

Infinite Lifting of an Action of Symplectomorphism Group on the set of bi-Lagrangian structures

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To Kira and Thierry Rothen's family.

Abstract

We consider a smooth $2n$ -manifold M endowed with a bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$. That is, ω is a symplectic form and $(\mathcal{F}_1, \mathcal{F}_2)$ is a pair of transversal Lagrangian foliations on (M, ω) . Such structures have an important geometric object called the Hess Connection. Among the many importance of these connections, they allow to classify affine bi-Lagrangian structures.

In this work, we show that a bi-Lagrangian structure on M can be lifted as a bi-Lagrangian structure on its trivial bundle $M \times \mathbb{R}^n$. Moreover, the lifting of an affine bi-Lagrangian structure is also an affine bi-Lagrangian structure. We define a dynamic on the symplectomorphism group and the set of bi-Lagrangian structures (that is an action of the symplectomorphism group on the set of bi-Lagrangian structures). This dynamic is compatible with Hess connections, preserves affine bi-Lagrangian structures, and can be lifted on $M \times \mathbb{R}^n$. This lifting can be lifted again on $(M \times \mathbb{R}^{2n}) \times \mathbb{R}^{4n}$, and coincides with the initial dynamic (in our sense) on $M \times \mathbb{R}^n$ for some bi-Lagrangian structures. Results still hold by replacing $M \times \mathbb{R}^{2n}$ with the

tangent bundle TM of M or its cotangent bundle T^*M for some manifolds M .

Keywords: Symplectic, Symplectomorphism, Bi-Lagrangian, Para-Kähler, Hess connection.

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1 Introduction

Let (M, ω) be a symplectic manifold. This means, ω is a symplectic form on M (that is, ω is a 2-form which is closed (the exterior differential (derivative) $d\omega$ vanishes) and nondegenerate as a bilinear form on the set of vector fields on M denoted $\mathfrak{X}(M)$), see [5, 11]. A bi-Lagrangian structure on (M, ω) is a pair $(\mathcal{F}_1, \mathcal{F}_2)$ of transversal Lagrangian foliations; or, a bi-Lagrangian structure on M is a triplet $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ where $(\mathcal{F}_1, \mathcal{F}_2)$ is a pair of transversal Lagrangian foliations on the symplectic manifold (M, ω) , see [1, 2, 3, 6, 7, 8, 9, 12]. In both cases, $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ is called a bi-Lagrangian manifold. Some details on Lagrangian foliations are given in §1.1.

Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. The Hess connection associated to $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ is the symplectic connection ∇ (that is, ∇ is a torsion-free connection parallelizing ω) which preserves the foliations, see [1, 2, 3, 6, 7, 8, 9, 12]. The existence and uniqueness of such a connection have been proved in [9], and it has been highlighted in [1, 2, 3]. Hess connections are particular cases of Bott connections (which are linear connections preserving the foliations, see [14, 15]). Bott connections are greatly used in the theory of the geometric quantization of real polarization (see [13] for example). Let us mention that a bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ on a manifold M corresponds one to one to a para-Kähler structure (G, F) on M (that is, G is a pseudo-Riemannian metric on M and F is a para-complex structure on M which permutes with G in the following sense: $G(F(\cdot), F(\cdot)) = -G(\cdot, \cdot)$). The three tensors ω , G and F are connected by the relation: $\omega(\cdot, \cdot) = G(F(\cdot), \cdot)$, see [6, 7, 8, 12]. Moreover, the Levi-Civita connection of G is the Hess connection of $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$, see [6, 7, 8, 12]. Therefore, bi-Lagrangian manifolds are at the interface of symplectic, semi-Riemannian and almost product (para-complex) manifolds. They are the areas of geometric quantization (see [9]) and of Koszul-Vinberg Cohomology (see [4]).

Before we can explain more precisely and prove our results, it is necessary to present some definitions, fix some notations and formulate some known results we need.

1.1 Definitions and notations

We assume that all the objects are smooth throughout this paper.

Let M be an m -manifold. By a p -dimensional, class C^r , $0 \leq r \leq \infty$ foliation \mathcal{F} of M we mean a decomposition of M into a union of disjoint connected subsets $\{\mathcal{F}_x\}_{x \in M}$, called the leaves of the foliation, with the following property: every point y in M has a neighborhood U and a system of local, class C^r coordinates $(y^1, \dots, y^m) : U \rightarrow \mathbb{R}^m$ such that for each leaf \mathcal{F}_x the components of $U \cap \mathcal{F}_x$ are described by the equations $y^{p+1} = \text{constant}, \dots, y^m = \text{constant}$, see [10].

The expressions $T\mathcal{F} \subset TM$ and $\Gamma(T\mathcal{F})$ (or $\Gamma(\mathcal{F})$) denote the tangent bundle to \mathcal{F} and the set of sections of $T\mathcal{F}$ respectively.

Let $\psi : M \rightarrow N$ be a diffeomorphism. The push forward $\psi_*\mathcal{F} = \{\psi_*\mathcal{F}_x\}_{x \in M}$ of \mathcal{F} by ψ is a foliation, and

$$\Gamma(\psi_*\mathcal{F}) := \{\psi_*X, X \in \Gamma(\mathcal{F})\} = \psi_*\Gamma(\mathcal{F}). \quad (1.1)$$

For every k -manifold M' , the set $M' \times \mathbb{R}^k$ is called the trivial bundle of M' . We say that a manifold is parallelizable when its tangent bundle is diffeomorphic to its trivial bundle. We denote by \mathcal{M}^π the set of parallelizable manifolds. Note that: every Lie group belongs to \mathcal{M}^π ; if M' is a k -manifold which can be covered by a single chart, then $M' \in \mathcal{M}^\pi$; every connected 1-manifold is an element of \mathcal{M}^π ; since the tangent bundle of the product of two manifolds and the product of their tangent bundles are diffeomorphic, then the product of two manifolds in \mathcal{M}^π also belongs to \mathcal{M}^π .

If the manifold M is endowed with a symplectic form ω (as a consequence, $m = 2n$), a foliation \mathcal{F} is Lagrangian if for every $X \in \Gamma(\mathcal{F})$, $\omega(X, Y) = 0$ if and only if $Y \in \Gamma(\mathcal{F})$. That is, the orthogonal section

$$\Gamma(\mathcal{F})^\perp = \{Y \in \mathfrak{X}(M) : \omega(X, Y) = 0, X \in \Gamma(\mathcal{F})\}$$

of $\Gamma(\mathcal{F})$ is equal to $\Gamma(\mathcal{F})$. A bi-Lagrangian structure on M consists on a pair $(\mathcal{F}_1, \mathcal{F}_2)$ of transversal Lagrangian foliations together with a symplectic form ω . As a consequence, $TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2$. We denote by $\mathcal{B}_l(M)$ the set of bi-Lagrangian structures on M .

Let $(\mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian structure on a symplectic $2n$ -manifold (M, ω) . Every point in M has an open neighborhood U which is the domain

of a chart whose local coordinates $(p^1, \dots, p^n, q^1, \dots, q^n)$ are such that

$$\begin{cases} \Gamma(\mathcal{F}_1)|_U = \left\langle \frac{\partial}{\partial p^1}, \dots, \frac{\partial}{\partial p^n} \right\rangle, \\ \Gamma(\mathcal{F}_2)|_U = \left\langle \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n} \right\rangle. \end{cases}$$

Such a chart, and such local coordinates, are said to be adapted to the bi-Lagrangian structure $(\mathcal{F}_1, \mathcal{F}_2)$. Moreover, if

$$\omega = \sum_{i=1}^n dq^i \wedge dp^i,$$

then such a chart, and such local coordinates, are said to be adapted to the bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$.

Let $\pi : M \times \mathbb{R}^{2n} \rightarrow M$ be the natural projection. We define $\Gamma(\mathcal{F}_1^\pi)$ and $\Gamma(\mathcal{F}_2^\pi)$ as follows

$$\begin{cases} \Gamma(\mathcal{F}_1^\pi) = \Gamma(\mathcal{F}_1) + \left\langle \frac{\partial}{\partial \xi_{n+1}}, \dots, \frac{\partial}{\partial \xi_{2n}} \right\rangle \subset \Gamma(T(M \times \mathbb{R}^{2n})), \\ \Gamma(\mathcal{F}_2^\pi) = \Gamma(\mathcal{F}_2) + \left\langle \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n} \right\rangle \subset \Gamma(T(M \times \mathbb{R}^{2n})) \end{cases}$$

where $\xi_i, i = 1, \dots, 2n$ are coordinates in \mathbb{R}^{2n} .

Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian $2n$ -manifold. Let us write

$$\text{Lift}(M, \omega, \mathcal{F}_1, \mathcal{F}_2) = (M \times \mathbb{R}^{2n}, \tilde{\omega}, \mathcal{F}_1^\pi, \mathcal{F}_2^\pi),$$

and

$$\text{Lift}^{k+1}(M, \omega, \mathcal{F}_1, \mathcal{F}_2) = \text{Lift}^k(M \times \mathbb{R}^{2n}, \tilde{\omega}, \mathcal{F}_1^\pi, \mathcal{F}_2^\pi), \quad k \in \mathbb{N}.$$

We show that, $\text{Lift}^k(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ exists for every $k \in \mathbb{N}$ (see Corollary 2.2); this means, $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ is infinitely liftable.

Note that the set $(\mathbb{R}^m)^*$ of linear forms on \mathbb{R}^m and \mathbb{R}^m are diffeomorphic. Depending on the context, \mathbb{R}^m will sometimes be considered as $(\mathbb{R}^m)^*$.

A symplectomorphism ψ between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a diffeomorphism $\psi : M_1 \rightarrow M_2$ such that $\psi^*\omega_2 = \omega_1$. Observe that the set $\text{Symp}(M_1, \omega_1)$ of all symplectomorphisms from (M_1, ω_1) to itself is a group.

Let $\text{Conn}(M)$ be the set of linear connections on M . Let $\nabla \in \text{Conn}(M)$. The torsion tensor T_∇ (or simply T if there is no ambiguity) and curvature tensor R_∇ (or simply R) are given respectively by

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in \mathfrak{X}(M)$$

and

$$R_{\nabla}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M)$$

where $[X, Y] := X \circ Y - Y \circ X$ is the Lie bracket of X and Y .

We say that a bi-Lagrangian structure is affine when its Hess connection ∇ is a curvature-free connection; that is, ∇ is flat. We denote by $\mathcal{B}_{lp}(M)$ the set of affine bi-Lagrangian structures on M . The set $\mathcal{B}_{lp}(M)$ is characterized in Theorem 1.5.

We say that a connection ∇

- parallelizes ω if $\nabla\omega = 0$; this means,

$$\omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) = X\omega(Y, Z), \quad X, Y, Z \in \mathfrak{X}(M); \quad (1.2)$$

- preserves \mathcal{F} if $\nabla\Gamma(\mathcal{F}) \subseteq \Gamma(\mathcal{F})$; more precisely,

$$\nabla_X Y \in \Gamma(\mathcal{F}), \quad (X, Y) \in \mathfrak{X}(M) \times \Gamma(\mathcal{F}). \quad (1.3)$$

Let $f, g \in C^\infty(M)$. The Poisson bracket $\{f, g\}$ of f and g is the smooth function defined by

$$\{f, g\} := \omega(X_f, X_g)$$

where X_f is the unique vector field verifying $\omega(X_f, Y) = -df(Y)$ for all $Y \in \mathfrak{X}(M)$. We call X_f the Hamiltonian vector field with Hamiltonian function f .

Einstein summation convention: an index repeated as sub and superscript in a product represents summation over the range of the index. For example,

$$\lambda^j \xi_j = \sum_{j=1}^n \lambda^j \xi_j.$$

In the same way,

$$X^j \frac{\partial}{\partial y^j} = \sum_{j=1}^n X^j \frac{\partial}{\partial y^j}.$$

Let $k \in \mathbb{N}$. Instead of $\{1, 2, \dots, k\}$ we will simply write $[k]$. The expression I_k stands for the $k \times k$ identity matrix in \mathbb{R} .

1.2 Technical tools

In this part, we present results that we will need in the following.

1.2.1 Symplectic manifolds

These manifolds provide ideal spaces for some dynamics. The collection of all symplectic manifolds forms a category where arrow or morphism set between two objects (symplectic manifolds) is the set of symplectomorphism between them. Among many results on this category, the cotangent bundle of a manifold is endowed with a so-called tautological 2-form, and a diffeomorphism between two manifolds lifts as a symplectomorphism on their cotangent bundles endowed with their respective tautological 2-forms. This part is devoted to the precise formulations of these results. For more familiarization with the concepts covered in this section, the reader is referred to [5, 11].

Let M be a m -manifold and let $q : T^*M \rightarrow M$ be the natural projection. The tautological 1-form or Liouville 1-form θ is defined by

$$\theta_{(x, \alpha_x)}(v) = \alpha_x(T_{(x, \alpha_x)}q(v)), \quad (x, \alpha_x) \in T^*M, v \in T_xM,$$

and its exterior differential $d\theta$ is called the canonical symplectic form or Liouville 2-form on the cotangent bundle T^*M .

Note that for any coordinate chart (U, x^1, \dots, x^m) on M , with associated cotangent coordinate chart $(T^*U, x^1, \dots, x^m, \xi_1, \dots, \xi_m)$ we have

$$\theta = \sum_1^m \xi_i dx_i,$$

and

$$d\theta = \sum_1^m d\xi_i \wedge dx_i.$$

Proposition 1.1. *Let M be a manifold. The cotangent bundle T^*M of M endowed with the canonical symplectic form $d\theta$ is a symplectic manifold.*

Proposition 1.2. *Let M_1 and M_2 be two diffeomorphic smooth manifolds, and let $\varphi : M_1 \rightarrow M_2$ be a diffeomorphism. The lift*

$$\hat{\varphi} : z = (x, \alpha_x) \mapsto (\varphi(x), (\varphi^{-1*}\alpha)_{\varphi(x)})$$

*of φ is a symplectomorphism from $(T^*M_1, d\theta_1)$ to $(T^*M_2, d\theta_2)$ where $d\theta_1$ and $d\theta_2$ are the canonical symplectic forms on T^*M_1 and T^*M_2 respectively.*

Remark 1.1. *Let M be a m -manifold. Proposition 1.1 and Proposition 1.2 still hold by replacing T^*M with $M \times \mathbb{R}^m$.*

1.2.2 Bi-Lagrangian (Para-Kähler) manifolds

These manifolds has been intensively explored in the past years, see [1, 2, 3, 4, 6, 7, 9, 12]. Among the many reasons to study them, they are the areas of geometric quantization and of Koszul-Vinberg Cohomology. In this part, we briefly give some needed results concerning Hess connections and affine bi-Lagrangian structures.

The Hess or bi-Lagrangian connection of a bi-Lagrangian structure is defined from the following theorem.

Theorem 1.3. *[9, Theor. 1] Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. There exists a unique torsion-free connection ∇ on M such ∇ parallelizes ω and preserves both foliations.*

Bi-Lagrangian connections are explicitly defined in the following result, see [1, p. 14], [2, p. 360], [3, p. 65].

Proposition 1.4. *Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. The Hess connection ∇ of $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ is*

$$\nabla_{(X_1, X_2)}(Y_1, Y_2) = (D(X_1, Y_1) + [X_2, Y_1]_1, D(X_2, Y_2) + [X_1, Y_2]_2) \quad (1.4)$$

where $D : \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$ is the map verifying

$$i_{D(X, Y)}\omega = L_X i_Y \omega, \quad (1.5)$$

and X_i is the \mathcal{F}_i -component of X for each $i \in [2]$.

The following result characterizes affine bi-Lagrangian structures.

Theorem 1.5. *[9, Theor. 2] Let $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian structure on a $2n$ -manifold M with ∇ as its Hess connection. Then the following assertions are equivalent.*

- a) *The connection ∇ is flat.*
- b) *Each point of M has a coordinate chart adapted to $(\omega, \mathcal{F}_1, \mathcal{F}_2)$.*

2 Statements and proofs of results

2.1 Statements of results

Our first result presents lifted bi-Lagrangian structures on the trivial bundles of some manifolds.

Theorem 2.1. *Let M be a $2n$ -manifold endowed with a bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$. Then $(\tilde{\omega}, \mathcal{F}_1^\pi, \mathcal{F}_2^\pi)$ is a bi-Lagrangian structure on $M \times \mathbb{R}^{2n}$ where $\tilde{\omega} = \pi^*\omega + d\theta$.*

Corollary 2.2. *A bi-Lagrangian $2n$ -manifold $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ is infinitely liftable; that is, $\text{Lift}^n(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ exists for every $n \in \mathbb{N}$.*

Before continuing to state our results, it is necessary to precise the following.

Remark 2.1. *Let (M, ω) be a symplectic manifold and let $\psi, \varphi : M \rightarrow M$ be two diffeomorphisms.*

Observe that

- $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$;
- the map $\nabla^\psi : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, $(X, Y) \mapsto \psi_* \nabla_{\psi_*^{-1} X} \psi_*^{-1} Y$ is an element of $\text{Conn}(M)$ for all $\nabla \in \text{Conn}(M)$, and

$$\begin{aligned} \nabla^{\psi \circ \varphi} &= (\psi \circ \varphi)_* \nabla_{(\psi \circ \varphi)_*^{-1}} (\psi \circ \varphi)_*^{-1} \\ &= \psi_* \circ \varphi_* \nabla_{\varphi_*^{-1} \circ \psi_*^{-1}} \varphi_*^{-1} \circ \psi_*^{-1} \\ &= (\nabla^\varphi)^\psi. \end{aligned}$$

Therefore, the symplectomorphism group $\text{Symp}(M, \omega)$ of (M, ω) acts on the left of

- $\mathfrak{X}(M)$ as follows

$$\begin{aligned} \text{Symp}(M, \omega) \times \mathfrak{X}(M) &\rightarrow \mathfrak{X}(M) \\ (\psi, X) &\mapsto \psi_* X \end{aligned}$$

- $\text{Conn}(M)$ as follows

$$\begin{aligned} \text{Symp}(M, \omega) \times \text{Conn}(M) &\rightarrow \text{Conn}(M) \\ (\psi, \nabla) &\mapsto \nabla^\psi \end{aligned}$$

In the following result, we show how a bi-Lagrangian structure can be pushed forward by a diffeomorphism. It will play a fundamental role in the proofs of Theorem 2.4 and Proposition 2.5.

Lemma 2.3. *Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold with ∇ as its Hess connection, and let N be a manifold which is diffeomorphic to M . Then for any diffeomorphism $\psi : M \rightarrow N$, $((\psi^{-1})^*\omega, \psi_*\mathcal{F}_1, \psi_*\mathcal{F}_2)$ is a bi-Lagrangian structure on N , with ∇^ψ as its Hess connection. Moreover, if $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ is affine, then so is $((\psi^{-1})^*\omega, \psi_*\mathcal{F}_1, \psi_*\mathcal{F}_2)$.*

Theorem 2.4. *Let M be a manifold endowed with a bi-Lagrangian structure. The map*

$$\begin{aligned} \triangleright : \text{Symp}(M, \omega) \times \mathcal{B}_l(M) &\longrightarrow \mathcal{B}_l(M) \\ (\psi, (\mathcal{F}_1, \mathcal{F}_2)) &\longmapsto (\psi_*\mathcal{F}_1, \psi_*\mathcal{F}_2) \end{aligned}$$

is a left group action. Moreover, for every $(\psi, (\mathcal{F}_1, \mathcal{F}_2)) \in \text{Symp}(M, \omega) \times \mathcal{B}_l(M)$, the Hess connection of $(\psi_*\mathcal{F}_1, \psi_*\mathcal{F}_2)$ is ∇^ψ where ∇ is that of $(\mathcal{F}_1, \mathcal{F}_2)$, and the inclusion $\triangleright(\text{Symp}(M, \omega) \times \mathcal{B}_{lp}(M)) \subset \mathcal{B}_{lp}(M)$ holds.

We claim that if $M \in \mathcal{M}^\pi$ is endowed with a bi-Lagrangian structure, then every bi-Lagrangian structure on M can be lifted on $M^\pi = TM$ or T^*M . The precise formulation of the claim is as follows:

Proposition 2.5. *Let M be a $2n$ -manifold endowed with a bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$, and belonging to \mathcal{M}^π . There exists a diffeomorphism $\Psi : M \times \mathbb{R}^{2n} \longrightarrow M^\pi$ such that $((\Psi^{-1})^*\tilde{\omega}, \Psi_*\mathcal{F}_1^\pi, \Psi_*\mathcal{F}_2^\pi)$ is a bi-Lagrangian structure on $M^\pi = TM$ or T^*M .*

Observe that every symplectomorphism on (M, ω) can be lifted as a symplectomorphism on $(M \times \mathbb{R}^{2n}, \tilde{\omega})$, this follows directly from Proposition 1.2 and Remark 1.1. Moreover, every bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ on M can be lifted as a bi-Lagrangian structure $(\tilde{\omega}, \mathcal{F}_1^\pi, \mathcal{F}_2^\pi)$ on $M \times \mathbb{R}^{2n}$ (Theorem 2.1). It is therefore natural to ask: How does \triangleright lift on $M \times \mathbb{R}^{2n}$? What is the relationship between $\hat{\triangleright}$ the lifting of \triangleright and $\tilde{\triangleright}$ the action (in the sense of Theorem 2.4) of $\text{Symp}(M \times \mathbb{R}^{2n}, \tilde{\omega})$ on $\mathcal{B}_l(M \times \mathbb{R}^{2n})$? By combing Theorem 2.1 and Theorem 2.4 we have the following result which is a powerful answer to the first question, and rightful the title of this paper.

Corollary 2.6. *The action \triangleright can be lifted infinitely in a similar sense as in Corollary 2.2.*

The set $\hat{\text{Symp}}(M, \omega)$ is defined as follows:

$$\hat{\text{Symp}}(M, \omega) = \{\hat{\psi} \in \text{Symp}(M \times \mathbb{R}^{2n}, \tilde{\omega}), \psi \in \text{Symp}(M, \omega)\},$$

see Proposition 1.2 and Remark 1.1 for more details.

Proposition 2.7. *Let $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian structure on a manifold M , and let ψ be a symplectomorphism on (M, ω) . If each point of M has a coordinate chart $(U, p^1, \dots, p^n, q^1, \dots, q^n)$ which is adapted to $(\mathcal{F}_1, \mathcal{F}_2)$ and verifying*

$$\hat{\psi}_* \frac{\partial}{\partial p^i} \in \Gamma((\psi_*\mathcal{F}_1)^\pi), \quad i \in [n], \quad (2.1)$$

then

$$\hat{\psi}\hat{\triangleright}(\mathcal{F}_1^\pi, \mathcal{F}_2^\pi) = \hat{\psi}\tilde{\triangleright}(\mathcal{F}_1^\pi, \mathcal{F}_2^\pi) := (\hat{\psi}_*\mathcal{F}_1^\pi, \hat{\psi}_*\mathcal{F}_2^\pi).$$

Remark 2.2. *Theorem 2.1, Theorem 2.4 and Proposition 2.7 which constitute the main results of this paper can be summarized as follows: Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. The diagram*

$$\begin{array}{ccc}
 (\mathcal{F}_1^\pi, \mathcal{F}_2^\pi) & \xrightarrow{\hat{\psi}_*} & ((\psi_* \mathcal{F}_1)^\pi, (\psi_* \mathcal{F}_2)^\pi) \\
 \uparrow \pi: \text{Lift} & & \uparrow \\
 (\mathcal{F}_1, \mathcal{F}_2) & \xrightarrow[\psi_*]{\triangleright: \text{Push forward}} & (\psi_* \mathcal{F}_1, \psi_* \mathcal{F}_2)
 \end{array}$$

exists. Moreover, the above diagram is commutative for some $(\mathcal{F}_1, \mathcal{F}_2)$.

2.2 Proofs of results

We start this section with the following observation.

Remark 2.3. *Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold with a curvature-free Hess connection ∇ . Let (G, F) be the para-Kähler structure associated to $(\omega, \mathcal{F}_1, \mathcal{F}_2)$. Each point of M has a coordinate chart $(U, p^1, \dots, p^n, q^1, \dots, q^n)$ such that for all $x \in U$*

$$\omega_x = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad F_x = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad \text{and } G_x = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Let $x \in U$. Since $R_\nabla = 0$, then by Theorem 1.5 there exists a coordinate chart $(U, p^1, \dots, p^n, q^1, \dots, q^n)$ such that

$$\Gamma(\mathcal{F}_1) = \left\langle \frac{\partial}{\partial p^1}, \dots, \frac{\partial}{\partial p^n} \right\rangle, \quad \Gamma(\mathcal{F}_2) = \left\langle \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n} \right\rangle \quad (2.2)$$

and

$$\omega_x = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

By (2.2), we have

$$F_x = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Thus since $G_x(X_x, Y_x) = \omega_x(F_x(X_x), Y_x)$, we obtain

$$G_x = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

2.2.1 Lifted bi-Lagrangian structures

Proof of Theorem 2.1. Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian $2n$ -manifold. We are going to show that $(M \times \mathbb{R}^{2n}, \tilde{\omega}, \mathcal{F}_1^\pi, \mathcal{F}_2^\pi)$ is a bi-Lagrangian manifold.

Let $(U, p^1, \dots, p^n, q^1, \dots, q^n)$ be a coordinate chart adapted to $(\mathcal{F}_1, \mathcal{F}_2)$, with $(U \times \mathbb{R}^{2n}, p^1, \dots, p^n, q^1, \dots, q^n, \xi_1, \dots, \xi_{2n})$ as its associated bundle coordinate chart. Then

$$\begin{cases} \Gamma(\mathcal{F}_1^\pi) = \left\langle \frac{\partial}{\partial p^1}, \dots, \frac{\partial}{\partial p^n}, \frac{\partial}{\partial \xi_{n+1}}, \dots, \frac{\partial}{\partial \xi_{2n}} \right\rangle, \\ \Gamma(\mathcal{F}_2^\pi) = \left\langle \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n} \right\rangle, \end{cases} \quad (2.3)$$

and the canonical symplectic form $d\theta$ is defined on $U \times \mathbb{R}^{2n}$ as follows

$$d\theta = \sum_1^{2n} d\xi_i \wedge dx_i.$$

Observe that $\tilde{\omega} = \pi^*\omega + d\theta$ is antisymmetric (as sum of two antisymmetric forms), closed (pull-backs commute with exterior derivatives) and non-degenerate (direct). That is, $\tilde{\omega}$ is a symplectic form on $M \times \mathbb{R}^{2n}$.

By (2.3), it follows that $(\mathcal{F}_1^\pi, \mathcal{F}_2^\pi)$ is a transversal pair of smooth Lagrangian distributions on $(M \times \mathbb{R}^{2n}, \tilde{\omega})$. Thus, it remains to show that $\mathcal{F}_i^\pi, i = 1, 2$ are completely integrable. Since the distributions \mathcal{F}_1^π and \mathcal{F}_2^π are similar, we only treat the case \mathcal{F}_1^π .

We are going to show that

$$d\theta([X, Y], Z) = 0, \quad X, Y, Z \in \Gamma(\mathcal{F}_1^\pi). \quad (2.4)$$

Note that

$$d\theta([X, Y], Z) = [X, Y]\theta(Z) - Z\theta([X, Y]) - \theta([X, Y], Z).$$

Let us write

$$\begin{cases} (y^i)_{i=1, \dots, 2n} = ((p^i)_{i=1, \dots, n}, (\xi_i)_{i=n+1, \dots, 2n}), \\ X = X^i \frac{\partial}{\partial y^i}, Y = Y^j \frac{\partial}{\partial y^j} \quad \text{and} \quad Z = Z^k \frac{\partial}{\partial y^k}. \end{cases}$$

Then

$$\begin{cases} [X, Y] = \mu^j \frac{\partial}{\partial y^j}, \\ [[X, Y], Z] = \lambda^j \frac{\partial}{\partial y^j}, \end{cases}$$

where

$$\begin{cases} \mu^j = X^i \frac{\partial Y^j}{\partial y^i} - Y^i \frac{\partial X^j}{\partial y^i}, \\ \lambda^j = \mu^i \frac{\partial Z^j}{\partial y^i} - Z^i \frac{\partial \mu^j}{\partial y^i}. \end{cases}$$

Thus,

$$[X, Y]\theta(z) = \mu^i \frac{\partial}{\partial y^i} (Z^k \xi_k), \quad (e_1)$$

$$\theta([X, Y], Z) = \lambda^j \xi_j, \quad (e_2)$$

$$Z\theta([X, Y]) = \frac{\partial}{\partial y^k} (\mu^i \xi_i). \quad (e_3)$$

Therefore

$$d\theta([X, Y], Z) = (e_1) - (e_2) - (e_3) = 0.$$

Observe that for every $X \in \mathfrak{X}(M \times \mathbb{R}^{2n})$, $\pi_* X$ depends only on components of X on M . Thus, by (2.4)

$$\tilde{\omega}([X, Y], Z) = 0, \quad X, Y, Z \in \Gamma(\mathcal{F}_1^\pi).$$

This completes the proof that $(\tilde{\omega}, \mathcal{F}_1^\pi, \mathcal{F}_2^\pi)$ is a bi-Lagrangian structure on $M \times \mathbb{R}^{2n}$.

This completes the proof of Theorem 2.1. \square

Corollary 2.2 follows by combining system (2.3) and Theorem 1.5.

2.2.2 Action of symplectomorphism group

Lemma 2.8. *Let (M, ω) be a symplectic manifold endowed with a Lagrangian foliation \mathcal{F} , and let N be a manifold such that M and N are diffeomorphic. Let $\psi : M \rightarrow N$ be a diffeomorphism. Then $\psi_* \mathcal{F}$ is a Lagrangian foliation on $(N, (\psi^{-1})^* \omega)$.*

Proof. Let $X = \psi_* X', Y = \psi_* Y' \in \Gamma(\psi_* \mathcal{F})$ where $X', Y' \in \Gamma(\mathcal{F})$.

On the one hand,

$$\begin{aligned} (\psi^{-1})^* \omega(X, Y) &= \omega(\psi_*^{-1} X, \psi_*^{-1} Y) \circ \psi^{-1} \\ &= \omega(X', Y') \circ \psi^{-1}. \end{aligned} \quad (2.5)$$

Since \mathcal{F} is a Lagrangian foliation, by (2.5) we get

$$(\psi^{-1})^* \omega(X, Z) = 0, \quad X \in \Gamma(\psi_* \mathcal{F}) \iff Z \in \Gamma(\psi_* \mathcal{F}). \quad (2.6)$$

That is, $\psi_* \mathcal{F}$ is a Lagrangian distribution on $(N, (\psi^{-1})^* \omega)$.

On the other hand,

$$[X, Y] = [\psi_* X', \psi_* Y'] = \psi_* [X', Y']. \quad (2.7)$$

By combining (2.7) and (2.6), we get $[X, Y] \in \Gamma(\psi_* \mathcal{F})$ for all $X, Y \in \Gamma(\psi_* \mathcal{F})$; this means, $\psi_* \mathcal{F}$ is completely integrable, so it is a Lagrangian foliation on $(N, (\psi^{-1})^* \omega)$ as claimed. \square

Proof of Lemma 2.3. Let $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian structure on a manifold M , let N be a manifold which is diffeomorphic to M , and let $\psi : M \rightarrow N$ be a diffeomorphism.

Since ψ is a diffeomorphism, then ψ_* is bijective. By combining this with Lemma 2.8, it follows that $(\mathcal{F}_1, \mathcal{F}_2)$ is a bi-Lagrangian foliation on $(N, (\psi^{-1})^* \omega)$. Now let ∇ be the Hess connection of $(\omega, \mathcal{F}_1, \mathcal{F}_2)$, we claim that ∇^ψ is that of $((\psi^{-1})^* \omega, \psi_* \mathcal{F}_1, \psi_* \mathcal{F}_2)$; more precisely,

1. ∇^ψ is a torsion-free connection:

$$(\nabla^\psi)_X Y - (\nabla^\psi)_Y X = [X, Y];$$

2. ∇^ψ parallelizes:

$$(\psi^{-1})^* \omega((\nabla^\psi)_X Y, Z) + (\psi^{-1})^* \omega(Y, (\nabla^\psi)_X Z) = X((\psi^{-1})^* \omega(Y, Z))$$

for any $X, Y, Z \in \mathfrak{X}(N)$;

3. ∇^ψ preserves both foliations:

$$(\nabla^\psi)_X Y \in \Gamma(\psi_* \mathcal{F}_i), (X, Y) \in \mathfrak{X}(N) \times \Gamma(\psi_* \mathcal{F}_i), i = 1, 2.$$

1. Let $X, Y \in \mathfrak{X}(N)$. Since ∇ is a torsion-free connection, we get

$$\begin{aligned} (\nabla^\psi)_X Y - (\nabla^\psi)_Y X &= \psi_*(\nabla_{\psi_*^{-1}X} \psi_*^{-1}Y - \nabla_{\psi_*^{-1}Y} \psi_*^{-1}X) \\ &= \psi_*[\psi_*^{-1}X, \psi_*^{-1}Y] \\ &= [X, Y]. \end{aligned}$$

2. Let $X, Y, Z \in \mathfrak{X}(N)$. Observe that

$$\begin{aligned} \Delta &:= (\psi^{-1})^* \omega(\nabla_X^\psi Y, Z) + (\psi^{-1})^* \omega(Y, \nabla_X Z) \\ &= (\psi^{-1})^* \omega(\psi_*(\nabla_{\psi_*^{-1}X} \psi_*^{-1}Y), Z) + (\psi^{-1})^* \omega(Y, \psi_*(\nabla_{\psi_*^{-1}X} \psi_*^{-1}Z)) \\ &= [\omega(\nabla_{\psi_*^{-1}X} \psi_*^{-1}Y, \psi_*^{-1}Z) + \omega(\psi_*^{-1}Y, \nabla_{\psi_*^{-1}X} \psi_*^{-1}Z)] \circ \psi^{-1} \\ &= [(\psi_*^{-1}X) (\omega(\psi_*^{-1}Y, \psi_*^{-1}Z))] \circ \psi^{-1} \tag{2.8} \\ &= [(\psi_*^{-1}X) ((\psi^{-1})^* \omega(Y, Z) \circ \psi)] \circ \psi^{-1} \\ &= X((\psi^{-1})^* \omega(Y, Z)). \end{aligned}$$

Note that (2.8) comes from the fact that ∇ parallelizes ω (see 1.2).

3. Let $X \in \mathfrak{X}(M)$ and $Y = \psi_* Y' \in \Gamma(\psi_* \mathcal{F}_i)$. We have

$$(\nabla^\psi)_X Y = \psi_*(\nabla_{\psi_*^{-1}X} \psi_*^{-1}Y) = \psi_*(\nabla_{\psi_*^{-1}X} Y'). \tag{2.9}$$

Since ∇ preserves \mathcal{F}_i (see 1.3), from (2.9) we have

$$\psi_*(\nabla_{\psi_*^{-1}X}Y') \in \Gamma(\psi_*\mathcal{F}_i).$$

That is,

$$(\nabla^\psi)_X Y \in \Gamma(\psi_*\mathcal{F}_i).$$

We use the following observation to end the proof.

Remark 2.4. *Observe that*

$$T_{\nabla^\psi}(X, Y) = \psi_*(T_{\nabla}(\psi_*^{-1}X, \psi_*^{-1}Y)), \quad X, Y \in \mathfrak{X}(N),$$

and

$$R_{\nabla^\psi}(X, Y)Z = \psi_*(R_{\nabla}(\psi_*^{-1}X, \psi_*^{-1}Y)\psi_*^{-1}Z), \quad X, Y, Z \in \mathfrak{X}(N).$$

Thus, if $R_{\nabla} = 0$, then $R_{\nabla^\psi} = 0$. As a consequence, \triangleright preserves affine bi-Lagrangian structures.

This completes the proof of Lemma 2.3. \square

Proof of Theorem 2.4. By Lemma 2.3, \triangleright is well defined, and by Remark 2.4, the inclusion $\triangleright(\text{Symp}(M, \omega) \times \mathcal{B}_{lp}(M)) \subset \mathcal{B}_{lp}(M)$ holds. The action properties of \triangleright come from those of the action of $\text{Symp}(M, \omega)$ on $\mathfrak{X}(M)$ (Remark 2.1). Theorem 2.4 is proved. \square

Proof of Proposition 2.5. By combining Theorem 2.1 and Lemma 2.3, Proposition 2.5 follows. \square

By equality (1.1) and Lemma 2.3, we get the following result.

Proposition 2.9. *Let $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian structure on a manifold M with (G, F) as its associated para-Kähler structure, and let $\psi : M \rightarrow N$ be a diffeomorphism. Then the paracomplex structure F^ψ associated to $((\psi^{-1})^*\omega, \psi_*\mathcal{F}_1, \psi_*\mathcal{F}_2)$ is*

$$F^\psi(X) = \psi_*F(\psi_*^{-1}X), \quad X \in \mathfrak{X}(N).$$

2.2.3 Lifting of \triangleright

Proof of Proposition 2.7. We start by defining $\hat{\triangleright}$ a lift of \triangleright , and its action properties will come from those of \triangleright .

Proposition 2.10. *Let (M, ω) be a symplectic manifold endowed with a bi-Lagrangian structure. Then a lift $\hat{\triangleright}$ of \triangleright can be defined by*

$$\hat{\psi} \hat{\triangleright} (\mathcal{F}_1^\pi, \mathcal{F}_2^\pi) = (\psi \triangleright (\mathcal{F}_1, \mathcal{F}_2))^\pi = ((\psi_* \mathcal{F}_1)^\pi, (\psi_* \mathcal{F}_2)^\pi)$$

for all $\psi \in \text{Symp}(M, \omega)$ and $(\omega, \mathcal{F}_1, \mathcal{F}_2) \in \mathcal{B}_l(M)$.

Proposition 2.11. *Let $\hat{\psi} \in \hat{\text{Symp}}(M, \omega)$ and $(\omega, \mathcal{F}_1, \mathcal{F}_2) \in \mathcal{B}_l(M)$ such that*

$$\hat{\psi}_* (\mathcal{F}_1^\pi) \subseteq (\psi_* \mathcal{F}_1)^\pi. \quad (2.10)$$

Then

$$\hat{\psi} \hat{\triangleright} (\mathcal{F}_1^\pi, \mathcal{F}_2^\pi) = \hat{\psi} \tilde{\triangleright} (\mathcal{F}_1^\pi, \mathcal{F}_2^\pi).$$

Proof. Note that the diagram

$$\begin{array}{ccc} M \times \mathbb{R}^{2n} & \xrightarrow{\hat{\psi}} & M \times \mathbb{R}^{2n} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\psi} & M \end{array}$$

is commutative. By lifting it on the tangent bundle, we get

$$\begin{array}{ccc} T(M \times \mathbb{R}^{2n}) & \xrightarrow{\hat{\psi}_*} & T(M \times \mathbb{R}^{2n}) \\ \pi_* \downarrow & & \downarrow \pi_* \\ TM & \xrightarrow{\psi_*} & TM \end{array}$$

and by the following decompositions

$$\begin{cases} \Gamma(TM) = \Gamma(\mathcal{F}_1) \oplus \Gamma(\mathcal{F}_2) = \Gamma(\psi_* \mathcal{F}_1) \oplus \Gamma(\psi_* \mathcal{F}_2) \\ \Gamma(T(M \times \mathbb{R}^{2n})) = \Gamma(\mathcal{F}_1^\pi) \oplus \Gamma(\mathcal{F}_2^\pi) = \Gamma((\psi_* \mathcal{F}_1)^\pi) \oplus \Gamma((\psi_* \mathcal{F}_2)^\pi) \end{cases}$$

we obtain

$$\begin{array}{ccc} \Gamma(\mathcal{F}_1^\pi) \oplus \Gamma(\mathcal{F}_2^\pi) & \xrightarrow{\hat{\psi}_*} & \Gamma((\psi_* \mathcal{F}_1)^\pi) \oplus \Gamma((\psi_* \mathcal{F}_2)^\pi) \\ \pi_* \downarrow & & \downarrow \pi_* \\ \Gamma(\mathcal{F}_1) \oplus \Gamma(\mathcal{F}_2) & \xrightarrow{\psi_*} & \Gamma(\psi_* \mathcal{F}_1) \oplus \Gamma(\psi_* \mathcal{F}_2) \end{array}$$

Thus, since $\hat{\psi}_*$ is bijective, then by (2.10) we obtain

$$\hat{\psi}_*(\mathcal{F}_1^\pi) = (\psi_*\mathcal{F}_1)^\pi \quad \text{and} \quad \hat{\psi}_*(\mathcal{F}_2^\pi) = (\psi_*\mathcal{F}_2)^\pi.$$

□

In the next result, we give a condition to obtain (2.10). We use the previous notations.

Proposition 2.12. *Let ψ be a symplectomorphism on (M, ω) . If each point of M has a coordinate chart $(U, p^1, \dots, p^n, q^1, \dots, q^n)$ adapted to $(\mathcal{F}_1, \mathcal{F}_2)$ such that*

$$\hat{\psi}_* \frac{\partial}{\partial p^i} \in \Gamma((\psi_*\mathcal{F}_1)^\pi), \quad i \in [n], \quad (2.11)$$

then

$$\hat{\psi}_*(\mathcal{F}_1^\pi) \subseteq (\psi_*\mathcal{F}_1)^\pi.$$

Proof. Let $(U \times \mathbb{R}^{2n}, p^1, \dots, p^n, q^1, \dots, q^n, \xi_1, \dots, \xi_{2n})$ be a bundle coordinate chart associated to $(U, p^1, \dots, p^n, q^1, \dots, q^n)$. We have

$$\Gamma(\hat{\psi}_*(\mathcal{F}_1^\pi)) = \left\langle \hat{\psi}_* \frac{\partial}{\partial p^1}, \dots, \hat{\psi}_* \frac{\partial}{\partial p^n}, \hat{\psi}_* \frac{\partial}{\partial \xi_{n+1}}, \dots, \hat{\psi}_* \frac{\partial}{\partial \xi_{2n}} \right\rangle.$$

Thus, by (2.11) it remains to show that

$$\hat{\psi}_* \frac{\partial}{\partial \xi_i} \in (\psi_*\mathcal{F}_1)^\pi, \quad i = n+1, \dots, 2n.$$

Let $i = 1, \dots, n$, $j = n+1, \dots, 2n$. We have

$$\tilde{\omega} \left(\hat{\psi}_* \frac{\partial}{\partial p^i}, \hat{\psi}_* \frac{\partial}{\partial \xi_j} \right) = \tilde{\omega} \left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial \xi_j} \right) \circ \hat{\psi}^{-1} = 0.$$

Then $\hat{\psi}_* \frac{\partial}{\partial \xi_i}$ belongs to $\Gamma(((\psi_*\mathcal{F}_1)^\pi)^\perp)$ which is equal to $\Gamma((\psi_*\mathcal{F}_1)^\pi)$.

This completes the proof of Proposition 2.12. □

By combining Proposition 2.12 and Proposition 2.11, Proposition 2.7 follows. □

3 Examples on (\mathbb{R}^2, ω)

We start this part by introducing Christoffel symbols. Let G be a pseudo-Riemannian metric in \mathbb{R}^2 defined as follows: $G(\partial_i, \partial_j) = G_{ij}$ where $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$. Let ∇ be the Levi-Civita connection of G . The Christoffel symbols Γ_{ij}^k ; $i, j, k = 1, 2$ of ∇ are defined as follows: $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$. More precisely,

$$\Gamma_{ij}^k = \frac{1}{2} G^{kl} (\partial_j G_{il} + \partial_i G_{lj} - \partial_l G_{ij}).$$

Our first examples are described on affine bi-Lagrangian structures. Suppose that $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ is an affine bi-Lagrangian structure on \mathbb{R}^2 . By Remark 2.3 there exists a system coordinate (x, y) such that

$$\omega = dy \wedge dx, \quad F = \frac{\partial}{\partial x} dx - \frac{\partial}{\partial y} dy \quad \text{and} \quad G = dx \otimes dy$$

where (G, F) is the associated para-Kähler structure of $(\omega, \mathcal{F}_1, \mathcal{F}_2)$. As a consequence, the Hess connection associated to $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ (which is the Levi-Civita connection of G , see [6, 7, 8, 12]) is trivial; that is, its Christoffel symbols vanish. That is why we will present a second example with a non-trivial Hess connection.

Remark 3.1. *Let M be a manifold. Every $X \in \mathfrak{X}(M)$ without singularity (this means, $X_z \neq 0$ for every $z \in M$) generates (induces) a foliation \mathcal{F}^X on M . In particular, if M is a 2-manifold endowed with a symplectic form ω , then \mathcal{F}^X is Lagrangian independently of the symplectic form ω . As a consequence, any $X, Y \in \mathfrak{X}(M)$ such that $\text{Dim} \langle X_z, Y_z \rangle = 2$ for every point $z \in M$ generates a bi-Lagrangian structure on (M, ω) , independently of the symplectic form ω .*

3.1 Case of $(\mathbb{R}^2, \omega = dy \wedge dx)$

3.1.1 Action of $\text{Symp}(\mathbb{R}^2, \omega)$ on $\mathcal{B}_l(\mathbb{R}^2)$

Symplectomorphism group on (\mathbb{R}^2, ω)

$$\text{Symp}(\mathbb{R}^2, \omega) := \{\psi \in \text{Diff}(\mathbb{R}^2) : \det T_x \psi = 1\}$$

where

$$\det T_x \psi := \frac{\partial \psi_1}{\partial x^1} \frac{\partial \psi_2}{\partial x^2} - \frac{\partial \psi_2}{\partial x^1} \frac{\partial \psi_1}{\partial x^2}.$$

For technical reasons, we describe our example on the subgroup $Symp_a(\mathbb{R}^2, \omega)$ of $Symp(\mathbb{R}^2, \omega)$ defined by:

$$Symp_a(\mathbb{R}^2, \omega) = \left\{ \psi_{AB} : (x, y) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + B, A \in SL_2(\mathbb{R}), B \in \mathbb{R}^2 \right\}$$

where $SL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) : \det A = 1\}$.

The action of $Symp_a(\mathbb{R}^2, \omega)$ on $\mathfrak{X}(\mathbb{R}^2)$ is: $Symp_a(\mathbb{R}^2, \omega) \times \mathfrak{X}(\mathbb{R}^2) \longrightarrow \mathfrak{X}(\mathbb{R}^2)$, $(\psi, X) \longmapsto \psi_*X$. More precisely, for any

$$(x, y) \in \mathbb{R}^2, \psi_{*(x,y)} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix},$$

$$\psi_{*(x,y)}X_{(x,y)} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} X^1(x, y) \\ X^2(x, y) \end{pmatrix} = \begin{pmatrix} \alpha X^1(x, y) + \beta X^2(x, y) \\ \gamma X^1(x, y) + \delta X^2(x, y) \end{pmatrix}.$$

Let $(\mathcal{F}^x, \mathcal{F}^y)$ be the pair of two decompositions of \mathbb{R}^2 constituted of all horizontal and vertical lines respectively. That is,

$$\mathcal{F}^x = \{\mathcal{F}_b^x = \mathbb{R} \times \{b\}\}_{b \in \mathbb{R}} \text{ and } \mathcal{F}^y = \{\mathcal{F}_a^y = \{a\} \times \mathbb{R}\}_{a \in \mathbb{R}}.$$

As a consequence,

$$\Gamma(\mathcal{F}^x) = \mathbb{R} \times \{0\} = \left\langle \frac{\partial}{\partial x} \right\rangle \text{ and } \Gamma(\mathcal{F}^y) = \{0\} \times \mathbb{R} = \left\langle \frac{\partial}{\partial y} \right\rangle.$$

By Remark 3.1, $(\mathcal{F}^x, \mathcal{F}^y)$ is a bi-Lagrangian structure on (\mathbb{R}^2, ω) . Before graphing it, we describe the action (in the sense of Theorem 2.4) of $Symp_a(\mathbb{R}^2, \omega)$ on $(\mathcal{F}^x, \mathcal{F}^y)$.

Let $\psi \in Symp_a(\mathbb{R}^2, \omega)$. Observe that

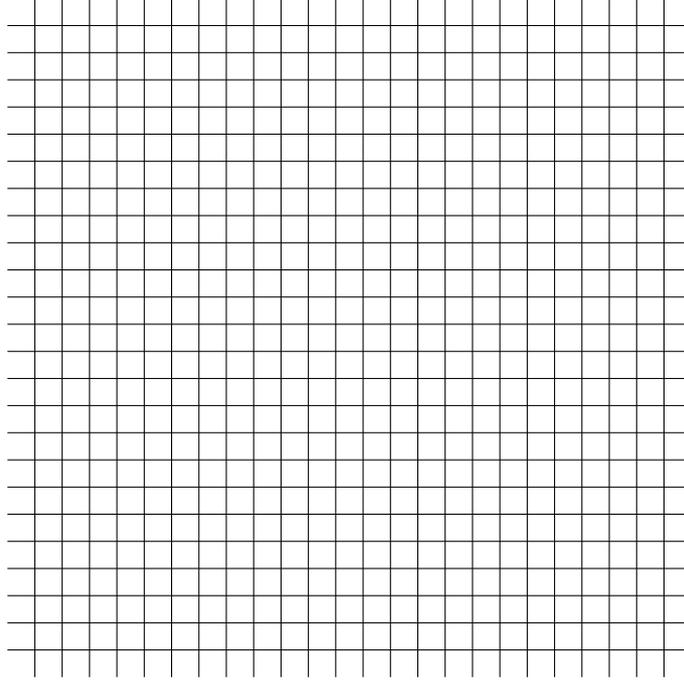
$$\left\{ \begin{array}{l} \psi_*\mathcal{F}_b^x : y = \frac{\gamma}{\alpha}x + \frac{b}{\alpha}, b \in \mathbb{R}, \\ \Gamma(\psi_*\mathcal{F}^x) = \left\langle \alpha \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y} \right\rangle, \end{array} \right. \text{ and } \left\{ \begin{array}{l} \psi_*\mathcal{F}_a^y : y = \frac{\delta}{\beta}x - \frac{a}{\beta}, a \in \mathbb{R}, \\ \Gamma(\psi_*\mathcal{F}^y) = \left\langle \beta \frac{\partial}{\partial x} + \delta \frac{\partial}{\partial y} \right\rangle. \end{array} \right.$$

The para-complex structure F^ψ associated to $(\psi_*\mathcal{F}^x, \psi_*\mathcal{F}^y)$ is

$$F^\psi(\psi_*\frac{\partial}{\partial x}) = \alpha \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y} \text{ and } F^\psi(\psi_*\frac{\partial}{\partial y}) = -\beta \frac{\partial}{\partial x} - \delta \frac{\partial}{\partial y}.$$

Similar results are obtained for another bi-Lagrangian structure belonging in the orbit

$$\mathcal{B}_0 = \{(\psi_*\mathcal{F}^x, \psi_*\mathcal{F}^y), \psi \in Symp_a(\mathbb{R}^2, \omega)\}$$

Figure 1 – The bi-Lagrangian structure $(\mathcal{F}^x, \mathcal{F}^y)$

of $(\mathcal{F}^x, \mathcal{F}^y)$ with respect to $\triangleright | \text{Symp}_a(\mathbb{R}^2, \omega) \times \mathcal{B}_l(\mathbb{R}^2)$.

The bi-Lagrangian structure $(\mathcal{F}^x, \mathcal{F}^y)$ can be represented as follows.

Now, we are going to apply Proposition 2.11 to $\triangleright | \text{Symp}_a(\mathbb{R}^2, \omega) \times \mathcal{B}_0$

Note that for any

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{R}) \text{ and } B = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2,$$

the map

$$\psi_{AB} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + B$$

is invertible with the explicit inverse

$$\psi_{AB}^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mapsto A^{-1} \begin{pmatrix} x \\ y \end{pmatrix} - A^{-1}B \quad (3.1)$$

Lifting of affine symplectomorphism

Proposition 3.1. *An affine symplectomorphism on (\mathbb{R}^2, ω) lifts as an affine symplectomorphism on $(\mathbb{R}^4, \tilde{\omega})$. That is, $\text{Sym}_a(\mathbb{R}^2) \subset \text{Sym}_a(\mathbb{R}^4, \tilde{\omega})$.*

Proof. Let $\psi \in \text{Symp}_a(\mathbb{R}^2, \omega)$. We have

$$\hat{\psi} : z = (p, \xi_p) \mapsto (\psi(p), (\psi^{-1*}\xi)_{\psi(p)}).$$

Let (x, y, s, t) be a coordinate system on \mathbb{R}^4 . Then $z = (x, y, s, t)$, $\xi = sdx + tdy$ and $\tilde{\omega} = dy \wedge dx + ds \wedge dx + dt \wedge dy$. Moreover, since

$$\psi(x, y) = (\alpha x + \beta y + a, \gamma x + \delta y + b)$$

for some $\alpha, \beta, \gamma, \delta, a, b \in \mathbb{R}$ verifying $\alpha\delta - \beta\gamma = 1$, then by (3.1), we get

$$\psi^{-1}(x, y) = (\delta x - \beta y + \delta a - \beta b, -\gamma x + \alpha y - \delta a + \alpha b).$$

As a consequence,

$$(\psi^{-1*}\xi)_{\psi(p)} = (s(p)\delta - t(p)\gamma)dx + (\alpha t(p) - \beta s(p))dy.$$

Then

$$\hat{\psi}(z) = (\alpha x + \beta y + a, \gamma x + \delta y + b, s\delta - t\gamma, -\beta s + \alpha t).$$

Therefore

$$T_z \hat{\psi} = \hat{\psi}_* z = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \delta & -\beta \\ 0 & 0 & -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \quad (3.2)$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

This ends the proof of Proposition 3.1. \square

Lifting of $(\mathcal{F}^x, \mathcal{F}^y)$.

Note that

$$\Gamma(\mathcal{F}^x) = \left\langle \frac{\partial}{\partial x} \right\rangle \text{ and } \Gamma(\mathcal{F}^y) = \left\langle \frac{\partial}{\partial y} \right\rangle.$$

Thus,

$$\Gamma((\mathcal{F}^x)^\pi) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right\rangle \text{ and } \Gamma((\mathcal{F}^y)^\pi) = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial s} \right\rangle.$$

Proposition 3.2. *Let $\psi \in \text{Symp}_a(\mathbb{R}^2, \omega)$. Then $\hat{\psi}_*((\mathcal{F}^y)^\pi) \subseteq (\psi_*\mathcal{F}^y)^\pi$.*

Proof. Let $\psi \in \text{Symp}_a(\mathbb{R}^2, \omega)$. By (3.2) we get

$$\hat{\psi}_* \frac{\partial}{\partial x} = \psi_* \frac{\partial}{\partial x} \in \Gamma((\psi_* \mathcal{F}^x)^\pi).$$

And by Proposition 2.7 we have the result. \square

Lifting of \mathcal{B}_0 .

We are going to explicit $((\psi_* \mathcal{F}^x)^\pi, (\psi_* \mathcal{F}^y)^\pi)$ for some $\psi \in \text{Symp}_a(\mathbb{R}^2, \omega)$.

Let $\psi \in \text{Symp}_a(\mathbb{R}^2, \omega)$, by Proposition 3.2 we get

$$\hat{\psi}_*((\mathcal{F}^x)^\pi) \subseteq (\psi_* \mathcal{F}^x)^\pi.$$

Thus, by Proposition 2.11 we obtain

$$((\psi_* \mathcal{F}^x)^\pi, (\psi_* \mathcal{F}^y)^\pi) = \hat{\psi}_*((\mathcal{F}^x)^\pi, (\mathcal{F}^y)^\pi),$$

and Proposition 3.1 implies that

$$\hat{\psi}_* = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix};$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Therefore

$$\begin{cases} \Gamma((\psi_* \mathcal{F}^y)^\pi) = \left\langle \hat{\psi}_* \frac{\partial}{\partial x}, \hat{\psi}_* \frac{\partial}{\partial t} \right\rangle \\ \Gamma((\psi_* \mathcal{F}^x)^\pi) = \left\langle \hat{\psi}_* \frac{\partial}{\partial y}, \hat{\psi}_* \frac{\partial}{\partial s} \right\rangle. \end{cases}$$

3.2 A bi-Lagrangian structure on $(\mathbb{R}^2, \omega = hdy \wedge dx)$

In this part, we present $(\mathcal{P}, \mathcal{F}^y)$ the bi-Lagrangian structure on (\mathbb{R}^2, ω) constituted of parabolas and vertical lines, and calculate its Hess connection.

3.2.1 Description de $(\mathcal{P}, \mathcal{F}^y)$

The foliation \mathcal{P} is described as follows:

$$\mathcal{P} = \{ \mathcal{P}_{(a,b)} : y = x^2 + b - a^2 \}_{(a,b) \in \mathbb{R}^2}.$$

Thus,

$$\begin{cases} \Gamma(\mathcal{P}) = \left\langle \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y} \right\rangle, \\ \Gamma(\mathcal{F}^x) = \left\langle \frac{\partial}{\partial y} \right\rangle. \end{cases}$$

Let us write

$$\begin{cases} U = \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}, \\ V = \frac{\partial}{\partial y}. \end{cases}$$

By Remark 3.1, $(\mathcal{P}, \mathcal{F}^y)$ is a bi-Lagrangian structure on (\mathbb{R}^2, ω) . It can be represented as follows.

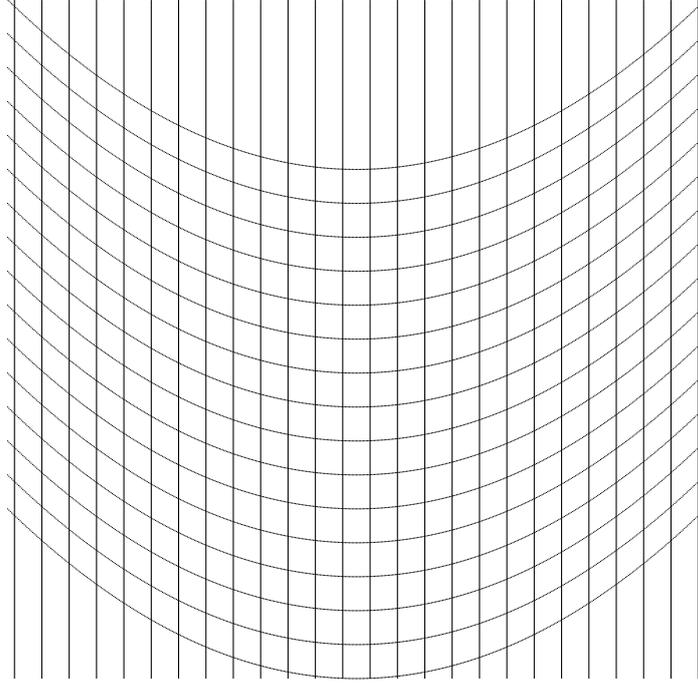


Figure 2 – The bi-Lagrangian structure $(\mathcal{P}, \mathcal{F}^y)$

3.2.2 The Hess connection of $(\mathcal{P}, \mathcal{F}^y)$

We are going to determine

$$\nabla_{(U,0)}(U, 0), \nabla_{(0,V)}(0, V), \nabla_{(U,0)}(0, V) \text{ and } \nabla_{(0,V)}(U, 0).$$

By (1.4) it is enough to calculate

$$D(U, U), D(V, V), D(U, 0), D(0, V).$$

Let us write $x^1 = x$ and $x^2 = y$.

Let $X, Y, Z \in \mathfrak{X}(\mathbb{R}^2)$. From (1.5), we get

$$\omega(D(X, Y), Z) = X\omega(Y, Z) - \omega(Y, [X, Z]).$$

Then

$$\begin{aligned} \omega(D(X, Y), Z) &= X[h(dx^2(Y)dx^1(Z) - dx^2(Z)dx^1(Y))] \\ &\quad - h(dx^2(Y)dx^1([X, Z]) - dx^2([X, Z])dx^1(Y)). \end{aligned}$$

Thus, on the one hand,

$$\omega\left(D(U, U), \frac{\partial}{\partial x^j}\right) = U[h(2x\delta_{1j} - \delta_{2j})] - 2h\delta_{1j}.$$

On the other hand,

$$\omega\left(D(U, U), \frac{\partial}{\partial x^j}\right) = h[\delta_{1j}dx^2(D(U, U) - \delta_{2j}dx^1(D(U, U))].$$

Then

$$\begin{cases} hdx^1(D(U, U)) = U(h), \\ hdx^2(D(U, U)) = U(2xh) - 2h. \end{cases}$$

Therefore,

$$D(U, U) = \frac{U(h)}{h}U.$$

In the same way as before,

$$D(V, V) = \frac{V(h)}{h}V.$$

Moreover, since $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$, then

$$[U, V] = \left[\frac{\partial}{\partial x^1} + 2x^1 \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2} \right] = 0.$$

Thus,

$$\begin{cases} \nabla_{(U,0)}(U, 0) = \left(\frac{U(h)}{h}, 0 \right), \\ \nabla_{(0,V)}(0, V) = \left(0, \frac{V(h)}{h} \right), \\ \nabla_{(U,0)}(0, V) = \nabla_{(0,V)}(U, 0) = (0, 0). \end{cases}$$

Therefore

$$\begin{cases} \Gamma_{11}^1 = \frac{U(h)}{h}, \\ \Gamma_{22}^2 = \frac{V(h)}{h}, \\ \Gamma_{22}^1 = \Gamma_{12}^1 = \Gamma_{21}^1 = 0, \\ \Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{11}^2 = 0. \end{cases} \quad (3.3)$$

3.2.3 The curvature tensor of ∇

Note that

$$R(U_i, U_j)U_k = R_{ijk}^l U_l, \quad i, j, k \in [2]$$

where $U_1 = U$, $U_2 = V$ and

$$R_{ijk}^l = U_i(\Gamma_{jk}^l) - U_j(\Gamma_{ik}^l) + \Gamma_{jk}^s \Gamma_{is}^l - \Gamma_{ik}^s \Gamma_{js}^l.$$

Thus by (3.3), we get

$$\begin{cases} R_{211}^1 = -R_{121}^1 = V(\Gamma_{11}^1), \\ R_{122}^2 = -R_{212}^2 = U(\Gamma_{22}^2), \\ \text{the other coefficients are zero.} \end{cases} \quad (3.4)$$

Remark 3.2. *By combining Theorem 1.5 and system (3.4), $(\omega, \mathcal{P}, \mathcal{F}^y)$ is an affine bi-Lagrangian structure on \mathbb{R}^2 when $V(\Gamma_{11}^1) = U(\Gamma_{22}^2) = 0$; in particular, when h is a constant map.*

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