, we have that $v^* \in L^2(\Sigma^R)$. Indeed, in the definition of the $L^2(M_t^R)$ -norm, we introduce a coordinate change $x \mapsto \varphi_t(x)$. The Jacobian is close to identity. We next consider the Gaussian weight, we have $e^{-\frac{|x+m_t(x)\mathbf{n}(x)|^2}{4}} = e^{-\frac{|x|^2}{4} - \frac{m_t(x)^2}{4} - \frac{m_t(x)}{2}x \cdot \mathbf{n}(x)}$. Since $x \in \Sigma$ and $\mathbf{n}(x)$ is the unit normal at x , we get $\frac{1}{2}x \cdot \mathbf{n}(x) = H(x)$. Restricted to B_R , we have $|H| < C$ and then we get the following estimate for $|m_t|$ sufficiently small over B_R

$$(1 - \varepsilon)e^{-\frac{|x|^2}{4}} \leq e^{-\frac{|x+m_t(x)\mathbf{n}(x)|^2}{4}} \leq (1 + \varepsilon)e^{-\frac{|x|^2}{4}}.$$

In this way, we have converted the fundamental solution $\mathcal{T}^R(t, s)$ to (2.4) over the RMCF M_t^R into a non autonomous dynamical system (3.1) on the fixed manifold Σ^R . Abusing notation slightly, we still use $\mathcal{T}^R(t, s)$ to denote the fundamental solution to the system (3.1).

Let $\alpha > 0$ be a positive number. We introduce the cone of functions:

$$\mathcal{K}^R(\alpha) := \{f \in L^2(\Sigma^R) \mid \|\pi_1 f\|_{L^2(\Sigma^R)} \geq \alpha \|\pi_2 f\|_{L^2(\Sigma^R)}\}$$

where π_1 means the $L^2(\Sigma^R)$ -projection to the direction of the first eigenfunction of L_Σ^R and π_2 is the $L^2(\Sigma^R)$ -orthogonal complement of π_1 . The larger α is, the narrower the cone is around ϕ_1^R , the leading eigenvector of L_Σ^R .

Proposition 3.2. (1) *For each α and R , there exists T sufficiently large such that for all $t > s > T$ we have $\mathcal{T}^R(t, s)\mathcal{K}^R(\alpha) \subsetneq \mathcal{K}^R(\alpha)$.*
(2) *For all $\varepsilon > 0, \alpha > 0$, and $R > 0$, there exists T sufficiently large, such that for all $t > s \geq T$ and all $v^* \in \mathcal{K}^R(\alpha)$ we have*

$$\|\pi_1(\mathcal{T}^R(t, s)v^*)\|_{L^2(\Sigma^R)} \geq e^{(t-s)(\lambda_1^R - \varepsilon)} \|\pi_1(v^*)\|_{L^2(\Sigma^R)},$$

where λ_1^R is the leading eigenvalue of L_Σ^R .

Proof. We first consider $t = s + 1$ and the initial condition to the equation (3.1) to be $v^*(0)$ at the time s . With the following Lemma 3.3, we get (denoting $\|\cdot\| = \|\cdot\|_{L^2(\Sigma^R)}$)

$$\|\pi_1 v^*(1)\| \geq e^{\lambda_1 - \delta} \|\pi_1 v^*(0)\|, \quad \|\pi_2 v^*(1)\| \leq e^{\lambda_2 + \delta} \|\pi_2 v^*(0)\|.$$

Taking quotient, we get that $\frac{\|\pi_1 v^*(1)\|}{\|\pi_2 v^*(1)\|} \geq e^{\lambda_1 - \lambda_2 - 2\delta} \frac{\|\pi_1 v^*(0)\|}{\|\pi_2 v^*(0)\|}$. The statement then follows by taking δ small and iterating the argument. \square

Lemma 3.3. *Let $v^*(t)$ be a solution to the non-autonomous system (3.1) with initial value $v^*(T)$ at the initial time $t = T$ and the Dirichlet boundary value. Then for any $\delta > 0$, there exists $T_0 > 0$ such that for all $T > T_0$ we have*

$$\|v^*(T+1) - e^{L_\Sigma^R} v^*(T)\|_{L^2(\Sigma^R)} \leq \delta \|v^*(T)\|_{L^2(\Sigma^R)}.$$

Proof. Suppose M_t^R is written as the graph of function m_t over Σ^R , then we have the bound (3.2), where the constant C is a constant depending on the geometry of Σ^R . Here we restrict to a fixed ball of radius R , so C is uniformly bounded. When t is large, we have that M_t is sufficiently close to Σ hence $\|m_t\|_{C^2}$ is very small.

To prove the lemma, we denote by $w(t) = v^*(t) - e^{L_\Sigma^R} v^*(t)$ where $\partial_t v^* = L_\Sigma^R v^* + P(v^*, t)$ and $\partial_t v^* = L_\Sigma^R v^*$ with the same initial value $v^*(T)$ at time $t = T$, then we have $\partial_t w = L_\Sigma^R w + P(v^*, t)$ and

$$\begin{aligned} \partial_t \int_{\Sigma^R} |w|^2 e^{-\frac{|x|^2}{4}} d\mu &= - \int_{\Sigma^R} |\nabla w|^2 e^{-\frac{|x|^2}{4}} d\mu + \int_{\Sigma^R} w P(v^*, t) e^{-\frac{|x|^2}{4}} d\mu \\ &\leq \int_{\Sigma^R} w^2 e^{-\frac{|x|^2}{4}} d\mu + \int_{\Sigma^R} P(v^*, t)^2 e^{-\frac{|x|^2}{4}} d\mu \\ e^{-1} \int_{\Sigma^R} |w(T+1)|^2 e^{-\frac{|x|^2}{4}} d\mu &\leq \int_0^1 e^{-t} \int_{\Sigma^R} P(v^*, T+t)^2 e^{-\frac{|x|^2}{4}} d\mu dt. \end{aligned}$$

Next, for any δ , we can choose t large enough so that $C\|m_t\|_{C^2} < \delta$. Then we get from (3.2) that $|P(v^*, t)| \leq \delta(|\text{Hess}_{v^*}| + |\nabla v^*| + |v^*|)$. Then following Corollary A.9 in [SX]

(also see [CM2] Lemma 5.4), we bound $\int_0^1 \int (|\text{Hess}_{v^*}| + |\nabla v^*|)^2 e^{-\frac{|x|^2}{4}} d\mu dt$ by a multiple of $\int |v^*(T)|^2 e^{-\frac{|x|^2}{4}} d\mu$. This completes the proof. \square

3.2. Feynman-Kac and the cone condition. With the Feynman-Kac formula, we give proof of our main result of this section.

Proof of Theorem 1.5. We express $v^*(x, t)$, the solution to $\partial_t v^* = L_{M_t} v^*$ with initial condition $v_0^* > 0$, by Feynman-Kac formula as

$$v^*(x, t) = \int_{\Omega} v_0^*(\omega(0)) \exp\left(\int_0^t V(\omega(s), s) ds\right) d\nu_{x,t}(\omega).$$

Here Ω is the set of all paths $\omega : [0, t] \rightarrow (M_s)_{0 \leq s \leq t}$ with endpoint $\omega(t) = x$.

We then pick a large T such that for $t \geq T$, M_t is sufficiently close to Σ over a ball B_R for some R large, and such that we have the preservation of the cone condition (see Proposition

3.2). We next modify $V(\cdot, t)$ to the function $\tilde{V}(\cdot, t) = \begin{cases} 0, & t < T, \\ V(\cdot, t), & t \geq T \end{cases}$. Then we get

$$\begin{aligned} \exp\left(\int_0^t V(\omega(s), s) ds\right) &\geq \exp\left(\int_0^t \tilde{V}(\omega(s), s) ds\right) = \exp\left(\int_T^t V(\omega(s), s) ds\right), \text{ and} \\ v^*(x, t) &\geq c_T \int_{\Omega} v_0^*(\omega(0)) \exp\left(\int_T^t V(\omega(s), s) ds\right) d\nu_{x,t}(\omega) \\ &\geq \frac{c_T \min v_0^*}{\|\phi_1(T)\|_{L^\infty}} \int_{\Omega} \phi_1(T)(\omega(0)) \exp\left(\int_T^t V(\omega(s), s) ds\right) d\nu_{x,t}(\omega) \end{aligned}$$

where $\phi_1(T)$ is the eigenfunction associated to the leading eigenvalue of the operator L_{M_T} .

By the cocycle property of the heat kernel applied to the time interval $[0, T]$, the above integral becomes

$$\int_{\Omega} \phi_1(T) \exp\left(\int_T^t V(\omega(s), s) ds\right) d\nu_{x,t}(\omega) = \int_{\Omega_1} \phi_1(T) \exp\left(\int_T^t V(\omega(s), s) ds\right) d\nu_{x,t}(\omega)$$

where Ω_1 is the set of paths $\omega : [T, t] \rightarrow (M_s)_{T \leq s \leq t}$. We next use Proposition 2.6 to obtain

$$\int_{\Omega_1} \phi_1(T) \exp\left(\int_T^t V(\omega(s), s) ds\right) d\nu_{x,t}(\omega) \geq \int_{\Omega_1} \phi_1(T) \exp\left(\int_T^t V(\omega(s), s) ds\right) d\nu_{x,t}^R(\omega).$$

By the Feynman-Kac formula, the RHS of the above formula is exactly the solution to the variational equation with initial condition $v^*(\cdot, T) = \phi_1(T)|_{M_T^R}$ and the Dirichlet boundary condition $v^*(t, \cdot)|_{M_t^R} = 0$ for all $t > T$.

Now we transplant the problem to Σ^R . By the cone preservation property Proposition 3.2(1), if $\phi_1(T)$ lies in the cone $\mathcal{K}(\alpha)$, then all its future orbit lies in the cone. This implies for all $\varepsilon > 0$, $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|v^*(t)\|_{L_R^2(\Sigma)} \geq \lambda_1^R - \varepsilon$ (see Proposition 3.2(2)). By Lemma 9.25 of [CM1] (see also Theorem 6.9 for Σ being asymptotically conical case), we have that $\lambda_1^R \rightarrow \lambda_1$

which is the first eigenvalue of L_Σ and the leading eigenfunction ϕ_1^R of L_Σ^R converges in the C_{loc}^∞ -sense to ϕ_1 , the leading eigenfunction of L_Σ . Thus $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|v^*(t)\|_{L^2(M_t)} \geq \lambda_1^R - \varepsilon$. On the other hand, we have

$$\frac{1}{2} \partial_t \int_{M_t} (v^*)^2 e^{-\frac{|x|^2}{4}} d\mu = \int_{M_t} \left(v^* L_{M_t} v^* - \tilde{H}^2 (v^*)^2 \right) e^{-\frac{|x|^2}{4}} d\mu \leq \lambda_1(t) \int_{M_t} (v^*)^2 e^{-\frac{|x|^2}{4}} d\mu,$$

where \tilde{H} is the rescaled mean curvature $\left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right)$ and $\lambda_1(t)$ is the leading eigenvalue of L_{M_t} . By assumption, we have $\limsup_t \lambda_1(t) \leq \lambda_1(\Sigma)$. This gives

$$\|v^*(t)\|_{L^2(M_t)} \leq e^{(\lambda_1(\Sigma) + \varepsilon)t} \|v_0^*\|_{L^2(M_0)}$$

for t sufficiently large, for any $\varepsilon > 0$. Thus we get item (1).

We next work on item (2) of the statement. For any time t , we decompose $v^*(t) = v_1^*(t) + v_2^*(t) + v_3^*(t)$, where v_1^* is the projection of v^* to the eigenspace corresponding to the leading eigenvalue of L_{M_t} , v_3^* consists of Fourier modes with negative eigenvalues and v_2^* consists of the rest, and write

$$g_1(t) \|v^*\|_{L^2(M_t)}^2 = \|v_1^*\|_{L^2(M_t)}^2, \quad g_2(t) \|v^*\|_{L^2(M_t)}^2 = \|v_2^*\|_{L^2(M_t)}^2,$$

$$g_3(t) \|v^*\|_{L^2(M_t)}^2 = - \int_{M_t} v_3^* L_{M_t} v_3^* e^{-\frac{|x|^2}{4}} d\mu.$$

By Pythagoras theorem we have $g_1(t) + g_2(t) \leq 1$. Notice that with this notion, we have

$$\|v^*\|_{Q(M_t)}^2 \leq (\Lambda + g_3(t)) \|v^*\|_{L^2(M_t)}^2.$$

Then we can write the time derivative of $\|v^*\|_{L^2(M_t)}^2$ as follows:

$$\begin{aligned} \frac{1}{2} \partial_t \|v^*\|_{L^2(M_t)}^2 &\leq \int_{M_t} v^* L_{M_t} v^* e^{-\frac{|x|^2}{4}} d\mu \\ &\leq \max\{0, \lambda_2(t)\} g_2(t) \|v^*(t)\|_{L^2(M_t)}^2 + \lambda_1(t) g_1(t) \|v^*(t)\|_{L^2(M_t)}^2 - g_3(t) \|v^*\|_{L^2(M_t)}^2 \\ &= [\max\{0, \lambda_2(t)\} g_2(t) + g_1(t) \lambda_1(t) - g_3(t)] \|v^*\|_{L^2(M_t)}^2. \end{aligned}$$

Then we claim that there exist $c > 0$ and infinitely many $t_i \rightarrow \infty$ such that $\frac{g_1(t_i)}{\Lambda + g_3(t_i)} \geq c$. In fact, there exists $c_1 > 0$ and infinitely many $t_i \rightarrow \infty$ such that

$$(\lambda_1(t_i) - \lambda_2(t_i)) g_1(t_i) - g_3(t_i) \geq c_1 > \lambda_1(t_i) - \lambda_2(t_i).$$

Otherwise it violates the exponential growth in item (1). Then $g_1(t_i) \geq 1 + (\lambda_1(t_i) - \lambda_2(t_i))^{-1} g_3(t_i)$. Thus we can pick c small to get the desired time sequences. This shows that exist $c > 0$ and infinitely many $t_i \rightarrow \infty$ such that $\frac{|\langle v^*(t_i), \phi_1(t_i) \rangle_{L^2(t_i)}|}{\|v^*(t_i)\|_{Q(t_i)}} > c$. \square

We need the following approximation of the RMCF equation and linearized RMCF equation in [SX].

Proposition 3.4 (Proposition 3.3 in [SX]). *Given an RMCF $\{M_t\}_{t \in [0, T]}$, there exists $\delta_0 > 0$, $\alpha' > 0$, $\varepsilon_0 > 0$ and C so that for $v_0 : M_0 \rightarrow \mathbb{R}$ satisfies*

$$|v_0| + |\nabla v_0| \leq \delta \leq \delta_0, \quad |\text{Hess}_{v_0}| \leq 1,$$

and $v : [0, T] \rightarrow \mathbb{R}$ satisfies the RMCF equation, and $v^ : [0, T] \rightarrow \mathbb{R}$ satisfies the linearized equation, with $v(\cdot, 0) = v^*(\cdot, 0) = v_0$. Then we have*

$$\|(v - v^*)(\cdot, T)\|_{C^{2, \alpha'}} \leq C\delta^{1 + \varepsilon_0}.$$

This proposition is a consequence of the fact that the nonlinear term in RMCF is quadratic. With this proposition, we can prove that if the initial positive perturbation is sufficiently small, then the solution to RMCF equation on M_t will also drift to the first eigenfunction direction.

Proposition 3.5. *Assume the setting of Theorem 1.5. Suppose $v_0 > 0$ is a C^2 positive function on M_0 . Then there exist $1 > c > 0$ and $C > 1$ and a sequence $t_i \rightarrow \infty$, such that for any fixed t_i , there exists $\varepsilon_i > 0$, such that for $\varepsilon < \varepsilon_i$, the perturbed RMCF starting from $\{x + \varepsilon v_0(x)\mathbf{n}(x) : x \in M_0\}$ can be written as a graph of function $v(\cdot, t)$ over M_t for $t \in [0, t_i]$, such that*

$$\|v\|_{Q(t_i)} \leq C\|v\|_{L^2(t_i)}, \quad \text{and} \quad \frac{|\langle v(t_i), \phi_1(t_i) \rangle_{L^2(t_i)}|}{\|v(t_i)\|_{Q(t_i)}} > c.$$

3.3. Cone condition with cutoff. In this section, we show that conclusion (2) in Theorem 1.5 holds also if we restrict to a bounded part of the manifold M_{t_i} . This will be useful in the next section when studying the dynamics in a neighborhood of Σ , considering that M_t can not be written as a global graph over Σ .

Theorem 3.6. *There exists $R_0 > 0$ and $C > 0$ such that for $R > R_0$, for the sequence of t_i in Theorem 1.5, we have $\|v^*(t_i)\|_{L^2(M_{t_i})} \leq C\|v^*(t_i)\|_{L^2(M_{t_i}^R)}$.*

Proof. Theorem 6.11 shows that there exists $R_0 > 0$ and $\eta > 0$ such that when $R > R_0$, $\|\phi_1(t) - \eta\phi_1\|_{L^2(\Sigma^R)} \rightarrow 0$ as $t \rightarrow \infty$. Here $\phi_1(t)$ is transplanted to Σ^R when t is sufficiently large. From Theorem 1.5, we have

$$\|v^*\|_{Q(t_i)} \leq C\|v^*\|_{L^2(t_i)}, \quad \text{and} \quad \frac{|\langle v^*(t_i), \phi_1(t_i) \rangle_{L^2(t_i)}|}{\|v^*(t_i)\|_{Q(t_i)}} > c.$$

Hence (we choose $\phi_1(t) > 0$)

$$\|v^*\|_{L^2(t_i)} < c\langle v^*(t_i), \phi_1(t_i) \rangle_{L^2(t_i)} = c\langle v^*(t_i), \phi_1(t_i)\chi_{B_R} \rangle_{L^2(t_i)} + c\langle v^*(t_i), \phi_1(t_i)(1 - \chi_{B_R}) \rangle_{L^2(t_i)}.$$

Corollary 6.10 implies that $\|\phi_1(t)(1 - \chi_R)\|_{L^2(M_t)} \leq CR^{-1}$ for sufficiently large R, t . Thus,

$$\|v^*\|_{L^2(t_i)} < c\langle v^*(t_i), \phi_1(t_i)\chi_{B_R} \rangle_{L^2(t_i)} + CR^{-1}\|v^*\|_{L^2(t_i)}.$$

So if R is sufficiently large,

$$\|v^*\|_{L^2(t_i)} \leq C\langle v^*(t_i), \phi_1(t_i)\chi_{B_R} \rangle_{L^2(t_i)} \leq C\|v^*\|_{L^2(M_{t_i}^R)}.$$

□

Corollary 3.7. *There exists $R_0 > 0$ and $C > 0$ with the following significance: in the setting of Theorem 1.5, we have*

$$\|v^*\|_{Q(M_{t_i}^R)} \leq C \|v^*\|_{L^2(M_{t_i}^R)}, \text{ and } \frac{|\langle v^*(t_i), \phi_1(t_i) \rangle_{L^2(M_{t_i}^R)}|}{\|v^*(t_i)\|_{Q(M_{t_i})}} > c.$$

4. DYNAMICS IN A NEIGHBORHOOD OF THE SHRINKER

In this section, we show that the RMCF in a neighborhood of the shrinker is approximated by the linear equation $\partial_t u = L_\Sigma u$.

Consider (M_t) is an RMCF converging to a conical shrinker Σ in the C_{loc}^∞ sense as $t \rightarrow \infty$ and we perturb the initial condition slightly to give a new RMCF (\widetilde{M}_t) . We choose the initial perturbation so small that the perturbed flow will stay close to the unperturbed flow for a long time until they both enter a δ -neighborhood of the shrinker in the $C^{2,\alpha}$ -norm over a large compact ball B_R . This is the red curve in Figure 1, which is controlled by the variational equation. The main body of this section is devoted to the blue curve that is the dynamics in a neighborhood of the shrinker to be approximated by the linear equation $\partial_t u = L_\Sigma u$. As we have discussed in the introduction, the main difficulty is created by the fact that M_t and \widetilde{M}_t cannot be written as global graphs over Σ so that a cutoff is not avoidable. The time span running the blue curve of local dynamics depends on the smallness ε of the initial perturbation (roughly $\log \varepsilon^{-1}$), so that the linear approximation has to be done over growing domain since otherwise the discarded information by the cutoff will accumulate large error over $\log \varepsilon^{-1}$ long time. In Section 4.1, we state a result on the exponential growth of the graphical domain to be proved in Section 5. In Section 4.2, we derive the equation governing the dynamics of the blue curve. In Section 4.3, we give the necessary cone condition and boundary condition to initiate the blue curve. In Section 4.4, we show that the dynamics on the blue curve can be approximated by the solution to the linear equation. In Section 4.5, we show that the cones become narrower and narrower under the dynamics, which enables us to complete the proof of the main theorem in Section 4.6. In Section 4.7, we show the perturbation can be generic and in Section 4.8, we consider ancient solutions.

4.1. Exponential growth of the graphical domain. Denote by \mathbb{A}_{r_1, r_2} the closure of the annulus region $B_{r_2} \setminus B_{r_1}$ for $r_2 > r_1$.

Definition 4.1. *Let us fix an integer $\ell > 3$ and $r > 0$, $\varepsilon_0 > 0$. We define the graphical scale $r(M_t)$ as the largest radius R such that M_t can be written as a graph of a function $u : \Sigma^R \rightarrow \mathbb{R}$ satisfying*

- (1) $\|u\|_{C^{2,\alpha}(\Sigma^r)} < \varepsilon_0$,
- (2) $\|\nabla^i u\|_{C^0(\Sigma \cap \mathbb{A}_{s-1, s})} < s^{-i+1} \varepsilon_0$, $i = 0, 1, 2$, $r < s < R$.

Here r , ε_0 and C_m are uniform constants independent of R .

We will fix ε_0 and C_m later, such that the following Proposition 4.2 holds. Proposition 4.2 will be proved in Section 5.1.

Proposition 4.2. *Fix $\ell > 3$. There exist ε_0 and $C_m > 0$ for $m = 0, 1, \dots, \ell$, $T > 0$ and $C > 0$ such that for all $t > T$ we have $r(M_t) \geq Ce^{t/2}$.*

We define $\mathbf{r}(M_t)$ to be the minimum of the above $r(M_t)$ and $Ce^{t/2}$. Sometimes we simply write $\mathbf{r}(t)$ for simplicity. On $\Sigma \cap B_{\mathbf{r}(t)}$, M_t can always be written as a graph with an appropriate decay rate.

4.2. The evolution equation governing the difference of two nearby RMCFs. We consider the following setting: Let $\{M_t\}_{t \in [0, \infty)}$ be an RMCF as in (\star) . Suppose \widetilde{M}_0 is a small perturbation of M_0 and \widetilde{M}_t is the rescaled MCF with initial condition \widetilde{M}_0 .

We first write \widetilde{M}_t as the normal graph of a function $v(\cdot, t)$ over M_t , i.e.

$$(4.1) \quad \widetilde{M}_t = \{x + v(x, t)\mathbf{n}(x) \mid x \in M_t\}.$$

Then taking the difference of the RMCF equations for \widetilde{M}_t and M_t we get

$$(4.2) \quad \partial_t v = L_{M_t} v + Q(v),$$

where $Q(v)$ is quadratically small in $\|v\|_{C^2}$ (see [CM2, Lemma 4.3] and [SX, Appendix A]).

Suppose t is sufficiently large so that the graphical scale of M_t can be defined as in Definition 4.1. We can introduce a diffeomorphism φ between $\Sigma^{\mathbf{r}(t)}$ and $M_t^{\mathbf{r}(t)}$ so that a function v on $M_t^{\mathbf{r}(t)}$ can be transplanted to a function on $\Sigma^{\mathbf{r}(t)}$ as the pullback φ^*v . Note that the difference between the two manifolds $M_t^{\mathbf{r}(t)}$ and $\Sigma^{\mathbf{r}(t)}$ grows linearly in the radial direction, since the C^0 norm in item (2) of Definition 4.1 does so. Instead of using the normal graphical function in Definition 4.1 to define the diffeomorphism φ , we adopt a polar-spherical coordinates approach. Let $\mathcal{C} := \{r\theta \mid r \geq 0, \theta \in \mathcal{S} \subset \mathbb{S}^n(1)\}$ be the cone such that $\lambda\Sigma \rightarrow \mathcal{C}$ as $\lambda \rightarrow 0_+$, where \mathcal{S} is a codimension 1 submanifold of $\mathbb{S}^n(1)$. On each spherical slice $\mathbb{S}^n(r)$, both $M_t \cap \mathbb{S}^n(r)$ and $\Sigma \cap \mathbb{S}^n(r)$ can be written as a normal graph over $r\mathcal{S}$.

Definition 4.3. *We define the diffeomorphism $\varphi_t : \Sigma^{\mathbf{r}(t)} \rightarrow M_t^{\mathbf{r}(t)}$ by mapping each point in $\Sigma^{\mathbf{r}(t)} \cap \mathbb{S}^n(r)$ to the point of $M_t^{\mathbf{r}(t)} \cap \mathbb{S}^n(r)$ on the same normal of $r\mathcal{S}$.*

Note that the diffeomorphism preserves the Gaussian weight $e^{-\frac{|x|^2}{4}}$. We next use φ_t to pull back functions on M_t to Σ to rewrite the equation as a nonautonomous system over $\Sigma^{\mathbf{r}(t)}$ as

$$(4.3) \quad \partial_t v^* = L_\Sigma v^* + \mathcal{Q}(v^*, t), \quad \text{with } v^* = (\varphi_t)^*v, \text{ and}$$

$$(4.4) \quad \mathcal{Q}(v^*, t) = P(v^*, t) + ((\varphi_t)^*Q)(v^*), \quad P(v^*, t) = (\varphi_t)^*(L_{M_t} v) - L_\Sigma v^*.$$

We emphasize that (4.3) is the equation satisfied by the restriction of $v^* = (\varphi_t)^*v$ to the graphical part $\Sigma^{\mathbf{r}(M_t)}$ where v solves (4.2). Equation (4.3) itself is not an autonomous evolutionary equation since the solution over the region $\Sigma^{\mathbf{r}(M_t)}$ is influenced by the part outside $B_{\mathbf{r}(M_t)}$ according to (4.2). The error term \mathcal{Q} is estimated in Appendix B (see Lemma B.2).

Similarly to Proposition 4.2, we have the following estimate for the perturbed RMCF.

Proposition 4.4. *For any large $r > 0$ and small $\delta_1 > 0$, there exists $\epsilon_2 > 0$ with the following significance. Suppose \widetilde{M}_t is a perturbed RMCF where $\widetilde{M}_0 = \{x + v_0(x)\mathbf{n}(x) \mid x \in M_0\}$. If $\|v_0\|_{C^{2,\alpha}} < \epsilon_2$, then there exist $T > 0$ such that $\widetilde{M}_t \cap B_{\mathbf{r}(t)}$ can be written as a graph of function $u(\cdot, t)$ on $\Sigma^{\mathbf{r}(t)}$ with $\|u(\cdot, t)\|_{C^2(\Sigma^r)} < \delta_1$ on Σ^r for $t \in [T, T + 1]$.*

Proof. M_t converging to Σ in C_{loc}^∞ sense implies that for any fixed r , when T is sufficiently large, M_t can be written as a small graph over Σ^r . If the initial perturbation is sufficiently small, then by the smooth dependence on the initial data of RMCF equation, we have \widetilde{M}_T can be written as a graph over M_T which is sufficiently small. Then Theorem A.1 implies that \widetilde{M}_T can be written as a small graph over Σ^r . The fact that $\widetilde{M}_t \cap B_{\mathbf{r}(t)}$ can be written as the graph of u over $\Sigma^{\mathbf{r}(t)}$ follows from the following Proposition 4.5(2) and Theorem A.1. \square

4.3. The cone condition to initiate the local dynamics.

Proposition 4.5. *Let $\delta > 0$ be a small number and r be a large number, then there exist T_\sharp and ϵ_1 with the following significance:*

- (1) *For $t \geq T_\sharp$, M_t can be written as the graph of a function $m(\cdot, t) : \Sigma^{\mathbf{r}(t)} \rightarrow \mathbb{R}$ with $\|m(\cdot, t)\|_{C^{2,\alpha}(\Sigma^r)} \leq \delta/2$;*
- (2) *For all $0 < \epsilon < \epsilon_1$, suppose \widetilde{M}_t is the RMCF written as the normal graph of $v(\cdot, t)$ over M_t as in (4.1) and with initial condition $v_0 > 0$ and $\|v_0\|_{C^{2,\alpha}(M_0)} \leq \epsilon$. Let v^* be the transplantation of v to $\Sigma^{\mathbf{r}(t)}$ which satisfies (4.3). Then at time T_\sharp , the transplanted function $v^*(\cdot, T_\sharp)$ is defined on $\Sigma^{\mathbf{r}(T_\sharp)}$, and*

$$\|v^*(T_\sharp)\chi_{\mathbf{r}(T_\sharp)}\|_Q \geq \sup_{t' \in [T_\sharp - 1, T_\sharp + 1]} \|v^*(t', \cdot)\|_{C^2(\mathbb{A}_{\mathbf{r}(t') - 1, \mathbf{r}(t')})} e^{-\frac{\mathbf{r}(t')^{3/2}}{4}};$$

$$(3) \frac{\left| \langle v^*(T_\sharp)\chi_{\mathbf{r}(T_\sharp)}, \phi_1 \rangle_{L^2(\Sigma)} \right|}{\|v^*(T_\sharp)\chi_{\mathbf{r}(T_\sharp)}\|_{Q(\Sigma)}} > c' > 0.$$

Proof. The item (1) follows from the definition of the graphical scale and $\mathbf{r}(t)$. We only need to choose ϵ_0 in Definition 4.1 to be $\delta/2$.

Next we prove the item (2). We first prove the desired inequality for v^* , the solution to the linearized RMCF equation. From Theorem 3.6 and Corollary 3.7 we know that there exists $R_0 > 0$ and a sequence of t_i such that the $Q_{R_0}(t_i)$ -norm of v^* dominates the $L^2(t_i)$ -norm of v^* , which grows exponentially by Theorem 1.5 (1). On the other hand, parabolic maximal principle shows that v^* grows exponentially, meanwhile $e^{-\mathbf{r}(t)^{3/2}}$ decays super exponentially since $\mathbf{r}(t)$ grows exponentially by Proposition 4.2. So we can choose a T_\sharp such that

$$\|v^*(T_\sharp)\chi_{R_0}\|_{Q(t_\sharp)} \geq \frac{1}{2} \sup_{t' \in [T_\sharp - 1, T_\sharp + 1]} \|v^*(t', \cdot)\|_{C^2(\mathbb{A}_{\mathbf{r}(t') - 3, \mathbf{r}(t') + 2})} e^{-\frac{\mathbf{r}(t')^{3/2}}{4}}.$$

Now we fix such T_{\sharp} . From Proposition 3.4, namely the approximation of v^* and v , we know that if ε is sufficiently small, v and v^* will be sufficiently close to each other, and thus

$$\|v(T_{\sharp})\chi_{R_0}\|_{Q(t_{\sharp})} \geq \frac{2}{3} \sup_{t' \in [T_{\sharp}-1, T_{\sharp}+1]} \|v(t', \cdot)\|_{C^2(\mathbb{A}_{\mathbf{r}(t')-2, \mathbf{r}(t')+1})} e^{-\frac{\mathbf{r}(t')^{3/2}}{4}}.$$

Finally, Lemma A.2 implies that after trasplantation, we will have

$$\|v^*(T_{\sharp})\chi_{R_0}\|_{Q(\Sigma)} \geq \sup_{t' \in [T_{\sharp}-1, T_{\sharp}+1]} \|v^*(t', \cdot)\|_{C^2(\mathbb{A}_{\mathbf{r}(t')-1, \mathbf{r}(t')})} e^{-\frac{\mathbf{r}(t')^{3/2}}{4}}.$$

When T_{\sharp} is sufficiently large, $\mathbf{r}(T_{\sharp}) > R_0$. Then we obtain item (2).

Item (3) follows from Theorem 3.6 and Corollary 3.7. Notice that the choice of T_{\sharp} satisfies $\frac{|\langle v^*(T_{\sharp})\chi_{R_0}, \phi_1 \rangle_{L^2(T_{\sharp})}|}{\|v^*(T_{\sharp})\|_{Q(T_{\sharp})}} > c$. Meanwhile, from Theorem 6.9, $\phi_1(t_i) \rightarrow \phi_1$ on B_R . So when T_{\sharp}

is chosen sufficiently large, we have $\frac{|\langle v^*(T_{\sharp})\chi_{R_0}, \phi_1 \rangle_{L^2(\Sigma)}|}{\|v^*(T_{\sharp})\chi_{\mathbf{r}(T_{\sharp})}\|_{Q(\Sigma)}} > \frac{1}{2}c$ applying (B.1). Again, if ε is sufficiently small, Proposition 3.4 and Lemma A.2 imply that $\frac{|\langle v^*(T_{\sharp})\chi_{\mathbf{r}(T_{\sharp})}, \phi_1 \rangle_{L^2(\Sigma)}|}{\|v^*(T_{\sharp})\chi_{\mathbf{r}(T_{\sharp})}\|_{Q(\Sigma)}} > \frac{1}{3}c = c'$. \square

4.4. Approximate the local dynamics by linear equation. We next consider the orbit of \widetilde{M}_t for $t \geq T_{\sharp}$. We will compare $v^*(t)$ with the solution of the autonomous equation $\partial_t v = L_{\Sigma}v$. Note that the linear equation is globally defined on Σ , but M_t is not a global graph, so we choose the initial condition $v(0)$ for the linear equation satisfying $v(0) = v^*(T_{\sharp} + n)\chi_R$, $R = \mathbf{r}(T_{\sharp} + n)$, where $\chi_R : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function that is 1 for $|x| \leq R - 1$, 0 for $|x| > R$ and $|\chi'_R| < 2$. Then we solve the initial value problem

$$(4.5) \quad \begin{cases} \partial_t v_n &= L_{\Sigma}v_n \\ v_n(0) &= v^*(T_{\sharp} + n)\chi_{\mathbf{r}(T_{\sharp}+n)}. \end{cases}$$

tions to this linear equation in Appendix D. The following lemma shows that the solution of the above equation (4.3) can be well-approximated by the linearized equation.

Proposition 4.6. *Let v^* , v_n and T_{\sharp} be as defined above. Suppose that $\|v^*(T_{\sharp} + n + t)\|_{C^{2,\alpha}(\Sigma^{\mathbf{r}(T_{\sharp}+n+t)})} < \delta$ for $t \in [0, 1]$. Then we get*

- (1) $\|v^*(T_{\sharp} + n + 1)\chi_{\mathbf{r}(T_{\sharp}+n+1)} - v_n(1)\|_Q \leq \delta \|v^*(T_{\sharp} + n)\chi_{\mathbf{r}(T_{\sharp}+n)}\|_Q$;
- (2) $\|v^*(T_{\sharp})\chi_{\mathbf{r}(T_{\sharp})}\|_Q \geq \sup_{t' \in [T_{\sharp}-1, T_{\sharp}+1]} \|v^*(t', \cdot)\|_{C^2(\mathbb{A}_{\mathbf{r}(t')-1, \mathbf{r}(t')})} e^{-\frac{\mathbf{r}(t')^{3/2}}{4}}$.

We postpone the proof to Appendix C. The proof is similar to that of Proposition 4.3 of [CM2]. The main difficulty created by the noncompactness is that the boundary terms behave badly when performing integration by parts. The key observation is that the graphical domain grows exponentially with respect to t (Proposition 4.2), so the Gaussian weight

decays superexponentially like $e^{-ce^{t/2}}$, and on the other hand, the difference between two nearby RMCFs grows at most exponentially (Proposition 4.4). Thus we use the item (2) in the definition of T_{\sharp} in Proposition 4.5 to absorb the boundary term.

4.5. Iterating the local dynamics. Let E be the Banach space of $C^{2,\alpha}$ functions on Σ . We consider an orthogonal decomposition $E = E_1 \oplus E_2$ with respect to the Q -norm, where $E_1 = \mathbb{R}\phi_1$ and E_2 the orthogonal complement of E_1 . Next, let κ be a positive constant. We define the cone

$$\mathcal{K}(\kappa) = \{u = (u_1, u_2) \in E_1 \oplus E_2 \mid \|u_1\|_Q \geq \kappa\|u_2\|_Q\}.$$

This is a cone containing $E_1 = \mathbb{R}\phi_1$ and larger κ implies narrower cone.

Proposition 4.6 implies the following cone preservation property.

Lemma 4.7. *Let $\kappa > 0$ be a fixed number. For all ε , there exists δ sufficiently small such that the following holds. Let v^* be as in (4.3) with $v^*(T_{\sharp})\chi_{\mathbf{r}(T_{\sharp})} := (v_1(0), v_2(0)) \in \mathcal{K}(\kappa)$ and $\|v^*(T_{\sharp} + t)\|_{C^{2,\alpha}(\mathbf{r}(T_{\sharp} + t))} < \delta$ for $t \in [0, m]$ for any $m \in \mathbb{N}$, then we have*

$$v^*(T_{\sharp} + n)\chi_{\mathbf{r}(T_{\sharp} + n)} := (v_1(n), v_2(n)) \in \mathcal{K}(\kappa), \quad \forall 0 \leq n \leq m.$$

Moreover, we have

- (1) $\|v_1(n)\|_Q \geq e^{(\lambda_1 - \varepsilon)n} \|v_1(0)\|_Q$;
- (2) $\frac{\|v_2(n+1)\|_Q}{\|v_1(n+1)\|_Q} \leq e^{(\lambda_2 - \lambda_1 + \varepsilon)n} \frac{\|v_2(n)\|_Q}{\|v_1(n)\|_Q} + \delta \left(\frac{\kappa+1}{\kappa}\right)$.

Proof. We have the following from Proposition 4.6 and the assumption $(v_1(0), v_2(0)) \in \mathcal{K}(\kappa)$

$$\begin{aligned} \|v_2(1)\|_Q &\leq e^{(\lambda_2 + \varepsilon/3)} \|v_2(0)\|_Q + \delta \left(\frac{1 + \kappa}{\kappa}\right) \|v_1(0)\|_Q, \\ \|v_1(1)\|_Q &\geq e^{(\lambda_1 - \varepsilon/3)} \|v_1(0)\|_Q - \delta \left(\frac{1 + \kappa}{\kappa}\right) \|v_1(0)\|_Q. \end{aligned}$$

Taking quotient, we get both items with $n = 1$, which implies $(v_1(1), v_2(1)) \in \mathcal{K}(\kappa)$. Then the lemma follows from iterations. \square

With this lemma, we prove the following result.

Proposition 4.8. *Let δ, ε and T_{\sharp} be as in Proposition 4.5. Then there exists time $T_{\dagger} (> T_{\sharp})$ of order $|\log(\delta/\varepsilon)|$ such that after the evolution of the RMCF for time T_{\dagger} , the function $v^*(T_{\dagger}, \cdot)$ is defined over $\Sigma^{\mathbf{r}(T_{\dagger})}$ satisfying*

- (1) $\|v^*(T_{\dagger}, \cdot)\chi_{\mathbf{r}(T_{\dagger})}\|_{C^{2,\alpha/2}} = \delta$;
- (2) $\|v^*(T_{\dagger}, \cdot)\chi_{\mathbf{r}(T_{\dagger})}\|_Q \geq \delta^d$, for some $d > 0$ independent of δ ;
- (3) $v^*(T_{\dagger}, \cdot)\chi_{\mathbf{r}(T_{\dagger})} \in \mathcal{K}(1/(C\delta))$.

Proof. The conclusions (1) and (2) will be proved in Section 5.3. Item (3) follows from Lemma 4.7(2). Indeed, the inequality in Lemma 4.7(2) gives that for large time n , the ratio estimate $\frac{\|v_2(n+1)\|_Q}{\|v_1(n+1)\|_Q} < C\delta$ stabilizes. This translates to the cone condition in item (3). \square

4.6. Proof of the main theorem. Now we prove the following theorem.

Theorem 4.9. *Let M_t and M_t be as in (\star) . There exists $\delta_0 > 0$, $\delta_1 > 0$ and $r > 0$ with the following significance: after a small initial positive perturbation on the initial data M_0 , there is a time T' such that at time T' , the perturbed RMCF \widetilde{M}_t can be written as a graph of function u on $\Sigma^{\mathbf{r}(T')}$, and decomposing $u = u_1 + u_2$ in $E = E_1 \oplus E_2$ we have*

- (1) $\|u\|_Q^d \geq C\|u\|_{C^{2,\alpha}(\Sigma \cap B_r)} \approx \delta_0$, where d is in Proposition 4.8;
- (2) $\|u_1\|_Q \geq C\delta_0^{-1}\|u_2\|_Q$ for some constant C independent of δ_0, δ_1 or r .

Proof. Step 1: Let us fix an integer $\ell \geq 4$. At first we will choose a large radius r such that $\Sigma \setminus B_r$ is very close to regular cones, say Σ can be written as a graph over these cones with C^ℓ -norm of the graph is less than $10^{-10}\delta_1$. See [CS, Section 2] for details. We first choose T sufficiently large such that M_T is a graph of a function f over $\Sigma \cap B_r$ with $\|f\|_{C^{2,\alpha}(\Sigma^r)} \leq \delta_1$, and we also assume T is sufficiently large such that item (3) in Theorem 1.5 holds for T (we choose $R(t) \equiv r$). We will also choose T sufficiently large such that after transplantation, $\overline{\phi}_1^r(T)$ is very close to ϕ_1 on Σ , in the sense that $\|\overline{\phi}_1^r(T) - \phi_1\|_{C^{2,\alpha}(\Sigma^r)} \leq \delta_1$. In the definition of \mathbf{r} (see Proposition 4.2) we can choose C so that $\mathbf{r}(T) = r$. We will fix δ_0 and δ_1 later, but they are both very small constants.

Let $u_0 > 0$ be a positive perturbation solution on M_0 . We pick ϵ small, to be determined. Let u be the solution to the perturbed RMCF on M_t and u^* be the solution to the linearized RMCF, both with initial data ϵu_0 . By the approximation of RMCF equation and linearized RMCF, we can pick ϵ sufficiently small, such that

$$\|u(\cdot, T) - u^*(\cdot, T)\|_{C^{2,\alpha}(M_T)} \leq C\delta_1^\epsilon \|u(\cdot, T)\|_{C^{2,\alpha}(M_T)}.$$

Step 2: By Theorem 1.5, we have $\frac{|\langle u^*(T), \phi_1^r(T) \rangle_{L^2(M_T^r)}|}{\|u^*(T)\|_{Q(M_T^r)}} > c'$. Then when δ_1 is sufficiently small, we can use triangle inequality to get $\frac{|\langle u(T), \phi_1^r(T) \rangle_{L^2(M_T^r)}|}{\|u(T)\|_{Q(M_T^r)}} > c'$, where C' and c' are some constants which may vary line to line. Let $\overline{u}(T)$ be the transplantation of $u(T)$ to Σ^R . Then by Lemma C.1 in [SX], we also have $\frac{|\langle \overline{u}(T), \overline{\phi}_1^r(T) \rangle_{L^2(M_T^r)}|}{\|\overline{u}(T)\|_{Q(M_T^r)}} > c'$. Finally, because we have assumed that T is sufficiently large so that $\|\overline{\phi}_1^r(T) - \phi_1\|_{L^2} \leq \delta_1$, we have $\frac{|\langle \overline{u}(T), \phi_1 \rangle_{L^2(M_T^r)}|}{\|\overline{u}(T)\|_{Q(M_T^r)}} > c'$,

This implies that $\overline{u}(T)$ lies in a cone $\mathcal{K}(\kappa)$ for a κ depending on δ_1 (See Section 4.5).

Step 3: Now we can use Proposition 4.8 to show that after a definite amount of time $T_\dagger \approx \log |\delta_0 / \|\overline{u}\chi_r\|_Q|$, we have

- $\|\overline{u}\chi_{\mathbf{r}(T_\dagger)}\|_{C^{2,\alpha}(B_r \cap \Sigma)} \approx \delta_0$, $\|\overline{u}\chi_{\mathbf{r}(T_\dagger)}\|_Q \approx \delta_0^{\alpha_1}$,
- $\overline{u}\chi_{\mathbf{r}(T_\dagger)} \in \mathcal{K}(1/(C\delta_0))$.

This implies the desired bound on $\overline{u}(T_\dagger)$.

Step 4: It remains to show that if we write the perturbed flow as a graph of function v over $\Sigma \cap B_{\mathbf{r}(T_\dagger)}$, v satisfies the same bound as \overline{u} . This is a consequence of the transplantation bound.

□

Theorem 4.9 can be used to describe the local feature of MCF after a positive perturbation. In particular, we have the following local description of MCF near a singularity modeled by an asymptotic conical self-shrinker: under the assumption (\star) , after a sufficiently small positive perturbation on initial data, there is a spacetime neighbourhood \mathcal{N} of $(0, 0)$ in which there is no singularity modeled by Σ .

Recall that the positive perturbation on a non-generic self-shrinker is not the only possible direction of perturbation that can decrease the Gaussian area of a non-generic self-shrinker. Colding-Minicozzi [CM1] have proved that infinitesimally translations (corresponding to eigenfunction $\langle x, \vec{e} \rangle$ for any unit vector \vec{e}) and infinitesimal dilation (corresponding to eigenfunction H) also decrease the Gaussian area of a non-generic self-shrinker.

For $\{\mathbf{M}_\tau\}$, the translations and dilation on the initial data will move the singularity $(0, 0)$ in the spacetime to somewhere else. We can show that the positive perturbation can perturb the asymptotic conical singularity better than translations and dilation. We are ready to give the proof of the conclusion of the main Theorem 1.1 for positive initial perturbations.

Proof of Theorem 1.1 for positive initial perturbations. Using the volume growth estimate $\frac{\text{Vol}(\widetilde{M}_t \cap B_R)}{R^n} \leq C$ (for all t large and all R) for RMCF (c.f. Lemma 2.9 of [CM1]), we get the estimate $F(M_t \setminus B_R) \leq CR^n e^{-R^2/4}$.

Fix δ ahead of time and let T_\dagger be as in Proposition 4.8. By choosing the $C^{2,\alpha}$ norm ε of the initial perturbation sufficiently small, we may allow T_\dagger sufficiently large and we pick $R = \mathbf{r}(M_{T_\dagger}) \simeq e^{T_\dagger/2}$, so that

$$F(M_{T_\dagger} \setminus B_R) \leq CR^n e^{-R^2/4} < \delta^3$$

and fix it in the following. Next we write $\widetilde{M}_{T_\dagger}^R$ as a graph of a function u^* over Σ^R , and suppose $\|u^*(T_\dagger, \cdot)\|_Q = \delta^{d'}$ with $d' \leq d$ by Proposition 4.8(2). Here we can guarantee that $d' > 1$ is sufficiently close to 1 (see the interpolation formula in Section 5.3).

By the second variation formula, we get

$$(4.6) \quad F(\widetilde{M}_{T_\dagger}^R) - F(\Sigma) = - \int_{\Sigma^R} u^* L_\Sigma u^* e^{-\frac{|x|^2}{4}} + O(\delta^3).$$

By Theorem 4.9(2), we get that the projection of u^* to the ϕ_1 component dominates, so we get

$$F(\widetilde{M}_{T_\dagger}^R) - F(\Sigma) \leq -(\lambda_1 - O(\delta))\delta^{2d'} + O(\delta^3).$$

Lemma 7.10 of [CM1] shows that the entropy of Σ is attained, so we have $F(\Sigma) = \lambda(\Sigma)$. It was proved in [CM1] (c.f. Theorem 4.30, 4.31 of [CM1]) that perturbation in the direction of ϕ_1 strictly decreases the F -functional, despite of translation or dilations. So if we have a translation or dilation \mathcal{R} of size δ , the same calculation as Theorem 4.30 of [CM1] gives the estimate $F(\mathcal{R}\widetilde{M}_{T_\dagger}^R) - F(\Sigma) \leq -\delta^{2.5}$.

This completes the proof.

□

Similarly, we have Corollary 1.2 holds for positive initial perturbations.

Proof of Corollary 1.2 for positive initial perturbations. By the avoidance principle of MCF, \widetilde{M}_t has distance at least $e^t\varepsilon$ to M_t for all $t > 0$ since $\lambda_1 > 1$. Meanwhile, the translation and dilation of the MCF has exactly scale $e^{t/2}\varepsilon$ and $e^t\varepsilon$ respectively, under RMCF. The conclusion then follows from the “Moreover” part of Theorem 1.1. □

4.7. Generic perturbations. In this section, we complete the proof of Theorem 1.1 and Corollary 1.2 by allowing generic initial perturbations that are not necessarily positive.

Completing the proof of Theorem 1.1 and Corollary 1.2. We only need to prove Theorem 4.9 holds. The proof is similar to [SX, Theorem 3.11] so we only sketch the proof here. The key is to prove Step 2 in Theorem 4.9 for an open dense subset \mathcal{S} of $\{u \in C^{2,\alpha}(M_0) \mid \|u\|_{C^{2,\alpha}} = 1\}$.

Let \mathcal{S} be the subset of $\{u \in C^{2,\alpha}(M_0) \mid \|u\|_{C^{2,\alpha}} = 1\}$ such that for any $u_0 \in \mathcal{S}$ Step 2 in Theorem 4.9 holds. By well-posedness of RMCF, openness of \mathcal{S} is straightforward. So we only need to prove the denseness of \mathcal{S} . In fact, for any u_0 with $\|u_0\|_{C^{2,\alpha}(M_0)} = 1$, let u be the solution to the linearized RMCF equation with initial data u_0 , there are two possible growth rate of u :

Case 1: $\|u\|_{L^2(M_t)}$ grows faster than $e^{(\lambda_1(\Sigma)-\epsilon)t}$ for any $\epsilon > 0$. Then the proof of Theorem 1.5 implies that Theorem 1.5 also holds for such u . This implies that such $u_0 \in \mathcal{S}$.

Case 2: $\|u\|_{L^2(M_t)}$ grows slower than $e^{(\lambda_1(\Sigma)-\epsilon)t}$ for some $\epsilon > 0$. Then we can add a small positive function to u_0 , and normalize it to get a nearby initial condition u'_0 . Because the positive function grows faster than $e^{(\lambda_1(\Sigma)-\epsilon)t}$ for any $\epsilon > 0$, it will dominate the whole function. Thus $u'_0 \in \mathcal{S}$.

Combining both cases above we know that \mathcal{S} is dense. □

4.8. Ancient solution. In the proof of the main theorem, we use only the finite time dynamics without establishing a stable/unstable manifold theorem. The existence of the stable manifold for noncompact shrinkers is an interesting open problem with the main difficulty again the failure of writing M_t as a graph over Σ . Similar argument as Theorem 4.1 of [SX] gives ancient solution dictated by Theorem 1.3. We refer readers to [SX] for more details.

5. GRAPHICAL ESTIMATES NEAR SHRINKER

In this section, we give necessary graphical estimates for an RMCF close to a shrinker on a large compact set. The proofs of Proposition 4.2 and Theorem 4.9(1) will be completed in this section. This section is organized as follows. In Section 5.1, we use pseudolocality to complete the proof of Proposition 4.2. In Section 5.2, we estimate the $C^{2,\alpha}$ -norm of the perturbed flow \widetilde{M}_t . In Section 5.3, we bound the $C^{2,\alpha}$ -norm by the Q -norm using an interpolation argument as well as the higher derivative estimate by Ecker-Huisken, hence complete the proof of Theorem 4.9(1).

5.1. Pseudolocality. In order to handle the non-compactness of the limit self-shrinker Σ , we use a pseudolocality lemma to extend the graphical scale of M_t . We use the formulation of [CS, Proposition 5.1], which is reformulated from Theorem 1.5 in [INS].

Lemma 5.1. *Given $\delta > 0$, there exists $\gamma > 0$ and a constant $\rho = \rho(n, \delta) > 0$ such that if a mean curvature flow $\{\mathbf{M}_t\}_{t \in [0,1]}$ satisfies that $\mathbf{M}_{-1} \cap B_\rho(x)$ is a Lipschitz graph over the plane $\{x_{n+1} = 0\}$ with Lipschitz constant less than γ and $0 \in \mathbf{M}_{-1}$, then $\mathbf{M}_t \cap B_\rho(x)$ intersects $B_\delta(x)$ and remains a Lipschitz graph over $\{x_{n+1} = 0\} \cap B_\delta(x)$ with Lipschitz constant less than δ for all $t \in [-1, 0]$.*

We next reformulate this Lemma in the context of RMCF. Note that the Lipschitz constant of a graph is invariant under dilation and that an RMCF (M_t) , $t \in [0, \infty)$ is related to an MCF (\mathbf{M}_τ) , $\tau \in [-1, 0]$ through $M_t = e^{\frac{t}{2}} \mathbf{M}_{-e^{-t}}$.

Lemma 5.2. *Given $\delta > 0$, there exist $\gamma > 0$ and a constant $\rho = \rho(n, \delta) > 0$ such that if an RMCF $\{M_t\}_{t \in [T, \infty)}$ satisfies that $M_T \cap B_{\sqrt{e^T} \rho}(x)$ is a Lipschitz graph over the plane L passing x with Lipschitz constant less than γ and $x \in M_T$, then $M_t \cap B_{\sqrt{e^t} \rho}(\sqrt{e^{t-T}} x)$ intersects $B_{\sqrt{e^t} \delta}(\sqrt{e^{t-T}} x)$ and remains a Lipschitz graph over $L \cap B_{\sqrt{e^t} \delta}(\sqrt{e^{t-T}} x)$ with Lipschitz constant less than δ for all $t \in [T, \infty)$.*

We will use Lemma 5.2 in the following way to control the gradient of graphs.

Corollary 5.3. *Suppose Σ is a conical shrinker, and M_t is an RMCF. Let R be sufficiently large. Then when T is sufficiently large, suppose M_T is a graph of $u(\cdot, T)$ on $\mathbb{A}_{R-1, R} \cap \Sigma$ with $|\nabla u(\cdot, T)| \leq \gamma$, then M_{T+t} is a graph of $u(\cdot, T+t)$ on $\mathbb{A}_{e^{t/2}(R-1), e^{t/2}R} \cap \Sigma$ with $|\nabla u(\cdot, t)| \leq \delta$ for $t > 0$.*

Proof. Suppose at time T very large, M_T is a graph of function $m(\cdot, T)$ over Σ in the ball of radius R , such that $|\nabla m(\cdot, T)| \leq \gamma/3$. Then we know that M_T is a graph on the tangent space $T_x \Sigma$ with Lipschitz constant less than $\gamma/2$. Then we can decompose $B_R \setminus B_{R-1}$ into several small balls $B_{\sqrt{e^{-T}} \rho}^j(x)$ with radius $\sqrt{e^{-T}} \rho$. On each $B^j(x)$ we use pseudolocality Lemma 5.2, we see that $M_{T+t} \cap \sqrt{e^{T+t}} B_{e^{t/2} \delta}^j(e^{t/2} x)$ is a graph on the same plane with Lipschitz constant δ for $t > 0$.

If we assume Σ is conical, then the tangent space $T_x \Sigma$ should be very close to the tangent space of Σ near $e^{t/2} x$. Moreover, this Lipschitz constant can be translated to the gradient of M_{T+t} as a graph m_{T+t} over the ball of radius roughly $e^{t/2} R$. So we obtain the corollary. \square

The above results are concerning the Lipschitz bound of the graph. Next we use Ecker-Huisken's interior estimate to get a higher order estimate of the graph. The following Lemma is a consequence of Ecker-Huisken's interior estimate of MCF (c.f. [EH, Section 4]).

Lemma 5.4. *Under the assumptions of Lemma 5.1, for any integer $\ell > 0$, there exists a constant C_ℓ depending on δ, γ and ρ , such that for all $t \in [-1/2, 0]$, we have that $\mathbf{M}_t \cap B_\delta$ is a C^ℓ graph with C^ℓ -norm less than C_ℓ .*

Similarly, this Lemma has a reformulation for RMCF, and has a consequence for RMCF.

Lemma 5.5. *Given $\delta > 0$, there exist $\gamma > 0$ and a constant $\rho = \rho(n, \delta) > 0$ such that if an RMCF $\{M_t\}_{t \in [T, \infty)}$ satisfies that $M_T \cap B_{\sqrt{e^T} \rho}(x)$ is a Lipschitz graph over the plane L passing x with Lipschitz constant less than γ and $x \in M_{-1}$, then for any integer $\ell > 0$, $M_t \cap B_{\sqrt{e^t} \rho}(\sqrt{e^{t-T}} x)$ intersects $B_{\sqrt{e^t} \delta}(\sqrt{e^{t-T}} x)$ and is a C^ℓ graph with ℓ -th derivative is bounded by $C_\ell (\sqrt{e^{t-T}})^{-\ell+1}$ for all $t \in [T + 1/2, \infty)$, where C_ℓ is a constant depending on δ, γ and ρ .*

Corollary 5.6. *Suppose Σ is a conical shrinker, and M_t is an RMCF converging to Σ . Let R be sufficiently large, and suppose M_T is a graph of $u(\cdot, T)$ on $\mathbb{A}_{R-1, R} \cap \Sigma$ with $|\nabla u(\cdot, T)| \leq \gamma$ for some T sufficiently large. Then for any integer $\ell > 0$, M_{T+t} is a graph of $u(\cdot, T+t)$ on $\mathbb{A}_{e^{t/2}(R-1), e^{t/2}R} \cap \Sigma$ with $\|\nabla^\ell u(\cdot, T+t)\|_{C^0} \leq (e^{t/2})^{-\ell+1} C_\ell$ for $t > 1/2$.*

Remark 5.7. *We give a remark about the Corollary 5.3 and Corollary 5.6. Note that in Lemma 5.2 and Lemma 5.5, the statement is about M_t being a graph over a fixed tangent space. Meanwhile in Corollary 5.6, the statement is about M_t being a graph over Σ . In fact, here we use the structure of an asymptotically conical self-shrinker. In particular, we use the fact that the curvature decays on Σ , and there exists $R_0 > 0$ such that for any $R \geq R_0$ and $C > 1$, any x on $\partial B_R \cap \Sigma$, $T_x \Sigma$ and $T_{C^\# x} \Sigma$ only different by a very small rotation θ (we use $C^\# x$ to denote the point on Σ whose projection to the asymptotic cone is C times the point on Σ which is the projection of x). The existence of such R_0 is a consequence of the structure of an asymptotically conical self-shrinker, see [BW] and [CS].*

We next work on the proof of Proposition 4.2.

Proof of Proposition 4.2. Suppose $\{M_t\}$ is an RMCF. Fix $R > 0$ from Corollary 5.6. We assume T is sufficiently large such that M_t is a graph of a function m_t over $\Sigma \cap B_{4R}$ with $\|m_t\|_{C^\ell(\Sigma \cap B_{4R})} \leq \gamma$, where γ comes from Corollary 5.6. Such T exists because M_t converges to Σ in C_{loc}^∞ sense.

Corollary 5.6 implies that M_{T+t} is a C^ℓ graph over $(\mathbb{A}_{Re^{t/2}, 2Re^{t/2}}) \cap \Sigma$ for $t > 1/2$. Meanwhile, for $t < 2$, we know that $\mathbb{A}_{Re^{t/2}, 2Re^{t/2}} \cap B_{4R}$ is always nonempty. So for $t \in [1/2, 2]$, M_{T+t} is a C^ℓ graph over $B_{2Re^{t/2}} \cap \Sigma$, with C^ℓ -norm satisfying the bound in Corollary 5.6.

Now we repeat the above process, starting from $t = 1, 2, 3, \dots$. Then the graphical region expands at the rate $2Re^{t/2}$. This concludes the proof. \square

5.2. Regularity estimates for the perturbed flow. In the last section, we see that once the RMCF is very close to Σ^R , the graphical region expands exponentially. For M_t converging to Σ in C_{loc}^∞ sense, it is always close to Σ^R , hence the expansion will last for all time. In our setting, the perturbed RMCF \widetilde{M}_t will not be close to Σ^R for all time. We need a quantitative characterization of the closeness of \widetilde{M}_t and Σ^R .

In this section we study some estimates of the RMCF as a graph over a part of Σ . In particular we generalize the Hölder estimates in [CM2] to non-compact asymptotically conical self-shrinker setting. We will always fix a conical self-shrinker Σ and R_* large such that on

$B_{R_*} \cap \Sigma$, there exists $C_\ell > 0$ such that $|\nabla^k A| \leq C(1 + |x|)^{-k-1}$ for $k = 0, 1, \dots, \ell + 1$, and $\Sigma \cap \partial B_{R_*}$ is the union of several submanifolds in $R_* \mathbb{S}^n$, which are sufficiently close to the cross sections of the asymptotic cone of the conical ends, and the tangent space on $\Sigma \cap \partial B_{R_*}$ is very close to the tangent space of the asymptotic cones.

The main goal of this section is to prove the following proposition:

Proposition 5.8. *Given $\epsilon_1 > 0$, there exist δ_1, C, ϵ and $\alpha > 0$, so that the following holds. Suppose $\{\widetilde{M}_t\}$ is an RMCF with $\mathbf{r}(\widetilde{M}_T) \geq R_*$ for some large T , and \widetilde{M}_{T+t} , $t \in [0, T^*]$, can be written as a graph of function u on Σ^{R_*} satisfying*

- (1) $|u(T)| + |\nabla u(T)| \leq \delta \leq \delta_1$, and $|\text{Hess}_{u(T)}| \leq \delta_1$.
- (2) $\|u(\cdot, T+t)\|_{C^2} \leq \delta_1$ on Σ^{R_*} for $t \leq T^*$,

then \widetilde{M}_{T+t} can be written as a graph of function u on $\Sigma^{\mathbf{r}(T+t)}$ for $t \in [0, T^*]$ and

- (1) on $\Sigma^{\mathbf{r}(T+t)}$ we have the estimates

$$|u(x, T+t)| \leq C^t(\delta + \epsilon_1)(1 + |x|), \quad |\nabla u(T+t)| \leq C^t \sqrt{(\delta + \epsilon_1)}, \quad |\text{Hess}_{u(T+t)}| \leq C^t \delta_1,$$
- (2) on Σ^{R_*} we have $\|u(\cdot, T+t)\|_{C^{2,\alpha}(\Sigma^{R_*})} \leq C^t(\delta + \epsilon_1)^\epsilon$,
- (3) for any integer ℓ , $\|u(\cdot, T+t)\|_{C^\ell(\Sigma^{R_*})} \leq C_\ell$ for some constant C_ℓ depending on δ_1 .

Lemma 5.9. *Suppose $\{\widetilde{M}_t\}$ is an RMCF. Given $\epsilon_1 > 0$, there exists $\delta_1 > 0$, $\tau > 0$, and $C > 0$ such that the following holds. Suppose that \widetilde{M}_T can be written as a graph of function $u(\cdot, T)$ over Σ^r , $r > R_*$ with $\|u(\cdot, T)\|_{C^1(\Sigma^r)} \leq \delta \leq \delta_1$, then for $t \in [0, \tau]$, we have*

$$\sup_{\Sigma^r} |u(\cdot, T+t)| \leq C \sup_{\Sigma^r} |u(\cdot, T)| + \epsilon_1.$$

Proof. First we prove this bound on $\mathbb{A}_{r-1,r} \cap \Sigma$. We use the pseudolocality argument. We choose sufficiently small $\delta_1 > 0$, such that we can use Corollary 5.3. More precisely, we pick δ in Lemma 5.2 to be ϵ_1 , and we pick T in Lemma 5.2 to be 0, and we pick $\rho < e^{-1}$ in Lemma 5.2. Corollary 5.3 applies to show that on $\mathbb{A}_{r-1,r} \cap \Sigma$, the RMCF \widetilde{M}_t can be written as a graph, whose supremum is bounded by $e^{t/2} \epsilon_1 \rho \leq \epsilon_1$.

Next we consider the bound on Σ^{r-1} . Let η be the cut-off function which is 1 on Σ^{r-1} and 0 outside Σ^r . Let us consider the supremum of ηu . Suppose the supremum of $\eta u(\cdot, t)$ is attained at p . If $|p| \geq r - 1$, then we can use the previous pseudolocality argument to prove that $\sup_{\Sigma^{r-1}} |u| \leq \epsilon_1$. Otherwise, we know that p is a local maximum of $u(\cdot, t)$. Then we use maximum principle (c.f. Lemma 3.13 of [CM2]) to show that $\sup_{\Sigma^{r-1}} |u(\cdot, T+t)| \leq C \sup_{\Sigma^{r-1}} |u(\cdot, T)| + \epsilon_1$. Combining two cases together we get the desired bound. \square

We next have the following local curvature estimate near a conical singularity.

Lemma 5.10. *For any $\epsilon_2 > 0$, there exists δ, η, τ and C_0 with the following significance. Suppose $\{\widetilde{\mathbf{M}}_t\}_{t \in [0, T]}$ is an MCF with $\max_{x \in \widetilde{\mathbf{M}}_0 \cap B_r} |A|^2(x, 0) \leq C < C_0$, $r > R_*$, and satisfies*

- $\widetilde{\mathbf{M}}_0 \cap B_\delta(x)$ is a graph over some hyperplane L , with Lipschitz constant less than η , for any $x \in \widetilde{\mathbf{M}}_0 \cap \mathbb{A}_{r,r+1}$.

Then for $t \in [\tau/2, \tau]$, we have $\max_{x \in \widetilde{\mathbf{M}}_t \cap B_{r+1}} |A|^2(x) \leq 2C + 2\epsilon_2$.

Proof. We extend the argument of [CM2, Corollary 3.26] to the noncompact setting. By pseudolocality argument, if η is sufficiently small, there exists $\tau > 0$ such that on $\widetilde{\mathbf{M}}_t \cap \mathbb{A}_{r+1/3, r+1/2}$, when $t \in [\tau/2, \tau]$, we have $|A|^2 < \epsilon_2$. Next we define $m(t) = \max_{x \in \widetilde{\mathbf{M}}_t \cap B_{r+1/2}} |A|^2(x, t)$. We have two cases: if $m(t) \leq \epsilon_2$ for all $t \in [\tau/2, \tau]$, then the inequality is proved. Otherwise, $m(t)$ is attained at somewhere on $\widetilde{\mathbf{M}}_t \cap B_{r+1/3}$. Using the inequality $(\partial_t - \Delta_{\widetilde{\mathbf{M}}_t})|A|^2 \leq |A|^4$, we can use the interior maximum principle to show that $m'(t) \leq m(t)^2$. This implies that $m(t) \leq \min\{2C, 2\epsilon_2\}$, if $t \in [0, \tau]$, where τ is a constant depending on C and ϵ_2 . \square

Proposition 5.11. *Given $\epsilon_1 > 0$, there exist $\tau > 0$, δ_1 , C , ε and $\alpha > 0$ so that the following holds. Suppose \widetilde{M}_t is an RMCF with $\mathbf{r}(T) \geq R_*$, and \widetilde{M}_{T+t} can be written as the graph of function w on $\Sigma \cap B_{\mathbf{r}(T)}$ satisfying $|w| + |\nabla w| \leq \delta \leq \delta_1$, and $|\text{Hess}_w| \leq \delta_1$, then \widetilde{M}_{T+t} can be written as a graph of function u on $\Sigma^{\mathbf{r}(T+t)}$ for $t \in [\tau/2, \tau]$ with $u(\cdot, 0) = w$ and*

(1) *On $\Sigma^{\mathbf{r}(T+t)}$ and for $t \in [\tau/2, \tau]$, we have*

$$|u(x, t)| \leq C(\delta + \epsilon_1)(1 + |x|), \quad |\nabla u(\cdot, t)| \leq C^{1/2} \sqrt{(\delta + \epsilon_1)}, \quad |\text{Hess}_{u(T+m-t)}| \leq C\delta_1,$$

(2) *on $\Sigma^{\mathbf{r}(T)}$ we have $\|u(\cdot, \tau)\|_{C^{2,\alpha}(\Sigma^{\mathbf{r}(T)})} \leq C(\delta + \epsilon_1)^\varepsilon$.*

Proof. Lemma 5.9 shows the first bound in (1) and Lemma 5.10 shows that third bound (with the relation between MCF and RMCF) in (1). The second bound in (1) comes from interpolation on $\Sigma^{\mathbf{r}(T+\tau/2)}$, and from pseudolocality on $\Sigma \cap (\mathbb{A}_{\mathbf{r}(T), \mathbf{r}(T+\tau/2)})$.

The above statement in (1) implies that u is a strictly parabolic equation on $[\tau/2, \tau]$, on $\Sigma^{\mathbf{r}(T)}$. Then (2) follows from similar argument as [CM2, Proposition 3.28], with the only difference being that we need to use interior Schauder estimate (c.f. Theorem 3.6 of [CS]). \square

We next give the proof of Proposition 5.8.

Proof of Proposition 5.8. Whenever $\|u(\cdot, T+t)\|_{C^2(\Sigma^r)} \leq \delta_1$, we can repeatedly use Proposition 5.11, each time in $\Sigma^{\mathbf{r}(T)}$. Outside $\Sigma^{\mathbf{r}(T)}$, we repeatedly use pseudolocality arguments. Notice that although at each step we only prove the desired bound for $t \in [\tau/2, \tau]$, we can start from $\tau/2$ to iterately use the argument, and therefore the desired bound can be obtained for all $t \in [1, T^*]$. For item (3), we use higher order estimate of curvature by Ecker-Huisken (c.f. [EH, Theorem 3.4]). More precisely, Ecker-Huisken proved that the higher order continuous norm of u is uniformly bounded by a constant depending on δ_1 . \square

5.3. Bounding the $C^{2,\alpha}$ -norm by the Q -norm. In application, we want to iterate Proposition 4.6 on a long period of time, which requires that $|u(\cdot, T+t)|_{C^{2,\alpha}} < \delta$ on Σ^r for t in that period of time. However, $|u(\cdot, T+t)|_{C^{2,\alpha}}$ may become very large if t is large, so Proposition 4.6 can only be used on a short time period $[T, T+\tau]$. It may happen that the Q -norm did not acquire sufficient growth over such time interval. To overcome this difficulty, in this section, we prove items (1) and (2) of Proposition 4.8. The key point is to interpolate the

higher derivative estimate of Ecker-Huisken with the Q -norm to bound the $C^{2,\alpha}$ -norm. Similar argument was used in [Sc], [CM3] and [CS]. We cite the following interpolation Lemma from [CM3, Appendix B]. It is mentioned in [CM3, Appendix B] that the result also extend to a hypersurface with scale-invariant curvature bound.

Lemma 5.12. *There exists C only depending on Σ and r such that, if u is a C^ℓ function on Σ^r , then we have for $a_{\ell,n} = \frac{\ell}{\ell+n}$, $b_{\ell,n} = \frac{\ell-1}{\ell+n}$, $c_{\ell,n} = \frac{\ell-2}{\ell+n}$*

$$\begin{aligned} \|u\|_{L^\infty(\Sigma^r)} &\leq C\{\|u\|_{(\Sigma^{r+1})} + \|u\|_{L^1(\Sigma^{r+1})}^{a_{\ell,n}} \|\nabla^\ell u\|_{L^\infty(\Sigma^{r+1})}^{1-a_{\ell,n}}\}, \\ \|\nabla u\|_{L^\infty(\Sigma^r)} &\leq C\{\|u\|_{(\Sigma^{r+1})} + \|u\|_{L^1(\Sigma^{r+1})}^{b_{\ell,n}} \|\nabla^\ell u\|_{L^\infty(\Sigma^{r+1})}^{1-b_{\ell,n}}\}, \\ \|\nabla^2 u\|_{L^\infty(\Sigma^r)} &\leq C\{\|u\|_{(\Sigma^{r+1})} + \|u\|_{L^1(\Sigma^{r+1})}^{c_{\ell,n}} \|\nabla^\ell u\|_{L^\infty(\Sigma^{r+1})}^{1-c_{\ell,n}}\}. \end{aligned}$$

We use χ_r to denote the characterizing function on Σ^r , i.e. $\chi_r = 1$ on Σ^r and 0 elsewhere.

Lemma 5.13. *If \widetilde{M}_t is a graph of function u on $\Sigma \cap B_{\mathbf{r}(t)}$ and $\|u(\cdot, t)\|_{C^\ell(\Sigma^r)} \leq C_\ell$. Then we have $\|u(\cdot, t)\|_{C^{2,\alpha}(\Sigma^r)} \leq C\|u(\cdot, t)\chi_{\mathbf{r}(t)}\|_Q^{\alpha(\ell)}$, where $\alpha(\ell) = \frac{\ell-3}{\ell+n}$ and C is a constant depending only on Σ , δ_1 and ℓ .*

Proof. The interpolation lemma 5.12 together with C^ℓ uniform upper bound proved in Proposition 5.11 imply that $\|u\|_{C^{2,\alpha}(\Sigma^r)} \leq C\|u\|_{L^2(\Sigma^{r+1})}^{\alpha(\ell)} \leq C\|u\chi_{\mathbf{r}(t)}\|_Q^{\alpha(\ell)}$. \square

Item (1) and (2) of Proposition 4.8 follow from this lemma.

6. CONVERGENCE OF GEOMETRIC QUANTITIES

In this section, we prove the convergence of eigenfunctions and eigenvalues under assumption (\star) , hence verifies the assumptions of Proposition 1.5 in the case of conical singularities. In Section 6.1, we prove the convergence of the leading eigenvalue $\lambda_1(M_t)$, hence verifies assumption (1) of Proposition 1.5. In Section 6.2, we prove the convergence of the leading eigenfunction. In Section 6.3, we prove the spectral gap, hence verifies assumption (2) of Proposition 1.5.

Given a hypersurface M (possibly with boundary), we define the first eigenvalue $\lambda_1(M)$ of the linearized operator $L_M := \Delta_M - \frac{1}{2}\langle x, \nabla_M \cdot \rangle + (|A|^2 + 1/2)$ to be the number

$$\lambda_1(M) := - \inf_{f \in W_w^{1,2}(M)} \frac{\int_M [|\nabla f|^2 - (|A|^2 + 1/2)f^2] e^{-\frac{|x|^2}{4}} d\mu}{\int_M f^2 e^{-\frac{|x|^2}{4}} d\mu},$$

where the $W_w^{1,2}(M)$ is the weighted $W^{1,2}$ space, i.e. the completion of the following space

$$\left\{ f \in C_c^\infty(M) : \int_M (f^2 + |\nabla f|^2) e^{-\frac{|x|^2}{4}} < \infty \right\}.$$

If M has a boundary, $\lambda_1(M)$ is called the first Dirichlet eigenvalue of L_M . Notice that if M is non-compact, then $\lambda_1(M)$ could be ∞ . We remind readers that compared with the many other contexts in MCF like [CM1] and [BW], our definition has a minus sign.

We first list some important properties of the first eigenvalue.

Proposition 6.1. *Suppose Σ is a self-shrinker with finite entropy.*

- ([CM1]) $\lambda_1(\Sigma) \geq 1$. If Σ is neither a sphere $S^n(\sqrt{2n})$ nor a generalized cylinder $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, then $\lambda_1(\Sigma) > 1$.
- ([BW]) If Σ is an asymptotically conical self-shrinker, then $\lambda_1(\Sigma) < +\infty$.

6.1. Convergence of the leading eigenvalues. In this section, we will always fix an asymptotically conical self-shrinker Σ , and M_t is an RMCF converging to Σ in C_{loc}^∞ sense as $t \rightarrow \infty$. We define $\lambda_1(t) := \lambda_1(M_t)$ to be the first eigenvalue of the linearized operator L_{M_t} on M_t , and we define $\lambda_1 = \lambda_1(\Sigma)$ to be the first eigenvalue of the linearized operator L_Σ on Σ . We will also localize the eigenvalues on the hypersurfaces. For any hypersurface M , and radius $R > 0$, we define $\lambda_1^R(M)$ to be the Dirichlet eigenvalue of L_M on $M \cap B_R$. By simple comparison argument we have $\lambda_1^R(M) \geq \lambda_1(M)$.

Our goal is to prove the following theorem:

Theorem 6.2. *Suppose (\star) , then $\lambda_1(M_t) \rightarrow \lambda_1(\Sigma)$ as $t \rightarrow \infty$.*

We need several lemmas to prove Theorem 6.2. The first Lemma shows that the RMCF M_t has a similar curvature bound as the self-shrinker Σ .

Lemma 6.3. *Suppose (\star) , then there exist constant $0 < T_0 < \infty$ and $0 < C < \infty$ only depending on the MCF \mathbf{M}_τ such that for $t > T_0$, $p \in M_t$, we have $|A|(p, t) \leq C(1 + |p|)^{-1}$.*

Proof. We divide \mathbb{R}^{n+1} into three parts. The first part is B_{R_0} for a sufficiently large R_0 . Then when T_0 is sufficiently large, on B_{R_0} , for $t > T_0$, M_t is a graph over Σ and the C^k -norm of the graph is small (depending on T_0). Then we can see that when $t > T_0$, inside B_{R_0} , M_t has uniformly bounded curvature $|A|$.

The second part is $B_{e^{t/2}\delta} \setminus B_{R_0}$, where $\delta > 0$ is a constant. Using the pseudolocality of RMCF, by the similar argument as Proposition 5.8 (also see [CS, Section 9]), inside this part $|A|$ is bounded by $C(1 + |x|)^{-1}$.

The third part is the domain outside $B_{e^{t/2}\delta}$. We fix this constant $\delta > 0$, and we consider the MCF \mathbf{M}_τ . Outside $B_\delta(0)$, \mathbf{M}_τ has no singularity when $\tau \leq 0$, thus $M_\tau \setminus B_\delta(0)$ has uniformly curvature upper bound (say C') when $\tau \leq 0$. Now we change the view back to RMCF, which says that outside $B_{e^{t/2}\delta}$, the curvature on M_t is bounded by $e^{-t/2}C'$. On the other hand, because \mathbf{M}_τ is an MCF of closed hypersurface, $\text{diam}(\mathbf{M}_\tau)$ is uniformly bounded (say C''). Hence M_t has diameter at most $C''e^{t/2}$. As a consequence, outside $B_{e^{t/2}\delta}(0)$, the curvature of M_t is also bounded by $C(1 + |x|)^{-1}$.

Combining three parts together gives us the desired curvature bound. \square

Next Lemma shows a uniform lower bound for the $\lambda_1(M_t)$.

Lemma 6.4. $\lambda_1(M_t) < C < +\infty$.

Proof. Suppose u is any smooth function on M_t , with $\int_{M_t} u^2 e^{-\frac{|x|^2}{4}} d\mu = 1$. By the uniform curvature upper bound, when $t > T_0$,

$$-\frac{\int_{M_t} [|\nabla u|^2 - (|A|^2 + 1/2)u^2] e^{-\frac{|x|^2}{4}} d\mu}{\int_{M_t} u^2 e^{-\frac{|x|^2}{4}} d\mu} \leq \frac{\int_{M_t} (C + 1/2)u^2 e^{-\frac{|x|^2}{4}} d\mu}{\int_{M_t} u^2 e^{-\frac{|x|^2}{4}} d\mu} \leq C + 1/2.$$

□

In the following we start to prove the convergence of $\lambda_1(M_t) \rightarrow \lambda_1(\Sigma)$. First we need some convergence properties for compact domains.

Proposition 6.5. *We have the following convergence properties:*

- (1) *Suppose M is a hypersurface with bounded entropy, then $\lambda_1^R(M) \rightarrow \lambda_1(M)$ as $R \rightarrow \infty$.*
- (2) *For fixed $R > 0$, $\lambda_1^R(M_t) \rightarrow \lambda_1^R(\Sigma)$ as $t \rightarrow \infty$.*

Proof. Item (1) was proved in Section 9 of [CM1]. Item (2) is a corollary of the following more general result. □

Proposition 6.6. *Suppose $(\Sigma, \partial\Sigma)$ is a manifold with boundary (in our case it is $\Sigma \cap B_R$), $\{g_t\}$ is a family of metrics on Σ , converging to a limit metric g_∞ in C^2 , and $\{V_t\}$ is a family of positive smooth functions converging to a limit function V_∞ in C^0 . Suppose $d\mu_t$ is the volume measure with respect to g_t . Then the leading eigenvalue of the self-adjoint operator L_t with respect to the functional $\int_\Sigma |\nabla u|_{g_t}^2 - V_t u^2 d\mu_t$ converges to the leading eigenvalue of $L = L_\infty$, as $t \rightarrow \infty$.*

Proof. Let $\lambda_1(t)$ be the first eigenvalue with respect to the time t moment, and λ_1 be the first eigenvalue with respect to the limit.

On one hand, suppose ϕ is an eigenfunction of λ_1 on the limit, then

$$\frac{\int_\Sigma |\nabla \phi|_{g_\infty}^2 - V_\infty \phi^2 d\mu_\infty}{\int_\Sigma \phi^2 d\mu_\infty} = -\lambda_1.$$

By our assumption and the minimizing property of the leading eigenvalue, we have

$$(6.1) \quad -\lambda_1(t) \leq \frac{\int_\Sigma |\nabla \phi|_{g_t}^2 - V_t \phi^2 d\mu_t}{\int_\Sigma \phi^2 d\mu_t} \rightarrow \frac{\int_\Sigma |\nabla \phi|_{g_\infty}^2 - V_\infty \phi^2 d\mu_\infty}{\int_\Sigma \phi^2 d\mu_\infty} = -\lambda_1$$

as $t \rightarrow \infty$. Thus we get $\limsup_{t \rightarrow \infty} -\lambda_1(t) \leq -\lambda_1$.

On the other hand, let ϕ_t be the leading eigenfunction with respect to g_t . We will assume that $\phi_t > 0$ and $\int_\Sigma \phi_t^2 d\mu_\infty = 1$. Then

$$-\lambda_1(t) = \frac{\int_\Sigma |\nabla \phi_t|_{g_\infty}^2 - V_\infty \phi_t^2 d\mu_\infty}{\int_\Sigma \phi_t^2 d\mu_\infty}.$$

Let us estimate

$$\left| \frac{\int_\Sigma |\nabla \phi_t|_{g_\infty}^2 - V_\infty \phi_t^2 d\mu_\infty}{\int_\Sigma \phi_t^2 d\mu_\infty} - \frac{\int_\Sigma |\nabla \phi_t|_{g_t}^2 - V_t \phi_t^2 d\mu_t}{\int_\Sigma \phi_t^2 d\mu_t} \right|.$$

Since $g_t \rightarrow g_\infty$ in C^2 , we have

- $d\mu_t \rightarrow d\mu_\infty$ in C^0 . Moreover, $\frac{d\mu_t}{d\mu_\infty} = 1 + o(t)$. When we write $o(t)$ we mean a quantity $\rightarrow 0$ as $t \rightarrow \infty$;
- $||\nabla\phi_t|_{g_t}^2 - |\nabla\phi_t|_{g_\infty}^2| \leq o(t)|\nabla\phi_t|_{g_\infty}^2$ where $o(t) \rightarrow 0$ in C^0 . Similarly, we can write $|\nabla\phi_t|_{g_t}^2 = |\nabla\phi_t|_{g_\infty}^2(1 + o(t))$;
- We can also write $V_t = V_\infty(1 + o(t))$ (note V_∞ is positive).

From the above items, we conclude that

$$\frac{\int_\Sigma |\nabla\phi_t|_{g_t}^2 - V_t\phi_t^2 d\mu_t}{\int_\Sigma \phi_t^2 d\mu_t} = \frac{\int_\Sigma |\nabla\phi_t|_{g_\infty}^2 - V_\infty\phi_t^2 d\mu_\infty}{\int_\Sigma \phi_t^2 d\mu_\infty} (1 + o(t)).$$

We remark that $o(t)$ is a quantity depending on how much g_t and V_t are close to g_∞ and V_∞ respectively, but does not depend on ϕ_t . In conclusion, we have $-\lambda_1(t) \geq -\lambda_1(1 + o(t))$. As a result, $\liminf_{t \rightarrow \infty} -\lambda_1(t) \geq -\lambda_1$. Thus we conclude that $\lim \lambda_1(t) = \lambda_1$. \square

Lemma 6.7. *For all $\epsilon > 0$, there exists $0 < T < \infty$, such that $\lambda_1(M_t) \geq \lambda_1(\Sigma) - \epsilon$ if $t > T$. As a consequence, we have $\liminf_{t \rightarrow \infty} \lambda_1(t) \geq \lambda_1(\Sigma)$.*

Proof. The proof is similar to the proof of Theorem 6.6. First, by (2) in Proposition 6.5, we can show $\lambda_1^R(M_t) \rightarrow \lambda_1^R(\Sigma)$ for any fixed R , as $t \rightarrow \infty$. Second, we have $\lambda_1^R(M_t) \leq \lambda_1(M_t)$, and (1) in Proposition 6.5 shows that $\lambda_1^R(\Sigma) \rightarrow \lambda_1(\Sigma)$ as $R \rightarrow \infty$. Thus we conclude this lemma. \square

Lemma 6.8. *For all $\epsilon > 0$, there exists $0 < T < \infty$, such that $\lambda_1(M_t) \leq \lambda_1(\Sigma) + \epsilon$ if $t > T$. As a consequence, we have $\limsup_{t \rightarrow \infty} \lambda_1(t) \leq \lambda_1(\Sigma)$.*

Proof. We choose two constants R and κ to be determined later, and we assume t is sufficiently large such that $\lambda_1^{2R}(M_t) \geq (1 - \kappa)\lambda_1^{2R}(\Sigma) \geq (1 - \kappa)\lambda_1(\Sigma)$.

Suppose ϕ_1^t is a first eigenfunction on M_t . Then

$$\lambda_1(M_t) = -\frac{\int_{M_t} [|\nabla\phi_1^t|^2 - (|A|^2 + 1/2)(\phi_1^t)^2] e^{-\frac{|x|^2}{4}} d\mu}{\int_{M_t} (\phi_1^t)^2 e^{-\frac{|x|^2}{4}} d\mu}.$$

We also assume η is a smooth cut-off function, which is constant 1 on B_R , and 0 outside B_{2R} , with $|\nabla\eta| \leq C/R$. Define $v = \phi_1^t \cdot \eta$ and $w = \phi_1^t \cdot (1 - \eta)$. Then v is supported on $M_t \cap B_R$, and w is supported on $M_t \setminus B_R$, and $v + w = \phi_1^t$. Then

$$\frac{\int_{M_t} [|\nabla v|^2 - (|A|^2 + 1/2)v^2] e^{-\frac{|x|^2}{4}} d\mu}{\int_{M_t} v^2 e^{-\frac{|x|^2}{4}} d\mu} \geq -\lambda_1^{2R}(M_t),$$

$$\frac{\int_{M_t} [|\nabla w|^2 - (|A|^2 + 1/2)w^2] e^{-\frac{|x|^2}{4}} d\mu}{\int_{M_t} w^2 e^{-\frac{|x|^2}{4}} d\mu} \geq \frac{\int_{M_t} [-(C(1 + |x|)^{-2} + 1/2)w^2] e^{-\frac{|x|^2}{4}} d\mu}{\int_{M_t} w^2 e^{-\frac{|x|^2}{4}} d\mu} \geq -C(1+R)^{-2} - 1/2.$$

Also,

$$\begin{aligned}
& \int_{M_t} [\nabla v \cdot \nabla w] e^{-\frac{|x|^2}{4}} d\mu = \int_{M_t} [(\phi_1^t \nabla \eta + \eta \nabla \phi_1^t) \cdot (-\phi_1^t \nabla \eta + (1 - \eta) \nabla \phi_1^t)] e^{-\frac{|x|^2}{4}} d\mu \\
(6.2) \quad & = \int_{M_t} [-(\phi_1^t)^2 |\nabla \eta|^2 + \eta(1 - \eta) |\nabla \phi_1^t|^2 + \phi_1^t(1 - 2\eta) \nabla \eta \cdot \nabla \phi_1^t] e^{-\frac{|x|^2}{4}} d\mu \\
& \geq \int_{M_t} [-C(\kappa)(\phi_1^t)^2 |\nabla \eta|^2 - \kappa |\nabla \phi_1^t|^2] e^{-\frac{|x|^2}{4}} d\mu \\
& \geq \int_{M_t} [-(C(\kappa) |\nabla \eta|^2 - \kappa(|A|^2 + 1/2)) (\phi_1^t)^2 \\
& \quad - \kappa (|\nabla \phi_1^t|^2 - (|A|^2 + 1/2) (\phi_1^t)^2)] e^{-\frac{|x|^2}{4}} d\mu \\
(6.3) \quad & \text{(note } |\nabla \eta| \leq CR^{-1}) \geq -(C(\kappa)R^{-2} + \kappa|A|^2 + \kappa/2 - \kappa\lambda_1(t)) \int_{M_t} (\phi_1^t)^2 e^{-\frac{|x|^2}{4}} d\mu \\
& \geq -C\kappa \int_{M_t} (\phi_1^t)^2 e^{-\frac{|x|^2}{4}} d\mu
\end{aligned}$$

In the last inequality we assume that R is sufficiently large. Let $m := \min\{-\lambda_1^{2R}(M_t), -C(1+R)^{-2} - 1/2\}$. Then $m < 0$ and when R is sufficiently large, $-\lambda_1^{2R}(\Sigma) \leq \frac{3}{2}(-C(1+R)^{-2} - 1/2)$, therefore when t is very large, $-\lambda_1^{2R}(M_t) \leq (-C(1+R)^{-2} - 1/2)$. Hence $m = -\lambda_1^{2R}(M_t)$. We have

$$\begin{aligned}
(6.4) \quad & \int_{M_t} [|\nabla v|^2 - (|A|^2 + 1/2)v^2] e^{-\frac{|x|^2}{4}} d\mu + \int_{M_t} [|\nabla w|^2 - (|A|^2 + 1/2)w^2] e^{-\frac{|x|^2}{4}} d\mu \geq \\
& m \left(\int_{M_t} v^2 e^{-\frac{|x|^2}{4}} d\mu + \int_{M_t} w^2 e^{-\frac{|x|^2}{4}} d\mu \right) = m \int_{M_t} (\phi_1^t)^2 e^{-\frac{|x|^2}{4}} d\mu.
\end{aligned}$$

Together with (6.2) and (6.4), we get

$$\begin{aligned}
-\lambda_1(M_t) & = \frac{\int_{M_t} [|\nabla \phi_1^t|^2 - (|A|^2 + 1/2)(\phi_1^t)^2] e^{-\frac{|x|^2}{4}} d\mu}{\int_{M_t} (\phi_1^t)^2 e^{-\frac{|x|^2}{4}} d\mu} \\
& \geq m - C\kappa = -\lambda_1^{2R}(M_t) - C\kappa \geq (1 - \kappa)\lambda_1(\Sigma) - C\kappa.
\end{aligned}$$

So for any $\epsilon > 0$, we can choose κ small such that $-\lambda_1(M_t) \geq -\lambda_1(\Sigma) - \epsilon$. This shows the desired result. \square

Proof of Theorem 6.2. Combine Lemma 6.7 and Lemma 6.8. \square

6.2. Convergence of the leading eigenfunctions. In this section, we prove the following theorem:

Theorem 6.9. *After normalization, the first eigenfunction ϕ_1^t on M_t converges to ϕ_1 as $t \rightarrow \infty$ in C_{loc}^∞ sense, where ϕ_1 is the first eigenfunction on Σ .*

Proof. By the standard elliptic theory, ϕ_1^t is smooth and positive on M_t . For any fixed ball of radius R , when t is very large, a part of M_t can be written as a graph over $\Sigma \cap B_R$. So if we fix a point P on Σ with $|P| \leq \sqrt{2n}$, and we denote by P_t the corresponding point on M_t (P_t is well-defined when t is sufficiently large). We can divide ϕ_1^t by a constant so that $\phi_1^t(P_t) = 1$. Then standard Harnack inequality and Schauder estimate show that ϕ_1^t converge locally smoothly to a limit function ϕ_1 . Finally, because $\lambda_1(t) \rightarrow \lambda_1$, ϕ_1 must satisfy the equation $L\phi_1 = \lambda_1(\Sigma)\phi_1$. Thus ϕ_1 is exactly the first eigenfunction if it belongs to the space $\left\{ u : \int_{\Sigma} u^2 e^{-\frac{|x|^2}{4}} d\mu < \infty \right\}$ (see [BW, Proposition 4.1]).

On M_t , ϕ_1^t satisfies the equation $\mathcal{L}_{M_t}\phi_1^t + (|A|^2 + 1/2 - \lambda_1(t))\phi_1^t = 0$. So by maximum principle, $\max \phi_1^t$ is attained at somewhere $|A|^2 + 1/2 \geq \lambda_1(t)$. By the decay of $|A|$ (see Lemma 6.3) and the convergence of the first eigenvalues, when t is sufficiently large, $\max \phi_1^t$ is attained only in $M_t \cap B_{r_0}$ with some fix $r_0 > 0$. Then by the Harnack inequality and entropy bound of the flow, we know that $\int_{M_t} (\phi_1^t)^2 e^{-\frac{|x|^2}{4}} d\mu$ is uniformly bounded. Hence $\int_{\Sigma} (\phi_1)^2 e^{-\frac{|x|^2}{4}} d\mu$ is finite. \square

We may also assume that the normalization is $\|\phi_1^t\|_{L^2(M_t)} = 1$, and similarly ϕ_1^t converges to a limit ϕ_1 , which is a first eigenfunction on Σ . At this moment, we do not know that whether $\|\phi_1\|_{L^2(\Sigma)} = 1$. Our goal is to show that $\|\phi_1\|_{L^2(\Sigma)} = 1$.

Corollary 6.10. *There exists $R_0 > 0$ and a function $\eta : [R_0, \infty) \rightarrow \mathbb{R}_+$ such that $\lim_{R \rightarrow \infty} \eta(R) = 0$, and $\|\phi_1^t(1 - \chi_R)\|_{L^2(M_t)} < CR^{-1}$.*

Proof. From the decay rate of ϕ_1 in [BW, Proposition 4.1], there exists $R_0 > 0$ such that for $p \in \Sigma$, $|p| > R \geq R_0$, $\phi_1(p) < C|p|^{-1}$. Then by the locally smooth convergence Theorem 6.9, $\phi_1^t \rightarrow \phi_1$ on $\Sigma \cap B_{R+1}$ for some constant $\eta > 0$. Thus, $\phi_1^t \leq C|p|^{-1}$ on $\Sigma \cap \partial B_{R_0}$ as well, when t is sufficiently large.

In the proof of Theorem 6.9, we have showed that by maximum principle, $\max \phi_1^t$ is attained at somewhere $|A|^2 + 1/2 \geq \lambda_1(t)$. This is also true is we replace maximum by local maximum. Thus, when R is sufficiently large, we know that $\phi_1^t \leq CR^{-1}$ on $M_t \setminus B_R$. This implies that

$$\|\phi_1^t(1 - \chi_R)\|_{L^2(M_t)} \leq \left(\int_{M_t \setminus B_R} CR^{-2} e^{-\frac{|x|^2}{4}} \right)^{1/2} \leq CR^{-1},$$

where in the last inequality we use the fact that the entropy of M_t is uniformly bounded. This yields the desired inequality. \square

As a consequence, we can show that ϕ_1^t actually converges to ϕ_1 with $\|\phi_1\|_{L^2(\Sigma)} = 1$. It is a consequence of the following theorem.

Theorem 6.11. *There exists $R_0 > 0$ with the following significance. Let us normalize ϕ_1^t such that $\|\phi_1^t\|_{L^2(M_t)} = 1$. Then for any $R > R_0$, $\|\overline{\phi_1^t} - \phi_1\|_{L^2(M_t \cap B_R)} \rightarrow 0$ as $t \rightarrow \infty$ and $\|\phi_1\|_{L^2(\Sigma)} = 1$. Here $\overline{\phi_1^t}$ is the function ϕ_1^t pulled back to Σ .*

Proof. The proof is the same as Theorem 6.9. Recall that $\max \phi_1^t$ is attained only in $M_t \cap B_{r_0}$ with some fix $r_0 > 0$. Also, $\|\phi_1^t\|_{L^2(M_t)} = 1$ implies that $\max \phi_1^t \geq \lambda(t)^{-1} \geq C$ for some constant C , inside $M_t \cap B_{r_0}$. Again, by Harnack inequality, we know that ϕ_1^t has an uniformly lower bound c_0 on $M_t \cap B_{r_0}$. As a consequence of Arzela-Ascoli theorem, ϕ_1^t smoothly converges to a limit on $M_t \cap B_{r_0}$. From Theorem 6.9, this limit ϕ_1 must be a first eigenfunction on Σ .

It remains to show $\|\phi_1\|_{L^2(\Sigma)} = 1$. Corollary 6.10 implies that $\|\phi_1^t(1 - \chi_R)\|_{L^2(M_t)} < CR^{-1}$ when t is sufficiently large. Thus $\|\phi_1^t \chi_R\|_{L^2(M_t)} > 1 - CR^{-1}$. This implies that $\|\phi_1\|_{L^2(\Sigma \cap B_R)} \geq 1 - CR^{-1}$, and as a consequence $\|\phi_1\|_{L^2(\Sigma)} \geq 1$. On the other hand, $\|\phi_1\|_{L^2(\Sigma)} \leq \limsup \|\phi_1^t\|_{L^2(M_t)} = 1$. So $\|\phi_1\|_{L^2(\Sigma)} = 1$. \square

6.3. Spectral gap between the first two eigenvalues. In this section, we study the gap between the first eigenvalue and the second eigenvalue. We denote by $\lambda_2(t) = \lambda_2(M_t)$ and $\lambda_2 = \lambda_2(\Sigma)$.

Theorem 6.12. *Suppose (\star) . Then there exists $C > 0$ such that*

$$\lambda_1(t) - \lambda_2(t) \geq C.$$

Proof. We prove by contradiction. Assume $\lambda_1(t) - \lambda_2(t) \rightarrow 0$ as $t \rightarrow \infty$. Then since $\lambda_1(t) \rightarrow \lambda_1(\Sigma)$, we have $\lambda_2(t) \rightarrow \lambda_1(\Sigma)$ as $t \rightarrow \infty$. Let ϕ_2^t be a second eigenfunction of L_{M_t} , namely ϕ_2^t satisfies the equation

$$\mathcal{L}_{M_t} \phi_2^t + (|A|^2 + 1/2 - \lambda_2(t)) \phi_2^t = 0.$$

Suppose p_t is the maximum point of ϕ_2^t and q_t is the minimum point of ϕ_2^t . By elliptic theory, ϕ_2^t must change sign, so $\phi_2^t(p_t) > 0$ and $\phi_2^t(q_t) < 0$. Moreover, by maximum principle, $|A|^2 + 1/2 - \lambda_2(t) \geq 0$ at p_t and q_t . When t is sufficiently large, $\lambda_2(t)$ is close to $\lambda_1(\Sigma) > 1$. Therefore Lemma 6.3 implies that $|p_t|$ and $|q_t|$ are bounded by some uniform constant $R_0 > 0$.

By multiplying a constant, we may assume $\phi_2^t(p_t) = 1$, and by the gradient estimate of ϕ_2^t on a bounded domain $M_t \cap B_{R_0}$, we have $\phi_2^t(q_t) \geq -C$ where C is a constant. This also implies a global L^∞ bound of ϕ_2^t , hence a weighted $W^{1,2}$ bound. Then elliptic theory shows that after passing to a subsequence, ϕ_2^t converges to a limit ϕ_2 on Σ in C_{loc}^∞ sense, and if $p_t \rightarrow p$, we have $\phi_2(p) = 1$. Moreover, ϕ_2 satisfies the equation $\mathcal{L}_\Sigma \phi_2 + (|A|^2 + 1/2 - \lambda_1) \phi_2 = 0$. As a consequence, ϕ_2 must be a first eigenvalue of L_Σ . This means that $\phi_2 > 0$ everywhere, which contradicts to the fact that $\phi_2^t(q_t) < 0$ for all t . \square

APPENDIX A. TRANSPLANTATION OF FUNCTIONS

In this section we fix a radius R sufficiently large, and we only consider t sufficiently large, such that a part of M_t can be written as a smooth graph of a function f on $\Sigma \cap B_R$. We denote this part by M_t^R . Recall that in Section 3.1 we introduce the notion called transplantation, namely for any function g defined on M_t^R , we defined a function $\varphi_t^* g$ defined on $\Sigma \cap B_R$, such that $\varphi_t^* g(x) = g(x + f(x)\mathbf{n})$. For simplicity, sometimes we just write \bar{g} to denote $\varphi_t^* g$.

Suppose N_t^R is a graph of function g over M_t^R . If f and g are sufficiently small (in C^ℓ norm for $\ell \geq 4$), then N_t^R can be viewed as a graph of function v where v is defined on Σ . The following lemma is similar to Theorem C.2 in [SX] and we omit the proof here.

Lemma A.1 (Theorem C.2 in [SX]). *For any $\varepsilon' > 0$, there exists $\varepsilon > 0$ with the following significance: if for $\|f\|_{C^\ell(\Sigma \cap B_{R+1})} \leq \varepsilon$, and $\|g\|_{C^\ell(\Sigma \cap B_{R+1})} \leq \varepsilon$, then*

$$\|v - (f + \bar{g})\|_{C^\ell(\Sigma \cap B_R)} \leq \varepsilon' \|\bar{g}\|_{C^\ell(\Sigma \cap B_{R+1})}$$

The above bound requires $\|f\|_{C^\ell}$ and $\|g\|_{C^\ell}$ are sufficiently small. On the asymptotically conical part, f and g may not have a small C^ℓ norm. So we also need the following estimate, in the spirit of pseudolocality theorem.

Lemma A.2. *For any $\varepsilon' > 0$ and integer $\ell > 0$, there exists $\varepsilon > 0$ with the following significance: suppose M_t and \widetilde{M}_t are two RMCFs as in Section 4.2, and $m(\cdot, t)$ is the graph function of M_t over Σ and $v(\cdot, t)$ is the graph function of \widetilde{M}_t over M_t . Suppose at $t = 0$, $\|v(\cdot, 0)\|_{C^{2,\alpha}(M_0^c)} \leq \varepsilon$ and $\|m(\cdot, 0)\|_{C^{2,\alpha}(\Sigma \cap B_r)} \leq \varepsilon$. Let $u(\cdot, t)$ be the graph function of \widetilde{M}_t on $\Sigma \cap B_{\mathbf{r}(M_t)}$. Then on $(B_{\mathbf{r}(M_t)} \setminus B_r) \cap \Sigma$, at time t , for $r + 1 < R < \mathbf{r}(M_t) - 2$, we have*

$$\|\nabla^k(u(\cdot, t) - (m(\cdot, t) + \bar{u}))\|_{C^0((B_{R+1} \setminus B_R) \cap \Sigma)} \leq C\varepsilon' \|\nabla^k \bar{u}\|_{C^0((B_{R+2} \setminus B_{R-1}) \cap \Sigma)}$$

for $k = 0, 1, 2, \dots, \ell$.

The proof uses Lemma A.1 on the asymptotically conical region of Σ . In fact, the asymptotically conical region is a part of MCF after rescaling. Before the rescaling, the MCF is a graph very close to a cone. Thus the desired estimate is just the usual transplantation after a rescaling.

APPENDIX B. POLAR-SPHERICAL TRANSPLANTATION AND THE L -OPERATOR

As we have explained in Section 4.2, we introduce a polar-spherical coordinates approach to transplant functions on $M_t^{\mathbf{r}(t)}$ to $\Sigma^{\mathbf{r}(t)}$. In this appendix, we shall study this transplantation in more details and in particular give the estimate of the error term \mathcal{Q} in (4.3) (see Lemma B.2 below).

Suppose Σ is a fixed asymptotically conical shrinker. Then Σ is very close to a cone $\mathcal{C} := \{r\theta, \theta \in \mathcal{S} \subset \mathbb{S}^n(1), r \geq 0\}$ far away from the origin, where \mathcal{S} is a codimension one submanifold of $\mathbb{S}^n(1)$. Then there is R sufficiently large such that $\Sigma \setminus B_R$ can be identified with $\mathcal{C} \setminus B_R$. In the following we will use (r, θ) to describe the points on $\Sigma \setminus B_R$, such that for $x = (r, \theta)$, $|x| = r$ and $x/|x|$ is perpendicular to θ on the unit sphere.

In the following, $\bar{r} = \bar{r}(r)$ is a smooth function which identifies with the r when $r \geq R + 1$, and equals to 0 when $r \leq R$. Then any function m defined on $\Sigma \setminus B_R$ can be written as $c(r, \theta)\bar{r} + f(x)$. Now suppose M is a hypersurface which is very close to Σ so that M can also be written as a graph over \mathcal{C} outside B_R (but possibly inside a much larger ball). Then we can write M as a graph of $m(x) = c(r, \theta)\bar{r} + f(x)$, where f is supported on B_{R+1} . Here m is defined as follows: inside B_R , M can be written as a graph of function f over Σ , and outside

B_{R+1} , m is the difference between them when they are both viewed as spherical graphs over \mathcal{C} .

We consider an RMCF M_t converging to a conical shrinker in the C_{loc}^∞ sense as $t \rightarrow \infty$. We next estimate the function c in the representation $c(r, \theta)r + f$ above. Let \mathbb{A}_{r_1, r_2} denote the annulus with inner radius $r_1 > R$ and outer radius r_2 . Suppose M is a graph of $m(x) = c(r, \theta)\bar{r}$ over $\mathcal{C} \cap \mathbb{A}_{r_1, r_2}$, with $\|m\|_{C^{2, \alpha}} \leq \varepsilon$. Then aM , the rescaling of M by a , is a graph of the function $c(r/a, \theta)\bar{r}$ over $\mathcal{C} \cap \mathbb{A}_{ar_1, ar_2}$. As a consequence, the $C^{2, \alpha}$ -norm of c is unchanged after rescaling (actually, the derivative terms become better). If M is a graph of $m(x) = c(r, \theta)\bar{r}$ over $\Sigma \cap \mathbb{A}_{r_1, r_2}$ rather than $\mathcal{C} \cap \mathbb{A}_{r_1, r_2}$, similar analysis holds true when R is sufficiently large. In fact, from [BW] and [CS] we know that the geometry of $\Sigma \cap \mathbb{A}_{ar_1, ar_2}$ converges to $\mathcal{C} \cap \mathbb{A}_{ar_1, ar_2}$ as $a \rightarrow \infty$.

As a consequence, the pseudolocality estimates for the graph of RMCF do hold for this setting. We have the following lemma arguing as Section 5.1:

Lemma B.1. *Suppose Σ is an asymptotically conical shrinker, and M_t is an RMCF converging to Σ . Let $R > R_0$ sufficiently large. Then when T sufficiently large, suppose M_T is a graph of function $u(\cdot, T) = c(\cdot, T)\bar{r}$ on $\mathbb{A}_{R-1, R} \cap \Sigma$ with $|\nabla c(\cdot, T)| \leq \gamma$, then for any integer $\ell \geq 0$, M_{T+t} is a graph of $u(\cdot, T+t)$ on $\mathbb{A}_{e^{t/2}(R-1), e^{t/2}R} \cap \Sigma$ with*

$$\|\nabla_r^\ell c(\cdot, T+t)\|_{C^0} \leq e^{-t/2} C_\ell, \quad \|\nabla_\theta^\ell c(\cdot, T+t)\|_{C^0} \leq C_\ell,$$

for $t > 1/2$. Moreover, $C_\ell \rightarrow 0$ as $\gamma \rightarrow 0$.

We next define φ as in Definition 4.3. It is clear that φ preserves the Gaussian weight. We next show that the weighted Sobolev norm behaves well under this pullback. Suppose M can be written as a graph of function $m(x) = c(r, \theta)\bar{r} + f(x)$ over $\Sigma^{r(t)}$, with $\|f\|_{C^{2, \alpha}} \leq \varepsilon$, $\|c\|_{C^{2, \alpha}} \leq \varepsilon$. This actually implies that $|D\varphi^{-1}| \leq (1 + C\varepsilon)$. Suppose v is a function over M and define $u^* = \varphi^*v$. Then we have

$$(B.1) \quad \begin{aligned} \int_{M^{r(t)}} |v(x)|^2 e^{-\frac{|x|^2}{4}} d\mu(x) &= \int_{\Sigma^{r(t)}} |u^*(y)|^2 e^{-\frac{|\varphi_t^{-1}(y)|^2}{4}} \det(D\varphi^{-1}) d\mu(y) \\ &\leq (1 + C\varepsilon) \int_{\Sigma^{r(t)}} |u^*|^2 e^{-\frac{|y|^2}{4}} d\mu(y). \end{aligned}$$

This implies that the weighted L^2 -norm is comparable after pulling back the functions on M to Σ . We have similar conclusions for weighted higher order Sobolev norms.

We next discuss the difference between the two operators $\varphi^*(L_M v)$ and $L_\Sigma(\varphi^*v)$ under the transplantation. We have the following Lemma.

Lemma B.2. *Suppose M satisfies*

- (1) *the function $m : \Sigma \rightarrow \mathbb{R}$ can be written as $m(x) = c(r, \theta)\bar{r} + f(x)$, where $\|c\|_{C^{2, \alpha}} < \varepsilon$ and $\|f\|_{C^{2, \alpha}} \leq \varepsilon$;*
- (2) *the second fundamental form decays as $|\nabla^j A|(x) \leq C \frac{1}{|x|^{j+1}}$, $j = 0, 1, 2$.*

then for all smooth $v : M \rightarrow \mathbb{R}$, we have the following pointwise bound

$$|\varphi^*(L_M v) - L_\Sigma(\varphi^* v)|(x) \leq C\varepsilon(1 + |\nabla\varphi^* v(x)| + |\nabla^2\varphi^* v(x)| + |x| \cdot |\nabla\varphi^* v(x)|).$$

Proof. For simplicity we denote by $u = \varphi^* v$. Inside B_{R+1} the calculation is the standard computations on the normal bundle, similar to (3.2).

Let us do the computations outside B_{R+1} , namely we only care about the situation where $m = c(r, \theta)\bar{r}$, which is the distance between M and Σ as two sections of the conical neighbourhood of \mathcal{C} .

Firstly, we consider a graph N over the cone \mathcal{C} , locally given by $c(r, \theta)r$. Temporarily we use φ to denote the retraction to \mathcal{C} . Let $r\theta \in \mathcal{C}$. We define $\gamma_\theta(t)$ to be the exponential curve of $\theta \in \mathcal{C} \cap S^n(1)$ inside the unit sphere, along the unit normal direction at θ . We also view it as a curve in \mathbb{R}^{n+1} . Then N is the submanifold given by $\{r\gamma_\theta(c(r, \theta))\}$. By rescaling, we may assume $(r, \theta) = (1, \theta)$.

Now we are going to use Fermi coordinate near $(r, \theta) \in \mathcal{C}$. For the definition of Fermi coordinate, see [LZ, Lemma A.2]. It is canonical to define a Fermi coordinate near $(r, \theta) \in \mathcal{C}$ by conically extending the Fermi coordinate of $\theta \in \mathcal{C} \cap S^n(1)$ in the unit sphere. With this coordinate, the term $|\varphi^*((\Delta_M + |A|^2 + 1/2)v) - (\Delta_M + |A|^2 + 1/2)(\varphi^* v)|(x)$ can be computed by using a slightly twisted metric of the Euclidean metric at (r, θ) , while the error is estimated in [LZ, Appendix A], implies that we have the desired bound. Similarly, $|\varphi^*(\langle v, x \rangle) - \langle \varphi^* v, x \rangle|(x)$ is straight forwardly bounded by $C\varepsilon|x| \cdot |\nabla\varphi^* v(x)|$.

To extend the above estimate to the situation that M being a graph over Σ , we can first write both of them as graphs over \mathcal{C} . Then the calculations are mostly verbatim. Thus we get the desired bound. \square

APPENDIX C. PROOF OF PROPOSITION 4.6

In this appendix, we give the proof of Proposition 4.6.

Proof of Proposition 4.6. We shall compare the solutions to the equation (4.3) and (4.5). The proof is similar to Proposition 4.3 of [CM2]. The main difficulty is that we do not have the Dirichlet boundary value condition for v^* . Thus, there is a boundary term when doing integration by parts that need to be absorbed.

For simplicity and without loss of generality, we consider only $n = 0$ and suppress the subscript of v_n . We consider first the L^2 bound. Define $w = v^* \chi - v$ where χ is a smooth function that is 1 on the set $\{|x| \leq \mathbf{r}(t) - 1\}$ and 0 on $\{|x| \geq \mathbf{r}(t)\}$ with $\mathbf{r}(t) := \mathbf{r}(M_{T_{\mathbb{H}}+t})$. Hence $\frac{\partial}{\partial t} \chi$ is of order $O(e^{t/2})$ and supported on the annulus $\mathbb{A}_{\mathbf{r}(t)} := \{\mathbf{r}(t) - 1 \leq |x| \leq \mathbf{r}(t)\}$ and $\nabla_x \chi$ and $\nabla_x^2 \chi$ are bounded by a constant and supported on the boundary of $B_{\mathbf{r}(t)}$.

We have $\partial_t v^* = L_\Sigma v^* + \mathcal{Q}(v^*)$ over $\Sigma^{\mathbf{r}(t)}$ (equation (4.3)) and $\partial_t v = L_\Sigma v$ over Σ (equation (4.5)), so we get

$$\begin{aligned} \partial_t w &= \chi \partial_t v^* + v^* \partial_t \chi - \partial_t v = \chi(L_\Sigma v^* + \mathcal{Q}(v^*)) + v^* \partial_t \chi - \partial_t v \\ (C.1) \quad &= L_\Sigma w + \chi \mathcal{Q}(v^*) + (-2\nabla \chi \cdot \nabla v^* - v^* \mathcal{L}_\Sigma \chi + v^* \partial_t \chi) \\ &:= L_\Sigma w + \chi \mathcal{Q}(v^*) + \mathcal{B}. \end{aligned}$$

Here the term \mathcal{B} is supported on the annulus $\mathbb{A}_{\mathbf{r}(t)}$ and is bounded by $O(\|v^*\|_{C^1(\mathbf{r}(t))}e^t)$.

We next estimate $\mathcal{Q}(v^*) = P(v^*) + \varphi_t^* \mathcal{Q}(v^*)$ over the ball $B_{\mathbf{r}(t)}$ (c.f. (4.4)). The estimate $|P(v^*)| \leq \delta(|\text{Hess}_{v^*}| + |\nabla v^*| + |v^*| + |x||\nabla v^*|)$ is given in Lemma B.2. We also have the same estimate for $\varphi_t^* \mathcal{Q}(v^*)$, following from Lemma 4.4 of [CM2]. In the following, we shall use the abbreviation $V = |A|^2 + \frac{1}{2}$. We will need the following estimate of Ecker (see also [BW] Lemma B.1)

$$(C.2) \quad \int_{\Sigma} |f|^2 |x|^2 e^{-\frac{|x|^2}{4}} d\mu \leq 4 \int_{\Sigma} (nf^2 + 4|\nabla f|^2) e^{-\frac{|x|^2}{4}} d\mu$$

to suppress the slow linear growth term $\delta|x||\nabla v^*|$ in \mathcal{Q} .

Then we compute

$$(C.3) \quad \begin{aligned} \partial_t \frac{1}{2} \int_{\Sigma} |w(t)|^2 e^{-\frac{|x|^2}{4}} &= \int_{\Sigma} w \partial_t w e^{-\frac{|x|^2}{4}} = \int_{\Sigma} w (L_{\Sigma} w + \chi \mathcal{Q} + \mathcal{B}) e^{-\frac{|x|^2}{4}} \\ &= \int_{\Sigma} (-|\nabla w|^2 + V w^2 + w \chi \mathcal{Q}(v^*)) e^{-\frac{|x|^2}{4}} + O(\|v^*, v\|_{C^1(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}) \\ &\leq C \int_{\Sigma} |w(t)|^2 e^{-\frac{|x|^2}{4}} + \int_{\Sigma} |\mathcal{Q}(v^*)|^2 \chi e^{-\frac{|x|^2}{4}} + O(\|v^*, v\|_{C^1(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}) \\ &\leq C \int_{\Sigma} |w(t)|^2 e^{-\frac{|x|^2}{4}} + \delta \int_{\Sigma} (|\text{Hess}_{v^*}| + |\nabla v^*| + |v^*|)^2 \chi e^{-\frac{|x|^2}{4}} + O(\|v^*, v\|_{C^1(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\frac{\mathbf{r}(t)^2}{4}}) \end{aligned}$$

where we use a constant C to bound V and in the last \leq , we use Lemma B.2 and (C.2).

We next have

$$(C.4) \quad \begin{aligned} \partial_t \frac{1}{2} \int_{\Sigma} (|\nabla w|^2 - V w^2) e^{-\frac{|x|^2}{4}} d\mu &= \int_{\Sigma} (\nabla w \cdot \nabla \partial_t w - V w \partial_t w) e^{-\frac{|x|^2}{4}} d\mu \\ &= \int_{\Sigma} -(L_{\Sigma} w)(L_{\Sigma} w + \chi \mathcal{Q}(v^*) + \mathcal{B}) e^{-\frac{|x|^2}{4}} d\mu \\ &\leq \int_{\Sigma} (-|L_{\Sigma} w|^2 + L_{\Sigma} w \cdot \mathcal{Q}(v^*)) \chi e^{-\frac{|x|^2}{4}} d\mu + O(\|v^*, v\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}) \\ &\leq \int_{\Sigma} 2|\mathcal{Q}(v^*)|^2 \chi e^{-\frac{|x|^2}{4}} d\mu + O(\|v^*, v\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}) \\ &\leq \int_{\Sigma} \delta (|\text{Hess}_{v^*}| + |\nabla v^*| + |v^*|)^2 \chi e^{-\frac{|x|^2}{4}} d\mu + O(\|v^*, v\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}), \end{aligned}$$

Combining (C.3) and (C.4), we get

$$\partial_t \|w\|_{\mathcal{Q}}^2 \leq C(A) \|w\|_{L^2}^2 + \int_{\Sigma} \delta (|\text{Hess}_{v^*}| + |\nabla v^*| + |v^*|)^2 \chi e^{-\frac{|x|^2}{4}} d\mu + O(\|v^*, v\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}).$$

Integrating over the time interval $[0, 1]$, we get (noting $\|w(0)\|_{\mathcal{Q}} = 0$)

$$\|w(1)\|_{\mathcal{Q}}^2 \leq \int_0^1 e^{C(A)t} \int_{\Sigma} \delta (|\text{Hess}_{v^*}| + |\nabla v^*| + |v^*|)^2 \chi e^{-\frac{|x|^2}{4}} d\mu dt + O(\|v^*, v\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}).$$

Let us next estimate $\int_{\Sigma} (|\text{Hess}_{v^*}| + |\nabla v^*| + |v^*|)^2 \chi e^{-\frac{|x|^2}{4}} d\mu$. First we recall that drifted Bochner formula $|\text{Hess}_u|^2 = \frac{1}{2} \mathcal{L}|\nabla u|^2 - \text{Ric}_{-|x|^2/4}(\nabla u, \nabla u) - \langle \nabla \mathcal{L}u, \nabla u \rangle$. Notice that the Bakry-Emery Ricci on the conical shrinker Σ is always bounded, so we have

$$\begin{aligned} \int_{\Sigma} |\text{Hess}_{v^*}|^2 \chi e^{-\frac{|x|^2}{4}} d\mu &\leq \int_{\Sigma} \left(\frac{1}{2} \mathcal{L}|\nabla v^*|^2 + C|\nabla v^*|^2 - \langle \nabla \mathcal{L}v^*, \nabla v^* \rangle \right) \chi e^{-\frac{|x|^2}{4}} d\mu \\ &= \int_{\Sigma} (C|\nabla v^*|^2 + (\mathcal{L}v^*)^2) \chi d\mu + \int_{\Sigma} \left(\frac{1}{2} \nabla|\nabla v^*|^2 + \mathcal{L}v^* \nabla v^* \right) \cdot \nabla \chi e^{-\frac{|x|^2}{4}} d\mu \\ &= \int_{\Sigma} (C|\nabla v^*|^2 + (\mathcal{L}v^*)^2) \chi e^{-\frac{|x|^2}{4}} d\mu + O(\|v^*\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}). \end{aligned}$$

Next we calculate

$$\begin{aligned} \partial_t \int_{\Sigma} |\nabla v^*|^2 \chi e^{-\frac{|x|^2}{4}} e^{-\frac{|x|^2}{4}} d\mu &\leq \int_{\Sigma} (\langle \nabla \partial_t v^*, \nabla v^* \rangle \chi + |\nabla v^*|^2 \frac{d}{dt} \chi) e^{-\frac{|x|^2}{4}} d\mu \\ &= \int_{\Sigma} (\langle \nabla \mathcal{L}v^*, \nabla v^* \rangle \chi e^{-\frac{|x|^2}{4}} d\mu + O(\|v^*\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4})) \\ &= \int_{\Sigma} (-|\mathcal{L}v^*|^2 + C|\nabla v^*|^2 + C|v^*|^2) \chi e^{-\frac{|x|^2}{4}} d\mu + O(\|v^*\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}). \end{aligned}$$

So by adding these two inequalities we get

$$\int_{\Sigma} |\text{Hess}_{v^*}|^2 \chi e^{-\frac{|x|^2}{4}} d\mu + \partial_t \int_{\Sigma} |\nabla v^*|^2 \chi e^{-\frac{|x|^2}{4}} d\mu \leq C \int_{\Sigma} |\nabla v^*| \chi e^{-\frac{|x|^2}{4}} d\mu + O(\|v^*\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}).$$

Integrate this, we get

$$\begin{aligned} &\int_T^{T+1} \int_{\Sigma} |\text{Hess}_{v^*}|^2 \chi e^{-\frac{|x|^2}{4}} d\mu dt + e^{-C(T+1)} \int_{\Sigma} |\nabla v^*(\cdot, T+1)|^2 \chi e^{-\frac{|x|^2}{4}} d\mu \\ &\leq e^{-CT} \int_{\Sigma} |\nabla v^*(\cdot, T)|^2 \chi e^{-\frac{|x|^2}{4}} d\mu + C \int_T^{T+1} \int_{\Sigma} |v^*|^2 \chi e^{-\frac{|x|^2}{4}} d\mu dt + O(\|v^*\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4}). \end{aligned}$$

We then absorb $O(\|v^*, v\|_{C^2(\mathbb{A}_{\mathbf{r}(t)})}^2 e^{-\mathbf{r}(t)^2/4})$ by $\|v^*\|_Q^2$ using the definition of T_{\sharp} (Proposition 4.5) as well as the Proposition D.5 on v . This completes the proof. \square

APPENDIX D. HEAT KERNEL OF L ON CONICAL SHRINKER

In this subsection, we sketch the existence of the heat kernel of the linearized operator L_{Σ} on a conical shrinker Σ and give the necessary gradient estimate for the solution to the heat equation $\partial_t v = L_{\Sigma} v$.

D.1. Existence of the heat kernel. We prove the following result on the existence of the heat kernel in this subsection.

Proposition D.1. *There exists heat kernel $\mathcal{H}(x, y, t) := \sum_i e^{-\mu_i t} u_i(x) u_j(x)$ defined on $\Sigma \times \Sigma \times (0, \infty)$, satisfies*

- (1) $\partial_t \mathcal{H}(x, y, t) = L_x \mathcal{H}(x, y, t)$,
- (2) $\mathcal{H}(x, y, t) = \mathcal{H}(y, x, t)$,
- (3) *the reproducing property (2.2),*
- (4) *the semi-group property (2.3).*

In Section 2.1, we have discussed the properties of heat kernel on a closed RMCF, where the existence comes from the classical theory of heat kernel of self-adjoint elliptic operators. For non-compact hypersurface, there is no classical theory, and we need to show the existence of the heat kernel.

The proof is very similar to Colding-Minicozzi's proof of existence of drifted heat kernel on a noncompact shrinker, see [CM5, Section 5]. We will only sketch the essential modifications here, and the rest of the steps are verbatim from [CM5].

The key step is to estimate the eigenfunctions of the linearized operator L . Suppose ϕ_i is the i -th eigenfunction on the conical Σ with $\int \phi_i^2 e^{-\frac{|x|^2}{4}} = 1$. In [BW], Bernstein-Wang proved that ϕ_1 decays at infinity. Here we show similar results for higher eigenfunctions.

Suppose ϕ satisfies $L\phi = \mu\phi$ on Σ . Note that by the elliptic spectral theory, there are only finitely many $\mu \geq 0$. Then we have $\mathcal{L}|\phi| + (|A|^2 + 1/2)|\phi| \geq \mu|\phi|$. Since the conical shrinker has uniformly bounded curvature, we conclude that $\mathcal{L}|\phi| \geq (\mu - C)|\phi| = \mu'|\phi|$. Then we write $\Sigma_t = \sqrt{-t}\Sigma$ to be the MCF associated to the shrinker Σ , and let

$$(D.1) \quad v(y, t) = (-t)^{-\mu'} |\phi| \left(\frac{y}{\sqrt{-t}} \right).$$

Then similar to the computations in [CM5, Lemma 2.4], we have $(\partial_t - \Delta_{\Sigma_t})v \leq 0$. Moreover, since in our case $v \geq 0$, [CM5, Lemma 2.11] still holds, hence the proof of [CM5, Theorem 2.1] is still true. In particular, we can show that

$$|\phi|^2(x) \leq C_n \lambda(\Sigma) \|\phi\|_{L^2(\Sigma)}^2 (4 + |x|^2)^{-2\mu'}.$$

Here C_n is a dimensional constant and $\lambda(\Sigma)$ is the entropy of Σ . Thus, we get the following pointwise estimate:

Lemma D.2. *Suppose ϕ satisfies $L\phi = \mu\phi$ on Σ , then $|\phi|(x) \leq C \|\phi\|_{L^2(\Sigma)} (4 + |x|^2)^{-(\mu-C)}$.*

From now on, we will assume $\{\phi_i\}_{i=1}^\infty$ are eigenfunctions of L on Σ , with $\|\phi_i\|_{L^2(\Sigma)} = 1$. With our notation, μ_i is non-increasing in i . Let us define the spectrum counting function $\mathcal{N}(\mu)$ to be the number of eigenvalues $\mu_i \geq \mu$ counted multiplicity. In order to study $\mathcal{N}(\mu)$, we introduce the space of functions defined on a ancient MCF M_t :

$$\tilde{\mathcal{P}}_d := \{u \mid (\partial_t - \Delta_{M_t})u \leq 0, |u(x, t)| \leq C(1 + |x|^d + |t|^{d/2}) \text{ for all } x \in M_t, t < 0\}.$$

Compare with the space \mathcal{P}_d defined in [CM5], $\tilde{\mathcal{P}}_d$ consists of subsolutions to the heat equation on the MCF. However, subsolutions still fit in [CM5]. Repeat Colding-Minicozzi's proof, we have the following theorem, which is [CM5, Theorem 0.5]:

Theorem D.3. *There exists a dimensional constant C_n so that for ancient MCF M_t with $\lambda(M_t) \leq \lambda_0$ and $d \geq 1$, then $\dim \tilde{\mathcal{P}}_d \leq C_n \lambda_0 d^n$. Here $\lambda(M_t)$ is the entropy of M_t .*

The proof is exactly the same as the proof in [CM5]. Although now the space consists of subsolutions, one can see that the discussions in [CM5] (like [CM5, Lemma 3.4], [CM5, Lemma 4.1] - actually in the paper cited there, the lemma was stated for subsolutions) are also valid for subsolutions.

Once we have Theorem D.3, let $M_t = \sqrt{-t}\Sigma$ and we notice that by (D.1) we can transfer an eigenfunction ϕ_i to a subsolution v_i of heat equation on $\sqrt{-t}\Sigma$ which belongs to $\tilde{\mathcal{P}}_{-2\mu+2C}$. Thus, a consequence of Theorem D.3 is the following Theorem, which is the analogy of Theorem [CM5, Theorem 0.7].

Theorem D.4. *There exists C_n so that $\mathcal{N}(\mu) \leq C_n \lambda(\Sigma) (-(\mu - C))^n$ for $-(\mu - C) \geq 1/2$.*

This theorem immediately shows the following properties of the spectrum of L on Σ :

- the spectrum of L on Σ is discrete,
- the eigenvalues $\mu_i \rightarrow -\infty$ as $i \rightarrow \infty$.

Now let us sketch the proof of Proposition D.1. The proof is the same as proof of [CM5, Theorem 5.3]. The only change is to replace \mathcal{L} with L . All the previous Theorems suggest that the proof is almost verbatim.

D.2. Estimate of v . Let v be the solution to the equation $\begin{cases} \partial_t v = L_\Sigma v \\ v|_{t=T} = v_0 \end{cases}$. Here $v_0 \geq 0$ is

supported on $B_{\mathbf{r}(M_T)}$. We can solve v by convoluting the initial data with the heat kernel of L_Σ . Equivalently, we can expand v into Fourier series in the weighted L^2 space, namely $v = \sum_{i=1}^{\infty} e^{\lambda_i t} a_i \phi_i$, if $v_0 = \sum_{i=1}^{\infty} a_i \phi_i$. We want to estimate v_0 near the boundary of $B_{\mathbf{r}(M_{T+t})}$ where $t \leq 1$.

Proposition D.5. *There exists a constant C only depends on the shrinker Σ such that*

$$\|v\|_{C^2(\Sigma)} \leq C e^{Ct} \|v_0\|_{C^1(B_{\mathbf{r}(M_T)})}.$$

Proof. First we prove $\|v\|_{C^0(\Sigma)} \leq C e^t \|v_0\|_{C^0(B_{\mathbf{r}(M_T)})}$. First we prove that $|v|(x) \rightarrow 0$ as $x \rightarrow \infty$. In fact, we can always choose $\pm A\phi_1$ as the upper barrier and lower barrier of the initial data v_0 , where A is a large constant. Then $v(x) \leq e^{\lambda_1 t} A\phi_1(x)$ (this can be seen from expressing the solutions to the linearized equation by convoluting with the heat kernel), and since $\phi_1(x) \rightarrow 0$ as $x \rightarrow \infty$ (see [BW]), $|v|(x) \rightarrow 0$ as $x \rightarrow \infty$.

Then the maximum of v is attained at some bounded region, so we can use maximum principle to show that $\partial_t(\max v) \leq (|A|^2 + 1/2) \max v$. We also have the similar estimate for minimum. Therefore, $\|v\|_{C^0(\Sigma)} \leq C e^t \|v_0\|_{C^0(B_{\mathbf{r}(M_T)})}$.

Next we estimate the gradient of v . $f := \frac{1}{2}|\nabla v|^2$ satisfies the equation:

$$\partial_t f = \mathcal{L}f + 2(|A|^2 + 1/2)f - |\nabla^2 v|^2 - (\text{Ric} + \nabla x)(\nabla v, \nabla v) + v\langle \nabla v, \nabla |A|^2 \rangle.$$

Observe that on Σ , $\text{Ric} + \nabla x$, $|A|^2$ and $\nabla |A|^2$ are all bounded from above. Moreover from previous discussion $\|v\|_{C^0}$ is bounded by $Ce^t\|v_0\|_{C^0}$. Therefore, $\partial_t f \leq \mathcal{L}f + Cf + Ce^t\|v_0\|_{C^0}$, where C only depends on Σ . As a consequence, $g := e^{-Ct}f - C'e^{Ct}\|v_0\|_{C^0}^2$ satisfies the equation $\partial_t g \leq \mathcal{L}g \leq Lg$. Then use the same argument as above for maximum shows that $g \leq Ce^t\|g\|_{C^0(B_{\mathbf{r}(M_T)})}$. Hence $f \leq Ce^{Ct}(\|f\|_{C^0(B_{\mathbf{r}(M_T)})} + \|v_0\|_{C^0(B_{\mathbf{r}(M_T)})}^2)$. Thus,

$$\|v\|_{C^1(\partial B_{\mathbf{r}(M_T+t)})} \leq Ce^{Ct}\|v_0\|_{C^1(B_{\mathbf{r}(M_T)})}.$$

Finally we prove the C^2 estimate of v . The idea is similar to C^1 estimate. Let $f = \frac{1}{2}|D^2v|^2$, then

$$\partial_t f = \langle D^2v, D^2\partial_t v \rangle = \langle D^2v, D^2\Delta v \rangle - \frac{1}{2}\langle D^2v, D^2\langle x, \nabla v \rangle \rangle + \langle D^2v, D^2(|A|^2 + 1/2)v \rangle.$$

Then similar to the calculation of previous C^1 case, we have

$$\partial_t f \leq \mathcal{L}f + Cf + C|v|^2 + C|\nabla v|^2 \leq \mathcal{L}f + Cf + C\|v\|_{C^1}^2.$$

Here C depends on the curvature and derivative of the curvature on the shrinker, and we know they are uniformly bounded. Then just like above, but now we choose $g := e^{-Ct}f - C'e^{Ct}\|v_0\|_{C^1}^2$, then g is an upper barrier and maximum principle shows that $g \leq Ce^t\|g\|_{C^0(B_{\mathbf{r}(M_T)})}$. So again we obtain $f \leq Ce^{Ct}(\|f\|_{C^0(B_{\mathbf{r}(M_T)})} + \|v_0\|_{C^1(B_{\mathbf{r}(M_T)})}^2)$, and as a consequence we get $\|v\|_{C^2(\partial B_{\mathbf{r}(M_T+t)})} \leq Ce^{Ct}\|v_0\|_{C^2(B_{\mathbf{r}(M_T)})}$. \square

We use the integration by parts formula in the proof of Proposition 4.6. To verify that we can do integration by parts of higher order derivatives of v , we need the following Lemma:

Lemma D.6. *We have $\int_{\Sigma} (|v|^2 + |\nabla v|^2 + |\mathcal{L}v|^2)e^{-\frac{|x|^2}{4}} d\mu < \infty$.*

Proof. Proposition D.5 implies that $\int_{\Sigma} (|v|^2 + |\nabla v|^2)e^{-\frac{|x|^2}{4}} d\mu < \infty$. So it only remains to show $\int_{\Sigma} |\mathcal{L}v|^2 e^{-\frac{|x|^2}{4}} d\mu < \infty$, and we only need to prove that

$$\int_{\Sigma} \|v\|_{C^2(\Sigma)} e^{-\frac{|x|^2}{4}} d\mu < \infty, \text{ and } \int_{\Sigma} |\langle x, \nabla v \rangle|^2 e^{-\frac{|x|^2}{4}} d\mu < \infty.$$

For the first integral, we use Proposition D.5 $\|v\|_{C^2(\Sigma)} \leq Ce^{Ct}\|v_0\|_{C^2(B_{\mathbf{r}(M_T)})}$, so the first integral is finite; for the second integral, we use Cauchy-Schwarz inequality to see that $|\langle x, \nabla v \rangle|^2 \leq |x|^2|\nabla v|^2$, so the second integral is finite. \square

As a consequence, we have the following corollary.

Corollary D.7. *For any u satisfies $\int_{\Sigma} (|u|^2 + |\nabla u|^2) e^{-\frac{|x|^2}{4}} d\mu < \infty$, we have*

$$\int_{\Sigma} \nabla u \cdot \nabla v e^{-\frac{|x|^2}{4}} d\mu = - \int_{\Sigma} u \mathcal{L} v e^{-\frac{|x|^2}{4}} d\mu = - \int_{\Sigma} v \mathcal{L} u e^{-\frac{|x|^2}{4}} d\mu, \quad \text{and}$$

$$\int_{\Sigma} \nabla u \cdot \nabla \mathcal{L} v e^{-\frac{|x|^2}{4}} d\mu = - \int_{\Sigma} \mathcal{L} u \cdot \mathcal{L} v e^{-\frac{|x|^2}{4}} d\mu.$$

Proof. This is a consequence of the previous lemma together with Corollary 3.10 of [CM1]. \square

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