Cherry Maps with Different Critical Exponents: Bifurcation of Geometry

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To the TIEMGNI DEFFO's family: the mother Carine and the children Abigail, Joakim, and Helza after the death of Richard the father of the family.

Abstract

We consider order preserving C^3 circle maps with a flat piece, irrational rotation number and critical exponents (ℓ_1, ℓ_2).

We detect a change in the geometry of the system. For $(\ell_1, \ell_2) \in [1, 2]^2$ the geometry is degenerate and it becomes bounded for $(\ell_1, \ell_2) \in [2, \infty)^2 \setminus \{(2, 2)\}$. When the rotation number is of the form $[abab\cdots]$; for some $a, b \in \mathbb{N}^*$, the geometry is bounded for (ℓ_1, ℓ_2) belonging above a curve defined on $]1, +\infty[^2$. As a consequence we estimate the Hausdorff dimension of the non-wandering set $K_f = \mathcal{S}^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(U)$. Precisely, the Hausdorff dimension of this set is equal to zero when the geometry is degenerate and it is strictly positive when the geometry is bounded.

Key words: Circle map, Irrational rotation number, Flat piece, Critical exponent, Geometry and Hausdorff Dimension.
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1 Introduction

We study a certain class of weakly order preserving, non-injective (on an interval exactly; called flat piece) circle maps which appear naturally in the study of Cherry flows on the two dimensions torus (see [8, 11, 14, 15]), non-invertible continuous circle map (see [7]) and of the dependence of the rotation interval on the parameter value for one-parameter families of continuous circle maps (see [16]). The dynamics of circle maps with a flat interval has been intensively explored in the past years, see [4, 5, 7, 14, 17, 18].

We discuss the geometry of the non-wandering set (set obtained by removing from the circle all pre-images of the flat piece). Where the geometry is concerned, we discover a dichotomy; which generalize the one found in [5]. Some of our maps show a "degenerate geometry", while others seem to be subject to the "bounded geometry".

Before we can explain more precisely our results, we introduce our class, adopt some notations and present basic lemmas.

1.1 The class of functions

We fix $\ell_1, \ell_2 \geq 1$ and we consider the class $\mathscr{L}$ of continuous circle maps f of degree one for which an arc U exists so that the following properties hold:

1. The image of U is one point.
2. The restriction of f to $\mathcal{S}^1 \setminus \overline{U}$ is a C^3-diffeomorphism onto its image.
3. Let (a, b) be a preimage of U under the projection of the real line to $\mathcal{S}^1$. On some right-sided neighborhood of b, f can be represented as
$$h_r((x-b)^{\ell_2});$$
where, h_r is C^3-diffeomorphism on a two-sided neighbourhood of b. Analogously, on a left-sided neighborhood of a, f equals
$$h_l((x-a)^{\ell_1});$$
In the following, we assume that $h_l(x) = h_r(x) = x$. In fact, it is possible to effect C^3 coordinate changes near a and b that will allow us to replace both h_l and h_r by the identity function.

Let F be a lift of f on the real line. The rotation number $\rho(f)$ of f is defined (independently of x and F) by
$$\rho(f) := \lim_{n\to\infty} \frac{F^n(x) - x}{n} (mod\, 1).$$
Let (q_n) be the sequence of denominators of the convergents of $\rho(f)$ (irrational) defined recursively by $q_1 = 1$, $q_2 = a_1$ and $q_{n+1} = a_n q_n + q_{n-1}$ for all $n \geq 3$; with,
$$\rho(f) = [a_0 a_1 \cdots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Additional Assumption

Let $f \in \mathscr{L}$. We say that Sf (the Schwarzian derivative of f) is negative if,
$$Sf(x) := \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\left(\frac{D^2 f(x)}{Df(x)}\right)^2 < 0; \quad \forall\, x,\ Df(x) \neq 0; \tag{A1}$$

With, $D^n f$ the n^{th} derivative of f; for $n \in \mathbb{N}$.

We will assume in the proof of the first part of Theorem 1 that f has a negative negative Schwarzian derivative.

1.2 Notations and Definitions

The fundamental notations are established in [5] (p.2-3). Let $f \in \mathscr{L}$.

1. For every $i \in \mathbb{Z}$, the writing $\underline{i}$ means $f^i(U)$.
2. Let I and J be two intervals. (I, J) is the interval between I and J. $[I, J) := I \cup (I, J)$ and $(I, J] := (I, J) \cup J$. $|I|$ is the length of the interval I, $|[I, J)| := |I| + |(I, J)|$ and $|(I, J]| := |J| + |(I, J)|$. We say that I and J are comparable when $|I|$ and $|J|$ are comparable. That means, there is $k > 0$ such that, $\frac{1}{k}|I| < |J| < k|I|$.
3. For any sequence Γ_n and for any real d, we adopt the writings:

$$\Gamma_n^{d(\ell_1,\ell_2)} := \begin{cases} \Gamma_n^{d\ell_1} & \text{if} \quad n \equiv 0[2] \\ \Gamma_n^{d\ell_2} & \text{if} \quad n \equiv 1[2] \end{cases} \qquad \Gamma_n^{d(\frac{1}{\ell_1},\frac{1}{\ell_2})} := \begin{cases} \Gamma_n^{d\frac{1}{\ell_1}} & \text{if} \quad n \equiv 0[2] \\ \Gamma_n^{d\frac{1}{\ell_2}} & \text{if} \quad n \equiv 1[2] \end{cases}$$

1.3 Discussion and statement of the results

Scaling ratios

The sequence

$$\alpha_n := \frac{|(\underline{-q_n}, \underline{0})|}{|[\underline{-q_n}, \underline{0})|} = \frac{|(f^{-q_n}(U), U)|}{|(f^{-q_n}(U), U)| + |f^{-q_n}(U)|}.$$

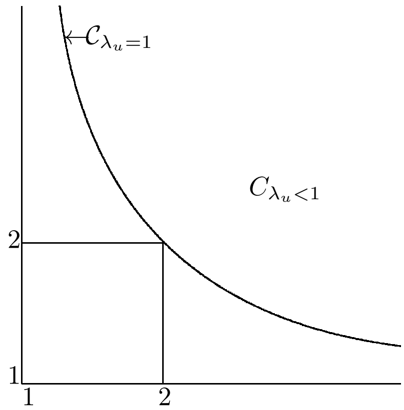
measure the geometry near a critical point. In fact, it serves as scaling relating the geometries of successive dynamic partitions.

The geometry is said degenerate when α_n goes to zero and the geometry is bounded, when α_n is bounded away from zero.

The study of this geometry is parametrised by rotation number and critical exponents. In [5], for pairs (ℓ, ℓ); $\ell > 1$ and ρ irrational number of bounded type (i.e. $\max_n a_n < \infty$), the authors found a transition between degenerate geometry and bounded geometry. In fact, they show under **(A1)** that, if $1 < \ell \leq 2$ and $\rho \in \mathbb{R} \setminus \mathbb{Q}$, then the geometry is degenerate and it is bounded (independently of **(A1)**) if $\ell > 2$ and ρ is irrational number of bounded type. In [4], for $\ell > 1$, the author proved that the class of function f of critical exponents $(1, \ell)$ or $(\ell, 1)$ have a degenerate geometry. In [15], the authors show that, for the maps in $\mathscr{L}$ with Fibonacci rotation number, when the critical exponents (ℓ_1, ℓ_2) belong in $(1, 2)^2$, the geometry is degenerate. Let us note that, Differently from other previous works, information on the geometry of the system is obtained by the study of the asymptotic behaviour of the renormalization operator.

In the present paper, we consider the cases where the critical exponents (ℓ_1, ℓ_2) belong in a subdomain (containing the previous domains) of $[1, \infty)^2$; also, the rotation number is not necessarily Fibonacci type and the results do not depend on the renormalization operator as in [9]. We use the formalism presented in [5] which is based on recursive inequalities analysis of α_n. For technical reason, in the case of bounded geometry, we introduce the vector sequence

$v_n := (-\ln \alpha_n, -\ln \alpha_{n-1})$ and, the new recursive inequality is controlled by a 2×2 matrix. When the rotation number is bi-periodic ($\rho = [abab\cdots]; a, b \in \mathbb{N}$), the 2×2 matrix has two eigenvalues (depending on rotation number and critical exponents) $\lambda_s \in (0,1)$ and $\lambda_u > 0$. The equation $\lambda_u = \lambda_u((a,b);(\ell_1,\ell_2)) = 1$ defines a curve $\mathcal{C}_{\lambda_u=1}$ (presented above) which separates the (ℓ_1, ℓ_2) plan into two components $C_{\lambda_u>1}$ (below the curve) and $C_{\lambda_u<1}$.



Main result Let $f \in \mathscr{L}$ with critical exponents (ℓ_1, ℓ_2). Then,

1. the scaling ratio α_n goes to zero when $(\ell_1, \ell_2) \in [1,2]^2$, $\rho \in \mathbb{R} \setminus \mathbb{Q}$ and **(A1)** holds.
2. the scaling ratio α_n bounded away from zero when $(\ell_1, \ell_2) \in [2,\infty)^2 \setminus \{(2,2)\}$ and ρ is bounded type.
3. the scaling ratio α_n bounded away from zero when $(\ell_1, \ell_2) \in C_{\lambda_u<1}$ and ρ is bi-periodic.

Estimation of Hausdorff Dimension of the non-wandering set. In the symmetric case ($\ell_1 = \ell_2 = \ell$), in [8], for $f \in \mathscr{L}$ with bounded type rotation number, the author shows that, if $\ell \in (1,2]$, then the Hausdorff dimension of the non-wandering set is equal to zero and that, if $\ell > 2$, then the Hausdorff dimension of the non-wandering set is strictly greater than zero. This result generalizes the one in [19] where the author treats the maps in $\mathscr{L}$ with critical exponents $(1,1)$. Let us note that, the results in [8] are more general (they only depend on geometry); precisely, if the rotation number is the bounded type, then, the Hausdorff Dimension of the non-wandering set is equal to zero when the geometry is degenerate and it is strictly greater than zero when the geometry is bounded; so we have the following result which is proved at the end of the paper.

Corollary 1.1. *Let $f \in \mathscr{L}$ with critical exponent (ℓ_1, ℓ_2) with bounded type rotation number. Then,*

1. *the Hausdorff dimension of the non-wandering set is equal to zero when $(\ell_1, \ell_2) \in [1,2]^2$ and **(A1)** holds and for the pairs $(\ell, 1)$; $\ell > 1$;*
2. *the Hausdorff dimension is strictly bigger than zero when $(\ell_1, \ell_2) \in [2,\infty)^2 \setminus \{(2,2)\}$;*
3. *the Hausdorff dimension is strictly bigger than zero when $(\ell_1, \ell_2) \in C_{\lambda_u<1}$ and ρ is bi-periodic.*

The following remark will simplify statements and proofs of results.

Remark 1.2. *Let us note that, our setting has some inherent symmetry. This will simplify statements and proof of our results.*

2 Tools

2.1 Cross-Ratio Inequalities

Notation 2.1. *We denote by $\mathbb{R}^4_<$, the subset of $\mathbb{R}^4$ defined by*

$$\mathbb{R}^4_< := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4,\ \textit{such that}\ x_1 < x_2 < x_3 < x_4\}.$$

The following result can be found in [1] (**Theorem 2**)

Proposition 2.2. *The Cross-Ratio Inequality (**CRI**).*
Let $f \in \mathscr{L}$. Let $(a, b, c, d) \in \mathbb{R}^4_<$. The cross-ratio $\boldsymbol{Cr}$ is defined by

$$\boldsymbol{Cr}(a, b, c, d) := \frac{|b-a||d-c|}{|c-a||d-b|}$$

and the cross-ratio $\boldsymbol{Poin}$ is defined by

$$\boldsymbol{Poin}(a, b, c, d) := \frac{|d-a||b-c|}{|c-a||d-b|}.$$

The distortion of the cross-ratio $\boldsymbol{Cr}$ and cross-ratio $\boldsymbol{Poin}$ are given respectively by

$$\mathcal{D}\boldsymbol{Cr}(a, b, c, d) := \frac{\boldsymbol{Cr}(f(a), f(b), f(c), f(d))}{\boldsymbol{Cr}(a, b, c, d)}$$

and

$$\mathcal{D}\boldsymbol{Poin}(a, b, c, d) := \frac{\boldsymbol{Poin}(f(a), f(b), f(c), f(d))}{\boldsymbol{Poin}(a, b, c, d)}.$$

Let us consider a set of $n+1$ quadruples $\{a_i, b_i, c_i, d_i\}$ with the following properties:

1. *Each point of the circle belongs to at most k intervals (a_i, d_i);*
2. *The intervals (b_i, c_i) do not intersect U.*

Then there is $C_k, C'_k > 0$ such that the following inequalities hold

$$\prod_{i=0}^{n} \mathcal{D}\boldsymbol{Cr}(a_i, b_i, c_i, d_i) \leq C_k$$

and

$$\prod_{i=0}^{n} \mathcal{D}\boldsymbol{Poin}(a_i, b_i, c_i, d_i) \geq C'_k.$$

Observe that, $\mathbf{Poin} + \mathbf{Cr} = 1$. Thus, by the **1.5 Lemma** in [2] we have the following result.

Proposition 2.3. *Let f be a C^3 function such that the Schwarzian derivative is negative. Then, $C'_k > 1$; that is, $C_k < 1$.*

Remark 2.4. *Let I and J be two intervals finite and non-zero length such that, $\bar{I} \cap \bar{J} = \emptyset$. We assume that, J is on the right of I and we put $I := [a, b]$ and $J := [c, d]$, then*

$$\boldsymbol{Cr}\,(I, J) := \frac{|I||J|}{|[I, J)||(I, J]|} = \boldsymbol{Cr}\,(a, b, c, d)$$

and

$$\boldsymbol{Poin}\,(I, J) := \frac{|(I, J)||[I, J]|}{|[I, J)||(I, J]|} = \boldsymbol{Poin}\,(a, b, c, d).$$

Fact 2.5. *Let $f \in \mathscr{L}$. Let $l(U)$ and $r(U)$ be the left and right endpoints of U (the flat piece of f) respectively. There are a left-sided neighborhood I^l of $l(U)$, a right-sided neighborhood I^r of $r(U)$ and three positive constants K_1, K_2, K_3 such that the following holds*

1. *If $y \in I^{l_i}$ with $l_1 := l$, $l_2 := r$, then*

$$K_1 |l_i(U) - y|^{l_i} \le |f(l_i(U)) - f(y)| \le K_2 |l_i(U) - y|^{l_i},$$
$$K_1 |l_i(U) - y|^{l_i - 1} \le \frac{df}{dx}(y) \le K_2 |l_i(U) - y|^{l_i - 1}.$$

2. *If $y \in (x, z) \subset I^{l_i}$, with z the closest point to the flat interval U then*

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le K_3 \frac{|x - y|}{|x - z|}.$$

.

The first part of **Fact 2.5** implies that,

$$f_{|I^{l_i}} \approx k_i x^{\ell_i};\ i = 1, 2. \quad \text{We assume that,} \quad k_1 = k_2. \tag{A2}$$

We need this assumption to prove that α_n goes to zero (**Lemma 2.9** and **Lemma 3.2**).

2.2 Basic Results

Proposition 2.6. *Let $n \ge 1$.*

- *The set of "long" intervals consists of the intervals*

$$\mathcal{A}_n := \{(\underline{i}, \underline{q_n + i});\ 0 \le i \le q_{n+1} - 1\}.$$

- *The set of "short" intervals consists of the intervals*

$$\mathcal{B}_n := \{(\underline{q_{n+1} + i}, \underline{i});\ 0 \le i \le q_n - 1\}.$$

The set $\mathcal{P}_n := \mathcal{A}_n \cup \mathcal{B}_n$ covers the circle modulo the end points and the flat piece and it is called the n^{th} dynamical partition.

The dynamical partition produced by the first $\underline{q_{n+1} + q_n}$ pre-images of U is denoted $\mathcal{P}^n$. It consists of

$$\wp_n := \{\underline{-i};\ 0 \le i \le q_{n+1} + q_n - 1\}$$

together with the gaps between these sets. As in the case of $\mathcal{P}_n$ there are two kinds of gaps, "long" and "short":

- *The set of "long" intervals consists of the intervals*

$$\mathcal{A}^n := \{(\underline{-q_n - i}, \underline{-i}) =: I_i^n;\ 0 \leq i \leq q_{n+1} - 1\}.$$

- *The set of "short" intervals consists of the intervals*

$$\mathcal{B}^n := \{(\underline{-i}, \underline{-q_{n+1} - i}) =: I_i^{n+1};\ 0 \leq i \leq q_n - 1\}.$$

Proposition 2.7. *The sequence $|(\underline{0}, \underline{q_n})|$ tends to zero at least exponentially fast.*

Proposition 2.8. *If A is a pre-image of U belonging to $\mathcal{P}^n$ and if B is one of the gaps adjacent to A, then $|A|/|B|$ is bounded away from zero by a constant that does not depend on n, A or B.*

Lemma 2.9. *The sequence*

$$f(\sigma_n) = \frac{|(\underline{1}, \underline{q_n + 1})|}{|(\underline{q_{n-1} + 1}, \underline{1})|}$$

is bounded.

The proofs of these results can be found in [5] (proof of the **Proposition 1**, **Proposition 2** and **Lemma 1.3**).

A proof of the following Proposition can be found in [3] **theorem:3.1 p.285**).

Proposition 2.10 (Koebe principle)**.** *Let $f \in \mathscr{L}$. For every ς, $\alpha > 0$, there exist a constant $\zeta(\varsigma, \alpha) > 0$, such that, the following holds. Let T and $M \subset T$ be two intervals and let S, D be the left and the right component of $T \setminus M$ and $n \in \mathbb{N}$. Suppose that:*

1. $\sum_{i=0}^{n-1} f^i(T) < \varsigma$,
2. $f^n : T \longrightarrow f^n(T)$ *is a diffeomorphism,*
3. $\dfrac{|f^n(M)|}{|f^n(S)|}, \dfrac{|f^n(M)|}{|f^n(D)|} < \alpha.$

Then,

$$\frac{1}{\zeta(\varsigma, \alpha)} \leq \frac{Df^n(x)}{Df^n(y)} \leq \zeta(\varsigma, \alpha), \quad \forall x, y \in M;$$

that is,

$$\frac{1}{\zeta(\varsigma, \alpha)} \cdot \frac{|A|}{|B|} \leq \frac{f^n(A)}{f^n(B)} \leq \zeta(\varsigma, \alpha) \cdot \frac{|A|}{|B|}, \quad \forall A, B \text{ (intervals)} \subseteq M;$$

where,

$$\zeta(\varsigma, \alpha) = \frac{1 + \alpha}{\alpha} e^{C\varsigma}$$

and $C \geq 0$ only depends on f.

Remark 2.11. *Let $f \in \mathscr{L}$. Given $n > 1$, T and M as before. $f^n : T \longrightarrow f^n(T)$ is diffeomorphism if only if, for all $0 \leq i \leq n - 1$, $f^i(T) \cap \overline{U} = \emptyset$; where, $\overline{U}$ designates the closure of U.*

3 Proof of results

Let us put together (parameter) sequences which are frequently used in this section.

$$\alpha_n = \frac{|(\underline{-q_n}, \underline{0})|}{|[\underline{-q_n}, \underline{0})|}, \quad \sigma_n = \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{q_{n-1}}, \underline{0}]|}, \quad s_n := \frac{|[\underline{-q_{n-2}}, \underline{0}]|}{|\underline{0}|}, \quad \tau_n := \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{q_{n-2}})|}$$

and

$$\beta_n(k) = \frac{|(\underline{-q_n + kq_{n-1}}, \underline{0})|}{|[\underline{-q_n + kq_{n-1}}, \underline{0})|}; \; k = 0, 1, \cdots a_{n-1}.$$

3.1 Proof of the first part of main result

3.1.1 A priori Bounds of α_n

Proposition 3.1. *Let $n \in \mathbb{N}$ and $(\ell_1, \ell_2) \in \Omega_0 = [1,2]^2$.*

For all α_n,

$$\alpha_n^{\frac{\ell_1,\ell_2}{2}} < 0.55;$$

for at least every other α_n

$$\alpha_n^{\frac{\ell_1,\ell_2}{2}} < 0.3.$$

If

$$\alpha_n^{\frac{\ell_1,\ell_2}{2}} > 0.3,$$

then either,

$$\alpha_n^{\frac{\ell_1,\ell_2}{2}} < 0.44 \quad or \quad \alpha_{n+1}^{\frac{\ell_1,\ell_2}{2}} < 0.16.$$

Proof. For every $n \in \mathbb{N}$ and $k = 0, 1, \cdots a_{n-1}$, we define the parameter sequences

$$\gamma_{1,n}(k) := |(\underline{-q_n + kq_{n-1}}, \underline{0})|, \quad \gamma_{1,n-1} := \gamma_{1,n-1}(0), \quad \gamma_n(k) = \frac{\gamma_{1,n}(k)}{\gamma_{1,n-1}}$$

$$\gamma_n^{(\ell_1|\ell_2)}(k) := \frac{\gamma_{1,n}^{\ell_1}(k)}{\gamma_{1,n-1}^{\ell_2}} \text{ if } n \in 2\mathbb{Z} \text{ and } \gamma_n^{(\ell_1|\ell_2)}(k) := \frac{\gamma_{1,n}^{\ell_2}(k)}{\gamma_{1,n-1}^{\ell_1}} \text{ if } n \in 2\mathbb{Z}+1.$$

These notations simplify the formalization of the following lemma which will play an important (essential) role in the proof of **Proposition 3.1**.

Lemma 3.2. *For n large enough and for every $k = 0, 1, \cdots a_{n-1} - 1$, the following inequality holds*

$$\frac{(\beta_n(k)^{\ell_1,\ell_2} + \alpha_{n-1}^{\ell_1,\ell_2}\gamma_n^{(\ell_1|\ell_2)}(k))(1 + \gamma_n^{(\ell_1|\ell_2)}(k))}{(1 + \alpha_{n-1}^{\ell_1,\ell_2}\gamma_n^{(\ell_1|\ell_2)}(k))(\beta_n(k)^{\ell_1,\ell_2} + \gamma_n^{(\ell_1|\ell_2)}(k))} \le s_n \beta_n(k+1). \tag{3.1}$$

Proof. Let n be an even non negative integer large enough. For fixed $k = 0, 1, \cdots a_{n-1} - 1$, according to the assumption **(A2)**, the left hand side of **(3.1)** is equal to the cross-ratio

$$\textbf{Poin}\,(\underline{-q_n + kq_{n-1} + 1}, \underline{-q_{n-1} + 1}).$$

Applying $f^{q_{n-1}-1}$, by expanding cross-ratio property, we get the inequality. ◻

The left hand side is a function of the three variables $\beta_n(k)^{\ell_1,\ell_2}$, $\alpha_{n-1}^{\ell_1,\ell_2}$ and $\gamma_n^{(\ell_1|\ell_2)}$. Observe that the function increases monotonically with each of the first two variables. However, relatively to the third variable, the function reaches a minimum. To see this, take the logarithm of the function and check that the first derivative is equal to zero only when

$$(\gamma_n^{(\ell_1|\ell_2)})^2 = \frac{\beta_n(k)^{\ell_1,\ell_2}}{\alpha_{n-1}^{\ell_1,\ell_2}}.$$

By substituting this value for $\gamma_n^{(\ell_1|\ell_2)}$ in **(3.1)**, we get that

$$\left(\frac{\beta_n(k)^{\frac{\ell_1,\ell_2}{2}} + \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}}{1+\beta_n(k)^{\frac{\ell_1,\ell_2}{2}}\alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}}\right)^2 \leq s_n\beta_n(k+1). \tag{3.2}$$

Put:

$$x_n(k) := \min\{\beta_n(k)^{\frac{\ell_1,\ell_2}{2}}, \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}\} \quad \text{and} \quad y_n(k) := \beta_n(k)^{\frac{\ell_1,\ell_2}{2}}.$$

Since $\beta_n(k+1) \leq y_n(k+1)$, substituting the above variable into **(3.2)** gives rise to a quadratic inequality in $x_n(k)$ whose only root in the interval $(0,1)$ is given by

$$\frac{\sqrt{s_n y_n(k+1)}}{1+\sqrt{1-s_n y_n(k+1)}};$$

that is,

$$x_n(k) = \min\{\beta_n(k)^{\frac{\ell_1,\ell_2}{2}}, \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}\} \leq \frac{\sqrt{s_n y_n(k+1)}}{1+\sqrt{1-s_n y_n(k+1)}}. \tag{3.3}$$

Lemma 3.3. *There is a subsequence of α_n including at least every other α_n, such that*

$$\limsup \alpha_n^{\frac{\ell_1,\ell_2}{2}} \leq 0.3.$$

Proof. We use the following elementary lemma in [5].

Lemma 3.4. *The function*

$$h_n(z) = \frac{\sqrt{s_n z}}{1+\sqrt{1-s_n z}}$$

moves points to the left, $h(z) < z$, if $z \geq 0.3$ and n is large enough.

We select the subsequence.

1. **The initial term:** there exists $n-2 \in \mathbb{N}$, such that $\alpha_{n-2}^{\frac{\ell_1,\ell_2}{2}} \leq 0.3$. This comes directly from the properties of the function h_n (**Lemma 3.4**) and from **(3.3)**.
2. **The next element:** suppose that α_{n-2} has been selected. If

$$x_n(k) = \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}} \quad \text{or} \quad \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}} \leq 0.3,$$

for some $k = 0, 1, \cdots a_{n-1} - 1$, then, we select α_{n-1} as the next term. Otherwise, α_n is the next term. Thus, the sequence is constructed.

□

Corollary 3.5. *For the whole sequence (α_n) we have*

$$\limsup \alpha_n^{\frac{\ell_1,\ell_2}{2}} \leq 0.3$$

Moreover,

- *if α_{n-1} does not belong to the subsequence (α_n) defined by the **Lemma 3.3** then either*

$$\alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}} < 0.44 \quad or \quad \alpha_n^{\frac{\ell_1,\ell_2}{2}} < 0.16$$

Proof. Observe that the function

$$H : (s,t) \in \mathbb{R}_+^2 \mapsto F(s,t) = \frac{s+t}{1+st}$$

is symmetric and for fixed s, the function $F(s,\cdot)$ reaches its minimum in zero by taking the value s. Therefore, for every $s,t \geq 0$,

$$s,t \leq \frac{s+t}{1+st}.$$

So,

$$\alpha_n^{\frac{\ell_1,\ell_2}{2}},\ \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}} \leq \frac{\alpha_n^{\frac{\ell_1,\ell_2}{2}} + \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}}{1+\alpha_n^{\frac{\ell_1,\ell_2}{2}}\alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}}. \tag{3.4}$$

Thus, according to that α_{n-2} is an element of the sequence and suppose that α_{n-1} do not belong to the previous subsequence of the **Lemma 3.3**, then, it follows from **(3.2)** that the right member of **(3.4)** is estimated as following

$$\frac{\alpha_n^{\frac{\ell_1,\ell_2}{2}} + \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}}{1+\alpha_n^{\frac{\ell_1,\ell_2}{2}}\alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}} \leq \sqrt{s_n\beta_n(1)^{\frac{\ell_1,\ell_2}{2}}} \approx \sqrt{\beta_n(1)^{\frac{\ell_1,\ell_2}{2}}} \leq \sqrt{0.3}. \tag{3.5}$$

Also,

$$\min\{\alpha_n^{\frac{\ell_1,\ell_2}{2}}, \alpha_{n-1}^{\frac{\ell_1,\ell_2}{2}}\} = \alpha_n^{\frac{\ell_1,\ell_2}{2}} \leq 0.3. \tag{3.6}$$

Thus, if $\alpha_n^{\frac{\ell_1,\ell_2}{2}} \geq 0.16$, then by combining this with **(3.5)** and **(3.6)**, we obtain the desired estimate. □

The **Proposition 3.1** is proved. □

3.1.2 Recursive formula of α_n

Proposition 3.6. *Let n be integer large enough,*

1. *if ℓ_1, $\ell_2 > 1$, we have*

$$\alpha_{2n}^{\ell_1} \leq M_{2n}(\ell_1)\alpha_{2n-2}^2 \quad and \quad \alpha_{2n+1}^{\ell_2} \leq M_{2n+1}(\ell_2)\alpha_{2n-1}^2; \tag{3.7}$$

where,

$$M_n(\ell) = s_{n-1}^2 \cdot \frac{2}{\ell} \cdot \frac{1}{1+\sqrt{1-\frac{2(\ell-1)}{\ell}s_{n-1}\alpha_{n-1}}} \cdot \frac{1}{1-\alpha_{n-2}} \cdot \frac{\sigma_n}{\sigma_{n-2}}.$$

2. *if $\ell_1 = \ell_2 = 1$, then*

$$\alpha_n \leq W_n^1 \cdot \frac{\sigma_n}{\sigma_{n-2}} \alpha_{n-2}. \tag{3.8}$$

Proof. We treat the case n even and the case n odd is treated in a similar way. Recall that,

$$\alpha_n = \frac{|(\underline{-q_n}, \underline{0})|}{|[\underline{-q_n}, \underline{0})|}.$$

For every n even large enough, by **Proposition 2.7** and point 2 of **Fact 2.5**, applying f to the equality, we get

$$\alpha_n^{\ell_1} = \frac{|(\underline{-q_n+1}, \underline{1})|}{|[\underline{-q_n+1}, \underline{1})|}.$$

which is certainly less than the cross-ratio

$$\mathbf{Poin}(\underline{-q_n+1}, (\underline{1}, \underline{-q_{n-1}+1}]).$$

Since the cross-ratio **Poin** is expanded by $f^{q_{n-1}-1}$, then,

$$\alpha_n^{\ell_1} < \delta_n(1) s_n(1); \tag{3.9}$$

with,

$$\delta_n(k) := \frac{|(\underline{-q_n + kq_{n-1}}, \underline{kq_{n-1}})|}{|[\underline{-q_n + kq_{n-1}}, \underline{kq_{n-1}})|}.$$

and

$$s_n(k) := \frac{|[\underline{-q_n + kq_{n-1}}, \underline{0}]|}{|(\underline{-q_n + kq_{n-1}}, \underline{0}]|}.$$

If $a_{n-1} = 1$, by multiplying and dividing the right member of **(3.9)** by α_{n-2}^2, we obtain directly **(3.11)**.

Suppose that $a_{n-1} > 1$ and estimate $\delta_n(k)$. By the Mean Value Theorem (Lagrange), f transforms the intervals defining the ratio $\delta_n(k)$ into a pair whose ratio is

$$\frac{u_k}{v_k} \delta_n(k)$$

with u_k being the derivative of $f\left(x^{\ell_1}\right)$ at a point in the interval

$$U_k := (\underline{-q_n + kq_{n-1}}, \underline{kq_{n-1}}),$$

and v_k being the derivative of $f\left(x^{\ell_1}\right)$ at a point in the interval

$$V_k := [\underline{-q_n + kq_{n-1}}, \underline{kq_{n-1}}).$$

Note that, for n sufficiently large,

$$u_1 < v_1 < u_2 < v_2 \cdots < v_{a_{n-1}}.$$

We see that the image of $\delta_n(k)$ by f is smaller than:

$$\mathbf{Poin}(\underline{-q_n + kq_{n-1} + 1}, (\underline{kq_{n-1} + 1}, \underline{-q_{n-1} + 1}]).$$

Once again, by expanding cross-ratio property $(f^{q_{n-1}-1})$, it follows that,

$$\frac{u_k}{v_k}\delta_n(k) \le s_n(k+1)\cdot\delta_n(k+1). \tag{3.10}$$

Multiplying **(3.10)** for $k = 0, ..., a_{n-1}-1$, and substituting the resulting estimate of $\delta_n(1)$ into **(3.9)** , we obtain:

$$\alpha_n^{\ell_1} \le \delta_n(a_{n-1})\frac{v_{a_{n-1}}}{u_1}s_n(1)\cdots s_n(a_{n-1}).$$

Observe that, $s_n(1)\cdots s_n(a_{n-1}) < s_n$ and

$$\frac{v_{a_{n-1}}}{u_1} \le \left(\frac{|(\underline{-q_{n-2}}, \underline{0})|}{|(\underline{q_{n-1}}, \underline{0})|}\right)^{\ell_1-1} \le \frac{|(\underline{-q_{n-2}}, \underline{0})|}{|(\underline{q_{n-1}}, \underline{0})|}.$$

Thus,

$$\alpha_n^{\ell_1} \le s_n\cdot\frac{|(\underline{-q_{n-2}}, \underline{0})|}{|(\underline{q_{n-1}}, \underline{0})|}\cdot\delta_n(a_{n-1});$$

which can be rewritten in the form

$$\alpha_n^{\ell_1} \le s_n\nu_{n-2}\mu_{n-2}\alpha_{n-2}^2; \tag{3.11}$$

with,

$$\nu_{n-2} := \frac{|[\underline{-q_{n-2}}, \underline{0})|}{|(\underline{q_{n-1}}, \underline{0})|}\cdot\frac{|[\underline{-q_{n-2}}, \underline{0})|}{|[\underline{-q_{n-2}}, \underline{a_{n-1}q_{n-1}})|}$$

and

$$\mu_{n-2} := \frac{|(\underline{-q_{n-2}}, \underline{a_{n-1}q_{n-1}})|}{|(\underline{-q_{n-2}}, \underline{0})|}.$$

It remains to estimate ν_{n-2} and μ_{n-2} to end this part. For ν_{n-2}, observe that,

$$|(\underline{-q_{n-2}}, \underline{0})| \le |(\underline{q_{n-3}}, \underline{0})|$$

so that,

$$\nu_{n-2} \le \frac{1}{\sigma_{n-1}\sigma_{n-2}}\cdot\frac{1}{1-\alpha_{n-2}} \tag{3.12}$$

The estimation of μ_{n-2} is facilitated by the following elementary lemma in [5].

Lemma 3.7. *Let $\ell \in (1, 2)$. For all numbers $x > y$, we have the following inequality:*

$$\frac{x^\ell - y^\ell}{x^\ell} \ge \left(\frac{x-y}{x}\right)\left[\ell - \frac{\ell(\ell-1)}{2}\left(\frac{x-y}{x}\right)\right].$$

Now, apply f into the intervals defining the ratio μ_{n-2}. By **Lemma 3.7**, the resulting ratio is larger than

$$\mu_{n-2}(\ell_1 - \frac{\ell_1(\ell_1-1)}{2}\mu_{n-2}).$$

The cross-ratio **Poin**

$$\textbf{Poin}\,(-q_{n-2}+1, (\underline{q_{n-1}+1}, \underline{1}));$$

that is,

$$\frac{|(\underline{-q_{n-2}+1}, \underline{q_{n-1}+1})||[\underline{-q_{n-2}+1}, \underline{1})|}{|[\underline{-q_{n-2}+1}, \underline{q_{n-1}+1})||(\underline{-q_{n-2}+1}, \underline{1})|}$$

is larger again. Thus, by expanding cross-ratio property on $f^{q_{n-2}}$, we obtain:

$$\mu_{n-2}(\ell_1 - \frac{\ell_1(\ell_1-1)}{2}\mu_{n-2}) \le s_{n-1}\sigma_n\sigma_{n-1}. \tag{3.13}$$

By solving this quadratic inequality, we obtain

$$\mu_{n-2} \ge \frac{1+\sqrt{1-\frac{2}{\ell_1}(\ell_1-1)s_{n-1}\sigma_n\sigma_{n-1}}}{\ell_1-1}. \tag{3.14}$$

Thus, by combining the **(3.13)** and **(3.14)**, we obtain

$$\mu_{n-2} < \frac{2}{\ell_1} \cdot \frac{1}{1+\sqrt{1-\frac{2(\ell_1-1)}{\ell_1}s_{n-1}\sigma_n\sigma_{n-1}}} s_{n-1}\sigma_n\sigma_{n-1}. \tag{3.15}$$

Since, $\sigma_n\sigma_{n-1} < \alpha_{n-1}$, the first inequality in **(3.7)** follows by combining the inequalities **(3.11)**, **(3.12)** and **(3.15)**. Likewise, the second inequality in **(3.7)** is obtained by following suitably the same reasoning as previously.

□

3.1.3 α_n go to zero

If $\ell_1 = \ell_2 = 1$, then by **Proposition 3.1**, $\prod_{k=2}^{k=n} W_k^1$ goes to zero; thus, by composing the inequality obtained by f, since $f(\sigma_n)$ is bounded (**Lemma 2.9**), then the result follows.

Note that, the cases where the critical exponents are of the form $(1,\ell)$ or $(\ell,1)$; with $\ell > 1$, are treated in [4].

Now, Let us suppose that $\ell_1, \ell_2 > 1$.

Technical reformulation of the **Proposition 3.6**.

Let W_n be a sequence defined by

$$M_n(\ell) = W_n(\ell)\frac{\sigma_n}{\sigma_{n-2}}.$$

Let

$$M'_n(\ell) := M_n(\ell)\alpha_{n-2}^{2-\ell} \quad \text{and} \quad W'_n(\ell) := W_n(\ell)\alpha_{n-2}^{2-\ell}$$

The recursive formula **(3.7)** can be written for n even in the form:

$$\alpha_n^{\ell_1} \le W'_n(\ell_1)\frac{\sigma_n}{\sigma_{n-2}}\alpha_{n-2}^{\ell_1}.$$

so,

$$\alpha_n^{\ell_1} \le \prod_{k=2}^{k=n} W'_k(\ell_1)\frac{\sigma_n}{\sigma_0}\alpha_0^{\ell_1}.$$

$\prod_{k=2}^{k=n} W'_k(\ell_1)$ goes to zero

Observe that, the size of $W_n'(\ell_1)$ is given by the study of the function

$$W_n'(x,y,\ell_1) = \frac{1}{\frac{\ell_1}{2} + \frac{\ell_1}{2}\sqrt{1 - \frac{2(\ell_1-1)}{\ell_1}x^{\frac{2}{\ell_2}}}} \cdot \frac{y^{\frac{4}{\ell_1}-2}}{1-y^{\frac{2}{\ell_1}}}$$

The meaning of variation of $W_n'(x,y,\ell_1)$ relative to the third variable is given by the following lemma in [5]).

Lemma 3.8. *For any $0 < y < \frac{1}{\sqrt{e}}$, $x \in (0,1)$ and $\ell_1 \in (1,2]$ the function $W_n'(x,y,\ell_1)$ is increasing with respect to ℓ_1.*

Analyse the asymptotic size of $W_i'(2)$

Since the hypotheses of the **Lemma 3.8** are satisfied (**Proposition 3.1**), the only remaining point is the verification of the convergence of $\prod_{i=1}^{n} W_i'(2)$.

- If $\alpha_{n-2} < (0.3)^{\ell_1}$, then $W'(2) < W'(0.55, 0.16, 2) < 0,9$.
- If not, then by the **Proposition 3.1**, $W'(2) < W'(0.3, 0.44, 2) < 0,98$ or else, $W_{n+1}'(2)W_n'(2) < W'(0.55, 0.16, 2)W'(0.16, 0.55, 2) < 0,85$

Corollary 3.9. *Let $\ell_1, \ell_2 \in [1,2]$. If $1 < \ell_1 < 2$ respectively $1 < \ell_2 < 2$, then α_{2n} respectively α_{2n+1} goes to zero least double exponentially fast. And, if $\ell_1 = 2$ or 1 respectively $\ell_2 = 2$ or 1, then α_{2n} respectively α_{2n+1} goes to zero least exponentially fast*

Proof. Let $n := 2p_n \in \mathbb{N}$. From the analysis of the asymptotic size of $W_i'(2)$, it follows that, when n goes to infinity, $\prod_{i=0}^{n} W_i'(\ell_1)$ goes to zero and α_n does so. Therefore,

$$\prod_{i=0}^{n} M_i(\ell_1)$$

goes to zero when n goes to infinity. Thus, by the **Proposition 3.6**, for n even, there is λ_0 such that

- if $1 < \ell_1 < 2$,

$$\alpha_n \leq \lambda_0^{\left(\frac{2}{\ell_1}\right)^{p_n}},$$

- and if $\ell_1 = 1, 2$,

$$\alpha_n \leq \lambda_0^n.$$

The case n odd is treated the same way. ☐

3.2 Proof of the second part of main result

In this section we find a bounded geometry domain.

3.2.1 Recursive Affine Inequality of order two on α_n

Let
$$\kappa_n := \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|}.$$

Remark 3.10. *Since the point $\underline{q_{n-2}}$ lies in the gap between $\underline{-q_{n-1}}$ and $\underline{-q_{n-1}+q_{n-2}}$ of the dynamical partition $\mathcal{P}_{n-2}$, then by the* ***Proposition 2.8****, τ_n/α_{n-1} and κ_n are comparable.*

Proposition 3.11. *For any bounded type rotation number, there is a uniform constant K so that*

$$\kappa_{2n} > K\left(\alpha_{2n-1}\right)^{\frac{1-\ell_2^{a_{2n}+1}}{\ell_2-1}} \quad \text{and} \quad \kappa_{2n+1} > K\left(\alpha_{2n}\right)^{\frac{1-\ell_1^{a_{2n+1}+1}}{\ell_1-1}}.$$

Proof of the Proposition: If $a_n = 1$, it comes down to showing that the sequence κ_n is bounded away from zero; which becomes relatively very simple. In fact, suppose that, $|(\underline{q_n}, \underline{-q_{n-1}})| \leq |(\underline{0}, \underline{q_n})|$, then

$$\kappa_n = \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|} \geq \frac{1}{2},$$

else, κ_n is greater than
$$\frac{|\underline{-q_{n+1}}|}{|[\underline{-q_{n+1}}, \underline{-q_{n-1}})|};$$
which by the **Proposition 2.8** is bounded away from zero.

In the following part of the proof, we suppose that $a_n > 1$ and the following lemma in [5] (**Lemma 4.1.**) is used.

Lemma 3.12. *The ratio*

$$1-\beta_n(i) = \frac{|\underline{-q_n+iq_{n-1}}|}{|[\underline{-q_n+iq_{n-1}}), \underline{0})|}$$

is bounded away from zero by a uniform constant for all $i = 0, \cdots, a_{n-1}$.

Lemma 3.13. *For every $n \in \mathbb{N}$ and for all $i = 0, \cdots, a_{n-1}$, there is a uniform constant K such that*
$$\beta_n(i)^{\ell_1,\ell_2} \geq \beta_n(i+1).$$

Proof. For n large enough and for fixed $i = 0, \cdots, a_{n-1}$, by the **Proposition 2.7** and the **Fact 2.5**, we have:

$$\beta_n(i)^{\ell_1,\ell_2} = \frac{|(\underline{-q_n+kq_{n-1}+1}, \underline{1})|}{|[\underline{-q_n+iq_{n-1}+1}, \underline{1})|};$$

which is greater than,

$$\mathbf{Cr}([\underline{-q_{n-2}+1}, \underline{-q_n+iq_{n-1}+1}), (\underline{-q_n+iq_{n-1}+1}, \underline{1})).$$

By applying the cross-ratio inequality under $f^{q_{n-2}-1}$, by the **Fact 2.5** the resulting ratio is greater than

$$\mathbf{Cr}([\underline{-q_{n-2}+1}, \underline{-q_n + iq_{n-1} + q_{n-2}+1}),\\ (\underline{-q_n + iq_{n-1} + q_{n-2}+1}, q_{n-2}+\underline{1}))$$

times a uniform constant. We now repeat this sequence of steps $a_{n-2}-1$ times: Apply $f^{q_{n-2}-1}$, discard the interval containing $\underline{0}$ and use the point **2** of the **Fact 2.5**, and replace the result by a cross-ratio spanning the intervals $\underline{-q_{n-2}+1}$. At the end, this will produce the cross ratio

$$\mathbf{Cr}([\underline{-q_{n-2}+1}, \underline{-q_n + iq_{n-1} + a_{n-2}q_{n-2}+1}),\\ (\underline{-q_n + iq_{n-1} + a_{n-2}q_{n-2}+1}, a_{n-2}q_{n-2}+\underline{1})).$$

And finally, by applying $f^{q_{n-3}-1}$, since by the **Lemma 3.12** the interval containing $\underline{-q_{n-2}+q_{n-3}}$ bounded away from zero, then resulting ratio is

$$\frac{|(\underline{-q_n+(i+1)q_{n-1}}, \underline{q_{n-1}})|}{|[\underline{-q_n+(i+1)q_{n-1}}, \underline{q_{n-1}})|}$$

times a uniform constant. Thus, since $\underline{-q_n+q_{n-1}}$ lies between $\underline{-q_n+(i+1)q_{n-1}}$ and $\underline{q_{n-1}}$, then, by the **Lemma 3.12**, this ratio is comparable to $\beta_n(i+1)$. $\square$

Back to the proof of the Proposition 3.11: Observe by the **Proposition 2.8** that

$$|(\underline{-q_n+(i+1)q_{n-1}}, \underline{0})|$$

and

$$|[\underline{-q_n+iq_{n-1}}, \underline{0})|$$

are comparable. Therefore, κ_{n-1} is comparable to the product

$$\beta_n(1)\cdots\beta_n(a_{n-1}-1).$$

By combining this with the **Lemma 3.13**, we have the **Proposition 3.11**.

Recursive Affine Inequality of order two on α_n

Proposition 3.14. *If $\rho(f)$ is of bounded type, then there is a uniform constant K so that,*

$$\alpha_{2n} \geq K\,(\alpha_{2n-1})^{\frac{\ell_2}{\ell_1}\cdot\frac{1-\ell_2^{-a_{2n}}}{\ell_2-1}}\,(\alpha_{2n-2})^{\ell_1^{-a_{2n-1}}}$$

and

$$\alpha_{2n+1} \geq K\,(\alpha_{2n})^{\frac{\ell_1}{\ell_2}\cdot\frac{1-\ell_1^{-a_{2n+1}}}{\ell_1-1}}\,(\alpha_{2n-1})^{\ell_2^{-a_{2n}}}.$$

Proof of the Proposition If n is even and large enough, then

$$\alpha_n^{\ell_1} = \frac{|(\underline{-q_n+1}, \underline{1})|}{|[\underline{-q_n+1}, \underline{1})|};$$

which in turn is larger than the product of two ratios

$$\xi_{1,n} = \frac{|(\underline{-q_n+1}, \underline{1})|}{|(\underline{-q_n+1}, \underline{-q_{n-1}+1})|} \quad \text{and} \quad \xi_{2,n} = \frac{|(\underline{-q_n+1}, \underline{-q_{n-1}+1})|}{|[\underline{-q_n+1}, \underline{-q_{n-1}+1})|}$$

Lemma 3.15. *For all n even large enough*

$$\xi_{1,n} \geq K\tau_n.$$

Proof. Observe that $\xi_{1,n}$ is greater than

$$\mathbf{Cr}\left((\underline{-q_n+1}, \underline{1}), \underline{-q_{n-1}+1}\right).$$

By applying **CRI** on $f^{q_{n-1}-1}$ and discarding the intervals containing $\underline{0}$. Repeat this $a_{n-1}-1$ times more: By the **Fact 2.5**, the resulting ratio is large than

$$\mathbf{Cr}\left((\underline{-q_{n-2}-q_{n-1}+1}, \underline{(a_{n-1}-1)q_{n-1}+1}), (\underline{1}, \underline{-q_{n-1}+1}])\right).$$

Apply $f^{q_{n-1}-1}$, and discard the intervals containing the flat interval. Apply f, replace the resulting by the cross-ratio

$$\mathbf{Cr}\left((\underline{-q_{n-2}+1}, \underline{q_{n-1}+1}), (\underline{1}, \underline{-q_{n-1}+1}])\right).$$

Thus, by **CRI** on $f^{q_{n-2}-1}$ and the inequalities above, we obtain

$$\xi_{1,n} > \frac{|(\underline{q_{n-2}}, \underline{-q_{n-3}}]|}{|(\underline{q_n}, \underline{-q_{n-3}}]|}\tau_n \tag{3.16}$$

The first factor on the right hand side of **3.16** is greater than

$$\frac{|\underline{-q_{n-3}}|}{|(\underline{0}, \underline{-q_{n-3}}]|};$$

which by the **Proposition 2.8** goes away from zero. The Lemma is proved. □

Lemma 3.16. *There is a uniform constant K so that, for all n*

$$\xi_{2,2n} \geq K\left(\alpha_{2n-2}\right)^{\ell_1^{-a_{2n-1}+1}} \quad \text{and} \quad \xi_{2,2n+1} \geq K\left(\alpha_{2n-1}\right)^{\ell_1^{-a_{2n}+1}}.$$

Proof. If $a_{n-1}=1$, then $\xi_{2,n}$ is greater than

$$\mathbf{Cr}\left([\underline{-q_{n-2}+1}, \underline{-q_n+1}), (\underline{-q_n+1}, \underline{-q_{n-1}+1})\right).$$

By applying **CRI** $(q_{n-2}-1)$, the ratio resulting is greater than

$$\mathbf{Cr}\left([\underline{-q_{n-2}+1}, \underline{-q_{n-1}+1}), (\underline{-q_{n-1}+1}, \underline{-q_{n-3}+1})\right)$$

times a uniform constant. Thus, by applying **CRI** to this ratio with $f^{q_{n-3}-1}$, inequalities above and the **Proposition 2.8** the result follows.

Now, suppose that $a_{n-1} > 1$ then $\xi_{2,n}$ is greater than

$$\mathbf{Cr}\left(\underline{[-q_n + q_{n-1} + 1}, \underline{-q_n + 1}), (\underline{-q_n + 1}, \underline{-q_{n-1} + 1})\right).$$

By applying $f^{q_{n-1}-1}$ to this ratio, it follows from the **Lemma 3.12** that

$$\xi_{2,n} \geq K'\beta_n(1).$$

And the lemma follows from this by using **Lemma 3.13** modulo the fact that

$$\beta_n(a_{n-1}) = \alpha_{n-2}.$$

□

Combining the **Lemma 3.15**, the **Lemma 3.16**, the **Proposition 3.11** and the **Remark 3.10**, the result of the **Proposition 3.14** follows.

Remark 3.17. *By the inequality obtained from the **Proposition 3.14**, the sequence ν_n defined by*

$$\nu_n = -\ln \alpha_n$$

verifies the following Recursive Affine Inequalities of order two for every $n > 0$

$$\nu_{2n} \leq \frac{\ell_2}{\ell_1} \cdot t_2(a_{2n})\nu_{2n-1} + \ell_1^{-a_{2n-1}}\nu_{2n-2} + \widetilde{K'} \tag{3.17}$$

and

$$\nu_{2n+1} \leq \frac{\ell_1}{\ell_2} \cdot t_1(a_{2n+1})\nu_{2n} + \ell_2^{-a_{2n}}\nu_{2n-1} + \widetilde{K'}; \tag{3.18}$$

with

$$t_i(j) = \frac{1 - \ell_i^{-j}}{\ell_i - 1}.$$

3.2.2 Analysis of Recursive Affine Inequality

We will prove that the sequence ν_n is bounded. Let us consider the sequence of vectors (v_n) defined by

$$v_n = \begin{pmatrix} \nu_n \\ \nu_{n-1} \end{pmatrix},$$

the vector given by

$$\overline{\kappa} = \begin{pmatrix} \widetilde{K'} \\ 0 \end{pmatrix}$$

and the sequence matrix

$$A_{\ell_1,\ell_2}(2n) = \begin{pmatrix} \dfrac{\ell_2}{\ell_1} \cdot t_2(a_{2n}) & \ell_1^{-a_{2n-1}} \\ 1 & 0 \end{pmatrix}$$

and

$$A_{\ell_1,\ell_2}(2n+1) = \begin{pmatrix} \dfrac{\ell_1}{\ell_2} \cdot t_1(a_{2n+1}) & \ell_2^{-a_{2n}} \\ 1 & 0 \end{pmatrix};$$

say associated matrix to the Recursive Affine Inequalities **(3.17)** and **(3.18)** respectively; in this sense that **(3.17)** and **(3.18)** can be rewritten respectively in the form:

$$v_{2n} \leq A_{\ell_1,\ell_2}(2n)v_{2n-1} + \overline{\kappa} \quad \text{and} \quad v_{2n+1} \leq A_{\ell_2,\ell_1}(2n+1)v_{2n} + \overline{\kappa}. \tag{3.19}$$

Therefore, for every $n := 2p_n + r_n$; with $p_n \in \mathbb{N}^*$ and $r_n \in \{0, 1\}$, we have

$$\begin{aligned} v_n \leq & \overline{A}_{\ell_1,\ell_2}(n)\overline{A}_{\ell_1,\ell_2}(n-2)\cdots\overline{A}_{\ell_1,\ell_2}(2+r_n)v_{2-r_n} + \\ & \left(Id + \sum_{i=4+r_n}^{n} \overline{A}_{\ell_1,\ell_2}(n)\overline{A}_{\ell_1,\ell_2}(n-2)\cdots\overline{A}_{\ell_1,\ell_2}(i)\right)\overline{\kappa'}. \end{aligned}$$

where,

$$\overline{A}_{\ell_1,\ell_2}(n) = A_{\ell_1,\ell_2}(n)A_{\ell_1,\ell_2}(n-1).$$

Observe that, if (ℓ_1, ℓ_2) is very close to an element of the set

$$\{(a,\infty),\ (\infty,b),\ (\infty,\infty),\ a,b \in \mathbb{R}\}$$

$\overline{A}_{\ell_1,\ell_2}(n)$ is diagonalizable with nonnegative eigenvalues and at most one is strictly positive; that is, $1/\ell_1$ or $1/\ell_2$ and as $\ell_1, \ell_2 > 2$, then, $\overline{A}_{\ell_1,\ell_2}(n)$ contracts the Euclidean metric; therefore, v_n (also ν_n and α_n) is bounded. In the following analysis, we suppose that $1 < \ell_1, \ell_2 < C < \infty$; for some C in $\mathbb{R}_+$.

Lemma 3.18. *Fix* $(\ell_1, \ell_2) \in [2, \infty)^2$. *The sequence*

$$\overline{A}^{\circ n}_{\ell_1,\ell_2} := \overline{A}_{\ell_1,\ell_2}(n)\overline{A}_{\ell_1,\ell_2}(n-2)\cdots\overline{A}_{\ell_1,\ell_2}(4)$$

is bounded (uniformly) by $\max\{\ell_1/\ell_2, \ell_2/\ell_1\}$.

Proof. Observe that, for all $n \in \mathbb{N}$,

$$\overline{A}_{\ell_1,\ell_2}(n) \leq \begin{pmatrix} (1-b_n(2))(1-b_{n-1}(2)) & \frac{\ell_1}{\ell_2}(1-b_n(2))b_{n-2}(2) \\ \frac{\ell_2}{\ell_1}(1-b_{n-1}(2)) & b_{n-2}(2) \end{pmatrix} =: \overline{B}_{\ell_1,\ell_2}(n);$$

with,

$$b_{2n}(\ell) = b_{2n} := \ell_2^{-a_{2n}} \quad \text{and} \quad b_{2n+1}(\ell) = b_{2n+1} := \ell_1^{-a_{2n+1}}.$$

Let be a sequence $(x_n)_{n\in\mathbb{N}}$ defined by:

$$(x_n)_{n\in\mathbb{N}} := \{1 - b_n(2), b_n(2); n \in \mathbb{N}\}.$$

Remark that for every $n \in \mathbb{N}$, $x_n \in [2^{-a}, 1 - 2^{-a}]$; where, $a := \max\{a_n, n \in \mathbb{N}\}$.

Thus, by setting

$$\overline{B}^{\circ n}_{\ell_1,\ell_2} = \begin{pmatrix} d^{1,n} & \frac{\ell_2}{\ell_1}d^{3,n} \\ \frac{\ell_1}{\ell_2}d^{2,n} & d^{4,n} \end{pmatrix},$$

it follows that for every $i \in \{1, 2, 3, 4\}$, there is $x_{k_{i,j}}$, $j = p_{n-2}, \cdots, n-2$ such that:

$$d^{i,n} \leq \sum_{j=p_{n-2}}^{n-2} x_{k_{i,1}}\cdots x_{k_{i,j}} \leq 1$$

This proves the lemma. □

Proposition 3.19. *When $(\ell_1, \ell_2) \in [2, \infty)^2 \setminus \{(2, 2)\}$, then $\overline{A}^{\circ n}_{\ell_1,\ell_2}$ contracts the Euclidean metric provided n is large enough. The scale of the contraction is bounded away from 1 independently of ℓ_1 and ℓ_2 and the particular sequence b_n, whereas the moment when the contraction starts depends on the upper bound of b_n.*

Proof. For fixed $n \in \mathbb{N}$, putting

$$\overline{A}^{\circ n}_{\ell_1,\ell_2} = \begin{pmatrix} d^{1,n}(z_1, z_2) & \frac{\ell_1}{\ell_2} d^{3,n}(z_1, z_2) \\ \frac{\ell_2}{\ell_1} d^{2,n}(z_1, z_2) & d^{4,n}(z_1, z_2) \end{pmatrix};$$

with $z_1 = 1/(\ell_1 - 1)$ and $z_2 = 1/(\ell_2 - 1)$ Thus, $d^{i,n}(z_1, z_2)$, $i = 1, 2, 3, 4$ are polynomials of respective degree $n-2$, $n-3$, $n-3$ and $n-4$; whose coefficients belong to the interval $[\ell^{-a}, 1 - \ell^{-a}]$; with, $\ell = \max\{\ell_1, \ell_2\}$. For fixed $i \in \{1, 2, 3, 4\}$, we denote by $d^{i,n}_j$ the coefficients of $d^{i,n}(z_1, z_2)$. Let us put:

$$\begin{cases} d_1(2) = 1 & \text{and} \quad d_1(i) = 0; \quad i = 1, 3, 4 \\ d_2(3) = 1 & \text{and} \quad d_2(i) = 0; \quad i = 1, 2, 4. \end{cases}$$

Then by the **Lemma 3.18**, the sums

$$\sum_{j=0}^{p_{n-2}} d^{i,j} z_1^{j+d_1(i)} z_2^{j+d_2(i)}; \quad i = 1, 2, 3, 4$$

are uniformly bounded. Therefore, for every $i \in \{1, 2, 3, 4\}$

$$\sum_{j=k}^{\infty} d^{i,n}_j z_1^{j+d_1(i)} z_2^{j+d_2(i)} \longrightarrow 0, \qquad \text{when } k \longrightarrow \infty$$

Lemma 3.20. *For every $i \in \{1, 2, 3, 4\}$, the sequence $(d^{i,n}_j)$ tends to zero at least exponentially.*

Proof. By a simple calculation, we have

$$d^{1,n}(0,0) = \prod_{i=1}^{p_{n-2}} b_{n-2i+1},\ d^{4,n}(0,0) = \prod_{i=1}^{p_{n-2}} b_{n-2i},\ d^{2,n}(0,0) = d^{3,n}(0,0) = 0.$$

Now, suppose that, for given $0 < n-1$ and $0 < j < n-1$ all the coefficients $d^{i,n-1}_j$ $i \in \{1, 2, 3, 4\}$, tend to zero at least exponentially fast. Then, since

$$\overline{A}^{\circ n}_{\ell_1,\ell_2} = \overline{A}_{\ell_1,\ell_2}(n) \overline{A}^{\circ n-1}_{\ell_1,\ell_2}$$

and by the form of the coefficients of $\overline{A}_{\ell_1,\ell_2}(n)$, the Lemma is proved and therefore, the **Proposition 3.19**. □

□

3.2.3 Particular case of Bounded Geometry

Proposition 3.21. *Let $f \in \mathscr{L}$ with critical exponents (ℓ_1, ℓ_2) and rotation number $\rho(f) = [abab\cdots]$; for some $a, b \in \mathbb{R}$. If the inequality*

$$\sqrt{(\ell_1^{-b} - \ell_2^{-a})^2 + (t_1(b)t_2(a) + 2(\ell_1^{-b} + \ell_2^{-a}))t_1(b)t_2(a)} + \\ + t_1(b)t_2(a) + \ell_1^{-b} + \ell_2^{-a} - 2 < 0;$$

holds, then the geometry of f is bounded.

Proof. If $\rho(f) = [abab\cdots]$, then

$$\overline{A}_{\ell_1,\ell_2} = \begin{pmatrix} t_1(b)t_2(a) + \ell_1^{-b} & \frac{\ell_2}{\ell_1}\ell_2^{-a}t_2(a) \\ \frac{\ell_1}{\ell_2}t_1(b) & \ell_2^{-a} \end{pmatrix}$$

and these eigenvalues λ_s and λ_u are defined as following

$$2\lambda_s = -\sqrt{(\ell_1^{-b} - \ell_2^{-a})^2 + (t_1(b)t_2(a) + 2(\ell_1^{-b} + \ell_2^{-a}))t_1(b)t_2(a)} + \\ + t_1(b)t_2(a) + \ell_1^{-b} + \ell_2^{-a}$$

and

$$2\lambda_u = \sqrt{(\ell_1^{-b} - \ell_2^{-a})^2 + (t_1(b)t_2(a) + 2(\ell_1^{-b} + \ell_2^{-a}))t_1(b)t_2(a)} + \\ + t_1(b)t_2(a) + \ell_1^{-b} + \ell_2^{-a};$$

Observe that, $\lambda_s \in (0, 1)$. Thus, if $\lambda_u < 1$, then $\overline{A}_{\ell_1,\ell_2}$ contracts the Euclidean metric.

This proves the proposition. □

3.3 Proof of Corollary

Lemma 3.22. *Let*

$$w_n(i) = \frac{|(\underline{-q_n + (i+1)q_{n-1}}, \underline{-q_n + iq_{n-1}})|}{|\underline{-q_n + iq_{n-1}}|}$$

be a parameter sequence with, $i = 0 \cdots a_{n-1} - 1$. $w_n(i)$, $i = 1 \cdots a_{n-1} - 1$ and $w_n^{\ell_1,\ell_2}(0)$ are comparable to α_{n-1}.

Proof. Suppose that $a_{n-1} > 1$ and let $i = 1, \cdots, a_{n-1} - 1$. We apply the **Proposition 2.10** to

- $T = [\underline{-q_n + (i+1)q_{n-1}}, \underline{-q_n + (i-1)q_{n-1}}]$,
- $M = (\underline{-q_n + (i+1)q_{n-1}}, \underline{-q_n + (i-1)q_{n-1}})$,
- $S = \underline{-q_n + (i+1)q_{n-1}}$,
- $D = \underline{-q_n + (i-1)q_{n-1}}$,
- $f^{q_n - (i-1)q_{n-1}}$.

1. For every $j < q_n - (i-1)q_{n-1}$, $f^j(T) \cap \overline{U} = \emptyset$; so $f^{q_n-(i-1)q_{n-1}}$ is diffeomorphism on T (**Remark 2.11**);

2. the set $\bigcup_{j=0}^{q_n-(i-1)q_{n-1}}(T)$ covers the circle at most two times;
3. for n large enough, and by the **Proposition 2.8**, we have

$$\frac{f^{q_n-(i-1)q_{n-1}}(M)}{f^{q_n-(i-1)q_{n-1}}(S)} < \frac{f^{q_n-(i-1)q_{n-1}}(M)}{f^{q_n-(i-1)q_{n-1}}(D)} = \frac{|(\underline{0}, \underline{-2q_{n-1}})|}{|\underline{-2q_{n-1}}|} < K.$$

Therefore, it follows from the **Proposition 2.10** and **Proposition 2.8** ($|\underline{-q_{n-1}}|$ and $|(\underline{0}, \underline{-q_{n-1}}]|$ are comparable) that $w_n(i)$ and α_{n-1} are comparable.

For $i = 0$ (which is the only case when $a = 1$), we apply the **Proposition 2.10** to

- $T = [\underline{-q_n + q_{n-1} + 1}, \underline{-q_n - q_{n-1} + 1}]$,
- $M = (\underline{-q_n + q_{n-1} + 1}, \underline{-q_n - q_{n-1} + 1})$,
- $S = \underline{-q_n + q_{n-1} + 1}$,
- $D = \underline{-q_n - q_{n-1} + 1}$,
- $f^{q_n - q_{n-1} - 1}$.

As before, the hypotheses are satisfied. And for n large enough,

$$w_n^{\ell_1,\ell_2}(0) = \frac{|(\underline{-q_n + q_{n-1} + 1}, \underline{-q_n + 1})|}{|\underline{-q_n + 1}|};$$

which is also uniformly comparable to α_{n-1}.

This concludes the proof. □

The rest of the proof of **Corollary** is as in [8] (Theorem 1.4 and Theorem 1.5).

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