

Construction of KdV flow -a unified approach-

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Abstract

A KdV flow is constructed on a space whose structure is described in terms of the spectrum of the underlying Schrödinger operators. The space includes the conventional decaying functions and ergodic ones. Especially any smooth almost periodic function can be initial data for the KdV equation.

1 Introduction

This article is a continuation of [11], where a KdV flow was constructed on a space of potentials with reflectionless property on an energy interval $[\lambda_1, \infty)$. Since the KdV equation is closely related with 1D Schrödinger operators, we use the terminology potentials to describe initial data for the KdV equation. When the previous paper was written, the author intended to remove this reflectionless property by approximating general potentials by reflectionless potentials, which made the procedure rather involved. However he has recognized that a direct extension is possible independently of the last paper. Therefore the present paper is readable without [11], although its knowledge would be very helpful for prompt understanding of the whole context.

Our approach to this problem is essentially based on Sato's philosophy [18], whose analytical version was given by Segal-Wilson [19]. From our point of view that is an analysis on eigen-spaces of underlying Schrödinger operators which seems quite natural due to GKMM and Lax.

To give perspective and state the main results several terminologies and notations have to be prepared. For positive odd integer n let Γ_n be

$$\Gamma_n = \{g = e^h; h \text{ is a real odd polynomial of degree } \leq n\}$$

and C be a simple smooth closed curve in $\mathbb{C} \cup \{\infty\}$ defined by

$$C = \{\pm\omega(y) + iy; y \in \mathbb{R}\}$$

with a smooth positive function ω on \mathbb{R} satisfying $\omega(y) = \omega(-y)$, hence C satisfies

$$C = -C, \quad C = \overline{C}.$$

D_{\pm} are the interior and exterior domains separated by the curve C defined by

$$D_+ = \{z \in \mathbb{C}; |\operatorname{Re} z| < \omega(\operatorname{Im} z)\}, \quad D_- = \{z \in \mathbb{C}; |\operatorname{Re} z| > \omega(\operatorname{Im} z)\}.$$

The curve C is chosen so that $g \in \Gamma_n$ remains bounded on D_+ , or more concretely

$$\omega(y) = O\left(y^{-(n-1)}\right) \text{ as } |y| \rightarrow \infty.$$

The **Hardy spaces** associated with curve C is defined by

$$\begin{cases} H(D_+) = \text{the closure in } L^2(C) \text{ of rational functions with no poles in } D_+ \\ H(D_-) = \text{the closure in } L^2(C) \text{ of rational functions with no poles in } D_- \end{cases}.$$

It is known that

$$L^2(C) = H(D_+) \oplus H(D_-) \quad (\text{not necessarily orthogonal}),$$

and elements of $H(D_\pm)$ can be extended as analytic functions on D_\pm respectively. The projections to $H(D_\pm)$ are given by

$$\begin{cases} \mathbf{p}_+ u(z) = \frac{1}{2\pi i} \int_C \frac{u(\lambda)}{\lambda - z} d\lambda & \text{for } z \in D_+ \\ \mathbf{p}_- u(z) = \frac{1}{2\pi i} \int_C \frac{u(\lambda)}{z - \lambda} d\lambda & \text{for } z \in D_- \end{cases} \quad \text{if } u \in L^2(C).$$

We enlarge the space $H(D_+)$ to admit polynomials. Namely for $N \in \mathbb{Z}_+$ set

$$H_N(D_+) = (z - b)^N H(D_+)$$

with $b \in D_-$, and define a norm in $H_N(D_+)$ by

$$\|u\|_N = \sqrt{\int_C |u(\lambda)|^2 |\lambda|^{-2N} |d\lambda|}.$$

Clearly $H_N(D_+)$ does not depend on the choice of b , and $z^m \in H_N(D_+)$ if $m \leq N - 1$.

In the previous paper we constructed the KdV flow as an action of Γ_n on a Grassmann manifold consisting of z^2 -invariant subspaces of $L^2(|z| = r)$. In the present case we construct an extension of the flow not on a Grassmann manifold of subspaces of $z^N L^2(C)$ but on a space of vector functions $\mathbf{a}(\lambda) = (a_1(\lambda), a_2(\lambda))$ on C . An analogue of a z^2 -invariant subspace is

$$W_{\mathbf{a}} = \{\mathbf{a}(\lambda) u(\lambda); u \in H_N(D_+)\},$$

where

$$\begin{cases} u_e(\lambda) = \frac{1}{2}(u(\lambda) + u(-\lambda)), \quad u_o(\lambda) = \frac{1}{2}(u(\lambda) - u(-\lambda)) \\ \mathbf{a}(\lambda) u(\lambda) = a_1(\lambda) u_e(\lambda) + a_2(\lambda) u_o(\lambda) \end{cases}.$$

In the present paper, however, spaces $W_{\mathbf{a}}$ will not appear explicitly.

For $L \in \mathbb{Z}_+$ a space of symbols of Toeplitz operators is introduced:

$$A_L(C) = \left\{ \begin{array}{l} a; a(\lambda) \text{ is bounded on } C \text{ and there exists a bounded analytic} \\ \text{function } f \text{ on } D_+ \text{ such that } \lambda^L (a(\lambda) - f(\lambda)) \text{ is bounded on } C \end{array} \right\}.$$

The number L is related to the degree of differentiability of the flow. The **Toeplitz operator** with symbol a is defined by

$$(T(a)u)(z) = f(z)u(z) + (\mathfrak{p}_+(a-f)u)(z)$$

for $u \in H_N(D_+)$, which is possible if $L \geq N$. This $T(a)$ does not depend on the choice of f and defines a bounded operator on $H_N(D_+)$. We have to treat vector symbols $\mathbf{a}(\lambda)$ and the vector version $\mathbf{A}_L(C)$ of $A_L(C)$ essentially due to the fact that the underlying Schrödinger operators are second order. The associated Toeplitz operator is defined by

$$(T(\mathbf{a})u)(z) = (T(a_1)u_e)(z) + (T(a_2)u_o)(z).$$

Let

$$\mathbf{A}_L^{inv}(C) = \{\mathbf{a} \in \mathbf{A}_L(C); T(\mathbf{a}) \text{ is invertible on } H_L(D_+)\}.$$

Since $1 \in H_1(D_+)$, $z \in H_2(D_+)$, one can define

$$u = T(\mathbf{a})^{-1}1 \in H_1(D_+), \quad v = T(\mathbf{a})^{-1}z \in H_2(D_+),$$

if $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ and $L \geq 2$. Set

$$\varphi_{\mathbf{a}}(z) = \mathbf{a}(z)u(z) - 1, \quad \psi_{\mathbf{a}}(z) = \mathbf{a}(z)v(z) - z \in H(D_-).$$

Then there exist a constant $\kappa_1(\mathbf{a})$ and $\phi_{\mathbf{a}} \in H(D_-)$ such that

$$\varphi_{\mathbf{a}}(z) = \kappa_1(\mathbf{a})z^{-1} + z^{-1}\phi_{\mathbf{a}}(z).$$

We call the functions $\{\varphi_{\mathbf{a}}, \psi_{\mathbf{a}}\}$ as **characteristic functions** for $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$, since \mathbf{a} is uniquely determined by them. Define

$$m_{\mathbf{a}}(z) = \frac{z + \psi_{\mathbf{a}}(z)}{1 + \varphi_{\mathbf{a}}(z)} + \kappa_1(\mathbf{a}) \quad (= z + o(1)).$$

Γ_n naturally acts on $\mathbf{A}_L(C)$, but not always on $\mathbf{A}_L^{inv}(C)$. Schrödinger operators and the KdV equation are obtained by applying the group Γ_n to $\mathbf{A}_L^{inv}(C)$. Let $e_x(z) = e^{xz} \in \Gamma_1$ and suppose $e_x\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ for any $x \in \mathbb{R}$. Then

$$f_{\mathbf{a}}(x, z) = e^{-xz}(1 + \varphi_{e_x\mathbf{a}}(z))$$

satisfies a Schrödinger equation

$$-\partial_x^2 f_{\mathbf{a}}(x, z) + q(x)f_{\mathbf{a}}(x, z) = -z^2 f_{\mathbf{a}}(x, z)$$

with $q(x) = -2\partial_x \kappa_1(e_x\mathbf{a})$. One can recover $m_{\mathbf{a}}(z)$ by

$$m_{\mathbf{a}}(z) = -\frac{\partial_x f_{\mathbf{a}}(x, z)|_{x=0}}{f_{\mathbf{a}}(0, z)}.$$

A solution to the KdV equation is obtained by another family of functions $e_{t,x}(z) = e^{xz+tz^3}$ of Γ_3 , namely

$$q(t, x) = -2\partial_x \kappa_1(e_{t,x}\mathbf{a})$$

satisfies

$$\partial_t q(t, x) = \frac{1}{4} \partial_x^3 q(t, x) - \frac{3}{2} q(t, x) \partial_x q(t, x) \quad (\text{KdV equation}). \quad (1)$$

Solutions to the higher order KdV equations can be obtained similarly. This is the core of Sato's theory.

The basic quantity $m_{\mathbf{a}}$ is closely related to the Weyl functions of Schrödinger operators. If q takes real values, one can associate a Schrödinger operator

$$L_q = -\partial_x^2 + q$$

with potential q . Throughout the paper we assume

$$(L_q u, u)_{L^2(\mathbb{R})} \geq \lambda_0 (u, u)_{L^2(\mathbb{R})} \quad \text{for any } u \in C_0^\infty(\mathbb{R}) \quad (2)$$

with some $\lambda_0 < 0$. Under this condition it is known that L_q has a unique self-adjoint extension, and there exist non-trivial functions $f_{\pm}(x, z) \in L^2(\mathbb{R}_{\pm})$ satisfying

$$-\partial_x^2 f_{\pm} + q f_{\pm} = -z^2 f_{\pm}.$$

These functions are unique up to constant multiple. The Weyl functions m_{\pm} are defined by

$$m_+(z) = \frac{\partial_x f_+(x, z)|_{x=0}}{f_+(0, z)}, \quad m_-(z) = -\frac{\partial_x f_-(x, z)|_{x=0}}{f_-(0, z)}.$$

m_{\pm} are analytic functions on $\mathbb{C} \setminus [\lambda_0, \infty)$ and satisfy

$$\frac{\text{Im } m_{\pm}(z)}{\text{Im } z} > 0.$$

Such an analytic function on \mathbb{C}_+ is called a **Herglotz function**. The functions m_{\pm} contain every information of the spectral properties of L_q . The simplest one is the coincidence of the $\text{sp}L_q$ with the domain of analyticity of m_{\pm} , hence m_{\pm} are analytic on $\mathbb{C} \setminus [\lambda_0, \infty)$ and the interior domain D_+ for the curve C is supposed to contain the interval $[-\mu_0, \mu_0]$ with $\mu_0 = \sqrt{-\lambda_0}$. One thing which should be stressed here is that m_{\pm} can be defined for any potential q regardless of decaying or oscillating. Moreover, since $f_{\mathbf{a}}(x, z) \in L^2(\mathbb{R}_{\pm})$ holds depending on $\text{Re } z \gtrless 0$ under a certain condition on \mathbf{a} , one see that $m_{\mathbf{a}}$ coincides with the Weyl functions m_{\pm} , that is,

$$m_{\mathbf{a}}(z) = \begin{cases} -m_+(-z^2) & \text{if } \text{Re } z > 0 \\ m_-(-z^2) & \text{if } \text{Re } z < 0 \end{cases}. \quad (3)$$

Hence in this case q is determined by $m_{\mathbf{a}}$ owing to the inverse spectral theory. We call $m_{\mathbf{a}}$ as **m -function** of \mathbf{a} , which will be the fundamental object in this paper, and call $f_{\mathbf{a}}(x, z)$ as **Baker-Akhiezer function** for L_q . For potentials q decaying sufficiently fast $f_{\mathbf{a}}(x, z)$ coincides with the Jost solution. It should be mentioned that R. Johnson [9] was the first who introduced the Weyl functions to Sato's theory.

As we have observed above the invertibility of $T(g_{\mathbf{a}})$ is crucial, which is verified with the aid of **tau-functions** in this paper. The tau-function was first

introduced by Hirota and its mathematical meaning was found by Sato. In our context it is defined as the Fredholm determinant of the operator

$$g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} : H_N(D_+) \rightarrow H_N(D_+),$$

that is

$$\tau_{\mathbf{a}}(g) = \det \left(g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} \right).$$

However to avoid a technical difficulty one version

$$\tau_{\mathbf{a}}^{(2)}(g) = \det_2 \left(g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} \right)$$

is employed, whose definition is possible when the operator $g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} - I$ is of Hilbert-Schmidt. The invertibility of $T(g\mathbf{a})$ is equivalent to $\tau_{\mathbf{a}}^{(2)}(g) \neq 0$. Any $g \in \Gamma_n$ can be approximated by rational functions r with the same number of zeros and poles in D_- . For such an r the image of $r^{-1}T(r\mathbf{a})T(\mathbf{a})^{-1}$ is finite dimensional and $m_{r\mathbf{a}}, \tau_{\mathbf{a}}(r)$ are computable by $\{\varphi_{\mathbf{a}}, m_{\mathbf{a}}\}$. Another key observation is

$$\begin{aligned} \tau_{\mathbf{a}}^{(2)}(g) \neq 0 \text{ for any } g \in \Gamma_n &\iff \tau_{\mathbf{a}}(r) \geq 0 \text{ for any real rational functions} \\ &\iff \frac{\operatorname{Im} m_{\mathbf{a}}(z)}{\operatorname{Im} z} > 0 \end{aligned}$$

if $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ satisfies $\mathbf{a}(\lambda) = \overline{\mathbf{a}(\bar{\lambda})}$ on C , which yields $g\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ for such an $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$. Keeping these facts in mind we define

$$\mathbf{A}_{L,+}^{inv}(C) = \left\{ \begin{array}{l} \mathbf{a} \in \mathbf{A}_L^{inv}(C); \mathbf{a}(\lambda) = \overline{\mathbf{a}(\bar{\lambda})} \text{ on } C, \tau_{\mathbf{a}}(r) \geq 0 \text{ for real rational} \\ \text{function } r \text{ with the same number of zeros and poles in } D_- \end{array} \right\}.$$

One can obtain concrete elements of $\mathbf{A}_{L,+}^{inv}(C)$ by defining \mathbf{a} directly from m_{\pm} . For a given potential q assume (2) and define m by (3). Then m is analytic on $\mathbb{C} \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R})$ ($\mu_0 = \sqrt{-\lambda_0}$) and satisfies

$$(M.1) \quad m(z) = \overline{m(\bar{z})} \text{ and}$$

$$\left\{ \begin{array}{ll} \frac{\operatorname{Im} m(z)}{\operatorname{Im} z} > 0 & \text{on } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \\ \frac{m(x) - m(-x)}{x} > 0 & \text{if } x \in \mathbb{R} \text{ and } |x| > \mu_0 \end{array} \right.$$

Assume further

$$(M.2) \quad m \text{ has an asymptotic behavior:}$$

$$m(z) = z + \sum_{1 \leq k \leq L-2} m_k z^{-k} + O(z^{-L+1}) \text{ on } D_-.$$

Then one has

Theorem 1 *If m satisfies (M.1), (M.2) for $L \geq 2$, then $\mathbf{m}(z) \equiv (1, m(z)/z) \in \mathbf{A}_{L,+}^{inv}(C)$ and the m -function $m_{\mathbf{m}}$ for \mathbf{m} is m .*

If $q \in C^{L-2}(-\delta, \delta)$, then it is known that the asymptotics of (M.2) holds in a sector

$$|\arg z| < \frac{\pi}{2} - \epsilon, \quad |\pi - \arg z| < \frac{\pi}{2} - \epsilon.$$

However the domain D_- is wider even for $n = 1$, and its boundary approaches to the axis $i\mathbb{R}$ if $n \geq 3$, therefore it is not trivial to find q satisfying (M.2) in D_- . Later in Theorems 3, 4 m associated with the Weyl functions m_{\pm} will be shown to fulfill (M.2) if q decays sufficiently fast or oscillates suitably.

Set

$$\mathcal{Q}_L(C) = \{q(x) = -2\partial_x \kappa_1(e_x \mathbf{a}); \mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)\}.$$

Then $m_{\mathbf{a}}$ is identified with m_{\pm} of q by (3) for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$, hence the inverse spectral theory show that $m_{\mathbf{a}}$ determines q . This makes it possible to define

$$(K(g)q)(x) = -2\partial_x \kappa_1(e_x g \mathbf{a}) \text{ with } q(x) = -2\partial_x \kappa_1(e_x \mathbf{a})$$

for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$, $g \in \Gamma_n$. One has

Theorem 2 *Suppose $L \geq \max\{n+1, 3\}$. Then $\{K(g)\}_{g \in \Gamma_n}$ defines a flow on $\mathcal{Q}_L(C)$. For a real odd polynomial h of degree n the function $(K(e^{th})q)(x)$ is C^1 in t and C^n in x and satisfies the $(n+1)/2$ th KdV equation. Especially $K(e^{tz^3})q(x)$ satisfies the KdV equation*

$$\partial_t q(t, x) = \frac{1}{4} \partial_x^3 q(t, x) - \frac{3}{2} q(t, x) \partial_x q(t, x)$$

if $q \in \mathcal{Q}_L(C)$ for $L \geq 4$.

We summarize the procedure to obtain $K(g)q$ for a given q as follows. Define m by (3) and assume the condition (M.2) for m . Solve the equation in $H_N(D_+)$ for $z \in D_+$

$$1 = e^{xz} g(z) \mathbf{f}(z) u(x, z) + \frac{1}{2\pi i} \int_C \frac{e^{x\lambda} g(\lambda) (\mathbf{m}(\lambda) - \mathbf{f}(\lambda)) u(x, \lambda)}{\lambda - z} d\lambda,$$

where $\mathbf{m}(z) = (1, m(z)/z)$ and $\mathbf{f}(z) = (1, f(z))$ with a bounded analytic function f on D_+ such that

$$m(z) - zf(z) = O(z^{-L+1}) \text{ on } C,$$

which is possible due to (M.2). The Baker-Akhiezer function is obtained by $f_{g\mathbf{m}}(x, z) = g(z) \mathbf{m}(z) u(x, z)$, and $\kappa_1(e_x g \mathbf{m})$ is determined by

$$e^{xz} f_{g\mathbf{m}}(x, z) = 1 + \kappa_1(e_x g \mathbf{m}) z^{-1} + o(z^{-1}).$$

Then we have $(K(g)q)(x) = -2\partial_x \kappa_1(e_x g \mathbf{m})$. Especially if $g = 1$, one can recover q from the Weyl functions m_{\pm} , which yields another way of the inverse spectral problem.

Any concrete example of initial data for the KdV flow is provided by Theorem 1. For a given m we have to verify the condition (M.2). There are two classes of potentials satisfying (M.2).

If $q^{(j)} \in L^1(\mathbb{R})$ for $j = 0, 1, \dots, L-2$, then (M.2) is valid in $\overline{\mathbb{C}}_+ \equiv \{z \in \mathbb{C}; \text{Im } z \geq 0\}$ for L (see [15]), which will be shown in Proposition 32.

The extended notion of reflection coefficients is defined by

$$R(z, q) = \frac{m_+(z) + \overline{m_-(z)}}{m_+(z) + m_-(z)}$$

where m_{\pm} are the Weyl functions for q . The modulus $|R(z, q)|$ coincides with that of the conventional reflection coefficient on \mathbb{R} if q decays sufficiently fast.

Theorem 3 *If $R(z, q)$ satisfies*

$$\int_0^{\infty} \lambda^M |R(\lambda, q)| d\lambda < \infty, \quad (4)$$

then $q \in \mathcal{Q}_L(C)$ with $L = M + 2 - (n + 1)/2$ holds.

If $R(\lambda, q) = 0$ for a.e. $\lambda \geq \lambda_1$ for some $\lambda_1 \leq \mathbb{R}$ (which means q is reflectionless on (λ_1, ∞)), the condition (4) is satisfied for any $M \geq 1$. This case was already treated in [12]. The resulting potential q is known to be meromorphic on \mathbb{C} and uniformly bounded on \mathbb{R} including all its derivatives.

Since $|R(\lambda, q)|$ is invariant under the flow $K(g)$, that is

$$|R(\lambda, q)| = |R(\lambda, K(g)q)| \quad \text{for a.e. } \lambda \in \mathbb{R}, \quad (5)$$

the condition (4) is supposed to play a significant role to investigate the flow $K(g)$ in future. (5) will be shown in a separate paper by using transfer matrices of $K(g)$.

On the other hand [10] showed for ergodic potential $q_{\omega}(x)$

$$\Sigma_{ac}(q_{\omega}) = \{\lambda \in \mathbb{R}; |R(\lambda, q_{\omega})| = 0\}.$$

Therefore in this case (4) is equivalent to

$$\int_{\mathbb{R}_+ \setminus \Sigma_{ac}(q_{\omega})} \lambda^M d\lambda < \infty. \quad (6)$$

In particular for periodic potentials $\Sigma_{ac}(q_{\omega}) = \Sigma(q_{\omega})$ (the spectrum of L_q) is valid, hence (6) means that the total length of spectral gaps is small, which can be estimated by the norms of derivatives of q .

For ergodic potentials the condition (6) requires the existence of rich ac spectrum, although it admits singular spectrum. This situation can be improved by replacing (4) by a similar condition on the curve $\widehat{C} = \{-z^2; z \in C, \operatorname{Re} z > 0\}$, which enables us to have

Theorem 4 *Let $\{q_{\omega}(x) = q(\theta_x \omega)\}$ be an ergodic process on $(\Omega, \mathcal{F}, P, \{\theta_x\}_{x \in \mathbb{R}})$. Suppose $q_{\omega} \in C_b^m(\mathbb{R})$. Then, $q_{\omega} \in \mathcal{Q}_L(C)$ holds for a.e. $\omega \in \Omega$ for $L \leq (m - 3(n - 1))/6$. In this case $(K(g)q_{\omega})(x) = f_g(\theta_x \omega)$ for $g \in \Gamma_n$ is valid with $f_g(\omega) = (K(g)q_{\omega})(0)$.*

Any almost periodic potentials can be considered as ergodic potentials and one can apply Theorem 3 or Theorem 4 to have solutions starting from almost periodic functions. Under the condition of Theorem 4 one has $q_{\omega} \in \mathcal{Q}_L(C)$ for every ω not for a.e. ω .

Rybkin [17] obtained solutions to the KdV equation with step like initial data, which is decaying on \mathbb{R}_+ and arbitrary on \mathbb{R}_- . He employed Hirota's tau-function and the Hankel transform on \mathbb{R} , which restricts the class of initial data to step like functions. In our approach the decaying condition on \mathbb{R}_+ can be removed, since we represent the solutions through information of the Weyl functions m_{\pm} on \mathbb{C}_+ not on \mathbb{R} . Our framework also admits step like initial data. For instance if $q \in C^{\infty}(\mathbb{R})$ is almost periodic on one axis \mathbb{R}_+ or \mathbb{R}_- and decaying on the opposite axis, since such a potential can be easily verified to satisfy (M.2).

For almost periodic initial data there are several papers. Egorova [4] treated limit periodic initial data. Damanik-Goldstein [3] and Eichinger-VandenBoom-Yuditskii [5] considered almost periodic potentials. Their approaches are different from ours and the associated Schrödinger operators must have only ac spectrum. Tsugawa [21] obtained solutions starting from quasi-periodic initial data without assuming pure ac spectrum, but he could not show the existence of global solutions in time.

One of the advantages of Sato's approach lies on the algebraic nature of the group Γ_n acting on the space of symbols. Especially the factor $q_{\zeta}(z) = (1 - \zeta^{-1}z)^{-1}$ plays a role of primes in number theory, which will be frequently used in the present paper.

Throughout the paper the following notations will be employed:

$$\left\{ \begin{array}{l} \mathbb{R} = \text{the set of all real numbers} \\ \mathbb{C} = \text{the set of all complex numbers} \\ \mathbb{Z} = \text{the set of all integers} \\ \mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}, \quad \mathbb{R}_- = \{x \in \mathbb{R}, x \leq 0\} \\ \mathbb{C}_+ = \{z \in \mathbb{C}, \text{Im } z > 0\}, \quad \mathbb{C}_- = \{z \in \mathbb{C}, \text{Im } z < 0\} \\ \mathbb{Z}_+ = \{n \in \mathbb{Z}, n \geq 0\}, \quad \mathbb{Z}_- = \{n \in \mathbb{Z}, n \leq 0\} \\ \bar{z} \text{ denotes the complex conjugate of } z: x + iy = x - iy \end{array} \right. .$$

2 Hardy spaces and Toeplitz operators

In the previous paper [11] we employed Segal-Wilson's version of Sato's theory, in which they constructed KdV flow on a Grassmann manifold in $H \equiv L^2(|z| = r)$. In his theory the Fourier space H is used as symbols space of pseudo-differential operators and the separation

$$H = \left(L^2\text{-closure of } \{z^k\}_{k \geq 0} \right) \oplus \left(L^2\text{-closure of } \{z^k\}_{k < 0} \right) \equiv H_+ \oplus H_-$$

is essential since the H_+ component exhibits the part of differential operators and he had to take out differential operators parts from pseudo-differential operators. Therefore the projection \mathbf{p}_+ to the Hardy space H_+ plays an essential role.

Since in this framework only a special class of solutions meromorphic on \mathbb{C} is possible to treat, the circle $|z| = r$ should be replaced by a certain unbounded curve to have more general solutions.

2.1 Hardy spaces and projections

Let C be a simple closed smooth curve passing ∞ in the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and oriented anti-clockwisely. We assume $C = -C$. The curve C separates \mathbb{C}_∞ into two domains D_\pm , where D_+ contains the origin 0. The situation was illustrated in (f1). The curve C is chosen so that $e^{h(z)}$ remains bounded on C , where $h(z)$ is a given real polynomial of odd degree.

Set

$$H(D_\pm) = L^2(C)\text{-closure of } \{\text{rational functions with no poles in } D_\pm\}.$$

Then

$$L^2(C) = H(D_+) \oplus H(D_-)$$

holds. For $f \in L^2(C)$ define

$$\mathfrak{p}_\pm f(z) = \pm \frac{1}{2\pi i} \int_C \frac{f(\lambda)}{\lambda - z} d\lambda \quad \text{for } z \in D_\pm,$$

and

$$(\Theta f)(z) = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{C \cap \{|\lambda - z| > \epsilon\}} \frac{f(\lambda)}{\lambda - z} d\lambda \quad \text{for } z \in C.$$

It is known that Θ is a bounded operator on $L^2(C)$ (see [1] and [2]) and \mathfrak{p}_\pm have a finite limit a.e. when z approaches to an element of C . They satisfy

$$\begin{cases} \mathfrak{p}_+ f(z) = f(z)/2 + (\Theta f)(z) \\ \mathfrak{p}_- f(z) = f(z)/2 - (\Theta f)(z) \end{cases} \quad \text{for } z \in C,$$

and \mathfrak{p}_\pm are projections from $L^2(C)$ onto $H(D_\pm)$ respectively. It should be noted that \mathfrak{p}_\pm are generally not orthogonal projections. If D_+ is a disc, $H(D_+)$ coincides with the conventional Hardy space.

To treat an analogue of z^2 -action on $L^2(C)$ for a function u on C we define the even part and the odd part of u by

$$\begin{cases} u_e(z) = \frac{1}{2}(u(z) + u(-z)) \\ u_o(z) = \frac{1}{2}(u(z) - u(-z)) \end{cases}.$$

The numbers ± 1 in front of z come from the solutions to $\omega^2 = 1$. Any function on C can be represented as $u = u_e + u_o$ and this yields an orthogonal decomposition in $L^2(C)$. It should be noted also that

$$\mathfrak{p}_+ : L_e^2(C) \rightarrow H(D_+) \cap L_e^2(C) \quad \text{and} \quad \mathfrak{p}_+ : L_o^2(C) \rightarrow H(D_+) \cap L_o^2(C), \quad (7)$$

where $L_e^2(C)$, $L_o^2(C)$ denote the even part and the odd part respectively.

2.2 Toeplitz operators

In the previous paper [11] we considered z^2 -invariant subspaces of $L^2(C)$ when C is a disc with center 0. If the curve C is unbounded, in place of subspaces of $L^2(C)$ we consider a family of bounded vector functions $\mathbf{a}(z) = (a_1(z), a_2(z))$ on C . The subspace corresponding to $\mathbf{a}(z)$ is

$$W_{\mathbf{a}} \equiv \{\mathbf{a}(z)u(z); u \in H(D_+)\} \subset L^2(C),$$

where

$$\mathbf{a}(z)u(z) \equiv a_1(z)u_e(z) + a_2(z)u_o(z). \quad (8)$$

This space will not appear explicitly in the sequel, but the Toeplitz operator with symbol $\mathbf{a}(z)$ plays an essential role.

Set

$$A(C) = \left\{ a(\lambda); \sup_{\lambda \in C} |a(\lambda)| < \infty \right\},$$

and

$$(T(a)u)(z) = \mathbf{p}_+(au)(z) \quad \text{for } a \in A(C).$$

Then $T(a)$ defines a bounded operator on $L^2(C)$ and is called a Toeplitz operator with symbol a .

To investigate the differentiability of solutions to the KdV equation we have to admit the multiplication operation by rational functions on $H(D_\pm)$. To realize such operations some modification of the spaces $H(D_\pm)$ is necessary. For an $N \in \mathbb{Z}_+$ and $b \in D_-$ set

$$H_N(D_+) = (z-b)^N H(D_+). \quad (9)$$

Clearly $H_N(D_+)$ does not depend on b , and

$$z^k \in H_N(D_+) \quad \text{for any integer } k \leq N-1.$$

For $u \in z^N L^2(C)$ we extend the definition of the projections by

$$\begin{cases} (\mathbf{p}_+u)(z) = \lim_{D_- \ni b', \operatorname{Re} b' \rightarrow \infty} \frac{(z-b')^N}{2\pi i} \int_C \frac{u(\lambda)}{\lambda-z} (\lambda-b')^{-N} d\lambda \quad \text{for } z \in D_+ \\ (\mathbf{p}_-u)(z) = \lim_{D_- \ni b', \operatorname{Re} b' \rightarrow \infty} \frac{(z-b')^N}{2\pi i} \int_C \frac{u(\lambda)}{z-\lambda} (\lambda-b')^{-N} d\lambda \quad \text{for } z \in D_- \end{cases}, \quad (10)$$

if they exist finitely. It should be noted that if they exist for an $N \geq 0$, then they exist also for any $N' \geq N$ and take the same values.

The extended \mathbf{p}_\pm are well-defined for a certain au .

Lemma 5 For $N \in \mathbb{Z}_+$ if $a \in A(C)$ satisfies

$$z^N (a(z) - f(z)) \text{ is bounded on } C \quad (11)$$

with a bounded analytic function f on D_+ , then, for $u \in H_N(D_+)$

$$\begin{cases} \mathbf{p}_+(au)(z) = f(z)u(z) + \frac{1}{2\pi i} \int_C \frac{(a(\lambda) - f(\lambda))u(\lambda)}{\lambda-z} d\lambda \in H_N(D_+) \\ \mathbf{p}_-(au)(z) = \frac{1}{2\pi i} \int_C \frac{(a(\lambda) - f(\lambda))u(\lambda)}{z-\lambda} d\lambda \in H(D_-) \end{cases} \quad (12)$$

hold. In particular for $u \in H(D_+)$ we have $\mathbf{p}_\pm(au) \in H(D_\pm)$ respectively, and they satisfy

$$\begin{cases} \mathbf{p}_+u = u & \text{if } u \in H_N(D_+) \\ \mathbf{p}_+u = 0 & \text{if } u \in H(D_-) \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{p}_-u = 0 & \text{if } u \in H_N(D_+) \\ \mathbf{p}_-u = u & \text{if } u \in H(D_-) \end{cases},$$

which implies $H_N(D_+) \cap H(D_-) = \{0\}$.

Proof. If $u \in H_N(D_+)$ and $b' \in D_-$,

$$\begin{aligned} & \frac{(z-b')^N}{2\pi i} \int_C \frac{a(\lambda)u(\lambda)}{\lambda-z} (\lambda-b')^{-N} d\lambda \\ &= \frac{1}{2\pi i} \int_C \frac{(a(\lambda)-f(\lambda))u(\lambda)}{\lambda-z} \left(\frac{z-b'}{\lambda-b'}\right)^N d\lambda + f(z)u(z) \end{aligned}$$

holds for $z \in D_+$ due to $fu(z-b')^{-N} \in H(D_+)$. Since $(a-f)u \in L^2(C)$

$$\lim_{b' \rightarrow \infty} \int_C \frac{(a(\lambda)-f(\lambda))u(\lambda)}{\lambda-z} \left(\frac{z-b'}{\lambda-b'}\right)^N d\lambda = \int_C \frac{(a(\lambda)-f(\lambda))u(\lambda)}{\lambda-z} d\lambda$$

is valid, which shows

$$\mathfrak{p}_+(au)(z) = f(z)u(z) + \frac{1}{2\pi i} \int_C \frac{(a(\lambda)-f(\lambda))u(\lambda)}{\lambda-z} d\lambda \in H_N(D_+).$$

On the other hand, due to $fu(z-b')^{-N} \in H(D_+)$

$$(z-b')^N \int_C \frac{a(\lambda)u(\lambda)}{z-\lambda} (\lambda-b')^{-N} d\lambda = \int_C \frac{(a(\lambda)-f(\lambda))u(\lambda)}{z-\lambda} \left(\frac{z-b'}{\lambda-b'}\right)^N d\lambda$$

holds for $z \in D_-$, and $(a-f)u \in L^2(C)$ implies

$$\begin{aligned} \mathfrak{p}_-(au)(z) &= \lim_{b' \rightarrow \infty} \frac{(z-b')^N}{2\pi i} \int_C \frac{a(\lambda)u(\lambda)}{z-\lambda} (\lambda-b')^{-N} d\lambda \\ &= \frac{1}{2\pi i} \int_C \frac{(a(\lambda)-f(\lambda))u(\lambda)}{z-\lambda} d\lambda, \end{aligned}$$

which shows (12). If $u \in H(D_-)$, then due to $au \in L^2(C)$ we easily have

$$\lim_{b' \rightarrow \infty} (z-b')^N \int_C \frac{a(\lambda)u(\lambda)}{\lambda-z} (\lambda-b')^{-N} d\lambda = \int_C \frac{a(\lambda)u(\lambda)}{\lambda-z} d\lambda$$

for $z \notin C$. Therefore, $\mathfrak{p}_\pm(au) \in H(D_\pm)$ respectively. The rest of the proof is clear. ■

Consequently the projections \mathfrak{p}_\pm can be extended to

$$L_N^2(C) \equiv H_N(D_+) \oplus H(D_-) \quad \left(\subset |\lambda|^N L^2(C) \right) \quad (13)$$

by (13) as projections. The norm in $L_N^2(C)$ is defined by

$$\|u\|_N^2 = \int_C \left| (\lambda-b)^{-N} \mathfrak{p}_+u(\lambda) \right|^2 |d\lambda| + \int_C |\mathfrak{p}_-u(\lambda)|^2 |d\lambda|.$$

Moreover this lemma enables us to extend the Toeplitz operator $T(a)$ as a bounded operator on $H_N(D_+)$ for a bounded function a satisfying (11). Subsequently a subset $A_L(C)$ of $A(C)$ for $L \in \mathbb{Z}_+$ is introduced as follows:

$$A_L(C) = \left\{ a \in A(C); \text{ there exists an analytic function } f \text{ on } D_+ \text{ such that } \sup_{z \in D_+} |f(z)| < \infty, \sup_{\lambda \in C} |\lambda^L (a(\lambda) - f(\lambda))| < \infty \right\}. \quad (14)$$

Lemma 5 enables us to define the Toeplitz operator on $H_N(D_+)$ by

$$T_N(a)u = \mathbf{p}_+(au) \in H_N(D_+).$$

Let $L \geq N' \geq N$. Then $\{T_N(a)\}_{N \geq 0}$ has the property that if $a \in A_L(C)$, then $T_{N'}(a)|_{H_N(D_+)} = T_N(a)$. Therefore we use the notation

$$T(a) = T_N(a).$$

The vector version of $A_L(C)$ and $T(a)$ are defined by

$$\begin{cases} \mathbf{A}_L(C) = \{\mathbf{a} = (a_1, a_2); a_1, a_2 \in A_L(C)\} \\ (T(\mathbf{a})u)(z) = T(a_1)u_e(z) + T(a_2)u_o(z) \end{cases}. \quad (15)$$

Set

$$\mathbf{A}_L^{inv}(C) = \{\mathbf{a} \in \mathbf{A}_L(C); T(\mathbf{a}) \text{ is invertible on } H_L(D_+)\}. \quad (16)$$

It should be noted that $\mathbf{A}_L^{inv}(C) \supset \mathbf{A}_{L'}^{inv}(C)$ holds if $L' \geq L$.

2.3 Characteristic functions and m -functions for $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$

In this section we define several quantities which will be necessary later when $T(\mathbf{a})^{-1}$ exists.

For $\mathbf{a} \in \mathbf{A}_2^{inv}(C)$ one can define two functions of $H(D_-)$ which characterize $W_{\mathbf{a}}$ and are closely related to the tau-function introduced later. For $\mathbf{a} \in \mathbf{A}_2^{inv}(C)$ set

$$\begin{cases} u(z) = (T(\mathbf{a})^{-1}1)(z) \in H_1(D_+) \\ v(z) = (T(\mathbf{a})^{-1}z)(z) \in H_2(D_+) \end{cases}, \quad (17)$$

which is possible due to $1 \in H_1(D_+)$, $z \in H_2(D_+)$, and

$$\begin{cases} \varphi_{\mathbf{a}}(z) = \mathbf{p}_-(\mathbf{a}u)(z) = a_1(z)u_e(z) + a_2(z)u_o(z) - 1 \in H(D_-) \\ \psi_{\mathbf{a}}(z) = \mathbf{p}_-(\mathbf{a}v)(z) = a_1(z)v_e(z) + a_2(z)v_o(z) - z \in H(D_-) \\ \Delta_{\mathbf{a}}(z) = \frac{(1 + \varphi_{\mathbf{a}}(-z))(\psi_{\mathbf{a}}(z) + z) - (1 + \varphi_{\mathbf{a}}(z))(\psi_{\mathbf{a}}(-z) - z)}{2z} \end{cases}. \quad (18)$$

Lemma 6 *If $\mathbf{a} \in \mathbf{A}_2^{inv}(C)$, $\{\varphi_{\mathbf{a}}, \psi_{\mathbf{a}}\}$ satisfies the following properties.*

(i) $\Delta_{\mathbf{a}}(b) \neq 0$ on D_- and

$$T(\mathbf{a})^{-1} \frac{1}{z+b} = \frac{(\varphi_{\mathbf{a}}(b) + 1)v - (\psi_{\mathbf{a}}(b) + b)u}{\Delta_{\mathbf{a}}(b)(z^2 - b^2)}. \quad (19)$$

(ii) $\{\varphi_{\mathbf{a}}, \psi_{\mathbf{a}}\}$ determines \mathbf{a} .

(iii) There exist $\kappa_1(\mathbf{a}) \in \mathbb{C}$ and $\phi_{\mathbf{a}} \in H(D_-)$ such that

$$\varphi_{\mathbf{a}}(z) = \kappa_1(\mathbf{a})z^{-1} + \phi_{\mathbf{a}}(z)z^{-1}. \quad (20)$$

Proof. Here the suffix \mathbf{a} is omitted. (18) implies that for $b \in D_-$ we have decompositions

$$\begin{cases} \frac{a_1(z)u_e(z) + a_2(z)u_o(z)}{z^2 - b^2} \\ = \frac{1}{2b} \left(\frac{\varphi(b) + 1}{z - b} - \frac{\varphi(-b) + 1}{z + b} \right) + \frac{1}{2b} \left(\frac{\varphi(z) - \varphi(b)}{z - b} - \frac{\varphi(z) - \varphi(-b)}{z + b} \right) \\ \frac{a_1(z)v_e(z) + a_2(z)v_o(z)}{z^2 - b^2} \\ = \frac{1}{2b} \left(\frac{\psi(b) + b}{z - b} - \frac{\psi(-b) - b}{z + b} \right) + \frac{1}{2b} \left(\frac{\psi(z) - \psi(b)}{z - b} - \frac{\psi(z) - \psi(-b)}{z + b} \right) \end{cases}$$

into elements of $H_2(D_+)$ and $H(D_-)$, hence

$$\begin{cases} T(\mathbf{a}) \frac{u}{z^2 - b^2} = \frac{1}{2b} \left(\frac{\varphi(b) + 1}{z - b} - \frac{\varphi(-b) + 1}{z + b} \right) \\ T(\mathbf{a}) \frac{v}{z^2 - b^2} = \frac{1}{2b} \left(\frac{\psi(b) + b}{z - b} - \frac{\psi(-b) - b}{z + b} \right) \end{cases}, \quad (21)$$

which yields

$$\begin{cases} T(\mathbf{a}) \frac{(\varphi(b) + 1)v - (\psi(b) + b)u}{z^2 - b^2} = \frac{\Delta(b)}{z + b} \\ T(\mathbf{a}) \frac{(\varphi(-b) + 1)v - (\psi(-b) - b)u}{z^2 - b^2} = \frac{\Delta(b)}{z - b} \end{cases}. \quad (22)$$

If $\Delta(b) = 0$, then the invertibility of $T(\mathbf{a})$ implies

$$\begin{cases} (\psi(b) + b)u - (\varphi(b) + 1)v = 0 \\ (\psi(-b) - b)u - (\varphi(-b) + 1)v = 0 \end{cases}.$$

Applying $T(\mathbf{a})$ we have

$$\begin{cases} (\psi(b) + b) - (\varphi(b) + 1)z = 0 \\ (\psi(-b) - b) - (\varphi(-b) + 1)z = 0 \end{cases},$$

which means

$$\psi(b) + b = \varphi(b) + 1 = \psi(-b) - b = \varphi(-b) + 1 = 0,$$

hence (21) implies $u = v = 0$. This contradicts $T(\mathbf{a})u = 1$, $T(\mathbf{a})v = z$, and we have $\Delta(b) \neq 0$. (19) can be deduced from (22).

Then, (22) shows that for any rational function $f(z)$ with simple poles only in D_- and of order $O(z^{-1})$ there exist two even rational functions $r_1(z)$, $r_2(z)$ with the same property such that

$$T(\mathbf{a})(r_1u + r_2v)(z) = f(z)$$

holds, hence

$$r_1(z)u(z) + r_2(z)v(z) = \left(T(\mathbf{a})^{-1} f \right)(z).$$

Since such rational functions are dense in $H_2(D_+)$, approximating $T(\mathbf{a})1$, $T(\mathbf{a})z \in H_2(D_+)$ by rational functions f_n and the continuity of $T(\mathbf{a})^{-1}$ show that $T(\mathbf{a})^{-1}f_n$ converges to 1, z compact uniformly on D_+ . Let

$$\mathcal{Z} = \left\{ z \in D_+; \det \begin{pmatrix} u_e(z) & v_e(z) \\ u_o(z) & v_o(z) \end{pmatrix} = 0 \right\}.$$

Then, the linear independence of $\{u, v\}$ implies the discreteness of \mathcal{Z} , and hence for $z \in D_+ \setminus \mathcal{Z}$ the associated $r_{n,1}(z)$, $r_{n,2}(z)$ converge to $\{r_1^{(1)}(z), r_2^{(1)}(z)\}$ and $\{r_1^{(2)}(z), r_2^{(2)}(z)\}$ depending on $T(\mathbf{a})^{-1}f_n \rightarrow 1$ or $T(\mathbf{a})^{-1}f_n \rightarrow z$. Since $r_{n,1}$, $r_{n,2}$ have no poles on D_+ , their limits $r_j^{(i)}(z)$ have no singularity on D_+ either, hence

$$\begin{cases} r_1^{(1)}(z)u(z) + r_2^{(1)}(z)v(z) = 1 \\ r_1^{(2)}(z)u(z) + r_2^{(2)}(z)v(z) = z \end{cases} \quad (23)$$

holds for $z \in D_+$. Since $\{r_j^{(i)}\}$ are constructed by $\{\varphi, \psi\}$, one see that $\{r_j^{(i)}\}$ depend only on $\{\varphi, \psi\}$, hence so does $\{u, v\}$. On the other hand (18) shows that $a_1(z)$, $a_2(z)$ are recovered from $\{\varphi, \psi\}$. We remark that

$$\det \begin{pmatrix} u_e(z) & v_e(z) \\ u_o(z) & v_o(z) \end{pmatrix} \neq 0 \text{ on } D_+.$$

(iii) is proved by applying (47) of Lemma 5 to $\mathbf{a} \in \mathcal{A}_2^{inv}(C)$. Namely we have

$$\begin{aligned} \varphi(z) &= \mathbf{p}_- (a_1(z)u_e(z) + a_2(z)u_o(z)) \\ &= \frac{1}{2\pi i} \int_C \frac{(a_1(\lambda) - f_1(\lambda))u_e(\lambda) + (a_2(\lambda) - f_2(\lambda))u_o(\lambda)}{z - \lambda} d\lambda. \end{aligned}$$

Since $a_j(\lambda) - f_j(\lambda) = O(\lambda^{-2})$ for $j = 1, 2$ and $u_e, u_o \in H_1(D_+)$,

$$\lambda(a_1(\lambda) - f_1(\lambda))u_e(\lambda), \lambda(a_2(\lambda) - f_2(\lambda))u_o(\lambda) \in L^2(C)$$

hold, hence

$$\begin{aligned} &\frac{1}{2\pi i} \int_C \frac{(a_1(\lambda) - f_1(\lambda))u_e(\lambda) + (a_2(\lambda) - f_2(\lambda))u_o(\lambda)}{z - \lambda} d\lambda \\ &= \kappa_1(\mathbf{a})z^{-1} + \phi_{\mathbf{a}}(z)z^{-1} \end{aligned}$$

is valid with

$$\begin{cases} \kappa_1(\mathbf{a}) = \frac{1}{2\pi i} \int_C ((a_1(\lambda) - f_1(\lambda))u_e(\lambda) + (a_2(\lambda) - f_2(\lambda))u_o(\lambda)) d\lambda \\ \phi_{\mathbf{a}}(z) = \frac{1}{2\pi i} \int_C \frac{\lambda((a_1(\lambda) - f_1(\lambda))u_e(\lambda) + (a_2(\lambda) - f_2(\lambda))u_o(\lambda))}{z - \lambda} d\lambda \end{cases}.$$

■

Owing to this Lemma we call $\{\varphi_{\mathbf{a}}, \psi_{\mathbf{a}}\}$ as the **characteristic functions** of \mathbf{a} or $W_{\mathbf{a}}$. This Lemma also implies the possibility of a kind of Riemann-Hilbert

factorization of $(a_1(z), a_2(z))$, namely (18), (23) yield a representation

$$\begin{pmatrix} a_{1,e}(z) & a_{1,o}(z) \\ a_{2,o}(z) & a_{2,e}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}^{-1} \begin{pmatrix} r_1^{(1)}(z) & r_1^{(1)}(z) \\ r_2^{(2)}(z) & r_2^{(2)}(z) \end{pmatrix} \begin{pmatrix} 1 + \varphi_e(z) & \varphi_o(z) \\ \psi_e(z) & z + \psi_o(z) \end{pmatrix},$$

where the second term is analytic on D_+ and the third term is analytic on D_- . The possibility of this factorization is very close to a sufficient condition for the invertibility of $T(\mathbf{a})$.

The m -function for $W_{\mathbf{a}}$ is defined by

$$m_{\mathbf{a}}(z) = \frac{z + \psi_{\mathbf{a}}(z)}{1 + \varphi_{\mathbf{a}}(z)} + \kappa_1(\mathbf{a}). \quad (24)$$

$\kappa_1(\mathbf{a})$ is added so that we have an asymptotic behavior

$$m_{\mathbf{a}}(z) = z + o(1) \text{ as } z \rightarrow \infty \text{ in } |\arg(\pm z)| < \pi/2 - \epsilon \quad (25)$$

(i) of (6) implies $1 + \varphi_{\mathbf{a}}(z)$ is not identically 0, hence $m_{\mathbf{a}}$ is meromorphic on D_- . Later we will see that $m_{\mathbf{a}}$ determines the potential q and is equal to the Weyl function under a certain condition on \mathbf{a} .

A good subset of $\mathbf{A}_L^{inv}(C)$ is given by $\mathbf{M}_L(C)$:

$$\begin{aligned} & \mathbf{M}_L(C) \\ &= \left\{ \mathbf{m}(z) = (m_1(z), m_2(z)); \mathbf{m} \text{ is analytic on } \mathbb{C} \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R}) \right. \\ & \quad \left. \text{with } \mu_0 = \sqrt{-\lambda_0} \text{ and satisfies (i), (ii) below:} \right\} \quad (26) \end{aligned}$$

(i) $\mathbf{m}(z) = \mathbf{1} + \sum_{1 \leq k < L} \mathbf{m}_k z^{-k} + O(z^{-L})$ on D_- with $\mathbf{1} = (1, 1)$, $\mathbf{m}_k \in \mathbb{C}^2$.

(ii) $m_1(z)m_2(-z) + m_1(-z)m_2(z) \neq 0$ on $\mathbb{C} \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R})$

For $\mathbf{m}, \mathbf{n} \in \mathbf{M}_L(C)$ define new elements by

$$\begin{cases} (\mathbf{m} \cdot \mathbf{n})(z) = (m_1(z)n_{1,e}(z) + m_2(z)n_{1,o}(z), m_1(z)n_{2,o}(z) + m_2(z)n_{2,e}(z)) \\ \widehat{\mathbf{m}}(z) = \left(\frac{2(m_{2,e}(z) - m_{1,o}(z))}{m_1(z)m_2(-z) + m_1(-z)m_2(z)}, \frac{2(m_{1,e}(z) - m_{2,o}(z))}{m_1(z)m_2(-z) + m_1(-z)m_2(z)} \right). \end{cases}$$

We have

Lemma 7 $\mathbf{M}_L(C)$ satisfies the group property:

$$\begin{cases} \mathbf{m} \cdot \mathbf{n}, \widehat{\mathbf{m}} \in \mathbf{M}_L(C) \\ \mathbf{m} \cdot \widehat{\mathbf{m}} = \widehat{\mathbf{m}} \cdot \mathbf{m} = \mathbf{1} \end{cases}. \quad (27)$$

Moreover it holds that

$$T(\mathbf{m} \cdot \mathbf{n}) = T(\mathbf{m})T(\mathbf{n}) \quad (28)$$

for $\mathbf{m}, \mathbf{n} \in \mathbf{M}_L(C)$. Consequently $\mathbf{M}_L(C) \subset \mathbf{A}_L^{inv}(C)$ is valid.

Proof. First note $\mathbf{M}_L(C) \subset \mathbf{A}_L(C)$. This is because for any $b \in D_-$ there exist $\widetilde{\mathbf{m}}_k \in \mathbb{C}^2$ such that

$$\sum_{1 \leq k < L} \mathbf{m}_k z^{-k} = \sum_{1 \leq k < L} \widetilde{\mathbf{m}}_k (z - b)^{-k} + O(z^{-L}).$$

The group property (27) is clear. To show (28) note $mH(D_-) \subset H(D_-)$ for any bounded analytic function m on D_- . Therefore, if scalars m_1, m_2 satisfy the condition (i) of (26), then for $u \in H_N(D_+)$

$$\mathfrak{p}_+(m_1 m_2 u) = \mathfrak{p}_+(m_1 \mathfrak{p}_+ m_2 u) + \mathfrak{p}_+(m_1 \mathfrak{p}_- m_2 u) = \mathfrak{p}_+(m_1 \mathfrak{p}_+ m_2 u)$$

holds. With this property in mind we have

$$\begin{aligned} T(\mathbf{m} \cdot \mathbf{n}) u &= \mathfrak{p}_+((m_1 n_{1,e} + m_2 n_{1,o}) u_e + (m_1 n_{2,o} + m_2 n_{2,e}) u_o) \\ &= \mathfrak{p}_+(m_1 \mathfrak{p}_+(n_{1,e} u_e + n_{2,o} u_o)) + \mathfrak{p}_+(m_2 \mathfrak{p}_+(n_{1,o} u_e + n_{2,e} u_o)) \\ &= T(m_1)(T(\mathbf{n}) u)_e + T(m_2)(T(\mathbf{n}) u)_o \\ &= T(\mathbf{m}) T(\mathbf{n}) u, \end{aligned}$$

which shows (28). ■

If $\mathbf{a}(z) = \mathbf{m}(z) = (m_1(z), m_2(z)) \in \mathbf{M}_2(C)$, then due to $\mathbf{m}(z) = \mathbf{1} + m_1 z^{-1} + O(z^{-2})$ we have

$$T(\mathbf{m})\mathbf{1} = \mathfrak{p}_+ m_1(z) = 1, \quad T(\mathbf{m})z = \mathfrak{p}_+ m_2(z)z = z + m_{12}$$

with $\mathbf{m}_1 = (m_{11}, m_{12})$, hence $u(z) = 1, v(z) = z - m_{12}$. Therefore

$$\begin{cases} \varphi_{\mathbf{a}}(z) = m_1(z) - 1 \\ \psi_{\mathbf{a}}(z) = -m_{12}m_1(z) + zm_2(z) - z \end{cases}$$

follows, which yields

$$m_{\mathbf{a}}(z) = \frac{-m_{12}m_1(z) + zm_2(z)}{m_1(z)} + m_{11} = \frac{zm_2(z)}{m_1(z)} + m_{11} - m_{12}. \quad (29)$$

3 Group action on $\mathbf{A}_L^{inv}(C)$

The KdV flow is described by a group action on $\mathbf{A}_L^{inv}(C)$. For $m \in \mathbb{Z}_-$ and odd $n \in \mathbb{Z}_+$ let $\Gamma_n^{(m)}$ be

$$\begin{cases} \Gamma_n^{(m)} = \left\{ \begin{array}{l} g = re^h; r \text{ is a rational function of order } m \text{ which} \\ \text{do not have poles nor zeros on } [-\mu_0, \mu_0] \cup i\mathbb{R}, \\ \text{and } h \text{ is a real odd polynomials of degree } \leq n \end{array} \right\} \\ \Gamma_n = \{g = e^h; h \text{ is a real odd polynomials of degree } \leq n\} \subset \Gamma_n^{(0)} \end{cases}, \quad (30)$$

where $\mu_0 = \sqrt{-\lambda_0}$ and the order m of a rational function r is defined by

$$m = \deg p - \deg q \text{ when } r = p/q \text{ with polynomials } p, q.$$

When we consider $g = re^h \in \Gamma_n^{(m)}$, the curve C is taken so that e^h remains bounded on D_+ . Therefore, C is parametrized as

$$C = \left\{ \begin{array}{l} \pm \omega(y) + iy; y \in \mathbb{R}, \omega(y) > 0, \omega(y) = \omega(-y), \omega \text{ is} \\ \text{smooth and satisfies } \omega(y) = O(y^{-(n-1)}) \text{ as } y \rightarrow \infty \end{array} \right\}. \quad (31)$$

In most of the cases $\Gamma_n^{(0)}$ will be treated. However, in some important cases we would like to consider $q_\zeta g, q_{\zeta_1} q_{\zeta_2} g$ with $g \in \Gamma_n^{(0)}$ to make arguments transparent, hence the numbers for m which are frequently appears are 0, -1 , -2 . Note that any rational function r can be represented as a product of finite numbers of $q_\zeta(z) = (1 - \zeta^{-1}z)^{-1}, q_\zeta(z)^{-1}$.

For $\mathbf{a} \in \mathbf{A}_L(C), g \in \Gamma_n^{(m)}$ a natural product $g\mathbf{a}$ is bounded on C due to $m \leq 0$, hence $g\mathbf{a} \in \mathbf{A}_L(C)$. For $a \in A_L(C)$ and $g \in \Gamma_n^{(m)}$ when $L \geq N + m \geq 0$, for $u \in H_N(D_+)$ an identity

$$T(g\mathbf{a})u = T(\mathbf{a})gu \in H_{N+m}(D_+) \quad (32)$$

and

$$T(g\mathbf{a}) : H_N(D_+) \rightarrow H_{N+m}(D_+) \subset H_N(D_+).$$

hold. It should be noted that generally one cannot expect (32) for $\mathbf{a} \in \mathbf{A}_L(C)$ and $g \in \Gamma_n^{(m)}$ unless g is even.

The invertibility of $T(g\mathbf{a})$ is crucial in this paper and this will be shown by using the tau-function, which is defined by the determinant of the difference between $T(g\mathbf{a})$ and $T(\mathbf{a})$, namely

$$g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1}.$$

The tau-function describes the $\Gamma_n^{(m)}$ action very well. To define the determinant we have to show the relevant operators are of Hilbert-Schmidt type.

For $a \in A_L(C)$ let f be an analytic function on D_+ such that

$$\sup_{z \in D_+} |f(z)| < \infty \quad \text{and} \quad \sup_{z \in C} |z^L(a(z) - f(z))| < \infty.$$

For $g_1 \in \Gamma_n^{(0)}, g_2 \in \Gamma_n^{(m)}$ and fixed $b \in D_-$ define

$$\begin{cases} S_{\mathbf{a}}u(z) = \frac{1}{2\pi i} \int_C \frac{\tilde{\mathbf{a}}(\lambda)}{z - \lambda} u(\lambda) d\lambda & \text{for } u \in H_N(D_+) \\ H_{g_2}u = \mathfrak{p}_+(g_2u) & \text{for } u \in H(D_-) \\ R_{\mathbf{a}}(g_1, g_2)u = \mathfrak{p}_+(g_2\mathfrak{p}_-g_1\mathbf{a}u) & \text{for } u \in H_N(D_+) \end{cases} \quad (33)$$

The domains and images for the above maps are as follows:

$$\begin{cases} T(g_2g_1\mathbf{a}) & H_N(D_+) \rightarrow H_{N+m}(D_+) & S_{g_1\mathbf{a}} & H_N(D_+) \rightarrow H(D_-) \\ T(g_1\mathbf{a}) & H_N(D_+) \rightarrow H_N(D_+) & H_{g_2} & H(D_-) \rightarrow H_{N+m}(D_+) \\ g_2 & H_N(D_+) \rightarrow H_{N+m}(D_+) & & \end{cases} .$$

H_{g_2} is akin to a Hankel operator if D_+ is the unit disc. Recall that the norms in $H_N(D_+)$ to $H(D_-)$ are respectively

$$\sqrt{\int_C |u(\lambda)|^2 |\lambda|^{-2N} |d\lambda|}, \quad \sqrt{\int_C |u(\lambda)|^2 |d\lambda|}.$$

Lemma 8 Assume $L \geq N \geq 0$. Then we have

(i) $S_{\mathbf{a}}$ defines a Hilbert-Schmidt class operator from $H_N(D_+)$ to $H(D_-)$ if

$$\int_{C^2} \left| \frac{z^N \tilde{\mathbf{a}}_j(z) - \lambda^N \tilde{\mathbf{a}}_j(\lambda)}{z - \lambda} \right|^2 |dz| |d\lambda| < \infty \text{ for } j = 1, 2. \quad (34)$$

(ii) Suppose $N + m \geq 0$. Then H_{g_2} is of Hilbert-Schmidt class from $H(D_-)$ to $H_{N+m}(D_+)$ if

$$\int_{C^2} \left| \frac{g_2(\lambda) - g_2(z)}{\lambda - z} \right|^2 |z|^{-2(N+m)} |dz| |d\lambda| < \infty. \quad (35)$$

(iii) Identities

$$\begin{cases} T(g_2 g_1 \mathbf{a}) = g_2 T(g_1 \mathbf{a}) + R_{\mathbf{a}}(g_1, g_2) \\ R_{\mathbf{a}}(g_1, g_2) = H_{g_2} S_{g_1 \mathbf{a}} \end{cases} \quad (36)$$

hold, and $R_{\mathbf{a}}(g_1, g_2)$ defines a trace class operator from $H_N(D_+)$ to $H_N(D_+)$ under the conditions (34), (35).

Proof. For $u \in H_N(D_+)$ it holds that

$$\begin{aligned} T(g_2 g_1 \mathbf{a}) u &= \mathbf{p}_+(g_2 \mathbf{p}_+ g_1 \mathbf{a} u) + \mathbf{p}_+(g_2 \mathbf{p}_- g_1 \mathbf{a} u) \\ &= g_2 T(g_1 \mathbf{a}) u + \mathbf{p}_+(g_2 \mathbf{p}_- g_1 \mathbf{a} u) \\ &= g_2 T(g_1 \mathbf{a}) u + H_{g_2} S_{g_1 \mathbf{a}} u. \end{aligned}$$

If $L \geq N$, we have

$$(\mathbf{p}_-(g_1 \mathbf{a} u))(z) = \frac{1}{2\pi i} \int_C \frac{g_1(\lambda) \tilde{\mathbf{a}}(\lambda)}{z - \lambda} u(\lambda) d\lambda$$

for $u \in H_N(D_+)$, $z \in D_-$. Note here an identity

$$\begin{aligned} (S_{g_1 \mathbf{a}}) u(z) &= \frac{1}{2\pi i} \int_C \frac{g_1(\lambda) (\lambda - b)^N \tilde{\mathbf{a}}(\lambda)}{z - \lambda} (\lambda - b)^{-N} u(\lambda) d\lambda \\ &= \frac{z^{-M}}{2\pi i} \int_C \frac{g_1(\lambda) (\mathbf{s}(\lambda) - \mathbf{s}(z))}{z - \lambda} (\lambda - b)^{-N} \lambda^{2N} u(\lambda) \lambda^{-2N} d\lambda \end{aligned}$$

for $z \in C$ with $\mathbf{s}(\lambda) = (\lambda - b)^N \tilde{\mathbf{a}}(\lambda)$ due to $u \in H_N(D_+)$. Since we can regard $S_{g_1 \mathbf{a}}$ as a map from $z^N L^2(C)$ to $L^2(C)$, we see that $S_{g_1 \mathbf{a}}$ is of Hilbert-Schmidt class from $H_N(D_+)$ to $H(D_-)$ if

$$\int_{C^2} \left| g_1(\lambda) \frac{\mathbf{s}(\lambda) - \mathbf{s}(z)}{z - \lambda} (\lambda - b)^{-N} \lambda^{2N} \right|^2 |\lambda|^{-2N} |dz| |d\lambda| < \infty,$$

which is equivalent to (34) if we replace $\mathbf{s}(\lambda)$ by $\lambda^N \tilde{\mathbf{a}}_j$ here.

On the other hand the assumption $m \leq 0$ implies $\sup_{z \in D_+} |g_2(z)| < \infty$, hence H_{g_2} defines an operator from $H(D_-)$ to $H(D_+)$. We find a condition for H_{g_2} also to be of Hilbert-Schmidt class. Note

$$H_{g_2} u(z) = \frac{1}{2\pi i} \int_C \frac{g_2(\lambda)}{\lambda - z} u(\lambda) d\lambda = \frac{1}{2\pi i} \int_C \frac{g_2(\lambda) - g_2(z)}{\lambda - z} u(\lambda) d\lambda$$

for $u \in H(D_-)$, $z \in D_+$. Then, H_{g_2} is of Hilbert-Schmidt class from $H(D_-)$ to $H_{N+m}(D_+)$ if

$$\int_{C^2} \left| \frac{g_2(\lambda) - g_2(z)}{\lambda - z} \right|^2 |z|^{-2(N+m)} |dz| |d\lambda|$$

which is (35). ■

For later purpose we find a sufficient condition on $g \in \Gamma_n^{(m)}$ under which (35) is satisfied. From now on we assume without loss of generality the curve C fulfills

$$\sup_{z \in C} \int_{|z-\lambda| \leq 1, \lambda \in C} |d\lambda| < \infty, \quad (37)$$

and there exists a neighborhood U of the closure of D_+ and $\epsilon > 0$ such that

$$z, \lambda \in C, |z - \lambda| \leq \epsilon \implies (z - \lambda)t + \lambda \in U \text{ for } t \in [0, 1]. \quad (38)$$

For $g \in \Gamma_n^{(m)}$ let $c > 0$ be a constant such that

$$\begin{cases} c^{-1} |z|^m \leq |g(z)| \leq c |z|^m \\ |g'(z)| \leq c |z|^{m+n-1} \end{cases} \text{ hold for } z \in U. \quad (39)$$

For $N \in \mathbb{Z}_+$ set

$$\Delta = \int_{C^2} \left| \frac{g(z) - g(\lambda)}{z - \lambda} \right|^2 |z|^{-2(N+m)} |dz| |d\lambda|,$$

which is the square of the Hilbert-Schmidt norm of the operator

$$H_g : H(D_-) \rightarrow H_{N+m}(D_+).$$

Lemma 9 *If $N \geq \max\{n, 1 - m\}$ hold, there exists a constant c_0 depending only on c such that $\Delta \leq c_0$.*

Proof. Let ϵ be $0 < \epsilon < 1$. We first show that there exists a constant c_1 depending on the constant c of (39) such that

$$\left| \frac{g(z) - g(\lambda)}{z - \lambda} \right| \leq c_1 \begin{cases} |z|^{m+n-1} & \text{if } |z - \lambda| \leq \epsilon |\lambda| \\ |z|^{-1} (|z|^m + |\lambda|^m) & \text{if } |z - \lambda| > \epsilon |\lambda| \end{cases} \quad (40)$$

holds for $z, \lambda \in C$. Since

$$\frac{g(z) - g(\lambda)}{z - \lambda} = \int_0^1 g'((\lambda - z)t + z) dt$$

for $z, \lambda \in C$, the properties (38), (39) show that there exists a constant c_1 such that

$$\left| \frac{g(z) - g(\lambda)}{z - \lambda} \right| \leq c_1 |z|^{m+n-1} I_{|z-\lambda| \leq \epsilon |\lambda|}.$$

The other estimate is clear and we have (40).

Note $\Delta \leq \Delta_1 + \Delta_2$ with

$$\begin{cases} \Delta_1 = c_1^2 \int_{|z-\lambda| \leq \epsilon|\lambda} |z|^{2(m+n-1)} |z|^{-2(N+m)} |dz| |d\lambda| \\ \Delta_2 = c_1^2 \int_{|z-\lambda| > \epsilon|\lambda} |\lambda|^{-2} (|z|^m + |\lambda|^m)^2 |z|^{-2(N+m)} |dz| |d\lambda| \end{cases}.$$

Since the exponent of the integrand of Δ_1 is equal to

$$2(m+n-1) - 2(N+m),$$

$\Delta_1 < \infty$ is valid if $N > n - 1/2$. On the other hand, there exists a constant c_2 such that

$$\Delta_2 \leq c_2 \int_{|z-\lambda| > \epsilon|\lambda} |\lambda|^{-2} (|z|^{2m} + |\lambda|^{2m}) |z|^{-2(N+m)} |dz| |d\lambda|.$$

The right side is finite if

$$-2N < -1, \quad -2(N+m) < -1$$

which is equivalent to $N+m \geq 1$. The above constants c_1, c_2 can be chosen depending on c , hence so does c_0 . ■

The dependence of the constant c_0 on the constant c will be used in the proof of the continuity of the tau-function later.

4 Derivation of Schrödinger operator and KdV equation

Schrödinger operators and solutions to the KdV equation can be obtained from $T(e^{xz} \mathbf{a})$, $T(e^{xz+tz^3} \mathbf{a})$ under their invertibility. This section is devoted to the rigorous derivation of these equations.

4.1 Differentiability

The KdV flow is constructed by one-parameter group $g_t(z) = e^{th(z)}$ with odd polynomial h , and for the construction the differentiability of $T(g_t \mathbf{a})$ with respect to t will be necessary. In this section we extend the definition $T(a)$. For a polynomial h of degree n

$$hu \in H_{N+n}(D_+) \quad \text{if } u \in H_N(D_+),$$

so for $a \in A_L(C)$ define

$$T(ha)u = \mathbf{p}_+(hau) = T(a)hu \in H_{N+n}(D_+),$$

which is possible if $L \geq N+n$. For $\mathbf{a} = (a_1, a_2) \in \mathbf{A}_L(C)$ we define

$$T(h\mathbf{a})u = T(ha_1)u_e + T(ha_2)u_o$$

Lemma 10 Let $\mathbf{a} \in \mathbf{A}_L(C)$ and $g_t(z) = e^{th(z)} \in \Gamma_n$. Assume $g_t \mathbf{a} \in \mathbf{A}_{N+n}^{inv}(C)$ for any $t \in \mathbb{R}$. Then if $L \geq N + n$, for any $u \in H_N(D_+)$

$$\begin{cases} \partial_t T(g_t \mathbf{a}) u = T(hg_t \mathbf{a}) u \in H_{N+n}(D_+) \\ \partial_t T(g_t \mathbf{a})^{-1} u = -T(g_t \mathbf{a})^{-1} T(hg_t \mathbf{a}) T(g_t \mathbf{a})^{-1} u \in H_{N+n}(D_+) \end{cases}$$

holds. Any higher derivative $\partial_t^k T(g_t \mathbf{a})^{-1} u$ exists if $L \geq N + kn$.

Proof. Let $N_1 = N + n$. Recall $T(\mathbf{a}) u = T(a_1) u_e + T(a_2) u_o$ if $\mathbf{a} = (a_1, a_2)$ (see (??)). The first identity follows easily from

$$\frac{T(g_t \mathbf{a}) u - T(g_s \mathbf{a}) u}{t - s} = \frac{1}{t - s} \int_s^t \mathfrak{p}_+(g_\tau h \mathbf{a} u) d\tau = \frac{1}{t - s} \int_s^t T(hg_\tau \mathbf{a}) u d\tau.$$

To show the second identity first we verify the continuity of $T(g_t \mathbf{a})^{-1} u$ in $H_{N_1}(D_+)$ with respect to t . Applying (ii) of Lemma 8 with $g_1 = 1$, $g_2 = g$ and replacing N by N_1 , we have

$$T(g_t \mathbf{a}) = g_t T(\mathbf{a}) + R_{\mathbf{a}}(1, g_t)$$

with

$$R_{\mathbf{a}}(1, g_t) = H_{g_t} S_{\mathbf{a}}.$$

Therefore

$$T(g_t \mathbf{a})^{-1} = \left(I + T(\mathbf{a})^{-1} g_t^{-1} R_{\mathbf{a}}(1, g_t) \right)^{-1} T(\mathbf{a})^{-1} g_t^{-1}$$

holds. We show $g_t^{-1} R_{\mathbf{a}}(1, g_t)$ is continuous in the Hilbert-Schmidt norm on $H_{N_1}(D_+)$, which is reduced to that of $g_t^{-1} H_{g_t}$ as an operator from $H(D_-)$ to $H_{N_1}(D_+)$. The HS-norm of $g_t^{-1} H_{g_t}$ is

$$\Delta = \int_{C^2} \left| \frac{g_t(\lambda) g_t(z)^{-1} - g_s(\lambda) g_s(z)^{-1}}{\lambda - z} \right|^2 |z|^{-2N_1} |dz| |d\lambda|. \quad (41)$$

The proof is carried out similarly to that of (35). Observe

$$\begin{aligned} \left| \frac{g_t(\lambda) g_t(z)^{-1} - g_s(\lambda) g_s(z)^{-1}}{\lambda - z} \right| &\leq \left| \frac{h(\lambda) - h(z)}{\lambda - z} \right| \int_s^t \left| e^{\tau(h(\lambda) - h(z))} \right| d\tau \\ &\leq c \left(|z|^{n-1} + |\lambda|^{n-1} \right) (t - s) \end{aligned}$$

for $z, \lambda \in C$. Then separating the integral (41) on $|\lambda - z| \leq \epsilon |\lambda|$ and $|\lambda - z| > \epsilon |\lambda|$, we have

$$\begin{aligned} \Delta &\leq c_1 (t - s)^2 \int_{|\lambda - z| \leq \epsilon |\lambda|} \left(|z|^{2(n-1)} + |\lambda|^{2(n-1)} \right) |z|^{-2N_1} |dz| |d\lambda| \\ &\quad + c_1 \int_{|\lambda - z| > \epsilon |\lambda|} |\lambda|^{-2} \left| g_t(\lambda) g_t(z)^{-1} - g_s(\lambda) g_s(z)^{-1} \right|^2 |z|^{-2N_1} |dz| |d\lambda|. \end{aligned}$$

The first term is dominated by $c_2 (t - s)^2$ if

$$2(n - 1) - 2N_1 < -1 \implies N_1 \geq n,$$

which is satisfied if $N \geq 0$. The second term tends to 0 as $s \rightarrow t$ if $-2N_1 < -1$, which is always valid if $N_1 \geq 1$. Therefore we have the continuity of $g_t^{-1}R_{\mathbf{a}}(1, g_t)$ in the HS-norm on $H_{N_1}(D_+)$, which implies $T(g_t\mathbf{a})^{-1}u$ is continuous in t for any fixed $u \in H_{N_1}(D_+)$ if $L \geq N + n$. Consequently noting the identity

$$\epsilon^{-1} \left(T(g_{t+\epsilon}\mathbf{a})^{-1} - T(g_t\mathbf{a})^{-1} \right) = T(g_{t+\epsilon}\mathbf{a})^{-1} \epsilon^{-1} (T(g_t\mathbf{a}) - T(g_{t+\epsilon}\mathbf{a})) T(g_t\mathbf{a})^{-1},$$

we have the Lemma. The existence of higher derivatives can be shown similarly. \blacksquare

4.2 Derivation of Schrödinger operator

First we derive a Schrödinger operator from $\mathbf{a} \in \mathbf{A}_1(C)$ and $g = e_x$ with

$$e_x(z) = e^{xz}.$$

The curve C is chosen so that $e_x(z)$ remains bounded for any fixed $x \in \mathbb{R}$, namely

$$C = \left\{ \begin{array}{l} \pm\omega(y) + iy, y \in \mathbb{R}; \omega(y) \text{ is a positive even smooth} \\ \text{function on } \mathbb{R} \text{ such that } \omega(y) = O(1) \text{ as } |y| \rightarrow \infty \end{array} \right\}. \quad (42)$$

Recall

$$\mathbf{a}(z) f(z) = a_1(z) f_e(z) + a_2(z) f_o(z)$$

for a vector function $\mathbf{a}(z) = (a_1(z), a_2(z))$ and a function $f(z)$ on C . For $L \geq 3$, $\mathbf{a} \in \mathbf{A}_L(C)$ assume $e_x\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ for any $x \in \mathbb{R}$. Let $u_x \in H_1(D_+)$ be

$$u_x = T(e_x\mathbf{a})^{-1} 1 \in H_1(D_+)$$

and set

$$w_x = \mathbf{p}_-(e_x\mathbf{a}u_x) \in H(D_-).$$

Then, for a bounded analytic vector $\mathbf{f}(z)$ on D_+ satisfying $\tilde{\mathbf{a}}(\lambda) = \mathbf{a}(\lambda) - \mathbf{f}(\lambda) = O(\lambda^{-L})$ on C

$$w_x(z) = \frac{1}{2\pi i} \int_C \frac{e^{x\lambda} \tilde{\mathbf{a}}(\lambda)}{z - \lambda} u_x(\lambda) d\lambda$$

holds. Since Lemma 10 implies $\partial_x^j u_x \in H_{j+1}(D_+)$ for $j \leq L - 1$,

$$\left\{ \begin{array}{l} \partial_x w_x(z) = \frac{1}{2\pi i} \int_C e^{x\lambda} \frac{\lambda \tilde{\mathbf{a}}(\lambda) u_x(\lambda) + \tilde{\mathbf{a}}(\lambda) \partial_x u_x(\lambda)}{z - \lambda} d\lambda \\ \partial_x^2 w_x(z) = \frac{1}{2\pi i} \int_C e^{x\lambda} \frac{\lambda^2 \tilde{\mathbf{a}}(\lambda) u_x(\lambda) + 2\lambda \tilde{\mathbf{a}}(\lambda) \partial_x u_x(\lambda) + \tilde{\mathbf{a}}(\lambda) \partial_x^2 u_x(\lambda)}{z - \lambda} d\lambda \end{array} \right. \quad (43)$$

Since $u_x \in H_1(D_+)$, the expansion

$$(z - \lambda)^{-1} = \sum_{1 \leq k \leq M} z^{-k} \lambda^{k-1} + z^{-M} \lambda^M (z - \lambda)^{-1}$$

shows

$$\begin{aligned}
w_x(z) &= \sum_{1 \leq k \leq L-1} z^{-k} \frac{1}{2\pi i} \int_C \lambda^{k-1} e^{x\lambda} \tilde{\mathbf{a}}(\lambda) u_x(\lambda) d\lambda \\
&\quad + z^{-L+1} \frac{1}{2\pi i} \int_C \frac{\lambda^{L-1} e^{x\lambda} \tilde{\mathbf{a}}(\lambda)}{z-\lambda} u_x(\lambda) d\lambda \\
&\equiv \sum_{1 \leq k \leq L-1} z^{-k} s_k(x) + z^{-L+1} \tilde{w}_x(z)
\end{aligned} \tag{44}$$

with $\tilde{w}_x \in H(D_-)$. Since $\partial_x u_x \in H_2(D_+)$, $\partial_x^2 u_x \in H_3(D_+)$, (43) shows similarly

$$\begin{cases} \partial_x w_x(z) = \sum_{1 \leq k \leq L-2} z^{-k} s'_k(x) + z^{-L+2} \tilde{w}_x^{(1)}(z) \\ \partial_x^2 w_x(z) = \sum_{1 \leq k \leq L-3} z^{-k} s''_k(x) + z^{-L+3} \tilde{w}_x^{(2)}(z) \end{cases} \in H(D_-). \tag{45}$$

with $\tilde{w}_x^{(1)}, \tilde{w}_x^{(2)} \in H(D_-)$. The notations in (18) and Lemma 6 imply

$$\begin{cases} w_x(z) = \varphi_{e_x \mathbf{a}}(z) \\ s_1(x) = \kappa_1(e_x \mathbf{a}) \end{cases}. \tag{46}$$

Set

$$f_{\mathbf{a}}(x, z) = \mathbf{a}(z) u_x(z) = e^{-xz} (1 + \varphi_{e_x \mathbf{a}}(z)). \tag{47}$$

Proposition 11 *Let $L \geq 3$ and assume $e_x \mathbf{a} \in \mathbf{A}_L^{inv}(C)$ for any $x \in \mathbb{R}$. Set*

$$q(x) = -2s'_1(x) = -2\partial_x \kappa_1(e_x \mathbf{a}).$$

Then, a Schrödinger equation

$$-\partial_x^2 f_{\mathbf{a}}(x, z) + q(x) f_{\mathbf{a}}(x, z) = -z^2 f_{\mathbf{a}}(x, z) \tag{48}$$

holds, and $\{s_k(x)\}_{2 \leq k \leq L-2}$ in (44) is determined by a recurrence relation

$$s''_k + 2s'_1 s_k - 2s'_{k+1} = 0, \quad (1 \leq k \leq L-3) \tag{49}$$

for given $s_1(x), \{s_k(0)\}_{2 \leq k \leq L-2}$.

Proof. The identity $w_x = e_x \mathbf{a} u_x - 1$ yields

$$\begin{cases} \partial_x w_x = e_x (z \mathbf{a} u_x + \mathbf{a} \partial_x u_x) \\ \partial_x^2 w_x = e_x (z^2 \mathbf{a} u_x + \mathbf{a} \partial_x^2 u_x + 2z \mathbf{a} \partial_x u_x) \end{cases},$$

which implies

$$\partial_x^2 w_x - 2z \partial_x w_x = e_x \mathbf{a} (\partial_x^2 u_x - z^2 u_x). \tag{50}$$

Here we have used the identity

$$z^2 \mathbf{a}(z) u(z) = \mathbf{a}(z) z^2 u(z).$$

Our strategy is to modify (50) so that the left hand side is an element of $H(D_-)$ and the right hand side is an element of $(e_x \mathbf{a}) H_3(D_+)$. From (45)

$$z \partial_x w_x = s'_1(x) + \sum_{2 \leq k \leq L-2} z^{-k+1} s'_k(x) + z^{-L+3} \tilde{w}_x^{(1)}(z) = s'_1(x) + v_x$$

follows with $v_x \in H(D_-)$, which combined with (50) yields

$$\begin{aligned}
& \partial_x^2 w_x + 2s_1' w_x - 2v_x \\
&= e_x \mathbf{a} (\partial_x^2 u_x - z^2 u_x) + 2z \partial_x u_x + 2s_1' (e_x \mathbf{a} u_x - 1) - 2v_x \\
&= e_x \mathbf{a} (\partial_x^2 u_x - z^2 u_x + 2s_1' u_x). \tag{51}
\end{aligned}$$

Note

$$\partial_x^2 w_x + 2s_1' w_x - 2v_x \in H(D_-), \quad \partial_x^2 u_x - z^2 u_x + 2s_1' u_x \in H_3(D_+).$$

Then, applying \mathfrak{p}_+ to (51) we have

$$0 = T(e_x \mathbf{a}) (\partial_x^2 u_x - z^2 u_x + 2s_1' u_x),$$

and the invertibility of $T(e_x \mathbf{a})$ on $H_3(D_+)$ yields (48).

$$\partial_x^2 u_x - z^2 u_x + 2s_1' u_x = 0.$$

Since

$$\begin{aligned}
0 &= \partial_x^2 w_x + 2s_1' w_x - 2v_x \\
&= \sum_{1 \leq k \leq L-3} (s_k'' + 2s_1' s_k - 2s_{k+1}') z^{-k} + z^{-L+3} \tilde{v}_x
\end{aligned}$$

with $\tilde{v}_x \in H(D_-)$, we have (49). ■

Remark 12 $q(x)$, $f_{\mathbf{a}}(x, z)$ themselves are well-defined as continuous functions if $L \geq 2$, and $\partial_x f_{\mathbf{a}}(x, z)$ exists as a continuous function. Although the Schrödinger equation (48) is satisfied if $L \geq 3$, it seems that $\partial_x^2 f_{\mathbf{a}}(x, z)$ exists even if $L = 2$ due to (48). However we have no rigorous proof.

Now the m -function $m_{\mathbf{a}}$ has another representation by $f_{\mathbf{a}}(x, z)$.

Corollary 13 *It holds that*

$$\left. \frac{\partial_x f_{\mathbf{a}}(x, z)}{f_{\mathbf{a}}(x, z)} \right|_{x=0} = -m_{\mathbf{a}}(z). \tag{52}$$

Proof. Set $b(z) = \partial_x u_x(z)|_{x=0}$, $\tilde{b}(z) = \partial_x f_{\mathbf{a}}(x, z)|_{x=0}$. Then

$$b(z) \in H_2(D_+) \quad \text{and} \quad \mathbf{a}(z) b(z) = \tilde{b}(z)$$

hold. Since $e^{xz} f_{\mathbf{a}}(x, z) = 1 + w_x(z)$, (45) shows

$$\begin{aligned}
\tilde{b}(z) &= -z(1 + w_0(z)) + \sum_{1 \leq k \leq L-2} z^{-k} s_k'(0) + z^{-L+2} \tilde{w}_0^{(1)}(z) \\
&= -z - s_1(0) + w(z)
\end{aligned}$$

with $w \in H(D_-)$. Applying \mathfrak{p}_+ to the identity

$$\mathbf{a}(z) b(z) = -z - s_1(0) + w(z)$$

we have

$$T(\mathbf{a})b(z) = -z - s_1(0),$$

hence

$$b(z) = -T(\mathbf{a})^{-1}z - s_1(0)T(\mathbf{a})^{-1}\mathbf{1},$$

which implies

$$\tilde{b}(z) = \mathbf{a}(z)b(z) = -(z + \psi_{\mathbf{a}}(z)) - s_1(0)(1 + \varphi_{\mathbf{a}}(z)).$$

Consequently we have

$$\begin{aligned} \left. \frac{\partial_x f_{\mathbf{a}}(x, z)}{f_{\mathbf{a}}(x, z)} \right|_{x=0} &= \frac{-(z + \psi_{\mathbf{a}}(z)) - s_1(0)(1 + \varphi_{\mathbf{a}}(z))}{1 + \varphi_{\mathbf{a}}(z)} \\ &= -m_{\mathbf{a}}(z) + \kappa_1(\mathbf{a}) - s_1(0) = -m_{\mathbf{a}}(z) \end{aligned}$$

due to $s_1(0) = \kappa_1(\mathbf{a})$. ■

4.3 Derivation of KdV equation

Our next task is to derive the KdV equation from $g = e_{t,x}$ with

$$e_{t,x}(z) = e^{xz+tz^3}.$$

In this case the curve C is determined by requesting $e_{t,x}(z)$ to be bounded on C , hence

$$C = \left\{ \begin{array}{l} \pm\omega(y) + iy, y \in \mathbb{R}; \omega \text{ is smooth positive even on } \mathbb{R}, \\ \text{and } \omega \text{ satisfies } \omega(y) = O(y^{-2}) \text{ as } y \rightarrow \infty. \end{array} \right\}.$$

For $L \geq 4$ let $\mathbf{a} \in \mathbf{A}_L(C)$ and assume $e_{t,x}\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ for any $t, x \in \mathbb{R}$. Let $u_{t,x} = T(e_{t,x}\mathbf{a})^{-1}\mathbf{1} \in H_1(D_+)$ and $w_{t,x} = \mathbf{p}_-(e_{t,x}\mathbf{a}f_{t,x}) \in H(D_-)$. Then, Lemma 10 implies

$$w_{t,x}(z) = \sum_{1 \leq k \leq L-1} z^{-k}s_k(t, x) + z^{-L+1}\tilde{w}_{t,x}(z) \quad \text{with } s_k \in \mathbb{C}, \tilde{w}_{t,x} \in H(D_-)$$

for $t, x \in \mathbb{R}$.

Proposition 14 *Let $L \geq 4$ and assume $e_{t,x}\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ for any $t, x \in \mathbb{R}$. Set*

$$q(t, x) = -2\partial_x s_1(t, x) = -2\partial_x \kappa_1(e_{t,x}\mathbf{a}).$$

Then q satisfies the KdV equation

$$\partial_t q(t, x) = \frac{1}{4}\partial_x^3 q(t, x) - \frac{3}{2}q(t, x)\partial_x q(t, x). \quad (53)$$

Proof. In this case $N = 1, n = 3$. Our strategy for the proof is similar to that of the last one. Since $w_{t,x} = e_{t,x}\mathbf{a}u_{t,x} - \mathbf{1} \in H(D_-)$

$$\begin{cases} \partial_x w_{t,x} = e_{t,x}(z\mathbf{a}u_{t,x} + \mathbf{a}\partial_x u_{t,x}) \\ \partial_t w_{t,x} = e_{t,x}(z^3\mathbf{a}u_{t,x} + \mathbf{a}\partial_t u_{t,x}) \end{cases},$$

which leads us to

$$\partial_t w_{t,x} = z^2 \partial_x w_{t,x} + e_{t,x} \mathbf{a} (\partial_t u_{t,x} - z^2 \partial_x u_{t,x}).$$

Since

$$z^2 \partial_x w_{t,x} \equiv z s'_1 + s'_2 + s'_3 z^{-1} \pmod{z^{-1} H(D_-)},$$

substituting $1 = e_{t,x} \mathbf{a} u_{t,x} - w_{t,x}$ and

$$\begin{aligned} z &= z e_{t,x} \mathbf{a} u_{t,x} - z w_{t,x} \\ &= \partial_x w_{t,x} - e_{t,x} \mathbf{a} \partial_x u_{t,x} - z w_{t,x} \\ &\equiv s'_1 z^{-1} - s_1 - s_2 z^{-1} - e_{t,x} \mathbf{a} \partial_x u_{t,x} \\ &\equiv (s'_1 - s_2) z^{-1} - s_1 (e_{t,x} \mathbf{a} u_{t,x} - w_{t,x}) - e_{t,x} \mathbf{a} \partial_x u_{t,x} \\ &\equiv (s'_1 - s_2 + s_1^2) z^{-1} - e_{t,x} \mathbf{a} (s_1 u_{t,x} + \partial_x u_{t,x}) \end{aligned}$$

(\equiv means $\pmod{z^{-1} H(D_-)}$) into the above identity yields

$$\begin{aligned} z^2 \partial_x w_{t,x} &\equiv s'_1 ((s'_1 - s_2 + s_1^2) z^{-1} - e_{t,x} \mathbf{a} (s_1 u_{t,x} + \partial_x u_{t,x})) + s'_2 + s'_3 z^{-1} \\ &= s'_2 (e_{t,x} \mathbf{a} f_{t,x} - w_{t,x}) + (s'_1 (s'_1 - s_2 + s_1^2) + s'_3) z^{-1} - e_{t,x} \mathbf{a} (s_1 u_{t,x} + \partial_x u_{t,x}) \\ &= -s'_2 w_{t,x} + (s'_1 (s'_1 - s_2 + s_1^2) + s'_3) z^{-1} + e_{t,x} \mathbf{a} ((s'_2 - s_1) u_{t,x} - \partial_x u_{t,x}) \end{aligned}$$

Therefore

$$\begin{aligned} \partial_t w_{t,x} &\equiv -s'_2 w_{t,x} + (s'_1 (s'_1 - s_2 + s_1^2) + s'_3) z^{-1} \\ &\quad + e_{t,x} \mathbf{a} ((s'_2 - s_1) u_{t,x} + \partial_t u_{t,x} - z^2 \partial_x u_{t,x} - \partial_x u_{t,x}) \end{aligned}$$

holds, and we have

$$\begin{cases} (s'_2 - s_1) u_{t,x} + \partial_t u_{t,x} - z^2 \partial_x u_{t,x} - \partial_x u_{t,x} = 0 \\ \partial_t w_{t,x} + s'_2 w_{t,x} - (s'_1 (s'_1 - s_2 + s_1^2) + s'_3) z^{-1} \equiv 0 \end{cases} \quad (54)$$

due to the invertibility of $T(e_{t,x} \mathbf{a})$ on $H_3(D_+)$. Since the coefficient of z^{-1} for the second identity vanishes, it follows that

$$\partial_t s_1 + s'_2 s_1 - (s'_1 (s'_1 - s_2 + s_1^2) + s'_3) = 0.$$

Here the identities for $k = 1, 2$ and $e^{tz^3} \mathbf{a}$ in (49) of Proposition 11

$$\begin{cases} s''_1 + 2s'_1 s_1 - 2s'_2 = 0 \\ s''_2 + 2s'_1 s_2 - 2s'_3 = 0 \end{cases}$$

allow us to have

$$\begin{aligned} \partial_t s_1 &= -s'_2 s_1 + (s'_1 (s'_1 - s_2 + s_1^2) + s'_3) \\ &= -s_1 (s''_1/2 + s'_1 s_1) + (s'_1)^2 + s_1^2 s'_1 - s'_1 s_2 + s''_2/2 + s'_1 s_2 \\ &= -s_1 (s''_1/2 + s'_1 s_1) + (s'_1)^2 + s_1^2 s'_1 + (s''_1/2 + s'_1 s_1)' / 2 \\ &= s'''_1/4 + 3(s'_1)^2 / 2, \end{aligned}$$

which is (53) by substituting $q(t, x) = -2\partial_x s_1(t, x)$. ■

The above calculation does not reveal explicitly the reason why the KdV equation appears. There is a hidden algebraic structure behind discovered by Sato [18].

The invertibility of $T(e_{t,x}\mathbf{a})$ plays an essential role in the above calculation. The non-existence of $T(e_{t,x}\mathbf{a})^{-1}$ at some point (t, x) means that the solution $q(t, x)$ has a singularity at (t, x) .

5 Tau-function

Hirota introduced an object $\tau_{\mathbf{a}}$ called tau-function whose mathematical meaning was discovered by Sato later. In this paper the tau-function will be used to prove the invertibility of $T(g\mathbf{a})$.

The tau-function defined in [11] is written in the present context as

$$\tau_{\mathbf{a}}(g) = \det(g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1}). \quad (55)$$

The operator $g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1}$ is a map on $H_N(D_+)$ and the determinant is well-defined if the operator $g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} - I$ is of trace class on $H_N(D_+)$. The identity (36) implies

$$g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} - I = g^{-1}R_{\mathbf{a}}(1, g)T(\mathbf{a})^{-1},$$

hence it is sufficient for this that $R_{\mathbf{a}}(1, g)$ is of trace class. Since $R_{\mathbf{a}}(1, g)$ is a product of H_g and $S_{\mathbf{a}}$, there are two cases where $R_{\mathbf{a}}(1, g)$ is of trace class.

- (i) H_g is of trace class.
- (ii) H_g and $S_{\mathbf{a}}$ are of Hilbert-Schmidt class.

(i) is the case for $g \in \Gamma_0^{(0)}$ (rational function of order 0) due to Lemma 37. However for other cases one has to impose an extra condition on \mathbf{a} . To avoid this inconvenience we use the modified determinant \det_2 , namely

$$\det_2(I + A) = \det(I + A)e^{-\text{tr}A}.$$

It is known that this determinant can be extended to any operator A of Hilbert-Schmidt (HS in short) class. Since $I + A$ is invertible if and only if $\det_2(I + A) \neq 0$, this \det_2 is sufficient to verify the existence of $T(g\mathbf{a})^{-1}$. Set

$$\tau_{\mathbf{a}}^{(2)}(g) = \det_2(g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1}) \text{ for } \mathbf{a} \in \mathbf{A}_L^{\text{inv}}(C), g \in \Gamma_n^{(m)}. \quad (56)$$

$\tau_{\mathbf{a}}^{(2)}(g)$ can be defined if $R_{\mathbf{a}}(1, g) = H_g S_{\mathbf{a}}$ is of HS class, which is valid if H_g is of HS-class as a map from $H(D_-)$ to $H_{N+m}(D_+)$, and Lemma 9 implies that H_g is of HS class if

$$N \geq \max\{n, 1 - m\}, \quad (57)$$

which implies

$$L \geq \max\{n, 1 - m\}. \quad (58)$$

Conversely the existence of N satisfying (57) follows from (58).

In the definitions of $\tau_{\mathbf{a}}(g)$, $\tau_{\mathbf{a}}^{(2)}(g)$ the operator $g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1}$ is a map on $H_N(D_+)$, hence $\tau_{\mathbf{a}}(g)$, $\tau_{\mathbf{a}}^{(2)}(g)$ may depend on N . However we have

Lemma 15 For $\mathbf{a} \in \mathbf{A}_L(C)$, $g \in \Gamma_n^{(m)}$ assume that $N, N' \in \mathbb{Z}_+$ satisfy (57) and the two operators

$$\begin{cases} T(\mathbf{a}) : H_N(D_+) \rightarrow H_N(D_+) \\ T(\mathbf{a}) : H_{N'}(D_+) \rightarrow H_{N'}(D_+) \end{cases}$$

are bijective. Then the determinants and the modified determinants on $H_N(D_+)$ and $H_{N'}(D_+)$ in (55) and (56) are equal, hence $\tau_{\mathbf{a}}(g)$, $\tau_{\mathbf{a}}^{(2)}(g)$ do not depend on N .

To show this lemma we use a metric free nature of determinant. For the necessary facts of determinant refer to [20].

Lemma 16 Let H, H_1 be two Hilbert spaces and H_1 be a subspace of H as vector spaces. Assume H_1 is dense. Suppose a linear operator A on H is of Hilbert-Schmidt class and satisfies

$$AH_1 \subset H_1 \quad \text{and} \quad A_{H_1} \equiv A|_{H_1} \quad \text{is of Hilbert-Schmidt class in } H_1.$$

Then

$$\det_2(I_H + A) = \det_2(I_{H_1} + A_{H_1})$$

holds. If A is of trace class, then $\det(I_H + A) = \det(I_{H_1} + A_{H_1})$ holds as well.

Proof. Let $\{e_n\}_{n \geq 1}$ be a complete orthonormal basis of H_1 and $\{f_n\}_{n \geq 1}$ be the orthonormal vectors in H generated from $\{e_n\}_{n \geq 1}$ by the Gram-Schmidt process. Since H_1 is dense in H , $\{f_n\}_{n \geq 1}$ turns to be complete in H . Let V_n be the n -dimensional subspace generated by $\{e_k\}_{1 \leq k \leq n}$ and P_n, Q_n be the orthogonal projections to V_n from H_1, H respectively. Then it is known that

$$\begin{cases} \det_2(I_{H_1} + P_n A_{H_1} P_n) \rightarrow \det_2(I_{H_1} + A_{H_1}) \\ \det_2(I_H + Q_n A Q_n) \rightarrow \det_2(I_H + A) \end{cases} \quad \text{as } n \rightarrow \infty. \quad (59)$$

Let B, E be $n \times n$ matrices whose entries are

$$f_i = \sum_{1 \leq j \leq n} b_{ij} e_j \rightarrow B = (b_{ij}), \quad \text{and} \quad E = ((e_i, e_j)_H).$$

If we denote

$$A_n^{H_1} = \left((Ae_i, e_j)_{H_1} \right)_{1 \leq i, j \leq n}, \quad A_n^H = \left((Af_i, f_j)_H \right)_{1 \leq i, j \leq n},$$

then the identity $A_n^H = B A_n^{H_1} E B^*$ yields

$$\begin{aligned} \det_2(I_H + Q_n A Q_n) &= \det_2(I_n + A_n^H) \\ &= \det_2(I_n + B A_n^{H_1} E B^*) \\ &= \det_2(I_n + B A_n^{H_1} B^{-1}) \\ &= \det_2(I_n + A_n^{H_1}) = \det_2(I_{H_1} + P_n A_{H_1} P_n) \end{aligned}$$

due to $BEB^* = I_n$. This together with (59) completes the proof for \det_2 . The proof for \det is similar. ■

Proof of Lemma 15. Note that for $N' \geq N$ an identity

$$g^{-1}R_{\mathbf{a}}^{(N')} (1, g) T_{N'}(\mathbf{a})^{-1} \Big|_{H_N(D_+)} = g^{-1}R_{\mathbf{a}}^{(N)} (1, g) T_N(\mathbf{a})^{-1} \quad (60)$$

holds on $H_N(D_+)$. On the other hand, for $u \in H(D_+)$ the identity

$$(b-z)^N u(z) = \lim_{\epsilon \rightarrow 0} (b-z)^N (1-\epsilon z)^{-N} u(z)$$

implies $H(D_+)$ is dense in $H_N(D_+)$. Then, applying Lemma 16 to

$$H = H_{N'}(D_+), H_1 = H_N(D_+), A = g^{-1}R_{\mathbf{a}}^{(N')} (1, g) T_{N'}(\mathbf{a})^{-1}$$

yields the lemma.

This Lemma yields flexibility in choosing N , namely N can be arbitrary if (57) is satisfied for given L, m, n . Therefore $\tau_{\mathbf{a}}^{(2)}(g)$ can be defined for $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$, $g \in \Gamma_n^{(m)}$ under the condition (58). However it should be noted that for any rational function $r \in \Gamma_0^{(m)}$ Lemma 37 shows $r^{-1}T(r\mathbf{a})T(\mathbf{a})^{-1} - I$ is of finite rank on any space $H_N(D_+)$ with N such that $-m \leq N \leq L$. Therefore, $\tau_{\mathbf{a}}^{(2)}(r)$, $\tau_{\mathbf{a}}(r)$ can be defined for $r \in \Gamma_0^{(m)}$, $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ if $L \geq -m$.

5.1 Cocycle property of tau-function

The tau-function is a key material to study the KdV flow and in this section we give fundamental properties for the tau-function.

Note

$$\mathbf{a} \in \mathbf{A}_L(C), g \in \Gamma_n^{(m)} \implies g\mathbf{a} \in \mathbf{A}_L(C),$$

since g is analytic and bounded on D_+ . Assume further

$$\mathbf{a} \in \mathbf{A}_L^{inv}(C), g_1 \in \Gamma_n^{(0)}, g_2 \in \Gamma_n^{(m)}.$$

We consider three tau-functions $\tau_{\mathbf{a}}^{(2)}(g_1g_2)$, $\tau_{\mathbf{a}}^{(2)}(g_1)$, $\tau_{g_1\mathbf{a}}^{(2)}(g_2)$ simultaneously, which is possible if $L \in \mathbb{Z}_+$ satisfies (26) and $g_1\mathbf{a} \in \mathbf{A}_L^{inv}(C)$.

For simplicity of notations set

$$\begin{aligned} E_{\mathbf{a}}(g_1, g_2) &= \text{tr} \left(\left((g_1g_2)^{-1} T(g_2g_1\mathbf{a}) T(g_1\mathbf{a})^{-1} g_1 - I \right) (g_1^{-1} T(g_1\mathbf{a}) T(\mathbf{a})^{-1} - I) \right) \\ &= \text{tr} \left((g_1g_2)^{-1} R_{\mathbf{a}}(g_1, g_2) T(g_1\mathbf{a})^{-1} R_{\mathbf{a}}(1, g_1) T(\mathbf{a})^{-1} \right). \end{aligned} \quad (61)$$

Lemma 17 *Assume L satisfies (58) and let N be one N of (57). Then we have the followings.*

(i) *The map $T(g_1\mathbf{a})$ is bijective on $H_N(D_+)$ if and only if $\tau_{\mathbf{a}}^{(2)}(g_1) \neq 0$. Similarly the map*

$$T(g_2g_1\mathbf{a}) : H_N(D_+) \rightarrow H_{N+m}(D_+)$$

is bijective if and only if $\tau_{\mathbf{a}}^{(2)}(g_1g_2) \neq 0$.

(ii) *If $\tau_{\mathbf{a}}^{(2)}(g_1) \neq 0$, then it holds that*

$$\tau_{\mathbf{a}}^{(2)}(g_1g_2) = \tau_{\mathbf{a}}^{(2)}(g_1) \tau_{g_1\mathbf{a}}^{(2)}(g_2) \exp(-E_{\mathbf{a}}(g_1, g_2)). \quad (62)$$

Additionally if $r_1 \in \Gamma_0^{(0)}$, $r_2 \in \Gamma_0^{(m)}$ and $\tau_{\mathbf{a}}(r_1) \neq 0$, then

$$\tau_{\mathbf{a}}(r_1 r_2) = \tau_{\mathbf{a}}(r_1) \tau_{\tau_{r_1 \mathbf{a}}}(r_2). \quad (63)$$

(iii) Suppose g_1 satisfies $g_1(z) = g_1(-z)$. Then, it holds that $T(g_2 g_1 \mathbf{a}) = T(g_2 \mathbf{a}) g_1$ and

$$\tau_{g_1 \mathbf{a}}^{(2)}(g_2) = \tau_{\mathbf{a}}^{(2)}(g_2).$$

Similarly, for rational functions $r_1 \in \Gamma_0^{(0)}$, $r_2 \in \Gamma_0^{(m)}$ satisfying $r_1(z) = r_1(-z)$ we have

$$\tau_{r_1 \mathbf{a}}(r_2) = \tau_{\mathbf{a}}(r_2), \quad \tau_{\mathbf{a}}(r_1 r_2) = \tau_{\mathbf{a}}(r_1) \tau_{\mathbf{a}}(r_2).$$

Proof. The identity (61) implies

$$g_2^{-1} T(g_2 g_1 \mathbf{a}) T(g_1 \mathbf{a})^{-1} = I + g_2^{-1} R_{\mathbf{a}}(g_1, g_2) T(g_1 \mathbf{a})^{-1}$$

with the Hilbert-Schmidt class operator $R_{\mathbf{a}}(g_1, g_2)$. Then general theory of Fredholm determinant shows that the operator $g_2^{-1} T(g_2 g_1 \mathbf{a}) T(g_1 \mathbf{a})^{-1}$ is bijective if and only if $\det_2(g_2^{-1} T(g_2 g_1 \mathbf{a}) T(g_1 \mathbf{a})^{-1}) \neq 0$, which implies the bijectivity of $T(g_2 g_1 \mathbf{a})$. The bijectivity of $T(g_1 \mathbf{a})$ follows by letting $g_2 = 1$.

The definition of the tau-function says

$$\tau_{\mathbf{a}}^{(2)}(g_1 g_2) = \det_2((g_1 g_2)^{-1} T(g_1 g_2 \mathbf{a}) T(\mathbf{a})^{-1}).$$

On the other hand it holds that

$$\begin{aligned} & (g_1 g_2)^{-1} T(g_1 g_2 \mathbf{a}) T(\mathbf{a})^{-1} \\ &= (g_1^{-1} (g_2^{-1} T(g_2 g_1 \mathbf{a}) T(g_1 \mathbf{a})^{-1}) g_1) (g_1^{-1} T(g_1 \mathbf{a}) T(\mathbf{a})^{-1}). \end{aligned} \quad (64)$$

Note the identity

$$\begin{cases} \det_2(G^{-1}(I+A)G) = \det_2(I+A) \\ \det_2((I+A)(I+B)) = \det_2(I+A) \det_2(I+B) e^{-\text{tr}(AB)} \end{cases}$$

for Hilbert-Schmidt operators A, B and a bounded operator G having bounded inverse. Then taking determinant in (64) yields

$$\begin{aligned} \tau_{\mathbf{a}}^{(2)}(g_1 g_2) &= \det_2(g_1^{-1} g_2^{-1} T(g_1 g_2 \mathbf{a}) T(g_2 \mathbf{a})^{-1} g_1) \det_2(g_1^{-1} T(g_1 \mathbf{a}) T(\mathbf{a})^{-1}) \\ &\quad \times \exp\left(-\text{tr}\left(g_1^{-1} g_2^{-1} R_{\mathbf{a}}(g_1, g_2) T(g_1 \mathbf{a})^{-1} g_1 g_1^{-1} R_{\mathbf{a}}(1, g_1) T(\mathbf{a})^{-1}\right)\right) \\ &= \tau_{\mathbf{a}}^{(2)}(g_1) \tau_{g_1 \mathbf{a}}^{(2)}(g_2) \exp(-E_{\mathbf{a}}(g_1, g_2)) \end{aligned}$$

For $r \in \Gamma_0^{(0)}$ Lemma 37 implies that $r^{-1} T(r \mathbf{a}) T(\mathbf{a})^{-1}$ is of finite rank, hence $\tau_{\mathbf{a}}(r)$ can be defined. Taking the determinant in (64) we easily have (63).

Suppose $g_1(z) = g_1(-z)$. For $u \in H_N(D_+)$, $\mathbf{a} = (a_1, a_2)$ we have

$$\begin{aligned} T(g_2 g_1 \mathbf{a}) u &= \mathfrak{p}_+(g_1 g_2 a_1) u_e + \mathfrak{p}_+(g_1 g_2 a_2) u_o \\ &= \mathfrak{p}_+(g_2 a_1) g_1 u_e + \mathfrak{p}_+(g_2 a_2) g_1 u_o \\ &= \mathfrak{p}_+(g_2 a_1) (g_1 u)_e + \mathfrak{p}_+(g_2 a_2) (g_1 u)_o \\ &= T(g_2 \mathbf{a})(g_1 u), \end{aligned}$$

which implies $T(g_2 g_1 \mathbf{a}) = T(g_2 \mathbf{a}) g_1$ on $H_N(D_+)$, hence

$$T(g_2 g_1 \mathbf{a}) T(g_1 \mathbf{a})^{-1} = T(g_2 \mathbf{a}) g_1 g_1^{-1} T(\mathbf{a})^{-1} = T(g_2 \mathbf{a}) T(\mathbf{a})^{-1}$$

is valid. This shows

$$\tau_{g_1 \mathbf{a}}^{(2)}(g_2) = \det_2(g_2^{-1} T(g_2 g_1 \mathbf{a}) T(g_1 \mathbf{a})^{-1}) = \det_2(g_2^{-1} T(g_2 \mathbf{a}) T(\mathbf{a})^{-1}) = \tau_{\mathbf{a}}^{(2)}(g_2).$$

■

5.2 Continuity of tau-functions

Since the determinant \det_2 is estimated by the HS-norm, the continuity of $\tau_{\mathbf{a}}^{(2)}(g)$ with respect to $g \in \Gamma_n^{(m)}$ follows from that of $g^{-1} R_{\mathbf{a}}(1, g)$ with respect to the HS-norm, which is reduced to the continuity of $H_g = \mathfrak{p}_+(g \cdot)$ on $H(D_-)$ with respect to the HS-norm due to

$$R_{\mathbf{a}}(1, g) = H_g S_{\mathbf{a}} \quad (\text{see Lemma 8}), \quad (65)$$

where

$$\begin{cases} R_{\mathbf{a}}(1, g) & : H_N(D_+) \rightarrow H_N(D_+) \\ H_g & : H(D_-) \rightarrow H_N(D_+) \\ S_{\mathbf{a}} & : H_N(D_+) \rightarrow H(D_-) \end{cases} .$$

The condition for L, n is (58), namely

$$L \geq \max\{n, 1 - m\}$$

and N is arbitrary if it satisfies (57). Denote

$$\begin{aligned} d_N(g_1, g_2) &= \left(\int_{C^2} \left| \frac{g_1(z)^{-1} g_1(\lambda) - g_2(z)^{-1} g_2(\lambda)}{z - \lambda} \right|^2 |z|^{-2N} |dz| |d\lambda| \right)^{1/2} \end{aligned}$$

if $g_1, g_2 \in \Gamma_n^{(m)}$, we have

Lemma 18 *Let $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ with L satisfying (58). Assume $g_1, g_2 \in \Gamma_n^{(m)}$ and $d_N(g_1, 1) \leq c_1$. Then there exists a constant $c_{\mathbf{a}}$ depending on c_1, \mathbf{a}, N such that*

$$\left| \tau_{\mathbf{a}}^{(2)}(g_1) - \tau_{\mathbf{a}}^{(2)}(g_2) \right| \leq c_{\mathbf{a}} d_N(g_1, g_2). \quad (66)$$

Proof. Recall the definition $\tau_{\mathbf{a}}^{(2)}(g) = \det_2(I + g^{-1} R_{\mathbf{a}}(1, g) T(\mathbf{a})^{-1})$ for $g \in \Gamma_n^{(m)}$, and

$$g^{-1} R_{\mathbf{a}}(1, g) T(\mathbf{a})^{-1} = g^{-1} H_g S_{\mathbf{a}} T(\mathbf{a})^{-1}.$$

The HS-norm of this operator is dominated by $d_N(g, 1)$, hence if $d_N(g, 1) < \infty$, then $\tau_{\mathbf{a}}^{(2)}(g)$ is defined finitely. Generally if $\|A\|_{HS}, \|B\|_{HS} \leq c_1$ there exists a constant c_2 depending only on c_1 such that

$$|\det_2(I + A) - \det_2(I + B)| \leq c_2 \|A - B\|_{HS}.$$

Therefore $\tau_{\mathbf{a}}^{(2)}(g_1) - \tau_{\mathbf{a}}^{(2)}(g_2)$ can be estimated by those of the HS-norms of $g_1^{-1} H_{g_1} - g_2^{-1} H_{g_2}$ on the space $H_N(D_+)$, which is just equal to $d_N(g_1, g_2)$. ■

For later purpose we give a sufficient condition for the convergence of $\tau_{\mathbf{a}}^{(2)}(g_k)$.

Lemma 19 Assume the following properties for $g_k, g \in \Gamma_n^{(0)}$:

$$\begin{cases} (i) \text{ there exist } c_1, c_2 > 0 \text{ such that for } z \in U \\ \quad c_1 \leq |g_k(z)| \leq c_2, \quad |g'_k(z)| \leq c_2 |z|^{n-1} \quad , \\ (ii) g_k(z) \rightarrow g(z) \text{ as } k \rightarrow \infty \text{ for any } z \in C \end{cases} \quad (67)$$

where U is a neighborhood of the closure of D_+ satisfying (38). Then, for $r \in \Gamma_0^{(m)}$ and $\mathbf{a} \in \mathbf{A}_L(C)$ with

$$L \geq \max\{n, 1 - m\},$$

it holds that

$$\tau_{\mathbf{a}}^{(2)}(rg_k) \rightarrow \tau_{\mathbf{a}}^{(2)}(rg).$$

Proof. Choose an integer $N \geq 0$ such that $L \geq N \geq \max\{n, 1 - m\}$. Set

$$\Delta_k(z, \lambda) = \frac{g(z)^{-1}g(\lambda) - g_k(z)^{-1}g_k(\lambda)}{z - \lambda}.$$

Since

$$d_{L,N}(rg_k, rg)^2 = \int_{C^2} |\Delta_k(z, \lambda)|^2 |z|^{-2N} |dz| |d\lambda|,$$

it is sufficient for $d_{L,N}(rg_k, rg) \rightarrow 0$ as $k \rightarrow \infty$ to show that there exists a function f integrable with respect to $|z|^{-2N} |dz| |d\lambda|$ such that

$$|\Delta_k(z, \lambda)|^2 \leq f(z, \lambda).$$

Note

$$\begin{aligned} \Delta_k(z, \lambda) &= r(z)^{-1}g(z)^{-1} \frac{r(\lambda)g(\lambda) - r(z)g(z)}{z - \lambda} \\ &\quad - r(z)^{-1}g_k(z)^{-1} \frac{r(\lambda)g_k(\lambda) - r(z)g_k(z)}{z - \lambda}. \end{aligned}$$

Then (40) implies

$$\left| \frac{r(z)g_k(z) - r(\lambda)g_k(\lambda)}{z - \lambda} \right| \leq c_1 \begin{cases} |z|^{m+n-1} & \text{if } |z - \lambda| \leq \epsilon |\lambda| \\ |z|^{-1}(|z|^m + |\lambda|^m) & \text{if } |z - \lambda| > \epsilon |\lambda| \end{cases}.$$

Set

$$f_1(z, \lambda) = |z|^{-m} \left(|z|^{m+n-1} I_{|z-\lambda| \leq \epsilon |\lambda|} + |\lambda|^{-1} (|z|^m + |\lambda|^m) I_{|z-\lambda| > \epsilon |\lambda|} \right).$$

Since $|r(z)^{-1}g_k(z)^{-1}| \leq c_2 |z|^{-m}$, it is sufficient to show the integrability of $f_1(z, \lambda)^2$. The rest of proof proceeds just as the proof of Lemma 9. The exponent of the first term is $2(n-1) - 2N$, which is less than if $N \geq n$. The integral of the second term is dominated by

$$\int_{C^2} |\lambda|^{-2} |z|^{-2m} (|z|^{2m} + |\lambda|^{2m}) |z|^{-2N} |dz| |d\lambda|,$$

which is finite if

$$-2N < -1, \quad -2m - 2N < -1 \implies N \geq 1, 1 - m.$$

■

6 Non-negativity condition of $\mathbf{A}_N^{inv}(C)$

Generally potentials arising from $\mathbf{a} \in \mathbf{A}_N^{inv}(C)$ are complex valued, so to obtain real potentials some sort of realness for \mathbf{a} and g is required. $\mathbf{a} \in \mathbf{A}_N(C)$, $g \in \Gamma_n^{(m)}$ are called **real** if they satisfy

$$\mathbf{a}(\lambda) = \overline{\mathbf{a}(\bar{\lambda})} \quad \text{for } \lambda \in C, \quad g(z) = \overline{g(\bar{z})} \quad \text{for } z \in \mathbb{C}. \quad (68)$$

If \mathbf{a} and g are real in this sense, then clearly we have

$$\begin{cases} \varphi_{\mathbf{a}}(z) = \overline{\varphi_{\mathbf{a}}(\bar{z})}, & \psi_{\mathbf{a}}(z) = \overline{\psi_{\mathbf{a}}(\bar{z})}, & m_{\mathbf{a}}(z) = \overline{m_{\mathbf{a}}(\bar{z})} \\ \tau_{\mathbf{a}}(g), \tau_{\mathbf{a}}^{(2)}(g) \in \mathbb{R} \end{cases},$$

and the associated potential takes real values.

Define a subclass of \mathbf{A}_L^{inv} :

$$\begin{aligned} \mathbf{A}_{L,+}^{inv}(C) &= \left\{ \mathbf{a} \in \mathbf{A}_L^{inv}(C); \tau_{\mathbf{a}}^{(2)}(r) \geq 0 \text{ for any real rational } r \in \Gamma_0^{(0)} \right\} \\ &= \left\{ \mathbf{a} \in \mathbf{A}_L^{inv}(C); \tau_{\mathbf{a}}(r) \geq 0 \text{ for any real rational } r \in \Gamma_0^{(0)} \right\}. \end{aligned}$$

The second identity follows from the identity

$$\tau_{\mathbf{a}}^{(2)}(r) = \tau_{\mathbf{a}}(r) \exp\left(-\text{tr}\left(r^{-1}T(r\mathbf{a})T(\mathbf{a})^{-1} - I\right)\right).$$

$\tau_{\mathbf{a}}(r)$ is well-defined for any rational function $r \in \Gamma_0^{(0)}$ since the relevant operator is of finite rank. Our strategy to show $\tau_{\mathbf{a}}^{(2)}(g) > 0$ for real $g \in \Gamma_n^{(0)}$ is as follows:

- (i) Show $\tau_{\mathbf{a}}^{(2)}(r) > 0$ for any real $r \in \Gamma_0^{(0)}$ and $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$.
- (ii) Approximate a general real $g \in \Gamma_n^{(0)}$ by a sequence of real $r_k \in \Gamma_0^{(0)}$.
- (iii) Use the continuity of $\tau_{\mathbf{a}}^{(2)}(\cdot)$ to have $\tau_{\mathbf{a}}^{(2)}(gr_k^{-1}) > 0$ for sufficiently large k and show $gr_k^{-1}\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$.
- (iv) Apply the cocycle property of $\tau_{\mathbf{a}}^{(2)}(\cdot)$ to have $\tau_{\mathbf{a}}^{(2)}(g) > 0$, namely

$$\tau_{\mathbf{a}}^{(2)}(g) = \tau_{\mathbf{a}}^{(2)}(gr_k^{-1}r_k) = \tau_{\mathbf{a}}^{(2)}(gr_k^{-1})\tau_{gr_k^{-1}\mathbf{a}}^{(2)}(r_k) \exp(-E_{\mathbf{a}}(gr_k^{-1}, r_k)) > 0.$$

This programme will be realized in the next section. At the same time a close connection of the m -function with the Weyl function for Schrödinger operators will be revealed.

6.1 Non-degeneracy of Tau-functions for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$

To investigate properties of $m_{\mathbf{a}}$ and $\tau_{\mathbf{a}}^{(2)}$ for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ we prepare several lemmas. In this section the curve C is parametrized by

$$C = \{\pm\omega(y) + iy; y \in \mathbb{R}\}$$

with a smooth function $\omega(y) > 0$ satisfying $\omega(y) = \omega(-y)$ and $\omega(y) = O(y^{-(n-1)})$.

Lemma 20 Let $a(x), b(x)$ be real valued analytic functions on an interval $I \subset \mathbb{R}$ satisfying

$$\begin{cases} \frac{a(y)b(x) - a(x)b(y)}{a(x)^2 + b(x)^2} \geq 0 \\ \frac{x-y}{a(x)^2 + b(x)^2} > 0 \\ a(x) \geq 0 \end{cases} \quad \text{for any } x, y \in I$$

Then we have either $a(x) = 0$ identically or $a(x) > 0$ for any $x \in I$.

Proof. Since the first assumption implies

$$a(y)b(x) - a(x)b(y) \geq 0 \quad \text{for any } x > y,$$

if $a(x)$ has a zero at $x_0 \in I$, setting $x = x_0$ or $y = x_0$ we have

$$\begin{cases} a(y)b(x_0) \geq 0 \quad \text{for any } y < x_0 \\ a(x)b(x_0) \leq 0 \quad \text{for any } x > x_0 \end{cases}. \quad (69)$$

The second assumption implies $b(x_0) \neq 0$, hence (69) together with the property $a(x) \geq 0$ shows $a(x) = 0$ on $(-\infty, x_0) \cap I$ or $(x_0, \infty) \cap I$. Then the analyticity of a yields the vanishing of $a(x)$ on I . ■

In what follows $\tau_{\mathbf{a}}(r)$ for $r \in \Gamma_0^{(0)}$ will be used instead of $\tau_{\mathbf{a}}^{(2)}(r)$. Recall that $\tau_{\mathbf{a}}(r) \geq 0$ holds for any real $r \in \Gamma_0^{(0)}$ if $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$. In what follows we have to assume $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ with $L \geq 2$ since in the proofs we use $\varphi_{\mathbf{a}}, m_{\mathbf{a}}$. Let

$$p_{\zeta}(z) = 1 + \zeta^{-1}z, \quad q_{\zeta}(z) = (1 - \zeta^{-1}z)^{-1} \quad \text{for } \zeta \in D_-.$$

Lemma 21 Let $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ with $L \geq 2$. Then the followings are valid.

- (i) $\text{Im } m_{\mathbf{a}}(z) / \text{Im } z > 0$ on $D_- \setminus \mathbb{R}$ and $m_{\mathbf{a}}(z)$ is analytic on D_- .
- (ii) $1 + \varphi_{\mathbf{a}}(z) \neq 0$ on D_- .
- (iii) $\tau_{\mathbf{a}}(q_{x_1}p_{x_2}) = (1 + \varphi_{\mathbf{a}}(x_1))(1 + \varphi_{\mathbf{a}}(x_2)) \frac{m_{\mathbf{a}}(x_1) - m_{\mathbf{a}}(x_2)}{\Delta_{\mathbf{a}}(x_2)(x_1 - x_2)} > 0$ for any $x_1, x_2 \in D_- \cap \mathbb{R}$ if $x_1 \neq x_2$.

Proof. In the formula in Lemma 37 setting $\zeta_1 = \zeta, \zeta_2 = \bar{\zeta}, \zeta'_1 = \eta, \zeta'_2 = \bar{\eta} \in D_-$, we see that $q_{\zeta}q_{\bar{\zeta}}p_{\eta}p_{\bar{\eta}}$ is a real element of $\Gamma_0^{(0)}$, hence $\tau_{\mathbf{a}}(q_{\zeta}q_{\bar{\zeta}}p_{\eta}p_{\bar{\eta}}) \geq 0$ if $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$. Lemma 37 implies that

$$\lim_{\eta \rightarrow \infty} \tau_{\mathbf{a}}(q_{\zeta}q_{\bar{\zeta}}p_{\eta}p_{\bar{\eta}}) = \tau_{\mathbf{a}}(q_{\zeta}q_{\bar{\zeta}}),$$

hence we have $\tau_{\mathbf{a}}(q_{\zeta}q_{\bar{\zeta}}) \geq 0$ for all $\zeta \in D_-$. On the other hand from Lemma 37 we have

$$\tau_{\mathbf{a}}(q_{\zeta}q_{\bar{\zeta}}) = |\varphi_{\mathbf{a}}(\zeta) + 1|^2 \frac{\text{Im } m_{\mathbf{a}}(\zeta)}{\text{Im } \zeta},$$

which implies $\text{Im } m_{\mathbf{a}}(\zeta) / \text{Im } \zeta \geq 0$ for $\zeta \in \{\zeta \in D_-; \varphi_{\mathbf{a}}(\zeta) + 1 \neq 0\} \equiv \mathcal{Z}_{\varphi}$. Note $\varphi_{\mathbf{a}}(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$, hence the set \mathcal{Z}_{φ} is discrete. Suppose $m_{\mathbf{a}}(\zeta_0) = 0$ for some $\zeta_0 \in \mathcal{Z}_{\varphi} \cap \mathbb{C}_+$. Then the maximum principle for the harmonic function $-\text{Im } m_{\mathbf{a}}(\zeta)$ shows $\text{Im } m_{\mathbf{a}}(\zeta) = 0$ identically there, which contradicts $m_{\mathbf{a}}(\zeta) = \zeta + O(\zeta^{-1})$ as $\zeta \rightarrow \infty$, hence > 0 on $\mathcal{Z}_{\varphi} \cap \mathbb{C}_+$. One has the same property also

in \mathbb{C}_- . On the other hand $m_{\mathbf{a}}(\zeta)$ has poles at $\zeta_0 \in D_- \setminus \mathcal{Z}_\varphi$ since $\varphi_{\mathbf{a}}(\zeta) + 1$ and $\psi_{\mathbf{a}}(\zeta) + \zeta$ do not vanish simultaneously due to $\Delta_{\mathbf{a}}(\zeta) \neq 0$. However this is impossible if we apply the same argument to $m_{\mathbf{a}}(\zeta)^{-1}$, which implies

$$m_{\mathbf{a}}(z) \text{ is analytic and } \frac{\operatorname{Im} m_{\mathbf{a}}(z)}{\operatorname{Im} z} > 0 \text{ holds on } D_- \setminus \mathbb{R}.$$

This in particular means $1 + \varphi_{\mathbf{a}}(z) \neq 0$ on $D_- \setminus \mathbb{R}$. The remaining problem is the existence or non-existence of poles of $m_{\mathbf{a}}(z)$ on $D_- \cap \mathbb{R}$. We rewrite the identity in Lemma 37 as

$$\tau_{\mathbf{a}}(q_{\zeta_1} p_{\zeta_2}) = \frac{(\psi_{\mathbf{a}}(\zeta_2) + \zeta_2)(\varphi_{\mathbf{a}}(\zeta_1) + 1) - (\varphi_{\mathbf{a}}(\zeta_2) + 1)(\psi_{\mathbf{a}}(\zeta_1) + \zeta_1)}{\Delta_{\mathbf{a}}(\zeta_2)(\zeta_1 - \zeta_2)},$$

which is valid for any $\zeta_1, \zeta_2 \in D_-$. Set $\zeta_1 = x_1, \zeta_2 = x_2 \in D_- \cap \mathbb{R}$ and

$$a(x) = \varphi_{\mathbf{a}}(x) + 1, \quad b(x) = \psi_{\mathbf{a}}(x) + x.$$

Then

$$\tau_{\mathbf{a}}(q_{x_1} p_{x_2}) = \frac{a(x_1)b(x_2) - a(x_2)b(x_1)}{\Delta_{\mathbf{a}}(x_2)(x_1 - x_2)}$$

holds, and the property $\Delta_{\mathbf{a}}(x) \neq 0, \Delta_{\mathbf{a}}(x) \rightarrow -1$ as $|x| \rightarrow \infty$ implies $\Delta_{\mathbf{a}}(x) < 0$ on $D_- \cap \mathbb{R}$, which together with $\tau_{\mathbf{a}}(q_{x_1} p_{x_2}) \geq 0$ (due to $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$) shows

$$\frac{a(x_1)b(x_2) - a(x_2)b(x_1)}{x_1 - x_2} \leq 0.$$

Moreover, in the inequality $\tau_{\mathbf{a}}(q_{x_1} p_{x_2}) \geq 0$ letting $x_2 \rightarrow \infty$, we have $\tau_{\mathbf{a}}(q_{x_1} p_{x_2}) \rightarrow \tau_{\mathbf{a}}(q_{x_1})$ just by the same argument as above and

$$a(x_1) = \varphi_{\mathbf{a}}(x_1) + 1 = \tau_{\mathbf{a}}(q_{x_1}) \geq 0$$

is valid. The condition $a(x)^2 + b(x)^2 > 0$ follows from $\Delta_{\mathbf{a}}(x) \neq 0$, hence one can apply Lemma 20 to have $a(x) > 0$, since $a(x) \rightarrow 1$ as $|x| \rightarrow \infty$. We have shown (i) and (ii). To show (iii) first assume $x_1 \neq x_2$ and suppose $\tau_{\mathbf{a}}(q_{x_1} p_{x_2}) = 0$. Then

$$0 = \frac{a(x_1)b(x_2) - a(x_2)b(x_1)}{x_1 - x_2} = a(x_1)a(x_2) \frac{c(x_2) - c(x_1)}{x_1 - x_2}$$

with $c(x) = b(x)/a(x)$. Observing $c(x)$ is analytic and $c'(x) \geq 0$, this identity implies $c(x)$ is identically constant, which contradicts $c(x) = x + o(1)$ as $|x| \rightarrow \infty$, hence we have $\tau_{\mathbf{a}}(q_{x_1} p_{x_2}) > 0$ if $x_1 \neq x_2$. ■

Lemma 22 *Let $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ with $L \geq 2$ and $r \in \Gamma_0^{(0)}$ be real. Then $\tau_{\mathbf{a}}(r) > 0$ holds, which in particular means $r\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$.*

Proof. First note that for $r_1, r_2 \in \Gamma_0^{(0)}$ and $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}$

$$\tau_{\mathbf{a}}(r_1) > 0, \tau_{\mathbf{a}}(r_2) > 0 \implies \tau_{\mathbf{a}}(r_1 r_2) > 0.$$

This is because $r_1 \mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ since the cocycle property implies

$$\tau_{r_1 \mathbf{a}}(r) = \frac{\tau_{\mathbf{a}}(r r_1)}{\tau_{\mathbf{a}}(r_1)} \geq 0 \text{ for any real } r \in \Gamma_0^{(0)},$$

and

$$\tau_{\mathbf{a}}(r_1 r_2) = \tau_{\mathbf{a}}(r_1) \tau_{r_1 \mathbf{a}}(r_2) > 0.$$

Generally real $r \in \Gamma_0^{(0)}$ is a product of

$$\begin{cases} (1) q_{\zeta} q_{\bar{\zeta}} p_{\eta} p_{\bar{\eta}} \text{ with } \eta, \zeta \in D_- \setminus \mathbb{R} \\ (2) q_s p_t \text{ with } s, t \in D_- \cap \mathbb{R} \\ (3) q_{\zeta} q_{\bar{\zeta}} p_s p_t \text{ with } \zeta \in D_- \setminus \mathbb{R}, s, t \in D_- \cap \mathbb{R} \\ (4) q_s q_t p_{\eta} p_{\bar{\eta}} \text{ with } \eta \in D_- \setminus \mathbb{R}, s, t \in D_- \cap \mathbb{R} \end{cases} \quad (70)$$

Therefore, if $\tau_{\mathbf{a}}(r) > 0$ is proved for any r of these 4 cases, we have $\tau_{\mathbf{a}}(r) > 0$ for any real $r \in \Gamma_0^{(0)}$.

We begin from the case (1) and let $r(z) = (q_{\zeta} q_{\bar{\zeta}} p_{\eta} p_{\bar{\eta}})(z)$. Then (113) of Lemma 37 implies

$$\begin{aligned} \tau_{\mathbf{a}}(r) &= \frac{|\eta + \zeta|^2 |\eta + \bar{\zeta}|^2 |\varphi_{\mathbf{a}}(\eta) + 1|^2 |\varphi_{\mathbf{a}}(\zeta) + 1|^2}{(4 \operatorname{Im} \eta \operatorname{Im} \zeta) |\Delta_{\mathbf{a}}(\eta)|^2} \\ &\quad \times \left(\left| \frac{m_{\mathbf{a}}(\eta) - \overline{m_{\mathbf{a}}(\zeta)}}{\eta^2 - \bar{\zeta}^2} \right|^2 - \left| \frac{m_{\mathbf{a}}(\eta) - m_{\mathbf{a}}(\zeta)}{\eta^2 - \zeta^2} \right|^2 \right) \end{aligned} \quad (71)$$

due to realness of \mathbf{a} . Note $\varphi_{\mathbf{a}}(z) + 1 \neq 0$ for any $z \in D_-$ due to Lemma 21. Owing to the symmetry of r with respect to ζ, η , one can assume $\operatorname{Im} \zeta > 0$, $\operatorname{Im} \eta > 0$. The condition $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ implies $\tau_{\mathbf{a}}(q_{\zeta} q_{\bar{\zeta}} p_{\eta} p_{\bar{\eta}}) \geq 0$ for any $\eta, \zeta \in D_- \cap \mathbb{C}_+$. Assume $\tau_{\mathbf{a}}(q_{\zeta_0} q_{\bar{\zeta}_0} p_{\eta_0} p_{\bar{\eta}_0}) = 0$ for some $\eta_0, \zeta_0 \in D_-$ and consider the analytic function

$$f(z) = \frac{m_{\mathbf{a}}(z) - m_{\mathbf{a}}(\zeta_0)}{m_{\mathbf{a}}(z) - \overline{m_{\mathbf{a}}(\zeta_0)}} \frac{z^2 - \bar{\zeta}_0^2}{z^2 - \zeta_0^2}.$$

The property $\tau_{\mathbf{a}}(q_{\zeta_0} q_{\bar{\zeta}_0} p_z p_{\bar{z}}) \geq 0$ shows

$$|f(z)| \leq 1 \quad \text{for any } z \in D_- \cap \mathbb{C}_+, \quad (72)$$

and the assumption implies the equality at (72) for $z = \eta_0$, which concludes $f(z) = e^{i\alpha}$ with $\alpha \in \mathbb{R}$ identically on $D_- \cap \mathbb{C}_+$. Then

$$m_{\mathbf{a}}(z) = \frac{(m_{\mathbf{a}}(\zeta_0) - e^{i\alpha} \overline{m_{\mathbf{a}}(\zeta_0)}) z^2 + e^{i\alpha} \overline{m_{\mathbf{a}}(\zeta_0)} \zeta_0^2 - m_{\mathbf{a}}(\zeta_0) \bar{\zeta}_0^2}{(1 - e^{i\alpha}) z^2 + e^{i\alpha} \zeta_0^2 - \bar{\zeta}_0^2}$$

holds, which contradicts $m_{\mathbf{a}}(z) = z + o(1)$ as $z \rightarrow \infty$. Therefore we have $|f(z)| < 1$ always, which is nothing but $\tau_{\mathbf{a}}(r) > 0$.

The case (2) is already proved in Lemma 20 if $s \neq t$. Suppose $s = t$ and let $s_n \in D_- \cap \mathbb{R}$ be a sequence converging to s . Then, $\tau_{\mathbf{a}}(q_{s_n} p_s) > 0$ is valid due to $s_n \neq s$ for any $n \geq 1$. Moreover one can show easily $\tau_{\mathbf{a}}(q_s q_{s_n}^{-1}) \rightarrow 1$ as $n \rightarrow \infty$. Taking sufficiently large n such that $\tau_{\mathbf{a}}(q_s q_{s_n}^{-1}) > 0$ and fixing it we see from the cocycle property

$$\tau_{\mathbf{a}}(q_s p_s) = \tau_{\mathbf{a}}(r_n(q_{s_n} p_s)) = \tau_{\mathbf{a}}(r_n) \tau_{r_n \mathbf{a}}(q_{s_n} p_s) > 0,$$

where $r_n = q_s q_{s_n}^{-1}$, since $r_n \mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ due to $\tau_{\mathbf{a}}(r_n) > 0$.

Similarly one can show $\tau_{\mathbf{a}}(q_\zeta q_{\bar{\zeta}} p_s p_t) > 0$ as a limiting case of (1). Setting $r_1 = q_\zeta q_{\bar{\zeta}} p_s p_t$ we have

$$\tau_{\mathbf{a}}(q_\zeta q_{\bar{\zeta}} p_s p_t) = \tau_{\mathbf{a}}(r_1 q_{-s} p_t) = \tau_{\mathbf{a}}(r_1) \tau_{r_1 \mathbf{a}}(q_{-s} p_t) > 0,$$

which shows the case (3). The case (4) can be shown similarly. ■

The next task is to approximate general $g \in \Gamma_n$ by rational functions.

Lemma 23 *Let $g \in \Gamma_n$. Then there exists a sequence of rational functions $\{r_k\}_{k \geq 1} \subset \Gamma_0^{(0)}$ such that $r_k \rightarrow g$ in the sense of (67).*

Proof. Let U be a neighborhood of \overline{D}_+ whose boundary is described by an equation $|x| = c|y|^{-(n-1)}$ for large $|y|$ with sufficiently large $c > 0$. For integer $k \geq 1$ let

$$\phi_k(z) = \left(\frac{1 + \frac{z}{2k}}{1 - \frac{z}{2k}} \right)^k.$$

Note $\lim_{k \rightarrow \infty} \phi_k(z) = e^z$. For a positive constant $a \leq k$ an inequality

$$e^{-2a} \leq |\phi_k(z)| \leq e^{2a} \quad \text{if } |\operatorname{Re} z| \leq a \quad (73)$$

holds. If $h(z) = c_1 z^n + \text{lower degree terms}$, then

$$c_2 \equiv \sup_{z \in U} |\operatorname{Re} h(z)| < \infty \quad (74)$$

is valid. Define real rational functions by

$$r_k(z) = \phi_k(h(z)).$$

The zeros and poles of $r_k(z)$ are determined by the equation $h(z) = \pm 2k$. If a is chosen so that $a > c_2$, then clearly there exist a constant $c_3 > 1$ such that

$$c_3^{-1} \leq |r_k(z)| \leq c_3 \quad \text{for } z \in U \text{ and } k \geq 1$$

holds. Moreover

$$|r'_k(z)| = |h'(z)| |\phi'_k(h(z))| = |h'(z)| \left| \frac{1 + \frac{h(z)}{2k}}{1 - \frac{h(z)}{2k}} \right|^{k-1} \left| 1 - \frac{h(z)}{2k} \right|^{-2}$$

shows

$$|r'_k(z)| \leq c_4 |z|^{n-1} \quad \text{for } z \in U \text{ and } k \geq 1.$$

Since $\lim_{k \rightarrow \infty} r_k(z) = e^{h(z)} = g(z)$, all conditions of Lemma 19 are satisfied. ■

Now we have

Proposition 24 *Let $g \in \Gamma_n^{(0)}$ be real and $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ with $L \geq \max\{n, 2\}$. Then, $\tau_{\mathbf{a}}^{(2)}(g) > 0$ holds, hence $g\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ is valid.*

Proof. $L \geq \max\{n, 1\}$ is necessary for the definition of $\tau_{\mathbf{a}}^{(2)}(g)$ for $g \in \Gamma_n^{(0)}$ and $L \geq 2$ is required to apply Lemma 22. First note that if $\tau_{\mathbf{a}}^{(2)}(g) > 0$ holds, then $g\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$. To show this let $\{r_k\}_{k \geq 1}$ be the sequence of Lemma 23 approximating g and r be any real rational function of $\Gamma_0^{(0)}$. Then $r_k \rightarrow g$ implies

$$\tau_{\mathbf{a}}^{(2)}(gr) = \lim_{k \rightarrow \infty} \tau_{\mathbf{a}}^{(2)}(r_k r) \geq 0$$

and

$$\tau_{g\mathbf{a}}^{(2)}(r) = \frac{\tau_{\mathbf{a}}^{(2)}(gr)}{\tau_{\mathbf{a}}^{(2)}(g)} \exp(E_{\mathbf{a}}(g, r)) \geq 0,$$

which means $g\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$.

Now $\{g_k = gr_k^{-1}\}_{k \geq 1}$ also satisfies the conditions of (67) with $g = 1$, hence Lemma 19 shows

$$\lim_{k \rightarrow \infty} \tau_{\mathbf{a}}^{(2)}(gr_k^{-1}) = \tau_{\mathbf{a}}^{(2)}(1) = 1.$$

Fix a sufficiently large $k \geq 1$ such that $\tau_{\mathbf{a}}^{(2)}(gr_k^{-1}) > 0$. Then the above argument shows $gr_k^{-1}\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$. Applying Lemma 22 to $gr_k^{-1}\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ and the rational function r_k we have $\tau_{gr_k^{-1}\mathbf{a}}^{(2)}(r_k) > 0$. The cocycle property of tau-functions implies

$$\tau_{\mathbf{a}}^{(2)}(g) = \tau_{\mathbf{a}}^{(2)}(gr_k^{-1}r_k) = \tau_{\mathbf{a}}^{(2)}(gr_k^{-1}) \tau_{gr_k^{-1}\mathbf{a}}^{(2)}(r_k) \exp(-E_{\mathbf{a}}(gr_k^{-1}, r_k)) > 0.$$

If $g = re^h$ with real $r \in \Gamma_0^{(0)}$, then

$$\tau_{\mathbf{a}}^{(2)}(g) = \tau_{\mathbf{a}}^{(2)}(r) \tau_{r\mathbf{a}}^{(2)}(e^h) \exp(-E_{\mathbf{a}}(r, e^h)) > 0,$$

which completes the proof. ■

6.2 m -function and Weyl function

Since for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ we know $e_x\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ ($e_x(z) = e^{xz}$) from Theorem 1, Proposition 11 shows that we have a Schrödinger equation

$$-\partial_x^2 f_{\mathbf{a}}(x, z) + q(x) f_{\mathbf{a}}(x, z) = -z^2 f_{\mathbf{a}}(x, z).$$

Since q is real valued, the Schrödinger operator

$$L_q = -\partial_x^2 + q$$

turns to be a symmetric operator on $L^2(\mathbb{R})$, one can consider the Weyl's classification of the boundaries $\pm\infty$ and the Weyl functions m_{\pm} if the boundaries are of limit point type. In this section we establish the connection between the m -function and the spectral theory of L_q initiated by Weyl. Necessary knowledge for this section will be obtained in [13].

Lemma 25 (*Boundary classification*) *For any $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\dim \{f \in L^2(\mathbb{R}_+); L_q f = z f\}$$

is independent of z . According to 1 or 2 of the dimension the boundary $+\infty$ is called limit point type or limit circle type respectively. It is also valid that if $+\infty$ is of limit circle type, then

$$\dim \{f \in L^2(\mathbb{R}_+); L_q f = z f\} = 2$$

for any $z \in \mathbb{C}$.

If the boundary $+\infty$ is of limit point type, then there exists a non-trivial solution $f_+(x, z)$ to $L_q f_+ = z f_+$ which is in $L^2(\mathbb{R}_+)$. f_+ is unique up to constant multiple. The Weyl function is defined by

$$m_+(z) = \frac{f'_+(0, z)}{f_+(0, z)}$$

The boundary $-\infty$ also has the same classification, and if it is of limit point type, the Weyl function at $-\infty$ is defined by

$$m_-(z) = -\frac{f'_-(0, z)}{f_-(0, z)}$$

with the $L^2(\mathbb{R}_-)$ non-trivial solution $f_-(x, z)$.

A general sufficient condition for the limit point type which is suitable for our purpose is known by [8]. A proof is given for completeness sake.

Lemma 26 *If there exists a positive solution f on \mathbb{R}_+ to $L_q f = \lambda_0 f$ for some $\lambda_0 \in \mathbb{R}$, then the boundary $+\infty$ is of limit point type.*

Proof. Define

$$u(x) = f(x) \int_0^x f(y)^{-2} dy.$$

Then u satisfies $L_q u = \lambda_0 u$. We show $u \notin L^2(\mathbb{R}_+)$, which implies that $+\infty$ is of limit point type due to Lemma 25. For this purpose set

$$s(x) = \int_0^x f(y)^{-2} dy \quad \text{and} \quad t(x) = (-s(x)^{-1})'.$$

Then $t(x) > 0$ and

$$-s(x)^{-1} = c + \int_0^x t(y) dy$$

with some constant c . Since $s(x)^{-1} > 0$, we have

$$\int_0^x t(y) dy < -c,$$

which implies

$$\int_0^\infty t(y) dy \leq -c < \infty. \tag{75}$$

On the other hand

$$\int_0^\infty u(x)^2 dx = \int_0^\infty s'(x)^{-1} s(x)^2 dx = \int_0^\infty \frac{dx}{(-s(x)^{-1})'} = \int_0^\infty \frac{dx}{t(x)}$$

holds. Since for any $t(x) > 0$

$$\int_0^\infty t(y) dy + \int_0^\infty t(y)^{-1} dy = \int_0^\infty (t(y) + t(y)^{-1}) dy \geq \int_0^\infty 2 dy = \infty,$$

(75) shows

$$\int_0^\infty u(x)^2 dx = \int_0^\infty \frac{dx}{t(x)} = \infty,$$

which completes the proof. ■

Suppose the boundaries $\pm\infty$ are of limit point type. Then, it is known that the symmetric operator L_q has a unique self-adjoint extension in $L^2(\mathbb{R})$ and $m_\pm(z)$ are analytic functions on $\mathbb{C} \setminus \text{sp}L_q$ satisfying $\text{Im } m_\pm(z) / \text{Im } z > 0$.

Now we consider relationship between $m_\mathbf{a}(z)$ and $m_\pm(z)$. The definition of $f_\mathbf{a}(x, z)$

$$f_\mathbf{a}(x, z) = \mathbf{a}(z) \left(T(e_x \mathbf{a})^{-1} \mathbf{1} \right) (z)$$

implies

$$f_{e_y \mathbf{a}}(x, z) = e_y(z) \mathbf{a}(z) \left(T(e_x e_y \mathbf{a})^{-1} \mathbf{1} \right) (z) = e_y(z) f_\mathbf{a}(x + y, z).$$

Therefore, Corollary 13 shows

$$-m_{e_y \mathbf{a}}(z) = \frac{\partial_x f_{e_y \mathbf{a}}(x, z)}{f_{e_y \mathbf{a}}(x, z)} \Big|_{x=0} = \frac{\partial_x f_\mathbf{a}(x + y, z)}{f_\mathbf{a}(x + y, z)} \Big|_{x=0} = \frac{\partial_y f_\mathbf{a}(y, z)}{f_\mathbf{a}(y, z)}, \quad (76)$$

hence it holds that

$$f_\mathbf{a}(x, z) = f_\mathbf{a}(0, z) \exp \left(- \int_0^x m_{e_y \mathbf{a}}(z) dy \right). \quad (77)$$

Proposition 27 Let $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}$ with $L \geq 3$ and q be the associated potential. Then the boundaries $\pm\infty$ are of limit point type for the Schrödinger operator L_q . The m -function $m_\mathbf{a}$ and the Weyl functions m_\pm are connected with $m_\mathbf{a}$ by

$$m_\mathbf{a}(z) = \begin{cases} -m_+(-z^2) & \text{if } \text{Re } z > 0 \\ m_-(-z^2) & \text{if } \text{Re } z < 0 \end{cases}. \quad (78)$$

Proof. The key ingredient for the proof is (76), which shows

$$\partial_x m_{e_x \mathbf{a}}(z) = -z^2 - q(x) + m_{e_x \mathbf{a}}(z)^2, \quad (79)$$

since $f_\mathbf{a}(x, z)$ satisfies $L_q f_\mathbf{a}(x, z) = -z^2 f_\mathbf{a}(x, z)$. (79) implies

$$\partial_x \text{Im } m_{e_x \mathbf{a}}(z) = -\text{Im } z^2 + 2 \text{Re } m_{e_x \mathbf{a}}(z) \text{Im } m_{e_x \mathbf{a}}(z),$$

which together with (77) yields

$$\begin{aligned} |f_\mathbf{a}(x, z)|^2 &= |f_\mathbf{a}(0, z)|^2 \exp \left(-2 \int_0^x \text{Re } m_{e_y \mathbf{a}}(z) dy \right) \\ &= |f_\mathbf{a}(0, z)|^2 \frac{\text{Im } m_\mathbf{a}(z)}{\text{Im } m_{e_x \mathbf{a}}(z)} \exp \left(- \int_0^x \frac{\text{Im } z^2}{\text{Im } m_{e_y \mathbf{a}}(z)} dy \right). \end{aligned}$$

Then an identity

$$\int_0^b |f_{\mathbf{a}}(x, z)|^2 dx = |f_{\mathbf{a}}(0, z)|^2 \frac{\operatorname{Im} m_{\mathbf{a}}(z)}{\operatorname{Im} z^2} \left(1 - \exp \left(- \int_0^b \frac{\operatorname{Im} z^2}{\operatorname{Im} m_{\epsilon_y \mathbf{a}}(z)} dy \right) \right)$$

follows. Since $\operatorname{Im} z^2 = 2 \operatorname{Re} z \operatorname{Im} z$ and $\operatorname{Im} m_{\mathbf{a}}(z) / \operatorname{Im} z > 0$ hold due to (i) of Lemma 21, if $\operatorname{Re} z > 0$, we have

$$\int_0^\infty |f_{\mathbf{a}}(x, z)|^2 dx \leq |f_{\mathbf{a}}(0, z)|^2 \frac{\operatorname{Im} m_{\mathbf{a}}(z)}{\operatorname{Im} z^2} < \infty. \quad (80)$$

On the other hand, if $z = \lambda \in D_- \cap \mathbb{R}$, then (77) implies $f_{\mathbf{a}}(x, \lambda) / f_{\mathbf{a}}(0, \lambda)$ is a positive solution to $L_q f = -\lambda^2 f$, hence the boundary $+\infty$ is of limit point type owing to Lemma 26. Since $f_{\mathbf{a}}(x, z) \in L^2(\mathbb{R}_+)$ if $\operatorname{Re} z > 0$ and $\operatorname{Im} z \neq 0$, the uniqueness of such a solution justifies $f_{\mathbf{a}}(x, z) = f_+(x, -z^2)$, which shows the identity $m_{\mathbf{a}}(z) = -m_+(-z^2)$ if $\operatorname{Re} z > 0$. The boundary $-\infty$ can be treated similarly, and we obtain $m_{\mathbf{a}}(z) = m_-(-z^2)$ if $\operatorname{Re} z < 0$, which completes the proof. ■

This Proposition says that for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}$ its m -function $m_{\mathbf{a}}(z)$ is analytically continuable up to $\mathbb{C} \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R})$ ($\mu_0 = \sqrt{-\lambda_0}$) although originally we knew its analyticity only on D_- .

The next issue is to show the converse statement of Proposition 27. This proposition and Lemma 21 implies that $m = m_{\mathbf{a}}$ for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ satisfies

$$\begin{cases} \frac{\operatorname{Im} m(z)}{\operatorname{Im} z} > 0 & \text{on } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \\ \frac{m(x) - m(-x)}{x} > 0 & \text{if } x \in \mathbb{R} \text{ and } |x| > \mu_0 \end{cases}. \quad (81)$$

It should be remarked that the analyticity of m on D_- implies $1 + \varphi_{\mathbf{a}}(z) \neq 0$ on D_- since $1 + \varphi_{\mathbf{a}}(z)$ and $z + \psi_{\mathbf{a}}(z)$ do not vanish simultaneously due to $\Delta_{\mathbf{a}}(z) \neq 0$. In the process of the proof we need an operation

$$(d_{\zeta} f)(z) = \frac{z^2 - \zeta^2}{f(z) - f(\zeta)} - f(\zeta)$$

for a function f on \mathbb{C} as long as they have meaning. Then $\{d_{\zeta}\}_{\zeta \in D_-}$ is commutative and $d_{\zeta} d_{-\zeta} = id$.

Proposition 28 *Let $L \geq 2$. For $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ suppose that $m_{\mathbf{a}}$ is analytic on $\mathbb{C} \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R})$ and satisfies (81). Then, $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ holds.*

Proof. We have to show $\tau_{\mathbf{a}}(r) \geq 0$ for any real rational function $r \in \Gamma_0^{(0)}$. Since such r is a product of the 4 types of rational functions of (70), first we prove $\tau_{\mathbf{a}}(r) \geq 0$ for r of (70). If r is of the type (1) with $\eta, \zeta \in D_- \cap \mathbb{C}_+$, then $\tau_{\mathbf{a}}(r)$ is given by (71) and $\tau_{\mathbf{a}}(r) > 0$ is equivalent to $|f(z)| < 1$ on $D_- \cap \mathbb{C}_+$ with

$$f(z) = \frac{m_{\mathbf{a}}(z) - m_{\mathbf{a}}(\zeta)}{m_{\mathbf{a}}(z) - \overline{m_{\mathbf{a}}(\zeta)}} \frac{z^2 - \overline{\zeta}^2}{z^2 - \zeta^2}.$$

Set $w = \zeta^2$. Then

$$f(\sqrt{z}) = \frac{m_{\mathbf{a}}(\sqrt{z}) - m_{\mathbf{a}}(\sqrt{w})}{m_{\mathbf{a}}(\sqrt{z}) - m_{\mathbf{a}}(\sqrt{w})} \frac{z - \bar{w}}{z - w}$$

holds. Since $m_{\mathbf{a}}(\sqrt{z})$ is analytic on \mathbb{C}_+ and $\text{Im } m_{\mathbf{a}}(\sqrt{z}) > 0$, Schwarz lemma implies $|f(\sqrt{z})| < 1$ for $z, \zeta \in \mathbb{C}_+$, unless $m_{\mathbf{a}}(\sqrt{z})$ is

$$m_{\mathbf{a}}(\sqrt{z}) = \frac{az + b}{cz + d}$$

with some constants a, b, c, d satisfying $ad - bc \neq 0$, which is impossible since $m_{\mathbf{a}}(\sqrt{z}) = \sqrt{z} + o(1)$ as $z \rightarrow \infty$. Therefore we have $|f(z)| < 1$ if $\text{Re } z, \text{Im } z > 0$, $\text{Re } \zeta, \text{Im } \zeta > 0$. On the other hand, when $z \in \mathbb{C}_-$ we use the identity

$$f(-\sqrt{z}) = \frac{m_{\mathbf{a}}(-\sqrt{z}) - m_{\mathbf{a}}(\sqrt{w})}{m_{\mathbf{a}}(-\sqrt{z}) - m_{\mathbf{a}}(\sqrt{w})} \frac{z - \bar{w}}{z - w}.$$

If $\text{Re } \zeta > 0, \text{Im } \zeta > 0$, then $\text{Im } w > 0$ implies

$$\left| \frac{z - \bar{w}}{z - w} \right| < 1$$

and

$$\text{Im } m_{\mathbf{a}}(-\sqrt{z}) > 0, \quad \text{Im } m_{\mathbf{a}}(\sqrt{w}) > 0$$

due to $\text{Im } \sqrt{z} < 0, \text{Im } \sqrt{w} > 0$, hence

$$\left| \frac{m_{\mathbf{a}}(-\sqrt{z}) - m_{\mathbf{a}}(\sqrt{w})}{m_{\mathbf{a}}(-\sqrt{z}) - m_{\mathbf{a}}(\sqrt{w})} \frac{z - \bar{w}}{z - w} \right| < \left| \frac{m_{\mathbf{a}}(-\sqrt{z}) - m_{\mathbf{a}}(\sqrt{w})}{m_{\mathbf{a}}(-\sqrt{z}) - m_{\mathbf{a}}(\sqrt{w})} \right| < 1,$$

which implies $|f(z)| < 1$ if $\text{Re } z < 0, \text{Im } z > 0, \text{Re } \zeta, \text{Im } \zeta > 0$. The rest of the cases can be proved by the symmetry and we have $\tau_{\mathbf{a}}(r) > 0$.

For the type (2) $r = q_s p_t$

$$\tau_{\mathbf{a}}(r) = \frac{(1 + \varphi_{\mathbf{a}}(s))(1 + \varphi_{\mathbf{a}}(-t))}{\Delta_{\mathbf{a}}(t)} \frac{m_{\mathbf{a}}(s) - m_{\mathbf{a}}(t)}{s - t}.$$

Recall

$$\frac{(1 + \varphi_{\mathbf{a}}(s))(1 + \varphi_{\mathbf{a}}(-t))}{\Delta_{\mathbf{a}}(t)} > 0 \quad \text{if } |s|, |t| > \mu_0.$$

On the other hand the property $\text{Im } m_{\mathbf{a}}(z) / \text{Im } z > 0$ implies

$$m'_{\mathbf{a}}(t) = \lim_{\epsilon \rightarrow 0} \frac{m_{\mathbf{a}}(t + i\epsilon) - m_{\mathbf{a}}(t - i\epsilon)}{2i\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{\text{Im } m_{\mathbf{a}}(t + i\epsilon)}{\text{Im}(t + i\epsilon)} \geq 0,$$

which shows

$$\frac{m_{\mathbf{a}}(s) - m_{\mathbf{a}}(t)}{s - t} \geq 0 \quad \text{if } s, t \in (-\infty, -\mu_0) \text{ or } (\mu_0, \infty).$$

If $s \in (-\infty, -\mu_0)$ and $t \in (\mu_0, \infty)$, then from $(m(x) - m(-x))/x > 0$ an inequality

$$m_{\mathbf{a}}(s) - m_{\mathbf{a}}(t) \leq 0$$

follows, hence we have $\tau_{\mathbf{a}}(r) \geq 0$ for the case (2). If $r = q_{\zeta} q_{\bar{\zeta}} p_s p_t$, the cocycle property implies

$$\tau_{\mathbf{a}}(r) = \tau_{\mathbf{a}}\left(q_{\zeta} p_{\bar{\zeta}} p_{\bar{\zeta}}^{-1} q_{\bar{\zeta}} p_s p_t\right) = \tau_{\mathbf{a}}\left(q_{\zeta} p_{\bar{\zeta}}\right) \tau_{q_{\zeta} p_{\bar{\zeta}} \mathbf{a}}\left(p_{\bar{\zeta}}^{-1} q_{\bar{\zeta}} p_s p_t\right).$$

Since the m -function for $q_{\zeta} p_{\bar{\zeta}} \mathbf{a}$ is $d_{\zeta} d_{\bar{\zeta}} m_{\mathbf{a}}$, which satisfies (81) due to Lemma 39, we have $\tau_{q_{\zeta} p_{\bar{\zeta}} \mathbf{a}}(p_s p_t) \neq 0$ in view of the last argument, and

$$\tau_{q_{\zeta} p_{\bar{\zeta}} \mathbf{a}}\left(p_{\bar{\zeta}}^{-1} q_{\bar{\zeta}} p_s p_t\right) = \tau_{q_{\zeta} p_{\bar{\zeta}} \mathbf{a}}\left(p_{\bar{\zeta}}^{-1} q_{\bar{\zeta}}\right) \tau_{q_{\zeta} p_{\bar{\zeta}} \mathbf{a}}(p_s p_t) = \Delta_{q_{\zeta} p_{\bar{\zeta}} \mathbf{a}}(\bar{\zeta}) \tau_{q_{\zeta} p_{\bar{\zeta}} \mathbf{a}}(p_s p_t) \neq 0$$

is valid. Therefore we have $\tau_{\mathbf{a}}(r) > 0$. The case (4) $r = q_s q_t p_{\eta} p_{\bar{\eta}}$ can be treated similarly, hence $\tau_{\mathbf{a}}(r) > 0$ for r of any type of (70).

The property $\tau_{\mathbf{a}}(r) \geq 0$ for general real $r \in \Gamma_0^{(0)}$ can be obtained by observing

$$\tau_{\mathbf{a}}(r_1 r_2) = \tau_{\mathbf{a}}(r_1) \tau_{r_1 \mathbf{a}}(r_2) \geq 0$$

if $\tau_{\mathbf{a}}(r_1) > 0$ and the m -function $m_{r_1 \mathbf{a}}$ satisfies (81) since $m_{r_1 \mathbf{a}}$ is obtained by repeating the operation $d_{\zeta} d_{\bar{\zeta}}$, d_t to $m_{\mathbf{a}}$. If $\tau_{\mathbf{a}}(r_1) = 0$, approximating r_1 by real rational functions r_n such that $\tau_{\mathbf{a}}(r_n) > 0$ one sees $\tau_{\mathbf{a}}(r_1 r_2) \geq 0$, which completes the proof. ■

It is certainly better to prove $\tau_{\mathbf{a}}(r) > 0$ directly by showing

$$\det\left(\frac{m_{\mathbf{a}}(\zeta_i) - m_{\mathbf{a}}(-\eta_j)}{\zeta_i^2 - \eta_j^2}\right) \neq 0 \quad (\text{see Lemma 37})$$

for $m_{\mathbf{a}}$ satisfying (81), however the author has no such proof.

6.3 Proof of Theorem 1

A more concrete criterion for an m to be in $\mathbf{A}_{L,+}^{inv}$ is given by the two conditions (M.1), (M.2) in Introduction. Recall the definitions. Suppose an analytic function m on $\mathbb{C} \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R})$ ($\mu_0 = \sqrt{-\lambda_0}$) satisfies

(M.1) $m(z)$ has the property (81), namely

$$\begin{cases} \frac{\operatorname{Im} m(z)}{\operatorname{Im} z} > 0 & \text{on } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \\ \frac{m(x) - m(-x)}{x} > 0 & \text{if } x \in \mathbb{R} \text{ and } |x| > \mu_0 \end{cases}. \quad (82)$$

(M.2) m has an asymptotic behavior:

$$m(z) = z + \sum_{1 \leq k \leq L-2} m_k z^{-k} + O(z^{-L+1}) \quad \text{on } D_-. \quad (83)$$

Proof of Theorem 1

Suppose m satisfies (M.1), (M.2) and set $\mathbf{m}(z) \equiv (1, m(z)/z)$. To see $\mathbf{m} \in \mathbf{M}_L(\mathbb{C})$ only the condition (ii) of (26), namely

$$m_1(z)m_2(-z) + m_1(-z)m_2(z) \neq 0 \quad \text{on } \mathbb{C} \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R}) \quad (84)$$

must be verified. For this $\mathbf{m}(z)$ the left hand side of (84) is $(m(z) - m(-z))/z$, which is not 0 since $\text{Im}(m(z) - m(-z)) > 0$ if $\text{Im} z > 0$ and $m(x) - m(-x) \neq 0$ if $|x| > \mu_0$. Since the identity $m = m_{\mathbf{m}}$ is clear, Proposition 28 implies $\mathbf{m} \in \mathbf{A}_{L,+}^{inv}(C)$, which proves the theorem.

It may be interesting to see to what extent $m_{\mathbf{a}}$ for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ has the property (M.2).

Proposition 29 *Let $C' = \sigma C$ with $\sigma > 1$. Then $m_{\mathbf{a}}$ for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ satisfies (M.2) on $D'_- = \sigma D_-$ replacing L by $L - 1 - (n - 1)/2$.*

Proof. To verify the property (M.2) recall the definition of the m -function $m_{\mathbf{a}}$ with $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$:

$$m_{\mathbf{a}}(z) = \frac{z + \psi_{\mathbf{a}}(z)}{1 + \varphi_{\mathbf{a}}(z)} + \kappa_1(\mathbf{a})$$

with

$$\begin{cases} \varphi_{\mathbf{a}}(z) = \mathbf{a}(z) T(\mathbf{a})^{-1} 1 - 1 \\ \psi_{\mathbf{a}}(z) = \mathbf{a}(z) T(\mathbf{a})^{-1} z - z \end{cases} \in H(D_-),$$

and $\kappa_1(\mathbf{a}) = \lim_{z \rightarrow \infty} z \varphi_{\mathbf{a}}(z)$. Set $u = T(\mathbf{a})^{-1} 1 \in H_1(D_+)$, $v = T(\mathbf{a})^{-1} z \in H_2(D_+)$. Since (12) implies

$$\mathbf{p}_-(\mathbf{a}u)(z) = \frac{1}{2\pi i} \int_C \frac{(\mathbf{a}(\lambda) - \mathbf{f}(\lambda)) u(\lambda)}{z - \lambda} d\lambda \in H(D_-)$$

for $u \in H_L(D_+)$, we have

$$\begin{cases} \varphi_{\mathbf{a}}(z) = \mathbf{p}_-(\mathbf{a}u)(z) = \frac{1}{2\pi i} \int_C \frac{(\mathbf{a}(\lambda) - \mathbf{f}(\lambda)) u(\lambda)}{z - \lambda} d\lambda \\ \psi_{\mathbf{a}}(z) = \mathbf{p}_-(\mathbf{a}v)(z) = \frac{1}{2\pi i} \int_C \frac{(\mathbf{a}(\lambda) - \mathbf{f}(\lambda)) v(\lambda)}{z - \lambda} d\lambda \end{cases}.$$

The expansion

$$\frac{1}{z - \lambda} = \sum_{k=1}^M z^{-k} \lambda^{k-1} + z^{-M} \frac{\lambda^M}{z - \lambda}$$

implies

$$\begin{cases} \varphi_{\mathbf{a}}(z) = \sum_{k=1}^{L-1} z^{-k} \ell_{k-1}(u) + z^{-L+1} \delta_1(z) \\ \psi_{\mathbf{a}}(z) = \sum_{k=1}^{L-2} z^{-k} \ell_{k-1}(v) + z^{-L+2} \delta_2(z) \end{cases}$$

with

$$\begin{cases} \ell_k(u) = \frac{1}{2\pi i} \int_C \lambda^k (\mathbf{a}(\lambda) - \mathbf{f}(\lambda)) u(\lambda) d\lambda \\ \delta_1(z) = \frac{1}{2\pi i} \int_C \frac{\lambda^{L-1} (\mathbf{a}(\lambda) - \mathbf{f}(\lambda)) u(\lambda)}{z - \lambda} d\lambda \in H(D_-) \\ \delta_2(z) = \frac{1}{2\pi i} \int_C \frac{\lambda^{L-2} (\mathbf{a}(\lambda) - \mathbf{f}(\lambda)) v(\lambda)}{z - \lambda} d\lambda \in H(D_-) \end{cases}.$$

Since $\lambda^{L-1} (\mathbf{a}(\lambda) - \mathbf{f}(\lambda)) u(\lambda) \in L^2(C)$, Schwarz inequality implies

$$|\delta_1(z)|^2 \leq \frac{1}{4\pi^2} \int_C \frac{|d\lambda|}{|z - \lambda|^2} \int_C |\lambda^{L-1} (\mathbf{a}(\lambda) - \mathbf{f}(\lambda)) u(\lambda)|^2 |d\lambda| \leq c |z|^{n-1}$$

for $z \in C'$ due to Lemma 42. Similarly we have $|\delta_2(z)|^2 \leq c|z|^{n-1}$. Therefore

$$\begin{cases} \varphi_{\mathbf{a}}(z) = \sum_{k=1}^{L'-1} z^{-k} \ell_{k-1}(u) + O(z^{-L'}) \\ \psi_{\mathbf{a}}(z) = \sum_{k=1}^{L'-2} z^{-k} \ell_{k-1}(v) + O(z^{-L'+1}) \end{cases}$$

holds on C' with $L' = L - 1 - (n - 1)/2$, which implies

$$m_{\mathbf{a}}(z) = z + \sum_{k=1}^{L'-2} c_k z^{-k} + O(z^{-L'+1}).$$

■

7 KdV flow

We are ready to construct the KdV flow.

7.1 Definition of the flow and Theorem 2

We define the KdV flow by making use of the m -functions and the continuity of $m_{\mathbf{a}}(\zeta)$ with respect to \mathbf{a} is necessary, which is shown by representing $m'_{\mathbf{a}}$ by the tau-function. The identities in (112) implies

$$\begin{aligned} m'_{\mathbf{a}}(\zeta) &= \frac{\tau_{\mathbf{a}}(q_{\zeta}^2)}{\tau_{\mathbf{a}}(q_{\zeta})^2} \\ &= \frac{\tau_{\mathbf{a}}^{(2)}(q_{\zeta}^2)}{\tau_{\mathbf{a}}^{(2)}(q_{\zeta})^2} \exp \operatorname{tr} \left(\left(q_{\zeta}^{-2} T(q_{\zeta}^2 \mathbf{a}) - 2q_{\zeta}^{-1} T(q_{\zeta} \mathbf{a}) + T(\mathbf{a}) \right) T(\mathbf{a})^{-1} \right). \end{aligned}$$

The cocycle property shows

$$\begin{aligned} m'_{g\mathbf{a}}(\zeta) &= \frac{\tau_{g\mathbf{a}}^{(2)}(q_{\zeta}^2)}{\tau_{g\mathbf{a}}^{(2)}(q_{\zeta})^2} \exp \operatorname{tr} \left(\left(q_{\zeta}^{-2} T(q_{\zeta}^2 g\mathbf{a}) - 2q_{\zeta}^{-1} T(q_{\zeta} g\mathbf{a}) + T(g\mathbf{a}) \right) T(g\mathbf{a})^{-1} \right) \\ &= \frac{\tau_{\mathbf{a}}^{(2)}(gq_{\zeta}^2)}{\tau_{\mathbf{a}}^{(2)}(gq_{\zeta})^2} \tau_{\mathbf{a}}^{(2)}(g) \exp \operatorname{tr} B_1 \end{aligned}$$

for $g \in \Gamma_n$ with

$$\begin{aligned} B_1 &= q_{\zeta}^{-2} T(gq_{\zeta}^2 \mathbf{a}) T(g\mathbf{a})^{-1} - 2q_{\zeta}^{-1} T(gq_{\zeta} \mathbf{a}) T(g\mathbf{a})^{-1} + I \\ &\quad + \left((gq_{\zeta}^2)^{-1} T(gq_{\zeta}^2 \mathbf{a}) T(g\mathbf{a})^{-1} g - I \right) \left(g^{-1} T(g\mathbf{a}) T(\mathbf{a})^{-1} - I \right) \\ &\quad - 2 \left((gq_{\zeta})^{-1} T(gq_{\zeta} \mathbf{a}) T(g\mathbf{a})^{-1} g - I \right) \left(g^{-1} T(g\mathbf{a}) T(\mathbf{a})^{-1} - I \right) \\ &= B_2 - g^{-1} B_2 g + g^{-1} B_2 T(g\mathbf{a}) T(\mathbf{a})^{-1}, \end{aligned}$$

where

$$B_2 = q_{\zeta}^{-2} T(gq_{\zeta}^2 \mathbf{a}) T(g\mathbf{a})^{-1} - 2q_{\zeta}^{-1} T(gq_{\zeta} \mathbf{a}) T(g\mathbf{a})^{-1} + I.$$

Then we have

$$\mathrm{tr}B_1 = \mathrm{tr}g^{-1} \left(q_\zeta^{-2}T(gq_\zeta^2\mathbf{a}) - 2q_\zeta^{-1}T(gq_\zeta\mathbf{a}) + T(g\mathbf{a}) \right) T(\mathbf{a})^{-1}.$$

Since for $v \in H_N(D_+)$

$$\begin{aligned} & q_\zeta^{-2}T(gq_\zeta^2\mathbf{a}) - 2q_\zeta^{-1}T(gq_\zeta\mathbf{a}) + T(g\mathbf{a})v(z) \\ &= \frac{1}{2\pi i} \int_C \frac{g(\lambda) \left(q_\zeta(z)^{-1} q_\zeta(\lambda) - 1 \right)^2 \tilde{\mathbf{a}}(\lambda) v(\lambda)}{\lambda - z} d\lambda \\ &= \frac{1}{2\pi i} \int_C \frac{(\lambda - z) g(\lambda) \tilde{\mathbf{a}}(\lambda) v(\lambda)}{(\zeta - \lambda)^2} d\lambda, \end{aligned}$$

the above operator is of rank 2, and the trace turns to

$$\mathrm{tr}B_1 = \frac{1}{2\pi i} \int_C \frac{\Theta(g, \lambda)}{(\zeta - \lambda)^2} d\lambda$$

with

$$\Theta(g, \lambda) = g(\lambda) \left(\lambda \tilde{\mathbf{a}}(\lambda) \left(T(\mathbf{a})^{-1} g^{-1} \right) (\lambda) - \tilde{\mathbf{a}}(\lambda) \left(T(\mathbf{a})^{-1} g^{-1} z \right) (\lambda) \right),$$

hence

$$m'_{g\mathbf{a}}(\zeta) = \frac{\tau_{\mathbf{a}}^{(2)}(gq_\zeta^2)}{\tau_{\mathbf{a}}^{(2)}(gq_\zeta)^2} \tau_{\mathbf{a}}^{(2)}(g) \exp \left(\frac{1}{2\pi i} \int_C \frac{\Theta(g, \lambda)}{(\zeta - \lambda)^2} d\lambda \right) \quad (85)$$

holds.

Lemma 30 For $L \geq \max\{n+1, 3\}$ let $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}_{L,+}^{inv}(C)$. Then

- (i) Suppose $m_{\mathbf{a}_1} = m_{\mathbf{a}_2}$. Then $m_{g\mathbf{a}_1} = m_{g\mathbf{a}_2}$ for any $g \in \Gamma_n$.
- (ii) Suppose $\partial_x \kappa_1(e_x \mathbf{a}_1) = \partial_x \kappa_1(e_x \mathbf{a}_2)$ for any $x \in \mathbb{R}$. Then $\partial_x \kappa_1(e_x g \mathbf{a}_1) = \partial_x \kappa_1(e_x g \mathbf{a}_2)$ for any $g \in \Gamma_n$ and $x \in \mathbb{R}$.

Proof. Propositions 24, 27, 28 provide necessary ingredients. Suppose $m_{\mathbf{a}_1} = m_{\mathbf{a}_2}$ for $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}_{L,+}^{inv}(C)$. Then Lemma 38 implies

$$m_{q_\zeta p_\eta \mathbf{a}_1}(z) = (d_\zeta d_\eta m_{\mathbf{a}_1})(z) = (d_\zeta d_\eta m_{\mathbf{a}_2})(z) = m_{q_\zeta p_\eta \mathbf{a}_2}(z).$$

Repeating this operation finite times one can show $m_{r\mathbf{a}_1} = m_{r\mathbf{a}_2}$ for any real rational function $r \in \Gamma_0^{(0)}$. For $g \in \Gamma_n$ we approximate it by real rational functions $r_k \in \Gamma_0^{(0)}$, which is possible by Lemma 23. We show the convergence $m'_{r_k \mathbf{a}}(\zeta) \rightarrow m'_{g\mathbf{a}}(\zeta)$ for each fixed $\zeta \in D_-$ by making use of (85). Lemma 19 shows

$$\lim_{k \rightarrow \infty} \frac{\tau_{\mathbf{a}}^{(2)}(r_k q_\zeta^2)}{\tau_{\mathbf{a}}^{(2)}(r_k q_\zeta)^2} = \frac{\tau_{\mathbf{a}}^{(2)}(g q_\zeta^2)}{\tau_{\mathbf{a}}^{(2)}(g q_\zeta)^2}.$$

On the other hand $\Theta(r_k, \lambda) \rightarrow \Theta(g, \lambda)$ in $L^2(C)$ is valid if $L \geq \max\{n+1, 3\}$, hence

$$\lim_{k \rightarrow \infty} m'_{r_k \mathbf{a}}(\zeta) = m'_{g\mathbf{a}}(\zeta) \quad (86)$$

follows. Consequently one sees

$$m'_{g\mathbf{a}_1}(\zeta) = \lim_{k \rightarrow \infty} m'_{r_k \mathbf{a}_1}(\zeta) = \lim_{k \rightarrow \infty} m'_{r_k \mathbf{a}_2}(\zeta) = m'_{g\mathbf{a}_2}(\zeta)$$

for $\zeta \in D_-$. Since generally $m_{g\mathbf{a}}(\zeta) = \zeta + o(1)$, we have $m_{g\mathbf{a}_1} = m_{g\mathbf{a}_2}$.

(ii) is proved by the uniqueness of the correspondence between the Weyl functions m_{\pm} and the potential q . That is, $m_{\mathbf{a}_1} = m_{\mathbf{a}_2}$ follows from

$$-2\partial_x \kappa_1(e_x \mathbf{a}_1) = -2\partial_x \kappa_1(e_x \mathbf{a}_2) = q(x).$$

Then (i) yields $m_{g\mathbf{a}_1} = m_{g\mathbf{a}_2}$, which implies again by the uniqueness

$$-2\partial_x \kappa_1(e_x g \mathbf{a}_1) = -2\partial_x \kappa_1(e_x g \mathbf{a}_2).$$

We have used the condition $L \geq \max\{n+1, 3\}$ to have the differentiability. ■

Set

$$\mathcal{Q}_L(C) = \{q; q(x) = -2\partial_x \kappa_1(e_x \mathbf{a}) \text{ with real } \mathbf{a} \in A_{L,+}^{inv}(C)\}.$$

Then this Lemma allows us to define

$$(K(g)q)(x) = -2\partial_x \kappa_1(e_x g \mathbf{a}) \quad \text{if } q = -2\partial_x \kappa_1(e_x \mathbf{a}) \in \mathcal{Q}_L(C)$$

for $g \in \Gamma_n$, and we have Theorem 2. We call the flow $\{K(g)\}_{g \in \Gamma_n}$ as **KdV flow**.

It might be helpful to remark that one can define an equivalent flow on the space of m -functions. Let

$$\mathcal{M}_L(C) = \{m; m(z) = m_{\mathbf{a}}(z) \text{ for } \mathbf{a} \in A_{L,+}^{inv}(C)\},$$

and define

$$g \cdot m_{\mathbf{a}} = m_{g\mathbf{a}} \quad \text{for } m_{\mathbf{a}} \in \mathcal{M}_L(C).$$

(i) of Lemma 6 justifies this definition. The set

$$\mathcal{M}_{\infty}(C) = \bigcap_{L \geq 1} \left(\bigcup_{\sigma > 1} \mathcal{M}_L(\sigma C) \right)$$

is equal to the set of all functions m satisfying (M.1), (M.2) on σD_- . Theorem 1 implies $g \cdot m \in \mathcal{M}_{\infty}(C)$ for $m \in \mathcal{M}_{\infty}(C)$.

7.2 Tau-function representation of the flow

Hirota introduced his tau-function as the function $u(t, x)$ such that $-2\partial_x^2 \log u(t, x)$ is a solution to the KdV equation and he tried to find an equation satisfied by $u(t, x)$. Sato discovered the intrinsic meaning of $u(t, x)$ and found that solutions to the KdV equation can be described by the tau-functions. Although the theorems in this paper can be proved without this representation, in view of its historical significance we show it in the present framework.

To define the tau-function $\tau_{\mathbf{a}}(g)$ we have to assume that the operator

$$g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} - I : H_N(D_+) \rightarrow H_N(D_+)$$

is of trace class. We assume in this section the condition (34) of Lemma 8 for $\lambda \mathbf{a}(\lambda)$, $N = 1$, that is

$$\int_{C^2} \left| \frac{z^2 \tilde{a}_j(z) - \lambda^2 \tilde{a}_j(\lambda)}{z - \lambda} \right|^2 |dz| |d\lambda| < \infty \text{ with } \tilde{a}_j(z) = a_j(z) - f_j(z), \quad (87)$$

which shows the operator $S_{\lambda \mathbf{a}}$ is of HS from $H_1(D_+)$ to $H(D_-)$. This condition can be verified in the same manner as Lemma 9 for $\mathbf{a}(z) = (1, m(z)/z)$ if m satisfies (M.1), (M.2) for sufficiently large L .

Let

$$e_x(z) = e^{xz} \in \Gamma_1, \quad e_{t,x}(z) = e^{xz+tz^3} \in \Gamma_3.$$

In Proposition 11 for $\mathbf{a} \in \mathbf{A}_3(C)$ satisfying $e_x \mathbf{a} \in \mathbf{A}_3^{inv}(C)$ for any $x \in \mathbb{R}$ we have introduced the potential q associated with \mathbf{a} of Schrödinger operator by $q(x) = -2\partial_x \kappa_1(e_x \mathbf{a})$. The $\kappa_1(\mathbf{a})$ is described by the characteristic functions as

$$\kappa_1(\mathbf{a}) = \lim_{\zeta \rightarrow \infty} \zeta \varphi_{\mathbf{a}}(\zeta).$$

Proposition 31 *Assume $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ and (87). Then an identity*

$$\kappa_1(e_x \mathbf{a}) = \partial_x \log \tau_{\mathbf{a}}(e_x)$$

holds, which yields

$$q(x) = -2\partial_x^2 \log \tau_{\mathbf{a}}(e_x),$$

if $L \geq 2$. Generally for $g \in \Gamma_n$ and $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ with $L \geq \max\{n, 2\}$

$$(K(g)q)(x) = -2\partial_x^2 \log \tau_{\mathbf{a}}(ge_x)$$

holds. Especially the solution $q(t, x)$ to the KdV equation starting from $q(x)$ is given by

$$q(t, x) = -2\partial_x^2 \log \tau_{\mathbf{a}}(e_{t,x}),$$

if $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ for $L \geq 4$. The condition $L \geq 4$ is necessary for the differentiability of $q(t, x)$ in t (see Proposition 14).

Proof. The definition of $\varphi_{\mathbf{a}}$ implies

$$\begin{aligned} \kappa_1(\mathbf{a}) &= \lim_{\zeta \rightarrow \infty} \zeta \varphi_{\mathbf{a}}(\zeta) = \lim_{\zeta \rightarrow \infty} \zeta \frac{1}{2\pi i} \int_C \frac{\varphi_{\mathbf{a}}(\lambda)}{\zeta - \lambda} d\lambda \\ &= \lim_{\zeta \rightarrow \infty} \zeta \frac{1}{2\pi i} \int_C \frac{\tilde{\mathbf{a}}(\lambda) \left(T(\mathbf{a})^{-1} 1 \right)(\lambda)}{\zeta - \lambda} d\lambda \\ &= \frac{1}{2\pi i} \int_C \tilde{\mathbf{a}}(\lambda) \left(T(\mathbf{a})^{-1} 1 \right)(\lambda) d\lambda. \end{aligned} \quad (88)$$

On the other hand the formal identity

$$\tau_{\mathbf{a}}(e_{\epsilon}) = \det \left(e_{\epsilon}^{-1} T(e_{\epsilon} \mathbf{a}) T(\mathbf{a})^{-1} \right) = \det \left(I + e_{\epsilon}^{-1} H_{e_{\epsilon}} S_{\mathbf{a}} T(\mathbf{a})^{-1} \right)$$

is justified by reproving below in a way different from that of Lemma 17 that $e_{\epsilon}^{-1} H_{e_{\epsilon}} S_{\mathbf{a}} T(\mathbf{a})^{-1}$ defines a trace class operator on $H_1(D_+)$. For $v \in H_1(D_+)$ it holds that

$$S_{\mathbf{a}} v(z) = \frac{1}{2\pi i} \int_C \frac{\tilde{\mathbf{a}}(\lambda) v(\lambda)}{z - \lambda} d\lambda = \ell_1(\mathbf{a}, v) z^{-1} + z^{-1} \left(S_{\mathbf{a}}^{(1)} v \right)(z)$$

with

$$\begin{cases} \ell_1(\mathbf{a}, v) = \frac{1}{2\pi i} \int_C \tilde{\mathbf{a}}(\lambda) v(\lambda) d\lambda \\ S_{\mathbf{a}}^{(1)} v(z) = \frac{1}{2\pi i} \int_C \frac{\lambda \tilde{\mathbf{a}}(\lambda) v(\lambda)}{z - \lambda} d\lambda \\ \quad = \frac{1}{2\pi i} \int_C \frac{(\lambda \tilde{\mathbf{a}}(\lambda) - z \tilde{\mathbf{a}}(z)) v(\lambda)}{z - \lambda} d\lambda \in H(D_-) \end{cases},$$

since $\tilde{\mathbf{a}}(\lambda) = O(\lambda^{-L})$ for $L \geq 2$. Hence for $z \in D_+$

$$\begin{aligned} e_\epsilon^{-1} H_{e_\epsilon} S_{\mathbf{a}} v(z) &= \frac{\ell_1(\mathbf{a}, v)}{2\pi i} \int_C \frac{e^{\epsilon(\lambda-z)}}{(\lambda-z)\lambda} d\lambda + e_\epsilon^{-1} H_{e_\epsilon} z^{-1} S_{\mathbf{a}}^{(1)} v(z) \\ &= \ell_1(\mathbf{a}, v) \frac{1 - e^{-\epsilon z}}{z} + e_\epsilon^{-1} H_{e_\epsilon} z^{-1} S_{\mathbf{a}}^{(1)} v(z) \end{aligned} \quad (89)$$

holds. Since $S_{\mathbf{a}}^{(1)}$ defines an HS operator from $H_1(D_+)$ to $H(D_-)$ under (87), $e_\epsilon^{-1} H_{e_\epsilon} z^{-1} S_{\mathbf{a}}^{(1)}$ turns to a trace class operator on $H_1(D_+)$, which makes it possible to define $\tau_{\mathbf{a}}(e_\epsilon)$ rigorously. Moreover in this case for $w \in H(D_-)$

$$e_\epsilon^{-1} H_{e_\epsilon} z^{-1} w(z) = \frac{1}{2\pi i} \int_C \left(\frac{e^{\epsilon(\lambda-z)} - 1}{\lambda - z} - \epsilon \right) \lambda^{-1} w(\lambda) d\lambda$$

holds due to

$$\frac{1}{2\pi i} \int_C \frac{1}{\lambda - z} \lambda^{-1} w(\lambda) d\lambda = \frac{1}{2\pi i} \int_C \lambda^{-1} w(\lambda) d\lambda = 0,$$

Hence the square of the HS-norm of $\epsilon^{-1} e_\epsilon^{-1} H_{e_\epsilon} z^{-1}$ is

$$\delta_\epsilon \equiv (2\pi)^{-2} \int_{C^2} \left| \frac{e^{\epsilon(\lambda-z)} - 1}{\epsilon(\lambda-z)} - 1 \right|^2 |\lambda|^{-2} |z|^{-2} |d\lambda| |dz|.$$

Since there exists a constant c such that

$$\left| \frac{e^{\epsilon(\lambda-z)} - 1}{\epsilon(\lambda-z)} - 1 \right| = \left| \int_0^1 (e^{t\epsilon(\lambda-z)} - 1) dt \right| \leq c$$

holds for $\epsilon \in \mathbb{R}$, $\lambda, z \in C$, the dominated convergence theorem shows $\lim_{\epsilon \rightarrow 0} \delta_\epsilon = 0$. Consequently we have

$$\left\| e_\epsilon^{-1} H_{e_\epsilon} z^{-1} S_{\mathbf{a}}^{(1)} T(\mathbf{a})^{-1} \right\|_{trace} \leq c_1 \epsilon \sqrt{\delta_\epsilon} = o(\epsilon). \quad (90)$$

The first term of (89) generates a rank 1 operator and

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \ell_1(\mathbf{a}, v) \frac{1 - e^{-\epsilon z}}{z} = \ell_1(\mathbf{a}, v) = \frac{1}{2\pi i} \int_C \tilde{\mathbf{a}}(\lambda) v(\lambda) d\lambda.$$

Noting an identity (see ([20]))

$$\det(I + A) = \exp(\text{tr} \log(I + A)) = \exp\left(\text{tr} A + O\left(\|A\|_{trace}^2\right)\right)$$

if $\|A\|_{trace} < 1$, we have

$$\begin{aligned}\tau_{\mathbf{a}}(e_\epsilon) &= \exp\left(\operatorname{tr}\left(e_\epsilon^{-1} H_{e_\epsilon} S_{\mathbf{a}} T(\mathbf{a})^{-1}\right) + o(\epsilon)\right) \\ &= \exp\left(\epsilon \frac{1}{2\pi i} \int_C \tilde{\mathbf{a}}(\lambda) \left(T(\mathbf{a})^{-1} \mathbf{1}\right)(\lambda) d\lambda + o(\epsilon)\right) \\ &= 1 + \epsilon \frac{1}{2\pi i} \int_C \tilde{\mathbf{a}}(\lambda) \left(T(\mathbf{a})^{-1} \mathbf{1}\right)(\lambda) d\lambda + o(\epsilon)\end{aligned}$$

for sufficiently small ϵ due to (90). Then (88) implies

$$\lim_{\epsilon \rightarrow 0} \frac{\tau_{\mathbf{a}}(e_\epsilon) - 1}{\epsilon} = \kappa_1(\mathbf{a}),$$

and the cocycle property shows

$$\lim_{\epsilon \rightarrow 0} \frac{\tau_{\mathbf{a}}(e_{x+\epsilon}) - \tau_{\mathbf{a}}(e_x)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{\tau_{e_x \mathbf{a}}(e_\epsilon) - 1}{\epsilon} \tau_{\mathbf{a}}(e_x) = \kappa_1(e_x \mathbf{a}) \tau_{\mathbf{a}}(e_x),$$

hence $\partial_x \log \tau_{\mathbf{a}}(e_x) = \kappa_1(e_x \mathbf{a})$ holds. The rest of the proof is automatic. ■

8 Sufficient conditions for $q \in \mathcal{Q}_L(C)$

A sufficient condition for $q \in \mathcal{Q}_L(C)$ was given in (83) in terms of its Weyl functions m_\pm (see Theorem 1). In this section we provide concrete examples of this class including the well-known cases. Throughout this section we treat $g = e^h$ with real odd polynomial h of degree n , hence the curve C is taken so that g is bounded on D_+ , namely

$$C = \left\{ \begin{array}{l} \pm\omega(y) + iy; \quad y \in \mathbb{R}, \omega(y) > 0, \omega(y) = \omega(-y), \\ \omega \text{ is smooth and satisfies } \omega(y) = O(y^{-(n-1)}) \text{ as } y \rightarrow \infty \end{array} \right\}.$$

8.1 Decaying potentials

If a potential q satisfies $q \in L^1(\mathbb{R}_+)$, it is known that for $0 \neq k \in \overline{\mathbb{C}}_+ \equiv \{z \in \mathbb{C}; \operatorname{Im} z \geq 0\}$ there exists the Jost solution $f_+(x, k)$ of

$$-\partial_x^2 f_+(x, k) + q(x) f_+(x, k) = k^2 f_+(x, k)$$

such that

$$\begin{cases} f_+(x, k) = e^{ixk} + o(1) \\ f'_+(x, k) = ik e^{ixk} + o(1) \end{cases} \quad \text{as } x \rightarrow \infty,$$

where $'$ denotes the derivative with respect to x . Therefore

$$m_+(z) = \frac{f'_+(0, \sqrt{z})}{f_+(0, \sqrt{z})}$$

and one can see that $m_+(z)$ is extendable to $\overline{\mathbb{C}}_+ \setminus \{0\}$ as a continuous function. $f_+(x, k)$ is obtained as a unique solution to an integral equation

$$e^{-ixk} f_+(x, k) = 1 + \int_x^\infty \frac{e^{2ik(s-x)} - 1}{2ik} q(s) e^{-iks} f_+(s, k) ds.$$

Rybkin ([15]) showed

$$e^{-ixk} f_+(x, k) = 1 + \sum_{j=1}^{N+1} f_j(x) (2ik)^{-j} + o(k^{-N-1}) \quad (91)$$

for q such that $q^{(j)} \in L^1(\mathbb{R})$ for $j = 0, 1, \dots, L$. The small o is uniform with respect to $x \geq 0$. The coefficients $\{f_j(x)\}$ are determined inductively by

$$\begin{cases} f_1(x) = -Q(x) \equiv -\int_x^\infty q(s) ds \\ f_{j+1}(x) = -f'_j(x) - \int_x^\infty q(s) f_j(s) ds, \quad (j \geq 1) \end{cases} .$$

Therefore one can show f_{j+1} is $L-j$ times differentiable and $f_{j+1}^{(L-j+1)} \in L^1(\mathbb{R}_+)$. Since

$$(e^{-ixk} f_+(x, k))' = \int_x^\infty e^{2ik(s-x)} q(s) e^{-iks} f_+(s, k) ds,$$

substituting (91) we have asymptotic behavior

$$(e^{-ixk} f_+(x, k))' = \sum_{j=1}^{L+1} g_j(x) (2ik)^{-j} + o(k^{-N-1}),$$

which leads us to

$$m_+(z) = -\sqrt{-z} + \sum_{j=1}^{L+1} c_j (-z)^{-j/2} + o(z^{-(L+1)/2}) \quad \text{if } z \rightarrow \infty \text{ on } \overline{\mathbb{C}}_+.$$

An analogous asymptotic behavior for $m_-(z)$ is possible if $q^{(j)} \in L^1(\mathbb{R})$ for $j = 0, 1, \dots, L$ by replacing c_j by $(-1)^{j+1} c_j$, which shows

Proposition 32 *If $q^{(j)} \in L^1(\mathbb{R})$ for $j = 0, 1, \dots, L$, then (M.2) is satisfied with $L+2$ for any curve C , and we have $q \in \mathcal{Q}_{L+1}(C)$. Therefore one can define the KdV flow $K(g)q$ for $g \in \Gamma_n$ if $L \geq 1$. For the KdV equation $L \geq 3$ is required to guarantee the differentiability.*

One cannot apply this proposition to the interesting case $q(x) = \varphi(x)/x$ with smooth periodic function φ satisfying $\varphi(0) = 0$, however there is a possibility of estimating directly m_\pm in this case by a sort of perturbation.

8.2 Reflection coefficients

To obtain another class of q satisfying (M.2) we prepare the necessary terminologies from the spectral theory of Schrödinger operators. Since $m_\pm(z)$ take values in \mathbb{C}_+ for $z \in \mathbb{C}_+$, we start from

Lemma 33 *For any complex numbers $m_\pm \in \mathbb{C}_+$ set*

$$m_1 = -\frac{1}{m_+ + m_-}, \quad m_2 = \frac{m_+ m_-}{m_+ + m_-}, \quad R = \frac{\overline{m_+} + m_-}{m_+ + m_-}.$$

Then, $m_1, m_2 \in \mathbb{C}_+$, $|R| \leq 1$ hold, and $\xi_j = (\arg m_j) / \pi \in [0, 1]$ ($j = 1, 2$) satisfy

$$\left| \xi_1 - \frac{1}{2} \right|, \quad \left| \xi_2 - \frac{1}{2} \right| \leq \frac{1}{2} |R|. \quad (92)$$

For $z \in \mathbb{C}_+$ set

$$\begin{cases} m_1(z) = -\frac{1}{m_+(z) + m_-(z)}, & m_2(z) = \frac{m_+(z)m_-(z)}{m_+(z) + m_-(z)} \\ \xi_j(z) = \frac{1}{\pi} \arg m_j(z) \left(= \frac{1}{\pi} \operatorname{Im} \log m_j(z) \right), & (j = 1, 2) \end{cases}. \quad (93)$$

Then, $\{m_j, j = 1, 2\}$ are Herglotz functions. $m(z)$ defined by

$$m(z) = \begin{cases} -m_+(-z^2) & \text{if } \operatorname{Re} z > 0 \\ m_-(-z^2) & \text{if } \operatorname{Re} z < 0 \end{cases}$$

satisfies the asymptotic behavior (83) if and only if m_{\pm} satisfy

$$\begin{cases} m_+(-z^2) = -z \left(1 - \sum_{k=2}^{L-1} c_k z^{-k} + O(z^{-L}) \right) \\ m_-(-z^2) = -z \left(1 - \sum_{k=2}^{L-1} c_k (-z)^{-k} + O(z^{-L}) \right) \end{cases}. \quad (94)$$

as $z \rightarrow \infty$ on $D_- \cap \{\operatorname{Re} z > 0\}$. It is known that if $q \in C^{L-2}([0, \delta])$ for some $\delta > 0$, then defining inductively the functions $c_j(x)$ by

$$\begin{cases} c_1(x) = 0, & c_2(x) = q(x)/2, \\ c_j(x) = (c'_{j-1}(x) - \sum_{\ell=1}^{j-1} c_{\ell}(x) c_{j-\ell}(x))/2, & j \geq 3 \end{cases}$$

one has $c_k = c_k(0)$. The coefficients for $m_-(-z^2)$ can be obtained by considering $q(-x)$ in place of $q(x)$. Then, if $q \in C^{L-2}(-\delta, \delta)$, (94) implies

$$\begin{cases} m_1(-z^2) = \frac{1}{2} z^{-1} \left(1 + \sum_{k=1}^M a_k z^{-2k} + O(z^{-L}) \right) \\ m_2(-z^2) = -\frac{1}{2} z \left(1 + \sum_{k=1}^M b_k z^{-2k} + O(z^{-L}) \right) \end{cases} \quad (95)$$

on $D_- \cap \{\operatorname{Re} z > 0\}$ with some $a_k, b_k \in \mathbb{R}$, where $M = [(L-1)/2]$ ($[x]$ denotes the integer part of x). m_{\pm} can be recovered from m_1, m_2 by

$$m_{\pm} = -\frac{1}{2m_1} \left(1 \pm \sqrt{1 + 4m_1 m_2} \right).$$

Observe

$$\begin{aligned} 1 + 4m_1(-z^2)m_2(-z^2) &= \left(\frac{m_+(-z^2) - m_-(-z^2)}{m_+(-z^2) + m_-(-z^2)} \right)^2 \\ &= \left(\frac{f_o(z) + O(z^{-L})}{1 - f_e(z) + O(z^{-L})} \right)^2 \end{aligned}$$

with $f(z) = \sum_{k=2}^{L-1} c_k z^{-k}$. Let N be the least number such that $(1 - (-1)^k) c_k \neq 0$. N should be odd and $N \geq 3$, since $f(z) = c_2 z^{-2} + O(z^{-3})$. Then one sees

$$1 + 4m_1(-z^2)m_2(-z^2) = -\sum_{k=N}^M d_k z^{-2k} + O(z^{-L-N}) \quad (96)$$

with $d_k \in \mathbb{R}$, $d_N \neq 0$. However one cannot expect to have (96) from (95).

We would like to have a partial converse:

Lemma 34 For Herglotz functions m_{\pm} suppose m_1, m_2 are given by (93) and satisfy (95) with M', L' such that $2M' < L'$. Then m satisfies the conditions (M.1), (M.2) in (83) with $L = 1 + [L'/2]$.

Proof. We have only to verify (M.2). Since

$$m_+ + m_- = -m_1^{-1}, \quad m_+ m_- = -m_1^{-1} m_2,$$

we have

$$m_{\pm} = \frac{1}{2} \left(-m_1^{-1} \mp \sqrt{m_1^{-2} + 4m_1^{-1}m_2} \right) = -\frac{1}{2m_1} (1 \pm \sqrt{1 + 4m_1 m_2}).$$

Then

$$\begin{aligned} \sqrt{\frac{1 + 4m_1(-z^2)m_2(-z^2)}{d_N}} &= z^{-N} \sqrt{1 + \sum_{k=N+1}^{M'} \frac{d_k}{d_N} z^{-2(k-N)} + O(z^{-L'+2N})} \\ &= z^{-N} \left(1 + \sum_{k=N+1}^{M'} d'_k z^{-2(k-N)} + O(z^{-L'+2N}) \right) \\ &= z^{-N} + \sum_{k=N+1}^{M'} d'_k z^{-2k+N} + O(z^{-L'+N}) \end{aligned}$$

holds with other constants d'_k . Since $L' - N \geq L' - M' > L'/2$, we have the lemma. ■

The asymptotic behavior of m_j is translated to that of ξ_j as follows. If $\text{Im } z > 0$, then $\text{Im } m_j(z) > 0$, hence $0 \leq \xi_j(z) \leq 1$ holds. $\log m_j$ are of Herglotz as well, since $\text{Im } \log m_j(z) = \pi \xi_j(z) \geq 0$. On the other hand (ii) of (71) implies that $m_1(z), m_2(z)$ take real values on $(-\infty, \lambda_0)$ and $m_1(\lambda) > 0$ there. Let $\lambda_1 \leq \lambda_0$ be a unique zero of $m_2(z)$ if it has, and set $\lambda_1 = \lambda_0$ if it has not. Assume in the sequel

$$\int_0^{\infty} \left| \xi_j(\lambda) - \frac{1}{2} \right| d\lambda < \infty \quad \text{for } j = 1, 2. \quad (97)$$

Then m_j are represented as

$$\begin{cases} m_1(z) = \frac{1}{2\sqrt{-z}} \exp \left(\int_{\lambda_0}^{\infty} \frac{\xi_1(\lambda) - I_{\lambda>0}/2}{\lambda - z} d\lambda \right) \\ m_2(z) = -\frac{\sqrt{-z}}{2} \frac{\lambda_1 - z}{-z} \exp \left(\int_{\lambda_0}^{\infty} \frac{\xi_2(\lambda) - I_{\lambda>0}/2}{\lambda - z} d\lambda \right) \end{cases}. \quad (98)$$

The function $\arg m_+(\lambda + i0)$ is intensively investigated by Gesztesy-Simon [6] in connection with inverse spectral problems, and they call $\arg m_+(\lambda + i0)/\pi$ as **xi-function**. To have (95) for some $L \geq 1$ it is sufficient that

$$\int_0^{\infty} \lambda^M \left| \xi_j(\lambda) - \frac{1}{2} \right| d\lambda < \infty, \quad (j = 1, 2) \quad (99)$$

hold for an $M \geq 1$. This is due to the expansion

$$-\int_{\lambda_0}^{\infty} \frac{f(\lambda)}{\lambda - z} d\lambda = \sum_{k=0}^{M-1} z^{-k-1} \int_{\lambda_0}^{\infty} \lambda^k f(\lambda) d\lambda + z^{-M} \int_{\lambda_0}^{\infty} \frac{\lambda^M f(\lambda)}{\lambda - z} d\lambda, \quad (100)$$

and the estimate

$$\left| \int_{\lambda_0}^{\infty} \frac{\lambda^M f(\lambda)}{\lambda - z} d\lambda \right| \leq \int_{\lambda_0}^{\infty} \frac{|\lambda^M f(\lambda)|}{|\lambda - z|} d\lambda \leq \frac{1}{|\operatorname{Im} z|} \int_{\lambda_0}^{\infty} |\lambda^M f(\lambda)| d\lambda. \quad (101)$$

We control $|\xi_j(\lambda) - 1/2|$ by another quantity. The **reflection coefficient** $R(z)$ is defined by

$$R(z) = \frac{\overline{m_+(z)} + m_-(z)}{m_+(z) + m_-(z)}.$$

This quantity was considered by Gesztesy-Simon, Rybkin and others as a generalization of the conventional reflection coefficient defined for decaying potentials.

8.3 Proof of Theorems 3, 4

Assume

$$\int_0^{\infty} \lambda^M |R(\lambda)| d\lambda < \infty. \quad (102)$$

Then Lemma 33 implies that (99) holds, and from (100), (101) one has

$$-\int_{\lambda_0}^{\infty} \frac{\xi_j(\lambda) - \frac{1}{2}}{\lambda - z} d\lambda = \sum_{k=0}^{M-1} z^{-k-1} \int_{\lambda_0}^{\infty} \lambda^k \left(\xi_j(\lambda) - \frac{1}{2} \right) d\lambda + O\left(z^{-M+n/2-1}\right)$$

if $z \in D_-$. Applying Lemma 34 with $L' = 2M - n + 2$ we see that this m satisfies (M.2) with

$$L = 1 + [L'/2] = 1 + [M - n/2 + 1] = M + 1 - (n - 1)/2,$$

which yields Theorem 3.

The condition (102) implies that the ac spectrum is large, which restricts the possible potential class strongly. We try to relax the condition (102) by replacing it with a condition on a curve surrounding $[\lambda_0, \infty)$.

We prepare two curves

$$\begin{cases} C = \{\pm i\omega(y) + iy; y \in \mathbb{R}\} \text{ with } \omega(y) = cy^{-(n-1)} \text{ for } |y| \geq 1 \\ C_1 = \{\pm i\omega_1(y) + iy; y \in \mathbb{R}\} \text{ with } \omega_1(y) = cy^{-(n_1-1)} \text{ for } |y| \geq 1 \end{cases}$$

for $n_1 > n$, and

$$\begin{cases} \widehat{C} = \{-z^2; z \in C, \operatorname{Re} z > 0\} = \{x \pm i\widehat{\omega}(x); x \in \mathbb{R}, x \geq \lambda_0\} \\ \widehat{C}_1 = \{-z^2; z \in C_1, \operatorname{Re} z > 0\} = \{x \pm i\widehat{\omega}_1(x); x \in \mathbb{R}, x \geq \lambda_0\} \end{cases}$$

with $\widehat{\omega}(x) = \widehat{c}x^{1-n/2}$, $\widehat{\omega}_1(x) = \widehat{c}_1x^{1-n_1/2}$ for $x \geq 1$. Then one can assume $D_- \subset D_{1,-}$.

Lemma 35 Let $M \in \mathbb{Z}$ be $M \geq n/2$. Assume $q \in C^{2M-n}(-\delta, \delta)$ and

$$\int_{\widehat{C}_1} |z^M R(z)| |dz| < \infty \quad (103)$$

holds. Then, the m satisfies (M.2) with $L \leq M + 1 - (n - 1)/2$ on the curve C .

Proof. Let ϕ be $\phi(z) = -\phi_k(-z)$ in Lemma 40 with $k = (n_1 - 1)/2$. Then ϕ maps $\mathbb{C} \setminus [0, \infty)$ onto $\widehat{D}_{1,-} = \{z; |\operatorname{Im} z| > \widehat{\omega}_1(\operatorname{Re} z), \operatorname{Re} z > \lambda_0\}$ conformally. Without loss of generality we can assume $-a_k^2 < \lambda_0$. (103) implies

$$\int_0^\infty \left| \phi(\lambda)^M R(\phi(\lambda)) \right| |d\phi(\lambda)| < \infty.$$

which is equivalent to

$$\int_0^\infty |\lambda^M R(\phi(\lambda))| d\lambda < \infty$$

due to $\phi'(\lambda) = 1 + o(1)$. Hence Lemma 33 implies

$$\int_0^\infty \lambda^M \left| \xi_j(\phi(\lambda)) - \frac{1}{2} \right| d\lambda < \infty. \quad (104)$$

Since $m_j(\phi(z))$ ($j = 1, 2$) are Herglotz functions and its argument on \mathbb{R} is $\pi \xi_j(\phi(\lambda))$, applying the formula (76) to $m_j(\phi(z))$ yields

$$\begin{cases} m_1(\phi(z)) = \frac{1}{2\sqrt{-z}} \exp\left(\int_0^\infty \frac{\xi_1(\phi(\lambda)) - 1/2}{\lambda - z} d\lambda\right) \\ m_2(\phi(z)) = -\frac{\sqrt{-z}}{2} \exp\left(\int_0^\infty \frac{\xi_2(\phi(\lambda)) - 1/2}{\lambda - z} d\lambda\right) \end{cases}.$$

We have assumed here $-a_k^2 \leq \lambda_1$ for simplicity. Then for $z \in \mathbb{C}_+$

$$m_1(z) = \frac{1}{2\sqrt{-\phi^{-1}(z)}} \exp\left(\int_0^\infty \frac{f(\lambda)}{\lambda - \phi^{-1}(z)} d\lambda\right)$$

with $f(\lambda) = \xi_1(\phi(\lambda)) - 1/2$. Lemma 40 implies

$$\phi^{-1}(z) = z - g_1(-z) - (-z)^{-(n_1-1)/2+1/2} g_2(-z) \quad (105)$$

with functions g_j analytic near $z = \infty$ taking real values on \mathbb{R} , hence

$$u(z) \equiv \sum_{k=0}^{M-1} \phi^{-1}(z^2)^{-k-1} \int_0^\infty \lambda^k f(\lambda) d\lambda$$

is analytic at $z = \infty$, and the identity

$$-\int_0^\infty \frac{f(\lambda)}{\lambda - \phi^{-1}(z)} d\lambda = u(\sqrt{z}) + \phi^{-1}(z)^{-M} \int_0^\infty \frac{\lambda^M f(\lambda)}{\lambda - \phi^{-1}(z)} d\lambda$$

holds. The estimate

$$\left| \int_{\lambda_0}^\infty \frac{\lambda^M f(\lambda)}{\lambda - \phi^{-1}(z)} d\lambda \right| \leq \frac{1}{|\operatorname{Im} \phi^{-1}(z)|} \int_{\lambda_0}^\infty |\lambda^M f(\lambda)| d\lambda$$

for $z \in \mathbb{C}_+$. Since $n_1 > n$, (105) shows for some $c > 0$

$$|\operatorname{Im} \phi^{-1}(z)| \geq c|z|^{-n/2+1} \quad \text{if } z \in \widehat{D}_-.$$

Therefore

$$m_1(z) = \frac{1}{2\sqrt{-z}} \left(\sum_{j=1}^{2M-1} \tilde{a}_j \sqrt{-z}^{-j} + O\left(z^{-M+n/2-1}\right) \right)$$

on \widehat{D}_- . However the assumption $q \in C^{2M-n}(-\delta, \delta)$ and (95) imply

$$m_1(z) = \frac{1}{2\sqrt{-z}} \left(1 + \sum_{k=1}^{M-(n-1)/2} a_k (-z)^{-k} + O\left(z^{-M+n/2-1}\right) \right)$$

on a sector $\{\epsilon < \arg z < \pi - \epsilon\}$, hence $\tilde{a}_j = 0$ for even j , which implies

$$m_1(-z^2) = \frac{1}{2z} \left(1 + \sum_{k=1}^{M-(n-1)/2} a_k z^{-2k} + O\left(z^{-2M+n-2}\right) \right)$$

on D_- . A similar calculation for $m_2(-z^2)$ is possible and one can obtain

$$m_2(-z^2) = -\frac{1}{2}z \left(1 + \sum_{k=1}^{M-(n-1)/2} b_k z^{-2k} + O\left(z^{-2M+n-2}\right) \right),$$

which together with Lemma 34 for $L = 2M + 2 - n$ completes the proof. ■

To apply Lemma 35 to ergodic potentials we need a lemma. The necessary terminologies can be found in Appendix.

Lemma 36 *Suppose the Lyapunov exponent $\gamma(\lambda)$ satisfies*

$$\int_0^\infty \lambda^m \gamma(\lambda) d\lambda < \infty \quad (106)$$

for some $m > 4$. Then, for a.e. $\omega \in \Omega$ the condition (103) is fulfilled on the curve \widehat{C}_1 by any integer M such that

$$M < \min \left\{ \frac{n_1}{4} - 1, \frac{m - n_1}{2} \right\}. \quad (107)$$

Proof. Set

$$\begin{cases} \rho(\lambda) = \sqrt{-\lambda} N(\lambda) I_{[\lambda_0, 0]}(\lambda) + \frac{1}{\pi} \sqrt{-\lambda} \gamma(\lambda) I_{(0, \infty)}(\lambda) \\ c = \mathbb{E}(q_\omega(0)/2) \\ w(z) = \mathbb{E}(m_\pm(z, \omega)) \end{cases}.$$

Since $\sqrt{-z}w(z)$ is of Herglotz, one has

$$w(z) = -\sqrt{-z} - \frac{c}{\sqrt{-z}} + \frac{1}{\sqrt{-z}} \int_{\lambda_0}^\infty \frac{\rho(\lambda)}{\lambda - z} d\lambda,$$

hence

$$w(z) = \sum_{k=-1}^{m_1} w_k(z) + \tilde{w}(z) \quad (108)$$

with

$$\begin{cases} w_k(z) = c_k (-z)^{-k-1/2} \\ c_{-1} = -1, c_0 = -c, c_k = (-1)^{k-1} \int_{\lambda_0}^{\infty} \lambda^{k-1} \rho(\lambda) d\lambda \\ \tilde{w}(z) = (-1)^{m_1} (-z)^{-m_1-1/2} \int_{\lambda_0}^{\infty} \frac{\lambda^{m_1} \rho(\lambda)}{\lambda - z} d\lambda \end{cases}$$

holds due to the assumption (106), where $m_1 = [m]$. Set

$$\begin{cases} \chi(z) = \frac{-\operatorname{Re} w(z)}{\operatorname{Im} z} - \operatorname{Im} w'(z) \\ \chi_k(z) = \frac{-\operatorname{Re} w_k(z)}{\operatorname{Im} z} - \operatorname{Im} w'_k(z) \\ \tilde{\chi}(z) = \frac{-\operatorname{Re} \tilde{w}(z)}{\operatorname{Im} z} - \operatorname{Im} \tilde{w}'(z) \end{cases} .$$

Noting

$$\begin{aligned} (-x - iy)^{-k-1/2} &= ix^{-k-1/2} (-1)^k (1 + iyx^{-1})^{-k-1/2} \\ &= ix^{-k-1/2} (-1)^k \left(1 - (k+1/2)iyx^{-1} + O(yx^{-1})^2\right) \end{aligned}$$

for $x \geq 1, 0 < y < 1$, we have

$$\chi_k(x + iy) = O\left(x^{-k-5/2}y\right).$$

On the other hand, the estimates

$$\left| \int_{\lambda_0}^{\infty} \frac{\lambda^{m_1} \rho(\lambda)}{(\lambda - z)^j} d\lambda \right| \leq y^{-j} \int_{\lambda_0}^{\infty} |\lambda|^{m_1} \rho(\lambda) d\lambda \quad (j = 1, 2)$$

yield a bound for the last term \tilde{w} of (108). Then, we have

$$\tilde{\chi}(x + iy) = O\left(y^{-2}x^{-m_1-1/2}\right),$$

imply

$$\chi(x + iy) = \sum_{k=-1}^{m_1} \chi_k(x + iy) + \tilde{\chi}(x + iy) = O\left(yx^{-3/2} + y^{-2}x^{-m_1-1/2}\right).$$

This together with $\operatorname{Im} w(x + iy) = O(x^{1/2})$ (due to $N(\lambda) \sim \sqrt{\lambda}$ as $\lambda \rightarrow \infty$) yields

$$\sqrt{2\chi(z) \operatorname{Im} w(z)} = O\left(yx^{-1} + y^{-2}x^{-m_1}\right)^{1/2}.$$

Therefore, if the curve is parametrized as $x + ix^{-(n_1/2-1)}$ near $x = \infty$, applying (137) we have

$$\begin{aligned} \mathbb{E} \left(\int_{\hat{C}} |z|^M |R(z, \omega)| |dz| \right) &\leq \int_{\hat{C}} |z|^M \sqrt{2\chi(z) \operatorname{Im} w(z)} |dz| \\ &\leq c \int_1^{\infty} x^M \left(x^{-(n_1/2-1)} x^{-1} + x^{(n_1-2)} x^{-m_1} \right)^{1/2} dx, \end{aligned}$$

which is finite for M such that

$$M - (n_1/2 - 1)/2 - 1/2 < -1 \text{ and } M + (n_1/2 - 1) - m_1/2 < -1.$$

Then, Fubini's theorem implies the condition (103). ■

Now one can prove Theorem 4. Suppose the Lyapunov exponent $\gamma(\lambda)$ satisfies

$$\int_0^\infty \lambda^m \gamma(\lambda) d\lambda < \infty. \quad (109)$$

Since and Lemma 35 and (107) require

$$L + (n - 1)/2 - 1 \leq M < \min\left(\frac{n_1}{4} - 1, \frac{m - n_1}{2}\right),$$

n_1 should satisfy

$$4L + 2n - 2 < n_1 < m - 2L - n + 3. \quad (110)$$

If

$$m - 2L - n + 3 - (4L + 2n - 2) = m - 6L - 3n + 5 > 2,$$

one can choose an odd integer n_1 satisfying (110). Then applying Lemma 35 and Lemma 36 we have $q_\omega \in \mathcal{Q}_L(C)$ for a.e. ω if $L < (m - 3(n - 1))/6$. On the other hand from

$$q_{\theta_x \omega}(y) = q_\omega(x + y) = (K(e_x)q_\omega)(y)$$

the identity

$$f_g(\theta_x \omega) = (K(g)q_{\theta_x \omega})(0) = (K(g)K(e_x)q_\omega)(0) = (K(e_x)K(g)q_\omega)(0) = K(g)q_\omega(x)$$

follows. Moreover Kotani-Krishna [11] showed that $q_\omega \in C_b^m(\mathbb{R})$ implies

$$\int_0^\infty \lambda^{m+1/2} \gamma(\lambda) d\lambda < \infty,$$

which is sufficient for (109) and completes the proof of Theorem 4.

9 Appendix

9.1 Calculation of $\tau_{\mathbf{a}}(r)$, $m_{r\mathbf{a}}(z)$ for rational functions r

This section is devoted to the calculation of $\tau_{\mathbf{a}}(r)$, $m_{r\mathbf{a}}(z)$ for general rational function r in terms of the characteristic functions $\{\varphi_{\mathbf{a}}, \psi_{\mathbf{a}}\}$ and m -function $m_{\mathbf{a}}$.

The simplest rational functions are

$$\begin{cases} p_\zeta(z) = 1 + z\zeta^{-1} \in \Gamma_0^{(1)} \\ q_\zeta(z) = (1 - z\zeta^{-1})^{-1} \in \Gamma_0^{(-1)} \end{cases} \text{ for } \zeta \in D_-.$$

Any rational function can be represented as a product of these simple functions.

We treat $r \in \Gamma_0^{(m)}$ with $m \leq 0$. Recall

$$\tau_{\mathbf{a}}(g) = \det(g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1})$$

for $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$, which is well-defined if $g^{-1}T(g\mathbf{a})T(\mathbf{a})^{-1} - I$ is of trace class in some space $H_N(D_+)$. Let $M = L - N \geq 0$ and observe for $u \in H_N(D_+)$

$$\begin{aligned} T(g\mathbf{a})T(\mathbf{a})^{-1}u &= \mathfrak{p}_+(g\mathfrak{p}_+\mathbf{a}T(\mathbf{a})^{-1}u) + \mathfrak{p}_+(g\mathfrak{p}_-\mathbf{a}T(\mathbf{a})^{-1}u) \\ &= gu + H_g\mathfrak{p}_-(\mathbf{a}T(\mathbf{a})^{-1}u) \end{aligned}$$

with $H_g : H_M(D_-) (= z^{-M}H(D_-)) \rightarrow H(D_+)$ defined by

$$H_g v(z) = \mathfrak{p}_+(gv)(z) = \frac{1}{2\pi i} \int_C \frac{g(\lambda)}{\lambda - z} v(\lambda) d\lambda. \quad (111)$$

Here $v \in H(D_-)$ is

$$v = \mathfrak{p}_-(\mathbf{a}T(\mathbf{a})^{-1}u) = \mathbf{a}T(\mathbf{a})^{-1}u - u.$$

The key identity is (19):

$$T(\mathbf{a})^{-1} \frac{1}{z+b} = \frac{(\psi_{\mathbf{a}}(b) + b)u - (\varphi_{\mathbf{a}}(b) + 1)v}{\Delta_{\mathbf{a}}(b)(z^2 - b^2)} \text{ for } b \in D_-.$$

Lemma 37 Let $\mathbf{a} \in \mathbf{A}_L^{inv}(C)$ and $r \in \Gamma_0^{(m)}$.

(i) For any N such that $-m \leq N \leq L$ the operator $r^{-1}T(r\mathbf{a})T(\mathbf{a})^{-1} - I$ has a finite rank on $H_N(D_+)$ not greater than the numbers of the poles of r . Hence $\tau_{\mathbf{a}}(r)$ is well-defined for any $r \in \Gamma_0^{(m)}$.

(ii) Suppose $L \geq 2$. For $\zeta, \zeta_1, \zeta_2 \in C \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R})$

$$\begin{cases} \tau_{\mathbf{a}}(q\zeta) = 1 + \varphi_{\mathbf{a}}(\zeta) \\ \tau_{\mathbf{a}}(q\zeta_1 q\zeta_2) = (1 + \varphi_{\mathbf{a}}(\zeta_1))(1 + \varphi_{\mathbf{a}}(\zeta_2)) \frac{m_{\mathbf{a}}(\zeta_1) - m_{\mathbf{a}}(\zeta_2)}{\zeta_1 - \zeta_2} \end{cases}. \quad (112)$$

(iii) Suppose $L \geq 2$. Suppose $r \in \Gamma_0^{(0)}$ has simple zeroes $\{\eta_j\}_{1 \leq j \leq n}$ and simple poles $\{\zeta_j\}_{1 \leq j \leq n}$ in D_- . Then H_r has a finite rank not greater than n , and it holds that

$$\begin{aligned} \tau_{\mathbf{a}}(r) &= \left(\prod_{j=1}^n \frac{(\varphi_{\mathbf{a}}(\zeta_j) + 1)(\varphi_{\mathbf{a}}(-\eta_j) + 1)}{\Delta_{\mathbf{a}}(\eta_j) r'(\eta_j) \hat{r}'(\zeta_j)} \right) \det \left(\frac{1}{\eta_i - \zeta_j} \right) \det \left(\frac{m_{\mathbf{a}}(\zeta_i) - m_{\mathbf{a}}(-\eta_j)}{\zeta_i^2 - \eta_j^2} \right), \end{aligned} \quad (113)$$

with $\hat{r}(z) = r(z)^{-1}$.

Proof. Let $\{\zeta_j\}_{j=1}^n$ be the poles of r . For simplicity assume they are simple. Then r can be expressed as

$$r(z) = r(\infty) + \sum_{j=1}^n \frac{r_j}{z - \zeta_j} \text{ with } r_j = \lim_{z \rightarrow \zeta_j} (z - \zeta_j) r(z) = \frac{1}{\hat{r}'(\zeta_j)},$$

hence for $v \in H(D_-)$

$$(H_r v)(z) = \frac{1}{2\pi i} \int_C \frac{r(\lambda)}{\lambda - z} v(\lambda) d\lambda = \sum_{j=1}^n \frac{r_j v(\zeta_j)}{z - \zeta_j}$$

and

$$r(z)^{-1} (H_r v)(z) = \sum_{j=1}^n f_j(z) v(\zeta_j) \quad \text{with} \quad f_j(z) = \frac{r_j}{(z - \zeta_j) r(z)}, \quad (114)$$

which means that the map $r^{-1} T(\mathbf{a}) T(\mathbf{a})^{-1} - I$ is of finite rank.

To compute $\tau_{\mathbf{a}}(r)$, we take the independent vectors f_j and obtain the coefficients of the image of $r^{-1} H_r$ for $u = T(\mathbf{a})^{-1} f_j$.

If $L \geq 2$, one can use the characteristic functions and m -function. Let $r = q_{\zeta}$. Then $n = 1$ and $f_1 = 1$, hence $r(z)^{-1} (H_r v)(z) = v(\zeta)$. For $u = 1$

$$v = \mathbf{a} T(\mathbf{a})^{-1} 1 - 1 = \varphi_{\mathbf{a}},$$

which yields $\tau_{\mathbf{a}}(q_{\zeta}) = 1 + \varphi_{\mathbf{a}}(\zeta)$. If $r = q_{\zeta_1} q_{\zeta_2}$, then $n = 2$ and

$$f_1(z) = \frac{r_1}{(z - \zeta_1) r(z)} = \frac{\zeta_2 - z}{\zeta_2 - \zeta_1}, \quad f_2(z) = \frac{r_2}{(z - \zeta_2) r(z)} = \frac{\zeta_1 - z}{\zeta_1 - \zeta_2}.$$

For $u = f_1$, $u = f_2$

$$\begin{cases} v_1 = \mathbf{a} T(\mathbf{a})^{-1} f_1 - f_1 = \frac{\zeta_2}{\zeta_2 - \zeta_1} \varphi_{\mathbf{a}} - \frac{1}{\zeta_2 - \zeta_1} \psi_{\mathbf{a}} \\ v_2 = \mathbf{a} T(\mathbf{a})^{-1} f_2 - f_2 = \frac{\zeta_1}{\zeta_1 - \zeta_2} \varphi_{\mathbf{a}} - \frac{1}{\zeta_1 - \zeta_2} \psi_{\mathbf{a}} \end{cases},$$

hence

$$\begin{aligned} \tau_{\mathbf{a}}(q_{\zeta_1} q_{\zeta_2}) &= \det \begin{pmatrix} 1 + v_1(\zeta_1) & v_1(\zeta_2) \\ v_2(\zeta_1) & 1 + v_2(\zeta_2) \end{pmatrix} \\ &= \frac{(\zeta_1 + \psi_{\mathbf{a}}(\zeta_1))(1 + \varphi_{\mathbf{a}}(\zeta_2)) - (1 + \varphi_{\mathbf{a}}(\zeta_1))(\zeta_2 + \psi_{\mathbf{a}}(\zeta_2))}{\zeta_1 - \zeta_2}, \end{aligned}$$

which is (112).

Now go back to (113). Since $f_j(z)$ is a rational function with poles at η_i and $f_j(\infty) = 0$, an identity

$$f_j(z) = \frac{r_j}{(z - \zeta_j) r(z)} = \sum_{i=1}^n \frac{r_{ij}}{(z - \zeta_j)(z - \eta_i)}$$

with

$$r_{ij} = \lim_{z \rightarrow \eta_i} \frac{r_j(z - \eta_i)}{(z - \zeta_j) r(z)} = \frac{r_j}{(\eta_i - \zeta_j) r'(\eta_i)}$$

is valid. Then (19) yields

$$T(\mathbf{a})^{-1} f_j = \sum_i r_{ij} T(\mathbf{a})^{-1} \frac{1}{z - \eta_i} = \sum_i r_{ij} \frac{(\varphi_{\mathbf{a}}(-\eta_i) + 1)v - (\psi_{\mathbf{a}}(-\eta_i) - \eta_i)u}{\Delta_{\mathbf{a}}(\eta_i)(z^2 - \eta_i^2)}$$

and

$$\begin{aligned} \mathbf{a} T(\mathbf{a})^{-1} f_j &= \sum_i r_{ij} \frac{(\varphi_{\mathbf{a}}(-\eta_i) + 1)(\psi_{\mathbf{a}} + z) - (\psi_{\mathbf{a}}(-\eta_i) - \eta_i)(\varphi_{\mathbf{a}} + 1)}{\Delta_{\mathbf{a}}(\eta_i)(z^2 - \eta_i^2)} \\ &= (\varphi_{\mathbf{a}} + 1) \sum_i r_{ij} (\varphi_{\mathbf{a}}(-\eta_i) + 1) \frac{m_{\mathbf{a}} - m_{\mathbf{a}}(-\eta_i)}{\Delta_{\mathbf{a}}(\eta_i)(z^2 - \eta_i^2)}. \end{aligned}$$

Therefore, noting $f_j(\zeta_i) = \delta_{ij}$, we have

$$\begin{aligned}
\tau_{\mathbf{a}}(r) &= \det \left((\delta_{ij} + (\mathbf{a}T(\mathbf{a})^{-1}f_j)(\zeta_i) - f_j(\zeta_i))_{1 \leq i, j \leq n} \right) \\
&= \det \left((\mathbf{a}T(\mathbf{a})^{-1}f_j)(\zeta_i) \right) \\
&= \det \left((\varphi_{\mathbf{a}}(\zeta_i) + 1) \sum_{k=1}^n r_{kj} \frac{m_{\mathbf{a}}(\zeta_i) - m_{\mathbf{a}}(-\eta_k)}{\zeta_i^2 - \eta_k^2} \frac{\varphi_{\mathbf{a}}(-\eta_k) + 1}{\Delta_{\mathbf{a}}(\eta_k)} \right). \\
&= \left(\prod_{j=1}^n \frac{(\varphi_{\mathbf{a}}(\zeta_i) + 1)(\varphi_{\mathbf{a}}(-\eta_j) + 1)}{\Delta_{\mathbf{a}}(\eta_j)} \right) \det(r_{ij}) \det \left(\frac{m_{\mathbf{a}}(\zeta_i) - m_{\mathbf{a}}(-\eta_j)}{\zeta_i^2 - \eta_j^2} \right),
\end{aligned}$$

where

$$\det(r_{ij}) = \det \left(\frac{1}{(\eta_i - \zeta_j) r'(\eta_i) \widehat{r}'(\zeta_j)} \right) = \left(\prod_{j=1}^n \frac{1}{r'(\eta_i) \widehat{r}'(\zeta_j)} \right) \det \left(\frac{1}{\eta_i - \zeta_j} \right),$$

which is (113). ■

It should be remarked that if $\eta_i \neq \zeta_j$, then

$$\det \left(\frac{1}{\eta_i - \zeta_j} \right) \neq 0.$$

This is because the identity

$$0 = \sum_{j=1}^n \frac{u_j}{\eta_i - \zeta_j} \quad \text{for any } i$$

implies the rational function $f(z) = \sum_{j=1}^n u_j (z - \zeta_j)^{-1}$ satisfies $f(\eta_i) = 0$ for $i = 1, 2, \dots, n$, which shows $f(z) = 0$ identically.

The next task is to compute $m_{r\mathbf{a}}$ for rational functions r . In principle the computation for general rational r is possible similarly as the previous lemma, however to grasp the picture it is enough to know the change of $m_{r\mathbf{a}}$ for $r = q_{\zeta}p_{\eta}$, since the formula of $m_{r\mathbf{a}}$ for general r can be obtained by iteration of $q_{\zeta}p_{\eta}$.

Lemma 38 *Let $\mathbf{a} \in \mathbf{A}_2^{inv}$ and $\zeta, \eta \in D_-$ and assume $\tau_{\mathbf{a}}(q_{\zeta}p_{\eta}) \neq 0$. Then*

$$m_{q_{\zeta}p_{\eta}\mathbf{a}}(z) = (d_{\zeta}d_{\eta}m_{\mathbf{a}})(z).$$

Proof. Let $r = q_{\zeta}p_{\eta}$ with $\zeta, \eta \in D_-$. First we have to compute $\varphi_{r\mathbf{a}}, \psi_{r\mathbf{a}}$. Then

$$r(z)^{-1} = r_1 + \frac{r_2}{z + \eta} \quad \text{with } r_1 = -\frac{\eta}{\zeta}, \quad r_2 = \eta \left(1 + \frac{\eta}{\zeta} \right) \quad (115)$$

holds. To compute $\varphi_{r\mathbf{a}}$ set $w_1 = T(r\mathbf{a})^{-1}1$. The definition implies $1 + \varphi_{r\mathbf{a}} = raw_1$, hence (115) yields

$$\begin{aligned}
(\mathbf{a}w_1)(z) &= r(z)^{-1} (1 + \varphi_{r\mathbf{a}}(z)) \\
&= r_1 + \frac{r_2 (1 + \varphi_{r\mathbf{a}}(-\eta))}{z + \eta} + \frac{r_2 (\varphi_{r\mathbf{a}}(z) - \varphi_{r\mathbf{a}}(-\eta))}{z + \eta},
\end{aligned} \quad (116)$$

which is a decomposition in $H_1(D_+) \oplus H(D_-)$. Applying \mathfrak{p}_+ , we have

$$T(\mathbf{a})w_1 = r_1 + \frac{r_2(1 + \varphi_{r\mathbf{a}}(-\eta))}{z + \eta}.$$

Therefore (19) implies

$$\begin{aligned} w_1 &= r_1 T(\mathbf{a})^{-1} 1 + r_2 (1 + \varphi_{r\mathbf{a}}(-\eta)) T(\mathbf{a})^{-1} \frac{1}{z + \eta} \\ &= r_1 u + \mu_1 \frac{(m_{\mathbf{a}}(\eta) - \kappa_1(\mathbf{a}))u - v}{z^2 - \eta^2} \end{aligned}$$

with

$$\mu_1 = \frac{r_2(1 + \varphi_{\mathbf{a}}(\eta))(1 + \varphi_{r\mathbf{a}}(-\eta))}{\Delta_{\mathbf{a}}(\eta)},$$

where $u = T(\mathbf{a})^{-1} 1$, $v = T(\mathbf{a})^{-1} z$, from which it follows that

$$\mathbf{a}w_1 = (1 + \varphi_{\mathbf{a}}) \left(r_1 - \mu_1 \frac{m_{\mathbf{a}} - m_{\mathbf{a}}(\eta)}{z^2 - \eta^2} \right). \quad (117)$$

The identity (116) shows the left hand side is meromorphic on D_- vanishing at $z = \zeta$, hence

$$\mu_1 \frac{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}{\zeta^2 - \eta^2} = r_1$$

holds, which together with (117) shows

$$1 + \varphi_{r\mathbf{a}} = r_1 r (1 + \varphi_{\mathbf{a}}) \left(1 - \frac{\zeta^2 - \eta^2}{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)} \frac{m_{\mathbf{a}} - m_{\mathbf{a}}(\eta)}{z^2 - \eta^2} \right). \quad (118)$$

This identity also shows

$$\kappa_1(r\mathbf{a}) = \lim_{z \rightarrow \infty} z \varphi_{r\mathbf{a}}(z) = \zeta + \eta + \kappa_1(\mathbf{a}) - \frac{\zeta^2 - \eta^2}{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}. \quad (119)$$

Similarly one has $\psi_{r\mathbf{a}}$, namely $w_2 = T(r\mathbf{a})^{-1} z$ satisfies

$$\begin{aligned} (\mathbf{a}w_2)(z) &= r(z)^{-1} (z + \psi_{r\mathbf{a}}(z)) \\ &= r_1 z + r_2 + r_2 \frac{-\eta + \psi_{r\mathbf{a}}(-\eta)}{z + \eta} + r_2 \frac{\psi_{r\mathbf{a}}(z) - \psi_{r\mathbf{a}}(-\eta)}{z + \eta}, \end{aligned}$$

hence

$$T(\mathbf{a})w_2 = r_1 z + r_2 + r_2 \frac{-\eta + \psi_{r\mathbf{a}}(-\eta)}{z + \eta},$$

which yields

$$w_2 = r_1 v + r_2 u + \mu_2 \frac{(m_{\mathbf{a}}(\eta) - \kappa_1(\mathbf{a}))u - v}{z^2 - \eta^2}$$

with

$$\mu_2 = \frac{r_2(\varphi_{\mathbf{a}}(\eta) + 1)(-\eta + \psi_{r\mathbf{a}}(-\eta))}{\Delta_{\mathbf{a}}(\eta)}.$$

Then

$$\mathbf{a}w_2 = r_1(z + \psi_{\mathbf{a}}) + (1 + \varphi_{\mathbf{a}}) \left(r_2 - \mu_2 \frac{m_{\mathbf{a}} - m_{\mathbf{a}}(\eta)}{z^2 - \eta^2} \right)$$

with

$$\mu_2 = \frac{r_2(\varphi_{\mathbf{a}}(\eta) + 1)(-\eta + \psi_{r\mathbf{a}}(-\eta))}{\Delta_{\mathbf{a}}(\eta)},$$

hence setting $z = \zeta$ we have

$$r_1(\zeta + \psi_{\mathbf{a}}(\zeta)) + (1 + \varphi_{\mathbf{a}}(\zeta)) \left(r_2 - \mu_2 \frac{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}{\zeta^2 - \eta^2} \right) = 0,$$

which yields

$$\mu_2 = r_2 \frac{\zeta^2 - \eta^2}{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)} \left(1 - \frac{m_{\mathbf{a}}(\zeta) - \kappa_1(\mathbf{a})}{\zeta + \eta} \right),$$

and

$$\begin{aligned} & z + \psi_{r\mathbf{a}}(z) \\ &= \frac{r_2}{r_1} \left(1 - \frac{m_{\mathbf{a}}(\zeta) - \kappa_1(\mathbf{a})}{\zeta + \eta} \right) (1 + \varphi_{r\mathbf{a}}(z)) - r_2 r (1 + \varphi_{\mathbf{a}}) \frac{m_{\mathbf{a}} - m_{\mathbf{a}}(\zeta)}{\zeta + \eta} \end{aligned} \quad (120)$$

Consequently $m_{r\mathbf{a}}$ is computed from (117), (119) and (120) as

$$\begin{aligned} m_{r\mathbf{a}}(z) &= \frac{z + \psi_{r\mathbf{a}}(z)}{1 + \varphi_{r\mathbf{a}}(z)} + \kappa_1(r\mathbf{a}) \\ &= \frac{(m_{\mathbf{a}}(z) - m_{\mathbf{a}}(\zeta)) \frac{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}{\zeta^2 - \eta^2}}{\frac{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}{\zeta^2 - \eta^2} - \frac{m_{\mathbf{a}}(z) - m_{\mathbf{a}}(\eta)}{z^2 - \eta^2}} + m_{\mathbf{a}}(\zeta) - \frac{m_{\mathbf{a}}(\zeta) - m_{\mathbf{a}}(\eta)}{\zeta^2 - \eta^2} \\ &= (d_{\zeta} d_{\eta} m_{\mathbf{a}})(z) \end{aligned}$$

■

This formula can be easily understood if we first show the identity $m_{q_{\zeta}\mathbf{a}} = d_{\zeta} m_{\mathbf{a}}$. The reason why we do not take this procedure is that we have defined $m_{\mathbf{a}}$ for \mathbf{a} such that $T(\mathbf{a})$ maps $H_N(D_+)$ to $H_N(D_+)$ bijectively, hence $m_{q_{\zeta}\mathbf{a}}$, $m_{p_{\zeta}\mathbf{a}}$ are out of the present framework. However, a slight modification of the definition of $m_{\mathbf{a}}$ might allow us to show

$$m_{q_{\zeta}\mathbf{a}} = m_{p_{\zeta}\mathbf{a}} = d_{\zeta} m_{\mathbf{a}},$$

and the identity of the Lemma would be more straightly understandable.

9.2 Properties of m -functions and Herglotz functions

The m -function $m_{\mathbf{a}}$ for $\mathbf{a} \in \mathbf{A}_{L,+}^{inv}(C)$ has the following properties:

$$\begin{aligned} & m_{\mathbf{a}} \text{ is analytic on } \mathbb{C} \setminus ([-\mu_0, \mu_0] \cup i\mathbb{R}) \text{ and } m_{\mathbf{a}}(z) = \overline{m_{\mathbf{a}}(\bar{z})} \\ & \frac{\operatorname{Im} m_{\mathbf{a}}(z)}{\operatorname{Im} z} > 0 \text{ for } z \in \mathbb{C} \setminus \mathbb{R}, \quad \frac{m_{\mathbf{a}}(x) - m_{\mathbf{a}}(-x)}{x} > 0 \text{ for } |x| > \mu_0. \end{aligned} \quad (121)$$

with $\mu_0 = \sqrt{-\lambda_0}$. For such function m define

$$m_+(z) = -m(\sqrt{-z}), \quad m_-(z) = m(-\sqrt{-z}) \quad \text{for } z \in \mathbb{C}_+.$$

Then m_{\pm} become Herglotz functions, namely they analytic functions on $\mathbb{C} \setminus [\lambda_0, \infty)$ satisfying

$$\begin{aligned} m_{\pm}(z) &= \overline{m_{\pm}(\bar{z})} \quad \text{and} \quad \frac{\operatorname{Im} m_{\pm}(z)}{\operatorname{Im} z} \geq 0 \quad \text{for any } z \in \mathbb{C} \setminus \mathbb{R} \\ m_+(x) + m_-(x) &< 0 \quad \text{for } x < \lambda_0 \end{aligned} .$$

A necessary and sufficient condition for m to be a Herglotz function is that m has a representation

$$m(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \sigma(d\xi)$$

with a real α , non-negative β and measure σ on \mathbb{R} satisfying

$$\int_{-\infty}^{\infty} \frac{1}{\xi^2 + 1} \sigma(d\xi) < \infty.$$

The present m_{\pm} are represented as

$$m_{\pm}(z) = \alpha_{\pm} + \beta_{\pm} z + \int_{\lambda_0}^{\infty} \left(\frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \sigma_{\pm}(d\xi).$$

The original m is recovered by m_{\pm} by

$$m(z) = \begin{cases} -m_+(-z^2) & \text{if } \operatorname{Re} z > 0 \\ m_-(-z^2) & \text{if } \operatorname{Re} z < 0 \end{cases} .$$

Lemma 39 *If m satisfies the property (121), then so do $d_s m$, $d_{\zeta} d_{\bar{\zeta}} m$ if $|s| > \mu_0$, $\zeta \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$.*

Proof. If $s > 0$, $\operatorname{Re} z > 0$, then setting $w \equiv -z^2 \in \mathbb{C}_-$, $u = -s^2 < \lambda_0$ we have

$$d_s m(z) = \frac{z^2 - s^2}{-m_+(-z^2) + m_+(-s^2)} + m_+(-s^2) = \frac{w - u}{m_+(w) - m_+(u)} + m_+(u).$$

Since m_+ is of Herglotz,

$$\frac{m_+(w) - m_+(u)}{w - u} = \beta_+ + \int_{\lambda_0}^{\infty} \frac{1}{(\lambda - w)(\lambda - u)} \sigma_+(d\lambda)$$

and we see $\operatorname{Im}(m_+(w) - m_+(u)) / (w - u) > 0$ due to $\operatorname{Im} w < 0$, $u < \lambda_0$, which implies $\operatorname{Im} d_s m(z) > 0$. If $s < 0$, $\operatorname{Re} z > 0$, then

$$d_s m_{\mathbf{a}}(z) = \frac{z^2 - s^2}{-m_+(-z^2) - m_-(-s^2)} - m_-(-s^2) = \frac{w - u}{m_+(w) + m_-(u)} - m_-(u).$$

Since

$$\frac{m_+(w) + m_-(u)}{w - u} = \frac{m_+(w) - m_+(u)}{w - u} + \frac{m_+(u) + m_-(u)}{w - u} \in \mathbb{C}_-$$

due to $m_+(u) + m_-(u) < 0$, we have $d_s m_a(z) \in \mathbb{C}_+$. The cases ($s < 0$, $\operatorname{Re} z < 0$), ($s > 0$, $\operatorname{Re} z < 0$) can be treated similarly.

On the other hand note

$$d_\zeta d_{\bar{\zeta}} m(z) = \frac{z^2 - \bar{\zeta}^2}{\frac{z^2 - \zeta^2}{m(z) - m(\zeta)} - \frac{\zeta^2 - \bar{\zeta}^2}{m(\zeta) - \overline{m(\zeta)}}} - \left(\frac{\zeta^2 - \bar{\zeta}^2}{m(\zeta) - \overline{m(\zeta)}} - m(\zeta) \right).$$

We can assume $\operatorname{Im} z, \operatorname{Im} \zeta > 0$. To compute the imaginary part the term $(\zeta^2 - \bar{\zeta}^2) / (m(\zeta) - \overline{m(\zeta)})$ can be neglected, and

$$\frac{z^2 - \bar{\zeta}^2}{\frac{z^2 - \zeta^2}{m(z) - m(\zeta)} - \frac{\zeta^2 - \bar{\zeta}^2}{m(\zeta) - \overline{m(\zeta)}}} + m(\zeta) = \frac{m(z)w - \overline{m(\zeta)}a}{w - a}$$

with

$$w = \frac{z^2 - \zeta^2}{m(z) - m(\zeta)}, \quad a = \frac{\zeta^2 - \bar{\zeta}^2}{m(\zeta) - \overline{m(\zeta)}}.$$

Hence

$$\begin{aligned} \operatorname{Im} d_\zeta d_{\bar{\zeta}} m(z) &= \operatorname{Im} \frac{(m(z)w - \overline{m(\zeta)}a)(\bar{w} - a)}{|w - a|^2} \\ &= \frac{|w|^2 \operatorname{Im} m(z) - a \operatorname{Im} w (m(z) - m(\zeta)) - a^2 \operatorname{Im} m(\zeta)}{|w - a|^2} \\ &= \frac{|w|^2 \operatorname{Im} m(z) - a \operatorname{Im} (z^2 - \zeta^2) - a^2 \operatorname{Im} m(\zeta)}{|w - a|^2} \end{aligned}$$

Suppose $\operatorname{Re} z, \operatorname{Re} \zeta > 0$, then

$$\begin{aligned} &|w|^2 \operatorname{Im} m(z) - a \operatorname{Im} (z^2 - \zeta^2) - a^2 \operatorname{Im} m(\zeta) \\ &= \frac{\operatorname{Im} v \operatorname{Im} u}{\operatorname{Im} m_+(v)} - \left| \frac{u - v}{m_+(u) - m_+(v)} \right|^2 \operatorname{Im} m_+(u) \end{aligned}$$

with $u = -z^2, v = -\zeta^2 \in \mathbb{C}_-$. The Herglotz representation for m_+ shows

$$\begin{cases} \frac{m_+(u) - m_+(v)}{u - v} = \beta_+ + \int_{\lambda_0}^{\infty} \frac{1}{(\lambda - u)(\lambda - v)} \sigma_+(d\lambda) \\ \operatorname{Im} m_+(u) = \beta_+ + \int_{\lambda_0}^{\infty} \frac{1}{|\lambda - u|^2} \sigma_+(d\lambda) \end{cases},$$

which implies

$$\frac{\operatorname{Im} v \operatorname{Im} u}{\operatorname{Im} m_+(v)} - \left| \frac{u - v}{m_+(u) - m_+(v)} \right|^2 \operatorname{Im} m_+(u) > 0$$

if $u, v \in \mathbb{C}_-$. The case $\operatorname{Re} z > 0, \operatorname{Re} \zeta < 0$ is computed similarly, that is

$$\begin{aligned} &|w|^2 \operatorname{Im} m(z) - a \operatorname{Im} (z^2 - \zeta^2) - a^2 \operatorname{Im} m(\zeta) \\ &= - \left| \frac{u - v}{m_+(u) + m_-(v)} \right|^2 \operatorname{Im} m_+(u) - \frac{\operatorname{Im} v \operatorname{Im} u}{\operatorname{Im} m_-(v)} > 0, \end{aligned}$$

since $\operatorname{Im} m_+(u) < 0, \operatorname{Im} m_-(v) > 0$ if $\operatorname{Im} u < 0, \operatorname{Im} v > 0$, which completes the proof. ■

9.3 Conformal maps

Although Riemann mapping theorem says that every simply connected domain on \mathbb{C} can be an image of a conformal map on \mathbb{C}_+ , sometimes a quantitative estimate of it is necessary. In this section we provide a model of conformal map from $\mathbb{C} \setminus (-\infty, 0]$ to D_ω^- of (43).

A conformal map ψ on \mathbb{C}_+ is easily obtained if $\text{Im } \psi'(z)$ has a definite sign on \mathbb{C}_+ . A simple such example is $\psi_\infty(z) = \sqrt{z}$, and a more general conformal map in this framework can be constructed for an integer $k \geq 1$ by an integral

$$\psi_k(z) = \frac{k-1/2}{k+1/2} \sqrt{z} + \int_0^\infty \sqrt{z+t} (1+t)^{-k-3/2} dt.$$

This ψ_k satisfies

$$\text{Re } \psi_k(z) > 0, \quad \text{Im } \psi_k(z) > 0, \quad \text{Re } \psi'_k(z) > 0, \quad \text{Im } \psi'_k(z) < 0$$

for $z \in \mathbb{C}_+$, hence ψ_k maps \mathbb{C}_+ to $\psi_k(\mathbb{C}_+) (\subset \mathbb{C}_+)$, and $\phi_k(z) = \psi_k(z)^2$ maps \mathbb{C}_+ to $\phi_k(\mathbb{C}_+) (\subset \mathbb{C}_+)$ conformally. Since $\psi_k(z)$ takes real values on $[0, \infty)$, $\psi_k(z)$ and $\phi_k(z)$ can be extended as conformal maps from $\mathbb{C} \setminus (-\infty, 0]$ to a domain in $\{\text{Re } z > 0\}$ and a domain in \mathbb{C} respectively. Set

$$\begin{cases} a_k = 2 \int_0^1 s^2 (1-s^2)^{k-1} ds = \frac{\sqrt{\pi} \Gamma(k)}{2\Gamma(k+3/2)} \\ b_k = 2a_k \left(2a_k^2 k (k+1/2)^2 / (k-1/2)^2 + 1 \right)^{-1/2} \end{cases}.$$

Lemma 40 *The image $\phi_k(\mathbb{C} \setminus (-\infty, 0])$ is described as follows:*

$$\phi_k(\mathbb{C} \setminus (-\infty, 0]) = \mathbb{C} \setminus \{z \in \mathbb{C}; |\text{Im } z| \leq \omega(\text{Re } z), \text{Re } z \leq a_k^2\}$$

with positive smooth function $\omega(x)$ on $(-\infty, a_k^2)$ such that

$$\omega(x) = \begin{cases} 2a_k (-x)^{-k+1/2} (1 + O(x^{-1})) & \text{as } x \rightarrow -\infty \\ b_k (a_k^2 - x)^{1/2} (1 + O(a_k^2 - x)) & \text{as } x \rightarrow a_k^2 - 0 \end{cases}. \quad (122)$$

Moreover, ϕ_k takes a form of

$$\phi_k(z) = z + f_1(z) + z^{-k+1/2} f_2(z) \quad (123)$$

with some real rational functions f_1, f_2 (that is, $f_j(z) = \overline{f_j(\bar{z})}$ for $j = 1, 2$) satisfying

$$\begin{cases} f_1(\infty) = (k^2 - 1/4)^{-1} \\ f_2(\infty) = 2(-1)^k a_k \end{cases}.$$

Conversely, $\phi_k^{-1}(w)$ has an expression

$$\phi_k^{-1}(w) = w + g_1(w) + w^{-k+1/2} g_2(w) \quad (124)$$

with real g_1, g_2 analytic in a neighborhood of ∞ . Moreover, it holds that

$$g_1(\infty) = -(k^2 - 1/4)^{-1}, \quad g_2(\infty) = -2(-1)^k a_k.$$

Proof. Setting $s = \sqrt{(z+t)/(1+t)}$, we have

$$\psi_k(z) = \frac{k-1/2}{k+1/2} \sqrt{z} + 2(z-1)^{-k} \int_1^{\sqrt{z}} s^2 (s^2-1)^{k-1} ds.$$

Since the integral $\int_0^z s^2 (s^2-1)^{k-1} ds$ is an odd polynomial of degree $2k+1$, the integral

$$p(z) = \frac{k-1/2}{k+1/2} (z-1)^k + 2\sqrt{z}^{-1} \int_0^{\sqrt{z}} s^2 (s^2-1)^{k-1} ds$$

defines a polynomial of degree k , and

$$\psi_k(z) = (z-1)^{-k} (\sqrt{z}p(z) - p(1)) \quad (125)$$

holds. It should be noted that $\sqrt{z}p(z) - p(1)$ has zero of degree k at $z=1$, so $\psi_k(z)$ has no singularity at $z=1$. Set

$$s(x) = \operatorname{Re} \psi_k(x+i0), \quad t(x) = \operatorname{Im} \psi_k(x+i0)$$

for $x \in \mathbb{R}$. Then, (125) implies

$$s(x) = \begin{cases} -p(1)(x-1)^{-k} & \text{for } x < 0 \\ (x-1)^{-k} (\sqrt{x}p(x) - p(1)) & \text{for } x \geq 0 \end{cases},$$

$$t(x) = \begin{cases} (x-1)^{-k} \sqrt{-x}p(x) & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases},$$

and their asymptotics are

$$s(x) = \begin{cases} a_k(1+kx+O(x^2)) & \text{as } x \rightarrow -0 \\ a_k(-x)^{-k}(1+kx^{-1}+O(x^{-2})) & \text{as } x \rightarrow -\infty \end{cases}$$

$$t(x) = \begin{cases} \frac{k-1/2}{k+1/2} \sqrt{-x}(1+O(x)) & \text{as } x \rightarrow -0 \\ \sqrt{-x}(1+O(x^{-1})) & \text{as } x \rightarrow -\infty \end{cases},$$

where we have used

$$\begin{cases} p(1) = (-1)^{k-1} a_k, & p(0) = \frac{k-1/2}{k+1/2} (-1)^k \\ p(z) = z^k - \left(k - \frac{2}{4k^2-1}\right) z^{k-1} + \dots \end{cases}.$$

From (125)

$$\phi_k(z) = \psi_k(z)^2 = z + f_1(z) + z^{-k+1/2} f_2(z)$$

follows, which yields (123) with

$$\begin{cases} f_1(z) = (z-1)^{-2k} (p(1)^2 + zp(z)^2) - z \\ f_2(z) = -2(z-1)^{-2k} p(1)p(z)z^k \end{cases},$$

and

$$\begin{cases} \operatorname{Re} \phi_k(x + i0) = x + f_1(x) + f_2(x) \times \begin{cases} x^{-k+1/2} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \\ \operatorname{Im} \phi_k(x + i0) = \begin{cases} 0 & \text{if } x > 0 \\ \sqrt{-x} x^{-k} f_2(x) & \text{if } x < 0 \end{cases} \end{cases} \quad (126)$$

is valid, hence (25) shows

$$\begin{cases} \operatorname{Re} \phi_k(x + i0) = x + (k^2 - 1/4)^{-1} + O((-x)^{-1}) \\ \operatorname{Im} \phi_k(x + i0) = 2a_k (-x)^{-k+1/2} + O((-x)^{-k-1/2}) \end{cases}$$

as $x \rightarrow -\infty$, and

$$\begin{cases} \operatorname{Re} \phi_k(x + i0) = a_k^2 + \left(2ka_k^2 + \left(\frac{k-1/2}{k+1/2} \right)^2 \right) x + O(x^2) \\ \operatorname{Im} \phi_k(x + i0) = 2 \frac{k-1/2}{k+1/2} a_k \sqrt{-x} + O((-x)^{3/2}) \end{cases}$$

as $x \rightarrow -0$. Since $\operatorname{Re} \phi_k(x + i0) = s(x)^2 - t(x)^2$, $\operatorname{Im} \phi_k(x + i0) = 2s(x)t(x)$, (22) implies

$$\begin{cases} \operatorname{Re} \phi_k(x + i0) \text{ is increasing and moving from } -\infty \text{ to } \infty \\ \operatorname{Im} \phi_k(x + i0) > 0 \text{ on } (-\infty, 0) \text{ and } 0 \text{ on } [0, \infty) \end{cases}.$$

Therefore, ω can be defined by an equation

$$\omega(\operatorname{Re} \phi_k(x + i0)) = \operatorname{Im} \phi_k(x + i0).$$

due to (24), and (126), (65), (24) show $\omega(x)$ satisfies (122).

We use (125) to show (124). Set $\vartheta(z) = z^2$. ϑ is a conformal map from $\{\operatorname{Re} z > 0\}$ to $\mathbb{C} \setminus (-\infty, 0]$ and define $\tilde{\psi}_k(s) = \psi_k(\vartheta(s))$. Then the function

$$\begin{aligned} F(s) &= \tilde{\psi}_k(s) - s \\ &= -p(1)(s^2 - 1)^{-k} + s \left((s^2 - 1)^{-k} p(s^2) - 1 \right). \end{aligned}$$

is a rational function whose poles only at $s = \pm 1$ and has expansion

$$F(s) = c_1 s^{-1} + c_2 s^{-3} + \dots + c_k s^{-2k+1} + c_{k+1} s^{-2k} +$$

at $s = \infty$ with $c_1 = (2k^2 - 1/2)^{-1}$ and $c_{k+1} = -p(1)$, namely the first coefficient of even order starts from $2k$. We consider an equation for a given t :

$$s + F(s) = t \quad (127)$$

and find a solution of a form

$$s = t + G(t).$$

Since the even coefficients of the power series of F vanish up to $2(k-1)$, Lemma 41 shows that there exists uniquely such G that G is real and analytic near $t = \infty$

and the odd coefficients of G vanishes up to $2(k-1)$. $\tilde{\psi}_k(z)$ is one-to-one on $\{|z| > r_1\}$ and its inverse is given by $w + G(w)$ on $\{|w| > r_2\}$. Since $\phi_k(z) = \vartheta(\psi_k(z))$ is a conformal map from $\mathbb{C} \setminus (-\infty, 0]$ to $\phi_k(\mathbb{C} \setminus (-\infty, 0])$, its inverse is given by $\phi_k^{-1}(w) = (\vartheta\tilde{\psi}^{-1}\vartheta^{-1})(w)$ for $w \in \phi_m(\mathbb{C} \setminus (-\infty, 0])$. Let

$$\begin{cases} G_1(t) = \frac{1}{2}(G(\sqrt{t}) + G(-\sqrt{t})) & (= G_e(\sqrt{t})) \\ G_2(t) = \frac{1}{2\sqrt{t}}(G(\sqrt{t}) - G(-\sqrt{t})) & \left(= \frac{1}{\sqrt{t}}G_o(\sqrt{t})\right) \end{cases}.$$

Then $G(t) = G_1(t^2) + tG_2(t^2)$, and we have

$$\begin{aligned} & (\vartheta\tilde{\psi}^{-1}\vartheta^{-1})(w) \\ &= (\sqrt{w} + G_1(w) + \sqrt{w}G_2(w))^2 \\ &= w + G_1(w)^2 + w((G_2(w) + 1)^2 - 1) + 2\sqrt{w}G_1(w)(G_2(w) + 1). \end{aligned}$$

Since G has the even coefficients vanishing up to $2(k-1)$, $w^k G_1(w)$ is analytic near $w = \infty$. Therefore, setting

$$\begin{cases} g_1(w) = G_1(w)^2 + wG_2(w)(G_2(w) + 2) \\ g_2(w) = 2w^k G_1(w)(G_2(w) + 1) \end{cases},$$

we have

$$\phi_k^{-1}(w) = w + g_1(w) + w^{-k+1/2}g_2(w)$$

with some g_1, g_2 analytic in a neighborhood of ∞ satisfying

$$g_1(\infty) = -(k^2 - 1/4)^{-1}, \quad g_2(\infty) = -2(-1)^k a_k,$$

which completes the proof. \blacksquare

Lemma 41 *Let F be a power series of s^{-1} given by $F(s) = \sum_{j=1}^{\infty} a_j s^{-j}$ and assume it has the positive radius of convergence and consider an equation:*

$$t = s + F(s). \quad (128)$$

(i) *This equation is uniquely solvable if $|t^{-1}|$ is sufficiently small and it has a form:*

$$s = t + G(t)$$

with a convergent power series of t^{-1} given by

$$G(t) = \sum_{j=1}^{\infty} x_j t^{-j}. \quad (129)$$

(ii) x_n is determined from $\{a_j\}_{j=1}^n$ for each $n \geq 1$. The first three coefficients are

$$x_1 = -a_1, \quad x_2 = -a_2, \quad x_3 = -a_1^2 - a_3.$$

(iii) *Suppose $F(s)$ has a form*

$$F(s) = \sum_{j=1}^k a_{2j-1} s^{-2j+1} + \sum_{j=2k}^{\infty} a_j s^{-j} \quad (130)$$

for an $k \geq 1$. Then, the coefficients x_j of $G(t)$ vanish for even j up to $2(k-1)$. Moreover, if $a_{2j} \neq 0$, then $x_{2j} = -a_{2j}$.

Proof. Replacing s by s^{-1} and t by t^{-1} we see the equation (128) is equivalent to

$$t = \frac{s}{1 + sF(s^{-1})}. \quad (131)$$

The condition on F implies

$$t(0) = 0, \quad \frac{dt}{ds}(0) = 1, \quad \frac{d^2t}{ds^2}(0) = 0,$$

hence the complex function theory shows the existence of the solution $s(t)$ of (131) in a neighborhood of 0 satisfying

$$s(0) = 0, \quad \frac{ds}{dt}(0) = 1, \quad \frac{d^2s}{dt^2}(0) = 0,$$

which implies the existence of G of the form of (129). One can show inductively that the coefficient x_n is determined from $\{a_j\}_{j=1}^n$. To show (iii) one can assume $a_j = 0$ for every $j \geq 2k$ owing to (ii). The relation between F, G is rewritten as

$$F(s) + G(F(s) + s) = 0.$$

If we define $\widehat{f}(s) = -f(-s)$, then the above equation turns to

$$\widehat{F}(s) + \widehat{G}(\widehat{F}(s) + s) = 0.$$

Therefore, if $\widehat{F}(s) = F(s)$, the uniqueness implies $\widehat{G}(s) = G(s)$, which shows the first part of (iii). To show the second part we note that if

$$\begin{cases} F(s) = \sum_{j=1}^{k-1} a_j s^{-j} + a_k s^{-k} \equiv F_1(s) + a_k s^{-k} \\ G(s) = \sum_{j=1}^{k-1} x_j s^{-j} + x_k s^{-k} + \sum_{j=k+1}^{\infty} x_j s^{-j} \\ \quad \equiv G_{(1)}(s) + x_k s^{-k} + G_{(2)}(s) \end{cases},$$

and with some b_m

$$F_1(s) + G_{(1)}(s + F_1(s)) = b_k s^{-k} + O(s^{-k-1})$$

holds, which is verified by induction, then the identity

$$\begin{aligned} F_1(s) + a_k s^{-k} + G_{(1)}(s + F_1(s) + a_k s^{-k}) + x_k (s + F_1(s) + a_k s^{-k})^{-k} \\ + G_{(2)}(s + F_1(s) + a_k s^{-k}) = 0 \end{aligned}$$

together with

$$G_{(1)}(s + F_1(s) + a_k s^{-k}) = G_{(1)}(s + F_1(s)) + O(s^{-k-2})$$

implies $x_k = -a_k - b_k$. Since, if k is even and $a_{2j} = 0$ for any $j \leq k/2$, then (ii) implies $x_k = 0$, and hence $b_k = 0$. However, clearly b_k is determined from $\{a_j\}_{1 \leq j \leq k-1}$, hence $b_k = 0$ is valid if $a_{2j} = 0$ for any $j \leq k/2 - 1$ regardless of the value a_k . Consequently we have $x_k = -a_k$ if k is even and $a_{2j} = 0$ for any $j \leq k/2 - 1$ holds. ■

9.4 Estimates of relevant integral

Suppose the curve C is of the form:

$$C = \left\{ \pm\omega(y) + iy; y \in \mathbb{R}, \omega(y) = O\left(y^{-(n-1)}\right) \right\}$$

with $\omega(y) > 0$, $\omega(y) = \omega(-y)$. Assume (37), namely

$$\sup_{z \in C} \int_{|z-\lambda| \leq 1, \lambda \in C} |d\lambda| < \infty.$$

Lemma 42 *Let $C' = \sigma C$ with $\sigma > 1$. Then*

$$\int_C \frac{1}{|\lambda - z|^2} |d\lambda| = O\left(|z|^{n-1}\right) \text{ for } z \in C'.$$

Proof. Let

$$\begin{aligned} \int_C \frac{1}{|\lambda - z|^2} |d\lambda| &= \int_{|\operatorname{Im}(\lambda - z)| \leq \delta |\operatorname{Im} z|} \frac{|d\lambda|}{|\lambda - z|^2} + \int_{|\operatorname{Im}(z - \lambda)| > \delta |\operatorname{Im} z|} \frac{|d\lambda|}{|\lambda - z|^2} \\ &\equiv I_1 + I_2 \end{aligned}$$

with $\delta \in (0, 1)$ specified later. Since C is parametrized as $\omega(t) + it$, we see

$$I_1 = \int_{|t - \operatorname{Im} z| \leq \delta |\operatorname{Im} z|} \frac{(1 + \omega'(t)^2)^{1/2}}{(t - \operatorname{Im} z)^2 + (\omega(t) - \operatorname{Re} z)^2} dt \leq c_1 \pi \rho(z)^{-1},$$

where

$$\begin{cases} \rho(z) = \inf_{\lambda \in C; |\operatorname{Im}(\lambda - z)| \leq \delta |\operatorname{Im} z|} |\operatorname{Re}(z - \lambda)| \\ c_1 = \sup_{t \in \mathbb{R}} (1 + \omega'(t)^2)^{1/2} \end{cases}.$$

I_2 is estimated as

$$\begin{aligned} I_2 &= \int_{|t - \operatorname{Im} z| > |\operatorname{Im} z| \delta} \frac{(1 + \omega'(t)^2)^{1/2} dt}{|(\omega(t) - \operatorname{Re} z) + i(t - \operatorname{Im} z)|^2} \\ &\leq c_1 \int_{|t - \operatorname{Im} z| > |\operatorname{Im} z| \delta} \frac{dt}{|t - \operatorname{Im} z|^2} = c_1 \int_{|x| > |\operatorname{Im} z| \delta} \frac{dx}{|x|^2} = c_2 |\operatorname{Im} z|^{-1}. \end{aligned}$$

Therefore, we have

$$\int_C \frac{|d\lambda|}{|\lambda - z|^2} \leq c_1 \pi \rho(z)^{-1} + c_2 |\operatorname{Im} z|^{-1}.$$

We have to show

$$\rho(z) \geq c_3 |\operatorname{Im} z|^{-(n-1)}, \quad (132)$$

if δ is chosen suitably. Assume $\operatorname{Re} z, \operatorname{Im} z > 0$. Note that in the region

$$\{\lambda \in C; |\operatorname{Im}(z - \lambda)| \leq \delta |\operatorname{Im} z|\} = \{\lambda \in C; (1 - \delta) \operatorname{Im} z \leq \operatorname{Im} \lambda \leq (1 + \delta) \operatorname{Im} z\}.$$

Since $\omega(t) = t^{-(n-1)}$ for sufficiently large t , inequalities

$$\begin{aligned} |\operatorname{Re}(z - \lambda)| &\geq \operatorname{Re} z - \operatorname{Re} \lambda \geq \sigma \omega(\sigma^{-1} |\operatorname{Im} z|) - \omega((1 - \delta) |\operatorname{Im} z|) \\ &= \left(\sigma^n - (1 - \delta)^{-(n-1)} \right) |\operatorname{Im} z|^{-(n-1)} \end{aligned}$$

are valid for $z \in C'$. Therefore, if

$$\sigma^n > (1 - \delta)^{-(n-1)},$$

we have (132). ■

9.5 Ergodic Schrödinger operators

This section provides several basic facts on 1D Schrödinger operators with ergodic potentials, which are necessary in this paper.

Let (Ω, \mathcal{F}, P) be a probability space and $\{\theta_x\}_{x \in \mathbb{R}}$ be a one parameter group of \mathcal{F} -measurable transformations on Ω which satisfies

$$P(\theta_x^{-1}A) = P(A) \text{ for any } x \in \mathbb{R} \text{ and } A \in \mathcal{F}. \text{ (stationarity)} \quad (133)$$

$(\Omega, \mathcal{F}, P, \{\theta_x\}_{x \in \mathbb{R}})$ is called ergodic if it satisfies

$$P(\theta_x^{-1}A \ominus A) = 0 \text{ for any } x \in \mathbb{R} \implies P(A) = 0 \text{ or } 1. \quad (134)$$

For an \mathcal{F} -measurable real valued function Q on Ω set

$$q_\omega(x) = Q(\theta_x \omega), \quad \omega \in \Omega.$$

Then we obtain an ergodic potential $\{q_\omega\}_{\omega \in \Omega}$. A simple but important example is quasi-periodic potentials. Set $\Omega = \mathbb{R}^n / \mathbb{Z}^n$ and for $\alpha \in \mathbb{R}^n$

$$\theta_x \omega = x\alpha + \omega, \quad P = \text{the Lebesgue measure on } \mathbb{R}^n / \mathbb{Z}^n.$$

This $(\Omega, \mathcal{F}, P, \{\theta_x\}_{x \in \mathbb{R}})$ is ergodic if α is rationally independent and the resulting $q_\omega(x)$ is a quasi-periodic function. If $n = 1$, we have a periodic function and for $n = \infty$ in a certain sense we have an almost periodic function. One has more random ergodic potentials. For a technical reason we assume

$$\mathbb{E}(|Q|) = \int_{\Omega} |Q(\omega)| P(d\omega) < \infty \text{ and } Q(\omega) \geq \lambda_0 \text{ for any } \omega \in \Omega. \quad (135)$$

\mathbb{E} denotes the expectation by P . Then one can consider the associated Schrödinger operator

$$L_\omega = -\partial_x^2 + q_\omega.$$

Under the condition (135) it is known that $\inf \text{sp } L_\omega \geq \lambda_0$ and the boundaries $\pm\infty$ are of limit point type for L_ω for a.e. $\omega \in \Omega$. One can apply the Weyl spectral theory to each L_ω .

The **Floquet exponent** is defined by

$$w(z) = \mathbb{E}(m_\pm(z, \omega)) \text{ (the two expectations coincide),} \quad (136)$$

by which the **Lyapounov exponent** and **integrated density of states** are obtained by

$$\gamma(z) = -\text{Re } w(z) \ (\geq 0), \quad N(\lambda) = \frac{1}{\pi} \text{Im } w(\lambda) \ (\lambda \in \mathbb{R}).$$

$N(\lambda)$ is non-negative, continuous and non-decreasing on \mathbb{R} . [10] found an identity

$$\frac{\gamma(z)}{\text{Im } z} - \text{Im } w'(z) = \frac{1}{4} \mathbb{E} \left(\left(\frac{1}{\text{Im } m_+(z, \omega)} + \frac{1}{\text{Im } m_-(z, \omega)} \right) |R(z, \omega)|^2 \right)$$

for $z \in \mathbb{C}_+$. Set

$$\chi(z) = \frac{\gamma(z)}{\text{Im } z} - \text{Im } w'(z) \geq 0.$$

Then applying the Schwarz inequality we have

$$\begin{aligned}
\mathbb{E}(|R(z, \omega)|) &\leq \sqrt{4\chi(z)} \sqrt{\mathbb{E} \left(\frac{1}{\operatorname{Im} m_+(z, \omega)} + \frac{1}{\operatorname{Im} m_-(z, \omega)} \right)^{-1}} \\
&\leq \sqrt{4\chi(z)} \sqrt{\mathbb{E} \left(\frac{\operatorname{Im} m_+(z, \omega) + \operatorname{Im} m_-(z, \omega)}{4} \right)} \\
&= \sqrt{2\chi(z) \operatorname{Im} w(z)} \quad (\text{due to (136)}). \tag{137}
\end{aligned}$$

It is also known that

$$\Sigma_{ac}^\omega = \{\lambda \in \mathbb{R}; \gamma(\lambda) = 0\} = \{\lambda \in \mathbb{R}; R(\lambda, \omega) = 0\} \text{ for a.e. } \omega \in \Omega. \tag{138}$$

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