Dynamical characterization of central sets along filter *

pintu debnath † Sayan Goswami ‡

Abstract

Using the notions of Topological dynamics, H. Furstenberg defined central sets and proved the Central Sets Theorem. Later V. Bergelson and N. Hindman characterized central sets in terms of algebra of the Stone-Čech Compactification of discrete semigroup. They found that central sets are the members of the minimal idempotents of βS, the Stone-Čech Compactification of a semigroup $(S, \cdot)$. We know that any closed subsemigroup of βS is generated by a filter. We call a set A to be a $\mathcal{F}$- central set if it is a member of a minimal idempotent of a closed subsemigroup of βS, generated by the filter $\mathcal{F}$. In this article we will characterize the $\mathcal{F}$-central sets dynamically.

Mathematics subject classification 2020: 37B05, 05D10

1 Introduction

H. Frustenberg introduced the notion of central sets [F81, Defination 8.3] and proved several combinatorial properties of such sets using topological dynamics. Later, V. Bergelson and N. Hindman in [BH90], established an algebraic characterization of central sets. For arbitrary semigroup, the interplay of central sets between algebra of the Stone-Čech Compactification and Topological dynamics was explored in [SY96].

Definition 1.1. A dynamical system is a pair $(X, \langle T_s \rangle_{s\in S})$ such that

1. X is a compact topological space;

2. S is a semigroup;

3. for each s $s \in S$, T_s is a continuous function from X to X; and

***Keywords:** Algebra in the Stone-Čech compactification, Topological dynamics, Central set

†Department of Mathematics, Basirhat College, Basirhat -743412, North 24th parganas, West Bengal, India. **pintumath1989@gmail.com**

‡Department of Mathematics, University of Kalyani, Kalyani-741235, Nadia, West Bengal, India **sayan92m@gmail.com**

4. For all $s, t \in S$, $T_s \circ T_t = T_{st}$.

For any discrete semigroup $(S, \cdot)$, central set was defined as the member of the minimal idempotents of its Stone-Čech Compactification, say βS. To state dynamical characterization of central sets, we need the following definitions.

Definition 1.2. Let S be a discrete semigroup and $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system.

1. Let A be a subset of S. Then the set A is called syndetic if and only if there exists a finite subset F of S such that $S = \cup_{t \in F} t^{-1} A$.

2. A point $x \in X$ is uniformly recurrent point if and only if for each neighbourhood U of x, $\{s \in S : T_s x \in U\}$ is syndetic.

3. $x, y \in X$ are called proximal if and only if for every neighbourhood U of the diagonal in $X \times X$, there exists $s \in F$ such that $(T_s(x), T_s(y) \in U$.

From [SY96, Theorem 2.4], we know the following theorem.

Theorem 1.3. *Let S be a semigroup and let $B \subseteq S$. Then B is central if and only if there exists a dynamical system $(X, \langle T_s \rangle_{s \in S})$, two points $x, y \in X$, and a neighbourhood U of y such that x and y are proximal, y is uniformly recurrent and $B = \{s \in S : T_s(x) \in U\}$.*

We will extend this result for certain filters $\mathcal{F}$, over any discrete semigroup S, which generates a closed subsemigroup in the space of ultrafilters, say βS (see preliminaries section). A sets is $\mathcal{F}$-central if it is a member of any minimal idempotent of the semigroup generated by $\mathcal{F}$ (see definition 2.2). Recently in [GP21], the Central Sets Theorem along some filters has been established. Before stating our main theorem, let us define the analogous notion of uniformly recurrent and proximality along a filter $\mathcal{F}$.

Definition 1.4. Let S be a semigroup, and $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. Let T be a closed subsemigroup of βS such that for filter $\mathcal{F}$, $\overline{\mathcal{F}} = T$.

1. $A \subseteq S$ is $\mathcal{F}$-syndetic if for every $F \in \mathcal{F}$, there is a finite set $G \subseteq F$ such that $G^{-1} A \in \mathcal{F}$.

2. A point $x \in X$ is $\mathcal{F}$-uniformly recurrent point if and only if for each neighbourhood U of x, $\{s \in S : T_s x \in U\}$ is $\mathcal{F}$-syndetic.

3. $x, y \in X$ are $\mathcal{F}$-proximal if and only if every neighbourhood U of the diagonal in $X \times X$ and for each $F \in \mathcal{F}$ there exists $s \in F$ such that $(T_s(x), T_s(y) \in U$.

The following is our one:

Theorem 1.5. *Let S be a semigroup and let $B \subseteq S$. Then B is $\mathcal{F}$-central if and only if there exists a dynamical system $(X, \langle T_s \rangle_{s \in S})$ and there exist $x, y \in X$ and a neighbourhood U of y such that x and y are $\mathcal{F}$- proximal, y is $\mathcal{F}$-uniformly recurrent and $B = \{s \in S : T_s(x) \in U\}$.*

2 Preliminaries

Let us first give a brief review of algebraic structure of the Stone-Čech compactification of any discrete semigroup S.

The set $\{\overline{A} : A \subset S\}$ is a basis for the closed sets of βS. The operation '$\cdot$' on S can be extended to the Stone-Čech compactification βS of S so that$(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for any is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). This is a famous Theorem due to Ellis that if S is a compact right topological semigroup then the set of idempotents $E\left(S\right) \neq \emptyset$. A non-empty subset I of a semigroup T is called a *left ideal* of S if $TI \subset I$, a *right ideal* if $IT \subset I$, and a *two sided ideal* (or simply an *ideal*) if it is both a left and right ideal. A *minimal left ideal* is the left ideal that does not contain any proper left ideal. Similarly, we can define *minimal right ideal* and *smallest ideal*.

Any compact Hausdorff right topological semigroup T has the smallest two sided ideal

$$\begin{aligned} K(T) &= \bigcup\{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } T\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal $K(T)$ of T. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if the set $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. See [HS12] for an elementary introduction to the algebra of βS and for any unfamiliar details.

Definition 2.1. Let S be a discrete semigroup and let C be a subset of S. Then C is central if and only if there is an idempotent p in $K\left(\beta S\right)$ such that $C \in p$.

For every filter $\mathcal{F}$, on the semigroup S, define $\overline{\mathcal{F}} \subseteq \beta S$ by $\overline{\mathcal{F}} = \cap_{F \in \mathcal{F}} \overline{F}$. Note that $\overline{\mathcal{F}}$ is closed subset of βS consisting of all ultrafilters on S that contain $\mathcal{F}$. Conversely, every closed subset of βS is uniquely represented in such a form. If $\mathcal{F}$ is idempotent filter, i.e. $\mathcal{F} \supset \mathcal{F} \cdot \mathcal{F}$, then $\overline{\mathcal{F}}$ becomes a semigroup, but the converse is not true always. In this article, without mentioned further, we will consider only those filters $\mathcal{F}$, which generates a closed subsemigroup of βS. For details readers can see [SZZ09]. For some new development in this area we refer [CJ21].

Definition 2.2 ($\mathcal{F}$-central set)**.** Let S be a discrete semigroup and let $\mathcal{F}$ generates a closed subsemigroup of βS. Then a set C is said to be $\mathcal{F}$-central if and only if there is an idempotent p in $K\left(\overline{\mathcal{F}}\right)$ such that $C \in p$.

Remark 2.3. For βS , we know that $\mathcal{F} = \{S\}$. Then $\mathcal{F}$-central sets are nothing but the usual central sets. Let S is a dense subsemigroup of $\left(\left(0, \infty\right), +\right)$, and

$0^{+}(S)=\{p\in\beta S:\text{ for any }\epsilon>0,S\cap(0,\epsilon)\in p\}$. Then $\mathcal{F}=\{(0,\epsilon)\cap S:\epsilon>0\}$ and $\overline{\mathcal{F}}=0^{+}(S)$. A set $C\subseteq S$ is central set near zero if and only if there is some idempotent $p\in K(0^{+}(S))$ with $C\in p$. So in this case $\mathcal{F}$-centrals are central sets near zero. See [HL99] for combinatorial results of central sets near zero and [P18, Theorem 2.12] for dynamical characterizations of central sets near zero.

If $(X,\langle T_s\rangle_{s\in S})$ be a dynamical system, then $\overline{\{T_s:s\in S\}}$ in X^X is a semigroup, which is referred as enveloping semigroup of the dynamical system. Now we recall the following theorem.

Theorem 2.4 ([HS12, Theorem 19.11])**.** *Let $(X,\langle T_s\rangle_{s\in S})$ be a dynamical system and define $\theta:S\to X^X$ by $\theta(s)=T_s$. Then is a continuous homomorphism from βS onto the enveloping semigroup of $(X,\langle T_s\rangle_{s\in S})$.($\widetilde{\theta}$ be a continuous extension of θ.)*

Let us recall the definition [HS12, Definition 19.12], which will be useful.

Definition 2.5. Let $(X,\langle T_s\rangle_{s\in S})$ be a dynamical system and define $\theta:S\to X^X$ by $\theta(s)=T_s$. For each $p\in\beta S$, let $T_p=\widetilde{\theta}(p)$.

As an immediate consequences of Theorem 8, we have the following remark [HS12, Remark 19.13].

Remark 2.6. Let $(X,\langle T_s\rangle_{s\in S})$ be a dynamical system and let $p,q\in\beta S$. Then $T_p\circ T_q=T_{pq}$ and for each $x\in X$, $T_p(x)=p-lim_{s\in S}T_s(x)$.

Strategy of our proof: In the next section, we will first establish the relation between $\mathcal{F}$-proximality and the ultrafilters containing $\mathcal{F}$. Then we will establish the relation between algebra and $\mathcal{F}$-uniformly recurrent point. Soon after we will deduce three lemma which will explore the relation between algebra, $\mathcal{F}$-proximal and $\mathcal{F}$ uniformly recurrent point. Then using these results we will obtain our desire result.

3 Dynamical characterization of $\mathcal{F}$-central set

From [HS12, Lemma 19.22], we get the characterization of proximality which states that for a dynamical system $(X,\langle T_s\rangle_{s\in S})$ and $x,y\in S$. Points x and y of X are proximal if and only if there is some $p\in\beta S$ such that $T_p(x)=T_p(y)$. We get an analogous result for $\mathcal{F}$-proximality by the following lemma, which will be very convenient for us.

Lemma 3.1. *Let S be a semigroup, and $(X,\langle T_s\rangle_{s\in S})$ be a dynamical system. Then $x,y\in X$ are $\mathcal{F}$-proximal if and only if there is some $p\in\overline{\mathcal{F}}$ such that $T_p(x)=T_p(y)$.*

Proof. Let $x,y\in X$ are $\mathcal{F}$-proximal. Let $\mathcal{N}$ be the set of all neighbourhoods of the diagonal in $X\times X$. For each $U\in\mathcal{N}$, let $B_U=\{s\in S:(T(x),T_s(y))\in U\}$. From definition 1.4, it follows that $\{B_U:U\in\mathcal{N}\}\cup\mathcal{F}$ has finite intersection property. Now choose $p \in \overline{\mathcal{F}}$ such that $\{B_U : U \in \mathcal{N}\} \cup \mathcal{F} \in p$. Let $z = T_p(x)$. To see that $z = T_p(y)$, let V be an open neighbourhood of z in X. Since X is compact Hausdorff, there exist disjoint open sets V_1, V_2 such that $z \in V_1$ and $X \setminus V \subseteq V_2$. Let $U = (V \times V) \cup (V_2 \times X)$. Then U is a neighbourhood of the diagonal in $X \times X$ such that $\pi_2\left(\pi_1^{-1}(V_1) \cap U\right) \subseteq V$, where π_1 and π_2 denote the first and second projections of $X \times X$ on to X respectively. Let $E = \{s \in S : T_s(x) \in V_1\}$ and $F = \{s \in S : (T_s(x), T_s(y)) \in U\}$. Then $E, F \in p$ and $E \cap F \subseteq \{s \in S : T_s(y) \in V\}$. Thus $\{s \in S : T_s(y) \in V\} \in p$ for every open neighbourhood V of z.

Conversely suppose, we have $p \in \overline{\mathcal{F}}$ such that $T_p(x) = T_p(y) = z$. Let U be a neighbourhood of the diagonal in $X \times X$. Choose an open neighbourhood V of z in X such that $V \times V \subseteq U$. Let $B = \{s \in S : T_s(x) \in V\}$ and $C = \{s \in S : T_s(y) \in V\}$. Then $B \cap C \in p$. For each $F \in \mathcal{F}$, choose $s \in F \cap B \cap C$. Then $(T_s(x), T_s(y)) \in V \times V \subseteq U$. □

In order to establish the equivalence of dynamical and algebraic notions of central sets, the following theorem is necessary.

Theorem 3.2. *Let S be a semigroup and $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and L be a minimal left ideal of $\overline{\mathcal{F}}$ and $x \in X$. The following statements are equivalent.*

(a) The points x is a $\mathcal{F}$-uniformly recurrent point of $(X, \langle T_s \rangle_{s \in S})$.

(b) There exists $u \in L$ such that $T_u(x) = x$.

(c) There exist $y \in X$ and an idempotent $u \in L$ such that $T_u(y) = x$.

(d) There exists an idempotent $u \in L$ such that $T_u(x) = x$.

Proof. $(a) \Rightarrow (b)$ Choose any $v \in L$. Let $\mathcal{N}$ be the set of neighbourhoods of x in X. For each $U \in \mathcal{N}$, let $B_U = \{s \in S : T_s(x) \in U\}$. Since x is a $\mathcal{F}$-uniformly recurrent point, each B_U is $\mathcal{F}$-syndetic set. So for every $F \in \mathcal{F}$, there is some finite set $F_{U,F} \subset F$ such that $F_{U,F}^{-1} B_U \in \mathcal{F} \subset v$. So for each $U \in \mathcal{N}$ and $F \in \mathcal{F}$ pick $t_{U,F} \in F_{U,F}$ such that $t_{U,F}^{-1} B_U \in v$. Given $U \in \mathcal{N}$, let $C_U = \{t_{V,F} : V \in \mathcal{N}, V \subseteq U \text{ and } F \in \mathcal{F}\}$. Then $\{C_U : U \in \mathcal{N}\} \cup \mathcal{F}$ has finite intersection property. So pick $w \in \overline{\mathcal{F}}$ such that $\{C_U : U \in \mathcal{N}\} \subseteq w$ and let $u = w \cdot v$. Since L is a left ideal of $\overline{\mathcal{F}}$, $u \in L$. To see that $T_u(x) = x$, we let $U \in \mathcal{N}$ and show that $B_U \in u$, for which it suffices that $C_U \subseteq \left\{t \in S : t^{-1} B_U \in v\right\}$. So let $t \in C_U$ and pick $V \in \mathcal{N}$ and $F \in \mathcal{F}$ such that $V \subseteq U$ and $t = t_{V,F}$. Then $t^{-1} B_V \in v$ and $t^{-1} B_V \subseteq t^{-1} B_U$.

$(b) \Rightarrow (c)$ Let $K = \{v \in L : T_v(x) = x\}$. It suffices to show that K is a compact subsemigroup of L, since then K has an idempotent. By the assumption, $K \neq \emptyset$. Further if $v \in L \setminus K$ there is some neighbourhood of x such that $B = \{s \in S : T_s(x) \in U\} \notin v$. Then $\overline{B} \cap L$ is a neighbourhood of v in βS which misses K. Finally, to see that K is a semigroup, let $v, u \in K$. Then $T_{v \cdot w}(x) = T_v(T_w(x)) = T_v(x) = x$.

$(c) \Rightarrow (d)$ Using remark 2.6, we have $T_u(x) = T_u(T_u(y)) = T_u(y) = x$.

$(d) \Rightarrow (a)$ Let U be a neighbourhood of x and let $B = \{s \in S : T_s(x) \in U\}$ and suppose that B is not $\mathcal{F}$-syndetic. Then there exists $F \in \mathcal{F}$ such that

$$\left\{S \setminus \cup_{t \in F_F} t^{-1}B : F_F \text{ is a finite nonempty subset of } F\right\} \cup \mathcal{F}$$

has finite intersection property. So pick some $w \in \overline{\mathcal{F}}$ such that

$$\left\{S \setminus \cup_{t \in F_F} t^{-1}B : F_F \text{ is a finite nonempty subset of } F\right\} \subseteq w.$$

Then $\left(\overline{\mathcal{F}} \cdot w\right) \cap \overline{B} = \emptyset$ (As $B \in v \cdot w$ implies $t^{-1}B \in w$ for some $t \in F$). Now $\left(\overline{\mathcal{F}} \cdot w\right)$ is a left ideal of $\overline{\mathcal{F}}$, so $\overline{\mathcal{F}} \cdot w \cdot u$ is a left ideal of $\overline{\mathcal{F}}$ which is contained in L, and hence $\overline{\mathcal{F}} \cdot w \cdot u = L$. Thus we may pick some $v \in \overline{\mathcal{F}} \cdot w$ such that $v \cdot u = u$. Again $T_v(x) = T_v(T_u(x)) = T_{v \cdot u}(x) = T_u(x) = x$, so in particular $B \in v$. But, $v \in \overline{\mathcal{F}} \cdot w$ and $\left(\overline{\mathcal{F}} \cdot w\right) \cap \overline{B} = \emptyset$, a contradiction. □

Lemma 3.3. *Let S be a semigroup and $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and let $x \in X$. Then for each $F \in \mathcal{F}$, there is a $\mathcal{F}$-uniformly recurrent point $y \in \overline{\{T_s(x) : s \in F\}}$ such that x and y are $\mathcal{F}$- proximal.*

Proof. Let L be any minimal left ideal of $\overline{\mathcal{F}}$ and pick an idempotent $u \in L$. Let $y = T_u(x)$. For each $F \in \mathcal{F}$, clearly $y \in \overline{\{T_s(x) : s \in F\}}$, as $\mathcal{F} \subseteq u$.By lemma 3.1, y is a $\mathcal{F}$-uniformly recurrent point of $(X, \langle T_s \rangle_{s \in S})$. By remark 2.6, we have $T_u(y) = T_u(T_u(x)) = T_{u \cdot u}(x) = T_u(x)$. So by Lemma 3.1, x and y are $\mathcal{F}$-proximal. □

Lemma 3.4. *Let S be a semigroup and $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and let $x, y \in X$. If x and y are $\mathcal{F}$- proximal, then there is a minimal left ideal L of $\overline{\mathcal{F}}$ such that $T_u(x) = T_u(y)$ for all $u \in L$.*

Proof. Pick $v \in \overline{\mathcal{F}}$ such that $T_v(x) = T_v(y)$ and pick a minimal left ideal L of $\overline{\mathcal{F}}$ such that $L \subseteq \overline{\mathcal{F}} \cdot v$. To see that L is as required, let $u \in L$ and choose $w \in \overline{\mathcal{F}}$ such that $u = w \cdot v$. Then by remark 2.6, we have $T_u(x) = T_{w \cdot v}(x) = T_w(T_v(x)) = T_w(T_v(y)) = T_{w \cdot v}(y) = T_u(y)$. □

Lemma 3.5. *Let S be a semigroup and $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system and let $x \in X$. Then there exists an idempotent u in $K\left(\overline{\mathcal{F}}\right)$ such that $T_u(x) = y$ if and only if both y is $\mathcal{F}$-uniformly recurrent and x and y are $\mathcal{F}$- proximal.*

Proof. Since u is a minimal idempotent of $\overline{\mathcal{F}}$, there is a minimal left ideal L of $\overline{\mathcal{F}}$ such that $u \in L$. By theorem 3.2, y is $\mathcal{F}$-uniformly recurrent and by remark 2.6, $T_u(y) = T_u(T_u(x)) = T_{u \cdot u}(x) = T_u(x)$. So x and y are $\mathcal{F}$- proximal.

Conversely, by lemma 3.4, pick a minimal left ideal L of $\overline{\mathcal{F}}$ such that $T_u(x) = T_u(y)$ for all $u \in L$. Pick by theorem 3.2, an idempotent $u \in L$ such that $T_u(y) = y$. Hence $T_u(x) = y$. □

We now give the dynamical characterization of $\mathcal{F}$-central sets.

Proof of Theorem 1.5: Let $G = S \cup \{e\}$, $X = \prod_{s \in G} \{0,1\}$ and for $s \in S$ define $T_s : X \to X$ by $T_s(x)(t) = x(t \cdot s)$ for all $t \in G$. Then by [HS12, Lemma 19.14] $(X, \langle T_s \rangle_{s \in S})$ is a dynamical system. Now let $x = \chi_B$, the characteristic function of B. Pick a minimal idempotent in $\overline{\mathcal{F}}$ such that $B \in u$ and let $y = T_u(x)$. Then by theorem 3.2 y is $\mathcal{F}$-uniformly recurrent and x and y are $\mathcal{F}$-proximal. Now let $U = \{z \in X : z(e) = y(e)\}$. Then U is a neighbourhood of $y \in X$. We note that $y(e) = 1$. Indeed, $y = T_u(x)$ so, $\{s \in S : T_s(x) \in U\} \in u$ and we may choose some $s \in B$ such that $T_s(x) \in U$. Then $y(e) = T_s(x)(e) = x(s \cdot e) = 1$. Thus given any $s \in S$, $s \in B \Longleftrightarrow x(s) = 1 \Longleftrightarrow T_s(x) \in U$.

Conversely, choose a dynamical system $(X, \langle T_s \rangle_{s \in S})$, points $x, y \in X$ and a neighbourhood U of y such that x and y are $\mathcal{F}$- proximal, y is $\mathcal{F}$-uniformly recurrent and $B = \{s \in S : T_s(x) \in U\}$. Choose by theorem 3.2, a minimal idempotent $u \in \overline{\mathcal{F}}$ such that $T_u(x) = y$. Then $B \in u$. $\square$

Acknowledgment: The author acknowledges the grant UGC-NET SRF fellowship with id no. 421333 of CSIR-UGC NET December 2016.

References

[BH90] V. Bergelson and N. Hindman, Nonmetrizable topological dynamics and Ramsay theory, Trans. Am. Math. Soc. 320(1990), 293-320.

[CJ21] C. Christopherson and J. H. Johnson Jr, Algebraic characterizations of some relative notions of size, arXiv:2105.09723

[F81] H. Furstenberg, Recurrence in Ergodic theory and combinatorial number theory, Princeton University press, Princeton, 1981.

[GP21] S. Goswami and J. Poddar, Central sets theorem along filters and some combinatorial consequences, arXiv:2106.14435.

[HL99] N. Hindman and I. Leader, The semigroup 0f ultrafilters near 0, Semigroup forum 59(1999), 33-55.

[HS12] N. Hindman and D. Strauss, Algebra in the Stone-Čech Compactification: Theory and Application, de gruyter, Berlin, 2012.

[P18] S.K.Patra, Dynamical characterizations of combinatorially rich sets near zero, Topology and its Applications 240(2018) 173-182.

[SY96] H. Shi and H. Young, Nonmetrizable topological dynamical characterization of central sets, Fundam. Math. 150 (1996) 1-9.

[SZZ09] O. Shuungula, Y. Zelenyuk and Y. Zelenyuk, The clouser of the smallest ideal of an ultrafilter semigroup, Semigroup Forum 79 (2009), 531-539.