ELEMENTARY AMENABILITY AND ALMOST FINITENESS

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Abstract. We show that every free continuous action of a countably infinite elementary amenable group on a finite-dimensional compact metrizable space is almost finite. As a consequence, the crossed products of minimal such actions are $\mathcal{Z}$-stable and classified by their Elliott invariant.

1. Introduction

A basic principle in the study of group actions and their operator-algebraic crossed products is that dynamical towers produce matricial structure, and that dynamical towers with Følner (i.e., approximately invariant) shapes produce approximately central matricial structure.

In the case of a free measure-preserving action of a countable amenable group on a standard probability space, a theorem of Ornstein and Weiss gives the existence of a family of disjoint Følner-shaped towers which cover all but a small piece of the space [31]. The smallness of this remainder, expressed in terms of the measure, means that the multimatrix algebras generated by the towers partition the unit of the von Neumann algebra crossed product up to a small error in trace norm. When combined with the approximate centrality that ensues from the Følnerness of the tower shapes, this yields local approximation by finite-dimensional *-subalgebras and hence hyperfiniteness of the crossed product[1], and if the action is ergodic one obtains the unique hyperfinite II_1 factor.

The analogous C*-theory involving actions of countable amenable groups on compact metrizable spaces is more complicated and, despite many significant advances, still incomplete. While a variety of different tools and techniques have been developed over the last four decades [34, 9, 23, 22, 35, 39], in large part stimulated by new ideas that have emerged from the Elliott classification program, a basic pattern has crystallized through the course of recent progress, which we can summarize as follows.

A major point of departure from the measure-theoretic setting is the presence of dimensionality, which means that matrix models coming from the dynamics will need to be continuous instead of discrete, or that they will need to be reconceptualized as order-zero maps from matrices into the algebra. The first of these options, which connects to ideas of dimension-rank ratio and dimension growth, is particularly powerful when dimensional regularity hypotheses on the space (e.g., finite covering dimension) are replaced by more general ones on the dynamics such as zero mean dimension or the small boundary property, which have so far only yielded to this approach. In this case the goal has been to show that the C*-algebra crossed product is $\mathcal{Z}$-stable, which has been achieved in the case of free minimal $\mathbb{Z}^d$-actions with zero mean dimension

Date: August 18, 2022.

[1]This conclusion was originally derived in a more directly operator-algebraic way by Connes as a consequence of his theorem on the equivalence of injectivity and hyperfiniteness [5].

[11, 30], although it seems to be a difficult problem to establish similarly general statements for other acting groups (see however [29]).

The second option can be developed in two different ways. In general one is confronted with the problem that, unlike for the trace norm approximations in the von Neumann algebraic setting, the operator norm approximations that are essential for unraveling C*-structure cannot be done by purely spatial means and instead must be implemented with the help of spectral constructions, even if the space is zero-dimensional. This is true both for the approximation of the unit in the tower configurations and for the approximate centrality demanded of the matrix models. One possibility is to drop the Følner requirement and allow enough overlap between towers so that the bump functions implementing approximate centrality will form a genuine partition of unity. Control on the multiplicity of this overlapping will lead to estimates on the nuclear dimension of the crossed product [42, 16, 36, 37], and can be formalized at the dynamical level through the notions of dynamic asymptotic dimension [15] and tower dimension [18]. Another idea, formalized in the definition of almost finiteness (see Section 2), is to insist on the Følnerness and disjointness of the towers and express the shortfall in the partitioning by means of a topological version of Cuntz subequivalence, which is then sufficient to imply that the crossed product is $\mathcal{Z}$-stable [18]. The two approaches are connected at the dynamical level by the observation that a free action on a space of finite covering dimension is almost finite if its dynamic asymptotic dimension or tower dimension is finite (see Corollary 6.2 of [20] and Theorem 5.14 of [18]). The first approach has proven to be very effective for certain classes of groups, as in the recent paper [4] where finite dynamic asymptotic dimension (and hence also almost finiteness) is established for free actions of many solvable groups, including polycyclic groups and the lamplighter group, on zero-dimensional compact metrizable spaces. On the other hand, such use of the dimensional idea of controlled overlapping from which a proof of finite nuclear dimension can be derived has invariably required the space to have finite covering dimension and the group to have finite asymptotic dimension, the latter being a property that excludes many amenable groups. From this perspective almost finiteness has turned out to be more broadly applicable, and indeed almost finiteness has been shown to hold for free minimal actions of groups with local subexponential growth on zero-dimensional compact metrizable spaces [6, 7], as well as for a generic free minimal action of any countably infinite amenable group on the Cantor set [3]. Moreover, by Theorem 7.6 of [20], for a fixed countably infinite group G, if every free action of G on a zero-dimensional compact metrizable space is almost finite, then every free action of G on a finite-dimensional compact metrizable space (and in fact every free action with the topological small boundary property) is almost finite.

Given this dynamical picture it comes as quite a surprise that, for unital simple nonelementary separable C*-algebras, finite nuclear dimension is actually equivalent to $\mathcal{Z}$-stability (the forward implication was proven in [41], while the backward implication was recently established in [1] after a string of partial results beginning with the breakthrough in [27]). The significance of these two regularity properties and their equivalence in this context is that the class of unital simple separable C*-algebras having finite nuclear dimension and satisfying the UCT is classified by the Elliott invariant (ordered K-theory paired with tracial states) [21, 33, 13, 10, 38], and every stably finite member of this class is an inductive limit of subhomogenous C*-algebras whose spectra have covering dimension at most two [8] (see Theorem 6.2(iii) in [38]). The crossed product of a free minimal action of a countable amenable group on a compact metrizable space will therefore be covered by these classification and structure results as soon as it is known to have finite nuclear dimension, or equivalently be $\mathcal{Z}$-stable (note that the UCT is automatic by [40], and that amenability implies the existence of an invariant Borel probability measure and hence of a tracial state, which ensures stable finiteness in view of simplicity). That $\mathcal{Z}$-stability does not always hold in this context, even for free minimal $\mathbb{Z}$-actions, was shown in [12].

In this paper we establish the following theorem, which generalizes the almost finiteness result from [4]. By definition, the class of elementary amenable groups is the smallest class of groups which contains all finite groups and Abelian groups and is closed under taking subgroups, quotients, extensions, and direct limits. This class includes all solvable groups, is closed under taking wreath products, and contains many groups with both exponential growth and infinite asymptotic dimension, such as $\mathbb{Z} \wr \mathbb{Z}$. A finitely generated infinite amenable group cannot be elementary amenable if it has intermediate growth [2], like the prototypical Grigorchuk group [14], or if it is simple, like the commutator subgroup of the topological full group of a minimal subshift [24, 17].

Theorem A. *Every free continuous action of a countably infinite elementary amenable group on a finite-dimensional compact metrizable space is almost finite.*

By Theorem 7.6 of [20], as mentioned three paragraphs above, it is enough to prove the theorem in the case of zero-dimensional compact metrizable spaces, which is what we will do, also without the assumption that the countable group be infinite (for finite groups, an action as in Theorem A is almost finite if and only if the space is zero-dimensional, and so Theorem A is actually false in this case, and Theorem 7.6 of [20] is only valid when the group is infinite). In other words, we will establish that every countable elementary amenable group G satisfies the following property:

($\star$) every free continuous action of G on a zero-dimensional compact metrizable space is almost finite.

As is clear from the quantification over finite subsets in the definition of almost finiteness, property ($\star$) is preserved under taking countable direct limits. In Theorem 3.1 we prove that property ($\star$) is preserved under finite extensions, while in Theorem 5.4, to which most of our efforts will be devoted, we show that property ($\star$) is preserved under extensions by $\mathbb{Z}$. Actually none of these three permanence properties require the zero-dimensionality hypothesis on the space, but in order to bootstrap our way to the final result we will rely on the fact that property ($\star$) holds for the trivial group, as can be seen from the definition of almost finiteness (see Section 2) by taking the tower bases therein to form a fine enough clopen partition of the space and the proportionally small subsets of the tower shapes to be empty. To conclude that property ($\star$) holds for all countable elementary amenable groups we can then appeal to a theorem of Osin [32] which, refining a result of Chou [2], characterizes this class as the smallest class of groups that contains the trivial group and is closed under taking countable direct limits and extensions by $\mathbb{Z}$ and finite groups. Note that, in view of [6, 7], we actually obtain property ($\star$) and hence also Theorem A for a broader class of groups, namely the smallest class that contains all countable groups of local subexponential growth and is closed under taking countable direct limits and extensions by $\mathbb{Z}$ and finite groups.

One of the novelties of the proof of Theorem 5.4 is that it integrates conceptual aspects from all three of the approaches that we sketched above (corresponding to the regularity properties of zero mean dimension, finite dynamic asymptotic dimension, and almost finiteness). The argument proceeds by applying a recursive disjointification procedure to an initial collection of overlapping open towers whose levels have boundaries of upper H-density zero. The shapes of these towers are Følner rectangles in the semidirect product $H \rtimes \mathbb{Z}$, and the towers generated by the restrictions of these shapes to H cover all but a piece of the space with small upper H-density, as can be arranged using the hypothesized almost finiteness of the H-action. When H is infinite these rectangles are chosen to be thin in the $\mathbb{Z}$ direction and tall (i.e, much larger) in the H direction, in which case the multiplicity of the overlapping of the towers is small in proportion to the size of their shape in the H direction, very much in the spirit of the small dimension-rank ratios that appear in the proof of $\mathcal{Z}$-stability from zero mean dimension in [11] and in the general study of inductive limits in classification theory. This allows us to disjointify, modulo a set of zero H-density, into open towers with smaller subrectangular shapes which are mostly Følner. The exceptional towers whose shapes fail to be Følner occupy a part of the space with small upper H-density and thus can be absorbed, via comparison, using the almost finiteness of the H-action. In fact the initial towers will themselves need to be shaved down a little bit at the outset in order to achieve the Følner disjointification (via the tiling argument captured in Lemma 4.1), but this loss will also be small in upper H-density and can likewise be absorbed.

By our discussion prior to the statement of Theorem A, we obtain the following corollary. The precise link to $\mathcal{Z}$-stability is given by Theorem 12.4 of [18], which asserts that, given an almost finite free minimal action $G \curvearrowright X$ of a countably infinite group on a compact metrizable space, the crossed product $C(X) \rtimes G$ is $\mathcal{Z}$-stable (note that almost finiteness implies that G is amenable and so the reduced and full crossed products coincide in this case).

Corollary B. *The crossed products of free minimal actions of countably infinite elementary amenable groups on finite-dimensional compact metrizable spaces are classified by their Elliott invariant and are simple inductive limits of subhomogeneous C^*-algebras whose spectra have covering dimension at most two.*

Theorem A also has consequences for topological full groups and homology. Let $G \curvearrowright X$ be free continuous action of a countably infinite elementary amenable group on the Cantor set. Denote by $[[G \curvearrowright X]]$ the topological full group of the action and by $[[G \curvearrowright X]]_0$ the subgroup of $[[G \curvearrowright X]]$ generated by the elements of finite order whose powers have clopen fixed point sets. In Section 7 of [25] Matui defines an index map I from $[[G \curvearrowright X]]$ to the first homology group $H_1(G \curvearrowright X)$ with integer coefficients. The fact that the action $G \curvearrowright X$ is almost finite implies, by Corollary 7.16 of [25], that I is surjective and has kernel $[[G \curvearrowright X]]_0$, so that it induces an isomorphism $H_1(G \curvearrowright X) \cong [[G \curvearrowright X]]/[[G \curvearrowright X]]_0$. If the action is in addition minimal then the commutator subgroup of $[[G \curvearrowright X]]$ is simple (by Theorem 4.7 of [26]) and equal to the alternating group $\mathsf{A}(G \curvearrowright X)$ (by Theorem 4.7 of [26] and Theorem 4.1 of [28]).

The main body of the paper begins in Section 2 with some general terminology and notation. The case of finite extensions (Theorem 3.1) is treated in Section 3, while Sections 4 and 5 are devoted to extensions by $\mathbb{Z}$ (Theorem 5.4). Section 4 contains two technical lemmas for use in the proof of Theorem 5.4, which occupies the bulk of Section 5.

Acknowledgements. The first author was partially supported by NSF grant DMS-1800633. Funding was also provided by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044-390685587, Mathematics Münster:

Dynamics–Geometry–Structure. Both authors were affiliated with Texas A&M University during the initial stages of this work. We thank Xin Ma, Brandon Seward, and Andrea Vaccaro for comments and corrections.

2. General terminology and notation

We write e for the identity element of a group.

For finite sets K and F of a group G, we define the K-boundary of F by

$$\partial_K F = \{t \in G : Kt \cap F \neq \emptyset \text{ and } Kt \cap (G \setminus F) \neq \emptyset\}.$$

For $\delta > 0$, we say that F is *(left)* (K,δ)*-invariant* if $|\partial_K F| \leq \delta |F|$. By the Følner characterization of amenability, the group G is amenable if and only if it admits a nonempty finite (K,δ)-invariant set for every finite set $K \subseteq G$ and $\delta > 0$.

Let $G \curvearrowright X$ be a free continuous action of a countable group on a compact metric space (we only consider free actions in this paper). By a *tower* we mean a pair (S, V) where S is a finite subset of G (the *shape*) and V is a subset of X (the *base*) such that the sets sV for $s \in S$ (the *levels*) are pairwise disjoint. The tower is *open* if V is open and *clopen* if V is clopen. The *footprint* of the tower is the set SV.

A *castle* is a finite collection $\{(S_i, V_i)\}_{i \in I}$ of towers such that the sets $S_i V_i$ for $i \in I$ are pairwise disjoint. The castle is *open* if each of the towers is open and *clopen* if each of the towers is clopen. The *footprint* of the castle is the set $\bigsqcup_{i \in I} S_i V_i$.

Let A and B be subsets of X. We say that A is *subequivalent* to B and write $A \prec B$ if for every closed set $C \subseteq A$ there are finitely many open sets $U_1, \dots, U_n$ which cover C and elements $s_1, \dots, s_n \in G$ such that the sets $s_i U_i$ for $i = 1, \dots, n$ are pairwise disjoint and contained in B. For a nonnegative integer m we write $A \prec_m B$ if for every closed set $C \subseteq A$ there are a finite collection $\mathscr{U}$ of open subsets of X which cover C, an $s_U \in G$ for every $U \in \mathscr{U}$, and a partition of $\mathscr{U}$ into subcollections $\mathscr{U}_0, \dots, \mathscr{U}_m$ such that for every $i = 0, ..., m$ the sets $s_U U$ for $U \in \mathscr{U}_i$ are pairwise disjoint and contained in B.

The action $G \curvearrowright X$ has *comparison* if $A \prec B$ for all nonempty open sets $A, B \subseteq X$ which satisfy $\mu(A) < \mu(B)$ for every G-invariant Borel probability measure μ on X. It has m*-comparison* for a nonnegative integer m if $A \prec_m B$ for all nonempty open sets $A, B \subseteq X$ which satisfy $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$. In these definitions one can equivalently take A to range over closed sets instead of open ones (Proposition 3.4 of [18]).

The action $G \curvearrowright X$ is *almost finite* if for every $n \in \mathbb{N}$, finite set $K \subseteq G$, and $\delta > 0$ there exist

(i) an open castle $\{(S_i, V_i)\}$ each of whose shapes is (K,δ)-invariant and each of whose levels has diameter less than δ, and
(ii) sets $S'_i \subseteq S_i$ such that $|S'_i| \leq |S_i|/n$ and $X \setminus \bigsqcup_{i \in I} S_i V_i \prec \bigsqcup_{i \in I} S'_i V_i$.

Note that the shape condition in (i) implies that G is amenable. In the case that G is finite, the action $G \curvearrowright X$ is almost finite if and only if X is zero-dimensional (notice that for $n > |G|$ the sets S'_i in the above definition will have to be empty).

It is worth noting (although we will not need this fact) that when X is zero-dimensional we can characterize almost finiteness by the existence, for every finite set $K \subseteq G$ and $\delta > 0$, of an open castle whose shapes are (K,δ)-invariant and whose footprint is the entire space X (Theorem 10.2 of [18]).

When G is amenable, the *upper* and *lower densities* (or *G-densities* if we wish to emphasize the acting group) of a set $A \subseteq X$ are defined by

$$\overline{D}_G(A) = \inf_F \sup_{x\in X} \frac{1}{|F|}\sum_{s\in F} 1_A(sx) \quad \text{and} \quad \underline{D}_G(A) = \sup_F \inf_{x\in X} \frac{1}{|F|}\sum_{s\in F} 1_A(sx)$$

where F ranges in both cases over the nonempty finite subsets of G. Writing $M_G(X)$ for the set of all G-invariant Borel probability measures on X, we can alternatively express the upper density as $\sup_{\mu\in M_G(X)} \mu(A)$ when A is closed and the lower density as $\inf_{\mu\in M_G(X)} \mu(A)$ when A is open (see Proposition 3.3 of [20]).

Almost finiteness in measure for the action $G \curvearrowright X$ is defined in the same way as almost finiteness except that condition (ii) is replaced by the requirement that $X \setminus \bigsqcup_{i\in I} S_i V_i$ have upper density less than δ (uniform smallness in measure). By Theorem 5.6 of [20], the action $G \curvearrowright X$ is almost finite in measure if and only if it has the *small boundary property*, which asks that X have a basis of open sets whose boundaries have zero upper H-density. By Theorem 6.1 of [20] the following are equivalent:

(i) the action is almost finite,
(ii) the action is almost finite in measure and has comparison,
(iii) the action is almost finite in measure and has m-comparison for some m.

3. Finite extensions

Let G be a finite extension of a countable group H. Let $G \curvearrowright X$ be a free continuous action on compact metrizable space.

Theorem 3.1. *Suppose that the restricted action $H \curvearrowright X$ is almost finite. Then the action $G \curvearrowright X$ is almost finite.*

Proof. Since the action $H \curvearrowright X$ is almost finite, by the results recalled in Section 2 it has the small boundary property. It is immediate from the definition of the latter that the action $G \curvearrowright X$ also has the small boundary property. Thus to show that $G \curvearrowright X$ is almost finite it suffices, again by the discussion in Section 2, to prove that it has m-comparison for some m. Suppose that for some open sets $A, B \subseteq X$ we have $\mu(A) < \mu(B)$ for every G-invariant Borel probability measure μ on X. Let $g_1, \dots, g_n$ be representatives for the left cosets of H in G with $g_1 = e$. Since the action $H \curvearrowright X$ is almost finite, it follows from the proof of Lemma 7.4 in [20] that the set A can be covered by $n+1$ open sets $A_1, \dots, A_{n+1}$ such that $\nu(A_i) < \frac{1}{n}\nu(A)$ for every i and every H-invariant Borel probability measure ν on X (to construct the $(n+1)$st set take the closed complement of the footprint of the open castle in the proof of Lemma 7.4 in [20] and enlarge it to an open set whose measure is only slightly larger for every H-invariant Borel probability measure, as is possible by Lemma 3.3 in [18]). Given such a measure ν, the Borel probability measure

$$\overline{\nu}(D) = \frac{1}{n}(\nu(g_1 D) + \nu(g_2 D) + \ldots + \nu(g_n D))$$

is G-invariant, and for every $i = 1, \dots, n+1$ we have

$$\begin{aligned}\nu(A_i) \leq n\overline{\nu}(A_i) < \overline{\nu}(A) < \overline{\nu}(B) &= \frac{1}{n}(\nu(g_1 B) + \ldots + \nu(g_n B)) \\ &\leq \nu(g_1 B \cup \ldots \cup g_n B).\end{aligned}$$

Given a closed subset C of A and taking closed sets $C_i \subseteq A_i$ such that $C = \bigcup_{i=1}^{n+1} C_i$, the fact that the action of H has comparison (by virtue of being almost finite) thus yields, for every i, pairwise disjoint open sets $U_{i,1}, \dots, U_{i,k_i} \subseteq g_1 B \cup \dots \cup g_n B$ and $h_{i,1}, \dots, h_{i,k_i} \in H$ such that $C_i \subseteq \bigcup_{k=i}^{k_i} h_{i,k} U_{i,k}$. For every $1 \leq i \leq n+1$ and $1 \leq j \leq n$ the sets $W_{i,j,k} := g_j^{-1}(g_j B \cap U_{i,k})$ for $k = 1, \dots, k_i$ are pairwise disjoint and contained in B, and we have $C \subseteq \bigcup_{i=1}^{n+1} \bigcup_{j=1}^{n} \bigcup_{k=1}^{k_i} h_{i,k} g_j W_{i,j,k}$. This shows that $G \curvearrowright X$ has $n(n+1)$-comparison. □

4. Two lemmas

We collect here two lemmas that will be needed for the proof of Theorem 5.4. The first concerns conditions under which the join of finitely many disjoint collections of subsets of a group, when restricted to an ambient Følner set S, will mostly consist of Følner sets, where the degree of approximate invariance is prescribed but necessarily much lower than that of S, and "mostly" is understood in the sense that the exceptional sets will have collective size proportionally small relative to $|S|$. The second lemma is a version of the implication (ii)⇒(i) in Theorem 6.1 of [20] in which the hypotheses are relativized to a subgroup.

Let $\mathscr{F}$ be a collection of subsets of G. We say that a set $A \subseteq G$ is *$\mathscr{F}$-tileable* if there is a $T \subseteq G$ and sets $F_t \in \mathscr{F}$ for $t \in T$ such that the sets $F_t t$ for $t \in T$ form a partition of A.

Lemma 4.1. *Let $n \in \mathbb{N}$. Let K be a finite subset of G and $\delta > 0$. Let $\mathscr{F}$ be a finite collection of $(K, \delta^3/(8|K|^2 n))$-invariant finite subsets of G, and writing $D = (\bigcup \mathscr{F})(\bigcup \mathscr{F})^{-1}$ let S be a $(D^2, \delta^2/(4|K|))$-invariant finite subset of G. For each $i = 1, \dots, n$ let $\{B_{i,1}, \dots, B_{i,m_i}\}$ be a finite disjoint collection of $\mathscr{F}$-tileable finite subsets of G. For every $I \subseteq \{1, \dots, n\}$ set $\Omega_I = \prod_{i \in I} \{1, \dots, m_i\}$ and for every $\omega \in \Omega_I$ set*

$$B_\omega = \Big(S \cap \bigcap_{i \in I} B_{i,\omega_i} \Big) \setminus \Big(\bigcup_{i \in I^c} \bigsqcup_{j=1}^{m_i} B_{i,j} \Big).$$

Then the set Ω_0 of all $\omega \in \Omega := \bigsqcup_{I \subseteq \{1,\dots,n\}} \Omega_I$ such that B_ω fails to be (K, δ)-invariant satisfies $|\bigcup_{\omega \in \Omega_0} B_\omega| \leq \delta |S|$.

Proof. Note that the sets B_ω are pairwise disjoint. Set $S_0 = S \cap \partial_{D^2} S$. Then $|S_0| \leq |\partial_{D^2} S| \leq \delta^2 (4|K|)^{-1} |S|$. By $\mathscr{F}$-tileability, for every $i = 1, \dots, n$ and $j = 1, \dots, m_i$ there are a $T_{i,j} \subseteq G$ and $F_{i,j,t} \in \mathscr{F}$ for $t \in T_{i,j}$ such that $B_{i,j} = \bigsqcup_{t \in T_{i,j}} F_{i,j,t} t$. Write $T'_{i,j}$ for the set of all $t \in T_{i,j}$ such that $F_{i,j,t} t \subseteq S$, and $T''_{i,j}$ for the set of all $t \in T_{i,j}$ such that $F_{i,j,t} t \subseteq S \setminus \partial_D S$. Observe that, since $FF^{-1}(S \setminus \partial_D S) \subseteq S$ for every $F \in \mathscr{F}$, if $F_{i,j,t} t \cap (S \setminus \partial_D S)$ is nonempty for some $t \in T_{i,j}$ then taking any element s in this intersection we have $F_{i,j,t} t \subseteq F_{i,j,t} F_{i,j,t}^{-1} s \subseteq S$ and hence $t \in T'_{i,j}$.

For every $I \subseteq \{1, \dots, n\}$ set $\Gamma_I = \prod_{i \in I} \{(j,t) : 1 \leq j \leq m_i,\ t \in T''_{i,j}\}$. For each $\gamma = ((j_i, t_i))_{i \in I} \in \Gamma_I$ define

$$E_\gamma = \Big(\bigcap_{i \in I} F_{i,j_i,t_i} t_i \Big) \setminus \Big(\bigcup_{i \in I^c} \bigsqcup_{j=1}^{m_i} B_{i,j} \Big) \subseteq S \setminus \partial_D S$$

and note that these sets over all $\gamma \in \Gamma := \bigsqcup_{I \subseteq \{1,\dots,n\}} \Gamma_I$ are pairwise disjoint. By the observation at the end of the first paragraph, for every $\gamma \in \Gamma$ we have

$$E_\gamma = \Big(\bigcap_{i \in I} F_{i,j_i,t_i} t_i \Big) \cap \Big(\bigcup_{i \in I^c} \bigsqcup_{j=1}^{m_i} \bigsqcup_{t \in T'_{i,j}} (G \setminus F_{i,j,t} t) \Big)$$

and therefore

$$\partial_K E_\gamma \subseteq \bigcup_{i=1}^{n} \bigsqcup_{j=1}^{m_i} \bigsqcup_{t \in T'_{i,j}} \partial_K F_{i,j,t} t. \tag{4.1}$$

Write Γ_0 for the set of all $\gamma \in \Gamma$ such that E_γ fails to be $(K, \delta/2)$-invariant. Since the sets E_γ are pairwise disjoint, each element of G belongs to $\partial_K E_\gamma$ for at most $|K|$ many γ, and so

$$\sum_{\gamma \in \Gamma} |\partial_K E_\gamma| \leq |K| \Big| \bigcup_{\gamma \in \Gamma} \partial_K E_\gamma \Big|. \tag{4.2}$$

Also, since the sets $F_{i,j,t} t$ for $t \in T'_{i,j}$ are subsets of S and each element of S is contained in at most n of them, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{t \in T'_{i,j}} |F_{i,j,t} t| \leq n|S|. \tag{4.3}$$

We therefore obtain

$$\begin{aligned}
\sum_{\gamma \in \Gamma_0} |E_\gamma| &< \frac{2}{\delta} \sum_{\gamma \in \Gamma} |\partial_K E_\gamma| \\
&\overset{(4.2)}{\leq} \frac{2|K|}{\delta} \Big| \bigcup_{\gamma \in \Gamma} \partial_K E_\gamma \Big| \\
&\overset{(4.1)}{\leq} \frac{2|K|}{\delta} \Big| \bigcup_{i=1}^{n} \bigsqcup_{j=1}^{m_i} \bigsqcup_{t \in T'_{i,j}} \partial_K F_{i,j,t} t \Big| \\
&\leq \frac{2|K|}{\delta} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{t \in T'_{i,j}} |\partial_K F_{i,j,t} t| \\
&\leq \frac{2|K|}{\delta} \cdot \frac{\delta^3}{8|K|^2 n} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{t \in T'_{i,j}} |F_{i,j,t} t| \\
&\overset{(4.3)}{\leq} \frac{\delta^2}{4|K|} |S|.
\end{aligned}$$

Set $S_1 = S_0 \cup \bigsqcup_{\gamma \in \Gamma_0} E_\gamma$. Then

$$|S_1| \leq |S_0| + \sum_{\gamma \in \Gamma_0} |E_\gamma| \leq \frac{\delta^2}{4|K|}|S| + \frac{\delta^2}{4|K|}|S| = \frac{\delta^2}{2|K|}|S|.$$

Writing Ω_1 for the set of all $\omega \in \Omega$ such that $|B_\omega \cap S_1| > \delta(2|K|)^{-1}|B_\omega|$, we thereby obtain

$$\Big|\bigsqcup_{\omega\in\Omega_1} B_\omega\Big| = \sum_{\omega\in\Omega_1} |B_\omega| < \frac{2|K|}{\delta}\sum_{\omega\in\Omega_1}|B_\omega \cap S_1| \le \frac{2|K|}{\delta}|S_1| \le \delta|S|.$$

To complete the proof it is therefore enough to verify that $\Omega_0 \subseteq \Omega_1$.

Let $\omega \in \Omega \setminus \Omega_1$. Then there is a $\Gamma_\omega \subseteq \Gamma$ such that $(S \setminus S_0) \cap B_\omega = (S \setminus S_0) \cap \bigsqcup_{\gamma\in\Gamma_\omega} E_\gamma$ and $E_\gamma \subseteq B_\omega$ for all $\gamma \in \Gamma_\omega$, in which case we can write B_ω as the union of $B_\omega \cap S_1$ and $\bigsqcup_{\gamma\in\Gamma_\omega\setminus\Gamma_0} E_\gamma$, so that

$$\begin{aligned}|\partial_K B_\omega| &\le |\partial_K(B_\omega \cap S_1)| + \sum_{\gamma\in\Gamma_\omega\setminus\Gamma_0} |\partial_K E_\gamma| \\ &\le |K||B_\omega \cap S_1| + \frac{\delta}{2}\sum_{\gamma\in\Gamma_\omega\setminus\Gamma_0}|E_\gamma| \\ &\le |K| \cdot \frac{\delta}{2|K|}|B_\omega| + \frac{\delta}{2}|B_\omega| \\ &= \delta|B_\omega|.\end{aligned}$$

This shows that $\omega \notin \Omega_0$ and hence that $\Omega_0 \subseteq \Omega_1$. □

Lemma 4.2. *Let G be a amenable group and H a subgroup of G. Let $G \curvearrowright X$ be a free action on a compact Hausdorff space. Suppose that (i) the restricted action $H \curvearrowright X$ has comparison, and (ii) for every finite set $K \subseteq G$ and $\delta > 0$ there is an open castle $\{S_i, V_i\}_{i\in I}$ for the G-action such that each V_i has diameter smaller than δ, each shape S_i is (K,δ)-invariant and the upper H-density of $X \setminus \bigsqcup_{i\in I} S_iV_i$ is less than δ. Then the action $G \curvearrowright X$ is almost finite.*

Proof. If H is finite then every subset of X of upper H-density less than $|H|^{-1}$ is empty, so that when $\delta < |H|^{-1}$ the castles in (ii) are clopen and have footprint equal to X, from which we deduce that $G \curvearrowright X$ is almost finite. We may thus assume that H is infinite. Let K be a finite subset of G and $0 < \delta < 1$. Choose a finite set $e \in K' \subseteq H$ with $|K'| > 1/\delta$. By assumption, there is an open castle $\{(S_i, V_i)\}_{i\in I}$ for the G-action whose shapes are $(K \cup K', \delta)$-invariant and the complement of whose footprint has upper H-density at most δ. In particular, it satisfies the first condition in the definition of almost finiteness with respect to K and δ. Choose a set R of representatives for the right cosets of H in G. For each $i \in I$ partition S_i into subsets of right cosets of H, i.e.,

$$S_i = \bigsqcup_{g\in R} B_{i,g}g$$

where each $B_{i,g}$ is contained in H. Note that left translation by K' preserves the right cosets of H. If $B_{i,g}$ for some $g \in R$ has cardinality less than $1/\delta$ then all of its elements belong to $\partial_{K'}B_{i,g}$ and so $|\partial_{K'}B_{i,g}| \ge |B_{i,g}|$. Writing L for the set of all $g \in R$ such that $0 < |B_{i,g}| < 1/\delta$, it follows that

$$\sum_{g\in L}|B_{i,g}| \le \sum_{g\in L}|\partial_{K'}B_{i,g}| \le |\partial_{K'}S_i| \le \delta|S_i|,$$

i.e., most elements of S_i share a coset with at least $1/\delta$ other elements. For each $i \in I$ and $g \in R$ choose a set $B'_{i,g} \subseteq B_{i,g}$ with cardinality equal to $\lceil \frac{\delta}{1-\delta}|B_{i,g}|\rceil$. Set $S'_i = \bigsqcup_{g\in R} B'_{i,g}g$ and note that when $|B_{i,g}| \geq 1/\delta$ we have $|B'_{i,g}| \leq \frac{\delta}{1-\delta}|B_{i,g}| + 1 \leq 2\delta|B_{i,g}|$, so that

$$|S'_i| \leq \sum_{g \in L} |B_{i,g}| + \sum_{g \in R \setminus L} |B'_{i,g}| \leq 3\delta|S_i|.$$

Let μ be any H-invariant Borel probability measure on X. By construction, the set $\bigsqcup_{i \in I} S'_i V_i$ has μ-measure at least $\frac{\delta}{1-\delta}\mu(\bigsqcup_{i \in I} S_i V_i)$, which is greater than or equal to δ. On the other hand, since the closed set $X \setminus \bigsqcup_{i \in I} S_i V_i$ has upper H-density less than δ its μ-measure is less than δ, and so our hypothesis that the H-action has comparison yields

$$X \setminus \bigsqcup_{i \in I} S_i V_i \prec \bigsqcup_{i \in I} S'_i V_i.$$

Since we can take δ to be as small as we wish, this shows that the action $G \curvearrowright X$ is almost finite. $\square$

5. Extensions by $\mathbb{Z}$

Our goal here is to prove Theorem 5.4. We will need a version of the Ornstein–Weiss quasitiling theorem [31], which we record as Corollary 5.2. The statement is a simple consequence of the following more usual version, which we reproduce in the form presented in [19]. A finite subset K of a group G is said to be ε-*quasitiled* by a finite collection $\mathscr{F} = \{F_1, \dots, F_n\}$ of finite subsets of G if there are sets $C_1, \dots, C_n \subseteq G$ and $F_{i,c} \subseteq F_i$ with $|F_{i,c}| \geq (1-\varepsilon)|F_i|$ for every $i = 1, \dots, n$ and $c \in C_i$ such that (i) the union $\bigcup_{i=1}^n F_i C_i$ is contained in K and has cardinality at least $(1-\varepsilon)|K|$, and (ii) the collection $\{F_{i,c}c : 1 \leq i \leq n,\ c \in C_i\}$ is disjoint. As in Section 4, we say that K is $\mathscr{F}$-*tileable* if there are sets $C_1, \dots, C_n \subseteq G$ such that the collection $\{F_i c : 1 \leq i \leq n,\ c \in C_i\}$ partitions K.

Theorem 5.1. *Let G be a group. Let $0 < \varepsilon < \frac{1}{2}$ and let $m \in \mathbb{N}$ be such that $(1 - \varepsilon/2)^m < \varepsilon$. Let $e \in F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m$ be finite subsets of G such that for each $k = 2, \dots, m$ the set F_k is $(F_{k-1}, \varepsilon/8)$-invariant. Then every $(F_m, \varepsilon/4)$-invariant finite subset of G is ε-quasitiled by $\{F_1, \dots, F_m\}$.*

Corollary 5.2. *Let G be an amenable group. Let $0 < \varepsilon < \frac{1}{2}$. Let K be a finite subset of G and $\delta > 0$. Then there exists a finite collection $\mathscr{F}$ of (K, δ)-invariant finite subsets of G containing e such that for every $(\bigcup \mathscr{F}, \varepsilon/4)$-invariant finite set $E \subseteq G$ there is an $\mathscr{F}$-tileable $E' \subseteq E$ satisfying $|E'| \geq (1 - \varepsilon)|E|$.*

Proof. Let $m \in \mathbb{N}$ be such that $(1 - \varepsilon/2)^m < \varepsilon$. Since G is amenable we can find $e \in F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m$ as in the statement of Theorem 5.1. Write $\mathscr{F}$ for the (finite) collection of all sets F such that for some $j = 1, \dots, m$ we have $F \subseteq F_j$ and $|F| \geq (1 - \varepsilon)|F_j|$. In view of the definition of ε-quasitiling, Theorem 5.1 then tells us that for every $(F_m, \varepsilon/4)$-invariant finite set $E \subseteq G$ there is an $\mathscr{F}$-tileable $E' \subseteq E$ such that $|E'| \geq (1 - \varepsilon)|E|$. As $F_m = \bigcup \mathscr{F}$ this yields the conclusion. $\square$

Let H be a countable group and α an automorphism of H, and form the corresponding semidirect product $H \rtimes \mathbb{Z}$. Inside $H \rtimes \mathbb{Z}$ we view $\mathbb{Z}$ multiplicatively as the group $\langle g \rangle$ with generator g satisfying $g^i t g^{-i} = \alpha^i(t)$ for all $i \in \mathbb{Z}$ and $t \in H$. When we say an *interval* in $\langle g \rangle$ we mean a set of the form $\{g^m, g^{m-1}, \dots, g^n\}$ for some integers $m \leq n$.

Lemma 5.3. *Let $H \rtimes \mathbb{Z} \curvearrowright X$ be a continuous action on a compact metrizable space. Then $\overline{D}_H(gA) = \overline{D}_H(A)$ for all $A \subseteq X$.*

Proof. For every $A \subseteq X$ we have, with F ranging over the nonempty finite subsets of H,

$$\begin{aligned}\overline{D}_H(A) = \inf_F \sup_{x\in X} \frac{1}{|F|}\sum_{s\in F} 1_A(sx) &= \inf_F \sup_{x\in X} \frac{1}{|F|}\sum_{s\in F} 1_{gA}(gsx)\\ &= \inf_F \sup_{x\in X} \frac{1}{|\alpha(F)|}\sum_{s\in F} 1_{gA}(\alpha(s)gx)\\ &= \inf_F \sup_{x\in X} \frac{1}{|\alpha(F)|}\sum_{t\in \alpha(F)} 1_{gA}(tgx)\\ &= \overline{D}_H(gA).\end{aligned}$$

□

Theorem 5.4. *Let $H \rtimes \mathbb{Z} \curvearrowright X$ be a free continuous action on a compact metric space. Suppose that the restricted action $H \curvearrowright X$ is almost finite. Then the action $H \rtimes \mathbb{Z} \curvearrowright X$ is almost finite.*

Proof. Let $\varepsilon > 0$. Let K be a finite subset of H and $0 < \delta < \frac{1}{12}$. Take an $r \in \mathbb{N}$ large enough so that any interval in $\langle g\rangle$ of cardinality at least $\lfloor \delta r\rfloor$ is $(\{g\}, \delta)$-invariant. Denote by A the symmetric interval $\{g^{-r}, \dots g^{r}\}$ and by A^+ the symmetric interval $\{g^{-(r+\lceil 3\delta r\rceil)}, \dots, g^{r+\lceil 3\delta r\rceil}\}$. Set $\beta = \min\{\varepsilon/(4r+1), \delta\}$.

Set $K' = \bigcup_{i=-2r}^{2r} \alpha^i(K)$ and $K'' = \bigcup_{i=-4r}^{4r} \alpha^i(K') = \bigcup_{i=-6r}^{6r} \alpha^i(K)$. By Corollary 5.2 there exists a finite collection $\mathscr{F}$ of $(K'', \beta^3/(8|K'|(8r+1)))$-invariant finite subsets of H containing e such that for every $(\bigcup\mathscr{F}, \varepsilon/4)$-invariant finite set $E \subseteq H$ there is an $\mathscr{F}$-tileable $E' \subseteq E$ satisfying $|E'| \geq (1-\varepsilon)|E|$.

Since $H \curvearrowright X$ is almost finite it is almost finite in measure, and so writing η for the minimum of $\beta^2/(4|K'|))$ and $\varepsilon/4$ we can find an open castle $\{(S_k, V_k)\}_{k=1}^n$ for this action whose shapes are $(((\bigcup\mathscr{F})(\bigcup\mathscr{F})^{-1})^2, \eta)$-invariant and whose footprint $\bigsqcup_{k=1}^n S_kV_k$ has lower H-density at least $1-\varepsilon$. The proof of Theorem 5.6 in [20] shows that we may assume the boundary of each level of each tower in the castle to have zero upper H-density. By Theorem 5.5 of [20], for any $k \in \{1, \dots, n\}$ we can find a finite disjoint collection $\mathscr{U}$ of open subsets of V_k whose diameters are as small as we wish such that the set $V_k \setminus \bigcup\mathscr{U}$ has zero upper H-density. Since the action $H \rtimes \mathbb{Z} \curvearrowright X$ is free, we may therefore furthermore assume, by replacing each tower (S_k, V_k) with a collection of towers with shape S_k whose bases are the members of a suitable collection of open subsets of V_k of the type just described, that (A^3S_k, V_k) is a tower for every $k = 1, \dots, n$. For each $k = 1, \dots, n$, since the shape S_k is $(\bigcup\mathscr{F}, \varepsilon/4)$-invariant we can find, by the previous paragraph, an $\mathscr{F}$-tileable $B_k \subseteq S_k$ satisfying $|B_k| \geq (1-\varepsilon)|S_k|$.

Our goal will be to develop a recursive disjointification process that uses the intersection patterns of the partial orbits AB_kx for $1 \leq k \leq n$ and $x \in V_k$ to build an open castle whose tower shapes are Følner subrectangles of the rectangles A^+B_k and whose footprint will be close to one in lower H-density. This footprint will be a subset of $\bigcup_{k=1}^n AB_kV_k$, and in general the containment will be proper since we will have to discard some of the subrectangles that the construction produces on account of their not being sufficiently Følner. For the disjointification at each stage of the recursion we cannot simply rely on the obvious complementation, but must subject this complementation to a game of give-and-take with some of the previously defined subrectangles (which thereby get redefined) in order to ensure that the width (in the A direction) of every subrectangle, without exception, is a sufficiently long interval. Note that when H is infinite the aspect ratio of the rectangular tower shapes AB_k is extremely large: the rectangles are very thin in the A direction and very tall (i.e., much larger) in the B_k direction. This guarantees that the multiplicity of overlap among the partial orbits AB_kx is small relative to the size of the sets B_k, which means that we do not need to partition the sets B_k too finely to produce our subrectangles, thereby ensuring via Lemma 4.1 that most of these subrectangles will be Følner.

Set $\mathscr{F}' = \{\alpha^i(F) : -4r \leq i \leq 4r,\ F \in \mathscr{F}\}$. Since each member of $\mathscr{F}$ is $(K'', \beta^3/(8|K'|(8r+1)))$-invariant, each member of $\mathscr{F}'$ is $(K', \beta^3/(8|K'|(8r+1)))$-invariant, as is easily checked using the fact that for finite sets $F \subseteq H$ one has $\partial_{K'}\alpha^i(F) = \alpha^i(\partial_{\alpha^{-i}(K')}F)$ for every i. Moreover, the fact that each B_k is $\mathscr{F}$-tileable implies that for every $k = 1, \dots, n$ and $i = -4r, \dots, 4r$ the set $\alpha^i(B_k)$ is $\mathscr{F}'$-tileable.

For the purposes of the next few paragraphs we let $1 \leq k \leq n$ and $x \in V_k$. Observe that if $y \in V_j$ for some $1 \leq j \leq n$ and $A^3B_kx \cap AB_jy \neq \emptyset$ then $y = cx$ for some $c \in (AB_j)^{-1}A^3B_k$, so that $c = g^it$ for some $i \in \{-4r, \dots, 4r\}$ and $t \in \alpha^{-i}(B_j^{-1})B_k$. In this case

$$\begin{aligned} A^3B_kx \cap AB_jy = (A^3B_k \cap AB_jg^it)x &= (A^3B_k \cap Ag^i\alpha^{-i}(B_j)t)x \\ &= (A^3 \cap Ag^i)(B_k \cap \alpha^{-i}(B_j)t)x. \end{aligned}$$

Notice moreover that, given a $b \in B_k$ and an $i \in \{-4r, \dots, 4r\}$, there are at most one $j \in \{1, \dots, n\}$ and one $y \in V_j$ of the form g^itx for some $t \in H$ such that $A^3bx \cap AB_jy \neq \emptyset$ (when $i = 0$ these j and y always exist and are equal to k and x, respectively). Indeed if $y_1 \in V_{j_1}$ and $y_2 \in V_{j_2}$ for $1 \leq j_1, j_2 \leq n$ are such that $A^3bx \cap AB_{j_1}y_1 \neq \emptyset$ and $A^3bx \cap AB_{j_2}y_2 \neq \emptyset$ and we have $y_1 = g^it_1x$ and $y_2 = g^it_2x$ for some $t_1, t_2 \in H$, then since

$$A^3bx \cap AB_{j_1}y_1 = (A^3 \cap Ag^i)bx \subseteq (A^3 \cap Ag^i)(B_k \cap \alpha^{-i}(B_{j_1})t_1)x$$

and

$$A^3bx \cap AB_{j_2}y_2 = (A^3 \cap Ag^i)bx \subseteq (A^3 \cap Ag^i)(B_k \cap \alpha^{-i}(B_{j_2})t_2)x$$

we have $\alpha^{-i}(B_{j_1})t_1 \cap \alpha^{-i}(B_{j_2})t_2 \neq \emptyset$, which implies (multiplying on the left by g^i and applying the action at x) that $B_{j_1}g^it_1x \cap B_{j_2}g^it_2x \neq \emptyset$, i.e., $B_{j_1}y_1 \cap B_{j_2}y_2 \neq \emptyset$. But this is only possible if $j_1 = j_2$ and $y_1 = y_2$ since $\{(B_j, V_j)\}_{j=1}^n$ is a castle. Let us call the points y which arise in this way but subject to the condition $-2r \leq i \leq 2r$ (i.e., as if we had substituted A for A^3 in the above argument) as the *base points associated to x and b*.

Notice that if one has two different points y as above, say y_1 and y_2, which belong to a common V_j, then the corresponding distinct values of i, say i_1 and i_2, must satisfy $|i_1 - i_2| \geq 2r + 1$, i.e., Ag^{i_1} and Ag^{i_2} must be disjoint. This is clear from the equalities in the last two displays of the previous paragraph, since $AB_jy_1 \cap AB_jy_2 = \emptyset$ given that (A^3B_j, V_j) is a tower. In particular, the only base point associated to x which belongs to V_k is x itself (corresponding to the case $i = 0$), since an i satisfying $|i| \geq 2r + 1$ lies outside of the range from $-2r$ to $2r$.

For all $i = -4r, \dots, 4r$ and $j = 1, \dots, n$ write $T_{x,i,j}$ for the set of all $t \in H$ such that g^itx belongs to V_j and the set $A^3B_k \cap AB_jg^it = (A^3 \cap Ag^i)(B_k \cap \alpha^{-i}(B_j)t)$ is nonempty. Note that for a given i the sets $\alpha^{-i}(B_j)t$ for $1 \leq j \leq n$ and $t \in T_{x,i,j}$ are pairwise disjoint, by observations made two paragraphs above. Also, the fact that $\{(B_j, V_j)\}_{j=1}^n$ is a castle implies that $T_{x,0,k} = \{e\}$ and $T_{x,0,j} = \emptyset$ for $j \neq k$.

Write $\mathscr{I}$ for the collection of all $I \subseteq \{-4r, \dots, 4r\}$ such that $0 \in I$. For $I \in \mathscr{I}$ define $\Lambda_{x,I}$ to be the set of all pairs $\lambda = (l, t)$ where l is a function $I \to \{1, \dots, n\}$ and t is a function $i \mapsto t_i$ on I such that $t_i \in T_{x,i,l_i}$ for all $i \in I$. We refer to I as the *domain* of λ. Since $T_{x,0,k} = \{e\}$ and $T_{x,0,j} = \emptyset$ for $j \neq k$, we are forced to have $l_0 = k$. Also, the paragraph before the previous one shows that $l_{i_1} \neq l_{i_2}$ whenever $|i_1 - i_2| \leq 2r + 1$, so that l takes each of its values at most four times, and takes the value k exactly once in the interval $\{-2r, \dots, 2r\}$, at 0. For each $\lambda = (l, t) \in \Lambda_{x,I}$ set

$$B_{x,\lambda} = \Big(\bigcap_{i \in I} \alpha^{-i}(B_{l_i}) t_i \Big) \setminus \Big(\bigcup_{i \in I^c} \bigsqcup_{j=1}^{n} \bigsqcup_{t \in T_{x,i,j}} \alpha^{-i}(B_j) t \Big), \tag{5.1}$$

which is a subset of B_k since $\alpha^{-i}(B_{l_i}) t_i$ is equal to B_k when $i = 0$. Set $\Lambda_x = \bigsqcup_{I \in \mathscr{I}} \Lambda_{x,I}$. The sets $B_{x,\lambda}$ for $\lambda \in \Lambda_x$ form a partition of B_k. Notice also that for each $I \in \mathscr{I}$ and $\lambda = (l, t) \in \Lambda_{x,I}$ the elements $y \in X$ such that for some $1 \leq j \leq n$ we have $y \in V_j$ and $A^3 B_{x,\lambda} x \cap A B_j y \neq \emptyset$ are precisely $g^i t_i x$ for $i \in I$, with $g^i t_i x \in V_{l_i}$ for each $i \in I$. Such points whose corresponding i belongs to $I \cap \{-2r, \dots, 2r\}$ are, for each $b \in B_{x,\lambda}$, the base points associated to x and b, and whenever $B_{x,\lambda} \neq \emptyset$ we will refer to these as the *base points associated to x and $B_{x,\lambda}$*.

For each $i = -4r, \dots, 4r$ and $j = 1, \dots, n$, the $\mathscr{F}'$-tileability of $\alpha^i(B_j)$ implies that each of the sets $\alpha^i(B_j)t$ for $t \in T_{x,i,j}$ is $\mathscr{F}'$-tileable. Since S_k is $(((\bigcup \mathscr{F})(\bigcup \mathscr{F})^{-1})^2, \beta^2/(4|K'|))$-invariant and the members of $\mathscr{F}'$ are $(K', \beta^3/(8|K'|(8r+1)))$-invariant, we can thus apply Lemma 4.1 (taking there n to be $8r + 1$, δ to be β, K to be K', S to be S_k, $\mathscr{F}$ to be $\mathscr{F}'$, and the sets $B_{i,j}$ to be the sets $\alpha^{-i}(B_j)t$ with $-4r \leq i \leq 4r$, $1 \leq j \leq n$, and $t \in T_{x,i,j}$ (subject to a reindexing that shifts i by $4r + 1$ and absorbs both the j and t in $\alpha^{-i}(B_j)t$ into the j in $B_{i,j}$), with the intersection with S in the definition of the sets B_ω being redundant) to obtain a $\Lambda'_x \subseteq \Lambda_x$ such that for every $\lambda \in \Lambda'_x$ the set $B_{x,\lambda}$ is (K', δ)-invariant (using the fact that $\beta \leq \delta$) and

$$\Big| \bigsqcup_{\lambda \in \Lambda'_x} B_{x,\lambda} \Big| \geq (1 - \beta) \Big| \bigsqcup_{\lambda \in \Lambda_x} B_{x,\lambda} \Big|. \tag{5.2}$$

We will recursively construct, for $k = 1, \dots, n$, intervals $A^{(k)}_{x,\lambda}$ in $\langle g \rangle$ for $x \in \bigsqcup_{j=1}^{k} V_j$ and $\lambda \in \Lambda_x$ such that

(i) each $A^{(k)}_{x,\lambda}$ is either empty or has cardinality at least $\lfloor \delta r \rfloor$,

(ii) each nonempty $A^{(k)}_{x,\lambda}$ is contained in A^+ and (with some exceptions) intersects A,

(iii) $\bigcup_{j=1}^{k} A B_j V_j = \bigsqcup_{x \in \bigcup_{j=1}^{k} V_j} \bigsqcup_{\lambda \in \Lambda_x} A^{(k)}_{x,\lambda} B_{x,\lambda} x$.

For every x it is possible that there are some λ for which $B_{x,\lambda} = \emptyset$, and in such cases it will be of no value for us to define $A^{(k)}_{x,\lambda}$, but we include these sets anyway in the construction in order to streamline our description of the process. In the end we will obtain a castle (via (iii) at the final stage $k = n$), but some of the tower shapes will not be sufficiently invariant. The sets Λ'_x, which will not play any role in the construction itself, will tell us which towers to retain in order to ensure approximate invariance.

At the first stage, when $k = 1$, we set $A^{(1)}_{x,\lambda} = A$ for all $x \in V_1$ and $\lambda \in \Lambda_x$.

Suppose now that $1 \leq k < n$ and that we have defined the intervals $A^{(k)}_{x,\lambda}$ for $x \in \bigsqcup_{j=1}^{k} V_j$ and $\lambda \in \Lambda_x$. Let $x \in V_{k+1}$ and let us define the sets $A^{(k+1)}_{x,\lambda}$ for $\lambda \in \Lambda_x$. The definition of these sets for points in $\bigsqcup_{j=1}^{k} V_j$ will occur in a piecemeal way during the course of the construction, or in some cases at the very end.

We begin with some notation. Write $\mathscr{I}^-$ for the collection of all $I \subseteq \{-2r, \dots, 2r\}$ such that $0 \in I$. For $I \in \mathscr{I}^-$ define $\Lambda_{x,I}^-$ to be the set of all pairs $\theta = (l, t)$ where l is a function $I \to \{1, \dots, n\}$ and t is a function $i \mapsto t_i$ on I such that $t_i \in T_{x,i,l_i}$ for all $i \in I$. As in the case of $\mathscr{I}$, we call I the *domain* of θ. Set $\Lambda_x^- = \bigsqcup_{I \in \mathscr{I}^-} \Lambda_{x,I}^-$.

We also define $\mathscr{I}^+$, $\Lambda_{x,I}^+$ for $I \in \mathscr{I}^+$, and Λ_x^+ exactly as their minus versions above except that we replace the interval $\{-2r, \dots, 2r\}$ with $\{-6r, \dots, 6r\}$.

For $\theta = (l, t) \in \Lambda_x^-$ with domain I we write Λ_θ for the set of all $\lambda = (m, u) \in \Lambda_x$ such that if J denotes the domain of λ then $J \cap \{-2r, \dots, 2r\} = I$, and $m_i = l_i$ and $u_i = t_i$ for all $i \in I$. Similarly, we write Λ_θ^+ for the set of all $\gamma = (m, u) \in \Lambda_x^+$ such that if J denotes the domain of γ then $J \cap \{-2r, \dots, 2r\} = I$, and $m_i = l_i$ and $u_i = t_i$ for all $i \in I$.

Let $I \in \mathscr{I}^-$ and $\theta = (l, t) \in \Lambda_{x,I}^-$. Over the course of the next few paragraphs we will construct an interval $A_\theta \subseteq A^+$, and once this is done we will set $A_{x,\lambda}^{(k+1)} = A_\theta$ for every $\lambda \in \Lambda_\theta$. Since the sets Λ_θ for $\theta \in \Lambda_x^-$ partition Λ_x, this will define $A_{x,\lambda}^{(k+1)}$ for every $\lambda \in \Lambda_x$. The type of operation that we will employ to construct the intervals A_θ will depend on the intersection patterns of the partial orbit $AB_{k+1}x$ with the sets AB_jV_j for $j = 1, \dots, k$, which is the information captured by θ. In some cases (what we call "synchronization" below) information about the history of the recursion is required to determine A_θ. Along the way we will also define some of the intervals $A_{y,\lambda'}^{(k+1)}$ for $y \in \bigsqcup_{j=1}^{k} V_j$ and $\lambda' \in \Lambda_y$, and, unlike for the sets $A_{x,\lambda}^{(k+1)}$, there will be dependence in this case on the elements of Λ_y and not merely on their restrictions to that part of their domain which lies in $\{-2r, \dots, 2r\}$ (as registered by the elements of Λ_y^-). We must be careful that each of these $A_{y,\lambda'}^{(k+1)}$ gets defined in an unambiguous way during the process of defining $A_{x,\lambda}^{(k+1)}$ both for different λ and for different x. As we will see, this is the reason for requiring the domains of elements of Λ_y for $y \in \bigsqcup_{j=1}^{n} V_j$ to be subsets of $\{-4r, \dots, 4r\}$ instead of $\{-2r, \dots, 2r\}$, even though the definition of A_θ, and hence of each $A_{x,\lambda}^{(k+1)}$, will only depend on restrictions to the smaller interval $\{-2r, \dots, 2r\}$. For each of the points $y \in \bigsqcup_{j=1}^{k} V_j$ that remains after the entire process has been completed for every $x \in V_{k+1}$, the interval $A_{y,\lambda'}^{(k+1)}$ will simply be defined as $A_{y,\lambda'}^{(k)}$.

Set $B_{x,\theta} = \bigsqcup_{\lambda \in \Lambda_\theta} B_{x,\lambda}$. If $AB_{x,\theta}x \subseteq \bigcup_{j=1}^{k} AB_jV_j$ then we set $A_{x,\theta}^{(k+1)} = \emptyset$, while if $AB_{x,\theta}x$ and $\bigcup_{j=1}^{k} AB_jV_j$ are disjoint then we set $A_{x,\theta}^{(k+1)} = A$.

Suppose then that $AB_{x,\theta}x \setminus \bigcup_{j=1}^{k} AB_jV_j$ is a nonempty proper subset of $AB_{x,\theta}x$. Set $I_0 = \{i \in I : 1 \leq l_i \leq k\}$, and note that for each $\lambda \in A_\theta$ the points $g^i t_{l_i} x$ for $i \in I_0$ are the base points associated to x and $B_{x,\lambda}$ which belong to V_j for some $1 \leq j \leq k$. By calculations similar to earlier ones, $AB_{x,\theta}x \setminus \bigcup_{j=1}^{k} AB_jV_j$ is equal to $A'B_{x,\theta}x$ where $A' = A \setminus \bigcup_{i \in I_0} Ag^i$. The set A' is an interval $\{g^p, \dots, g^q\}$ where $-r \leq p \leq q \leq r$, and we have three possible scenarios depending on whether this interval shares the same left endpoint as A, shares the same right endpoint as A, or lies strictly inside of A. Enumerating the elements of I_0 in increasing order as $i_1 < \dots < i_d$, we can describe these three mutually exclusive possibilities more precisely as follows:

(I) $0 < i_1$, in which case $p = -r$ and $q = i_1 - r - 1$,

(II) $i_d < 0$, in which case $p = i_d + r + 1$ and $q = r$,

(III) there exists a (necessarily unique) $1 \leq w < d$ such that $i_{w+1} - i_w > 2r + 1$, in which case $p = i_w + r + 1$ and $q = i_{w+1} - r - 1$.

This description will be useful below, but our construction will actually proceed by subdividing into a different trio of cases, namely

(1) $q - p \geq \delta r$,

(2) $q - p < \delta r$ and $p > -\delta r$,

(3) $q - p < \delta r$, $p \leq -\delta r$, and $q < \delta r$.

Note that if $q - p < \delta r$ then at least one of the inequalities $p > -\delta r$ and $q < \delta r$ and must hold, and so these three cases do indeed exhaust the possibilities. Each of the cases (2) and (3) will be further split into three subcases.

In treating case (2) (and implicitly also case (3)) we will need the following notation. Given a $J \subseteq \{-6r, \dots, 6r\}$, a $\gamma = (m, u) \in \Lambda_x^+$ with domain J, and an $i \in J \cap \{-2r, \dots, 2r\}$, and writing $y = g^i u_i x$ and $J' = \{-4r, \dots, 4r\} \cap (J - i)$, we define γ_i to be the element of $\Lambda_{y,J'}$ given by $i' \mapsto (m_{i'+i}, \alpha^i(u_{i'+i}u_i^{-1}))$ for all $i' \in J'$, and if $i = i_w$ for some $1 \leq w \leq d$ then for brevity we write $A_{w,\gamma}^{(j)}$ for the set $A_{y,\gamma_{i_w}}^{(j)}$. The usual calculations show that, when the set B_{y,γ_i} (as defined in (5.1)) is nonempty, the points $g^{i'} u_{i'} x$ for $i' \in J' + i$ are precisely the base points associated to y and B_{y,γ_i}.

(1) Case $q - p \geq \delta r$. We set $A_{x,\theta}^{(k+1)} = A'$ (**filling**).

(2) Case $q - p < \delta r$ and $p > -\delta r$. In this situation there is a $1 \leq w \leq d$ such that $p = i_w + r + 1$ (we are in one of the scenarios (II) and (III), but which one doesn't matter for the construction to follow).

For each $\gamma \in \Lambda_\theta^+$, adjacent to the interval A' on the left (within $\langle g \rangle \cong \mathbb{Z}$) is an interval $A_{w_\gamma,\gamma}^{(k)} g^{i_{w_\gamma}}$ for some $1 \leq w_\gamma \leq w$ with $p - 1 - (r + \lceil 3\delta r \rceil) \leq i_{w_\gamma}$, as follows from our recursive hypothesis in (ii) that $A_{y,\lambda'}^{(k)} \subseteq A^+$ for all $y \in \bigcup_{j=1}^k V_j$ and $\lambda' \in \Lambda_y$. If it happens that $w_\gamma = w$ then $A_{w_\gamma,\gamma}^{(k)}$ is a subinterval of A sharing the same right endpoint as A, while if $w_\gamma < w$ then $A_{w_\gamma,\gamma}^{(k)}$ is a subinterval of A^+ which intersects $A^+ \setminus A$ (as well as A, as we will observe below).

It can happen that a bunch of the sets $A_{w_\gamma,\gamma}^{(k)}$ for different $\gamma \in \Lambda_\theta^+$ are equal to a common $A_{y,\lambda'}^{(k)}$ for some y and $\lambda' \in \Lambda_y$, as γ contains more information than is needed to determine y and λ' (it is a function of a wider range of coordinates). This will mean that $A_{y,\lambda'}^{(k+1)}$ for such y and λ' will possibly get defined several times below as γ ranges through Λ_θ^+, but a careful tracking of the argument will make it clear that this definition does not in fact depend on the particular γ, so that there is no ambiguity. Such a $A_{y,\lambda'}^{(k+1)}$ will also only get defined for this particular θ, since λ' contains enough information to determine θ. Indeed this is the reason for having these sets (along with the corresponding sets $B_{y,\lambda'}$) depend on the broader range of coordinates from $-4r$ to $4r$ and not merely on the range from $-2r$ to $2r$, which is sufficient to register the intersection patterns of the original towers. It will also become clear that there are no competing definitions of $A_{y,\lambda'}^{(k+1)}$ across different x, as any time $A_{y,\lambda'}^{(k+1)}$ gets defined differently from $A_{y,\lambda'}^{(k)}$ this will take place deep enough inside an imbrication of towers so as to prevent a similar discrepancy from occurring for a different x.

Write Γ_0 for the set of all $\gamma \in \Lambda_\theta^+$ such that $|A_{w_\gamma,\gamma}^{(k)}| < 2\delta r$, and set $\Gamma_1 = \Lambda_\theta^+ \setminus \Gamma_0$.

We divide into three subcases: (2a) $\Gamma_1 = \emptyset$, (2b) $\Gamma_0 = \emptyset$, and (2c) $\Gamma_0, \Gamma_1 \neq \emptyset$.

(2a) Case $\Gamma_1 = \emptyset$. Define $A^{(k+1)}_{x,\theta}$ to be the empty set and for each $\gamma \in \Lambda^+_\theta$ define $A^{(k+1)}_{w_\gamma,\gamma}$ to be the interval $A^{(k)}_{w_\gamma,\gamma} \sqcup A' g^{-i_{w_\gamma}}$ (**left donation**). Note in this case that $|A^{(k+1)}_{w_\gamma,\gamma}| \geq |A^{(k)}_{w_\gamma,\gamma}| \geq \delta r$, so that $A^{(k+1)}_{w_\gamma,\gamma}$ satisfies (i). By our recursive hypothesis we have $A^{(k)}_{w_\gamma,\gamma} \subseteq A^+$ and $A^{(k)}_{w_\gamma,\gamma} \cap A \neq \emptyset$. (as indicated in (ii) there are exceptions to the latter holding, but this will not be the case under the current circumstances, as will become clear through the condition ($\bullet$) below and its implicit counterpart in the case (3c)). Together with the fact that $|A^{(k)}_{w_\gamma,\gamma}| < 2\delta r$ and $p - q < \delta r$, this implies that $A^{(k+1)}_{w_\gamma,\gamma} \subseteq A^+$. The nonemptiness of $A^{(k)}_{w_\gamma,\gamma} \cap A$ also implies that $A^{(k+1)}_{w_\gamma,\gamma} \cap A$ is nonempty, and so $A^{(k)}_{w_\gamma,\gamma}$ satisfies (ii).

(2b) Case $\Gamma_0 = \emptyset$. Define $A^{(k+1)}_{x,\theta}$ to be the interval $\{g^{p-\lfloor \delta r \rfloor}, \dots, g^{p-1}\} \sqcup A'$ and for each $\gamma \in \Lambda^+_\theta$ define $A^{(k+1)}_{w_\gamma,\gamma}$ to be the interval $A^{(k)}_{w_\gamma,\gamma} \setminus \{g^{-i_{w_\gamma}+p-\lfloor \delta r \rfloor}, \dots, g^{-i_{w_\gamma}+p-1}\}$ (**left appropriation**). Then $A^{(k+1)}_{x,\theta} \subseteq A$ (since $p > -\delta r$ and $\delta < \frac{1}{2}$ and hence $p - \lfloor \delta r \rfloor \geq -r$) and $|A^{(k+1)}_{x,\theta}| \geq \delta r$. For each $\gamma \in \Lambda^+_\theta$, since $|A^{(k)}_{w_\gamma,\gamma}| \geq 2\delta r$ (by our hypothesis that $\Gamma_0 = \emptyset$) we have

$$|A^{(k+1)}_{w_\gamma,\gamma}| \geq |A^{(k)}_{w_\gamma,\gamma}| - \lfloor \delta r \rfloor \geq 2\delta r - \delta r = \delta r.$$

Moreover, since we defined $A^{(k+1)}_{w_\gamma,\gamma}$ by removing from $A^{(k)}_{w_\gamma,\gamma}$ a subinterval which shares the same right endpoint and has cardinality smaller than $|A^+ \setminus A|/2$, and since $A^{(k)}_{w_\gamma,\gamma}$ is contained in A^+ and intersects A by hypothesis (as in the case of (2a) it will become clear that this is not one of the exceptional situations in (ii)), we see that $A^{(k+1)}_{w_\gamma,\gamma}$ must also intersect A and be contained in A^+, so that $A^{(k+1)}_{w_\gamma,\gamma}$ satisfies (i) and (ii).

(2c) Case $\Gamma_0, \Gamma_1 \neq \emptyset$. Here we apply a process of **left synchronization** that will effectively return us to one of the cases (2a) and (2b). We will begin by arguing that any two successive numbers in the sequence $i_1 < \cdots < i_w$ are less than $2\delta r$ apart. Suppose that this is not the case. Take the largest $\rho \in \{1, \dots, w-1\}$ such that $i_{\rho+1} - i_\rho \geq 2\delta r$. Let $\sigma \in \{\rho, \dots, w\}$ be such that l_{i_σ} is smallest among $l_{i_\rho}, \dots, l_{i_w}$ (these numbers are distinct by an observation made earlier, since their indices are all at most $2r$ apart). Suppose that $\rho < \sigma$. Then for all $\gamma \in \Lambda^+_\theta$ the set $A^{(l_{i_\sigma})}_{\sigma,\gamma}$ must contain the interval $\{g^{r-\lceil 2\delta r \rceil}, \dots, g^r\}$, since at the l_{i_σ}th stage of the recursion we will have applied the filling operation in case (1), unless there was no intersection at all with previous towers (e.g., if $l_{i_\sigma} = 0$), in which case $A^{(l_{i_\sigma})}_{\sigma,\gamma}$ is simply equal to A. But this fact alone then determines, for each $v = \sigma+1, \dots, w$, the sets $A^{(j)}_{v,\gamma}$ for $j = l_{i_v}, \dots, k$ and $\gamma \in \Lambda^+_\theta$, since these sets are defined by application of the left donation or appropriation operations in cases (2a) or (2b) at previous stages of the recursion (to make this assertion one also has to rule out the possibility that the right synchronization in case (3c) was never applied to any of these intervals at previous stages, but, as will be clear once we have a picture of the entire recursive construction, this follows from the fact that the intervals $Ag^{i_1}, \dots, Ag^{i_d}$ intersect each other and are bounded on the right by A', so that there is not enough of a history stretching to the right to produce the kind of unsynced situation that triggers (3c)). Therefore we must in fact be in one of the cases (2a) and (2b) at the current stage, a contradiction. So we must have $\sigma = \rho$. Let τ be such that l_{i_τ} is smallest among $l_{i_{\rho+1}}, \dots, l_{i_\omega}$. Then for all $\gamma \in \Lambda^+_\theta$ the set $A^{(l_{i_\tau})}_{\tau,\gamma}$ must contain the interval $\{g^{r-(i_\tau - i_{\rho+1})+1}, \dots, g^r\}$, whose cardinality is $i_\tau - i_{\rho+1}$, which is at least $2\delta r$. Like before, this fact alone then determines, for each $v = \tau+1, \dots, w$, the sets $A^{(j)}_{v,\gamma}$ for $j = l_{i_\rho}, \dots, k$ and $\gamma \in \Lambda^+_\theta$, which puts us into one of the cases (2a) and (2b), a contradiction. We have thus verified that any two successive numbers in the sequence $i_1 < \cdots < i_w$ are less than $2\delta r$ apart. An argument along similar lines shows that we must also have $i_1 + 2r < 2\delta r$.

Let $w' \in \{1, \dots, w\}$, to be determined, and define the function $\pi : \{1, \dots, w'\} \to \{1, \dots, w\}$ recursively by taking $\pi(1)$ so that $l_{i_{\pi(1)}}$ is smallest among $l_{i_1}, \dots, l_{i_w}$ and for $\kappa > 1$ taking $\pi(\kappa)$ so that $l_{i_{\pi(\kappa)}}$ is smallest among $l_{i_{\pi(\kappa-1)+1}}, \dots, l_{i_w}$. The number w' is then the last stage for this process can be carried out, and we will necessarily have $\pi(w') = w$. Note that $l_{i_{\pi(1)}} < \cdots < l_{i_{\pi(w')}}$ since the numbers $l_{i_1}, \dots, l_{i_w}$ are distinct (by the same observation as in the previous paragraph). The argument in the previous paragraph actually shows that $i_{\pi(\kappa)} - i_{\pi(\kappa-1)} < 2\delta r$ for all $\kappa = 2, \dots, w'$ and $i_{\pi(1)} + 2r < 2\delta r$. In particular, w' will be quite a bit larger than 1.

As will be clear once one has a complete picture of how the whole construction works under successive recursive steps, and in particular of how synchronization works at each step in conjunction with the donation and appropriation operations, we have the following, where $\mathscr{D}$ denotes the disjoint collection of all nonempty intervals of the form $A^{(k)}_{\pi(\kappa),\gamma} g^{i_{\pi(\kappa)}}$ for $\kappa = 2, \dots, w'$:

- ($\bullet$) for every $\gamma \in \Lambda^+_\theta$ the interval $\{g^{i_1+r+\lfloor 3\delta r\rfloor+1}, \dots, g^{i_w+r} = g^{p-1}\}$ is covered by $\mathscr{D}$ and intersects every member of $\mathscr{D}$, each member of $\mathscr{D}$ has cardinality at most $2\delta r$, and the last (i.e., rightmost) member of $\mathscr{D}$ intersects A.

The problem we need to make adjustments for is that while the numbers $i_{\pi(\kappa)}$ for $\kappa = 2, \dots, w'$ do not depend on γ, the sets $A^{(k)}_{\pi(\kappa),\gamma}$ do in general. We want to erase this dependence for values of κ close to w, i.e., we will define the sets $A^{(k+1)}_{\kappa,\gamma}$ for such κ so that they do not depend on γ. This will allow us to define $A^{(k+1)}_{x,\theta}$ by using one of the left donation and appropriation procedures in cases (2a) and (2b).

Choose a $\gamma_0 \in \Lambda^+_\theta$. We will replicate part of the interval pattern in ($\bullet$) associated to this fixed element in order to produce the desired uniformization across all $\gamma \in \Lambda^+_\theta$. By ($\bullet$) we have a $\kappa_1 \in \{1, \dots, w'\}$ such that $A^{(k)}_{\pi(\kappa_1),\gamma_0} g^{i_{\pi(\kappa_1)}}$ contains $g^{\lfloor -\frac{1}{4}r\rfloor}$ (we use the fraction $\frac{1}{4}$ in ($\bullet$) so that when applying the right-sided version of the synchronization argument below in (3c) for a different point x (and possibly at a later stage of the recursion) there is no possible interference coming from the right direction that will render the construction ambiguous, and also vice versa with left and right interchanged; for this we also need to ensure that δ is small enough). Let i be such that g^i is the left endpoint of the interval $A^{(k)}_{\pi(\kappa_1),\gamma_0} g^{i_{\pi(\kappa_1)}}$. We assume that γ_0 was chosen so as to maximize κ_1 and, subject to this condition, furthermore chosen so that i is minimized.

Let $\gamma \in \Lambda^+_\theta$. By ($\bullet$) there exists a $\kappa_0 \in \{1, \dots, w'\}$ such that the interval $A^{(k)}_{\pi(\kappa_0),\gamma} g^{i_{\pi(\kappa_0)}}$ contains g^i. Then $\kappa_0 \leq \kappa_1$, for otherwise the left endpoint of $A^{(k)}_{\pi(\kappa_0),\gamma} g^{i_{\pi(\kappa_0)}}$ would be to the right of $g^{\lfloor -\frac{1}{4}r\rfloor}$ by the maximality of κ_1, and hence to the right of g^i. For $\kappa = 1, \dots, \kappa_0 - 1$ define $A^{(k+1)}_{\pi(\kappa),\gamma}$ to be $A^{(k)}_{\pi(\kappa),\gamma}$. Next define A_0 to be the interval in $\langle g\rangle$ whose left endpoint is the left endpoint of $A^{(k)}_{\pi(\kappa_0),\gamma} g^{i_{\pi(\kappa_0)}}$ and whose right endpoint is the right endpoint of $A^{(k)}_{\pi(\kappa_1),\gamma_0} g^{i_{\pi(\kappa_1)}}$. If A_0 has cardinality at most $2\delta r$, then we set

$$A^{(k+1)}_{\pi(\kappa_1),\gamma} = A_0 g^{-i_{\pi(\kappa_1)}},$$

and

$$A^{(k+1)}_{\pi(\kappa),\gamma} = \begin{cases} \emptyset & \kappa = \kappa_0, \dots, \kappa_1 - 1 \\ A^{(k)}_{\pi(\kappa),\gamma_0} & \kappa = \kappa_1 + 1, \dots, w'. \end{cases}$$

All of the sets we have just defined, when nonempty, have cardinality at least $\lfloor \delta r \rfloor$ and at most $2\delta r$, and they are contained in A^+.

Suppose now that $|A_0| > 2\delta r$. Since the cardinalities of $A^{(k)}_{\pi(\kappa_0),\gamma}$ and $A^{(k)}_{\pi(\kappa_1),\gamma_0}$ are at most $2\delta r$ by ($\bullet$), the left endpoint of $A^{(k)}_{\pi(\kappa_0),\gamma} g^{i_{\pi(\kappa_0)}}$ cannot be g^i, and hence must be g^j for some $j < i$. By the minimality of i in our choice of γ_0, this implies that $\kappa_0 < \kappa_1$. Write A_0 as $\{g^M, \dots, g^N\}$. Then we can find an $M < i' < N$ such that the intervals

$$\begin{aligned} A^{(k+1)}_{\pi(\kappa_0),\gamma} &= \{g^{M - i_{\pi(\kappa_0)}}, \dots, g^{i' - i_{\pi(\kappa_0)}}\} \\ A^{(k+1)}_{\pi(\kappa_1),\gamma} &= \{g^{i'+1 - i_{\pi(\kappa_1)}}, \dots, g^{N - i_{\pi(\kappa_1)}}\}. \end{aligned}$$

both have cardinality at most $2\delta r$ and at least $\lfloor \delta r \rfloor$ and the first one is contained in $A^{(k)}_{\pi(\kappa_0),\gamma}$ and the second one in $A^{(k)}_{\pi(\kappa_1),\gamma_0}$, in which case both are contained in A^+. It could happen that $A^{(k+1)}_{\pi(\kappa_1),\gamma}$ does not intersect A while $A^{(k)}_{\pi(\kappa_1),\gamma}$ does, creating one of the exceptions to the second part of condition (ii) (there may also already be some such exceptions from the kth stage that carry over to the definition of the sets $A^{(k+1)}_{\pi(\kappa),\gamma}$). However, as will become clear in the next paragraph, this will not affect the validity of the last item in ($\bullet$) when applying donation and appropriation procedures at later stages. Finally, define

$$A^{(k+1)}_{\pi(\kappa),\gamma} = \begin{cases} \emptyset & \kappa = \kappa_0 + 1, \dots, \kappa_1 - 1 \\ A^{(k)}_{\pi(\kappa),\gamma_0} & \kappa = \kappa_1 + 1, \dots, w'. \end{cases}$$

These sets, when nonempty, have cardinality at least $\lfloor \delta r \rfloor$ and at most $2\delta r$, and they are contained in A^+.

With the above definitions, common to all γ there is a $w'' \in \{\kappa_1 + 1, \dots, w'\}$ (which is in fact quite a bit larger than κ_1 depending on how small δ is) such that $A^{(k+1)}_{\pi(\kappa),\gamma}$ is nonempty when $\kappa = w''$ and empty when $w'' < \kappa \le w'$, and the intervals $A^{(k+1)}_{\pi(w''),\gamma}$ for $\gamma \in \Lambda^+_\theta$ are all the same and, by ($\bullet$), intersect A. This means that we are effectively in one of the scenarios covered by (2a) and (2b), with the common (nonempty) interval $A^{(k+1)}_{\pi(w''),\gamma} g^{i_{\pi(w'')}}$ being immediately adjacent, to the left, of the interval $A' = \{g^p, \dots, g^q\}$. We thus now consider the intervals $A^{(k+1)}_{\pi(w''),\gamma}$ to be only provisionally defined, and proceed as we did in (2a) and (2b) according to the common length of the intervals to arrive at the final definitions of $A^{(k+1)}_{\pi(w''),\gamma}$, along with the definition of $A^{(k+1)}_{x,\theta}$.

(3) $q - p < \delta r$, $p \le -\delta r$, and $q < \delta r$. In this case we symmetrize the procedure from (2) the obvious way, replacing right with left and similarly treating three subcases with the operations of (3a) **right donation**, (3b) **right appropriation**, and (3c) **right synchronization**.

As indicated at the outset, we now set $A^{(k+1)}_{x,\lambda} = A_\theta$ for every $\lambda \in \Lambda_\theta$. These sets satisfy (i) and (ii) by observations made in the course of the above construction. This completes the definition of the sets $A^{(k+1)}_{x,\lambda}$ for all $\lambda \in \Lambda_x$.

Observe that, for every $A^{(k+1)}_{y,\lambda'}$ with $y \in \bigcup_{j=1}^k V_j$ that gets defined in the above procedure, the symmetric difference between the partial orbits $A^{(k+1)}_{y,\lambda'} B_{y,\lambda'} y$ and $A^{(k)}_{y,\lambda'} B_{y,\lambda'} y$ is contained in one of the partial orbits $AB_{k+1}x$, i.e., passing from $A^{(k)}_{y,\lambda'}$ to $A^{(k+1)}_{y,\lambda'}$ involves a reconfiguration that occurs inside of $AB_{k+1}x$. In particular, if the set $A^{(k+1)}_{y,\lambda'}$ gets defined then it does so in the course of the construction for only one particular x, and hence in an unambiguous way. If $A^{(k+1)}_{y,\lambda'}$ does not get defined then we simply set it to be equal to $A^{(k)}_{y,\lambda'}$. It is then readily seen from the construction that the sets $A^{(k+1)}_{y,\lambda'} B_{y,\lambda'} y$ over all $y \in \bigcup_{j=1}^k V_j$ are pairwise disjoint and their (disjoint) union with the pairwise disjoint sets $A^{(k+1)}_{x,\lambda} B_{x,\lambda} x$ for $x \in V_{k+1}$ and $\lambda \in \Lambda_x$ coincides with the union of $AB_{k+1}V_{k+1}$ with the sets $A^{(k)}_{y,\lambda'} B_{y,\lambda'} y$ for $y \in \bigcup_{j=1}^k V_j$. The equality in (iii) is thus satisfied for $k+1$ given that it is satisfied for k by our recursive hypothesis. This completes the recursive step.

By Lemma 5.3, the collection of subsets of X with upper H-density zero is $H \rtimes \mathbb{Z}$-invariant, and so the boundaries of the levels of the towers $(A^3 S_k, V_k)$ all belong to this collection given that the boundary of each V_k does. Since this collection is also an algebra, we therefore deduce, in view of the way that the above construction proceeded based on intersection patterns, that for each $k = 1, \dots, n$ there are a finite disjoint collection $\mathscr{U}_k$ of open subsets of V_k such that the set $\overline{V}_k \setminus \bigcup \mathscr{U}_k$, along with each of its images under elements of $H \rtimes \mathbb{Z}$, has zero upper H-density and for each $U \in \mathscr{U}_k$ sets Λ_U and Λ'_U and, for all $\lambda \in \Lambda_U$, sets $A_{U,\lambda}$ and $B_{U,\lambda}$ such that for every $x \in U$ we have $\Lambda_U = \Lambda_x$, $\Lambda'_U = \Lambda'_x$, $A_{U,\lambda} = A^{(n)}_{x,\lambda}$, and $B_{U,\lambda} = B_{x,\lambda}$. We then consider the open castle $\mathscr{C}$ consisting of the towers $(A_{U,\lambda} B_{U,\lambda}, U)$ with $\lambda \in \Lambda'_U$, but discarding the ones for which $A_{U,\lambda} B_{U,\lambda}$ is empty (note that each base U will be shared by many towers in general, as indexed by λ, even though the footprints of these towers are pairwise disjoint). Since the intervals $A_{U,\lambda}$ all have cardinality at least $\lfloor \delta r \rfloor$ (ignoring the empty ones) they are $(\{g\}, \delta)$-invariant, and so each of the shapes $A_{U,\lambda} B_{U,\lambda}$ is $(\{g\}, \delta)$-invariant. Moreover, the fact that each $B_{U,\lambda}$ with $\lambda \in \Lambda'_U$ is (K', δ)-invariant implies that each of the shapes $A_{U,\lambda} B_{U,\lambda}$ is (K, δ)-invariant, since for all $g^i \in A^{(k)}_{U,\lambda}$ we have $\partial_K (g^i B_{U,\lambda}) = g^{-i} \partial_{\alpha^{-i}(K)} B_{U,\lambda}$ and hence

$$|\partial_K (A_{U,\lambda} B_{U,\lambda})| \leq \sum_{i \,:\, g^i \in A_{U,\lambda}} |\partial_{\alpha^{-i}(K)} B_{U,\lambda}| \leq \sum_{i \,:\, g^i \in A_{U,\lambda}} \delta |B_{U,\lambda}| = \delta |A_{U,\lambda} B_{U,\lambda}|.$$

Given that the set $H \cup \{g\}$ generates $H \rtimes \mathbb{Z}$, a standard exercise then shows that we can make the shapes $A_{U,\lambda} B_{U,\lambda}$ with $\lambda \in \Lambda'_U$ as left invariant as we wish by making an appropriate choice of K and taking δ small enough. Note also that the bases of the towers in $\mathscr{C}$ can be made to have as small a diameter as we wish by taking the diameters of the bases $V_1, \dots, V_n$ of the initial towers to be sufficiently small.

We finally verify that our castle $\mathscr{C}$ has footprint of lower H-density at least $1 - 3\varepsilon$. Let μ be an H-invariant Borel probability measure on X. For each $k = 1, \dots, n$ the sets $\alpha^i(s) g^i V_k = g^i s V_k$ for $s \in S_k$ and $i = -2r, \dots, 2r$ are pairwise disjoint since $(A^2 S_k, V_k)$ is a tower, and so for every $B \subseteq S_k$, $U \subseteq V_k$, and $i = -2r, \dots, 2r$ we have $\mu(\alpha^i(B) g^i U) = |B| \mu(g^i U)$. Setting

$W_0 = \bigsqcup_{k=1}^n \bigsqcup_{U\in\mathscr{U}_k} \bigsqcup_{\lambda\in\Lambda_U\setminus\Lambda'_U} B_{U,\lambda}U$, we therefore have, for every $i\in\mathbb{Z}$,

$$\begin{aligned}
\mu(g^iW_0) &= \sum_{k=1}^n \sum_{U\in\mathscr{U}_k} \sum_{\lambda\in\Lambda\setminus\Lambda'_U} \mu(\alpha^i(B_{U,\lambda})g^iU) \\
&= \sum_{k=1}^n \sum_{U\in\mathscr{U}_k} \sum_{\lambda\in\Lambda\setminus\Lambda'_U} |B_{U,\lambda}|\mu(g^iU) \\
&\overset{(5.2)}{\leq} \sum_{k=1}^n \sum_{U\in\mathscr{U}_k} \beta \sum_{\lambda\in\Lambda} |B_{U,\lambda}|\mu(g^iU) \\
&= \beta \sum_{k=1}^n \sum_{U\in\mathscr{U}_k} \sum_{\lambda\in\Lambda} \mu(\alpha^i(B_{U,\lambda})g^iU) \\
&= \beta \sum_{k=1}^n \sum_{U\in\mathscr{U}_k} \sum_{\lambda\in\Lambda} \mu(g^iB_{U,\lambda}U) \\
&= \beta\mu\bigg(g^i\bigg(\bigsqcup_{k=1}^n \bigsqcup_{U\in\mathscr{U}_k} \bigsqcup_{\lambda\in\Lambda} B_{U,\lambda}U\bigg)\bigg) \\
&\leq \beta.
\end{aligned}$$

and hence $\mu(A^+W_0) \leq \sum_{i=-2r}^{2r} \mu(g^iW_0) \leq (4r+1)\beta \leq \varepsilon$. Since $|B_k|/|S_k| \geq 1-\varepsilon$ for every k and the castle footprint $\bigsqcup_{k=1}^n S_kV_k$ has lower H-density at least $1-\varepsilon$, the μ-measure of the open set $\bigsqcup_{k=1}^n B_kV_k$ is at least $(1-\varepsilon)^2$. Since for every k the images of the set $\overline{V}_k \setminus \bigcup \mathscr{U}_k$ under elements of $H\rtimes\mathbb{Z}$ all have μ-measure zero, the subset $W := \bigsqcup_{k=1}^n \bigsqcup_{U\in\mathscr{U}_k} \bigsqcup_{\lambda\in\Lambda} B_{U,\lambda}U$ of $\bigsqcup_{k=1}^n B_kV_k$ has the same μ-measure as $\bigsqcup_{k=1}^n B_kV_k$, and so we obtain

$$\mu(W\setminus A^+W_0)) \geq \mu(W) - \mu(A^+W_0) \geq (1-\varepsilon)^2 - \varepsilon \geq 1-3\varepsilon.$$

By the $k=n$ instance of (iii) we see that the footprint of the castle $\mathscr{C}$ contains $W\setminus A^+W_0$ and hence, by virtue of what we have just shown, has lower H-density at least $1-3\varepsilon$. Since by hypothesis the restricted action $H \curvearrowright X$ is almost finite and hence has comparison, it follows by Lemma 4.2 that the action $H\rtimes\mathbb{Z} \curvearrowright X$ is almost finite, as desired. □

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