# TAKAGI TYPE FUNCTIONS AND DYNAMICAL SYSTEMS: THE SMOOTHNESS OF THE SBR MEASURE AND THE EXISTENCE AND SMOOTHNESS OF LOCAL TIME

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Abstract. We investigate Takagi-type functions with roughness parameter $\gamma$ that are Hölder continuous with coefficient $H = \frac{\log \gamma}{\log \frac{1}{2}}$. Analytical access is provided by an embedding into a dynamical system related to the baker transform where the graphs of the functions are identified as their global attractors. They possess stable manifolds hosting Sinai-Bowen-Ruelle (SBR) measures. We identify these measures with the laws of certain symmetric Bernoulli convolutions. Dually, where duality is related to "time" reversal, we give a representation of the Takagi-type curves centered around fibers of the associated stable manifold in terms of Bernoulli convolutions. Duality also relates SBR to occupation measure. As opposed to SBR measure - Bernoulli convolutions belong to the first chaos - occupation measure turns out to be a functional in the second Rademacher chaos, in terms of this non-Gaussian Malliavin calculus. Using a Fourier analytic criterion and variants of Weyl's equidistribution theorem, we prove for smoothness parameters $\gamma = 2^{-\frac{1}{m}}, m \in \mathbb{N}$, that the Takagi-type curves possess square integrable local times with $m-2$ smooth derivatives.

## 1. Introduction

The interest in the subject of this paper, rough Takagi-type curves, arose from a two dimensional example of such functions studied in the context of the Fourier analytic approach of rough path analysis or rough integration theory laid out in [19] and [20]. In [20], the construction of a Stratonovich type integral of a rough function $f$ with respect to another rough function $g$ is based on the notion of paracontrol of $f$ by $g$. This Fourier analytic concept generalizes the original notion of control introduced by Gubinelli [18]. In search of a good example of two-dimensional functions for which no component is controlled by the other one, in [26] we come up with a pair of Weierstrass functions $W = (W_1, W_2)$. One of them fluctuates on all dyadic scales in a sinusoidal manner, the other one in a cosinusoidal one. Hence while the first one has minimal increments, the second one has maximal ones, and vice versa. This is seen to mathematically underpin in a rigorous way the fact that they are mutually not controlled. It is also seen that the Lévy areas of the approximating finite sums of the representing series do not converge. This geometric pathology motivated us to look for further geometric properties of the pair, or of its single components. Here we look at a relative of the Weierstrass curves, Takagi-type curves with similar regularity parameters. In contrast to the former, they are more easily accessible to the analysis we employ for the

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investigation of their geometric properties. They are given by

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \gamma^n \Phi(2^n x), \quad x \in [0,1],$$

with $\Phi(x) = d(x, \mathbb{Z})$, the distance of $x$ to the closest point in $\mathbb{Z}$, and a roughness parameter $\gamma \in ]\frac{1}{2}, 1[$ (see Figure 1). They are Hölder continuous with Hurst parameter $H = \frac{\log \gamma}{\log \frac{1}{2}}$.

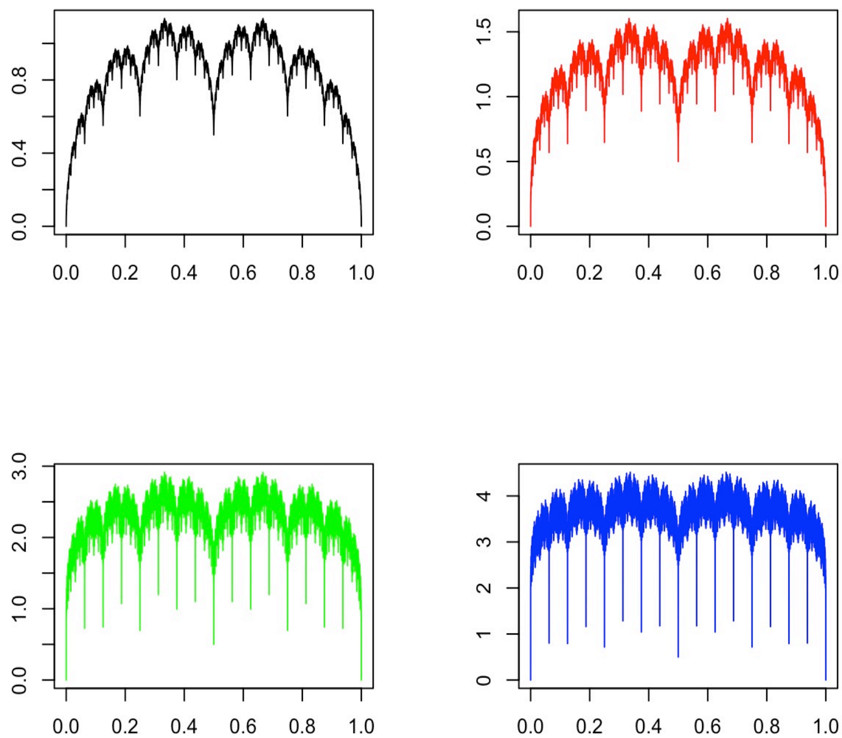


FIGURE 1. Takagi curve for $\gamma = 2^{-\frac{1}{2}}$, $\gamma = 2^{-\frac{1}{3}}$, $\gamma = 2^{-\frac{1}{6}}$, $\gamma = 2^{-\frac{1}{10}}$

We continue the study of geometric properties of such functions by asking the questions: under which condition on $\gamma$ does $\mathcal{T}$ possess a local time? How smooth is this local time? In fact, we shall answer these questions first for a modification $H$ of $\mathcal{T}$ defined in (5.1), obtained by perturbation of $\mathcal{T}$ with a very smooth path, naturally given by geometric properties of the associated dynamical system our analysis is based on, and subsequently also for the unperturbed $\mathcal{T}$. The answer to these questions resonates back to rough path analysis, as is impressively shown in [12]. There it is proved that curves possessing smooth local times have a regularizing effect, if added to an ill-posed ODE. Regularization problems of this kind in the sense of path by path uniqueness of solutions to SDE was already investigated in [14] assuming that the vector field is bounded and measurable and the perturbation $\mathcal{T}$ is a Brownian motion. The notion of regularization by noise has received a lot of attention is the past decades (see for example [1, 17, 21, 34, 35] in the ODE case, [7, 11] in the PDE case and [8, 9] for the two-parameter case. To find regularization criteria, in [12] the notion of $(\rho, \gamma)$-irregularity is introduced. It is proved that adding a $(\rho, \gamma)$-irregular function to an ill-posed ODE typically gives rise to a well-posed equation. This notion of irregularity is based on a Sobolev smoothness of the occupation measure given in terms of the decay of its Fourier modes. It is worth noting that in all the above mentioned works, the perturbation $\mathcal{T}$ is a stochastic process. As pointed out in [16], it would be interesting to replace the stochastic noise by a deterministic function that is "strongly spread". Concepts describing the strength of spreading may be more naturally expressed by means of local times, i.e. densities of occupation measures.

It had been noticed in a series of papers (see [23], [4], [5], [6], [3], [27], [36]) on one-dimensional Weierstrass type curves that the number of iterations of the expansion by a real factor can be taken as a starting point in interpreting their graphs as pullback attractors of dynamical systems in which a baker transformation defines the dynamics. This observation marks, in many of the papers quoted, the point of departure for determining the Hausdorff dimension of graphs of one dimensional Weierstrass type functions. For a historical survey of this work the reader may consult [6]. For our curve we use the same metric dynamical system based on a suitable baker transformation as a starting point. This is done by introducing, besides a variable $x$ that encodes expansion by the factor 2 forward in time, an auxiliary variable $\xi$ describing contraction by the factor $\frac{1}{2}$ in turn, forward in time as well. The operation of expansion-contraction in both variables is described by the baker transformation $B = (B_1, B_2)$. Backward in time, the sense of expansion and contraction is interchanged. The action of applying forward expansion in one step just corresponds to stepping from one term in the series expansion of $\mathcal{T}$ to the following one. This indicates that $\mathcal{T}$ is an attractor of a three dimensional hyperbolic dynamical system $F$ that, besides contracting a leading variable by the factor $\gamma$, adds the first term of the series to the result. So by definition of $F$, $\mathcal{T}$ is its attractor. Since $\frac{1}{2}$, the factor $x$ in the forward fiber motion, is the smallest Lyapunov exponent of the linearization of $F$, there is a stable manifold related to this Lyapunov exponent. It is spanned by the vector which is given as another Weierstrass type series

$$S(\xi, x) = -\sum_{n=1}^{\infty} \kappa^n \Phi'\big(B_2^n(\xi, x)\big),$$

where $\kappa = \frac{1}{2\gamma} \in ]\frac{1}{2}, 1[$ is a roughness parameter dual to $\gamma$. This will be explained below. The pushforward of the Lebesgue measure by $S(\cdot, x)$ for $x \in [0,1]$ fixed, is the $x$-marginal of the Sinai-Bowen-Ruelle measure of $F$. The definition of $F$ as a linear transformation added to a very smooth function may be understood as conveying the concept of *self-affinity* for the Takagi curve. Self-affinity can be seen as a concept providing the magnifying lens to zoom out microscopic properties of the underlying geometric object to a macroscopic scale. Our main tool of *telescoping relations* translates this rough idea into mathematical formulas, quite in the sense of Keller's paper [27]. Our telescoping is done in both time directions, forward and backward, and in doing this, we can, roughly, relate the Sinai-Bowen-Ruelle measure and the occupation measure underlying local time by duality through the operation of time reversal. More formally, we investigate the doubly infinite series

$$H(\xi, x) = \sum_{n \in \mathbb{Z}} \gamma^{-n} \left[\Phi\big(B_2^{-n}(\xi, x)\big) - \Phi\big(B_2^{-n}(\xi, 0)\big)\right], \quad \xi, x \in [0,1].$$

A key equation relates $H, \mathcal{T}$ and the stable process $S$ by the formula

$$H(\xi, y) - H(\xi, x) = \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y S(\xi, z)\mathrm{d}z.$$

For a geometric interpretation of the increments of $H$, define the stable fiber through a point $(x, \mathcal{T}(x))$ of the graph of $\mathcal{T}$ by solutions of the initial value problem of the ODE

$$\frac{d}{dv} l_{(\xi,x,w)}(v) = S(\xi, v), \quad l_{(\xi,x,w)}(x) = w,$$

where we set $w = \mathcal{T}(x)$. Then vertical distances on different stable fibers are just given by the increments of $H$:

$$l_{(\xi,y,\mathcal{T}(y))}(y) - l_{(\xi,x,\mathcal{T}(x))}(y) = H(\xi, y) - H(\xi, x), \quad \xi, x, y \in [0, 1].$$

To study the Sinai-Bowen-Ruelle measure, we describe the stable manifold function $S(\xi, \cdot)$ by a decomposition along *jumps* of the dyadic components of $\xi$ from 0 to 1. This decomposition turns out to be identical to a symmetric Bernoulli convolution. Absolute continuity vs. singularity of Bernoulli convolutions depending on the roughness parameter $\kappa$ have been investigated intensively for almost a century.

For the investigation of the occupation measure, in a dual step, we give $H(\cdot, x)$ a decomposition along jumps of the dyadic components of $x$ from 0 to 1. It leads to a surprising formula in which the essential part of $H(\cdot, x)$ is represented by a functional composed of a mixture of integrals in the first and second Rademacher chaos. The Rademacher version of Malliavin's calculus was developed in [32] and follow-up papers. In Rademacher calculus, the role of multiple integrals of Brownian motion is taken by multiple integrals with respect to a sequence $(X_n)_{n\in\mathbb{N}}$ of i.i.d. Bernoulli variables. The classical Bernoulli convolution (for instance the SBR function $S(\cdot, 0)$) can be seen as a typical integral $I_1(f) = \sum_{n\in\mathbb{N}} f(n)X_n$, for an $\ell_2$−sequence $f(n)_{n\in\mathbb{N}}$, in the first Rademacher chaos, while the second chaos is generated by double integrals of the form $I_2(g) = \sum_{n,m\in\mathbb{N}} g(n, m)X_nX_m$, where $g(n, m)_{n,m\in\mathbb{N}}$ is a double sequence in $\ell_2$ that vanishes on the diagonal. By computing mixed moments of Rademacher integrals of the first and second chaos, i.e. $E(I_1(f)^n I_2(g)^m), n, m \in \mathbb{N}$, using the rules of this calculus, we finally arrive at a closed formula for the Fourier transform of the essential parts of $H$ which is given by an infinite product of cosine functions which, for nonnegative arguments $u$ take the form

$$\cos(\frac{u}{8}\frac{\gamma}{1-\gamma}) \cdot \prod_{k\geq 1} \cos(g_k(u)),$$

where

$$g_k(u) = \big[u\gamma^k + \sqrt{\frac{u}{2}}\big\{(\gamma)^{\frac{k}{2}} \frac{1 + (\frac{\kappa}{2})^{\frac{1}{2}}}{1 - (\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1 - (\frac{\kappa}{2})^{\frac{1}{2}}}\big\}\big].$$

To prove square integrability of this function, we essentially have to deal with integrating

$$\prod_{k\geq 1} \cos^2(g_k(u)) = \exp(\sum_{k\geq 1} \ln \cos^2(g_k(u))).$$

Since $\ln \cos^2$ is periodic with period $\pi$, this problem of integrability is tackled by means of the equidistribution properties of the sequence $h_k(u) = g_k(u)(\mathrm{mod}\pi)$. This leads us directly into metric number theory, in particular variants of Weyl's (see [41]) Fourier analytic equidistribution criterion, due to Koksma [29], and the quantitative estimate of *discrepancy* between empiric averages and the average with respect to the limiting (Lebesgue) measure that are due to Erdös, Turan (see Kuipers, Niederreiter [30]). In our main result we prove that equidistribution is guaranteed for the choice $\gamma = 2^{-\frac{1}{m}}, m \geq 2$, the cases for which Wintner [42] proved smoothness of the corresponding Bernoulli convolution in the first Rademacher chaos. It turns out that local times exist in these cases that are $m - 2$ times continuously differentiable with derivatives that are Hölder continuous of any order $\rho \in ]0, 1[$, both for the *drifted* and *non drifted* Takagi curves. We conjecture that smooth local times exist for a.a. $\gamma \in ]\frac{1}{2}, 1[$, but do not exclude the possibility that there are singular values $\gamma$ such as the Pisot numbers for Bernoulli convolutions (see Erdös [15] for which occupation measures have no densities.

The paper is organized along these lines of reasoning in the following way. In Section 2, repeating [4], [23] or [27], we explain the interpretation of our Takagi-type curve in terms of dynamical systems based on the baker transform. In Section 3, we describe the measures related to the SBR measure, deduce telescoping relationships between them, and representation formulas for $S(\xi,\cdot)$ using the jump times $\tau_n, n\in\mathbb{N}$ for the dyadic components of $\xi$ from 0 to 1. In Section 4, we just give a brief summary of the long history of research on the smoothness of Bernoulli convolutions. Dually, in the long Section 5, we investigate smoothness of the occupation measure of $H$ and $\mathcal{T}$. We start by deriving a representation of $H(\cdot,x)$ by means of Rademacher integrals in the first and second chaos, along the jump times $\sigma_n, n\in\mathbb{N}$ of the dyadic components of $x$. In subsection 5.1 we compute mixed moments of integrals in the first and second Rademacher chaos, using the rules of the Rademacher version of Malliavin's calculus. These results are used in Subsection 5.2 to deduce closed formulas for the Fourier transform of the essential parts of the occupation measure. In Subsection 5.3 we finally prove that these Fourier transforms are square integrable in the cases $\gamma = 2^{-\frac{1}{m}}, m\geq 2$, and consequently smooth local times exist.

## 2. The curve as the attractor of a dynamical system

Let $\gamma\in]\frac12,1[$. Our aim is to investigate the fine structure geometry of the one-dimensional Takagi type curves given by

$$\mathcal{T}(x)=\sum_{n=0}^{\infty}\gamma^n\Phi(2^n x),\quad x\in[0,1], \tag{2.1}$$

where $\Phi(y)=d(y,\mathbb{Z}), y\in\mathbb{R}$. Let us first determine the Hölder exponent of $x\mapsto\mathcal{T}(x)$ (see [3] for an overview).

**Proposition 2.1.** *$\mathcal{T}$ is Hölder continuous with exponent $-\frac{\log\gamma}{\log 2}$.*

*Proof.* Let $x,y\in[0,1]$ and choose an integer $k\geq 0$ such that

$$2^{-(k+1)}\leq |x-y|\leq 2^{-k}.$$

Then we have, using the Lipschitz continuity of the distance function

$$\begin{aligned}
|\mathcal{T}(x)-\mathcal{T}(y)| &\leq \sum_{n=1}^{k}\gamma^n|d(2^n x,\mathbb{Z})-d(2^n y,\mathbb{Z})|+2\sum_{n=k+1}^{\infty}\gamma^n\\
&\lesssim \sum_{n=1}^{k}(2\gamma)^n|x-y|+\gamma^k\lesssim(2\gamma)^k\ 2^{-k}+\gamma^k\simeq\gamma^k=2^{-k\frac{\log\gamma}{\log\frac12}}\\
&\lesssim |x-y|^{-\frac{\log\gamma}{\log 2}}.
\end{aligned}$$

This shows that $\frac{\log\gamma}{\log\frac12}$ is an upper bound for the Hölder exponent of $\mathcal{T}$. To see that it is also a lower bound, for $n\in\mathbb{N}$ choose $x_n=0, y_n=2^{-n}$. Then we may write

$$\begin{aligned}
|\mathcal{T}(x_n)-\mathcal{T}(y_n)| &= \Big|\sum_{k=1}^{\infty}\gamma^k d(2^{k-n},\mathbb{Z})\Big|\\
&= \sum_{k=1}^{n-1}\gamma^k 2^{k-n}\simeq 2^{-n\frac{\log\gamma}{\log\frac12}}=|x_n-y_n|^{-\frac{\log\gamma}{\log 2}}.
\end{aligned}$$

Since $|x_n - y_n| \to 0$ as $n \to \infty$, this shows that $-\frac{\log \gamma}{\log 2}$ is also a lower bound for the Hölder exponent of $\mathcal{T}$. The argument can be extended to the other points in the interval. $\square$

Our access to the analysis and geometry of $\mathcal{T}$ is via the theory of dynamical systems. In fact, we shall describe a dynamical system on $[0,1]^2$, alternatively $\Omega = \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ the attractor of which is given by the graph of the function. For elements of $\Omega$ we write for convenience $\omega = ((\omega_{-n})_{n\geq 0}, (\omega_n)_{n\geq 1})$; one understands $\Omega$ as the space of 2-dimensional sequences of Bernoulli random variables. Denote by $\theta$ the canonical shift on $\Omega$, given by

$$\theta : \Omega \to \Omega, \quad \omega \mapsto (\omega_{n+1})_{n\in\mathbb{Z}}.$$

$\Omega$ is endowed with the product $\sigma$-algebra, and the infinite product $\iota = \otimes_{n\in\mathbb{Z}}(\frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}})$ of Bernoulli measures on $\{0,1\}$. We recall that $\theta$ is $\iota$-invariant.

Now let

$$T = (T_1, T_2) : \Omega \to [0,1]^2, \quad \omega \mapsto (\sum_{n=0}^{\infty} \omega_{-n} 2^{-(n+1)}, \sum_{n=1}^{\infty} \omega_n 2^{-n}).$$

Let us denote by $T_1$ the first component of $T$, and by $T_2$ the second one. It is well known that $\iota$ is mapped by the transformation $T$ to $\lambda^2$ (i.e. $\iota = \lambda^2 \circ T$), the 2-dimensional Lebesgue measure. It is also well known that the inverse of $T$, the dyadic representation of the two components from $[0,1]^2$, is uniquely defined apart from the dyadic pairs. For these we define the inverse to map to the sequences not converging to 0. Let

$$B = (B_1, B_2) = T \circ \theta \circ T^{-1}.$$

We call $B = (B_1, B_2)$ the *baker's transformation*. The $\theta$-invariance of $\iota$ directly translates into the $B$-invariance of $\lambda^2$:

$$\lambda^2 \circ B^{-1} = (\lambda^2 \circ T) \circ \theta^{-1} \circ T^{-1} = (\iota \circ \theta^{-1}) \circ T^{-1} = \iota \circ T^{-1} = \lambda^2. \tag{2.2}$$

For $(\xi, x) \in [0,1]^2$ let us note

$$T^{-1}(\xi, x) = \big((\overline{\xi}_{-n})_{n\geq 0}, (\overline{x}_n)_{n\geq 1}\big).$$

Let us calculate the action of $B$ and its entire iterates on $[0,1]^2$.

**Lemma 2.2.** *Let $(\xi, x) \in [0,1]^2$. Then for $k \geq 0$*

$$B^k(\xi, x) = \Big(2^k \xi \,(mod\ 1), \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \frac{x}{2^k}\Big),$$

*for $k \geq 1$*

$$B^{-k}(\xi, x) = \Big(\frac{\xi}{2^k} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, 2^k x (mod\ 1)\Big).$$

**Proof:** By definition of $\theta^k$ for $k \geq 0$

$$B^k(\xi, x) = \Big(\sum_{n\geq 0} \overline{\xi}_{-n+k} 2^{-(n+1)}, \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \sum_{n\geq 1} \overline{x}_n 2^{-(k+n)}\Big).$$

Now we can write

$$\sum_{n\geq 0} \overline{\xi}_{-n+k} 2^{-(n+1)} = 2^k \xi (\text{mod } 1) \quad \text{and} \quad \sum_{n\geq 1} \overline{x}_n 2^{-(k+n)} = \frac{x}{2^k}.$$

This gives the first formula. For the second, note that by definition of $\theta^{-k}$ for $k \geq 1$

$$B^{-k}(\xi, x) = \Big(\sum_{n\geq 0} \overline{\xi}_{-n} 2^{-(n+1+k)} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, \sum_{n\geq 1} \overline{x}_{n+k} 2^{-n}\Big).$$

Again, we identify

$$\sum_{n\geq 1} \overline{x}_{n+k} 2^{-n} = 2^k x (\text{mod } 1) \qquad \text{and} \qquad \sum_{n\geq 0} \overline{\xi}_{-n} 2^{-(n+1+k)} = \frac{\xi}{2^k}.$$

□

For $k \in \mathbb{Z}, (\xi, x) \in [0,1]^2$ we abbreviate the $k$-fold iterate of the baker transform of $(\xi, x)$ as

$$B^k(\xi, x) = \big(B_1^k(\xi, x), B_2^k(\xi, x)\big) = (\xi_k, x_k),$$

where for $k \geq 0$

$$\xi_k = 2^k \xi (\text{mod } 1), \quad \text{and} \quad x_k = \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \frac{x}{2^k},$$

and for $k \geq 1$

$$\xi_{-k} = \frac{\xi}{2^k} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, \quad \text{and} \quad x_{-k} = 2^k x (\text{mod } 1).$$

Following Baranski [4, 5, 6], Shen [36], Hunt [23] and [25], we will next interpret the Takagi curve $\mathcal{T}$ by a transformation on our base space $[0,1]^2$. Let

$$\begin{aligned} F : [0,1]^2 \times \mathbb{R} &\rightarrow [0,1]^2 \times \mathbb{R}, \\ (\xi, x, y) &\mapsto \Big(B(\xi, x), \gamma y + \Phi(B_2(\xi, x))\Big). \end{aligned}$$

Here we note $B = (B_1, B_2)$ for the two components of the baker transform $B$.

For convenience, we extend $\mathcal{T}$ from $[0,1]$ to $[0,1]^2$ by setting

$$\mathcal{T}(\xi, x) = \mathcal{T}(x), \quad \xi, x \in [0,1].$$

To see that the graph of $\mathcal{T}$ is an attractor for $F$, the skew-product structure of $F$ with respect to $B$ plays a crucial role.

**Lemma 2.3.** *For any $\xi, x \in [0,1]$ we have*

$$F\big(\xi, x, \mathcal{T}(\xi, x)\big) = \Big(B(\xi, x), \mathcal{T}\big(B(\xi, x)\big)\Big).$$

**Proof:** By the definition of the baker's transform we may write

$$\mathcal{T}(\xi, x) = \sum_{n=0}^{\infty} \gamma^n \Phi\Big(B_2^{-n}(\xi, x)\Big), \quad \xi, x \in [0,1].$$

Hence, setting $k = n - 1$, for $\xi, x \in [0,1]$

$$\begin{aligned} \mathcal{T}\big(B_2(\xi, x)\big) &= \sum_{n=0}^{\infty} \gamma^n \Phi\Big(B_2^{-n+1}(\xi, x)\Big) \\ &= \Phi\big(B_2(\xi, x)\big) + \gamma \sum_{k=0}^{\infty} \gamma^k \Phi\Big(B_2^{-k}(\xi, x)\Big) \\ &= \Phi\big(B_2(\xi, x)\big) + \gamma \mathcal{T}(x). \end{aligned}$$

Hence by definition of $F$

$$\Big(B(\xi, x), \mathcal{T}(B(\xi, x))\Big) = \Big(B(\xi, x), \mathcal{T}(B_2(\xi, x))\Big) = F\Big(\xi, x, \mathcal{T}(\xi, x)\Big).$$

□

To assess the stability properties of the dynamical system generated by $F$, we calculate its Jacobian. We obtain for $\xi, x \in [0,1], y \in \mathbb{R}$

$$DF(\xi,x,y) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\Phi'\big(B_2(\xi,x)\big) & \gamma \end{bmatrix}.$$

Hence the Lyapunov exponents of the dynamical system associated with $F$ are given by $2, \frac{1}{2}$, and $\gamma$. The corresponding invariant vector fields are given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X(\xi,x) = \begin{pmatrix} 0 \\ 1 \\ -\sum_{n=1}^{\infty} \left(\frac{1}{2\gamma}\right)^n \Phi'\big(B_2^n(\xi,x)\big) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as is straightforwardly verified. Note that $X$ is well defined, since by our choice of $\gamma$ we have $2\gamma > 1$. Hence we have in particular for $\xi, x \in [0,1], y \in \mathbb{R}$

$$DF(\xi,x,y)X(\xi,x) = \frac{1}{2}\, X\big(B(\xi,x)\big).$$

Note that the vector $X$ spans an invariant stable manifold and does not depend on $y$.

## 3. The Sinai-Bowen-Ruelle measure

Abbreviate $\kappa = \frac{1}{2\gamma} \in ]0,1[$. In Tsujii [39] the problem of the absolute continuity of the Sinai-Bowen-Ruelle (SBR) measure on the stable manifold described by

$$S(\xi,x) = \sum_{n=1}^{\infty} \kappa^n \Phi'\big(B_2^n(\xi,x)\big), \quad \xi, x \in [0,1],$$

with respect to Lebesgue measure has been treated. It has been related to the *transversality* of the map $x \mapsto S(\xi,x) - S(\eta,x)$ for $\xi, \eta \in [0,1]$. Here, we shall use the relative simplicity of the definition of $\mathcal{T}$ to represent $S$ in terms of a symmetric Bernoulli convolution, smoothness properties of which have been investigated intensively over a long time period.

To recall the SBR measure of $F$, let us first calculate the action of $S$ on the $\lambda^2$-measure preserving map $B$. For $\xi, x \in [0,1]$ we have

$$\begin{aligned} S(B(\xi,x)) &= \sum_{n=1}^{\infty} \kappa^n \Phi'\Big(B_2^n\big(B_2(\xi,x)\big)\Big) \\ &= \sum_{n=1}^{\infty} \kappa^n \Phi'\big(B_2^{n+1}(\xi,x)\big) \\ &= \kappa^{-1} \sum_{k=1}^{\infty} \kappa^k \Phi'\big(B_2^k(\xi,x)\big) - \Phi'\big(B_2(\xi,x)\big) \\ &= 2\gamma S(\xi,x) - \Phi'\big(B_2(\xi,x)\big). \end{aligned}$$

So we may define the Anosov skew product

$$\begin{aligned} \Gamma : [0,1]^2 \times \mathbb{R} &\to [0,1]^2 \times \mathbb{R}, \\ (\xi,x,v) &\mapsto \Big(B(\xi,x), 2\gamma v - \Phi'\big(B_2(\xi,x)\big)\Big). \end{aligned}$$

Then the equation just obtained yields the following result (compare with Lemma 2.3).

**Lemma 3.1.** *For $\xi, x \in [0,1]$ we have*

$$\Gamma\big(\xi,x,S(\xi,x)\big) = \big(B(\xi,x), S(B(\xi,x))\big).$$

*The push-forward of the Lebesgue measure in* $\mathbb{R}^2$ *to the graph of* $S$ *given by*

$$\psi = \lambda^2 \circ (id, S)^{-1}$$

*on* $\mathcal{B}([0,1]^2) \otimes \mathcal{B}(\mathbb{R})$ *is* $\Gamma$*-invariant.*

**Proof:** The first equation has been verified above. The $\Gamma$-invariance of $\psi$ is a direct consequence of the $B$-invariance of $\lambda^2$. □

Define $\pi_2 : [0,1]^2 \to [0,1], (\xi, x) \mapsto x$ and define the measure

$$\mu = \lambda^2 \circ (\pi_2, S)^{-1} \tag{3.1}$$

on $\mathcal{B}([0,1]^2)$. The measure $\mu$ is called the *Sinai-Bowen-Ruelle measure* of $\Gamma$. Its marginals in $x \in [0,1]$ are denoted $\mu_x = \lambda \circ S(\cdot, x)^{-1}$.

We now define a map on our probability space that exhibits certain increments of $S$ in a self similar way. Let

$$G(\xi, x) = \sum_{n\in\mathbb{Z}} \kappa^{-n} \big[\Phi'\big(B_2^{-n}(\xi, x)\big) - \Phi'\big(B_2^{-n}(0, x)\big)\big], \quad \xi, x \in [0,1].$$

Then we have the following simple relationship between $G$ and $S$.

**Lemma 3.2.** *For* $x, \xi, \eta \in [0,1]$ *we have*

$$G(\xi, x) - G(\eta, x) = S(\xi, x) - S(\eta, x).$$

**Proof:** For $x, \xi, \eta \in [0,1]$, the equation

$$\begin{aligned} G(\xi, x) - G(\eta, x) &= \sum_{n\in\mathbb{Z}} \kappa^{-n} \big[\Phi'\big(B_2^{-n}(\xi, x)\big) - \Phi'\big(B_2^{-n}(\eta, x)\big)\big] \\ &= \sum_{k=1}^{\infty} \kappa^{k} \big[\Phi'\big(B_2^{k}(\xi, x)\big) - \Phi'\big(B_2^{k}(\eta, x)\big)\big] \\ &= S(\xi, x) - S(\eta, x), \end{aligned}$$

holds. Here we used that the first sum for non-negative integers $n$ is zero.

This completes the proof. □

The following result describes the scaling properties of $G$.

**Lemma 3.3.** *For* $\xi, x \in [0,1]$ *we have*

$$G\big(B^{-1}(\xi, x)\big) = \kappa G(\xi, x).$$

**Proof:** Note that by definition, setting $n + 1 = k$, for $\xi, x \in [0,1]$

$$\begin{aligned} G\big(B^{-1}(\xi, x)\big) &= \sum_{n\in\mathbb{Z}} \kappa^{-n} \big[\Phi'\big(B^{-n-1}(\xi, x)\big) - \Phi'\big(B^{-n}(B_1^{-1}(\xi, x), 0)\big)\big] \\ &+ \sum_{n\in\mathbb{Z}} \kappa^{-n} \big[\Phi'\big(B^{-n-1}(\xi, x)\big) - \Phi'\big(B^{-n-1}(\xi, 0)\big)\big] \\ &+ \sum_{n\in\mathbb{Z}} \kappa^{-n} [\Phi'\big(B_2^{-n-1}(\xi, 0)\big) - \Phi'\big(B_2^{-n}(B_1^{-1}(\xi, x), 0)\big)] \\ &= \kappa G(\xi, x) + \sum_{k=1}^{\infty} \kappa^{k} [\Phi'(B_2^{k-1}(\xi, 0)) - \Phi'(B_2^{k}(B_1^{-1}(\xi, x), 0))] \\ &= \kappa G(\xi, x). \end{aligned}$$

For the last equality, note that $\Phi'$ is constant on the two halves of the unit interval, and that

$$B_2^{k-1}(\xi,0)=\frac{\overline{\xi}_0}{2^{k-1}}+\cdots+\frac{\overline{\xi}_{-k+2}}{2}$$

and

$$B_2^k(B_1^{-1}(\xi,x),0)=\frac{\overline{x}_1}{2^k}+\frac{\overline{\xi}_0}{2^{k-1}}+\cdots+\frac{\overline{\xi}_{-k+2}}{2}$$

belong to the same half. This provides the claimed identity.□

We finish this section by giving a representation which will be the starting point for our subsequent approach of transversality of $S$. To this end, fix $\xi,\eta\in[0,1]$. We recursively define the following sequence of times of disagreement of dyadic components of $\xi$ and $\eta$. For $n\in\mathbb{N}$ let

$$\tau_1=\inf\{\ell\geq 0:\bar{\xi}_{-\ell}\neq\bar{\eta}_{-\ell}\},\ \text{and}\ \tau_{n+1}=\inf\{\ell>\tau_n:\bar{\xi}_{-\ell}\neq\bar{\eta}_{-\ell}\},\tag{3.2}$$

and for $x\in[0,1]$

$$\begin{aligned}g(x):=&\sum_{m=0}^{\infty}\kappa^m\Big[\Phi'\Big(B_2^m(0,\frac{1+x}{2})\Big)-\Phi'\Big(B_2^m(0,\frac{x}{2})\Big)\Big]\\=&\sum_{m=0}^{\infty}\kappa^m\Big[\Phi'\Big(\frac{1+x}{2^{m+1}}\Big)-\Phi'\Big(\frac{x}{2^{m+1}}\Big)\Big].\end{aligned}$$

We have the following result.

**Proposition 3.4.** *Let $\xi,\eta,x\in[0,1]$. Then*

$$S(\xi,x)-S(\eta,x)=\sum_{\ell=1}^{\infty}\kappa^{\tau_\ell+1}\,(-1)^{(1-\bar{\xi}_{-\tau_\ell})}g\big(B_2^{\tau_\ell}(\xi,x)\big).\tag{3.3}$$

**Proof:** It follows from the definition of $\tau_n,n\in\mathbb{N}$, that $\xi$ can be written

$$\xi=(\overline{\eta}_0,\ldots,\overline{\eta}_{-\tau_1+1},\overline{\xi}_{-\tau_1},\overline{\eta}_{-\tau_1-1},\cdots,\overline{\eta}_{-\tau_2+1},\overline{\xi}_{-\tau_2},\overline{\eta}_{-\tau_2-1},\cdots\cdots,\overline{\eta}_{-\tau_n+1},\overline{\xi}_{-\tau_n},\overline{\eta}_{-\tau_n-1},\cdots).$$

For $n\in\mathbb{N}$ let $\xi^n$ be the sequence which up to $\tau_n$ represents the dyadic expansion of $\xi$, and then switches to the representing sequence of $\eta$. Then for $n\in\mathbb{N}$ we have

$$\xi^n=(\overline{\eta}_0,\ldots,\overline{\eta}_{-\tau_1+1},\overline{\xi}_{-\tau_1},\overline{\eta}_{-\tau_1-1},\cdots,\overline{\eta}_{-\tau_2+1},\overline{\xi}_{-\tau_2},\overline{\eta}_{-\tau_2-1},\cdots\cdots,\overline{\eta}_{-\tau_n+1},\overline{\xi}_{-\tau_n},\overline{\eta}_{-\tau_n-1},\cdots,\overline{\eta}_{-m},\cdots).$$

Note that $\lim_{n\to\infty}\xi^n=\xi$. We can therefore rewrite the left hand side of (3.3) in the telescoping form

$$S(\xi,x)-S(\eta,x)=\sum_{l=1}^{\infty}\Big(S(\xi^\ell,x)-S(\xi^{\ell-1},x)\Big),$$

where $\xi^0=\eta$.

For $\ell\in\mathbb{N}$ let us calculate $S(\xi^\ell,\cdot)-S(\xi^{\ell-1},\cdot)$. Since $\overline{\xi}^{\ell}_{-k}=\overline{\xi}^{\ell-1}_{-k}$ for $k\leq\tau_\ell-1$, we have

$$\begin{aligned}S(\xi^\ell,x)-S(\xi^{\ell-1},x)=&\sum_{n=1}^{\infty}\kappa^n\Big[\Phi'\big(B_2^n(\xi^\ell,x)\big)-\Phi'\big(B_2^n(\xi^{\ell-1},x)\big)\Big]\\=&\sum_{n=\tau_\ell+1}^{\infty}\kappa^n\Big[\Phi'\big(B_2^n(\xi^\ell,x)\big)-\Phi'\big(B_2^n(\xi^{\ell-1},x)\big)\Big]\\=&\kappa^{\tau_\ell}\sum_{m=1}^{\infty}\kappa^m\Big[\Phi'\Big(B_2^m\big(B_2^{\tau_\ell}(\xi^\ell,x)\big)\Big)-\Phi'\Big(B_2^m\big(B_2^{\tau_\ell}(\xi^{\ell-1},x)\big)\Big)\Big].\end{aligned}$$

Now
$$B_2^{\tau_\ell}(\xi^\ell, x) = B_2^{\tau_\ell}(\xi, x) = B_2^{\tau_\ell}(\xi^{\ell-1}, x).$$
And in case $\overline{\xi}_{-\tau_\ell} = 1$ we have
$$B_2^1\big(B_2^{\tau_\ell}(\xi^\ell, x)\big) = B_2^{\tau_\ell+1}(\xi^\ell, x) = \frac{1 + B_2^{\tau_\ell}(\xi^\ell, x)}{2} = \frac{1 + B_2^{\tau_\ell}(\xi, x)}{2},$$
while
$$B_2^1\big(B_2^{\tau_\ell}(\xi^{\ell-1}, x)\big) = B_2^{\tau_\ell+1}(\xi^{\ell-1}, x) = \frac{B_2^{\tau_\ell}(\xi, x)}{2}.$$
In case $\overline{\xi}_{-\tau_\ell} = 0$, we have in contrast
$$B_2^1\big(B_2^{\tau_\ell}(\xi^\ell, x)\big) = B_2^{\tau_\ell+1}(\xi^\ell, x) = \frac{B_2^{\tau_\ell}(\xi^\ell, x)}{2} = \frac{B_2^{\tau_\ell}(\xi, x)}{2},$$
while
$$B_2^1\big(B_2^{\tau_\ell}(\xi^{\ell-1}, x)\big) = B_2^{\tau_\ell+1}(\xi^{\ell-1}, x) = \frac{1 + B_2^{\tau_\ell}(\xi^\ell, x)}{2} = \frac{1 + B_2^{\tau_\ell}(\xi, x)}{2}.$$
So we may write by definition of $g$
$$\begin{aligned} S(\xi^\ell, x) - & S(\xi^{\ell-1}, x) \\ &= (-1)^{(1-\overline{\xi}_{-\tau_\ell})}\, \kappa^{\tau_\ell} \sum_{m=1}^\infty \kappa^m \Big[\Phi'\Big(B_2^{m-1}\Big(\frac{1 + B_2^{\tau_\ell}(\xi, x)}{2}\Big)\Big) - \Phi'\Big(B_2^{m-1}\Big(\frac{B_2^{\tau_\ell}(\xi, x)}{2}\Big)\Big)\Big] \\ &= \kappa^{\tau_\ell+1}\, (-1)^{(1-\overline{\xi}_{-\tau_\ell})}\, g\big(B_2^{\tau_\ell}(\xi, x)\big). \end{aligned}$$
Hence we obtain the claimed representation
$$S(\xi, x) - S(\eta, x) = \sum_{\ell=1}^\infty \kappa^{\tau_\ell+1}\, (-1)^{(1-\overline{\xi}_{-\tau_\ell})}\, g\big(B_2^{\tau_\ell}(\xi, x)\big), \;\; \xi, \eta, x \in [0, 1].$$
□

Let us calculate $g$. Here, as opposed to the trigonometric case in the stable manifold of Weierstrass curves (see [24]), the simplicity of $\phi$ implies the following surprising identity.

**Lemma 3.5.** *We have $g(x) = -2, x \in [0, 1]$.*

**Proof:** Let us inspect the first term in the series decomposition of $g$. Here we have $B_2^0(0, \frac{1+x}{2}) = \frac{1+x}{2} \in [\frac{1}{2}, 1]$, while $B_2^0(0, \frac{x}{2}) = \frac{x}{2} \in [0, \frac{1}{2}]$. Hence the contribution of the first term is $-1 - 1 = -2$. For $m \geq 1$ we have in contrast that both $B_2^m(0, \frac{1+x}{2}) = \frac{1+x}{2^{m+1}}$ and $B_2^m(0, \frac{x}{2}) = \frac{x}{2^{m+1}}$ belong to $[0, \frac{1}{2}]$. Hence the contribution of terms of order $m \geq 1$ vanishes. This implies the claimed identity.□

Lemma 3.5 gives the following simplification of the representation formula of Proposition 3.4.

**Corollary 3.6.** *Let $\xi, \eta, x \in [0, 1]$. Then*

$$S(\xi, x) - S(\eta, x) = -2 \sum_{\ell=1}^\infty \kappa^{\tau_\ell+1}\, (-1)^{(1-\bar{\xi}_{-\tau_\ell})}. \tag{3.4}$$

**Proof:** This follows by combining Lemma 3.5 and Proposition 3.4. □

Let us finally extend this property to $S(\xi, .)$ for $\xi \in [0,1]$. First observe that for any $x \in [0,1]$ we have

$$S(0,x) = \sum_{n=1}^{\infty} \kappa^n \Phi'(\frac{x}{2^n}) = \sum_{n=1}^{\infty} \kappa^n = \frac{\kappa}{1-\kappa}. \tag{3.5}$$

Now denote by $\tau_n^0, n \in \mathbb{N}$, the sequence of stopping times described above for the particular case $\eta = 0$. Then we obtain

**Corollary 3.7.** *Let $\xi, x \in [0,1]$. Then*

$$S(\xi,x) = \frac{\kappa}{1-\kappa} - 2\sum_{\ell=1}^{\infty} \kappa^{\tau_\ell^0+1} = \sum_{n=0}^{\infty} \big(1_{\{\bar{\xi}_{-n}=0\}} - 1_{\{\bar{\xi}_{-n}=1\}}\big)\kappa^{n+1}. \tag{3.6}$$

**Proof:** Combine Corollary 3.6 with (3.5), and note that by definition $\bar{\xi}_{-\tau_k^0} = 1$ for all $k \geq 1$. $\square$

The final equality in the preceding Corollary shows that $S$ qualifies as a Bernoulli convolution in the sense of Erdös [15].

## 4. Smoothness of the SBR measure

In Corollary 3.7 we have identified $S$ with a Bernoulli convolution. Objects of this type have been studied intensively for a very long time, starting in the 1930ies with remarkable papers such as Wintner [42], Kershner, Wintner [28], or Erdös [15], and in later years by Brauer [10], Solomyak [38], or Shmerkin [37]. The research has even gained more intensity over the recent three decades. For a survey see Varju [40], or Peres [33]. Predominant interest in the research has been addressing the question of absolute continuity of the random variables given by the infinite convolution

$$Z(\kappa) = \sum_{n\in\mathbb{N}} X_n \kappa^n$$

with i.i.d. Bernoulli sequences $(X_n)_{n\in\mathbb{N}}$, and $\kappa \in ]0,1[$. From the early papers on, this problem has been tackled by a well known Fourier analytic criterion. The Fourier transform of the random variable $Z(\kappa)$ is given by the infinite product

$$\phi_Z(u) = \prod_{n=1}^{\infty} \cos(u\kappa^n), \quad u \in \mathbb{R}.$$

The criterion states that absolute continuity with square integrable density holds provided $\phi_Z^2$ is integrable on $\mathbb{R}$. Surprisingly, the answer to this question strongly depends on algebraic properties of $\kappa$. Early discoveries about this problem lead to the conclusion that either the law is absolutely continuous or singular with respect to Lebesgue measure. It has been shown that for $\kappa = (\frac{1}{2})^{\frac{1}{n}}, n \in \mathbb{N}$, the convolution is absolutely continuous (see Wintner [42], or Varju [40], section 2.2). This fact will prove to become important in our treatment of local time below. Erdös [15] has contributed the remarkable result that absolute continuity is ruled out if $\frac{1}{\kappa}$ is a *Pisot number*, i.e. its Galois conjugates, the roots of its minimal polynomial over the complex field are all contained inside the unit disc. It is known that absolute continuity holds for almost all $\kappa \in [\frac{1}{2}, 1]$, but that $\frac{1}{2}$ is a cluster point of values $\kappa$ for which the law of the convolution is singular. We refrain here from giving a more complete account of the state of knowledge on the sets of $\kappa$ for which absolute continuity resp. singularity hold, and refer to Varju [40].

## 5. The existence of a local time for $\mathcal{T}$

In this section we use a similar criterion as in the preceding one to show that the occupation measure associated with $\mathcal{T}$ possesses a square integrable density. This will be done in an indirect way. We first establish an intrinsic link between the Takagi curve as the attractor of an underlying dynamical system and its stable manifold spanned by $S$. It will identify $H(\xi, x) = \mathcal{T}(x) - \int_0^x S(\xi, z)\mathrm{d}z = H(\xi, x) - xS(\xi, 0)$ for $x, \xi \in [0, 1]$ as a function having much in common with the function $G$ of the preceding sections. In the following key lemma we establish the link between $\mathcal{T}$ and the stable manifold of $F$. For this purpose, we define

$$H(\xi, x) = \sum_{n\in\mathbb{Z}} \gamma^n \big[\Phi(B_2^{-n}(\xi, x)\big) - \Phi\big(B_2^{-n}(\xi, 0)\big)\big], \quad \xi, x \in [0, 1]. \tag{5.1}$$

In a certain sense, $H$ looks into the inverse direction of the integer order of $\mathbb{Z}$ than $G$. We have the following relationships between $H$ and $S$.

**Lemma 5.1.** *For $x, y, \xi \in [0, 1]$ we have*

$$H(\xi, y) - H(\xi, x) = \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y S(\xi, z)\mathrm{d}z.$$

*For $x, \xi, \eta \in [0, 1]$ we have*

$$H(\eta, x) - H(\xi, x) = \int_0^x \big(S(\xi, z) - S(\eta, z)\big)\mathrm{d}z = x(S(\xi, 0) - S(\eta, 0)).$$

**Proof:** For $x, y, \xi \in [0, 1]$ we have indeed (recall $\kappa = \frac{1}{2\gamma}$)

$$\begin{aligned}
H(\xi, y) - H(\xi, x) &= \sum_{n\in\mathbb{Z}} \gamma^n \big[\Phi\big(B_2^{-n}(\xi, y)\big) - \Phi\big(B_2^{-n}(\xi, x)\big)\big] \\
&= \sum_{n=0}^{\infty} \gamma^n \big[\Phi\big(B_2^{-n}(\xi, y)\big) - \Phi\big(B_2^{-n}(\xi, x)\big)\big] \\
&\qquad + \sum_{k=1}^{\infty} \gamma^{-k} \big[\Phi\big(B_2^{k}(\xi, y)\big) - \Phi\big(B_2^{k}(\xi, x)\big)\big] \\
&= \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y \sum_{k=1}^{\infty} (2\gamma)^{-k} \Phi'\big(B_2^k(\xi, z)\big)\mathrm{d}z \\
&= \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y \sum_{k=1}^{\infty} \kappa^k \Phi'\big(B_2^k(\xi, z)\big)\mathrm{d}z \\
&= \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y S(\xi, z)\mathrm{d}z.
\end{aligned}$$

To argue for the second equation, note that for $x, \xi, \eta \in [0,1]$ we have

$$\begin{aligned} H(\eta,x) - H(\xi,x) &= \sum_{n\in\mathbb{Z}} \gamma^n \Big\{ \big[\Phi\big(B_2^{-n}(\eta,x)\big) - \Phi\big(B_2^{-n}(\eta,0)\big)\big] \\ &\qquad - \big[\Phi\big(B_2^{-n}(\xi,x)\big) - \Phi\big(B_2^{-n}(\xi,0)\big)\big] \Big\} \\ &= \sum_{k=1}^{\infty} \gamma^{-k} \Big\{ \big[\Phi\big(B_2^{k}(\eta,x)\big) - \Phi\big(B_2^{k}(\eta,0)\big)\big] \\ &\qquad - \big[\Phi\big(B_2^{k}(\xi,x)\big) - \Phi\big(B_2^{k}(\xi,0)\big)\big] \Big\} \\ &= -\int_0^x \sum_{k=1}^{\infty} (2\gamma)^{-k} \big[\Phi'\big(B_2^k(\eta,z)\big) - \Phi'\big(B_2^k(\xi,z)\big)\big] \mathrm{d}z \\ &= -\int_0^x \sum_{k=1}^{\infty} \kappa^{k} \big[\Phi'\big(B_2^k(\eta,z)\big) - \Phi'\big(B_2^k(\xi,z)\big)\big] \mathrm{d}z \\ &= \int_0^x [S(\xi,z) - S(\eta,z)] \mathrm{d}z. \end{aligned}$$

This completes the proof of the second equation. □

We next address the scaling properties of $H$.

**Lemma 5.2.** *For $\xi, x \in [0,1]$ we have*

$$H\big(B(\xi,x)\big) = \gamma H(\xi,x) + \frac{\overline{\xi}_0}{2} S(2\xi,0).$$

*Consequently, for $\xi, x, y \in [0,1]$*

$$H(B(\xi,y)) - H(B(\xi,x)) = \gamma[H(\xi,y) - H(\xi,x)].$$

**Proof:** Let $\xi, x \in [0,1]$. By definition and setting $n-1=k$ we obtain, defining $\hat{\xi}$ to be represented by the dyadic sequence $(0, \overline{\xi}_{-1}, \overline{\xi}_{-2}, \cdots)$,

$$\begin{aligned} H(B(\xi,x)) &= \sum_{n\in\mathbb{Z}} \gamma^n \big[\Phi\big(B^{-n+1}(\xi,x)\big) - \Phi\big(B^{-n}(B_1(\xi,x),0)\big)\big] \\ &= \sum_{n\in\mathbb{Z}} \gamma^n \big[\Phi\big(B^{-n+1}(\xi,x)\big) - \Phi\big(B^{-n+1}(\xi,0)\big)\big] \\ &+ \sum_{n\in\mathbb{Z}} \gamma^n \big[\Phi\big(B^{-n+1}(\xi,0)\big) - \Phi\big(B^{-n}(B_1(\xi,x),0)\big)\big] \\ &= \gamma \sum_{k\in\mathbb{Z}} \gamma^k \big[\Phi\big(B^{-k}(\xi,x)\big) - \Phi\big(B^{-k}(\xi,0)\big)\big] \\ &+ \sum_{k=1}^{\infty} \gamma^{-k} [\Phi\big(B_2^{k+1}(\xi,0)\big) - \Phi\big(B_2^{k+1}(\hat{\xi},0)\big)\big] \\ &= \gamma H(\xi,x) + \overline{\xi}_0 \sum_{k=1}^{\infty} \gamma^{-k} 2^{-k-1} \Phi'(B_2^{k+1}(\hat{\xi},0)) \\ &= \gamma H(\xi,x) + \frac{\overline{\xi}_0}{2} \sum_{k=1}^{\infty} \kappa^k \Phi'(B_2^k(2\xi,0)) \\ &= \gamma H(\xi,x) + \frac{\overline{\xi}_0}{2} S(2\xi,0). \end{aligned}$$

The last equation follows from the fact that the second term in the first formula only depends on $\xi$. $\square$

We finally give a representation of increments of $H$ that can be considered dual to the representation of Proposition 3.4. It will serve as an entrance to investigating the smoothness of the occupation measure, i. e. the existence of local time. To this end, fix $x, y \in [0,1]$. We recursively define the following sequence of times of disagreement of dyadic components of $x$ and $y$. For $n \in \mathbb{N}$ let

$$\sigma_1 = \inf\{\ell \geq 1 : \bar{x}_\ell \neq \bar{y}_\ell\}, \text{ and } \sigma_{n+1} = \inf\{\ell > \sigma_n : \bar{x}_\ell \neq \bar{y}_\ell\}. \tag{5.2}$$

In case $x = 0$ we denote this sequence by $(\sigma^0_\ell)_{\ell \in \mathbb{N}}$.

In analogy to the increment functions $g$ in Section 3 we define the following function

$$h(\xi, x) := \sum_{m=0}^{\infty} \gamma^{-m-1} \big[\Phi(B_2^m(\xi, \frac{1+x}{2})) - \Phi(B_2^m(\xi, \frac{x}{2}))\big], \quad \xi, x \in [0,1].$$

We have the following result.

**Proposition 5.3.** *Let $\xi, x, y \in [0,1]$. Then*

$$H(\xi, y) - H(\xi, x) = \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell} (-1)^{(1-\bar{y}_{\sigma_\ell})} h(B_1^{-\sigma_\ell}(\xi, y), B_2^{-\sigma_\ell}(\xi, x)). \tag{5.3}$$

**Proof:** It follows from the definition of $\sigma_n, n \in \mathbb{N}$, that $y$ can be written

$$y = (\overline{x}_1, \ldots, \overline{x}_{\sigma_1 - 1}, \overline{y}_{\sigma_1}, \overline{x}_{\sigma_1+1}, \cdots, \overline{x}_{\sigma_2-1}, \overline{y}_{\sigma_2}, \overline{x}_{\sigma_2+1}, \cdots\cdots, \overline{x}_{\sigma_n-1}, \overline{y}_{\sigma_n}, \overline{x}_{\sigma_n+1}, \cdots).$$

For $n \in \mathbb{N}$ let $y^n$ be the sequence which up to $\sigma_n$ represents the dyadic expansion of $y$, and then switches to the representing sequence of $x$. Then for $n \in \mathbb{N}$ we have

$$y^n = (\overline{x}_1, \ldots, \overline{x}_{\sigma_1 - 1}, \overline{y}_{\sigma_1}, \overline{x}_{\sigma_1+1}, \cdots, \overline{x}_{\sigma_2-1}, \overline{y}_{\sigma_2}, \overline{x}_{\sigma_2+1}, \cdots\cdots, \overline{x}_{\sigma_n-1}, \overline{y}_{\sigma_n}, \overline{x}_{\sigma_n+1}, \cdots, \overline{x}_m, \cdots).$$

Note that $\lim_{n\to\infty} y^n = y$. Hence the left hand side of (5.3) has the telescoping representation

$$H(\xi, y) - H(\xi, x) = \sum_{l=1}^{\infty} \Big(H(\xi, y^\ell) - H(\xi, y^{\ell-1})\Big),$$

where $y^0 = x$.

For $\ell \in \mathbb{N}$ let us calculate $H(\cdot, y^\ell) - H(\cdot, y^{\ell-1})$. Since $B_2^{-n}(\xi, y^\ell) = B_2^{-n}(\xi, y^{\ell-1})$ for $n \geq \sigma_\ell$ we have

$$\begin{aligned} H(\xi, y^\ell) - H(\xi, y^{\ell-1}) &= \sum_{n\in\mathbb{Z}} \gamma^n \Big[\Phi\big(B_2^{-n}(\xi, y^\ell)\big) - \Phi\big(B_2^{-n}(\xi, y^{\ell-1})\big)\Big] \\ &= \sum_{n<\sigma_\ell} \gamma^n \Big[\Phi\big(B_2^{-n}(\xi, y^\ell)\big) - \Phi\big(B_2^{-n}(\xi, y^{\ell-1})\big)\Big] \\ &= \gamma^{\sigma_\ell} \sum_{m=1}^{\infty} \gamma^{-m} \Big[\Phi\Big(B_2^m\big(B^{-\sigma_\ell}(\xi, y^\ell)\big)\Big) - \Phi\Big(B_2^m\big(B^{-\sigma_\ell}(\xi, y^{\ell-1})\big)\Big)\Big] \\ &= \gamma^{\sigma_\ell} \sum_{k=0}^{\infty} \gamma^{-k-1} \Big[\Phi\Big(B_2^k\big(B_2^1(B^{-\sigma_\ell}(\xi, y^\ell))\big)\Big) - \Phi\Big(B_2^k\big(B_2^1(B^{-\sigma_\ell}(\xi, y^{\ell-1}))\big)\Big)\Big]. \end{aligned}$$

Now note that by definition $B^{-\sigma_\ell}(\xi, y^\ell)$ corresponds to the doubly infinite sequence

$$(\cdots, \overline{\xi}_0, \overline{y}_1, \cdots, \overline{y}_{\sigma_\ell - 1}, \overline{y}_{\sigma_\ell}, \overline{x}_{\sigma_\ell+1}, \cdots),$$

while $B^{-\sigma_\ell}(\xi, y^{\ell-1})$ is represented by

$$(\cdots, \overline{\xi}_0, \overline{y}_1, \cdots, \overline{y}_{\sigma_\ell - 1}, \overline{x}_{\sigma_\ell}, \overline{x}_{\sigma_\ell+1}, \cdots),$$

where $\overline{y}_{\sigma_\ell}$ resp. $\overline{x}_{\sigma_\ell}$ take position 0 in the sequences. Hence in case $\overline{y}_{\sigma_\ell} = 1$ we may continue to write

$$\begin{aligned} & H(\xi, y^\ell) - H(\xi, y^{\ell-1}) \\ & \quad \gamma^{\sigma_\ell}\,(-1)^{(1-\overline{y}_{\sigma_\ell})}\,h(B_1^{-\sigma_\ell}(\xi, y), B_2^{-\sigma_\ell}(\xi, x)), \end{aligned}$$

while in case $\overline{y}_{\sigma_\ell} = 0$ the sign on the rhs just changes. This provides the claimed formula. □

Let us next calculate $h$.

**Lemma 5.4.** *We have*

$$h(\xi, x) = 2\kappa[\Phi(\frac{1+x}{2}) - \Phi(\frac{x}{2})] + \kappa S(\xi, 0) = -2\kappa\, x + \kappa S(\xi, 0), \quad \xi, x \in [0,1].$$

**Proof:** We may write

$$\begin{aligned} h(\xi, x) &= 2\kappa\,[\Phi(\frac{1+x}{2}) - \Phi(\frac{x}{2})] \\ &+ \sum_{m=1}^{\infty} \gamma^{-m-1}\big[\Phi(B_2^m(\xi, \frac{1+x}{2})) - \Phi(B_2^m(\xi, \frac{x}{2}))\big] \\ &= 2\kappa\,[\Phi(\frac{1+x}{2}) - \Phi(\frac{x}{2})] \\ &+ \sum_{m=1}^{\infty} \gamma^{-m-1}\big[\Phi(\frac{1+x}{2^{m+1}} + \frac{\overline{\xi}_0}{2^m} + \cdots + \frac{\overline{\xi}_{-m+1}}{2}) - \Phi(\frac{x}{2^{m+1}} + \frac{\overline{\xi}_0}{2^m} + \cdots + \frac{\overline{\xi}_{-m+1}}{2})\big] \\ &= 2\kappa[\Phi(\frac{1+x}{2}) - \Phi(\frac{x}{2})] \\ &+ \sum_{m=1}^{\infty} \gamma^{-m-1} 2^{-m-1}\,\Phi'(\frac{\overline{\xi}_0}{2^m} + \cdots + \frac{\overline{\xi}_{-m+1}}{2}) \\ &= 2\kappa\,[\Phi(\frac{1+x}{2}) - \Phi(\frac{x}{2})] + \kappa S(\xi, 0). \end{aligned}$$

Finally note that for $x \in [0,1]$ we have $\Phi(\frac{1+x}{2}) - \Phi(\frac{x}{2}) = -x$. (This completes the proof.□

Using Lemma 5.4, we may write the formula of Proposition 5.3 in a more explicit way.

**Proposition 5.5.** *Let* $\xi, x, y \in [0,1]$. *Then*

$$\begin{aligned} H(\xi, y) - H(\xi, x) &= -2\kappa \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell}\,(-1)^{(1-\bar{y}_{\sigma_\ell})} B_2^{-\sigma_\ell}(\xi, x) \\ & \quad + \kappa \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell}\,(-1)^{(1-\bar{y}_{\sigma_\ell})} S(B_1^{-\sigma_\ell}(\xi, y), 0). \end{aligned} \tag{5.4}$$

*In particular*

$$H(\xi, y) = \kappa \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell^0}\, S(B_1^{-\sigma_\ell^0}(\xi, y), 0). \tag{5.5}$$

Let us finally use the representation formulas for $S$ from section 3 to describe $H$ in still more detail. We prefer to continue with the simpler second identity in Proposition 5.5. In anticipation of the calculation of local time for *tilted* curves $\mathcal{T}$, let us give a representation of $H(\xi, y) + \beta y S(\xi, 0)$ for any $\beta \in \mathbb{R}$.

**Corollary 5.6.** *Let $\xi, y \in [0,1]$. Let $X_n = 1_{\{\overline{y}_n = 0\}} - 1_{\{\overline{y}_n = 1\}}, R_n = 1_{\{\overline{y}_n = 1\}}$ for $n \geq 1$. Then we have*

$$H(\xi, y) + \beta y S(\xi, 0) = \frac{\kappa + \beta}{2} S(\xi, 0) - \frac{\kappa^2}{2} \frac{\gamma}{1-\gamma} + \sum_{n=1}^{\infty} X_n \left[ (\kappa + \beta) S(\xi, 0) (\frac{1}{2})^{n+1} + \kappa^2 \gamma^n (1 - \sum_{\ell=1}^{\infty} (\frac{1}{2})^{\ell+1} X_{n+\ell}) \right]. \tag{5.6}$$

**Proof:** According to Proposition 5.3 we have to calculate $S(B_1^{-\sigma_\ell^0}(\xi, y), 0)$ for $\xi, y \in [0,1], \ell \in \mathbb{N}$ using the formulas provided by Corollaries 3.6, 3.7. To exploit them, we first note that the dyadic sequence associated with $B_1^{-\sigma_\ell}(\xi, y)$ is given by $(\cdots, \overline{\xi}_{-1}, \overline{\xi}_0, \overline{y}_1, \overline{y}_2, \cdots, \overline{y}_{\sigma_\ell^0 - 1}, \overline{y}_{\sigma_\ell^0})$. We therefore can write

$$\begin{aligned} S(B_1^{-\sigma_\ell^0}(\xi, y), 0) &= \sum_{m=0}^{\sigma_\ell^0 - 1} \kappa^{m+1} \left[ 1_{\{\overline{y}_{\sigma_\ell^0 - m} = 0\}} - 1_{\{\overline{y}_{\sigma_\ell^0 - m} = 1\}} \right] + \kappa^{\sigma_\ell^0} S(\xi, 0) \\ &= \kappa^{\sigma_\ell^0} \sum_{n=1}^{\sigma_\ell^0} \kappa^{-n+1} X_n + \kappa^{\sigma_\ell^0} S(\xi, 0). \end{aligned}$$

Now insert this formula into the representation of Proposition 5.3, and observe that $\gamma\kappa = \frac{1}{2}$ and that $R_\ell = \frac{1 - X_\ell}{2}$ to obtain

$$\begin{aligned} & H(\xi, y) + \beta y S(\xi, 0) \\ &= \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell^0} \Big[ \kappa^{\sigma_\ell^0} \sum_{n=1}^{\sigma_\ell^0} \kappa^{-n+2} X_n + \kappa^{\sigma_\ell^0} (\kappa + \beta) S(\xi, 0) \Big] \\ &= \sum_{\ell=1}^{\infty} (\frac{1}{2})^{\ell+1} (1 - X_\ell) \left[ \sum_{n=1}^{\ell} \kappa^{-n+2} X_n + (\kappa + \beta) S(\xi, 0) \right] \\ &= \frac{\kappa + \beta}{2} S(\xi, 0) - (\kappa + \beta) S(\xi, 0) \sum_{n=1}^{\infty} X_n (\frac{1}{2})^{n+1} \\ &\quad + \sum_{n=1}^{\infty} \kappa^{-n+2} X_n (\frac{1}{2})^{n+1} (1 - X_n) + \sum_{n=1}^{\infty} \kappa^{-n+2} X_n \sum_{\ell=n+1}^{\infty} (\frac{1}{2})^{\ell+1} (1 - X_\ell) \\ &= \frac{\kappa + \beta}{2} S(\xi, 0) - (\kappa + \beta) S(\xi, 0) \sum_{n=1}^{\infty} X_n (\frac{1}{2})^{n+1} + \sum_{n=1}^{\infty} \frac{\kappa^2}{2} \gamma^n X_n - \frac{\kappa^2}{2} \frac{\gamma}{1-\gamma} \\ &\quad + \sum_{n=1}^{\infty} \kappa^2 \gamma^n X_n \sum_{\ell=1}^{\infty} (\frac{1}{2})^{\ell+1} (1 - X_{n+\ell}) \\ &= \frac{\kappa + \beta}{2} S(\xi, 0) - \frac{\kappa^2}{2} \frac{\gamma}{1-\gamma} \\ &\quad + \sum_{n=1}^{\infty} X_n \left[ (\kappa + \beta) S(\xi, 0) (\frac{1}{2})^{n+1} + \kappa^2 \gamma^n (1 - \sum_{\ell=1}^{\infty} (\frac{1}{2})^{\ell+1} X_{n+\ell}) \right]. \end{aligned}$$

This gives the desired representation. □

5.1. **Moments of iterated integrals in Rademacher calculus.** The formula representing $H$ in Corollary 5.6 exhibits a surprising link to Malliavin's calculus for non-Gaussian stochastic bases. In fact, the essential part of $H(\xi,\cdot)+\beta\cdot S(\xi,0)$ is composed of constant multiples of the expressions

$$\sum_{n\geq 1}\gamma^n X_n \qquad \text{and} \qquad \sum_{n\geq 1}\sum_{m\geq 1} 1_{\{n<m\}}\kappa^{-n}2^{-m}X_m X_n,$$

where $(X_n)_{n\geq 1}$ is a sequence of i.i.d Bernoulli-$\frac{1}{2}$ variables, also called *Rademacher sequence*. In other words, we face multiple integrals in the first and second Rademacher chaos, in terms of a calculus that has many similarities with the Gaussian Malliavin calculus. Its theory has been investigated by Nourdin, Peccati, Reinert [32] for applications originating in statistics. Since we will investigate the questions of existence of occupation densities for $H(\xi,\cdot)$ by means of Fourier transforms, involving exponentials of linear combinations of the first and second chaos Rademacher integrals, the main task we will have to face is the estimation of moments of products of these integrals. Their special integrands will be denoted by

$$f(i)=\gamma^i, \quad h(i,j)=\kappa^{-i}2^{-j}1_{\{i<j\}}, \quad i,j\in\mathbb{N},$$

the multiple integrals by

$$I_1(f)=\sum_{n\geq 1}f(n)X_n, \quad I_2(h)=\sum_{n\geq 1}\sum_{m\geq 1}h(n,m)X_nX_m.$$

It will be more convenient to work with the symmetrization of the function $h$. We therefore start by giving a representation for this function.

**Lemma 5.7.** *Let $g$ be the symmetrization of $h$, i.e. $g(i,j)=\frac{1}{2}(h(i,j)+h(j,i)), i,j\in\mathbb{N}$. Then, as in classical Malliavin calculus, $I_2(g)=I_2(h)$, and for $i,j\in\mathbb{N}$ we have*

$$g(i,j)=\frac{1}{2}\gamma^{\frac{i+j}{2}}\left(\frac{\kappa}{2}\right)^{\frac{|i-j|}{2}}1_{\{i\neq j\}}.$$

**Proof:** We may write

$$\begin{aligned}
g(i,j) &= \frac{1}{2}(\kappa^{-i}2^{-j}\,1_{\{i<j\}}+\kappa^{-j}2^{-i}1_{\{j<i\}})\\
&= \frac{1}{2}\kappa^{-i\wedge j}2^{-i\vee j}1_{\{i\neq j\}}=\frac{1}{2}\kappa^{-i\wedge j}(\gamma\kappa)^{i\vee j}1_{\{i\neq j\}}\\
&= \frac{1}{2}\gamma^{i\vee j}\,\kappa^{i\vee j-i\wedge j}\,1_{\{i\neq j\}}=\frac{1}{2}\gamma^{i\vee j}\,\kappa^{|i-j|}\,1_{\{i\neq j\}},
\end{aligned}$$

and alternatively

$$\begin{aligned}
g(i,j) &= \frac{1}{2}\kappa^{-i\wedge j}2^{-i\vee j}1_{\{i\neq j\}}=\frac{1}{2}(2\gamma)^{i\wedge j}2^{-i\vee j}1_{\{i\neq j\}}\\
&= \frac{1}{2}\gamma^{i\wedge j}2^{-(i\vee j-i\wedge j)}1_{\{i\neq j\}}=\frac{1}{2}\gamma^{i\wedge j}2^{-|i-j|}1_{\{i\neq j\}}.
\end{aligned}$$

Taking roots of the alternative expressions obtained, we finally have

$$\begin{aligned}
g(i,j) &= \frac{1}{2}\gamma^{\frac{i\vee j}{2}}\kappa^{\frac{|i-j|}{2}}\;\gamma^{\frac{i\wedge j}{2}}2^{-\frac{|i-j|}{2}}1_{\{i\neq j\}}\\
&= \frac{1}{2}\gamma^{\frac{i+j}{2}}\left(\frac{\kappa}{2}\right)^{\frac{|i-j|}{2}}1_{\{i\neq j\}}.
\end{aligned}$$

This is the desired formula. □

In order to calculate the Fourier transform of (constants multiplied to $f$ resp. $g$ are unessential)

$$\mathbb{R} \ni u \mapsto \exp(iu(I_1(f) + I_2(g))),$$

we will mainly be occupied with computing the moments

$$E(I_1(f)^n I_2(g)^m)$$

for $n, m \geq 0$. In fact, it will be sufficient to confine our attention to even $n$, since the transformation $X \mapsto -X$ for the sequence $X = (X_i)_{i\in\mathbb{N}}$ is measure preserving, and via this transformation $I_1(f)$ maps to $-I_1(f)$, while $I_2(g)$ maps to $I_2(g)$. So for odd $n$, we get $E(I_1(f)^n I_2(g)^m) = -E(I_1(f)^n I_2(g)^m)$, proving that

$$E(I_1(f)^n I_2(g)^m) = 0 \quad \text{for odd} \quad n \in \mathbb{N}.$$

For the calculus of the moments, let us briefly recall some notation well known from combinatorics, and encountered for instance in Malliavins calculus. We denote the set of all nonnegative integer valued sequences for which almost all components vanish by $S_0$. For $p = (p_i)_{i\in\mathbb{N}} \in S_0$ we write

$$|p| = \sum_{i\geq 1} p_i, \quad p! = \prod_{i\geq 1} p_i!, \quad \binom{|p|}{p} = \frac{|p|}{p!} = \frac{|p|!}{\prod_{i\geq 1} p_i!}.$$

5.1.1. *Moments of Rademacher integrals in the first chaos.* Let us first suppose $m = 0$ and compute $E(I_1(f)^{2n})$ for $n \geq 0$. We may simply write

$$E(I_1(f)^{2n}) = \sum_{j_1,\cdots,j_{2n}=1}^{\infty} \gamma^{\sum_{i=1}^{2n} j_i} E(\prod_{i=1}^{2n} X_{j_i}).$$

Now for $k \in \mathbb{N}$ and a $2n$ tuple of indices $(j_1, \cdot, j_{2n})$ let $T_k = \{i : 1 \leq i \leq 2n, j_i = k\}$. Then $(T_k)_{k\in\mathbb{N}}$ is a partition of $\{1, \cdots, 2n\}$, and for almost all $k \in \mathbb{N}$ we have $T_k = \emptyset$. So we may write

$$E(\prod_{i=1}^{2n} X_{j_i}) = E(\prod_{k\geq 1} X_k^{|T_k|}),$$

and from the definition of our Bernoulli variables it is immediately clear that $E(\prod_{k\geq 1} X_k^{|T_k|}) = 1$, if $|T_k|$ is even for any $k \in \mathbb{N}$, while $E(\prod_{k\geq 1} X_k^{|T_k|}) = 0$ if this is not the case. Consequently we have

$$\begin{aligned} &E(I_1^{2n}(f)) \\ &= \sum_{(T_i)_{i\geq 1} \text{ partition of } \{1,\cdots,2n\}, |T_i| \text{ even}} \prod_{k\geq 1} \gamma^{k|T_k|} \\ &= \sum_{p\in S_0 \text{ even}, |p|=2n} |\{(T_i)_{i\in\mathbb{N}} \text{ partition of } \{1, \cdots, 2n\}, |T_i| = p_i \text{ for } i \geq 1\}| \prod_{k\geq 1} \gamma^{kp_k} \\ &= \sum_{p\in S_0 \text{ even}, |p|=2n} \binom{2n}{p} \prod_{k\geq 1} \gamma^{kp_k}. \end{aligned} \tag{5.7}$$

For the last equation in (5.7) we use the well known fact from combinatorics that the number of partitions of a set of $2n$ elements into subsets of $p = (p_i)_{i\geq 1}$ elements is given by the multinomial coefficient $\binom{2n}{p}$ for multiindices $p \in S_0$ such that $|p| = 2n$. We can summarize our findings in the following statement.

**Proposition 5.8.** *For $n \geq 1$ we have*

$$E(I_1^{2n}(f)) = \sum_{p \in S_0 \text{ even}, |p|=2n} \binom{2n}{p} \prod_{k \geq 1} \gamma^{kp_k}.$$

Let us finally use the result of Proposition 5.8 to countercheck the *dual* of the well known result from Section 4. We obtain

$$\begin{aligned} E(\cos(uI_1(f))) &= \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!} E(I_1(f)^{2n}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!} \sum_{p \in S_0 \text{ even}, |p|=2n} \binom{2n}{p} \prod_{k \geq 1} \gamma^{kp_k} \\ &= \sum_{p \in S_0 \text{ even}} \frac{(-1)^{\frac{|p|}{2}}}{|p|!} \binom{|p|}{p} \prod_{k \geq 1} (u\gamma^k)^{p_k} \\ &= \sum_{p \in S_0 \text{ even}} \frac{(-1)^{\frac{|p|}{2}}}{p!} \prod_{k \geq 1} (u\gamma^k)^{p_k} \\ &= \sum_{p \in S_0 \text{ even}} \prod_{k \geq 1} \frac{(-1)^{\frac{p_k}{2}}}{p_k!} (u\gamma^k)^{p_k} \\ &= \prod_{k \in \mathbb{N}} \cos(u\gamma^k). \end{aligned} \tag{5.8}$$

This formula could of course have been obtained more directly, without the detour of the calculation of even moments of $I_1(f)$ by taking into account the independence of the $X_i, i \geq 1$, and elementary trigonometric identities.

5.1.2. *Moments of Rademacher integrals in the second chaos.* Let us next assume $n = 0$ and compute $E(I_2(h)^m)$ for $m \geq 0$ by a similar line of reasoning. We first write

$$E(I_2(g)^m) = \sum_{j_1, \cdots, j_{2m}=1}^{\infty} 1_{\{j_1 \neq j_2, \cdots, j_{2m-1} \neq j_{2m}\}} \gamma^{\frac{\sum_{i=1}^{2m} j_i}{2}} \prod_{r=1}^{m} (\frac{\kappa}{2})^{\frac{|j_{2r-1} - j_{2r}|}{2}} E(\prod_{i=1}^{2m} X_{j_i}).$$

We therefore now have to reduce our space of admissible $2m$-tuples of integers $(j_1, \cdots, j_{2m})$ to the set

$$\Delta_m = \{(j_1, j_2, \cdots, j_{2m-1}, j_{2m}) : j_{2i-1} \neq j_{2i} \text{ for all } 1 \leq i \leq m\}.$$

For $k \in \mathbb{N}$ and a $2m$-tuple of indices $(j_1, \cdots, j_{2m}) \in \Delta_m$ let

$$T_k = \{i : 1 \leq i \leq 2m, j_i = k\}.$$

Again, $(T_k)_{k \in \mathbb{N}}$ is a partition of $\{1, \cdots, 2m\}$, and for almost all $k \in \mathbb{N}$ we have $T_k = \emptyset$. Moreover, all partition sets belong to

$$D_m = \{T : T \subset \{1, \cdots, 2m\}, \{2i-1, 2i\} \not\subset T \text{ for all } 1 \leq i \leq m\}.$$

For a partition $(T_k)_{k \in \mathbb{N}}$ in $D_m$ we have, as before,

$$E(\prod_{i=1}^{2m} X_{j_i}) = E(\prod_{k \geq 1} X_k^{|T_k|}),$$

and from the definition of our Bernoulli variables it is immediately clear that $E(\prod_{k \geq 1} X_k^{|T_k|}) = 1$, if $|T_k|$ is even for any $k \in \mathbb{N}$, while $E(\prod_{k \geq 1} X_k^{|T_k|}) = 0$ if this is not the case. To describe the moments, we need one more notation. For $h, k \in \mathbb{N}, h \neq k$, let

$$q_{hk} = |\{l : 1 \leq l \leq m, \{2l-1, 2l\} \subset T_h \cup T_k\}|.$$

In these terms, we may write

$$\begin{aligned} &E(I_2(g)^m) \\ &= (\frac{1}{2})^m \sum_{(T_i)_{i\geq 1}\subset D_m \text{ even partition of } \{1,\cdots,2m\}} \prod_{k\geq 1} \gamma^{\frac{k}{2}|T_k|} \cdot \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}} \\ &= (\frac{1}{2})^m \sum_{p\in S_0 \text{ even}, |p|=2m} \prod_{k\geq 1} \gamma^{\frac{k}{2} p_k} \\ &\qquad \cdot \sum_{(T_i)_{i\geq 1}\subset D_m \text{ even partition of } \{1,\cdots,2m, |T_i|=p_i \text{ for } i\in\mathbb{N}\}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}}. \end{aligned}$$

We denote the interaction term appearing in the last line of the preceding inequality by

$$L_m(p) = \sum_{(T_i)_{i\geq 1}\subset D_m \text{ even partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}},$$

for an even multiindex $p \in S_0$ such that $|p| = 2m$.

To further compute the function $L_m(p)$, even for $p \in S_0$ not necessarily even, it will be simpler to argue for the complement of $\Delta_m$ in the set if all tuples $(j_1, \cdots, j_{2m})$ of integers. For this purpose, for $H \subset \{1, \cdots, m\}$ let

$$\Delta_m(H) = \{(j_i, \cdots, j_{2m}) : j_{2l-1} = j_{2l} \text{ for } l \in H, \quad j_{2l-1} \neq j_{2l} \text{ for } l \notin H\},$$

and correspondingly

$$D_m(H) = \{T : T \subset \{1, \cdots, 2m\}, \{2l-1, 2l\} \subset T \text{ for } l \in H, \{2l-1, 2l\} \not\subset T \text{ for } l \notin H\}.$$

Then, by definition, $(\Delta_m(H))_{H\subset\{1,\cdots,m\}}$ is a pairwise disjoint partition of the set of all $(j_1, \cdots, j_{2m}) \in \mathbb{N}^{2m}$, and we may identify $D_m(\emptyset) = D_m$. Therefore we can write

$$\begin{aligned} &L_m(p) \\ &= \sum_{(T_i)_{i\geq 1}\subset D_m \text{ partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}} \\ &= \sum_{(T_i)_{i\geq 1} \text{ partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}} \\ &\quad - \sum_{\emptyset\neq H\subset\{1,\cdots,2m\}} \sum_{(T_i)_{i\geq 1}\subset D_m(H) \text{ partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}}. \end{aligned} \tag{5.9}$$

Now define

$$N_m(p) = \sum_{(T_i)_{i\geq 1} \text{ partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}}.$$

Now note that the quantities appearing in the last line of (5.9) are invariant for permutation of index pairs $(j_{2l-1}, j_{2l})$. Hence we may abbreviate for $0 \leq s \leq m$ $D_m(s) = D_m(\{1, \cdots, 2s\})$ and have in continuation of (5.9)

$$\begin{aligned} L_m(p) &= N_m(p) \\ &\quad - \sum_{1\leq s\leq m} \binom{m}{s} \sum_{(T_i)_{i\geq 1}\subset D_m(s) \text{ partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}}. \end{aligned} \tag{5.10}$$

Recall that partitions $(T_l)_{l\geq 1}$ for some $(j_1,\cdots,j_{2m})\in\mathbb{N}_0^{\mathbb{N}}$ just describe the sets

$$T_l=\{i: j_i=l\}.$$

In this setting, for a partition $(T_l)_{l\geq 1}$ in $D_m(s)$ let us define $\pi_s^1((j_1,\cdots,j_{2m}))=(j_1,\cdots,j_{2s})$, and $\pi_s^2((j_1,\cdots,j_{2m}))=(j_{2s+1},\cdots,j_{2m})$,

$$U_l(s)=\{i: j_i=l,\ \ i\in\pi^1(D_m(s))\},\quad V_l(s)=\{i: j_i=l,\ \ i\in\pi^2(D_m(s))\}.$$

In these terms, we may identify further, with $\tilde q_{hk}$ defined as $q_{hk}$, but referring to the partition $(V_l(s))_{l\geq 1}$

$$
\begin{aligned}
&L_m(p) \qquad (5.11)\\
&=N_m(p)-\sum_{1\leq s\leq m}\binom{m}{s}\sum_{q+r=p,q\,\text{even}}\ \sum_{\substack{(V_l(s))_{l\geq 1}\,\text{partition in}\,\pi^2(D_m(s)),|V_l(s)|=r_l\\ (U_l(s))_{l\geq 1}\,\text{partition in}\,\pi^1(D_m(s)),|U_l(s)|=q_l,}}\ \prod_{h<k}(\frac{\kappa}{2})^{\frac{k-h}{2}\tilde q_{hk}}\\
&=N_m(p)-\sum_{1\leq s\leq m}\binom{m}{s}\sum_{q+r=p,|q|=2s,\,q\,\text{even}}\frac{1}{2^s}\frac{s!}{(\frac{q}{2})!}\sum_{(V_l(s)_{l\geq 1}\,\text{partition in}\,\pi^2(D_m(s)),|V_l(s)|=r_l}\ \prod_{h<k}(\frac{\kappa}{2})^{\frac{k-h}{2}\tilde q_{hk}}\\
&=N_m(p)-\sum_{1\leq s\leq m}\binom{m}{s}\sum_{q+r=p,|q|=2s,\,q\,\text{even}}\frac{1}{2^s}\frac{s!}{(\frac{q}{2})!}L_{m-s}(r)\\
&=N_m(p)-\sum_{1\leq\frac{|q|}{2}\leq m,\,q\,\text{even}}\frac{1}{2^{\frac{|q|}{2}}}\frac{m!}{(m-\frac{|q|}{2})!\,(\frac{q}{2})!}L_{m-\frac{|q|}{2}}(p-q).
\end{aligned}
$$

From (5.11) it becomes clear that we now can argue recursively to compute $L_m(p)$. Indeed, writing $q_1,r_1$ instead of $q,r$ we obtain in the next step

$$
\begin{aligned}
(5.12)\qquad & N_m(p)-\sum_{1\leq\frac{|q_1|}{2}\leq m,\,q_1\,\text{even}}\frac{1}{2^{\frac{|q_1|}{2}}}\frac{m!}{(m-\frac{|q_1|}{2})!\,(\frac{q_1}{2})!}L_{m-\frac{|q_1|}{2}}(p-q_1)\\
&=N_m(p)-\sum_{1\leq\frac{|q_1|}{2}\leq m,\,q_1\,\text{even}}\frac{1}{2^{\frac{|q_1|}{2}}}\frac{m!}{(m-\frac{|q_1|}{2})!\,(\frac{q_1}{2})!}\times\Big[N_{m-\frac{|q_1|}{2}}(p-q_1)\\
&\quad-\sum_{1\leq\frac{|q_2|}{2}\leq m-\frac{|q_1|}{2}}\frac{1}{2^{\frac{|q_2|}{2}}}\frac{(m-\frac{|q_1|}{2})!}{(m-\frac{|q_1|}{2}-\frac{|q_2|}{2})!\,(\frac{q_2}{2})!}L_{m-\frac{|q_1|}{2}-\frac{|q_2|}{2}}(p-q_1-q_2)\Big]\\
&=N_m(p)-\sum_{1\leq\frac{|q_1|}{2}\leq m,\,q_1\,\text{even}}\frac{1}{2^{\frac{|q_1|}{2}}}\frac{m!}{(m-\frac{|q_1|}{2})!\,(\frac{q_1}{2})!}N_{m-\frac{|q_1|}{2}}(p-q_1)\\
&\quad+\sum_{2\leq\frac{|q_1|}{2}+\frac{|q_2|}{2}\leq m,\,q_1,q_2\,\text{even}}\frac{1}{2^{\frac{|q_1|}{2}+\frac{|q_2|}{2}}}\frac{m!}{(m-\frac{|q_1|}{2}-\frac{|q_2|}{2})!\,(\frac{q_1}{2})!\,(\frac{q_2}{2})!}L_{m-\frac{|q_1|}{2}-\frac{|q_2|}{2}}(p-q_1-q_2).
\end{aligned}
$$

Since we interpret $\frac{|q_1|}{2}$ as cardinality of partitions of $\{1,\cdots,\frac{|q_1|}{2}\}$, we may accordingly consider $\frac{|q_2|}{2}$ as cardinality of partitions of $\{\frac{|q_1|}{2}+1,\cdots,\frac{|q_1|}{2}+\frac{|q_2|}{2}\}$. For convenience, we take the union of the two families of disjoint partitions, and simply write $\frac{|q_2|}{2}$ for the cardinality of the unions. This means that we replace $\frac{|q_1|}{2}+\frac{|q_2|}{2}$ simply by $\frac{|q_2|}{2}$, and the multiindex $q_1+q_2$ by $q_2$. With these modifications, (5.12) becomes

$$
\begin{aligned}
(5.13) \qquad & N_m(p) - \sum_{1\le \frac{|q_1|}{2}\le m,\, q_1 \text{ even}} \frac{1}{2^{\frac{|q_1|}{2}}} \frac{m!}{(m-\frac{|q_1|}{2})!\,(\frac{q_1}{2})!} N_{m-\frac{|q_1|}{2}}(p-q_1) \\
& + \sum_{2\le \frac{|q_1|}{2}+\frac{|q_2|}{2}\le m,\, q_1,q_2 \text{ even}} \frac{1}{2^{\frac{|q_1|}{2}+\frac{|q_2|}{2}}} \frac{m!}{(m-\frac{|q_1|}{2}-\frac{|q_2|}{2})!\,(\frac{q_1}{2})!\,(\frac{q_2}{2})!} L_{m-\frac{|q_1|}{2}-\frac{|q_2|}{2}}(p-q_1-q_2) \\
&= N_m(p) - \sum_{1\le \frac{|q_1|}{2}\le m,\, q_1 \text{ even}} \frac{1}{2^{\frac{|q_1|}{2}}} \frac{m!}{(m-\frac{|q_1|}{2})!\,(\frac{q_1}{2})!} N_{m-\frac{|q_1|}{2}}(p-q_1) \\
& + \sum_{2\le \frac{|q_2|}{2}\le m,\, q_2 \text{ even}} \frac{1}{2^{\frac{|q_2|}{2}}} \frac{m!}{(m-\frac{|q_2|}{2})!\,(\frac{q_2}{2})!} L_{m-\frac{|q_2|}{2}}(p-q_2)
\end{aligned}
$$

After at most $m+1$ iterations we get an expression composed of terms, the $l$th one of which for $l\ge 1$ reads

$$
(5.14) \qquad (-1)^l \sum_{l\le \frac{|q_l|}{2}\le m,\, q_l \text{ even}} \frac{1}{2^{\frac{|q_l|}{2}}} \frac{m!}{(m-\frac{|q_l|}{2})!\,(\frac{q_l}{2})!} N_{m-\frac{|q_l|}{2}}(p-q_l)
$$

For $l=0$ the expression in (5.12) evidently reduces to $N_m(p)$. Therefore we obtain

**Proposition 5.9.** *For $p\in\mathbb{N}_0^{\mathbb{N}}$ such that $|p|=2m$ let*

$$
L_m(p) = \sum_{(T_i)_{i\ge 1}\subset D_m \text{ partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} \left(\frac{\kappa}{2}\right)^{\frac{k-h}{2} q_{hk}},
$$

*and*

$$
N_m(p) = \sum_{(T_i)_{i\ge 1} \text{ partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} \left(\frac{\kappa}{2}\right)^{\frac{k-h}{2} q_{hk}}.
$$

*Then we have*

$$
\begin{aligned}
(5.15) \qquad & L_m(p) \\
&= \sum_{l=0}^m (-1)^l \sum_{l\le \frac{|q_l|}{2}\le m,\, q_1 \text{ even}} \frac{1}{2^{\frac{|q_l|}{2}}} \frac{m!}{(m-\frac{|q_l|}{2})!\,(\frac{q_l}{2})!} N_{m-\frac{|q_l|}{2}}(p-q_l) \\
&= \sum_{0\le \frac{|q|}{2}\le m,\, q \text{ even}, |q|\in 4\mathbb{N}} \frac{1}{2^{\frac{|q|}{2}}} \frac{m!}{(m-\frac{|q|}{2})!\,(\frac{q}{2})!} N_{m-\frac{|q|}{2}}(p-q).
\end{aligned}
$$

**Proof:** The last line of (5.13) follows from the cancellation of terms for which $\frac{|q|}{2}$ is odd, as seen from the previous line. □

It remains to compute $N_m(p)$ for partitions $(T_i)_{i\ge 1}$ of $\{1,\cdots,2m\}$, $m\in\mathbb{N}$. This will turn out much simpler than the direct computation of the corresponding $L_m(p)$. In fact, for such a partition we let for $i\in\mathbb{N}$

$$
U_i = T_i\cap\{2,4,\cdots,2m\}, \quad V_i = T_i\cap\{1,3,\cdots,2m-1\}.
$$

Then $|U_i|+|V_i| = |T_i| = p_i, i\in\mathbb{N}$, and $|U_i|,|V_i|\le m$. In these terms, by definition, we have

$$
q_{kh} = |(U_k-1)\cap V_h| + |(U_h-1)\cap V_k|, \quad k,h\in\mathbb{N}.
$$

But this may be simplified further. By writing $(j_1, j_3, \cdots, j_{2m-1}), (j_2, j_4, \cdots, j_{2m})$ as $(f_1, \cdots, f_m), (g_1, \cdots, g_m)$ and sampling $(U_i)_{i\geq 1}$ from the tuples $(f_1, \cdots, f_m)$ and $(V_i)_{i\geq 1}$ from $(g_1, \cdots, g_m)$, re-interpreting $q_{kh} = |U_k \cap V_h| + |U_h \cap V_k|$, we can rewrite

$$\begin{aligned} &N_m(p) \\ &= \sum_{u+v=p} \sum_{(U_i)_{i\geq 1}, (V_i)_{i\geq 1} \text{ partitions of } \{1,\cdots,m\}, |U_i|=u_i, |V_i|=v_i} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}}. \end{aligned} \tag{5.16}$$

Now let $r_{kh} = |U_k \cap V_h|$. Then $q_{kh} = r_{kh} + r_{hk}, h, k \in \mathbb{N}$. We know that $\sum_{h\in\mathbb{N}} r_{kh} = u_k, \sum_{k\in\mathbb{N}} r_{kh} = v_h$. The cardinality of the set of partitions $(U_i)_{i\geq 1}$ of $\{1, \cdots, m\}$ is given by $\binom{m}{u}$. Now each one of the sets of this partition $U_k$ gets subdivided into sets of cardinalities $r_{kh}, h \in \mathbb{N}$. The cardinalities of these sets of subdivisions are given by $\binom{u_k}{(r_{kh})_{h\in\mathbb{N}}}$. Hence we obtain a total of

$$\sum_{\sum_{h\in\mathbb{N}}(r_{kh}+r_{hk})=p_k, k\in\mathbb{N}} \binom{m}{u} \cdot \prod_{k\in\mathbb{N}} \binom{u_k}{(r_{kh})_{h\in\mathbb{N}}} = \sum_{\sum_{h\in\mathbb{N}}(r_{kh}+r_{hk})=p_k, k\in\mathbb{N}} \binom{m}{(r_{kh})_{k,h\in\mathbb{N}}}$$

configurations. So we finally obtain

$$N_m(p) = \sum_{\sum_{h\in\mathbb{N}} q_{kh}=p_k, k\in\mathbb{N}} \binom{m}{(r_{kh})_{k,h\in\mathbb{N}}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{kh}}.$$

We may summarize our findings in

**Proposition 5.10.** *For $p \in \mathbb{N}_0^{\mathbb{N}}$ such that $|p| = 2m$ let*

$$N_m(p) = \sum_{(T_i)_{i\geq 1} \text{ partition of } \{1,\cdots,2m\}, |T_i|=p_i \text{ for } i\in\mathbb{N}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}}.$$

*Then we have*

$$N_m(p) = \sum_{\sum_{h\in\mathbb{N}} q_{kh}=p_k, k\in\mathbb{N}} \binom{m}{(r_{kh})_{k,h\in\mathbb{N}}} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{kh}}. \tag{5.17}$$

The following summarizes our findings from Propositions 5.9 and 5.10 about the moments of $I_2(g)$.

**Proposition 5.11.** *For $m \geq 1$ we have*

$$\begin{aligned} &E(I_2(g)^m) = \\ &\sum_{p,q,r\in S_0\, p,q \text{ even}, |p|=2m, |q|\in 4\mathbb{N}, \sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})+q=p} \prod_{k\geq 1} \gamma^{\frac{k}{2} p_k} \cdot (\frac{1}{2})^{\frac{|p|}{2}+\frac{|q|}{2}} \binom{m}{\frac{q}{2}\ (r_{kh})_{k,h\in\mathbb{N}}} \cdot \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2}(r_{kh}+r_{hk})}. \end{aligned} \tag{5.18}$$

5.1.3. *Moments of products of Rademacher integrals in the first and second chaos.* Let us finally compute the moments of $I_1(f)^{2n} I_2(g)^m$ for $n, m \in \mathbb{N}$. According to the cases discussed above we are allowed to write

$$\begin{aligned} &I_1(f)^{2n} I_2(g)^m \\ &= (\frac{1}{2})^m \sum_{j_1,\cdots,j_{2n},j_{2n+1},\cdots,j_{2n+2m}=1}^{\infty} \gamma^{\sum_{i=1}^{2n} j_i + \frac{\sum_{l=1}^{2m} j_{2n+l}}{2}} 1_{\{j_{2n+1}\neq j_{2n+2},\cdots,j_{2n+2m-1}\neq j_{2n+2m}\}} \\ &\qquad \cdot \prod_{r=1}^{m} (\frac{\kappa}{2})^{\frac{|j_{2n+2r-1}-j_{2n+2r}|}{2}} E(\prod_{i=1}^{2n+2m} X_{j_i}). \end{aligned}$$

In natural extensions of notation used above let

$$\Delta_{n,m} = \{(j_1, \cdots, j_{2n}, j_{2n+1}, \cdots, j_{2n+2m}) : j_{2n+2i-1} \neq j_{2n+2i} \text{ for all } 1 \leq i \leq m\},$$

for $(j_1, \cdots, j_{2n+2m}) \in \Delta_{n,m}$ and $k \in \mathbb{N}$ let

$$T_k = \{i : 1 \le i \le 2n+2m, j_i = k\},$$

$$D_{n,m} = \{T : T \subset \{1, \cdots, 2n+2m\}, \{2n+2i-1, 2n+2i\} \not\subset T \text{ for all } 1 \le i \le m\},$$

and finally for $h, k \in \mathbb{N}, h \neq k$ let

$$q_{kh} = |\{l : 1 \le l \le m, \{2n+2l-1, 2n+2l\} \subset T_h \cup T_k\}|.$$

Again, $(T_k)_{k\ge1}$ is a partition of $\{1, \cdots, 2n+2m\}$, and the condition that $E(\prod_{i=1}^{2n+2m} X_{j_i})$ either vanish or be 1 leads to the conclusion that only *even* partitions $(T_k)_{k\ge1}$ of $\{1, \cdots, 2n+2m\}$ give nonzero contributions. So we obtain the formula

$$\begin{aligned}
(5.19)\qquad & E(I_1(f)^{2n} I_2(g)^m) \\
&= (\frac{1}{2})^m \sum_{(T_i)_{i\ge1}\subset D_{n,m} \text{ even partition of} \{1,\cdots,2n+2m\}} \\
&\qquad \cdot \prod_{k\ge1} \gamma^{k|T_k\cap\{1,\cdots,2n\}|+\frac{k}{2}|T_k\cap\{2n+1,\cdots,2n+2m\}|} \cdot \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{hk}}.
\end{aligned}$$

(5.19) stipulates the use of separate multiindices to denote partitions of $\{1, \cdots, 2n\}$ and $\{2n+1, \cdots, 2n+2m\}$. We note

$$U_i = T_i \cap \{1, \cdots, 2n\}, \quad V_i = T_i \cap \{2n+1, \cdots, 2n+2m\}, \quad i \in \mathbb{N}.$$

Then we can give a multiindex version of (5.19) by

$$\begin{aligned}
(5.20)\qquad & E(I_1(f)^{2n} I_2(g)^m) \\
&= \sum_{q,r\in S_0, q+r \text{ even}, |q|=2n, |r|=2m} (\frac{1}{2})^{\frac{|r|}{2}} \cdot \prod_{k\ge1} \gamma^{kq_k+\frac{k}{2}r_k} \\
&\qquad \sum_{(U_i)_{i\in\mathbb{N}} \text{ part. of} \{1,\cdots,2n\}, (V_i-2n)_{i\in\mathbb{N}}\subset D_m \text{ part. of} \{1,\cdots,2m\}, |U_i|=q_i, |V_i|=r_i} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{kh}} \\
&= \sum_{q,r\in S_0, q+r \text{ even}, |q|=2n, |r|=2m} (\frac{1}{2})^{\frac{|r|}{2}} \cdot \prod_{k\ge1} \gamma^{kq_k+\frac{k}{2}r_k} \\
&\qquad \binom{2n}{q} \sum_{(V_i-2n)_{i\in\mathbb{N}}\subset D_m \text{ part. of} \{1,\cdots,2m\}, |V_i|=r_i} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{kh}} \\
&= \sum_{q,r\in S_0, q+r \text{ even}, |q|=2n, |r|=2m} (\frac{1}{2})^{\frac{|r|}{2}} \cdot \prod_{k\ge1} \gamma^{kq_k+\frac{k}{2}r_k} \binom{2n}{q} L_m(r).
\end{aligned}$$

The simplification leading to the second equation follows from the combinatorial arguments given in the discussion of the moments of $I_1(f)^{2n}$. To identify $\sum_{(V_i-2n)_{i\in\mathbb{N}}\subset D_m \text{ part. of} \{1,\cdots,2m\}, |V_i|=r_i} \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2} q_{kh}}$ with $L(r)$ (without the condition that the related partitions be even) already defined in the discussion of the moments of $I_2(g)^m$, we just have to realize that taking intersections with $\{2n+1, \cdots, 2n+2m\}$ and shifting by $2n$ produces a partition sequence of $\{1, \cdots, 2m\}$ in $D_m$, if we start with a partition sequence of $\{1, \cdots, 2n+2m\} \subset D_{n,m}$. Moreover, $q_{kh}$ defined above correspond exactly to the coefficients defined in the discussion of the moments of $I_2(g)^m$, if we take those with respect to the partition sequence of $\{2n+1, \cdots, 2n+2m\}$ shifted by $2n$. In summary we obtain

**Proposition 5.12.** *For $n, m \in \mathbb{N}$ we have*

$$E(I_1(f)^{2n} I_2(g)^m) = \sum_{q,r\in S_0, q+r \text{ even}, |q|=2n, |r|=2m} (\frac{1}{2})^{\frac{|r|}{2}} \cdot \prod_{k\ge1} \gamma^{kq_k+\frac{k}{2}r_k} \binom{2n}{q} L_m(r).$$

Note that while $q+r$ in the formula of Lemma 5.12 is an even partition, $q$ and $r$ need not be even. This can be seen by the following Example.

**Ex. 1**: Let $m=3, n=2$, and take the following 10-tuple of indices:

$$j_1 = j_5 = j_7 = j_9 = 1, j_2 = j_6 = 2, j_3 = j_8 = 3, j_4 = j_{10} = 4.$$

This tuple obviously belongs to $D_{2,3}$. Then we have

$$T_1 = \{1,5,7,9\}, T_2 = \{2,6\}, T_3 = \{3,8\}, T_4 = \{4,10\}, T_k = \emptyset \text{ for } k \geq 5.$$

And so $(T_k)_{k\geq 1}$ is even, while

$$U_1 = T_1 \cap \{1,\cdots,4\} = \{1\}, V_1 = T_1 \cap \{5,\cdots,10\} = \{5,7,9\},$$

hence neither $(U_k)_{k\geq 1}$ nor $(V_k)_{k\geq 1}$ is even.

Using the results of subsection 5.1.2, and recalling that the $L_m(r)$ of 5.12 is a simple extension of its analogue in 5.1.2 we can give a still more explicit expression for the formula in 5.12.

**Proposition 5.13.** *For $n,m \in \mathbb{N}$ we have*

$$E(I_1(f)^{2n} I_2(g)^m) = \sum_{o,p,q,r\in S_0, o+p,q \text{ } even, |o|=2n, |p|=2m, |q|\in 4\mathbb{N}, \sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})+q=p} (\frac{1}{2})^{\frac{|p|}{2}+\frac{|q|}{2}} \cdot \prod_{k\geq 1} \gamma^{ko_k+\frac{k}{2}p_k} \binom{2n}{o} \binom{m}{\frac{q}{2} \; (r_{kh})_{k,h\in\mathbb{N}}} \cdot \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2}(r_{kh}+r_{hk})}. \tag{5.21}$$

**Proof:** This is a combination of Propositions 5.11 and 5.12. $\square$

5.2. **The Fourier transform.** Let us finally apply Proposition 5.13 to compute the Fourier transforms of our occupation time functional, recalling from Corollary 5.6 that we have to deal with linear combinations of components from the first and second Rademacher chaos. We obtain

**Proposition 5.14.** *For $u \in \mathbb{R}$, and $a,b \in \mathbb{R}$ we have*

$$E[\exp\{iu(aI_1(f)+bI_2(g))\}] = \sum_{o,q,r\in S_0, o+\sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot}), q \text{ } even, |q|\in 4\mathbb{N}} (-1)^{\frac{|o|}{2}} \prod_{k\geq 1} \frac{(u^2a^2\gamma^{2k})^{\frac{o_k}{2}}}{o_k!} (-1)^{\frac{|q|}{4}} \prod_{k\geq 1} \frac{(\frac{u^2b^2}{16}\gamma^{2k})^{\frac{q_k}{4}}}{(\frac{q_k}{2})!} (-1)^{\frac{|r|}{2}} \prod_{h<k} \frac{(\frac{u^2b^2}{16}\gamma^{2k}(\frac{\kappa}{2})^{2(k-h)})^{\frac{r_{hk}+r_{kh}}{4}}}{(r_{kk})!(r_{kh})!(r_{hk})!}.$$

**Proof:** Indeed, for $u \in \mathbb{R}$ the Lemma yields

$$
\begin{aligned}
(5.22) \qquad & E[\exp\{iu(aI_1(f)+bI_2(g))\}] \\
&= \sum_{n,m=0}^{\infty} \frac{(iu)^{n+m}a^nb^m}{(n)!(m)!} E(I_1(f)^n I_2(g)^m) \\
&= \sum_{n,m=0}^{\infty} \frac{(-1)^{\frac{n+m}{2}}}{(n)!(m)!} u^{n+m}a^nb^m \cdot \sum_{o,p,q,r\in S_0, o+p,q\,\text{even}, |o|=n, |p|=2m, |q|\in 4\mathbb{N}, \sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})+q=p} \\
&\qquad (\frac{1}{2})^{\frac{|p|}{2}+\frac{|q|}{2}} \cdot \prod_{k\geq 1} \gamma^{ko_k+\frac{k}{2}p_k} \binom{n}{o} \binom{m}{\frac{q}{2}\ (r_{kh})_{k,h\in\mathbb{N}}} \cdot \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2}(r_{kh}+r_{hk})} \\
&= \sum_{o,p,q,r\in S_0, o+p,q\,\text{even}, |q|\in 4\mathbb{N}, \sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})+q=p} \\
&\qquad (\frac{1}{2})^{\frac{|p|}{2}+\frac{|q|}{2}} \cdot (-1)^{\frac{|o|}{2}} \prod_{k\geq 1} \frac{(u^2a^2\gamma^{2k})^{\frac{o_k}{2}}}{o_k!} (-1)^{\frac{|p|}{4}} \prod_{k\geq 1} \frac{(u^2b^2\gamma^{2k})^{\frac{p_k}{4}}}{(\frac{q_k}{2})!(r_{k\cdot})!} \cdot \prod_{h<k} (\frac{\kappa}{2})^{\frac{k-h}{2}(r_{kh}+r_{hk})}.
\end{aligned}
$$

Now we use the equality $\frac{|p|}{2} = \frac{|q|}{2} + |r|$, to distribute the appearing powers of $\frac{1}{2}$ and $-1$ to the last two products. We remark that given $q$ is even, the condition $o+p$ even readily translates into the condition that $o + \sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})$ is even. Finally note that the seemingly awkward notation in the products $\prod_{k\geq 1}\cdots$ is necessary, since $\frac{o_k}{2}$ and $\frac{p_k}{4}$ need not be integer. This is the desired formula. $\square$

The formula produced by Proposition 5.14 can be given a much simpler and concise form. To derive it, we shall start giving a simpler formula for the central part in the following Lemma.

**Lemma 5.15.** *We have for $b \in \mathbb{R}$*

$$
\sum_{q\ \textit{even},\, |q|\in 4\mathbb{N}} (-1)^{\frac{|q|}{4}} \prod_{k\geq 1} \frac{(\frac{u^2b^2}{16}\gamma^{2k})^{\frac{q_k}{4}}}{(\frac{q_k}{2})!} = \cos(\frac{ub}{4}\frac{\gamma}{1-\gamma}).
$$

**Proof:** Using $i^2=-1$, we may rewrite

$$
\begin{aligned}
\sum_{q\,\text{even},\, |q|\in 4\mathbb{N}} (-1)^{\frac{|q|}{4}} \prod_{k\geq 1} \frac{(\frac{u^2b^2}{16}\gamma^{2k})^{\frac{q_k}{4}}}{(\frac{q_k}{2})!} &= \sum_{q\,\text{even},\, |q|\in 4\mathbb{N}} \prod_{k\geq 1} \frac{(i\frac{ub}{4}\gamma^k)^{\frac{q_k}{2}}}{(\frac{q_k}{2})!} \\
= \sum_{p\in S_0,\, |p|\in 2\mathbb{N}} \prod_{k\geq 1} \frac{(i\frac{ub}{4}\gamma^k)^{p_k}}{p_k!} &= \lim_{N\to\infty} \sum_{p_1,\cdots,p_N\geq 0, |p|\in 2\mathbb{N}} \prod_{k=1}^{N} \frac{(i\frac{ub}{4}\gamma^k)^{p_k}}{p_k!}.
\end{aligned}
$$

Now fix $N \in \mathbb{N}$. We can write

$$
\begin{aligned}
&\sum_{p_1,\cdots,p_N \geq 0, |p| \in 2\mathbb{N}} \prod_{k=1}^{N} \frac{(i\frac{ub}{4}\gamma^k)^{p_k}}{p_k!} \\
&= \sum_{p_1,\cdots,p_{N-1} \geq 0, |p| \in 2\mathbb{N}} \prod_{k=1}^{N-1} \frac{(i\frac{ub}{4}\gamma^k)^{p_k}}{p_k!} \\
&\qquad\qquad \cdot \big[ \sum_{k=0}^{\infty} \frac{(i\frac{ub}{4}\gamma^N)^{2k}}{(2k)!} 1_{2\mathbb{N}}(p_1 + \cdots + p_{N-1}) + \sum_{k=0}^{\infty} \frac{(i\frac{ub}{4}\gamma^N)^{2k+1}}{(2k+1)!} 1_{2\mathbb{N}+1}(p_1 + \cdots + p_{N-1}) \big] \\
&= \sum_{p_1,\cdots,p_{N-1} \geq 0, |p| \in 2\mathbb{N}} \prod_{k=1}^{N-1} \frac{(i\frac{ub}{4}\gamma^k)^{p_k}}{p_k!} \\
&\qquad\qquad \cdot \big[ \cos(\frac{ub}{4}\gamma^N) 1_{2\mathbb{N}}(p_1 + \cdots + p_{N-1}) + i \sin(\frac{ub}{4}\gamma^N) 1_{2\mathbb{N}+1}(p_1 + \cdots + p_{N-1}) \big] \\
&= \sum_{p_1,\cdots,p_{N-1} \geq 0, |p| \in 2\mathbb{N}} \prod_{k=1}^{N-1} \frac{(i\frac{ub}{4}\gamma^k)^{p_k}}{p_k!} \\
&\qquad\qquad \cdot \frac{1}{2} \big[ \exp(i\frac{ub}{4}\gamma^N) + (-1)^{p_1+\cdots+p_{N-1}} \exp(-i\frac{ub}{4}\gamma^N) \big] \\
&= \frac{1}{2} \big[ \sum_{p_1,\cdots,p_{N-1} \geq 0, |p| \in 2\mathbb{N}} \prod_{k=1}^{N-1} \frac{(i\frac{ub}{4}\gamma^k)^{p_k}}{p_k!} \exp(i\frac{ub}{4}\gamma^N) \\
&\qquad\qquad + \sum_{p_1,\cdots,p_{N-1} \geq 0, |p| \in 2\mathbb{N}} \prod_{k=1}^{N-1} \frac{(-i\frac{ub}{4}\gamma^k)^{p_k}}{p_k!} \exp(-i\frac{ub}{4}\gamma^N) \big] \\
&= \frac{1}{2} \big[ \exp(i\frac{ub}{4} \sum_{k=1}^{N} \gamma^k) + \exp(-i\frac{ub}{4} \sum_{k=1}^{N} \gamma^k) \big] \\
&= \cos(\frac{ub}{4} \sum_{k=1}^{N} \gamma^k).
\end{aligned}
$$

It remains to let $N \to \infty$ to obtain the desired result. □

The evaluation of the remainder of the expression provided by Proposition 5.14 is done in the following Lemma. Arguments are similar to the ones used in the preceding Lemma.

**Lemma 5.16.** *We have for $u, a, b \geq 0$ (for $a, b, u < 0$ in the formulas $u, a, b$ has to be replaced by $|u|t, |a|, |b|$)*

$$\sum_{o,r\in S_0, o+\sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})\text{ } even} (-1)^{\frac{|o|}{2}} \prod_{k\geq 1} \frac{(u^2a^2\gamma^{2k})^{\frac{o_k}{2}}}{o_k!} (-1)^{\frac{|r|}{2}} \prod_h \frac{(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{r_{kh}}{2}}}{(r_{kh})!}$$

$$= \prod_k \{ \cos(ua\gamma^k)\cos[\frac{\sqrt{ub}}{2}\{(\gamma)^{\frac{k}{2}}\frac{1+(\frac{\kappa}{2})^{\frac12}}{1-(\frac{\kappa}{2})^{\frac12}} - (\frac12)^k\frac{1}{1-(\frac{\kappa}{2})^{\frac12}}\}]$$

$$- \sin(ua\gamma^k)\sin[\frac{\sqrt{ub}}{2}\{(\gamma)^{\frac{k}{2}}\frac{1+(\frac{\kappa}{2})^{\frac12}}{1-(\frac{\kappa}{2})^{\frac12}} - (\frac12)^k\frac{1}{1-(\frac{\kappa}{2})^{\frac12}}\}]\}$$

$$= \prod_k \cos[ua\gamma^k + \sqrt{ub}\{(\gamma)^{\frac{k}{2}}\frac{1+(\frac{\kappa}{2})^{\frac12}}{1-(\frac{\kappa}{2})^{\frac12}} - (\frac12)^k\frac{1}{1-(\frac{\kappa}{2})^{\frac12}}\}]$$

**Proof:** For $u \in \mathbb{R}$ we have

$$\sum_{o,r\in S_0, o+\sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})\text{ even}} (-1)^{\frac{|o|}{2}} \prod_{k\geq 1} \frac{(u^2a^2\gamma^{2k})^{\frac{o_k}{2}}}{o_k!} (-1)^{\frac{|r|}{2}} \prod_{h,k} \frac{(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{r_{kh}}{2}}}{(r_{kh})!}$$

$$= \sum_{o,r\in S_0, o+\sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})\text{ even}} \prod_{k\geq 1} \frac{(iua\gamma^k)^{o_k}}{o_k!} \prod_{h,k} (-1)^{\sum_{k,h}\frac{r_{kh}}{2}} \frac{(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{r_{kh}}{2}}}{(r_{kh})!}$$

$$= \sum_{o,r\in S_0, o+\sum_{h\in\mathbb{N}}(r_{\cdot h}+r_{h\cdot})\text{ even}} \prod_{k\geq 1} \frac{(iua\gamma^k)^{o_k}}{o_k!} \prod_h \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{\frac{r_{kh}}{2}}}{(r_{kh})!} \prod_h \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{\frac{r_{hk}}{2}}}{(r_{hk})!}$$

$$= \lim_{N\to\infty}\prod_{k=1}^N \{ \sum_{o_k, \sum_{h\in\mathbb{N}}(r_{kh}+r_{hk})\text{ even}} \frac{(iua\gamma^k)^{o_k}}{o_k!} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{kh}}}{(r_{kh})!} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{hk}}}{(r_{hk})!}$$

$$+ \sum_{o_k, \sum_{h\in\mathbb{N}}(r_{kh}+r_{hk})\text{ odd}} \frac{(iua\gamma^k)^{o_k}}{o_k!} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{kh}}}{(r_{kh})!} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{hk}}}{(r_{hk})!}\}$$

$$= I_1 + I_2.$$

Let us now fix $N$, and $h \leq N$. Assume that $N$ is big enough so that $o_k$ and $\sum_{h\leq N}(r_{kh}+r_{hk})$ are both even (which will be the case for large $N$). Then

$$I_1 = \sum_{o_k, \sum_{h\in\mathbb{N}}(r_{kh}+r_{hk})\text{ even}} \frac{(iua\gamma^k)^{o_k}}{o_k!} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{kh}}}{(r_{kh})!} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{hk}}}{(r_{hk})!}$$

$$= \sum_{o_k\text{ even}} \frac{(iua\gamma^k)^{o_k}}{o_k!}\Big\{ \sum_{\sum_{h\in\mathbb{N}} r_{kh}\text{ even}} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{kh}}}{(r_{kh})!} \sum_{\sum_{h\in\mathbb{N}} r_{hk}\text{ even}} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{hk}}}{(r_{hk})!}$$

$$+ \sum_{\sum_{h\in\mathbb{N}} r_{kh}\text{ odd}} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{kh}}}{(r_{kh})!} \sum_{\sum_{h\in\mathbb{N}} r_{hk}\text{ odd}} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac12}]^{r_{hk}}}{(r_{hk})!}\Big\}$$

Let us now focus on the different parts of the above sum. We first have

$$\sum_{o_k\text{ even}} \frac{(iua\gamma^k)^{o_k}}{o_k!} = \sum_{l=0}^{\infty} \frac{(iua\gamma^k)^{2l}}{(2l)!} = \cos(ua\gamma^k).$$

Moreover

$$
\begin{aligned}
&\sum_{\sum_{h\in\mathbb{N}} r_{kh}\ \text{even}} \prod_{h=1}^{N} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \\
&= \sum_{r_{k1},\cdots,r_{kN-1}\geq 0} \prod_{h=1}^{N-1} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \Big\{ \sum_{l=0}^{\infty} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}]^{2l}}{(2l)!} 1_{2\mathbb{N}}(r_{k1}+\ldots+r_{kN-1}) \\
&\quad + \sum_{l=0}^{\infty} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}]^{2l+1}}{(2l+1)!} 1_{2\mathbb{N}+1}(r_{k1}+\ldots+r_{kN-1})\Big\} \\
&= \sum_{r_{k1},\cdots,r_{kN-1}\geq 0} \prod_{h=1}^{N-1} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \Big\{ \cos[(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}] 1_{2\mathbb{N}}(r_{k1}+\ldots+r_{kN-1}) \\
&\quad + i\sin[(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}] 1_{2\mathbb{N}+1}(r_{k1}+\ldots+r_{kN-1})\Big\} \\
&= \sum_{r_{k1},\cdots,r_{kN-1}\geq 0} \prod_{h=1}^{N-1} \frac{(i\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{\frac{|k-h|}{2}})^{r_{kh}}}{(r_{kh})!} \\
&\quad \Big\{\frac{1}{2}\Big(\exp\big[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}\big] + (-1)^{r_{k1}+\ldots+r_{kN-1}} \exp\big[-i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}\big]\Big)\Big\} \\
&= \Big\{\frac{1}{2}\Big(\exp\big[i(\frac{ub}{4}\gamma^k)^{\frac{1}{2}}\sum_{h=1}^{N}(\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big] + \exp\big[-i(\frac{ub}{4}\gamma^k)^{\frac{1}{2}}\sum_{h=1}^{N}(\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big]\Big)\Big\} \\
&= \cos\big[(\frac{ub}{4}\gamma^k)^{\frac{1}{2}}\sum_{h=1}^{N}(\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big].
\end{aligned}
$$

Taking the limit as $N\to\infty$ we get

$$\cos\big[(\frac{ub}{4}\gamma^k)^{\frac{1}{2}}\sum_{h=1}^{\infty}(\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big].$$

Now let $\alpha = (\frac{\kappa}{2})^{\frac{1}{2}}$ and let us further calculate $\sum_{h=1}^{\infty}\alpha^{|k-h|}$. We have

$$
\begin{aligned}
\sum_{h=1}^{\infty}\alpha^{|k-h|} &= \sum_{h=1}^{k}\alpha^{k-h} + \sum_{h=k+1}^{\infty}\alpha^{h-k} = \sum_{p=0}^{k-1}\alpha^p + \alpha\sum_{p=0}^{\infty}\alpha^p \\
&= \frac{\alpha^k-1}{1-\alpha} + \frac{\alpha}{1-\alpha} = \frac{1+\alpha}{1-\alpha} - \alpha^k\frac{1}{1-\alpha}.
\end{aligned}
$$

Substituting this into the equation above gives

$$
\begin{aligned}
\cos\big[(\frac{ub}{4}\gamma^k)^{\frac{1}{2}}\sum_{h=1}^{\infty}(\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big] &= \cos\big[(\frac{ub}{4}\gamma^k)^{\frac{1}{2}}\Big\{\frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{\kappa}{2})^{\frac{k}{2}}\frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\Big\}\big] \\
&= \cos\big[\frac{\sqrt{ub}}{2}\Big\{(\gamma)^{\frac{k}{2}}\frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k\frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\Big\}\big].
\end{aligned}
$$

On the other hand, we have, again for $N$ chosen large enough,

$$\begin{aligned}
&\sum_{\sum_{h\in\mathbb{N}} r_{kh}\ \mathrm{odd}} \prod_{h=1}^{N} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \\
&\quad= \sum_{r_{k1},\cdots,r_{kN-1}\ge 0} \prod_{h=1}^{N-1} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \\
&\quad\quad \Big\{ \sum_{l=0}^{\infty} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}]^{2l+1}}{(2l+1)!} 1_{2\mathbb{N}}(r_{k1}+\ldots+r_{kN-1}) \\
&\quad\quad + \sum_{l=0}^{\infty} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}]^{2l}}{(2l)!} 1_{2\mathbb{N}+1}(r_{k1}+\ldots+r_{kN-1}) \Big\} \\
&\quad= \sum_{r_{k1},\cdots,r_{kN-1}\ge 0} \prod_{h=1}^{N-1} \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \\
&\quad\quad \Big\{ i\sin[(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}] 1_{2\mathbb{N}}(r_{k1}+\ldots+r_{kN-1}) \\
&\quad\quad + \cos[(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}] 1_{2\mathbb{N}+1}(r_{k1}+\ldots+r_{kN-1}) \Big\} \\
&\quad= \sum_{r_{k1},\cdots,r_{kN-1}\ge 0} \prod_{h=1}^{N-1} \frac{(i\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{\frac{|k-h|}{2}})^{r_{kh}}}{(r_{kh})!} \\
&\quad\quad \Big\{ \frac{1}{2}\Big( \exp\big[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}\big] - (-1)^{r_{k1}+\ldots+r_{kN-1}} \exp\big[-i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-N|})^{\frac{1}{2}}\big] \Big) \Big\} \\
&\quad= \Big\{ \frac{1}{2}\Big( \exp\big[i(\frac{ub}{4}\gamma^k)^{\frac{1}{2}} \sum_{h=1}^{N} (\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big] - \exp\big[-i(\frac{ub}{4}\gamma^k)^{\frac{1}{2}} \sum_{h=1}^{N} (\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big] \Big) \Big\} \\
&\quad= i\sin\big[(\frac{ub}{4}\gamma^k)^{\frac{1}{2}} \sum_{h=1}^{N} (\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big]
\end{aligned}$$

As $N \to \infty$, similar computations as before result in

$$\sin\big[(\frac{ub}{4}\gamma^k)^{\frac{1}{2}} \sum_{h=1}^{\infty} (\frac{\kappa}{2})^{\frac{|k-h|}{2}}\big] = \sin\big[\frac{\sqrt{ub}}{2}\big\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\big\}\big].$$

Using symmetry, we obtain

$$\begin{aligned}
I_1 \quad &= \cos(ua\gamma^k)\Big\{ \cos^2\big[\frac{\sqrt{ub}}{2}\big\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\big\}\big] \\
&\quad - \sin^2\big[\frac{\sqrt{ub}}{2}\big\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\big\}\big]\Big\} \\
&= \cos(ua\gamma^k)\cos\big[\sqrt{ub}\big\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\big\}\big].
\end{aligned}$$

To turn to $I_2$, we first calculate

$$\sum_{o_k\ \mathrm{odd}} \frac{(iua\gamma^k)^{o_k}}{o_k!} = \sum_{l=0}^{\infty} \frac{(iua\gamma^k)^{2l+1}}{(2l+1)!} = i\sin(ua\gamma^k).$$

Furthermore

$$
\begin{aligned}
I_2 \quad &= \sum_{o_k, \sum_{h\in\mathbb{N}} r_{kh}+r_{hk} \text{ odd}} \frac{(iua\gamma^k)^{o_k}}{o_k!} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \prod_{h=1}^N \frac{[i(\frac{u}{4b}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{hk}}}{(r_{hk})!} \\
&= \sum_{o_k \text{ odd}} \frac{(iua\gamma^k)^{o_k}}{o_k!} \Big\{ \sum_{\sum_{h\in\mathbb{N}} r_{kh} \text{ even}} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \sum_{\sum_{h\in\mathbb{N}} r_{hk} \text{ odd}} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{hk}}}{(r_{hk})!} \\
&\quad + \sum_{\sum_{h\in\mathbb{N}} r_{kh} \text{ odd}} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{kh}}}{(r_{kh})!} \sum_{\sum_{h\in\mathbb{N}} r_{hk} \text{ even}} \prod_{h=1}^N \frac{[i(\frac{ub}{4}\gamma^k(\frac{\kappa}{2})^{|k-h|})^{\frac{1}{2}}]^{r_{hk}}}{(r_{hk})!} \Big\} \\
&= i\sin(ua\gamma^k)\Big\{ 2i\cos\big[\frac{\sqrt{ub}}{2}\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\}\big] \\
&\quad \times \sin\big[\frac{\sqrt{ub}}{2}\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\}\big]\Big\} \\
&= -\sin(ua\gamma^k)\sin\big[\sqrt{ub}\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\}\big].
\end{aligned}
$$

The result follows. □

We finally combine the results of the preceding Lemmas and Proposition 5.14, to obtain a simpler formula for the Fourier transform of our occupation time functional.

**Proposition 5.17.** *For $u, a, b \in \mathbb{R}_+$ (for $u, a, b < 0$ we have to replace $u, a, b$ by $|u|, |a|, |b|$ in the following formula) we have*

$$
\begin{aligned}
&E(\exp(iu[aI_1(f) + bI_2(g)])) \\
&\quad = \cos(\frac{ub}{4}\frac{\gamma}{1-\gamma}) \cdot \prod_{k\geq 1} \cos\big[ua\gamma^k + \sqrt{ub}\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\}\big].
\end{aligned}
$$

**Proof:** The formulas follow by combining Proposition 5.14 with Lemmas 5.15, 5.16. □

5.3. **The existence of local time for the Wintner cases.** In this subsection we will investigate local times for $\mathcal{T}$. As usual, we shall employ a Fourier analytic criterion. For this, we start with the formulas given by Proposition 5.17. Following the idea of proof by Wintner for Bernoulli convolutions, we shall argue for the roots $(\frac{1}{2})^{\frac{1}{n}}, n \geq 1$. Let us first recall the formula from Corollary 5.6, on order to specify the numbers $a$ and $b$ arising in the linear combinations for which we calculated the Fourier transform. First of all, we choose $\beta = -\kappa$ in Corollary 5.6. With this choice, an inspection of the representation formula of Corollary 5.6 easily reveals that $a = \kappa^2, b = -\frac{\kappa^2}{2}$. Observe that the common factor $\kappa^2$ may be eliminated by just rescaling the argument $u$ of the Fourier transform. Assuming this done, we effectively have to deal with $a = 1, b = -\frac{1}{2}$. Hence the special case with which we have to work is the Fourier transform

$$
\phi(u) = \cos(\frac{u}{8}\frac{\gamma}{1-\gamma}) \cdot \prod_{k\geq 1} \cos\big[u\gamma^k + \sqrt{\frac{u}{2}}\{(\gamma)^{\frac{k}{2}} \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\}\big], \quad u \geq 0. \tag{5.23}
$$

We first simplify the arguments of the cos terms. We set $A := \frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}$, and note that for $v := A^{-2}u$ we get

$$
\begin{aligned}
(5.24)\qquad & u\gamma^k + \sqrt{\frac{u}{2}}\{(\gamma)^{\frac{k}{2}}\frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} - (\frac{1}{2})^k \frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}} \\
&= u\gamma^k + \sqrt{\frac{u}{2}\gamma^k}\; A\;(1 - \frac{(\frac{\kappa}{2})^{\frac{k}{2}}}{1+(\frac{\kappa}{2})^{\frac{1}{2}}}) \\
&= A^2[v\gamma^k + \sqrt{\frac{v}{2}\gamma^k}\;(1 - \frac{(\frac{\kappa}{2})^{\frac{k}{2}}}{1+(\frac{\kappa}{2})^{\frac{1}{2}}})].
\end{aligned}
$$

Let us abbreviate $c_k = 1 - \frac{(\frac{\kappa}{2})^{\frac{k}{2}}}{1+(\frac{\kappa}{2})^{\frac{1}{2}}}$, $k \geq 1$, and define

$$g_k(v) := A^2[v + \sqrt{\frac{v}{2}}c_k], \quad v \geq 0.$$

Assume from now on that $\gamma = 2^{-\frac{1}{m}}$ for some $m \geq 2$. Then we have to prove integrability of the squared Fourier transform (which we now write as a function of $v = \frac{u}{A^2}$)

$$
\begin{aligned}
(5.25)\qquad \phi^2(v) &= \prod_{k\geq 1}\cos^2(g_k(2^{-\frac{k}{m}}v)) \\
&= \prod_{q=0}^{m-1}\prod_{k\geq 1}\cos(g_k(2^{-k}2^{-\frac{q}{m}}v)) \\
&= \exp(\sum_{q=0}^{m-1}\sum_{k\geq 1}\ln\cos^2(g_k(2^{-k}2^{-\frac{q}{m}}v)).
\end{aligned}
$$

Let us further define

$$\psi(w) := \prod_{k\geq 1}\cos^2(g_k(2^{-k}w)) = \exp(\sum_{k\geq 1}\ln\cos^2(g_k(2^{-k}w)), \quad w \geq 0,$$

so that we get

$$\phi^2(v) = \prod_{q=0}^{m-1}\psi(2^{-\frac{q}{m}}v).$$

We concentrate the following discussion on $\psi$, the argument of which we continue writing $v$.

Let us now explain our argument leading to integrability of $\phi^2$. We shall first separate two parts of the infinite product: the first part consists of the factors with small index $k$, the second one of those with large $k$. For the second one, we shall use a very rough general argument. For the more subtle treatment of the first part, we shall roughly use Weyl's equidistribution theorem and variants thereof, see for instance Kuipers, Niederreiter [30].

To separate the two parts, for $v \geq 0$ let

$$L(v) = \inf\{k \geq 1 : g_k(2^{-k}v) \leq \frac{\pi}{4}\}.$$

It is easy to see that for all $v \geq 0$ $(\sqrt{2^{-k}v}c_k)_{k\geq 1}$ is a decreasing sequence converging to 0, and therefore this also holds for $(g_k(2^{-k}v))_{k\geq 1}$. For this reason, $L(v)$ is well defined. Now we simply estimate the second part by using boundedness of cos. So

$$\prod_{k\geq L(v)} \cos^2(g_k(2^{-k}v)) \leq 1. \tag{5.26}$$

Let us estimate $L(v)$ for $v \geq 0$. For $k \geq 1$

$$g_k(2^{-k}v)) \geq \frac{\pi}{4} > g_{k+1}(2^{-k-1}v)$$

means

$$A^2(2^{-k}v + \sqrt{2^{-k}\frac{v}{2}c_k}) \geq \frac{\pi}{4} > A^2(2^{-k-1}v + \sqrt{2^{-k-1}\frac{v}{2}c_k}).$$

Since for $c > 0$ the inverse of the strictly increasing function $v \mapsto A^2(v + \sqrt{\frac{v}{2}}c)$ is given by $s \mapsto (\sqrt{\frac{s}{A^2} + \frac{c^2}{2}} - \frac{c}{\sqrt{2}})^2$, the above inequalities are equivalent to

$$2^{-k}v \geq (\sqrt{\frac{\pi}{4A^2} + \frac{c_k^2}{2}} - \frac{c_k}{\sqrt{2}})^2 > 2^{-k-1}v,$$

in other words

$$2^k(\sqrt{\frac{\pi}{4A^2} + \frac{c_k^2}{2}} - \frac{c_k}{\sqrt{2}})^2 \leq v < 2^{k+1}(\sqrt{\frac{\pi}{4A^2} + \frac{c_k^2}{2}} - \frac{c_k}{\sqrt{2}})^2.$$

And consequently by definition

$$2^{L(v)}(\sqrt{\frac{\pi}{4A^2} + \frac{c_{L(v)}^2}{2}} - \frac{c_{L(v)}}{\sqrt{2}})^2 \leq v < 2^{L(v)+1}(\sqrt{\frac{\pi}{4A^2} + \frac{c_{L(v)}^2}{2}} - \frac{c_{L(v)}}{\sqrt{2}})^2. \tag{5.27}$$

To get an estimate independent on the $c_{L(v)}$-terms in the preceding inequalities, note that

$$b := \frac{1}{1 + 2^{-\frac{2m-1}{2m}}} \leq c_{L(v)} \leq 1$$

and that $c \mapsto (\sqrt{\frac{s}{A^2} + \frac{c^2}{2}} - \frac{c}{\sqrt{2}})^2$ is decreasing on $\mathbb{R}_+$. Therefore, taking logarithms in (5.27) yields the estimate

$$L(v)\ln(2) + 2\ln(\sqrt{\frac{\pi}{4A^2} + \frac{1}{2}} - \frac{1}{\sqrt{2}}) \leq \ln(v) < (L(v) + 1)\ln(2) + 2\ln(\sqrt{\frac{\pi}{4A^2} + \frac{b^2}{2}} - \frac{b}{\sqrt{2}}).$$

Recalling that $A = \frac{1+2^{-\frac{2m-1}{2m}}}{1-2^{-\frac{2m-1}{2m}}}$, and defining

$$c_1 := 2\ln(\sqrt{\frac{\pi}{4A^2} + \frac{b^2}{2}} - \frac{b}{\sqrt{2}}), \quad c_2 := 2\ln(\sqrt{\frac{\pi}{4A^2} + \frac{1}{2}} - \frac{1}{\sqrt{2}}),$$

we obtain the inequalities

$$\frac{\ln(v) - c_1}{\ln(2)} < L(v) \leq \frac{\ln(v) - c_2}{\ln(2)}. \tag{5.28}$$

We now enter into the estimation of the part of the infinite product of our Fourier transform associated with factors of small indices, namely

$$\rho(v) := \prod_{1\leq k<L(v)} \cos^2(g_k(2^{-k}v)) = \exp(\sum_{1\leq k<L(v)} \ln\cos^2 g_k(2^{-k}v)). \tag{5.29}$$

For $v \in [2^{p-1}, 2^p]$ we have $(p-1)\ln(2) \le \ln(v) \le p\ln(2)$. Therefore, according to (5.28), the summation in $k$ in (5.29) has to extend from 1 to $p - \frac{c_2}{\ln(2)}$. If we use the transformation $v := 2^p w$ for $w \in [\frac{1}{2}, 1]$, we have

$$g_k(2^{-k}v) = g_k(2^{p-k}w) = g_{p-l}(2^l w), \quad l = p-k,$$

with $-\frac{c_2}{\ln(2)} \le l < p - \frac{c_2}{\ln(2)}$. Since $\ln\cos^2$ is periodic with period $\pi$, we can define

$$x_l^p(w) := g_{(p-l)\vee 1}(2^l w) \quad (\mathrm{mod}\ \pi)), \quad l \ge 1, p \in \mathbb{N},$$

to identify

$$\sum_{-\frac{c_2}{\ln(2)} \le l < p - \frac{c_2}{\ln(2)}} \ln\cos^2(g_{p-l}(2^l w)) = \sum_{-\frac{c_2}{\ln(2)} \le l < p - \frac{c_2}{\ln(2)}} \ln\cos^2(x_l^p(w)).$$

To discuss the asymptotic behaviour of these sums as $p \to \infty$, we will employ equidistribution of the averaged sums of the $x_l^p(w), l \ge 1$. Since these averages do not depend on finite fixed numbers of summands, we shall from now on study the asymptotic behaviour of the (averaged) sums

$$\sum_{1 \le l \le p} \ln\cos^2(x_l^p(w)), \quad \text{and associated measures} \quad \mu_p = \frac{1}{p}\sum_{l=1}^p \delta_{x_l^p(w)}.$$

Equidistribution is derived by means of a variant of Weyl's theorem, called Koksma's general metric theorem (see Kuipers, Niederreiter [30], p. 34).

**Lemma 5.18.** *For each $p \in \mathbb{N}$, the sequence $(x_l^p(w))_{l \ge 1}$ is equidistributed on $[0, \pi]$, i.e. $(\mu^p)_{p\in\mathbb{N}}$ converges to equidistribution on $[0, \pi]$ for (Lebesgue) a.a. $w \in [\frac{1}{2}, 1]$.*

**Proof:** Let for $w \in [\frac{1}{2}, 1], p, l \in \mathbb{N}$

$$u_l^p(w) := g_{(p-l)\vee 1}(2^l w).$$

We have to show that for each $p \in \mathbb{N}$, the sequence of functions $(u_l^p)_{l \ge 1}$ satisfies the criteria of Koksma's general metric theorem (see Kuipers, Niederreiter [30], p. 34). They are formulated in terms of the (existing) derivatives of the functions. In fact, we have

$$(u_l^p)'(w) = A^2(2^l + 2^{\frac{l-1}{2}} \cdot \frac{1}{2}\frac{1}{\sqrt{w}} c_{(p-l)\vee 1}).$$

We claim that even the second part of the preceding derivative is increasing in $l$ for fixed $w$. To prove this, we have to show that for $l > k$

$$2^{\frac{l-1}{2}} c_{(p-l)\vee 1} > 2^{\frac{k-1}{2}} c_{(p-k)\vee 1},$$

in other words that

$$2^{\frac{l-k}{2}} > \frac{c_{(p-k)\vee 1}}{c_{(p-l)\vee 1}}. \tag{5.30}$$

Let $x := (\frac{\kappa}{2})^{\frac{1}{2}}$. We have to show that

$$\frac{1 + x - x^{p-k}}{1 + x - x^{p-l}} < 2^{\frac{l-k}{2}}.$$

We omit the simpler case where $l = p$. We can write

$$\frac{1 + x - x^{p-k}}{1 + x - x^{p-l}} = 1 + \frac{x^{p-l}(1 - x^{l-k})}{1 + x - x^{p-l}}.$$

But, by monotonicity, $\frac{x^{p-l}}{1+x-x^{p-l}} \le x$, and therefore it is sufficient to prove

$$1 + x - x^{l-k+1} < 2^{\frac{l-k}{2}}.$$

But an analysis of the function $h : x \mapsto 1 + x - x^{l-k+1}$ yields that it has a global maximum at $x_0 = (\frac{1}{l-k+1})^{\frac{1}{l-k}}$ given by the value $h(x_0) = x_0$, and clearly $x_0 < 2^{\frac{l-k}{2}}$. So we derive that for $l > k$ we have

$$(u_l^p)'(w) - (u_k^p)'(w) \geq A^2(2^l - 2^k) \geq 2A^2 > 0, \tag{5.31}$$

while monotonicity of $(u_l^p)' - (u_k^p)'$ follows from the monotonicity in $l$ of the second parts of the derivatives, and the monotonicity (in $w$) of $w \mapsto \frac{1}{\sqrt{w}}$. Hence the sequences $(u_l^p)_{l\geq 1}$ satisfy Koksma's conditions for each $p \in \mathbb{N}$, and our claim follow from Koksma's general metric theorem.□

In our study of equidistribution we can go one step ahead, and use the essential steps of the proof of Lemma 5.18 to give an estimate for the exponential deviation from functional empirical averages and the integral of the function with respect to the limiting Lebesgue measure on $[0, \pi]$. More precisely, we will estimate for a function $h : [0, \pi] \to \mathbb{R}$ of bounded variation $V(h)$ the asymptotic behavior in $p$ of the integral in $w$ of

$$\begin{aligned} &\exp(p[\sum_{q=0}^{m-1} \frac{1}{p} \sum_{1\leq l\leq p} h(g_{(p-l)\vee 1}(2^l 2^{-\frac{q}{m}} w)) - m\frac{1}{\pi}\int_0^\pi h(x)dx]) \\ &\exp(p[\sum_{q=0}^{m-1} \frac{1}{p} \sum_{1\leq l\leq p} h(x_l^p(2^{-\frac{q}{m}} w)) - m\frac{1}{\pi}\int_0^\pi h(x)dx]). \end{aligned} \tag{5.32}$$

This in fact will turn out to turn out to depend essentially on estimating the *discrepancy* of the $x_l^p(w), 1 \leq l \leq p$. The discrepancy of $x_l^p(w), 1 \leq l \leq p$, is given by

$$D_p := \sup_{0\leq a<b\leq\pi} |\frac{1}{p} \sum_{1\leq l\leq p} 1_{[a,b[}(\delta_{x_l^p(w)}) - (b-a)|.$$

According to Erdös-Turan (see Kuipers, Niederreiter [30], p. 112), we have for any $q \in \mathbb{N}$

$$D_p \leq \frac{6}{q+1} + \frac{4}{\pi}\sum_{k=1}^{q}(\frac{1}{k} - \frac{1}{q+1})|\frac{1}{p}\sum_{l=1}^{p}\exp(2\pi i x_l^p(w)k)|. \tag{5.33}$$

Not to overload the notation, let us state the following Lemma for the simple case of just one summand in the exponential of (5.32).

**Lemma 5.19.** *Let $p, q \in \mathbb{N}$, and $h : [0, \pi] \to \mathbb{R}$ be a function of bounded variation $V(h)$. Then there exists a constant $c > 0$ such that*

$$\int_{\frac{1}{2}}^{1} \exp(p[\frac{1}{p}\sum_{1\leq l\leq p} h(x_l^p(w)) - \frac{1}{\pi}\int_0^\pi h(x)dx])dw \leq \exp(cV(h)p^{\frac{1}{2}}\ln(p)). \tag{5.34}$$

**Proof:** In the following constants $c$ appearing in our inequalities are universal for the essential parameters of the expressions and may change their values from line to line. We first employ the inequality of Koksma (Kuipers, Niederreiter [30], p. 143, and p. 91) and then the inequality of Erdös-Turan (see Kuipers, Niederreiter [30], p. 112), to obtain for $p, q \in \mathbb{N}$

$$\begin{aligned}(5.35)\qquad &\int_{\frac{1}{2}}^{1} \exp(p\big[\frac{1}{p}\sum_{1\le l\le p} h(x_l^p(w)) - \frac{1}{\pi}\int_0^{\pi} h(x)dx\big])dw \\ &\le \int_{\frac{1}{2}}^{1} \exp(V(h)pD_p)dw \\ &\le \exp(6V(h)\frac{p}{q}) \int_{\frac{1}{2}}^{1} \exp(\frac{4}{\pi}V(h)p[\sum_{k=1}^{q}\frac{1}{k}|\frac{1}{p}\sum_{l=1}^{p}\exp(2\pi i x_l^p(w)k)|])dw.\end{aligned}$$

We continue estimating the integral in the last line of (5.35), starting with even moments of the exponent. In fact, for $m \in \mathbb{N}$, using the notation of Lemma 5.18 and $\sum_{k=1}^{q}\frac{1}{k} \le \ln(q)$, we have

$$\begin{aligned}(5.36)\qquad &(\sum_{k=1}^{q}\frac{1}{k}\frac{1}{p}|\sum_{1\le l\le p}\exp(2\pi i x_l^p(w)k))|)^{2m} = (\sum_{k=1}^{q}\frac{1}{k}\frac{1}{p}|\sum_{1\le l\le p}\exp(2\pi i u_l^p(w)k))|)^{2m} \\ &\le \big(\ln(q)\sum_{k=1}^{q}\frac{1}{k}\frac{1}{p^2}[p + \sum_{1\le h<l\le p}\exp(2\pi i(u_l^p(w)-u_h^p(w))k)]\big)^m \\ &= (\ln(q)^m p^{-2m}\big(\sum_{r=0}^{m}\binom{m}{r}(p\ln(q))^r(\sum_{k=1}^{q}\frac{1}{k}\sum_{1\le h<l\le p}\exp(2\pi i(u_l^p(w)-u_h^p(w))k))^{m-r}\big)\end{aligned}$$

We continue estimating the last factor in (5.36), employing again elements of the argument leading to the proof of Lemma 5.18. In fact, we have

$$\begin{aligned}&\sum_{k=1}^{q}\frac{1}{k}\sum_{1\le h<l\le p}\exp(2\pi i(u_l^p(w)-u_h^p(w))k))^{m-r} \\ &\quad = \sum_{1\le k_1,\cdots,k_{m-r}\le q}\frac{1}{k_1\cdots k_{m-r}}\frac{1}{p^{2(m-r)}}\sum_{1\le h_s<j_s\le p}\exp(2\pi i\sum_{s=1}^{m-r}(u_{j_s}^p(w)-u_{h_s}^p(w))k_s)).\end{aligned}$$

Recall from the proof of Lemma 5.18 the monotonicity properties of the functions $u_j^p - u_h^p$ for $h < j$ to define the monotone functions

$$f_{h_1,j_1,\cdots,h_{m-r},j_{m-r}}(z) = \sum_{s=1}^{m-r}(u_{j_s}^p(w)-u_{h_s}^p(w))k_s), \quad 1\le h_s<j_s\le p, 1\le s\le m-r.$$

We next integrate (5.36) in $w$ over the interval $[\frac{1}{2},1]$, change variables w.r.t. the monotone functions $f_{h_1,j_1,\cdots,h_{m-r},j_{m-r}}$ and use the fact that the integral of $\exp(2\pi i z)$ over bounded intervals is bounded, to estimate (5.36) by

$$\begin{aligned}&\sum_{1\le k_1,\cdots,k_{m-r}\le q}\frac{1}{k_1\cdots k_{m-r}} \\ &\quad\sum_{1\le h_s<j_s\le p}\int\frac{1}{\sum_{s=1}^{m-r}((u_{j_s}^p)'-u_{h_s}^p)')(f^{-1}_{h_1,j_1,\cdots,h_{m-r},j_{m-r}}(z))k_s}\exp(2\pi i z)dz.\end{aligned}$$

Now we use that the integral of $\exp(2\pi i z)$ in $z$ is bounded, and that (see the proof of Lemma 5.18) for $1\le h_s<j_s\le p, 1\le s\le m$, we have

$$\sum_{s=1}^{m-r}((u_{j_s}^p)'-(u_{h_s}^p)')(f^{-1}_{h_1,j_1,\cdots,h_{m-r},j_{m-r}}(z))k_s \ge A^2\sum_{s=1}^{m-r}(2^{j_s}-2^{h_s}) = A^2\sum_{s=1}^{m-r}2^{h_s}(2^{j_s-h_s}-1).$$

It is easy to see that

$$\sum_{1\le h_s<j_s\le p} \frac{1}{\sum_{s=1}^{m-r} 2^{h_s}(2^{j_s-h_s}-1)}$$

is bounded in $p$. Hence, using again that $\sum_{1\le k\le q}\frac{1}{k}\le \ln(q)$, we may bound the $w$-integral of (5.36) by

$$(5.37)\quad \int_{\frac{1}{2}}^{1}(\ln(q)^m p^{-2m}\Big(\sum_{r=0}^{m}\binom{m}{r}(p\ln(q))^r(\sum_{k=1}^{q}\frac{1}{k}\sum_{1\le h<l\le p}\exp(2\pi i(u_l^p(w)-u_h^p(w))k))^{m-r}\Big)dw$$

$$\le c(p\ln(q)^{2m}\sum_{r=0}^{m}\binom{m}{r}p^r = c(p\ln(q)^{2m}(p+1)^m\le c(\frac{\ln(q)}{p^{\frac{1}{2}}})^{2m}.$$

We finally estimate odd moments by first employing Cauchy-Schwarz' inequality, with an analogous result. So we finally have, continuing (5.35)

$$\exp(6V(h)\frac{p}{q})\ \int_{\frac{1}{2}}^{1}\exp(\frac{4}{\pi}V(h)p[\sum_{k=1}^{q}\frac{1}{k}|\frac{1}{p}\sum_{l=1}^{p}\exp(2\pi i x_l^p(w)k)|])dw$$

$$\le \exp(6V(h)\frac{p}{q})\ \sum_{m=0}^{\infty}\frac{1}{(2m)!}(V(h)cp)^m(\frac{\ln(q)}{p^{\frac{1}{2}}})^m = \exp(cV(h)[\frac{p}{q}+p^{\frac{1}{2}}\ln(q)]).$$

Now take $q=p$ to arrive at the claimed inequality. □

We can finally state our main result.

**Theorem 5.20.** *Let $\gamma = 2^{-\frac{1}{m}}$ for some $m\in\mathbb{N}$. Then for $\alpha < m-\frac{1}{2}$ we have $u\mapsto |u|^{2\alpha}\phi^2(u)$ is integrable on $\mathbb{R}$. Therefore $y\mapsto H(\xi,y)-\kappa y S(\xi,0)$ possesses a square integrable occupation density which possesses $m-2$ continuous square integrable derivatives which are Hölder continuous of order $\rho$ for any $\rho\in]0,1[$.*

**Proof:** We have to estimate the integral of $v\mapsto v^{2\alpha}\phi^2(v)$ on $\mathbb{R}_+$, remarking that the evenness of cos trivially extends this to all of $\mathbb{R}$. We obtain, with a constant $C$ standing for the contribution of large indices in the infinite product describing the Fourier transform, and a constant $p_0$ marking the domain for which small index contributions become relevant

$$(5.38)\quad \int_0^\infty v^{2\alpha}\phi^2(v)dv$$

$$= C+\sum_{p=p_0}^{\infty}\int_{2^{p-1}}^{2^p} v^{2\alpha}\phi^2(v)dv$$

$$= C+\sum_{p=p_0}^{\infty}\int_{\frac{1}{2}}^{1} 2^{p(1+2\alpha)}w^{2\alpha}[\exp(p\sum_{q=0}^{m-1}[\frac{1}{p}\sum_{1\le l\le p}\ln\cos^2(x_l^p(2^{-\frac{q}{m}}w))]])dw$$

$$= C+\sum_{p=p_0}^{\infty}\int_{\frac{1}{2}}^{1} w^{2\alpha}\exp(p\sum_{q=0}^{m-1}[\frac{1}{p}\sum_{1\le l\le p}\ln\cos^2(x_l^p(2^{-\frac{q}{m}}w))] - mp\frac{1}{\pi}\int_0^\pi \ln\cos^2(x)dx)$$

$$\exp(p\ln(2)[1+2\alpha)+mp\frac{1}{\pi}\int_0^\pi\ln\cos^2(x)dx])dw.$$

The Lebesgue integral of the $\pi$-periodic $\ln\cos^2$ is well known. We have

$$(5.39)\quad \frac{1}{\pi}\int_0^\pi \ln\cos^2(x)dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\ln\cos(x)dx = -2\ln(2).$$

Hence the last line of (5.38) simply becomes

$$\exp(p\ln(2)[1+2\alpha)-2m]). \tag{5.40}$$

We have to be a little careful with estimating the contributions of the first line on the rhs of (5.38) by means of Lemma 5.19, since $\ln\cos^2$ is not of bounded variation on $[0,\pi]$. Given $\epsilon>0$, choose a function of bounded variation $h:[0,\pi]\to\mathbb{R}$ such that $\frac{1}{\pi}\int|h(x)-\ln\cos^2(x)|dx<\epsilon$. Using $h$ instead of $\ln\cos^2$ in (5.38) forces us to modify (5.40) to

$$\exp(p\ln(2)[1+2\alpha)-2(m-\epsilon)]). \tag{5.41}$$

We apply Lemma 5.19 with $h$ to obtain an estimate for the contributions in the first line on the rhs of (5.38) which is given by

$$\exp(cV(h)p^{\frac{1}{2}}\ln(p)). \tag{5.42}$$

But the product of (5.41) and (5.42) remains summable, since $p^{\frac{1}{2}}\ln(p)$ is dominated by $p$ as $p\to\infty$. This proves the claim for $\alpha<m-\frac{1}{2}-\epsilon$. But $\epsilon$ is arbitrary. The regularity statement for the occupation density follows from a standard embedding theorem of Fourier analysis, see Bahouri [2], p. 44. □

Let us finally address the problem of occupation densities for the non drifted Takagi function. Consulting Corollary 5.6, we have too choose $\beta=1$ there, and have to face the Fourier transform of

$$\begin{aligned}\mathcal{T}(y)&-\mathcal{T}(0)\\ &=\frac{1+\kappa}{2}S(\xi,0)-\frac{\kappa^2}{2}\frac{\gamma}{1-\gamma}\\ &+\sum_{n=1}^{\infty}X_n\left[(1+\kappa)S(\xi,0)(\frac{1}{2})^{n+1}+\kappa^2\gamma^n(1-\sum_{\ell=1}^{\infty}(\frac{1}{2})^{\ell+1}X_{n+\ell})\right].\end{aligned} \tag{5.43}$$

Its square, the integrability properties of which we have to assess, will not contain the terms $\frac{\kappa}{2}S(\xi,0)-\frac{\kappa^2}{2}\frac{\gamma}{1-\gamma}$. Hence we have to deal with the Fourier transform of

$$\begin{aligned}&\sum_{n=1}^{\infty}X_n\left[(1+\kappa)S(\xi,0)(\frac{1}{2})^{n+1}+\kappa^2\gamma^n(1-\sum_{\ell=1}^{\infty}(\frac{1}{2})^{\ell+1}X_{n+\ell})\right]\\ &\quad=\sum_{n=1}^{\infty}\gamma^nX_n\left[\kappa^2+\frac{(1+\kappa)\kappa^n}{2}S(\xi,0)\right]-\frac{\kappa^2}{2}\sum_{n<m}\kappa^{-n}2^{-m}X_nX_m\\ &\quad=I_1(f_n)+I_2(\tilde{g}),\end{aligned} \tag{5.44}$$

with

$$f_n=\gamma^n[\kappa^2+\frac{(1+\kappa)\kappa^n}{2}S(\xi,0)],\quad \tilde{g}=-\frac{\kappa^2}{2}g,$$

and $g$ as chosen above. Scaling $u$ with the common factor $\kappa^2$ as above (and still denoting the re-scaled variable by $u$) the results of Subsection 5.1 will produce A Fourier transform which for $u\geq 0$ has the form

(5.45)
$$\phi(u)=\cos(\frac{u}{8}\frac{\gamma}{1-\gamma})\cdot\prod_{k\geq 1}\cos\left[u\gamma^k(1+\frac{(1+\kappa)\kappa^{k-2}}{2}S(\xi,0))+\sqrt{\frac{u}{2}}\{(\gamma)^{\frac{k}{2}}\frac{1+(\frac{\kappa}{2})^{\frac{1}{2}}}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}-(\frac{1}{2})^k\frac{1}{1-(\frac{\kappa}{2})^{\frac{1}{2}}}\}\right].$$

(5.23) differs from (5.45) only in the factor of $\gamma^k$ which is 1 in the first case, and $1+\frac{(1+\kappa)\kappa^{k-2}}{2}S(\xi,0)$ in the second case, an asymptotically vanishing small perturbation of 1. If we choose again $\gamma = 2^{-\frac{1}{m}}$ for $m \geq 2$, we know that $\kappa \leq \frac{1}{\sqrt{2}}$, and so $|S(\xi, 0)|$ is bounded by $\frac{1}{\sqrt{2}-1}$, and therefore $1 + \frac{(1+\kappa)\kappa^{k-2}}{2} S(\xi, 0) \geq 0$ for all $k \geq 1$. We can adopt a small modification of the arguments of the preceding and the present subsections, to show that $|u|^{2\alpha}\phi^2(u)$ is integrable over $\mathbb{R}$ provided $\alpha < m - \frac{1}{2}$ as before. The arguments are just somewhat more lengthy. Since this paper is already rather long, we refrain from giving further details here, and just state the conclusion.

**Theorem 5.21.** *Let $\gamma = 2^{-\frac{1}{m}}$ for some $m \in \mathbb{N}$. Then for $\alpha < m - \frac{1}{2}$ we have $u \mapsto |u|^{2\alpha}\phi^2(u)$ is integrable on $\mathbb{R}$. Therefore $y \mapsto \mathcal{T}(y)$ possesses a square integrable occupation density which possesses $m - 2$ continuous square integrable derivatives which are Hölder continuous of order $\rho$ for any $\rho \in ]0, 1[$.*

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