# ISOMETRIES OF LATTICES AND AUTOMORPHISMS OF K3 SURFACES

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ABSTRACT. The aim of this paper is to give necessary and sufficient conditions for an integral polynomial to be the characteristic polynomial of a semi-simple isometry of some even unimodular lattice of given signature. This result has applications to automorphisms of K3 surfaces; in particular, we show that every Salem number of degree 4, 6, 8, 12, 14 or 16 is the dynamical degree of an automorphism of a non-projective K3 surface.

#### 0. Introduction

Let  $r, s \ge 0$  be integers such that  $r \equiv s \pmod{8}$ ; this congruence condition is equivalent to the existence of an even, unimodular lattice with signature (r, s). When  $r, s \ge 1$ , such a lattice is unique up to isomorphism (see for instance [S 77], chap. V); we denote it by  $\Lambda_{r,s}$ . In [GM 02], Gross and McMullen raise the following question (see [GM 02], Question 1.1) :

Question. What are the possibilities for the characteristic polynomial F(X) = det(X - t) of an isometry  $t \in SO(\Lambda_{r,s})$ ?

The aim of this paper is to answer this question for semi-simple isometries.

The condition  $t \in SO(\Lambda_{r,s})$  implies that  $F(X) = X^{\deg(F)}F(X^{-1})$ , hence F is a symmetric polynomial (cf. §2). Let  $2n = \deg(F)$ , and let 2m(F) be the number of roots of F outside the unit circle. As shown in [GM 02], we have the further necessary conditions :

(C 1) |F(1)|, |F(-1)| and  $(-1)^n F(1)F(-1)$  are squares.

(C 2)  $r \ge m(F)$ ,  $s \ge m(F)$ , and if moreover  $F(1)F(-1) \ne 0$ , then  $m(F) \equiv r \equiv s \pmod{2}$ .

Gross and McMullen prove that if  $F \in \mathbb{Z}[X]$  is an irreducible, symmetric and monic polynomial satisfying condition (C 2) and such that |F(1)F(-1)| = 1, then there exists  $t \in SO(\Lambda_{r,s})$  with characteristic polynomial F (see [GM 02], Theorem 1.2). They speculate that conditions (C 1) and (C 2) are sufficient for a monic *irreducible* polynomial to be realized as the characteristic polynomial of an isometry of  $\Lambda_{r,s}$ ; this is proved in [BT 20], Theorem A. More generally, Theorem A of [BT 20] implies that if a monic, irreducible and symmetric polynomial F satisfies conditions (C 1) and (C 2), then *there exists* an even,

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unimodular lattice of signature (r, s) having an isometry with characteristic polynomial F. This is also the point of view of the present paper - we treat the definite and indefinite cases simultaneously.

On the other hand, Gross and McMullen show that these conditions do not suffice in the case of *reducible* polynomials (see [GM 02], Proposition 5.2); several other examples are given in [B 20]. Another example is the following :

**Example 1.** Let  $F(X) = (X^4 - X^2 + 1)(X - 1)^4$ , and let (r, s) = (8, 0); conditions (C 1) and (C 2) hold, but there does not exist any positive definite, even, unimodular lattice of rank 8 having an isometry with characteristic polynomial F; note that this amounts to saying that the lattice  $E_8$  does not have any isometry with characteristic polynomial F.

All these examples are *counter-examples to a Hasse principle*. Indeed, the first result of the present paper is that conditions (C 1) and (C 2) are sufficient *locally*. If p is a prime number, we say that a  $\mathbf{Z}_p$ -lattice (L, q) is even if  $q(x, x) \in 2\mathbf{Z}_p$  for all  $x \in L$ ; note that if  $p \neq 2$ , then every lattice is even, since 2 is a unit in  $\mathbf{Z}_p$ . The following is proved in Theorem 7.2 and Proposition 7.1:

**Theorem 1.** Let  $F \in \mathbb{Z}[X]$  be a monic, symmetric polynomial of even degree.

(a) Condition (C 1) holds if and only if for all prime numbers p, there exists an even, unimodular  $\mathbf{Z}_p$ -lattice having a semi-simple isometry with characteristic polynomial F.

(b) The group  $SO_{r,s}(\mathbf{R})$  contains a semi-simple element having characteristic polynomial F if and only if condition (C 2) holds.

The next result is a necessary and sufficient condition for the local-global principle to hold. We start by defining an obstruction group (see §11). Let us write  $F(X) = F_1(X)(X-1)^{n_+}(X+1)^{n_-}$  and assume that  $n_+ \neq 2$ ,  $n_- \neq 2$ ; in this case, the group only depends on the polynomial F; we denote it by  $III_F$ .

Let us now assume that condition (C 2) holds, and let  $t \in SO_{r,s}(\mathbf{R})$  be a semi-simple isometry with characteristic polynomial F; such an isometry exists by part (b) of the above theorem (or Proposition 7.1). Assume moreover that condition (C 1) also holds. In §14, we define a homomorphism

$$\epsilon_t : \coprod_F \to \mathbf{Z}/2\mathbf{Z}$$

and prove the following (see Theorem 14.1):

**Theorem 2.** The isometry  $t \in SO_{r,s}(\mathbf{R})$  preserves an even, unimodular lattice if and only if  $\epsilon_t = 0$ .

**Example 2.** Let  $F(X) = (X^4 - X^2 + 1)(X - 1)^4$ ; we have  $III_F \simeq \mathbb{Z}/2\mathbb{Z}$ . If  $t_1 \in SO_{4,4}(\mathbb{R})$ , we have  $\epsilon_{t_1} = 0$ , and if  $t_2 \in SO_{8,0}(\mathbb{R})$ , then  $\epsilon_{t_2} \neq 0$ . Hence  $\Lambda_{4,4}$  has a semi-simple isometry with characteristic polynomial F, but the lattice  $E_8$  does not have such an isometry.

**Corollary 1.** Let  $G \in \mathbb{Z}[X]$  be a monic, irreducible, symmetric polynomial such that |G(1)| is not a square, and suppose that |G(-1)| is a square. Let

 $m \ge 4$  be an even integer, and set

$$F(X) = G(X)(X-1)^m.$$

Assume that condition (C 2) holds for F. Then every semi-simple isometry  $t \in SO_{r,s}(\mathbf{R})$  with characteristic polynomial F preserves an even, unimodular lattice.

Indeed, Condition (C 1) holds for F since |G(-1)| is a square, and one can check that  $\coprod_F = 0$ ; therefore Theorem 2 implies the corollary.

For polynomials F without linear factors, Theorem 2 is proved in [B 20], Theorem 27.4. However, it turns out that including linear factors is very useful in the applications to K3 surfaces, which we now describe.

The second part of the paper gives applications to automorphisms of K3 surfaces, inspired by a series of papers of McMullen (see [McM 02], [McM 11], [McM 16]).

Recall that a monic, irreducible, symmetric polynomial  $S \in \mathbb{Z}[X]$  of degree  $\geq 4$  is a *Salem polynomial* if S has exactly two roots outside the unit circle, both positive real numbers. A real number is called a *Salem number* if it is the unique real root > 1 of a Salem polynomial; it is an algebraic unit.

If  $T : \mathcal{X} \to \mathcal{X}$  is an automorphism of a complex K3 surface, then  $T^* : H^2(\mathcal{X}, \mathbb{C}) \to H^2(\mathcal{X}, \mathbb{C})$  respects the Hodge decomposition

$$H^{2}(\mathcal{X}, \mathbf{C}) = H^{2,0}(\mathcal{X}) \oplus H^{1,1}(\mathcal{X}) \oplus H^{0,2}(\mathcal{X});$$

since dim $(H^{2,0}) = 1$ , the automorphism  $T^*$  acts on it by multiplication with a complex number, denoted by  $\delta(T)$ ; we have  $|\delta(T)| = 1$ . Moreover,  $T^* : H^2(\mathcal{X}, \mathbf{Z}) \to H^2(\mathcal{X}, \mathbf{Z})$  preserves the intersection pairing. The above properties imply that the characteristic polynomial of  $T^*$  is a product of at most one Salem polynomial and of a finite number of cyclotomic polynomials, it satisfies condition (C 1), and  $\delta(T)$  is a root of this polynomial.

Moreover, assume that the characteristic polynomial is equal to SC, where S is a Salem polynomial of degree d with  $4 \leq d \leq 22$  and C is a product of cyclotomic polynomials; then  $\mathcal{X}$  is projective if and only if  $\delta(T)$  is a root of C (see [R 17], Theorem 2.2). Such a polynomial is called a *complemented Salem* polynomial (see Definition 15.1).

Let F be a complemented Salem polynomial, and let  $\delta$  be a root of F. We wish to decide whether F is the characteristic polynomial of an isomorphism  $T^*$  as above, with  $\delta(T) = \delta$ . We start with the simplest case, where the cyclotomic factor is a nontrivial power of X - 1.

**Theorem 3.** Let S be a Salem polynomial of degree d with  $4 \leq d \leq 18$  and let  $\delta$  be a root of S with  $|\delta| = 1$ . Let

$$F(X) = S(X)(X-1)^{22-d},$$

assume that condition (C 1) holds for F, and that |S(1)| is not a square. Then there exists a non-projective K3 surface  $\mathcal{X}$  and an automorphism  $T: \mathcal{X} \to \mathcal{X}$ such that

- F is the characteristic polynomial of  $T^*|H^2(\mathcal{X})$ .
- $T^*$  acts on  $H^{2,0}(\mathcal{X})$  by multiplication by  $\delta$ .

This is proved using Corollary 1, as well as some results of McMullen ([McM 11], Mc3) and Brandhorst ([Br 20]). For polynomials S with |S(1)| = 1, see Theorem 15.4; in this case, the answer depends on the congruence class of d modulo 8.

The dynamical degree of an automorphism  $T: \mathcal{X} \to \mathcal{X}$  is by definition the spectral radius of  $T^*$ ; since the characteristic polynomial of  $T^*$  is the product of a Salem polynomial and of a product of cyclotomic polynomial, the dynamical degree is a Salem number. We say that a Salem number is *realizable* if  $\alpha$  is the dynamical degree of an automorphism of a K3 surface.

Let  $\alpha$  be a Salem number of degree d with  $4 \leq d \leq 20$ , and let S be the minimal polynomial of  $\alpha$ . In §16 we prove an analog of Theorem 3 for  $F(X) = S(X)(X+1)^2(X-1)^{20-d}$  or  $S(X)(X^2-1)(X-1)^{20-d}$ , and show that if d = 4, 6, 8, 12, 14 or 16, then  $\alpha$  is realizable (see Corollary 16.7).

The aim of §17 is to prove that the second smallest known Salem number,  $\lambda_{18} = 1.1883681475...$ , is not realizable as the dynamical degree of an automorphism of a *non-projective* K3 surface. By contrast, McMullen proved that  $\lambda_{18}$  is the dynamical degree of an automorphism of a *projective* K3 surface (see [McM 16], Theorem 8.1).

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#### 1. Equivariant Witt groups

We start by recalling some notions and results from [BT 20], §3 and §4.

#### The equivariant Witt group

Let  $\mathcal{K}$  be a field, let  $\mathcal{A}$  be a  $\mathcal{K}$ -algebra and let  $\sigma : \mathcal{A} \to \mathcal{A}$  be a  $\mathcal{K}$ -linear involution. An  $\mathcal{A}$ -bilinear form is a pair (V, b) consisting of an  $\mathcal{A}$ -module Vthat is a finite dimensional  $\mathcal{K}$ -vector space, and a non-degenerate symmetric  $\mathcal{K}$ -bilinear form  $b : V \times V \to K$  such that  $b(ax, y) = b(x, \sigma(a)y)$  for all  $a \in \mathcal{A}$ and all  $x, y \in V$ .

The associated Witt group is denoted by  $W_{\mathcal{A}}(\mathcal{K})$  (see [BT 20], §3). If M is a simple  $\mathcal{A}$ -module, we denote by  $W_{\mathcal{A}}(\mathcal{K}, M)$  the subgroup of  $W_{\mathcal{A}}(\mathcal{K})$  generated by the classes of  $\mathcal{A}$ -bilinear forms (M, b). Every class in  $W_{\mathcal{A}}(\mathcal{K})$  is represented by an  $\mathcal{A}$ -bilinear form whose underlining  $\mathcal{A}$ -module is semisimple, and we have

$$W_{\mathcal{A}}(\mathcal{K}) = \bigoplus_{M} W_{\mathcal{A}}(\mathcal{K}, M),$$

where M ranges over the isomorphism classes of simple A-modules (see [BT 20], Corollary 3.11 and Theorem 3.12).

#### Discrete valuation rings and residue maps

Let O be a discrete valuation ring with field of fractions K, residue field and uniformizer  $\pi$ . Let  $(A, \sigma)$  be an O-algebra with involution, and set  $A_K = A \otimes_O K$ ,  $A_k = A \otimes_O k$ . An *A*-lattice in an  $A_K$  bilinear form V is an *A*-submodule L which is finitely generated as an *O*-module and satisfies KL = V. If L is an *A*-lattice, then so is its dual

$$L^{\sharp} = \{ x \in V \mid b(x, L) \subset O \}.$$

We say that L is unimodular if  $L^{\sharp} = L$  and almost unimodular if  $\pi L^{\sharp} \subset L \subset L^{\sharp}$ . If L is almost unimodular, then b induces an  $A_k$ -bilinear form  $L^{\sharp}/L \times L^{\sharp}/L \to k$ (see [BT 20], definition 4.1).

An  $A_K$ -bilinear form is said to be *bounded* if it contains an A-lattice. We denote by  $W^b_{A_K}(K)$  the subgroup of  $W_{A_K}(K)$  generated by the classes of bounded forms. The following result is proved in [BT 20]:

**Theorem 1.1.** (i) Every bounded  $A_K$ -bilinear form contains an almost unimodular A-lattice L.

(ii) The class of  $L^{\sharp}/L$  in  $W_{A_k}(k)$  only depends on the class of V in  $W_A(K)$ .

(iii) The map  $\partial: W_{A_K}(K) \to W_{A_k}(k)$  given by  $[V] \to [L^{\sharp}/L]$  is a homomorphism.

(iv) V contains a unimodular A-lattice if and only if V is bounded and  $\partial[V] = 0$  in  $W_{A_k}(k)$ .

**Proof.** See [BT 20], Theorem 4.3.

## 2. Symmetric polynomials and $\Gamma$ -modules

We recall some notions from [M 69] and [B 15]. Let K be a field. If  $f \in K[X]$  is a monic polynomial such that  $f(0) \neq 0$ , set  $f^*(X) = f(0)X^{\deg(f)}f(X^{-1})$ ; we say that f is symmetric if  $f^* = f$ . Recall the following definition from [B 15]:

**Definition 2.1.** Let  $f \in k[X]$  be a monic, symmetric polynomial. We say that f is of

• type 0 if f is a product of powers of X - 1 and of X + 1;

• type 1 if f is a product of powers of monic, symmetric, irreducible polynomials in k[X] of even degree;

• type 2 if f is a product of polynomials of the form  $gg^*$ , where  $g \in k[X]$  is monic, irreducible, and  $g \neq \pm g^*$ .

The following is well-known :

**Proposition 2.2.** Every monic symmetric polynomial is a product of polynomials of type 0, 1 and 2.

**Proof.** See for instance [B 15], Proposition 1.3.

Let J be the set of irreducible factors of F, and let us write  $F = \prod_{f \in J} f^{n_f}$ . Let  $I_1 \subset J$  be the subset of irreducible factors of type 1, and let  $I_0 \subset J$  be the set of irreducible factors of type 0; set  $I = I_0 \cup I_1$ . For all  $f \in I$ , set  $M_f = [K[X]/(f)]^{n_f}$ . Set  $M^0 = \bigoplus_{f \in I_0} M_f$ , and  $M^1 = \bigoplus_{f \in I_1} M_f$ . If  $f \in J$  such that  $f \neq f^*$ , set  $M_{f,f^*} = [K[X]/(f) \oplus K[X]/(f^*)]^{n_f}$ , and let  $M^2 = \bigoplus_{\substack{(f,f^*) \\ (f,f^*)}} M_{f,f^*}$ , where the sum runs over the pairs  $(f, f^*)$  with  $f \in J$  and  $f \neq f^*$ . Set  $M = M^0 \oplus M^1 \oplus M^2$ .

Let  $\Gamma$  be the infinite cyclic group, and let  $\gamma$  be a generator of  $\Gamma$ . Setting  $\gamma(m) = Xm$  for all  $m \in M$  endows M with a structure of semi-simple  $K[\Gamma]$ -module; we say that M is the semi-simple  $K[\Gamma]$ -module associated to the polynomial F.

Let us write  $F = F_0 F_1 F_2$ , where  $F_i$  is the product of the irreducible factors of type *i* of *F*. We have  $F_0 = (X-1)^{n^+} (X+1)^{n^-}$  for some integers  $n^+, n^- \ge 0$ . Set  $M^+ = [K[X]/(X-1)^{n^+}$  and  $M^- = [K[X]/(X+1)^{n^-}$ . The  $K[\Gamma]$ -module  $M^0$  splits as

$$M^0 = M^+ \oplus M^-.$$

#### 3. Isometries of quadratic forms

We recall some results from [M 69] and [B 15]. Let K be a field of characteristic  $\neq 2$ , let V be a finite dimensional K-vector space, and let  $q: V \times V \to K$ be a non-degenerate quadratic form. An *isometry* of (V,q) is by definition an isomorphism  $t: V \to V$  such that q(tx,ty) = q(x,y) for all  $x, y \in V$ . Let  $t: V \to V$  be an isometry, and let  $F \in K[X]$  be the characteristic polynomial of t. It is well-known that F is a symmetric polynomial (see for instance [B 15], Proposition 1.1). The following property is also well-known :

**Lemma 3.1.** If  $t: V \to V$  is an isometry of the quadratic form (V,q) and if the characteristic polynomial F of t satisfies  $F(1)F(-1) \neq 0$ , then

$$\det(q) = F(1)F(-1)$$
 in  $K^{\times}/K^{\times 2}$ .

**Proof.** See for instance [B 15], Corollary 5.2.

Recall that  $\Gamma$  is the infinite cyclic group, and let  $\sigma : K[\Gamma] \to K[\Gamma]$  be the *K*-linear involution such that  $\sigma(\gamma) = \gamma^{-1}$  for all  $\gamma \in \Gamma$ . An isometry  $t : V \to V$ endows *V* with a  $K[\Gamma]$ -module structure, and if moreover *t* is semi-simple with characteristic polynomial *F*, then this module is isomorphic to the semi-simple  $K[\Gamma]$ -module M = M(F) associated to the polynomial *F* (see §2). Hence *M* also carries a non-degenerate quadratic form, that we also denote by *q*. Note that (M, q) is a  $K[\Gamma]$ -bilinear form, and gives rise to an element [M, q] of the Witt group  $W_{K[\Gamma]}(K)$ . To simplify notation, set  $W_{\Gamma}(K) = W_{K[\Gamma]}(K)$ .

Let us write  $M = M^0 \oplus M^1 \oplus M^2$  as in §2, and let  $q^i$  denote the restriction of q to  $M^i$ ; this gives rise to an orthogonal decomposition  $(M, q) = (M^0, q^0) \oplus$   $(M^1, q^1) \oplus (M^2, q^2)$ , and  $(M^2, q^2)$  is hyperbolic, hence its class in  $W_{\Gamma}(K)$  is trivial (see for instance [M 69], Lemma 3.1). With the notation of §2, we have the further orthogonal decompositions

$$(M^0, q^0) = \bigoplus_{f \in I_0} (M_f, q_f) \text{ and } (M^1, q^1) = \bigoplus_{f \in I_1} (M_f, q_f),$$

where  $q_f$  is the restriction of q to  $M_f$  (see for instance [M 69], §3, or [B 15], Propositions 3.3 and 3.4). Note that if  $f \in I_0$ , then f(X) = X - 1 or X + 1, and we have the orthogonal decomposition  $(M^0, q^0) = (M^+, q^+) \oplus (M^-, q^-)$ , with  $q^+ = q_{X-1}$  and  $q^- = q_{X+1}$ .

## 4. Local fields and unimodular $\Gamma$ -lattices

Let K be a non-archimedean local field of characteristic 0, let O be its ring of integers, and let k be its residue field. If  $a \in O$ , set v(a) = 1 if  $v_K(a)$  is odd, and v(a) = 0 if  $v_K(a)$  is even or a = 0 (in other words, v(a) is the valuation of  $a \pmod{2}$  if  $a \neq 0$ , and v(0) = 0).

**Theorem 4.1.** Let  $F \in O[X]$  be a monic, symmetric polynomial. There exists a unimodular O-lattice having a semi-simple isometry with characteristic polynomial F if and only if one of the following holds

- (i) char(k)  $\neq 2$ , and v(F(1)) = v(F(-1)) = 0.
- (ii)  $\operatorname{char}(k) = 2$ , and v(F(1)F(-1)) = 0.

We start with a preliminary result, and some notation.

**Notation 4.2.** Let  $E_0$  be an étale K-algebra of finite rank, and let E be an étale  $E_0$ -algebra which is free of rank 2 over  $E_0$ . Let  $\sigma : E \to E$  be the involution fixing  $E_0$ . If  $\lambda \in E_0^{\times}$ , we denote by  $b_{\lambda}$  the quadratic form  $b_{\lambda} : E \times E \to K$  such that  $b_{\lambda}(x, y) = \operatorname{Tr}_{E/K}(\lambda x \sigma(y))$ .

**Proposition 4.3.** Let  $E_0$  be an étale K-algebra of finite rank, and let E be an étale  $E_0$ -algebra which is free of rank 2 over  $E_0$ . Let  $\sigma : E \to E$  be the involution fixing  $E_0$ . Let  $\alpha \in E_0^{\times}$  be such that  $\alpha\sigma(\alpha) = 1$ , and that the characteristic polynomial f of  $\alpha$  over K belongs to O[X]. Let  $\deg(f) = 2d$ , and assume that  $f(1)f(-1) \neq 0$ . Let  $u_+, u_- \in O^{\times}$ .

Let  $V = V^+ \oplus V^-$  be a finite dimensional K-vector space, and let  $\epsilon = (\epsilon^+, \epsilon^-) : V \to V$  be the isomorphism given by  $\epsilon^{\pm} : V^{\pm} \to V^{\pm}$ ,  $\epsilon^{\pm} = \pm id$ . Set  $n^+ = \dim(V^+)$  and  $n^- = \dim(V^-)$ .

If char(k)  $\neq 2$ , assume that if  $n^{\pm} = 0$ , then  $v(f(\pm 1)) = 0$ .

If char(k) = 2, assume that if  $n^+ = n^- = 0$ , then v(f(1)) + v(f(-1)) = 0.

If moreover  $K = \mathbf{Q}_2$ , assume that

- $n^+$  and  $n^-$  are both even,
- if  $n^+ = n^- = 0$ , then  $(-1)^d f(1) f(-1) = 1$  in  $\mathbf{Q}_2^{\times} / \mathbf{Q}_2^{\times 2}$ ,
- if  $n^{\pm} = 0$ , then  $v(f(\pm 1)) = 0$ ,
- $u_+u_- = (-1)^n$ , where  $2n = \dim(E \oplus V)$ .

Then there exists  $\lambda \in E_0^{\times}$  and non-degenerate quadratic forms

$$q^+: V^+ \times V^+ \to K, \quad q^-: V^- \times V^- \to K$$

such that, for  $q = q^+ \oplus q^-$ , we have

(i)

$$\partial[E \oplus V, b_{\lambda} \oplus q, \alpha \oplus \epsilon] = 0$$

in  $W_{\Gamma}(k)$ .

- (ii) If char(k)  $\neq 2$  then det( $q^{\pm}$ ) =  $u_{\pm}f(\pm 1)$  in  $K^{\times}/K^{\times 2}$ .
- (iii) If moreover  $K = \mathbf{Q}_2$ , then
- If  $n_{-} \neq 0$ , then  $v(\det(q^{-})) = v(f(-1))$ ,
- If  $n_{+} \neq 0$  and  $n_{-} \neq 0$ , then  $\det(q_{\pm}) = u_{\pm}f(\pm 1)$  in  $\mathbf{Q}_{2}^{\times}/\mathbf{Q}_{2}^{\times 2}$ ,

• det $(E \oplus V, b \oplus q) = (-1)^n$ , and  $(E \oplus V, b \oplus q)$  contains an even, unimodular  $\mathbb{Z}_2$ -lattice.

**Proof.** The proof depends on the values of v(f(1)) and v(f(-1)). We are in one of the following cases

(a) 
$$v(f(1)) = 0, v(f(-1)) = 0$$

- (b) v(f(1)) = 1, v(f(-1)) = 0,
- (c) v(f(1)) = 0, v(f(-1)) = 1,

(d) 
$$v(f(1)) = 1, v(f(-1)) = 1.$$

The algebra  $E_0$  decomposes as a product of fields  $E_0 = \prod_{v \in S} E_{0,v}$ . For all  $v \in S$ , set  $E_v = E \otimes_{E_0} E_{0,v}$ .

Assume first that the characteristic of k is  $\neq 2$ . The algebra  $E_v$  is of one of the following types

- (sp)  $E_v = E_{0,v} \times E_{0,v};$
- (un)  $E_v$  is an unramified extension of  $E_{0,v}$ ;

(+)  $E_v$  is a ramified extension of  $E_{0,v}$ , and the image  $\overline{\alpha}$  of  $\alpha$  in the residue field  $\kappa_v$  of  $E_v$  is 1;

(-)  $E_v$  is a ramified extension of  $E_{0,v}$ , and the image  $\overline{\alpha}$  of  $\alpha$  in the residue field  $\kappa_v$  of  $E_v$  is -1.

This gives a partition  $S = S_{sp} \cup S_{un} \cup S_+ \cup S_-$ .

Let  $\gamma$  be a generator of  $\Gamma$ , and let  $\chi_{\pm} : \Gamma \to \{\pm\}$  be the character sending  $\gamma$  to  $\pm 1$ .

Let us choose  $\lambda = (\lambda_v)_{v \in S}$  in  $E_0^{\times} = \prod_{v \in S} E_{0,v}^{\times}$  such that for every  $v \in S_{un}$ , we have  $\partial [E_v, b_{\lambda_v}, \alpha] = 0$  in  $W_{\Gamma}(k)$ ; this is possible by [BT 20], Proposition 6.4. The choices for  $v \in S_+$  and  $S_-$  depend on which of the cases (a), (b), (c) or (d) we are in. Let  $\overline{u}$  be the image of u in k.

Assume that we are in case (a) : then by hypothesis v(f(-1)) = v(f(1)) = 0. For  $v \in S_+ \cup S_-$ , we choose  $\lambda_v$  such that

$$\sum_{v \in S_+} \partial [E_v, b_{\lambda_v}, \alpha] = 0 \text{ in } W(k) = W(k, \chi_+) \subset W_{\Gamma}(k), \text{ and}$$
$$\sum_{v \in S_-} \partial [E_v, b_{\lambda_v}, \alpha] = 0 \text{ in } W(k) = W(k, \chi_-) \subset W_{\Gamma}(k).$$

This is possible by [BT 20], Proposition 6.6; indeed, by [BT 20], Lemma 6.8 we have  $\sum_{v \in S_{=}} [\kappa_v : k] \equiv v(f(1)) \pmod{2}$ ,  $\sum_{v \in S_{-}} [\kappa_v : k] \equiv v(f(-1)) \pmod{2}$ , and v(f(1)) = v(f(-1)) = 0 by hypothesis; therefore  $\partial[E, b_{\lambda}, \alpha] = 0$  in  $W_{\Gamma}(k)$ . Taking for  $q^{\pm}$  the zero form if  $n^{\pm} = 0$ , and a unimodular form of determinant  $u^{\pm}f(\pm 1)$  otherwise, we get

$$\partial [E \oplus V, b_{\lambda} \oplus q, (\alpha, \epsilon)] = 0$$

in  $W_{\Gamma}(k)$ . This implies (i) and (ii), and completes the proof in case (a).

Assume now that we are in case (b); then by hypothesis v(f(-1)) = 0 and v(f(1)) = 1. For  $v \in S_{-}$  we choose  $\lambda_v$  such that

 $\sum_{v \in S_{-}} \partial [E_v, b_{\lambda_v}, \alpha] = 0 \text{ in } W(k) = W(k, \chi_{-}) \subset W_{\Gamma}(k). \text{ This is possible by}$ 

[BT 20], Proposition 6.6; indeed, by [BT 20], Lemma 6.8 (ii) we have

$$\sum_{v \in S_{-}} [\kappa_v : k] \equiv v(f(-1)) \pmod{2},$$

and v(f(-1)) = 0 by hypothesis.

We now come to the places in  $S_+$ . Recall that by [BT 20], Lemma 6.8 (i) we have  $\sum_{v \in S_+} [\kappa_v : k] \equiv v(f(1)) \pmod{2}$ . Since v(f(1)) = 1 by hypothesis, this implies that  $\sum_{v \in S_+} [\kappa_v : k] \equiv 1 \pmod{2}$ . Therefore there exists  $w \in S_+$  such that  $[\kappa_w : k]$  is odd. By [BT 20], Proposition 6.6, we can choose  $\lambda_w$  such that  $\partial [E_w, b_{\lambda_w}, \alpha]$  is either one of the two classes of  $\gamma \in W(k) = W(k, \chi_+) \subset W_{\Gamma}(k)$ with  $\dim(\gamma) = 1$ . Let us choose the class of determinant  $-\overline{u_+}$ , and set  $\partial [E_w, b_{\lambda_w}, \alpha] = \delta$ .

Since v(f(1)) = 1, by hypothesis we have  $n^+ \ge 1$ . Let  $(V^+, q^+)$  be a nondegenerate quadratic form over K such that  $\det(q^+) = u_+ f(1)$ , and that

$$\partial[V^+, q^+, id] = -\delta$$
 in  $W(k) = W(k, \chi_+) \subset W_{\Gamma}(k)$ .

Let  $S'_+ = S_+ - \{w\}$ ; we have  $\sum_{v \in S'_+} [\kappa_v : k] \equiv 0 \pmod{2}$ , hence by

[BT 20], Proposition 6.6, for all  $v \in S'_+$  there exists  $\lambda_v \in E_{0,v}^{\times}$  such that  $\sum_{v \in S'_+} \partial[E_v, b_{\lambda_v}, \alpha] = 0$  in  $W(k) = W(k, \chi_+) \subset W_{\Gamma}(k)$ . We have

$$\partial[E \oplus V^+, b_\lambda \oplus q^+, (\alpha, id)] = 0$$

in  $W_{\Gamma}(k)$ . Taking for  $(V^{-}, q^{-})$  a quadratic form over O of determinant  $u_{-}f(-1)$  and setting  $q = q^{+} \oplus q^{-}$ , we get

$$\partial [E \oplus V, b \oplus q, (\alpha, \epsilon)] = 0$$

in  $W_{\Gamma}(k)$ . This completes the proof in case (b). The proof is the same in case (c), exchanging the roles of  $S_+$  and  $S_-$ .

Assume that we are in case (d), that is, v(f(1)) = v(f(-1)) = 1. By [BT 20], Lemma 6.8 (i) and (ii), we have

$$\sum_{v \in S_{+}} [\kappa_{v} : k] \equiv v(f(1)) \pmod{2}, \text{ and } \sum_{v \in S_{-}} [\kappa_{v} : k] \equiv v(f(-1)) \pmod{2}.$$

Therefore  $\sum_{v \in S_{\pm}} [\kappa_v : k] \equiv \sum_{v \in S_{\pm}} [\kappa_v : k] \equiv 1 \pmod{2}$ . Hence there exist  $w_{\pm} \in S_{\pm}$  such that  $[\kappa_{w_{\pm}} : k]$  and  $[\kappa_{w_{\pm}} : k]$  are odd. By [BT 20], Proposition 6.6, we can choose  $\lambda_{w_{\pm}}$  such that  $\partial[E_{w_{\pm}}, b_{\lambda_{w_{\pm}}}, \alpha]$  is either one of the two classes of  $\gamma \in W(k) = W(k, \chi_{\pm}) \subset W_{\Gamma}(k)$  with  $\dim(\gamma) = 1$ . Let us choose  $\lambda_{w_{\pm}}$  such that  $\partial[E_{w_{\pm}}, b_{\lambda_{w_{\pm}}}, \alpha]$  is represented by a form of dimension 1 and determinant  $\overline{u}_{\pm}$ , and set

$$\delta_{\pm} = \partial [E_{w_{\pm}}, b_{\lambda_{w_{\pm}}}, \alpha].$$

By hypothesis, we have  $n^+ \ge 1$  and  $n^- \ge 1$ . Let  $(V^{\pm}, q^{\pm})$  be non-degenerate quadratic forms over K such that  $\det(q^{\pm}) = u_{\pm}f(\pm 1)$  and that

$$\partial[V^{\pm}, q^{\pm}, \epsilon^{\pm}] = -\delta_{\pm} \text{ in } W(k) = W(k, \chi_{\pm}) \subset W_{\Gamma}(k).$$

Let  $S'_{+} = S_{+} - \{w_{+}\}$ ; we have  $\sum_{v \in S'_{+}} [\kappa_{v} : k] \equiv 0 \pmod{2}$ , hence by [BT 20], Proposition 6.6, for all  $v \in S'_{+}$  there exists  $\lambda_{v} \in E_{0,v}^{\times}$  such that  $\sum_{v \in S'_{+}} \partial [E_{v}, b_{\lambda_{v}}, \alpha] = 0$  in  $W(k) = W(k, \chi_{+}) \subset W_{\Gamma}(k)$ . Similarly, set  $S'_{-} = S_{+} - \{w_{-}\}$ ; we have  $\sum_{v \in S'_{-}} [\kappa_{v} : k] \equiv 0 \pmod{2}$ , hence by [BT 20], Proposition 6.6, for all  $v \in S_{-}$  there exists  $\lambda_{v} \in E_{0,v}^{\times}$  such that  $\sum_{v \in S'_{-}} \partial [E_{v}, b_{\lambda_{v}}, \alpha] = 0$  in  $W(k) = W(k, \chi_{+}) \subset W_{\Gamma}(k)$ . Set  $q = q^{+} \oplus q^{-}$ , and note that  $\partial [E \oplus V, b_{\lambda} \oplus q, (\alpha, \epsilon)] = 0$ 

in  $W_{\Gamma}(k)$ . This completes the proof when the characteristic of k is  $\neq 2$ .

Assume now that the characteristic of k is 2. The algebra  $E_v$  is of one of the following types

(sp)  $E_v = E_{0,v} \times E_{0,v};$ 

(un)  $E_v$  is an unramified extension of  $E_{0,v}$ ;

(r)  $E_v$  is a ramified extension of  $E_{0,v}$ .

This gives a partition  $S = S_{sp} \cup S_{un} \cup S_r$ .

If  $\lambda = (\lambda_v)_{v \in S}$  is an element of  $E_0^{\times} = \prod_{v \in S} E_{0,v}^{\times}$ , note that by Lemma 3.1 we have

$$\operatorname{disc}(b_{\lambda}) = (-1)^d f(1) f(-1)$$

in  $K^{\times}/K^{\times 2}$ .

We choose  $\lambda = (\lambda_v)_{v \in S}$  in  $E_0^{\times}$  such that for every  $v \in S_{un}$ , we have  $\partial[E_v, b_{\lambda_v}, \alpha] = 0$  in  $W_{\Gamma}(k)$ ; this is possible by [BT 20], Proposition 6.4.

Assume first that we are in case (a) or (d), and note that v(f(1)) + v(f(-1)) = 0. By [BT 20], Lemma 6.8 and Proposition 6.7, we can choose  $\lambda_v$  such that

$$\sum_{v \in S_r} \partial [E_v, b_{\lambda_v}, \alpha] = 0 \text{ in } W(k) = W(k, 1) \subset W_{\Gamma}(k).$$

Therefore  $\partial[E, b_{\lambda}, \alpha] = 0$  in  $W_{\Gamma}(k)$ .

If we are in case (a) or case (d) and  $K \neq \mathbf{Q}_2$ , take for q the unit quadratic form. We have

$$\partial[E \oplus V, b_{\lambda} \oplus q, \alpha \oplus \epsilon] = 0$$

in  $W_{\Gamma}(k)$ ; this concludes the proof in cases (a) and (d) when  $K \neq \mathbf{Q}_2$ .

Suppose that  $K = \mathbf{Q}_2$  and that we are in case (a). Suppose first that  $n^+ = n^- = 0$ . We already know that  $\partial[E, b_\lambda, \alpha] = 0$  in  $W_{\Gamma}(\mathbf{F}_2)$ , hence (i) holds. Since  $n^+ = n^- = 0$ , by hypothesis  $(-1)^d f(1) f(-1) = 1$  in  $\mathbf{Q}_2^{\times} / \mathbf{Q}_2^{\times 2}$ , therefore  $\operatorname{disc}(b_\lambda) = 1$  in  $\mathbf{Q}_2^{\times} / \mathbf{Q}_2^{\times 2}$ . Let us choose  $\lambda$  such that the quadratic form  $(E, b_\lambda)$  contains an even, unimodular  $\mathbf{Z}_2$ -lattice. If  $S_r = \emptyset$ , this is automatic; indeed, in that case the trace map  $E \to E_0$  is surjective, and hence every  $\mathbf{Z}_2$ -lattice of the shape  $(E, b_\lambda)$  is even. If not, by [BT 20] Propositions 8.4 and 5.4 we can choose  $\lambda$  having this additional property. This implies that (iii) holds as well.

We continue supposing that  $K = \mathbf{Q}_2$  and that we are in case (a); assume now that  $n^+ \neq 0$  and  $n^- = 0$ . Let us choose  $q^+$  such that  $\det(q^*) = (-1)^n f(1)f(-1)$ ; since  $\det(b_{\lambda}) = f(1)f(-1)$ , this implies that

$$\det(E \oplus V, b_{\lambda} \oplus q^+) = (-1)^n.$$

Moreover, let us choose the Hasse-Witt invariant of  $q^+$  in such a way that the quadratic form  $(E \oplus V, b_{\lambda} \oplus q^+ \oplus q^-)$  contains an even, unimodular  $\mathbb{Z}_{2^-}$ lattice; this is possible by [BT 20] Proposition 8.4. Therefore condition (iii) holds. Note that since  $v(\det(q^+)) = 0$ , we have  $\partial[V, q^+] = 0$  in  $W(\mathbb{F}_2)$ , hence  $\partial[E \oplus V, b_{\lambda} \oplus q, \alpha \oplus \epsilon] = 0$  in  $W_{\Gamma}(\mathbb{F}_2)$ ; therefore condition (i) also holds.

Assume now that  $n^+ = 0$  and  $n^- \neq 0$ . Let us choose  $q^-$  such that  $\det(q^-) = (-1)^n f(1) f(-1)$ , and note that since v(f(1)) = v(f(-1)) = 0, this implies that  $v(\det(q^-)) = v(f(-1)) = 0$ . As in the previous case, we see that  $\det(E \oplus V, b_\lambda \oplus q^-) = (-1)^n$ , and we choose the Hasse-Witt invariant of  $q^-$  so that  $(E \oplus V, b_\lambda \oplus q^-)$  contains an even, unimodular  $\mathbb{Z}_2$ -lattice; this is possible by [BT 20] Proposition 8.4. As in the previous case, we conclude that conditions (i) and (iii) are satisfied.

Suppose that  $n^+ \neq 0$  and  $n^- \neq 0$ . Let us choose  $q^+$  such that  $\det(q^+) = u_+f(1)$  and  $q^-$  such that  $\det(q^-) = u_-f(-1)$ . Since  $u_+u_- = (-1)^n$  and  $\det(b_{\lambda}) = f(1)f(-1)$ , this implies that

$$\det(E \oplus V, b_{\lambda} \oplus q^{+} \oplus q^{-}) = (-1)^{n}.$$

Note that since  $v(u_-) = v(f(-1)) = 0$ , we have  $v(\det(q^-)) = v(f(-1)) = 0$ . As in the previous cases, we can choose  $q^+$  and  $q^-$  such that  $(E \oplus V, b_\lambda \oplus q^-)$  contains an even, unimodular  $\mathbb{Z}_2$ -lattice, and that  $\partial[E \oplus V, b_\lambda \oplus q, \alpha \oplus \epsilon] = 0$ in  $W_{\Gamma}(\mathbf{F}_2)$ , hence conditions (i) and (iii) hold.

Assume now that  $K = \mathbf{Q}_2$  and that we are in case (d); note that the hypothesis implies that  $n^+, n^- \ge 2$ , and that both  $n^+$  and  $n^-$  are even. With our previous choice of  $\lambda$ , we have  $\partial[E, b_{\lambda}, \alpha] = 0$  in  $W_{\Gamma}(\mathbf{F}_2)$ . Let us choose  $q^+$  and  $q^-$  such that  $\det(q^{\pm}) = u_{\pm}f(\pm 1)$ , and note that this implies that  $v(\det(q^-)) = v(f(-1))$ , and that  $\det(E \oplus V, b_{\lambda} \oplus q^+ \oplus q^-) = (-1)^n$ . Moreover, choose the Hasse-Witt invariants of  $q^+$  and  $q^-$  so that  $(E \oplus V, b_{\lambda} \oplus q^+ \oplus q^-)$ contains an even, unimodular  $\mathbf{Z}_2$ -lattice; this is possible by [BT 20] Proposition 8.4. Therefore condition (iii) holds; moreover, we have  $\partial(V, q, \epsilon) = 0$  in  $W_{\Gamma}(\mathbf{F}_2)$ and  $\partial[E \oplus V, b_{\lambda} \oplus q, \alpha \oplus \epsilon] = 0$  in  $W_{\Gamma}(\mathbf{F}_2)$ , hence condition (i) is also satisfied. This concludes the proof in cases (a) and (d).

Suppose that we are in case (b) or case (c), and note that in both cases, we have v(f(1)) + v(f(-1)) = 1. Hence Proposition 6.7 and Lemma 6.8 imply that  $\sum_{v \in S_r} \partial [E_v, b_{\lambda_v}, \alpha]$  is the unique non-trivial element of  $W(k) = W(k, 1) \subset W_{\Gamma}(k)$ . Suppose first that  $K \neq \mathbf{Q}_2$ . We have either  $n^+ \geq 1$  or  $n^- \geq 1$ ; choose  $q^{\pm}$  such that  $\partial [V^{\pm}, q^{\pm}, \pm id]$  is also the unique non-trivial element of  $W(k) = W(k, 1) \subset W(k, 1) \subset W_{\Gamma}(k)$ . We have

$$\partial [E \oplus V, b_{\lambda} \oplus q, \alpha \oplus \epsilon] = 0$$

in  $W_{\Gamma}(k)$ . This settles cases (b) and (c) when  $K \neq \mathbf{Q}_2$ .

Assume now that  $K = \mathbf{Q}_2$ , and that we are in case (b), namely v(f(1)) = 1and v(f(-1)) = 0; then  $n^+ \geq 2$ , and is even. If  $n^- \neq 0$ , then choose  $q^$ such that  $\det(q^-) = u_-f(-1)$ , and note that this implies that  $v(\det(q^-)) =$ v(f(-1)) = 0; choose  $q^+$  such that  $\det(q^+) = u_+f(1)$ . Since v(f(1)) = 1, this implies that  $\partial[V^+, q^+, id]$  is the unique non-trivial element of  $W(\mathbf{F}_2) =$  $W_{\Gamma}(\mathbf{F}_2, 1) \subset W_{\Gamma}(\mathbf{F}_2)$ . Note that  $\det(E \oplus V, b_{\lambda} \oplus q^+ \oplus q^-) = (-1)^n$  in  $\mathbf{Q}_2^{\times}/\mathbf{Q}^{\times 2}$ . Moreover, choose the Hasse-Witt invariants of  $q^+$  and  $q^-$  such that the quadratic form  $(E \oplus V, b_{\lambda} \oplus q^+ \oplus q^-)$  contains an even, unimodular  $\mathbf{Z}_2$ -lattice; this is possible by [BT 20] Proposition 8.4. Hence condition (iii) holds, and condition (i) follows from the fact that  $\partial[E, b_{\lambda}, \alpha]$  and  $\partial[V^+, q^+, id]$  are both equal to the unique non-trivial element of  $W(\mathbf{F}_2) = W_{\Gamma}(\mathbf{F}_2, 1)$ , which is a group of order 2. Therefore  $\partial[E \oplus V, b_{\lambda} \oplus q, \alpha \oplus \epsilon] = 0$  in  $W_{\Gamma}(\mathbf{F}_2)$ , and hence condition (i) is also satisfied.

Suppose now that  $K = \mathbf{Q}_2$ , and that we are in case (c). Then v(f(1)) = 0and v(f(-1)) = 1, hence  $n^- \geq 2$ , and is even. If  $n^+ \neq 0$ , then choose  $q^+$  such that  $\det(q^+) = u_+f(1)$ . Choose  $q^-$  such that  $\det(q^-) = u_-f(-1)$ , and note that this implies that  $v(\det(q^-)) = v(f(-1)) = 1$ , and that  $\partial[V^-, q^-, -id]$  is the unique non-trivial element of  $W(\mathbf{F}_2) = W_{\Gamma}(\mathbf{F}_2, 1) \subset W_{\Gamma}(\mathbf{F}_2)$ . We conclude as in case (b). This settles cases (b) and (c), and hence the proof of the proposition is complete.

We now show that the conditions of Theorem 4.1 are sufficient. In the case where  $char(k) \neq 2$ , we obtain a more precise result (see part (ii) of the

following result; for  $K = \mathbf{Q}_2$  an analogous result is given in Theorem 5.1). We use the notation of §2.

**Theorem 4.4.** Let  $F \in O[X]$  be a monic, symmetric polynomial.

If char(k)  $\neq 2$ , assume that v(F(1)) = v(F(-1)) = 0.

If char(k) = 2, assume that v(F(1)F(-1)) = 0.

(i) Then there exists a unimodular O-lattice having a semi-simple isometry with characteristic polynomial F.

(ii) Assume in addition that  $\operatorname{char}(k) \neq 2$  and let  $u_+, u_- \in O^{\times}$ . If  $M^{\pm} \neq 0$ , then there exists a unimodular O-lattice having a semi-simple isometry with characteristic polynomial F such that the associated  $K[\Gamma]$ -bilinear form  $(M^{\pm}, q^{\pm})$  is such that

$$\det(q^{\pm}) = u_{\pm}F_1(\pm 1)$$

in  $K^{\times}/K^{\times 2}$ .

**Proof.** Let us write  $F = F_0F_1F_2$ , where  $F_i$  is the product of the irreducible factors of F of type i. The hyperbolic O-lattice of rank deg $(F_2)$  has an isometry with characteristic polynomial  $F_2$ , therefore it is enough to prove the theorem for  $F = F_0F_1$ .

From now on, we assume that  $F = F_0 F_1$ , in other words, all the irreducible factors of F are symmetric, of type 0 or 1. Let  $I_1$  be the set of irreducible factors of type 1 of F. We have  $F_1 = \prod_{f \in I_1} f^{n_f}$ ; note that  $F_1(1)F_1(-1) \neq 0$ . Let us write  $F(X) = F_1(X)(X-1)^{n^+}(X+1)^{n^-}$  for some integers  $n^+, n^-$  such that

us write  $F'(X) = F_1(X)(X-1)^{n^+}(X+1)^n$  for some integers  $n^+, n^-$  such that  $n^+, n^- \ge 0$ .

For all  $f \in I_1$ , set  $E_f = K[X]/(f)$ . Let  $\sigma_f : E_f \to E_f$  be the involution induced by  $X \mapsto X^{-1}$ , and let  $(E_f)_0$  be the fixed field of  $\sigma$  in  $E_f$ . Let  $M_f$  be an extension of degree  $n_f$  of  $(E_f)_0$ , linearly disjoint from  $E_f$  over  $(E_f)_0$ . Set  $\tilde{E}_f = E_f \otimes_{(E_f)_0} M_f$ . Let  $\alpha_f$  be a root of f in  $E_f$ . The characteristic polynomial of the multiplication by  $\alpha_f$  on  $\tilde{E}_f$  is  $f^{n_f}$ , and its minimal polynomial is f. Set  $\tilde{E} = \prod_{f \in I_1} \tilde{E}_f$ , and  $\tilde{M} = \prod_{f \in I_1} \tilde{M}_f$ . Let  $\tilde{\sigma} : \tilde{E} \to \tilde{E}$  be the involution of  $\tilde{E}$ induced by the involutions  $\sigma : E_f \to E_f$ . Set  $\tilde{\alpha} = (\alpha_f)_{f \in I_1}$ , and let us denote by  $\tilde{\alpha} : \tilde{E} \to \tilde{E}$  the multiplication by  $\tilde{\alpha}$ . Note that  $\tilde{\alpha}$  is semi-simple, with characteristic polynomial  $F_1$ .

Let  $V = V^+ \oplus V^-$  be a K-vector space with  $\dim(V^+) = n^+$  and  $\dim(V^-) = +n^-$ . Applying Proposition 4.3 (i) with  $E_0 = \tilde{M}$ ,  $E = \tilde{E}$ ,  $\sigma = \tilde{\sigma}$ ,  $\alpha = \tilde{\alpha}$  and  $f = F_1$ , we see that there exists  $\lambda \in \tilde{M}^{\times}$  and a non-degenerate quadratic form  $q: V \times V \to K$  such that

$$\partial[E \oplus V, b_{\lambda} \oplus q, \alpha \oplus \epsilon] = 0$$

in  $W_{\Gamma}(k)$ . By Theorem 1.1 (iv) this implies that there exists a unimodular O-lattice having a semi-simple isometry with characteristic polynomial F, proving part (i) of the theorem. Similarly, Proposition 4.3 (ii) implies part (ii) of the theorem.

13

To show that the conditions of Theorem 4.1 are necessary, we start with some notation and a preliminary result.

Extending the scalars to K, an even, unimodular lattice having a semisimple isometry with characteristic polynomial F gives rise to a  $K[\Gamma]$ -bilinear form on the semi-simple  $K[\Gamma]$ -module associated to F (see §2), and this form has an orthogonal decomposition  $M = M^0 \oplus M^1 \oplus M^2$ , cf. §3. The  $K[\Gamma]$ -form  $M^0$  has the further orthogonal decomposition  $M^0 = (M^+, q^+) \oplus (M^-, q^-)$ .

Notation 4.5. Let  $\gamma$  be a generator of  $\Gamma$ . Let  $N_{\pm}$  be the simple  $k[\Gamma]$ -module such that  $\dim_k(N_+) = 1$ , and that  $\gamma$  acts on  $N_{\pm}$  by  $\pm id$ ; note that  $N_+ = N_-$  if char(k) = 2.

**Lemma 4.6.** Let  $F \in O[X]$  be a monic, symmetric polynomial, and suppose that there exists a unimodular lattice having a semi-simple isometry with characteristic polynomial F. Let  $M = M^0 \oplus M^1 \oplus M^2$  be the corresponding orthogonal decomposition of  $K[\Gamma]$ -bilinear forms. Let us write  $F(X) = F_1(X)(X-1)^{n^+}(X+1)^{n^-}$  for some integers  $n^+, n^-$  such that  $n^+, n^- \ge 0$ , and such that  $F_1(1)F_1(-1) \ne 0$ . Then we have

(i) Assume that  $\operatorname{char}(k) \neq 2$ . Then the component of  $\partial[M^1]$  in  $W_{\Gamma}(k, N_+) \simeq W(k)$  is represented by a quadratic form of dimension  $v(F_1(1))$  over k.

(i) Assume that  $\operatorname{char}(k) \neq 2$ . Then the component of  $\partial[M^1]$  in  $W_{\Gamma}(k, N_-) \simeq W(k)$  is represented by a quadratic form of dimension  $v(F_1(-1))$  over k.

(i) Assume that char(k) = 2. Then the component of  $\partial[M^1]$  in  $W_{\Gamma}(k, N_+) = W_{\Gamma}(k, N_-) \simeq W(k)$  is represented by a quadratic form of dimension  $v(F_1(1)) + v(F_1(-1))$  over k.

**Proof.** Since M is extended from a unimodular lattice, we have  $\partial[M] = 0$  (see Theorem 1.1). Let  $M = M^0 \oplus M^1 \oplus M^2$  be the orthogonal decoposition of §3. We have  $\partial[M^2] = 0$ , hence  $\partial([M^0] + [M^1]) = 0$ .

From now on, we assume that  $M = M^0 \oplus M^1$ ; equivalently, all the irreducible factors of F are of type 0 or 1. Let us write  $F_1 = \prod_{f \in I_1} f^{n_f}$ . We have an arthograph decomposition

orthogonal decomposition

$$M^1 = \bigoplus_{f \in I} M_f,$$

where  $M_f = [K[X]/(f)]^{n_f}$  (see [M 69], §3, or [B 15], Propositions 3.3 and 3.4). For all  $f \in I$ , set  $E_f = K[X]/(f)$ , and let  $\sigma : E_f \to E_f$  be the K-linear involution induced by  $X \mapsto X^{-1}$ . By a well-known transfer property (see for instance [M 69], Lemma 1.1 or [B 15], Proposition 3.6) the  $K[\Gamma]$ -bilinear form  $M_f$  is the trace of a non-degenerate hermitian form over  $(E_f, \sigma)$ , hence it is an orthogonal sum of forms of the type  $b_{\lambda}$ , see notation 4.2.

By [BT 20], Lemma 6.8 (i) and Proposition 6.6, the component of  $\partial[M^1]$  in  $W_{\Gamma}(k, M_+)$  is represented by a form of dimension  $v(F_1(1))$ , and this implies (i). Similarly, applying [BT 20], Lemma 6.8 (ii) and Proposition 6.6 implies (ii), and [BT 20], Lemma 6.8 (ii) and Proposition 6.7) implies (iii).

**Proposition 4.7.** Let  $F \in O[X]$  be a monic, symmetric polynomial, and suppose that there exists a unimodular lattice having a semi-simple isometry with characteristic polynomial F. Then we have

If 
$$char(k) \neq 2$$
, then  $v(F(1)) = v(F(-1)) = 0$ .

If char(k) = 2, then v(F(1)F(-1)) = 0.

**Proof.** Suppose first that  $\operatorname{char}(k) \neq 2$ . If  $n^+ > 0$  and  $n^- > 0$ , then F(1) = F(-1) = 0, so there is nothing to prove. Assume that  $n^+ = 0$ . Then the component of  $\partial[M^0]$  in  $W_{\Gamma}(k, M_+)$  is trivial, and note that  $F(1) = F_1(1)$ . By Lemma 4.6 (i), the component of  $\partial[M^1]$  in  $W_{\Gamma}(k, M_+)$  is represented by a form of dimension  $v(F_1(1)) = v(F(1))$ , hence v(F(1)) = 0. Similarly,  $n^- = 0$  implies that v(F(-1)) = 0. This completes the proof of the proposition in the case where  $\operatorname{char}(k) \neq 2$ .

Assume now that  $\operatorname{char}(k) = 2$ . If  $n^+ > 0$  or  $n^- > 0$ , then F(1)F(-1) = 0, so there is nothing to prove. Assume that  $n^+ = n^- = 0$ . The component of  $\partial[M^1]$ in  $W_{\Gamma}(k, M_+) = W_{\Gamma}(k, M_-)$  is represented by a form of dimension v(F(1)) + v(F(-1)) (cf. Lemma 4.6 (iii)). Since  $n^+ = n^- = 0$  we have  $M = M^1$ , hence  $\partial[M^1] = 0$ , and we also have  $F = F_1$ ; therefore v(F(1)) + v(F(-1)) = 0. This completes the proof of the proposition.

**Proof of Theorem 4.1.** The theorem follows from Theorem 4.4 and Proposition 4.7.

#### 5. Even, unimodular $\Gamma$ -lattices over $\mathbf{Z}_2$

We keep the notation of §4, with  $K = \mathbf{Q}_2$  and  $O = \mathbf{Z}_2$ . Recall that if  $a \in \mathbf{Z}_2$ , we set v(a) = 0 if a = 0 or if the 2-adic valuation of a is even, and v(a) = 1 if the 2-adic valuation of a is odd.

If  $F \in \mathbf{Z}_2[X]$  is a monic, symmetric polynomial, we write  $F = F_0F_1F_2$ , where  $F_i$  is the product of the irreducible factors of type *i* of *F*. Recall that  $M = M^0 \oplus M^1 \oplus M^2$  is the semi-simple  $\mathbf{Q}_2[\Gamma]$ -module associated to *F*, and that  $M_0 = M^+ \oplus M^-$ .

**Theorem 5.1.** Let  $F \in \mathbb{Z}_2[X]$  be a monic, symmetric polynomial of even degree such that F(0) = 1, and set  $2n = \deg(F)$ . Let  $u_+, u_- \in \mathbb{Z}_2^{\times}$  such that  $u_+u_- = (-1)^n$ . Assume that the following conditions hold :

- (a) v(F(1)) = v(F(-1)) = 0.
- (b) If  $F(1)F(-1) \neq 0$ , then  $(-1)^n F(1)F(-1) = 1$  in  $\mathbf{Q}_2^{\times}/\mathbf{Q}_2^{\times 2}$ .

Then we have

(i) There exists an even, unimodular  $\mathbb{Z}_2$ -lattice having a semi-simple isometry with characteristic polynomial F.

(ii) If  $M^+ \neq 0$ ,  $M^- \neq 0$  and  $M^1 \neq 0$ , then there exists an even, unimodular  $\mathbb{Z}_2$ -lattice having a semi-simple isometry with characteristic polynomial F such that the associated  $\mathbb{Q}_2[\Gamma]$ -bilinear form  $(M^{\pm}, q^{\pm})$  is such that

$$\det(q^{\pm}) = u_{\pm}F_1(\pm 1)$$

15

in  $\mathbf{Q}_2^{\times}/\mathbf{Q}_2^{\times 2}$ .

**Proof.** Let *I* be the set of irreducible factors of *F* of type 1, and set  $F_1 = \prod_{f \in I_1} f^{n_f}$ . The hyperbolic  $\mathbb{Z}_2$ -lattice of rank deg $(F_2)$  has a semi-simple isometry

with characteristic polynomial  $F_2$ , therefore it is enough to prove the theorem for  $F = F_0 F_1$ .

From now on, we assume that all the irreducible factors of F are symmetric, of type 0 or 1; we have  $F = F_1(X - 1)^{n^+}(X + 1)^{n^-}$  for some integers  $n^+ \ge 0$ ,  $n^- \ge 0$ . Note that since deg(F) is even and F(0) = 1 by hypothesis,  $n^+$  and  $n^-$  are both even.

For all  $f \in I_1$ , set  $E_f = \mathbf{Q}_2[X]/(f)$ . Let  $\sigma : E_f \to E_f$  be the involution induced by  $X \mapsto X^{-1}$ , and let  $(E_f)_0$  be the fixed field of  $\sigma$  in  $E_f$ . Let  $M_f$  be an extension of degree  $n_f$  of  $(E_f)_0$ , linearly disjoint from  $E_f$  over  $(E_f)_0$ . Set  $\tilde{E}_f = E_f \otimes_{(E_f)_0} M_f$ . Let  $\alpha_f$  be a root of f in  $E_f$ . The characteristic polynomial of the multiplication by  $\alpha_f$  on  $\tilde{E}_f$  is  $f^{n_f}$ , and its minimal polynomial is f. Set  $\tilde{E} = \prod_{f \in I_1} \tilde{E}_f$ , and  $\tilde{M} = \prod_{f \in I_1} \tilde{M}_f$ . Let  $\tilde{\sigma} : \tilde{E} \to \tilde{E}$  be the involution of  $\tilde{E}$ induced by the involutions  $\sigma : E_f \to E_f$ . Set  $\tilde{\alpha} = (\alpha_f)_{f \in I_1}$ , and let us denote by  $\tilde{\alpha} : \tilde{E} \to \tilde{E}$  the multiplication by  $\tilde{\alpha}$ . Note that  $\tilde{\alpha}$  is semi-simple, with characteristic polynomial  $F_1$ .

We apply Theorem 8.1 of [BT 20] and Proposition 4.3 with  $E_0 = \tilde{M}, E = \tilde{E}, \sigma = \tilde{\sigma}$  and  $\alpha = \tilde{\alpha}$ .

Let  $V^{\pm}$  be a  $\mathbf{Q}_2$ -vector spaces of dimension  $n^{\pm}$ , and set  $V = V^+ \oplus V^-$ . Note that if  $n^+ = n^- = 0$ , then  $F_1 = F$ , hence the class of  $(-1)^n F_1(1) F_1(-1) = 1$ in  $\mathbf{Q}_2^{\times} / \mathbf{Q}_2^{\times 2}$  by hypothesis; therefore the hypotheses of Proposition 4.3 are satisfied. Proposition 4.3 (i) and (iii) imply that there exist  $\lambda \in \tilde{M}^{\times}$  and a non-degenerate quadratic form  $q: V \times V \to \mathbf{Q}_2$  such that

$$\partial[E \oplus V, b_{\lambda} \oplus q, \tilde{\alpha} \oplus \epsilon] = 0$$

in  $W_{\Gamma}(k)$ , that  $(\tilde{E} \oplus V, b_{\lambda} \oplus q)$  contains an even, unimodular  $\mathbb{Z}_2$ -lattice, and that  $v(\det(q^-) = v(F_1(-1))$ . By Theorem 1.1 (iv), this implies that there exists a unimodular lattice in  $\tilde{E} \oplus V$  stable by  $\tilde{\alpha} \oplus \epsilon$ , hence a unimodular lattice having a semi-simple isometry with characteristic polynomial F; therefore conditions (i) and (ii) of [BT 20], Theorem 8.1 hold. Moreover, since  $v(\det(q^-) = v(F_1(-1)))$ , Theorem 8.4 of [BT 20] implies that condition (iii) of [BT 20], Theorem 8.1 is also satisfied. This implies that there exists an even, unimodular  $\mathbb{Z}_2$ -lattice having a semi-simple isometry with characteristic polynomial F, and this completes the proof of (i). Part (ii) of the theorem also follows from Proposition 4.3, part (iii).

**Theorem 5.2.** Let  $F \in \mathbb{Z}_2[X]$  be a monic, symmetric polynomial of even degree such that F(0) = 1, and set  $2n = \deg(F)$ . Assume that there exists an even, unimodular  $\mathbb{Z}_2$ -lattice having a semi-simple isometry with characteristic polynomial F. Then we have

(a) 
$$v(F(1)) = v(F(-1)) = 0.$$

(b) If  $F(1)F(-1) \neq 0$ , then the class of  $(-1)^n F(1)F(-1)$  in  $\mathbf{Q}_2^{\times}/\mathbf{Q}_2^{\times 2}$  lies in  $\{1, -3\}$ .

**Proof.** Let L be an even, unimodular lattice having a semi-simple isometry with characteristic polynomial F. The lattice L gives rise to a  $\mathbf{Q}_2[\Gamma]$ -bilinear form M on a bounded module. Let us consider the orthogonal decomposition of  $\mathbf{Q}_2[\Gamma]$ -bilinear forms

$$M = M^0 \oplus M^1 \oplus M^2$$

(cf. §3). Since L is unimodular we have  $\partial[M] = 0$ ; note that  $\partial[M^2] = 0$ , hence we have  $\partial([M^0] + [M^1]) = 0$ .

From now on, we assume that  $M = M^0 \oplus M^1$ ; equivalently, all the irreducible factors of F are of type 0 or 1. Let I be the set of irreducible factors of F of type 1, and set  $F_1 = \prod_{f \in I_1} f^{n_f}$ . We have  $F = F_1(X - 1)^{n^+}(X + 1)^{n^-}$  for some integers  $n^+ \ge 0$ ,  $n^- \ge 0$ .

Further, we have an orthogonal decomposition  $M^1 = \bigoplus_{f \in I_1} M_f$ , where

$$M_f = [\mathbf{Q}_2[X]/(f)]^{n_f}$$

(see [M 69], §3, or [B 15], Propositions 3.3 and 3.4). For all  $f \in I_1$ , set  $E_f = \mathbf{Q}_2[X]/(f)$ , and let  $\sigma : E_f \to E_f$  be the  $\mathbf{Q}_2$ -linear involution induced by  $X \mapsto X^{-1}$ . By a well-known transfer property (see for instance [M 69], Lemma 1.1 or [B 15], Proposition 3.6) the  $\mathbf{Q}_2[\Gamma]$ -bilinear form  $M_f$  is the trace of a non-degenerate hermitian form over  $(E_f, \sigma)$ , hence it is an orthogonal sum of forms of the type  $b_{\lambda}$ , see notation 4.2.

The component of  $\partial[M^1]$  in  $W_{\Gamma}(k, N_{\pm})$  is represented by a form of dimension  $v(F(1)) + v(F(-1)) \pmod{2}$  (cf. [BT 20], Lemma 6.8 (ii) and Proposition 6.7).

Suppose that  $n^+ = n^- = 0$ . Then  $M = M^1$ , hence we have  $\partial[M^1] = 0$ ; by the above argument this implies that  $v(F(1)) + v(F(-1)) \pmod{2}$ . By [BT 20], Proposition 8.6 and Theorem 8.5, we have v(F(-1)) = 0, hence (a) holds. Since L is even and unimodular, the class of  $(-1)^n F(1)F(-1)$  in  $\mathbf{Q}_2^{\times/2}/\mathbf{Q}_2^{\times 2}$  lies in  $\{1, -3\}$ ; this shows that (b) holds as well.

Let  $M^0 = V^+ \oplus V^-$ , and let  $q^{\pm}$  the the quadratic form on  $V^{\pm}$ .

Suppose that  $n^+ \neq 0$ , and  $n^- = 0$ . Then F(1) = 0, hence v(F(1)) = 0. Since  $n^- = 0$ , the quadratic form  $q^-$  is the zero form, and  $v(\det(q^-)) = 0$ . By [BT 20], Theorem 8.5 and Proposition 8.6, we have  $v(\det(q^-)) = v(F(-1))$ , hence v(F(-1)) = 0. This implies (a), and (b) is obvious since F(1) = 0.

Assume now that  $n^+ = 0$  and  $n^- \neq 0$ ; then F(-1) = 0, hence (b) holds. By [BT 20], Theorem 8.5 and Proposition 8.6, we have  $v(\det(q^-)) = v(F(-1))$ ; since F(-1) = 0, this implies that  $v(\det(q^-)) = 0$ . Therefore  $\partial[M^0] = 0$ . This implies that  $\partial[M^1] = 0$ , and hence v(F(1)) + v(F(-1)) = 0. Since we already know that v(F(-1)) = 0, we obtain v(F(1)) = 0, and this implies (a).

Finally, if  $n^+ \neq 0$  and  $n^- \neq 0$ , then F(1) = F(-1) = 0, and hence both (a) and (b) hold. This concludes the proof of the theorem.

#### 6. Milnor signatures and Milnor indices

The aim of this section is to recall from [B 20] some notions of signatures and indices, inspired by Milnor's paper [M 68].

Let  $F \in \mathbf{R}[X]$  be a monic, symmetric polynomial. If (V,q) is a nondegenerate quadratic form over  $\mathbf{R}$  and if  $t: V \to V$  is a semi-simple isometry of q with characteristic polynomial F, we associate to each irreducible, symmetric factor  $\mathcal{P}$  of F an index  $\tau(\mathcal{P})$  and a signature  $\mu(\mathcal{P})$  as follows. Let  $V_{\mathcal{P}(t)}$  be the  $\mathcal{P}(t)$ -primary subspace of V, consisting of all  $v \in V$  with  $\mathcal{P}(t)^N v = 0$  for Nlarge. The *Milnor index*  $\tau(\mathcal{P})$  is by definition the index of the restriction of q to the subspace  $V_{\mathcal{P}(t)}$ , and we define the *Milnor signature*  $\mu(\mathcal{P})$  at  $\mathcal{P}$  as the signature of the restriction of q to  $V_{\mathcal{P}(t)}$ .

Let  $\operatorname{Irr}_{\mathbf{R}}(F)$  be the set of irreducible, symmetric factors of  $F \in \mathbf{R}[X]$ ; if  $\mathcal{P} \in \operatorname{Irr}_{\mathbf{R}}(F)$ , then either  $\operatorname{deg}(\mathcal{P}) = 2$ , or  $\mathcal{P}(X) = X \pm 1$ . If (r, s) is the signature of q, we have

$$\sum_{\mathcal{P}} \tau(\mathcal{P}) = r - s,$$

where the sum runs over  $\mathcal{P} \in \operatorname{Irr}_{\mathbf{R}}(F)$ .

If  $\mathcal{P} \in \operatorname{Irr}_{\mathbf{R}}(F)$ , let  $n_{\mathcal{P}} > 0$  be the integer such that  $\mathcal{P}^{n_{\mathcal{P}}}$  is the power of  $\mathcal{P}$  dividing F.

We denote by  $\operatorname{Mil}(F)$  the set of maps  $\tau : \operatorname{Irr}_{\mathbf{R}}(F) \to \mathbf{Z}$  such that the image of  $\mathcal{P} \in \operatorname{Irr}_{\mathbf{R}}(f)$  belongs to the set  $\{-\operatorname{deg}(\mathcal{P})n_{\mathcal{P}}, \ldots, \operatorname{deg}(\mathcal{P})n_{\mathcal{P}}\}$ . For all integers  $r, s \ge 0$ , let  $\operatorname{Mil}_{r,s}(F)$  be the subset of  $\operatorname{Mil}(F)$  such that  $\sum_{\mathcal{P}} \tau(\mathcal{P}) = r - s$ , where the sum runs over  $\mathcal{P} \in \operatorname{Irr}_{\mathbf{R}}(F)$ .

**Proposition 6.1.** Sending a semi-simple element  $SO_{r,s}(\mathbf{R})$  with characteristic polynomial F to its Milnor index induces a bijection between the conjugacy classes of semi-simple elements of  $SO_{r,s}(\mathbf{R})$  and  $Mil_{r,s}(F)$ .

**Proof.** See [B 20], §6.

#### 7. Local conditions for even, unimodular $\Gamma$ -lattices

Let  $F \in \mathbf{Z}[X]$  be a monic, symmetric polynomial, and let  $r, s \geq 0$  be integers such that  $r + s = \deg(F)$ . The aim of this section is to give local conditions for the existence of an even, unimodular lattice of signature (r, s) having a semi-simple isometry with characteristic polynomial F. More precisely, given a Milnor index  $\tau \in \operatorname{Mil}_{r,s}(F)$ , we give a necessary and sufficient condition for an even, unimodular lattice having a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau$  to exist everywhere locally.

Let m(F) be the number of roots z of F with |z| > 1 (counted with multiplicity).

**Proposition 7.1.** There exist an **R**-vector space V and a non-degenerate quadratic form q of signature (r, s) having a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau \in \text{Mil}_{r,s}(F)$  if and only if  $r \ge m(F)$ ,  $s \ge m(F)$ , and if moreover  $F(1)F(-1) \ne 0$ , then  $m(F) \equiv r \equiv s \pmod{2}$ . **Proof.** This follows from [B 15], Proposition 8.1. Indeed, the necessity of the conditions follows immediately from [B 15], Proposition 8.1. To prove the sufficiency, note that while the statement of [B 15], Proposition 8.1 only claims the existence of a non-degenerate quadratic form q of signature (r, s) having a semi-simple isometry with characteristic polynomial F, the proof shows the existence of such a form having a semi-simple isometry with a given Milnor index  $\tau \in \operatorname{Mil}_{r,s}(F)$ .

If p is a prime number, we say that a  $\mathbb{Z}_p$ -lattice (L, q) is even if  $q(x, y) \in 2\mathbb{Z}_p$ ; note that if  $p \neq 2$ , then every lattice is even, since 2 is a unit in  $\mathbb{Z}_p$ .

Assume moreover that F has even degree, and that F(0) = 1. Set  $2n = \deg(F)$ .

**Theorem 7.2.** There exists an even, unimodular  $\mathbb{Z}_p$ -lattice having a semisimple isometry with characteristic polynomial F for all prime numbers p if and only if |F(1)|, |F(-1)| and  $(-1)^n F(1)F(-1)$  are all squares.

**Proof.** This follows from Theorems 4.1, 5.1 and 5.2. Indeed, if p is a prime number  $\neq 2$ , the existence of a unimodular  $\mathbb{Z}_p$ -lattice having a semi-simple isometry with characteristic polynomial F implies that either F(1) = 0, or  $v_p(F(1))$  is even; similarly, either F(-1) = 0, or  $v_p(F(-1))$  is even (see Theorem 4.1). The existence of an even, unimodular  $\mathbb{Z}_2$ -lattice implies the same property for p = 2 by Theorem 5.2. This implies that |F(1)| and |F(-1)|are both squares, and therefore |F(1)F(-1)| is a square. If F(1)F(-1) = 0, we are done. If not, Theorem 5.2 implies that the class of  $(-1)^n F(1)F(-1)$ in  $\mathbb{Q}_2^{\times}/\mathbb{Q}_2^{\times 2}$  lies in  $\{1, -3\}$ ; since |F(1)F(-1)| is a square, this implies that  $(-1)^n F(1)F(-1)$  is a square. The converse is an immediate consequence of Theorems 4.1 and 5.1.

#### 8. The local-global problem

The aim of this section is to reformulate the local conditions of §7, and to give a framework for the local-global problem of the next sections. We also introduce some notation that will be used in the following sections,

Let  $F \in \mathbf{Z}[X]$  be a monic, symmetric polynomial of even degree such that F(0) = 1; set  $2n = \deg(F)$ . Let J be the set of irreducible factors of F, and let us write  $F = \prod_{f \in J} f^{n_f}$ . Let  $I_1 \subset J$  be the subset of irreducible factors of type 1, and let  $I_0 \subset J$  be the set of irreducible factors of type 0.

Let  $M = M^0 \oplus M^1 \oplus M^2$  be the semi-simple  $\mathbf{Q}[\Gamma]$ -module associated to the polynomial F (see §2).

Let  $r, s \ge 0$  be integers such that  $r + s = \deg(F)$  and that  $r \equiv s \pmod{8}$ . Let  $(V, q) = (V_{r,s}, q_{r,s})$  be the diagonal quadratic form over  $\mathbf{Q}$  with r entries 1 and s entries -1.

**Proposition 8.1.** The following properties are equivalent

#### EVA BAYER-FLUCKIGER

(i) For all prime numbers p there exists an even, unimodular  $\mathbf{Z}_p$ -lattice having a semi-simple isometry with characteristic polynomial F.

(ii) |F(1)|, |F(-1)| and  $(-1)^n F(1)F(-1)$  are all squares.

(iii) For all prime numbers p, the quadratic form  $(V, q) \otimes_{\mathbf{Q}} \mathbf{Q}_p$  has a semisimple isometry with characteristic polynomial F that stabilizes an even, unimodular lattice.

(iv) For all prime numbers p, the quadratic form  $(V,q) \otimes_{\mathbf{Q}} \mathbf{Q}_p$  has an isometry with module  $M \otimes_{\mathbf{Q}} \mathbf{Q}_p$ , giving rise to a class  $[M \otimes_{\mathbf{Q}} \mathbf{Q}_p, q]$  in  $W_{\Gamma}(\mathbf{Q}_p)$  such that  $\partial_p[M \otimes_{\mathbf{Q}} \mathbf{Q}_p, q] = 0$  in  $W_{\Gamma}(\mathbf{F}_p)$  and that  $v_2(\det(q_-)) \equiv v_2(F_1(-1)) \pmod{2}$ .

**Proof.** The equivalence of (i) and (ii) is Proposition 7.2, and it is clear that (iii) implies (i). Let us show that (i) implies (iii). Set  $u = (-1)^s$ . Since  $r + s \equiv 2n$  and  $r \equiv s \pmod{8}$ , we have  $n \equiv s \pmod{8}$ , hence  $u = (-1)^n$ . By (i), there exists an even, unimodular  $\mathbf{Z}_p$ -lattice having a semi-simple isometry with characteristic polynomial F. If  $F(1)F(-1) \neq 0$ , then the class of the determinant of this lattice in in  $\mathbf{Q}_p^{\times}/\mathbf{Q}_p^{\times 2}$  is equal to F(1)F(-1), and F(1)F(-1) = u by (ii); if F(1)F(-1) = 0, then by Theorem 4.4 (ii) and Theorem 5.1 (ii) we can assume that the determinant of this lattice in  $\mathbf{Q}_p^{\times}/\mathbf{Q}_p^{\times 2}$  is equal to u. Therefore the lattice is isomorphic to the diagonal  $\mathbf{Z}_p$ -lattice  $\langle 1, \ldots, u \rangle$  of determinant u if  $p \neq 2$  (cf. [O'M 73], 92:1), and to the orthogonal sum of n hyperbolic planes if p = 2 (see for instance [BT 20], Proposition 8.3). Let  $q^p$  be the quadratic form over  $\mathbf{Q}_p$  obtained from this lattice by extension of scalars; then the Hasse-Witt invariant of  $q^p$  is trivial if  $p \neq 2$ , and is equal to the Hasse-Witt invariant of the orthogonal sum of n hyperbolic planes if p = 2; its determinant is equal to  $u = (-1)^n$  in  $\mathbf{Q}_p^{\times}/\mathbf{Q}_p^{\times 2}$ . This implies that  $q^p$  and  $(V,q) \otimes \mathbf{Q}_p$  are isomorphic as quadratic forms over  $\mathbf{Q}_p$ . Since  $q^p$  has a semi-simple isometry with characteristic polynomial F that stabilizes a unimodular lattice, property (iii) holds. Finally, the equivalence of (iii) and (iv) follows from Theorem 1.1 (iv) and from [BT 20], Theorems 8.1 and 8.5.

**Terminology.** We say that the *local conditions for* F hold at the finite places if the equivalent conditions of Proposition 8.1 are satisfied.

Recall that m(F) is the number of roots z of F with |z| > 1 (counted with multiplicity).

**Proposition 8.2.** Let  $\tau \in \operatorname{Mil}_{r,s}(F)$  be a Milnor index. The following properties are equivalent :

(i) The quadratic form  $(V,q) \otimes_{\mathbf{Q}} \mathbf{R}$  has a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau$ .

(ii)  $r \ge m(F)$ ,  $s \ge m(F)$ , and if moreover  $F(1)F(-1) \ne 0$ , then  $m(F) \equiv r \equiv s \pmod{2}$ .

(iii) The quadratic form  $(V, q) \otimes_{\mathbf{Q}} \mathbf{R}$  has an isometry with module  $M \otimes_{\mathbf{Q}} \mathbf{R}$ and Milnor index  $\tau$ . **Proof.** The equivalence of (i) and (ii) follows from Proposition 7.1, and (iii) is a reformulation of (i).

**Terminology.** We say that the *local conditions for* F and  $\tau$  hold at the infinite place if the equivalent conditions of Proposition 8.2 are satisfied.

We consider the following conditions

(C 1) |F(1)|, |F(-1)| and  $(-1)^n F(1)F(-1)$  are all squares.

(C 2)  $r \ge m(F)$ ,  $s \ge m(F)$ , and if moreover  $F(1)F(-1) \ne 0$ , then  $m(F) \equiv r \equiv s \pmod{2}$ .

Note that the local conditions for F at the finite places hold if and only if condition (C 1) is satisfied, and that the local conditions for F and  $\tau$  hold if and only if condition (C 2) is satisfied.

**Terminology.** Let M and q be as above, and let p be a prime number. A  $\Gamma$ -quadratic form  $(M \otimes_{\mathbf{Q}} \mathbf{Q}_p, q)$  such that  $\partial_p[(M \otimes_{\mathbf{Q}} \mathbf{Q}_p, q] = 0$  in  $W_{\Gamma}(\mathbf{F}_p)$  and that  $v_2(\det(q_-)) \equiv v_2(F_1(-1)) \pmod{2}$  if p = 2 is called a *local solution* for F at the prime number p.

#### 9. $\mathbf{Q}[\Gamma]$ -forms, signatures and determinants

Let  $F \in \mathbf{Z}[X]$  be a monic, symmetric polynomial, and let us write  $F = F_0F_1F_2$ , where  $F_i$  is the product of the irreducible factors of type *i* of *F*. Let  $r, s \geq 0$  be integers such that  $r + s = \deg(F)$  and that  $r \equiv s \pmod{8}$ , and let  $\tau \in \operatorname{Mil}_{r,s}(F)$  be a Milnor index. Let (L, q) be an even, unimodular lattice having a semi-simple isometry with characteristic polynomial *F* and Milnor index  $\tau$ , and let (M, q) be the corresponding  $\mathbf{Q}[\Gamma]$ -form, and let

$$M = M^0 \oplus M^1 \oplus M^2$$

and

$$M^0 = M^+ \oplus M^-$$

be the associated orthogonal decompositions (cf. §3). Note that the Milnor index  $\tau$  and the degrees of the polynomials determine the signatures of the factors. We have  $\operatorname{sign}(M) = (r, s)$ . Set  $\operatorname{sign}(M^1) = (r_1, s_1)$ , and  $\operatorname{sign}(M^2) = (r_2, s_2)$ ; note that  $r_2 = s_2 = \operatorname{deg}(F_2)/2$ , since  $M^2$  is hyperbolic, and set

$$sign(M^+) = (r^+, s^+), sign(M^-) = (r_-, s_-).$$

We have  $\det(M) = (-1)^s$ ,  $\det(M^1) = F_1(1)F_1(-1) = (-1)^{s_1}|F_1(1)F_1(-1)|$ , and  $\det(M^2) = (-1)^{s_2}$ .

#### **Proposition 9.1.** We have

$$\det(M^+) = (-1)^{s_+} |F_1(1)|, \quad \det(M^-) = (-1)^{s_-} |F_1(-1)|$$

in  $\mathbf{Q}^{\times}/\mathbf{Q}^{\times 2}$ .

**Proof.** The sign of det $(M^{\pm})$  is  $(-1)^{s_{\pm}}$ . Let p be a prime,  $p \neq 2$ . By Lemma 4.6, the component of  $\partial_p[M^1]$  in  $W_{\Gamma}(\mathbf{F}_p, N_{\pm}) \simeq W(\mathbf{F}_p)$  is represented by a quadratic form of dimension  $v(F_1(\pm 1))$  over  $\mathbf{F}_p$ . If  $v(F_1(\pm 1)) = 0$ , then this component of  $\partial_p[M^1]$  is trivial, hence  $\partial_p[M^{\pm}]$  is also trivial. This implies that  $v(\det(M^{\pm}) = 0)$ . Assume now that  $v(F_1(\pm 1)) = 1$ . Then the component of  $\partial_p[M^1]$  in  $W_{\Gamma}(\mathbf{F}_p, N_{\pm}) \simeq W(\mathbf{F}_p)$  is represented by a quadratic form of dimension 1. Since  $\partial_p[M] = 0$ , this implies that  $\partial_p[M^{\pm}]$  is represented by a form of dimension 1, and therefore  $v(\det(M^{\pm}) = 1$ . Hence in this case too, we have  $v(\det(M^{\pm})) = v(F_1(\pm 1))$ .

Assume that p = 2. The component of  $\partial_2[M^1]$  in  $W_{\Gamma}(\mathbf{F}_2, N_{\pm}) \simeq W(\mathbf{F}_2)$  is represented by a quadratic form of dimension  $v(F_1(1)) + v(F_1(-1))$  over  $\mathbf{F}_2$ (see Lemma 4.6). If  $M^+ = 0$  and  $M^- = 0$ , there is nothing to prove. Assume that  $M^+ \neq 0$ , and  $M^- = 0$ . Then  $F_1(-1) = F(-1)$ , and by Theorem 5.2 (a), we have  $v(F_1(-1)) = 0$ . Hence  $\partial_2(M^1)$  is represented by a form of dimension  $v(F_1(1))$ . If  $v(F_1(1)) = 0$ , then  $\partial_2(M^1) = 0$ , and hence  $\partial_2(M^+) = 0$ ; therefore  $v(\det(M^+)) = 0$ . If  $v(F_1(1)) = 1$ , then  $\partial_2(M^1)$  is represented by a form of dimension 1 over  $\mathbf{F}_2$ , hence  $\partial_2(M^+)$  is also represented by a form of dimension 1 over  $\mathbf{F}_2$ . This implies that  $v(\det(M^+)) = 1$ . Therefore  $v(\det(M^+)) =$  $v(F_1(1))$ . The same argument shows that if  $M^+ = 0$  and  $M^- \neq 0$ , then  $v(\det(M^{-})) = v(F_1(-1))$ . Suppose now that  $M^+ \neq 0$  and  $M^- \neq 0$ . By [BT 20], Theorem 8.5 and Proposition 8.6, we have  $v(\det(M^{-})) = v(F_1(-1))$ . If  $v(F_1(1)) = v(F_1(-1))$ , then  $\partial_2(M^1) = 0$ . Therefore  $\partial_2(M^+ \oplus M^-) = 0$ . Since  $v(\det(M^{-})) = v(F_{1}(-1))$ , this implies that  $v(\det(M^{+})) = v(F_{1}(1))$ . If  $v(F_1(1)) + v(F_1(-1)) = 1$ , then  $\partial_2(M^1) \neq 0$ , and hence  $\partial_2(M^+ \oplus M^-) \neq 0$ . Therefore  $v(\det(M^+)) + v(\det(M^-)) = 1$ . Since  $v(\det(M^-)) = v(F_1(-1))$ , we have  $v(\det(M^+)) = v(F_1(1))$ . This completes the proof of the proposition.

## 10. Local decomposition

Let  $F \in \mathbf{Z}[X]$  be a monic, symmetric polynomial of even degree with F(0) = 1; set  $2n = \deg(F)$ . Let  $r, s \ge 0$  be integers such that  $r + s = \deg(F)$  and that  $r \equiv s \pmod{8}$ , let  $\tau \in \operatorname{Mil}_{r,s}(F)$  be a Milnor index. If the local conditions (C 1) and (C 2) hold, then we obtain a local solution everywhere (see §8). The aim of this section is to define local decompositions that will be useful in the following sections.

We start by introducing some notation. Let  $M = M^0 \oplus M^1 \oplus M^2$  be the semi-simple  $\mathbf{Q}[\Gamma]$ -module associated to F as in §2, with

$$M^1 = \bigoplus_{i \in I} M_f$$
 and  $M^0 = M^+ \oplus M^-$ .

If  $f \in I_1$ , set  $E_f = \mathbf{Q}[X]/(f)$  and let  $\sigma_f : E_f \to E_f$  be the involution induced by  $X \mapsto X^{-1}$ . Let  $(E_f)_0$  be the fixed field of  $\sigma_f$ , and let  $d_f \in (E_f)_0$  be such that  $E_f = (E_f)_0(\sqrt{d_f})$ . Note that  $M_f$  is an  $E_f$ -vector space of dimension  $n_f$ . Let

$$Q_f: M_f \times M_f \to \mathbf{Q}$$

be the orthogonal sum of  $n_f$  copies of the quadratic form  $E_f \times E_f \to \mathbf{Q}$  defined by  $(x, y) \mapsto \operatorname{Tr}_{E_f/\mathbf{Q}}(x\sigma_f(y))$ .

The Milnor index  $\tau \in \operatorname{Mil}_{r,s}(F)$  determines the signatures of  $M^+$  and  $M^-$ , as follows. Recall that  $\dim(M^+) = n_+$  and  $\dim(M^+) = n_+$ .

Let  $s_{\pm}$  and  $s_{\pm}$  be as in §9, and set  $D_{\pm} = (-1)^{s_{\pm}} F_1(\pm 1)$ ; let  $Q_{\pm}$  be the diagonal quadratic form of dimension  $n_{\pm}$  over **Q** defined by  $Q_{\pm} = \langle D_{\pm}, 1, \ldots, 1 \rangle$ . Let Q be the orthogonal sum

$$Q = \bigoplus_{f \in I_1} Q_f \oplus Q_+ \oplus Q_-.$$

$$sign(M^+) = (r^+, s^+), \ sign(M^-) = (r_-, s_-).$$

Recall form §8 that we denote by  $q = q_{r,s}$  the diagonal quadratic form over **Q** having r diagonal entries 1 and s diagonal entries -1.

Assume that conditions (C 1) and (C 2) hold. If p is a prime number, then  $(M, q) \otimes_{\mathbf{Q}} \mathbf{Q}_p$  has a structure of a  $\mathbf{Q}_p[\Gamma]$ -quadratic form (see §8), and we have the orthogonal decomposition (cf. §3).

$$(M,q) \otimes_{\mathbf{Q}} \mathbf{Q}_p = \bigoplus_{f \in I_1} (M_f^p, q_f^p) \oplus (M_+^p, q_+^p) \oplus (M_-^p, q_-^p) \oplus (M_2^p, q_2^p),$$

where  $M_f^p = M_f \otimes_{\mathbf{Q}} \mathbf{Q}_p$ ,  $M_+^p = M^+ \otimes_{\mathbf{Q}} \mathbf{Q}_p$ ,  $M_-^p = M^- \otimes_{\mathbf{Q}} \mathbf{Q}_p$ , and  $M_2^p = M^2 \otimes_{\mathbf{Q}} \mathbf{Q}_p$ . The  $\mathbf{Q}_p[\Gamma]$ -quadratic form  $(M_2^p, q_2^p)$  is hyperbolic.

For  $f \in I_1$ , set  $E_f^p = E_f \otimes_{\mathbf{Q}} \mathbf{Q}_p$  and  $(E_f)_0^p = (E_f)_0 \otimes_{\mathbf{Q}} \mathbf{Q}_p$ . There exists a unique non-degenerate hermitian form  $(M_f^p, h_f^p)$  over  $(E_f^p, \sigma_f)$  such that

$$q_f^p(x,y) = \operatorname{Tr}_{E_f^p/\mathbf{Q}_p}(h_f^p(x,y)),$$

see for instance [M 69], Lemma 1.1 or [B 15], Proposition 3.6.

Set  $\lambda_f^p = \det(h_f^p) \in (E_f^p)_0^{\times}/N_{E_f^p/(E_f^p)_0}$ . Note that the hermitian form  $h_f^p$  is isomorphic to the  $n_f$ -dimensional diagonal hermitian form  $\langle \lambda_f^p, 1, \ldots, 1 \rangle$  over  $E_f^p$ . Hence  $q_f^p$  is determined by  $\lambda_f^p$ .

Notation 10.1. With the notation above, set

$$\partial_p(\lambda_f^p) = \partial_p[q_f^p] \in W_{\Gamma}(\mathbf{F}_p).$$

**Proposition 10.2.** We have  $\dim(q_f^p) = \deg(f)n_f$ ,  $\det(q_f^p) = [f(1)f(-1)]^{n_f}$ , and the Hasse-Witt invariant of  $q_f^p$  satisfies

$$w_2(q_f^p) + w_2(Q_f) = \operatorname{cor}_{(E_f)_0^p/\mathbf{Q}_p}(\det(h_f^p), d_f)$$

in  $\operatorname{Br}_2(\mathbf{Q}_p)$ .

**Proof.** The assertion concerning the dimension is clear, the one on the determinant follows from Lemma 3.1, and the property of the Hasse-Witt invariants from [B 20], Proposition 12.8.

**Proposition 10.3.** (a)  $\dim(q_{\pm}^p) = n_{\pm}$  and

 $\det(q_{+}^{p})\det(q_{+}^{p}) = (-1)^{s_{+}+s_{-}}|F_{1}(1)F_{1}(-1)|.$ 

(b) If  $n_+ \neq 0$  and  $n_- \neq 0$ , then we can choose  $q_+^p$  and  $q_-^p$  such that  $\det(q_{\pm}^p) = (-1)^{s_{\pm}} |F_1(\pm 1)|$ .

(c) If  $n_{\pm} \neq 0$ , then the Hasse-Witt invariant of  $q_{\pm}^p$  can take either of the two possible values of  $\{0, 1\} = Br_2(\mathbf{Q}_p)$ .

(d) If  $p \neq 2$ , then  $\partial_p[q_{\pm}^p]$  can be either of the two possible classes of dimension  $v_p(\det(q_{\pm}^p))$  of  $W_{\Gamma}(\mathbf{F}_p, N^{\pm}) \simeq W(\mathbf{F}_p)$ .

**Proof.** (a) is clear, (b) follows from Theorem 4.4 (ii) and Theorem 5.1 (ii); (c) and (d) are straightforward to check.

We also need the following

**Lemma 10.4.** Let p be a prime number,  $p \neq 2$ , and let  $b_1$  and  $b_2$  be two quadratic forms over  $\mathbf{Q}_p$  with  $\dim(b_1) = \dim(b_2)$  and  $\det(b_1) = \det(b_2)$ . Then we have

$$w_2(b_1) = w_2(b_2)$$
 in  $\operatorname{Br}_2(\mathbf{Q}_p) \iff \partial_p[b_1] = \partial_p[b_2]$  in  $W(\mathbf{F}_p)$ .

**Proof.** The proof is straightforward.

Similarly, we have

$$(M,q) \otimes_{\mathbf{Q}} \mathbf{R} = \bigoplus_{f \in I_1} (M_f^{\infty}, q_f^{\infty}) \oplus (M_+^{\infty}, q_+^{\infty}) \oplus (M_-^{\infty}, q_-^{\infty}) \oplus (M_2^{\infty}, q_2^{\infty}),$$

where  $M_f^{\infty} = M_f \otimes_{\mathbf{Q}} \mathbf{R}, \ M_+^{\infty} = M^+ \otimes_{\mathbf{Q}} \mathbf{R}, \ M_-^{\infty} = M^- \otimes_{\mathbf{Q}} \mathbf{Q}_p$ , and  $M_2^{\infty} = M^2 \otimes_{\mathbf{Q}} \mathbf{Q}_p$ . The  $\mathbf{R}[\Gamma]$ -quadratic form  $(M_2^{\infty}, q_2^p)$  is hyperbolic.

The  $\mathbf{R}[\Gamma]$ -quadratic forms  $(M_f^{\infty}, q_f^{\infty})$  and  $(M_{\pm}^{\infty}, q_{\pm}^{\infty})$  are determined by the Milnor index  $\tau \in \operatorname{Mil}_{r,s}(F)$ .

**Proposition 10.5.** We have  $\dim(q_f^{\infty}) = \deg(f)n_f$ ,  $\det(q_f^{\infty}) = [f(1)f(-1)]^{n_f}$ , and the Hasse-Witt invariant of  $q_f^{\infty}$  satisfies

$$w_2(q_f^{\infty}) + w_2(Q_f) = \operatorname{cor}_{(E_f)_0^{\infty}/\mathbf{R}}(\det(h_f^{\infty}), d_f)$$

in  $\operatorname{Br}_2(\mathbf{R})$ .

**Proof.** The assertion concerning the dimension is clear, the one on the determinant follows from Lemma 3.1, and the property of the Hasse-Witt invariants from [B 20], Proposition 12.8.

For  $f \in I_1$ , set  $E_f^{\infty} = E_f \otimes_{\mathbf{Q}} \mathbf{R}$  and  $(E_f)_0^{\infty} = (E_f)_0 \otimes_{\mathbf{Q}} \mathbf{R}$ . There exists a unique non-degenerate hermitian form  $(M_f^{\infty}, h_f)$  over  $(E_f^{\infty}, \sigma_f)$  such that

$$q_f^{\infty}(x,y) = \operatorname{Tr}_{E_f^{\infty}/\mathbf{R}}(h_f^{\infty}(x,y)),$$

see for instance [M 69], Lemma 1.1 or [B 15], Proposition 3.6. Set

$$\lambda_f^{\infty} = \det(h_f^{\infty}) \in (E_f^{\infty})_0^{\times} / \mathcal{N}_{E_f^{\infty}/(E_f^{\infty})_0}.$$

#### 11. Obstruction group

We keep the notation of the previous sections. The aim of this section is to define a finite elementary abelian 2-group that will play an important role in the Hasse principle (see §13). Recall that J is the set of irreducible factors of the polynomial F, that  $I_0 \subset J$  is the set of factors of type 0,  $I_1 \subset J$  is the set of factors of type 1, and  $I = I_0 \cup I_1$ .

Notation 11.1. If  $f \in \mathbb{Z}[X]$  is an irreducible, symmetric polynomial of even degree, set  $E_f = \mathbb{Q}[X]/(f)$ , let  $\sigma_f : E_f \to E_f$  be the involution induced by  $X \mapsto X^{-1}$ , and let  $(E_f)_0$  be the fixed field of  $\sigma$  in  $E_f$ . Let  $\alpha \in E_f$  be the image of X.

**Definition 11.2.** Let  $f \in \mathbb{Z}[X]$  be an irreducible, symmetric polynomial of even degree, and let p be a prime number. We say that f is *ramified at* p if there exists a place w of  $(E_f)_0$  above p that is ramified in  $E_f$ ; otherwise, we say that f is *unramified* at p. We denote by  $\Pi_f^r$  the set of prime numbers p such that f is ramified at p.

Let  $p \in \Pi_f^r$  be an odd prime number. If w is a place of  $(E_f)_0$  above p that is ramified in  $E_f$ , we denote by  $\kappa_w$  the residue field of w, and by  $\overline{\alpha}$  be the image of  $\alpha$  in  $\kappa_w$ ; we denote by  $S_+$  the set of places w above p such that  $\overline{\alpha} = 1$ , and by  $S_-$  the set of places w above p such that  $\overline{\alpha} = -1$ . We denote by  $\Pi_f^{r,+}$  the set of prime numbers p such that there exists a place w above p with  $w \in S_+$ , and by  $\Pi_f^{r,-}$  the set of prime numbers p such that there exists a place w above p with  $w \in S_-$ .

Notation 11.3. If  $f, g \in \mathbb{Z}[X]$  are monic, irreducible, symmetric polynomials of even degree, we denote by  $\Pi_{f,g}$  the set of prime numbers p such that one of the following conditions holds :

(a) The polynomial f has a symmetric, irreducible factor  $f' \in \mathbf{Z}_p[X]$ , the polynomial g has a symmetric, irreducible factor  $g' \in \mathbf{Z}_p[X]$ , such that  $f' \pmod{p}$  and  $g' \pmod{p}$  have a common irreducible, symmetric factor in  $\mathbf{F}_p[X]$ .

(b)  $p \in \Pi_f^r \cap \Pi_g^r$ , and the polynomials  $f \pmod{p}$  and  $g \pmod{p}$  are both divisible by X - 1 in  $\mathbf{F}_p[X]$ .

(c)  $p \in \Pi_f^r \cap \Pi_g^r$ , and the polynomials  $f \pmod{p}$  and  $g \pmod{p}$  are both divisible by X + 1 in  $\mathbf{F}_p[X]$ .

$$F_1 = \prod_{f \in I_1} f^{n_f}$$
 and  $F_0(X) = (X-1)^{n^+} (X+1)^{n^-}$ 

for some integers  $n^+, n^- \ge 0$ .

For all prime numbers p, let  $D^p_+, D^p_- \in \mathbf{Q}_p^{\times}/\mathbf{Q}_p^{\times 2}$ .

**Notation 11.4.** If  $f \in I_1$ , let  $\Pi_{f,X-1}$  be the set of prime numbers p such that  $p \in \Pi_f^r$ , that  $f \pmod{p}$  is divisible by X - 1 in  $\mathbf{F}_p[X]$ , and that if  $n^+ = 2$ , then  $D_+^p \neq -1$ .

Let  $\Pi_{f,X+1}$  be the set of prime numbers p such that  $p \in \Pi_f^r$ , and that  $f \pmod{p}$  is divisible by X + 1 in  $\mathbf{F}_p[X]$ , and that if  $n^- = 2$ , then  $D_-^p \neq -1$ .

Let  $\Pi_{X-1,X+1} = \{2\}$  if the following conditions hold :  $n^+ \neq 0$ ,  $n^- \neq 0$ , and if  $n^+ = 2$ , then  $D^2_+ \neq -1$ ; if  $n^- = 2$ , then  $D^2_- \neq -1$ . Otherwise, set  $\Pi_{X-1,X+1} = \emptyset$ .

We denote by C(I) the set of maps  $I \to \mathbb{Z}/2\mathbb{Z}$ .

Notation 11.5. If  $f, g \in I$ , let  $c_{f,g} \in C(I)$  be such that

$$c_{f,g}(f) = c_{f,g}(g) = 1$$
, and  $c_{f,g}(h) = 0$  if  $h \neq f, g$ .

Let  $(f,g): C(I) \to C(I)$  be the map map sending c to  $c + c_{f,g}$ .

**Notation 11.6.** Let  $C_0(I)$  be the set of  $c \in C(I)$  such that

$$c(f) = c(g)$$
 if  $\Pi_{f,g} \neq \emptyset$ ,

and we denote by  $\operatorname{III}_F(D_+, D_-)$  the quotient of the group  $C_0(I)$  by the subgroup of constant maps.

In general, the group depends on  $D_+ = (D_+^p)$  and  $D_- = (D_-^p)$ . If  $n^+ \neq 2$ and  $n^- \neq 2$ , then  $\coprod_F (D_+, D_-)$  only depends on F, and we denote it by  $\coprod_F$ .

## 12. Local data

We keep the notation of §10. Assume that conditions (C 1) and (C 2) of §8 hold, and recall that this implies the existence of a "local solution" everywhere. This leads, for all prime numbers p, to an orthogonal decomposition of the associated  $\mathbf{Q}_p[\Gamma]$ -bilinear form (see §10). We obtain in this way a collection of  $\mathbf{Q}_p[\Gamma]$ -bilinear forms, one for each irreducible, symmetric factor of the characteristic polynomial. The dimensions and determinants of the bilinear forms are always the same, but their Hasse-Witt invariants vary.

The aim of this section is to give a combinatorial encoding of the possible Hasse-Witt invariants, called "local data".

We identify  $\operatorname{Br}_2(\mathbf{R})$  and  $\operatorname{Br}_2(\mathbf{Q}_p)$ , where p is a prime number, with  $\{0, 1\} = \mathbf{Z}/2\mathbf{Z}$ . Let  $\mathcal{V}$  be the set of all places of  $\mathbf{Q}$ , and let  $\mathcal{V}'$  be the set of finite places.

If p is a prime number, let  $q_f^p$  for  $f \in I_1$ ,  $q_+^p$  and  $q_-^p$  be as in §10; recall that if  $n_+ \neq 0$  and  $n_- \neq 0$ , we choose  $q_{\pm}^p$  such that  $\det(q_{\pm}^p) = (-1)^{s_{\pm}} |F_1(\pm 1)|$  (see Proposition 10.3 (b)).

Let  $a^p \in C(I)$  be the map defined as follows :

$$a^{p}(f) = w_{2}(q_{f}^{p}) + w_{2}(Q_{f})$$

if  $f \in I_1$ , set

$$a^{p}(X \pm 1) = w_{2}(q_{\pm}^{p}) + w_{2}(Q_{\pm}).$$

Let  $\mathcal{C}^p$  be the set of maps  $a^p \in C(I)$  obtained in this way.

**Proposition 12.1.** For almost all prime numbers p, the zero map belongs to the set  $C^p$ .

**Proof.** Let S be the set of prime numbers such that p is ramified in the extension  $E_f/\mathbf{Q}$  for some  $f \in I_1$ , or  $w_2(q) \neq w_2(Q)$  in  $\operatorname{Br}_2(\mathbf{Q}_p)$ ; this is a finite set. We claim that if  $p \notin S$ , then the zero map belongs to  $\mathcal{C}^p$ . Indeed, set  $q_f^p = Q_f^p$  for all  $f \in I_1$ , and  $q_{\pm}^p = Q_{\pm}^p$ . We have  $\det(q) = \det(Q)$  in  $\mathbf{Q}^{\times}/\mathbf{Q}^{\times 2}$ , and if  $p \notin S$  we have  $w_2(q) = w_2(Q)$  in  $\operatorname{Br}_2(\mathbf{Q}_p)$ , therefore, for  $p \notin S$ , we have

$$(M,q)\otimes_{\mathbf{Q}}\mathbf{Q}_p = \bigoplus_{f\in I_1} (M_f^p, q_f^p) \oplus (M_+^p, q_+^p) \oplus (M_-^p, q_-^p) \oplus (M_2^p, q_2^p).$$

If p is unramified in all the extensions  $E_f/\mathbf{Q}$  for  $f \in I_1$ , by [B 20], Lemma 11.2 we have  $\partial_p[M_f^p, q_f^p] = 0$  in  $W_{\Gamma}(\mathbf{F}_p)$ ; moreover,  $v_p(D_{\pm}) = 0$ , hence  $\partial[M_{\pm}^p, q_{\pm}^p] = 0$  in  $W_{\Gamma}(\mathbf{F}_p)$ .

The above arguments show that if  $p \notin S$ , then the choice of  $q_f^p = Q_f^p$  for all  $f \in I_1$  and  $q_{\pm}^p = Q_{\pm}^p$  gives rise to the element  $a^p = 0$  of  $\mathcal{C}^p$ ; therefore the zero map is in  $\mathcal{C}^p$ , as claimed. This completes the proof of the proposition.

Proposition 10.2 implies that if  $f \in I_1$ , then  $a^p(f)$  is determined by  $\det(h_f^p)$ . Set  $\lambda_f^p = \det(h_f^p) \in (E_f^p)_0^{\times}/\mathcal{N}_{E_f^p/(E_f^p)_0}$ . Set  $E^p = \prod_{f \in I_1} E_f^p$  and  $E_0^p = \prod_{f \in I_1} (E_f^p)_0$ ; the map  $a^p$  is determined by  $\lambda^p \in (E_0^p)^{\times}/\mathcal{N}_{E^p/E_0^p}(E^p)^{\times})$ , and the quadratic forms  $q_{\pm}^p$ .

Notation 12.2. With the above notation, we set  $a^p = a^p[\lambda^p, q^p_{\pm}] = a[\lambda^p, q^p_{+}, q^p_{-}]$ . Notation 12.3. If  $f, g \in I$ , let  $c_{f,g} \in C(I)$  be such that

$$c_{f,g}(f) = c_{f,g}(g) = 1$$
 and  $c_{f,g}(h) = 0$  if  $h \neq f, g$ .  
Let  $(f,g): C(I) \to C(I)$  be the map map sending  $c$  to  $c + c_{f,g}$ 

Recall that for all  $f, g \in I$ , the set  $\Pi_{f,g}$  consists of the prime numbers p such that  $f \pmod{p}$  and  $g \pmod{p}$  have a common symmetric factor in  $\mathbf{F}_p[X]$ .

If p is a prime number, let us consider the equivalence relation on C(I) generated by the elementary equivalence

$$a \sim b \iff b = (f, g)a$$
 with  $p \in \prod_{f,g}$ .

We denote by  $\sim_p$  this equivalence relation.

**Proposition 12.4.** The set  $C^p$  is a  $\sim_p$ -equivalence class of C(I).

**Proof.** Set  $A^p = w_2(q) + w_2(Q)$  in  $\operatorname{Br}_2(\mathbf{Q}_p) = \mathbf{Z}/2\mathbf{Z}$ , and note that for all  $a^p \in \mathcal{C}^p$ , we have  $\sum_{f \in J} a^p(f) = A^p$ .

We start by proving that the set  $\mathcal{C}^p$  is stable by the maps (f,g) for  $p \in \Pi_{f,g}$ . Let  $a^p[\lambda^p, q^p_{\pm}] \in \mathcal{C}^p$ , let  $f, g \in J$  be such that  $p \in \Pi_{f,g}$ , and let us show that  $(f,g)(a^p[\lambda^p, q^p_{\pm}]) \in \mathcal{C}^p$ . Note that if  $f \in I_1$ , then  $p \in \Pi_{f,g}$  implies that  $(E_f^p)^{\times}_0/N_{E_f^p/(E_f^p)_0}(E_f^p) \neq 0$ . Assume first that  $f, g \in I$ . There exist  $\mu_f, \mu_g \in (E_f^p)^{\times}_0/N_{E_f^p/(E_f^p)_0}(E_f^p)$  such that  $\operatorname{cor}_{(E_f)^p_0/\mathbf{Q}_p}(\mu_f, d_f) \neq \operatorname{cor}_{(E_f)^p_0/\mathbf{Q}_p}(\lambda_f, d_f)$  and  $\operatorname{cor}_{(E_f)^p_0/\mathbf{Q}_p}(\mu_g, d_g) \neq \operatorname{cor}_{(E_f)^p_0/\mathbf{Q}_p}(\lambda_g, d_g)$ . Let  $\mu^p \in E_0^p$  be obtained by replacing  $\lambda_f^p$  by  $\mu_f^p$ ,  $\lambda_f^p$  by  $\mu_f^p$ , and leaving the other components unchanged. We have  $a^p[\mu^p, q_{\pm}^p] = (f, g)(a^p[\lambda^p, q_{\pm}^p])$ . Using the arguments of [B 20], Propositions 16.5 and 22.1 we see that  $a^p[\mu^p, q_{\pm}^p] \in \mathcal{C}^p$ . Assume now that  $f \in I_1$  and g = X - 1. In this case, the hypothesis  $p \in \prod_{f,g}$  implies that there exists a place w of  $(E_f)_0$  above p that ramifies in  $E_f$ , and such that the w-component  $\lambda^w$  of  $\lambda^p$  is such that with the notation of [B 20], §22,  $\partial_p(\lambda^w)$  is in  $W_{\Gamma}(\mathbf{F}_p, N_+)$ . We modify the w-component of  $\lambda^p$  to obtain  $\mu_f \in (E_f^p)_0^{\times}/N_{E_f^p/(E_f^p)_0}(E_f^p)$  such that  $\operatorname{cor}_{(E_f)_0^p/\mathbf{Q}_p}(\mu_f, d_f) \neq \operatorname{cor}_{(E_f)_0^p/\mathbf{Q}_p}(\lambda_f, d_f)$ , and let  $b^p$  be a quadratic form over  $\mathbf{Q}_p$  with  $\dim(b^p) = \dim(q_+^p)$ ,  $\det(b^p) = \det(q_+^p)$ , and  $w_2(b^p) = w_2(q_+^p) + 1$ . We have  $a^p[\mu^p, b^p, q_-^p] = (f, g)(a^p[\lambda^p, q_+^p])$ . The arguments of [B 20], Propositions 16.5 and 22.1 show that  $a^p[\mu^p, b^p, q_-^p] \in \mathcal{C}^p$ .

Conversely, let us show that if  $a^p[\lambda^p, q_{\pm}^p]$  and  $a^p[\mu^p, b_{\pm}^p]$  are in  $\mathcal{C}^p$ , then  $a^p[\lambda^p, q_{\pm}^p] \sim_p a^p[\mu^p, b_{\pm}^p]$ . Let J' be the set of  $f \in J$  such that  $a^p[\lambda^p, q_{\pm}^p](f) \neq a^p[\mu^p, b_{\pm}^p](f)$ . Since  $\sum_{h \in J} a^p(h) = A^p$  for all  $a^p \in \mathcal{C}^p$ , the set J' has an even number of elements.

Assume first that  $p \neq 2$ . This implies that for all  $f \in J'$ , we have  $\partial_p(\lambda_f^p) \neq \partial_p(\mu_f^p)$  and that if  $f(X) = X \pm 1$ , then  $\partial_p(q_{\pm}^p) \neq \partial_p(b_{\pm}^p)$ . Hence there exist  $f, g \in J'$  with  $f \neq g$  such that  $\partial_p(W_{\Gamma}(\mathbf{Q}_p, M_f^p))$  and  $\partial_p(W_{\Gamma}(\mathbf{Q}_p, M_g^p))$  have a non-zero intersection. This implies that  $p \in \Pi_{f,g}$ . The element  $f, g)(a^p[\lambda^p, q_{\pm}^p])$  differs from  $a^p[\mu^p, b_{\pm}^p]$  in less elements than  $a^p[\lambda^p, q_{\pm}^p]$ . Since J' is a finite set, continuing this way we see that  $a^p[\lambda^p, q_{\pm}^p] \sim_p a^p[\mu^p, b_{\pm}^p]$ .

Suppose now that p = 2. Let J'' be the set of  $f \in J'$  such that  $\partial_2(\lambda_f^2) \neq \partial_2(\mu_f^2)$ , and note that J'' has an even number of elements. The same argument as in the case  $p \neq 2$  shows that applying maps (f, g), we can assume that  $J'' = \emptyset$ . If  $f \in J'$  and  $f \notin J''$ , then  $\partial_2(\lambda_f^2)$  belongs to  $W_{\Gamma}(\mathbf{F}_2, 1) \subset W_{\Gamma}(\mathbf{F}_2)$ . Therefore  $f, g \in J'$  and  $f, g \notin J''$ , then  $2 \in \Pi_{f,g}$ . The number of these elements is also even, hence after a finite number of elementary equivalences we see that  $a^p[\lambda^p, q_{\pm}^p] \sim_p a^p[\mu^p, b_{\pm}^p]$ . This completes the proof of the proposition.

Notation 12.5. Let  $a^p \in \mathcal{C}^p$ , and let  $c \in C(I)$ . Set

$$\epsilon_{a^p}(c) = \sum_{f \in I} c(f) a^p(f)$$

Recall from §11 that  $C_0(I)$  is the set of  $c \in C(I)$  such that

$$c(f) = c(g)$$
 if  $\Pi_{f,g} \neq \emptyset$ .

**Lemma 12.6.** Let  $a^p$ ,  $b^p$  be two elements of  $\mathcal{C}^p$ , and let  $c \in C_0(I)$ . Then

$$\epsilon_{a^p}(c) = \epsilon_{b^p}(c).$$

**Proof.** By Proposition 12.4, we have  $a^p \sim_p b^p$ ; we can assume that  $b^p = (f,g)a^p$  with  $p \in \prod_{f,g}$ . By definition, we have  $b^p(h) = a^p(h)$  if  $h \neq f, g$ ,  $b^p(f) = a^p(f) + 1$  and  $b^p(g) = a^p(g) + 1$ . Since  $c \in C_0(I)$  and  $\prod_{f,g} \neq \emptyset$ , we have c(f) = c(g), and this shows that  $\epsilon_{a^p}(c) = \epsilon_{b^p}(c)$ , as claimed.

Since  $\epsilon_{a^p}(c)$  does not depend on the choice of  $a^p \in \mathcal{C}^p$ , we set  $\epsilon^p(c) = \epsilon_{a^p}(c)$  for some  $a^p \in \mathcal{C}^p$ , and obtain a map

$$\epsilon^p: C_0(I) \to \mathbf{Z}/2\mathbf{Z}.$$

By Proposition 12.1, we have  $\epsilon^p = 0$  for almost all prime numbers p.

Let  $\epsilon^{\text{finite}} = \sum \epsilon^p$ , where the sum is taken over all the prime numbers p; this is a finite sum. Note that if (X-1)(X+1) does not divide F, then  $\epsilon^{\text{finite}}$  only depends on F; it does not depend of the choice of the Milnor signature  $\tau$ . We have a homomorphism

$$\epsilon^{\text{finite}} : C_0(F) \to \mathbf{Z}/2\mathbf{Z}.$$

Let  $v_{\infty} \in \mathcal{V}$  be the unique infinite place. Recall that the forms  $q_f^{\infty}$  and  $q_{\pm}^{\infty}$  are uniquely determined by the choice of the Milnor index  $\tau \in \operatorname{Mil}_{r,s}(F)$ . Let  $a^{\infty} \in C(I)$  be the map defined as follows:

if 
$$f \in I_1$$
,  
 $a^{\infty}(f) = w_2(q_f^{\infty}) + w_2(Q_f)$   
 $a^{\infty}(X \pm 1) = w_2(q_{\pm}^{\infty}) + w_2(Q_{\pm}),$ 

and

$$a^{\infty}(f) = 0$$
 if  $f \in J$  with  $f \notin I, f \neq X \pm 1$ .

We obtain a map

$$\epsilon^{\infty}_{\tau}: C(I) \to \mathbf{Z}/2\mathbf{Z}$$

by setting

$$\epsilon^{\infty}_{\tau}(c) = \sum_{f \in J} c(f) a^{\infty}(f).$$

For  $v \in \mathcal{V}$ , set  $\epsilon^v = \epsilon^p$  if  $v = v_p$ , and  $\epsilon^v = \epsilon_\tau^\infty$  if  $v = v_\infty$ . Set

$$\epsilon_{\tau}(c) = \sum_{v \in \mathcal{V}} \epsilon^{v}(c).$$

Since  $\epsilon^v = 0$  for almost all  $v \in \mathcal{V}$  (cf. Proposition 12.1), this is a finite sum. We have  $\epsilon_\tau = \epsilon^{\text{finite}} + \epsilon_\tau^\infty$ . We obtain a homomorphism

$$\epsilon_{\tau}: C_0(I) \to \mathbf{Z}/2\mathbf{Z}.$$

Recall from §11 that  $\operatorname{III}_F(D_+, D_-)$  is the quotient of  $C_0(I)$  by the constant maps.

**Proposition 12.7.** The homomorphism  $\epsilon_{\tau} : C_0(I) \to \mathbb{Z}/2\mathbb{Z}$  induces a homomorphism

$$\epsilon_{\tau}: \operatorname{III}_{F}(D_{+}, D_{-}) \to \mathbf{Z}/2\mathbf{Z}.$$

**Proof.** It suffices to show that if c(f) = 1 for all  $f \in J$ , then  $\epsilon(c) = 0$ . For all  $v \in \mathcal{V}$ , set  $A^v = w_2(q) + w_2(Q)$  in  $\operatorname{Br}_2(\mathbf{Q}_v) = \mathbf{Z}/2\mathbf{Z}$ , where  $\mathbf{Q}_v$  is either  $\mathbf{R}$  or  $\mathbf{Q}_p$ , for a prime number p. Note that  $A^v = 0$  for almost all  $v \in \mathcal{V}$ , and that  $\sum_{v \in \mathcal{V}} A^v = 0$ . Moreover, for all  $a^v \in \mathcal{C}^v$ , we have by definition  $\sum_{f \in J} a^v(f) = A^v$ .

Let  $c \in C(I)$  be such that c(f) = 1 for all  $f \in J$ . We have

$$\epsilon_{\tau}(c) = \sum_{v \in \mathcal{V}} \sum_{f \in J} c(f) \ a^{v}(f) = \sum_{v \in \mathcal{V}} \sum_{f \in J} a^{v}(f) = \sum_{v \in \mathcal{V}} A^{v} = 0.$$

#### 13. Hasse Principle

We keep the notation of the previous sections; in particular,  $F \in \mathbb{Z}[X]$  is a monic, symmetric polynomial of even degree such that F(0) = 1 and we set  $2n = \deg(F)$ . Let  $r, s \ge 0$  be integers such that  $r + s = \deg(F)$  and that  $r \equiv s \pmod{8}$ , and let  $\tau \in \operatorname{Mil}_{r,s}(F)$  be a Milnor index. We assume that conditions (C 1) and (C 2) hold.

Recall from \$12 that we have a homomorphism

$$\epsilon_{\tau} : \mathrm{III}_F(D_+, D_-) \to \mathbf{Z}/2\mathbf{Z}.$$

**Theorem 13.1.** There exists an even, unimodular lattice having a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau$  if and only if  $\epsilon_{\tau} = 0$ .

**Proof.** Assume that there exists an even, unimodular lattice (L, q) having a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau$ , and let (M, q) be the associated  $\mathbf{Q}[\Gamma]$ -quadratic form. Let  $M^0 \oplus M^1 \oplus M^2$  the corresponding orthogonal decomposition of §9. We have the further orthogonal decompositions  $(M^1, q^1) = \bigoplus_{f \in I_1} (M_f, q_f)$ , and  $(M^0, q^0) = (M^+, q^+) \oplus (M^-, q^-)$ 

(see §3). For all prime numbers p, this gives rise to a local decomposition as in §10, and to an element  $a^p \in \mathcal{C}^p$  given by  $a^p(f) = w_2(q_f^p) + w_2(Q_f)$  if  $f \in I_1$ , by  $a^p(X \pm 1) = w_2(q_{\pm}^p) + w_2(Q_{\pm})$ , and  $a^p(f) = 0$  if  $f \in J$  with  $f \notin I$ ,  $f \neq X \pm 1$  (see §12). Similarly, we have the element  $a^{\infty} \in C(I)$  given by  $a^{\infty}(f) = w_2(q_f^{\infty}) + w_2(Q_f)$  if  $f \in I_1$ ,  $a^{\infty}(X \pm 1) = w_2(q_{\pm}^{\infty}) + w_2(Q_{\pm})$ , and  $a^{\infty}(f) = 0$  if  $f \in J$  with  $f \notin I$ ,  $f \neq X \pm 1$ . Since  $q_f^p = q_f \otimes_{\mathbf{Q}} \mathbf{Q}_p$  and  $q_f^{\infty} = q_f \otimes_{\mathbf{Q}} \mathbf{R}$  for all  $f \in J$ , we have

$$\sum_{v \in \mathcal{V}} a^v(f) = 0 \text{ for all } f \in J.$$

This implies that  $\epsilon_{\tau} = 0$ .

Conversely, assume that  $\epsilon_{\tau} = 0$ . By [B 20], Theorem 13.5 this implies that for all  $v \in \mathcal{V}$  there exists  $b^v \in \mathcal{C}^v$  such that for all  $f \in J$ , we have  $\sum_{v \in \mathcal{V}} b^v(f) = 0$ .

If  $v \in \mathcal{V}'$  with  $v = v_p$  where p is a prime number, let us write  $b^v = a[\lambda^p, q^p_+, q^p_-]$ , for some  $\lambda^p \in (E^p_0)^{\times}/\mathcal{N}_{E^p/E^p_0}(E^p)^{\times}$ , and some quadratic forms  $q^p_+, q^p_-$  over  $\mathbf{Q}_p$ , as in notation 12.2.

Note that since  $v_{\infty}$  does not belong to any of the sets  $\Pi_{f,g}$ , we have  $b^{\infty}(f) = a^{\infty}(f) = w_2(q_f^{\infty}) + w_2(Q_f)$  if  $f \in I_1$ ,  $b^{\infty}(f) = a^{\infty}(X \pm 1) = w_2(q_{\pm}^{\infty}) + w_2(Q_{\pm})$ , and  $b^{\infty}(f) = a^{\infty}(f) = 0$  if  $f \in J$  with  $f \notin I$ ,  $f \neq X \pm 1$ . Recall that the forms  $q_f^{\infty}$  and  $q_{\pm}^{\infty}$  are uniquely determined by the choice of the Milnor index  $\tau \in \operatorname{Mil}_{r,s}(F)$ .

If  $f \in I_1$ , we have

$$b^{v_p}(f) = a[\lambda^p, q^p_+, q^p_-](f) = \operatorname{cor}_{(E_f)^p_0/\mathbf{Q}_p}(\lambda^p_f, d_f),$$

and

$$b^{v_{\infty}}(f) = a[\lambda^{\infty}, q_{+}^{\infty}, q_{-}^{\infty}](f) = \operatorname{cor}_{(E_{f})_{0}^{\infty}/\mathbf{R}}(\det(h_{f}^{\infty}), d_{f})$$

(cf. Propositions 10.2 and 10.5).

Since  $\sum_{v \in \mathcal{V}} b^v(f) = 0$ , we have

$$\sum_{v \in \mathcal{V}} \operatorname{cor}_{(E_f)_0^v/\mathbf{Q}_v}(\lambda_f^v, d_f) = 0,$$

where  $\mathbf{Q}_v = \mathbf{Q}_p$  if  $v = v_p$  and  $\mathbf{Q}_v = \mathbf{R}$  if  $v = v_\infty$ . This implies that

$$\sum_{w \in \mathcal{W}} (\lambda_f^w, d_f) = 0,$$

where  $\mathcal{W}$  is the set of primes of  $E_0$ . Therefore there exists  $\lambda_f \in E_0^{\times}/N_{E/E_0}(E^{\times})$ mapping to  $\lambda_f^w$  for all  $w \in \mathcal{W}$  (see for instance [B 20], Theorem 10.1). In particular, we have  $(\lambda_f, d_f) = (\lambda_f^w, d_f)$  in  $\operatorname{Br}_2(E_0^w)$  for all  $w \in \mathcal{W}$ .

Note that  $\tau(f)$  is an even integer. Let  $h_f: M_f \times M_f \to E_f$  be a hermitian form such that  $\det(h_f) = \lambda_f$ , and that the index of  $h_f$  is equal to  $\frac{\tau(f)}{2}$ ; such a hermitian form exists (see for instance [Sch 85], 10.6.9). Let us define

$$q_f: M_f \times M_f \to \mathbf{Q}$$

by

$$q_f(x,y) = \operatorname{Tr}_{E_f/\mathbf{Q}}(h_f(x,y)).$$

Let  $f = X \pm 1$ . We have  $\sum_{v \in \mathcal{V}} b^v(f) = 0$ , hence by the Brauer-Hasse-Noether theorem there exists  $a(\pm) \in \operatorname{Br}_2(\mathbf{Q})$  mapping to  $b^v(f)$  in  $\operatorname{Br}_2(\mathbf{Q}_v)$  for all  $v \in \mathcal{V}$ . Let  $q_{\pm}$  be a quadratic form over  $\mathbf{Q}$  of dimension  $n_{\pm}$ , determinant  $D_{\pm}$ , Hasse-Witt invariant  $w_2(q_{\pm}) = a(\pm) + w_2(Q_{\pm})$  and index  $\tau(X \pm 1) = r_{\pm} - s_{\pm}$ . Such a quadratic form exists; see for instance [S 77], Proposition 7.

Let  $q': M \times M \to \mathbf{Q}$  be the quadratic form given by

$$(M,q') = \bigoplus_{f \in I_1} (M_f,q_f) \oplus \bigoplus_{f \in I_0} (M_f,q_f) \oplus (M^2,q^2),$$

where  $(M^2, q^2)$  is hyperbolic. By construction, (M, q') has the same dimension, determinant, Hasse-Witt invariant and signature as (M, q), hence the quadratic forms (M, q') and (M, q) are isomorphic.

Let  $t : M \to M$  be defined by  $t(m) = \gamma m$ , where  $\gamma$  is a generator of  $\Gamma$ . By construction, t is an isometry of (M, q') and it is semi-simple with

characteristic polynomial F. By hypothesis, conditions (C 1) and (C 2) hold, hence  $(M, q') \otimes_{\mathbf{Q}} \mathbf{Q}_p$  contains an even, unimodular  $\mathbf{Z}_p$  lattice  $L_p$  stable by the isometry t. Let

$$L = \{ x \in M \mid x \in L_p \text{ for all prime numbers } p \}.$$

(L, q') is an even, unimodular lattice having a semi-simple isometry with characteristic polynomial F. This completes the proof of the theorem.

**Corollary 13.2.** Assume that conditions (C 1) and (C 2) hold, and that  $III_F = 0$ . Then there exists an even, unimodular lattice having a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau$ .

# 14. Even, unimodular lattices preserved by a semi-simple element of $SO_{r,s}(\mathbf{R})$

In this section, we reformulate the Hasse principle result of §13, and prove a result stated in the introduction. We keep the notation of §13. In particular  $F \in \mathbb{Z}[X]$  is a monic, symmetric polynomial of even degree such that F(0) = 1, and  $r, s \ge 0$  are integers such that  $r + s = \deg(F)$  and that  $r \equiv s \pmod{8}$ .

Let us now assume that condition (C 2) holds, and let  $t \in SO_{r,s}(\mathbf{R})$  be a semi-simple isometry with characteristic polynomial F. Let  $\tau = \tau(t) \in$  $\operatorname{Mil}_{r,s}(F)$  be the Milnor index associated to t in Proposition 6.1.

Assume that condition (C 1) also holds, and let  $\epsilon_{\tau}$  :  $\coprod_F(D_+, D_-) \rightarrow \mathbb{Z}/2\mathbb{Z}$  be the homomorphism defined in §13; set  $\epsilon_t = \epsilon_{\tau}$ . The following is a reformulation of Theorem 13.1 :

**Theorem 14.1.** The isometry  $t \in SO_{r,s}(\mathbf{R})$  preserves an even, unimodular lattice if and only if  $\epsilon_t = 0$ .

**Corollary 14.2.** If  $\epsilon_{\tau}$ :  $\coprod_{F}(D_{+}, D_{-}) \rightarrow \mathbb{Z}/2\mathbb{Z} =$ , the isometry  $t \in SO_{r,s}(\mathbb{R})$  preserves an even, unimodular lattice.

#### 15. Automorphisms of K3 surfaces

Which Salem numbers occur as dynamical degrees of automorphisms of complex analytic K3 surfaces? This question was raised by Curt McMullen in [McM 02], and was studied in many other papers (see for instance [GM 02], [O 10], [McM 11], [R 12], [BGA 16], [McM 16], [R 17], [Z 19], [Br 20]).

We refer to [H 16] and [Ca 14] for background on complex K3 surfaces (henceforth K3 surfaces, for short) and their automorphisms.

Let  $\mathcal{X}$  be a K3 surface, and let  $T : \mathcal{X} \to \mathcal{X}$  be an automorphism; it induces an isomorphism  $T^* : H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}, \mathbb{Z})$ . The *dynamical degree* of T is by definition the spectral radius of  $T^*$ ; it is either 1 or a Salem number. The characteristic polynomial of  $T^*$  is a product of at most one Salem polynomial and of a finite number of cyclotomic polynomials (see [McM 02], Theorem 3.2).

Let  $H^2(\mathcal{X}, \mathbf{C}) = H^{2,0}(\mathcal{X}) \oplus H^{1,1}(\mathcal{X}) \oplus H^{0,2}(\mathcal{X})$  be the Hodge decomposition of  $H^2(\mathcal{X}, \mathbf{C})$ . Since the subspace  $H^{2,0}(\mathcal{X})$  is one dimensional,  $T^*$  acts on it by multiplication by a scalar, denoted by  $\delta(T)$ , and called the *determinant* of T; we have  $|\delta(T)| = 1$ . Moreover,  $\delta(T)$  is a root of unity if  $\mathcal{X}$  is projective (cf. [McM 02], Theorem 3.5).

The intersection form of  $H^2(\mathcal{X}, \mathbf{Z})$  is an even, unimodular lattice of signature (3,19), hence it is isomorphic to  $\Lambda_{3,19}$ , and an automorphism of  $\mathcal{X}$  induces an isometry of that form. Therefore a necessary condition for a Salem number  $\alpha$  to occur as the dynamical degree of such an automorphism is that  $\Lambda_{3,19}$  has an isometry with characteristic polynomial SC, where S is the minimal polynomial of  $\alpha$ , and C is a (possibly empty) product of cyclotomic polynomials.

**Definition 15.1.** A complemented Salem polynomial is by definition a degree 22 polynomial that is the product of a Salem polynomial and of a (possibly empty) product of cyclotomic polynomials.

Recall from §8 that a monic, symmetric polynomial  $F \in \mathbf{Z}[X]$  satisfies condition (C 1) if and only if

$$|F(1)|, |F(-1)|$$
 and  $(-1)^n F(1)F(-1)$  are squares,

where  $2n = \deg(F)$ , and that this condition is *necessary* for F to be the characteristic polynomial of an isometry of an even, unimodular lattice.

If F is a complemented Salem polynomial, then m(F) = 1, since F has exactly two roots that are not on the unit circle. This implies that condition (C 2) holds for (r, s) = (3, 19).

**Definition 15.2.** Let F be a complemented Salem polynomial, and let  $\delta$  be a root of F with  $|\delta| = 1$ . We say that  $(F, \delta)$  is *realizable* (resp. projectively realizable) if there exists a K3 surface (resp. a projective K3 surface)  $\mathcal{X}$  and an automorphism  $T : \mathcal{X} \to \mathcal{X}$  such that

- F is the characteristic polynomial of  $T^*|H^2(\mathcal{X})$ .
- $T^*$  acts on  $H^{2,0}(\mathcal{X})$  by multiplication by  $\delta$ .

Let S be a Salem polynomial of degree d with  $4 \leq d \leq 20$ , and set

$$F(X) = S(X)(X-1)^{22-d}$$

Let us consider Salem polynomials S with |S(1)| = 1. In this case,  $\Pi_{S(X),X-1} = \emptyset$ , hence the obstruction group  $\coprod_F$  is not trivial, and not all Milnor indices are realized. We start by introducing some notation.

**Notation 15.3.** If  $\delta$  is a root of S with  $|\delta| = 1$ , let  $\tau_{\delta} \in \operatorname{Mil}_{3,19}(F)$  be such that

$$au_{\delta}(\mathcal{P}) = 2$$
 if  $\mathcal{P}(X) = (X - \delta)(X - \delta^{-1}),$ 

that

$$\tau_{\delta}(\mathcal{Q}) = -2$$
 for all  $\mathcal{Q} \in \operatorname{Irr}_{\mathbf{R}}(S)$  with  $\mathcal{Q} \neq \mathcal{P}$ ,

and that

$$\tau_{\delta}(X-1) = d - 22.$$

Let  $\tau_1 \in \operatorname{Mil}_{3,19}(F)$  be such that  $\tau_1(\mathcal{Q}) = -2$  for all  $\mathcal{Q} \in \operatorname{Irr}_{\mathbf{R}}(S)$ , and that  $\tau_1(X-1) = d-20$ .

**Theorem 15.4.** Let S be a Salem polynomial of degree d with  $4 \le d \le 20$ , set  $F(X) = S(X)(X-1)^{22-d}$ .

Assume that condition (C 1) holds for F and that |S(1)| = 1. Let  $\tau \in Mil_{3,19}(F)$ .

Then the lattice  $\Lambda_{3,19}$  has an isometry with characteristic polynomial F and Milnor index  $\tau$  if and only if one of the following holds

(i)  $d \equiv -2 \pmod{8}$  and  $\tau = \tau_{\delta}$  where  $\delta$  is a root of S with  $|\delta| = 1$ .

(ii)  $d \equiv 2 \pmod{8}$  and  $\tau = \tau_1$ .

**Proof.** The polynomials S and X - 1 are relatively prime over  $\mathbb{Z}$ . This implies that if the lattice  $\Lambda_{3,19}$  has a semi-simple isometry with characteristic polynomial F, then  $\Lambda_{3,19} \simeq L_1 \oplus L_2$  where  $L_1$  and  $L_2$  are even, unimodular lattices, such that  $L_1$  has an isometry with characteristic polynomial S, and  $L_2$  has a semi-simple isometry with characteristic polynomial  $(X - 1)^{22-d}$ .

Note that every  $\tau \in \text{Mil}_{3,19}(F)$  is either equal to  $\tau_1$ , or to  $\tau_\delta$ , where  $\delta$  is a root of S with  $|\delta| = 1$ . Assume first that  $\Lambda_{3,19}$  has a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau_1$ , and let  $\Lambda_{3,19} \simeq L_1 \oplus L_2$  be as above. The signature of  $L_1$  is (1, d - 1), and since  $L_1$  is unimodular and even, this implies that  $d \equiv 2 \pmod{8}$ .

Suppose now that  $\Lambda_{3,19}$  has a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau_{\delta}$ , where  $\delta$  is a root of S with  $|\delta| = 1$ . Let  $\Lambda_{3,19} \simeq L_1 \oplus L_2$  be as above. The signature of  $L_1$  is then (3, d-3), and since  $L_1$  is unimodular and even, we have  $d \equiv -2 \pmod{8}$ .

This implies that if  $\Lambda_{3,19}$  has a semi-simple isometry with characteristic polynomial F, then we are in one of the cases (i) or (ii).

Let us show the converse. Suppose first that we are in case (i). We have  $d \equiv -2 \pmod{8}$ ; this means that d = 6 or d = 14. Let (r, s) = (3, 3) if d = 6 and (r, s) = (3, 11) if d = 14; note that condition (C 2) holds for S and (r, s), and that  $r \equiv s \pmod{8}$ . By hypothesis, condition (C 1) holds for F; since F(-1) = S(-1), this implies that |S(-1)| is a square. Moreover, |S(1)| = 1 by hypothesis. We claim that condition (C 1) also holds for S. Since S is a Salem polynomial, we have S(1) < 0 and S(-1) > 0; we have  $d \equiv 2 \pmod{4}$ , therefore  $(-1)^{d/2}S(1)S(-1)$  is a square. This implies that condition (C 1) holds for S. Moreover, S is irreducible, hence  $III_S = 0$ .

Let  $\tau' \in \operatorname{Mil}_{r,s}(S)$  be the restriction of  $\tau_{\delta}$  to  $\operatorname{Mil}_{r,s}(S)$ . We have seen that conditions (C 1) and (C 2) hold for S, and that  $\operatorname{III}_{S} = 0$ . By Corollary 13.2 the even, unimodular lattice  $\Lambda_{r,s}$  has an isometry with characteristic polynomial Sand Milnor index  $\tau'$ . The identity is a semi-simple isometry of the lattice  $-E_8$ with characteristic polynomial  $(X-1)^8$ . Since  $\Lambda_{3,19} = \Lambda_{3,3} \oplus (-E_8) \oplus (-E_8) =$  $\Lambda_{3,11} \oplus (-E_8)$ , the lattice  $\Lambda_{3,19}$  has a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau_{\delta}$ , as claimed.

Suppose now that we are in case (ii). We have  $d \equiv 2 \pmod{8}$ ; that is, d = 10 or d = 18. Let (r, s) = (1, d - 1); note that  $r \equiv s \pmod{8}$ , and that

condition (C 2) holds for S and (r, s). We show as in case (i) that condition (C 1) holds for S and that  $\coprod_S = 0$ .

Let  $\tau'' \in \operatorname{Mil}_{r,s}(S)$  be the restriction of  $\tau_1$  to  $\operatorname{Mil}_{r,s}(S)$ . By Corollary 13.2 the even, unimodular lattice  $\Lambda_{r,s}$  has an isometry with characteristic polynomial S and Milnor index  $\tau''$ . The identity is a semi-simple isometry of the lattice  $\Lambda_{2,20-d}$  with characteristic polynomial  $(X-1)^{22-d}$ . Since  $\Lambda_{3,19} = \Lambda_{1,d-1} \oplus \Lambda_{2,20-d}$ , the lattice  $\Lambda_{3,19}$  has a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau_1$ .

**Proposition 15.5.** Let S be a Salem polynomial of degree d with  $4 \leq d \leq 22$ and  $d \equiv 6 \pmod{8}$ . and let  $\delta$  be a root of S with  $|\delta| = 1$ . Suppose that |S(1)|and S(-1) are both squares, and set  $F(X) = S(X)(X-1)^{22-d}$ . Then  $(F, \delta)$ is realizable.

**Proof.** The argument of Theorem 15.4 implies that the lattice  $\Lambda_{3,19}$  has a semi-simple isometry with characteristic polynomial F and Milnor index  $\tau_{\delta}$ . Applying [Br 20], Lemma 3.3 (1) of Brandhorst gives the desired result.

**Example 15.6.** If  $a \ge 0$  is an integer, the polynomial

 $S_a(X) = X^6 - aX^5 - X^4 + (2a - 1)X^3 - X^2 - aX + 1$ 

is an Salem polynomial (see [GM 02], page 284, Example 1), and S(1) = -1. Part (iii) of Theorem 15.5 implies that if  $\delta_a$  is a root of  $S_a$  with  $|\delta_a| = 1$  and  $F_a(X) = S_a(X)(X-1)^{16}$ , then  $(F_a, \delta_a)$  is realizable.

The polynomials  $S_a$  also appear in §4 of [McM 02] : for every integer  $a \ge 0$ , McMullen gives a geometric construction of an automorphism of a nonprojective K3 surface such that the dynamical degree and the determinant of the automorphisms are roots of  $S_a$  (see [McM 02], Theorem 4.1); this construction uses complex tori.

**Example 15.7.** Let S be a Salem polynomial of degree 14, and assume that |S(1)| = 1. Let  $\delta$  be a root of S, and set  $F(X) = S(X)(X-1)^8$ . If moreover S(-1) is a square, then condition (C 1) holds for F, and Theorem 15.5 (iii) implies that  $(F, \delta)$  is realizable.

Salem polynomials of degree 14 with |S(1)| = |S(-1)| = 1 were considered in several papers. Oguiso proved that the third smallest known Salem number  $\lambda_{14}$  is the dynamical degree of an automorphism of a non-projective K3 surface (see [O 10], Proposition 3.2). If a Salem number is a root of a Salem polynomial S of degree 14 with |S(1)| = |S(-1)| = 1, then this was shown by Reschke (see [R 12], Theorem 1.2).

#### 16. Realizable Salem numbers

The aim of this section is to show that if  $\alpha$  is a Salem number of degree d with d = 4, 6, 8, 12, 14 or 16, then  $\alpha$  is the dynamical degree of an automorphism of a *non-projective* K3 surface; partial results are given for the other values of d as well.

**Notation 16.1.** Let S be a Salem polynomial of degree d with  $4 \leq d \leq 20$ , and let  $\delta$  be a root of S with  $|\delta| = 1$ . Let F be a complemented Salem polynomial with Salem factor S. We define the Milnor index  $\tau_{\delta} \in \text{Mil}_{3,19}(F)$ as follows :

- $\tau_{\delta}((X-\delta)(X-\delta^{-1}))=2;$
- $\tau_{\delta}(\mathcal{P}) < 0$  for all  $\mathcal{P} \in \operatorname{Irr}_{\mathbf{R}}(F)$  such that  $\mathcal{P}(X) \neq (X \delta)(X \delta^{-1})$ .

**Theorem 16.2.** Let S be a Salem polynomial of degree d, and let  $\delta$  be a root of S with  $|\delta| = 1$ . Suppose that  $d \leq 16$  and  $d \equiv 0, 4$  or  $6 \pmod{8}$ . Then there exists a complemented Salem polynomial F with Salem factor S such that  $\Lambda_{3,19}$  has a realizable isometry with Milnor index  $\tau_{\delta}$ .

Notation 16.3. If  $f \in \mathbb{Z}[X]$  is an irreducible, symmetric polynomial of even degree, set  $E_f = \mathbb{Q}[X]/(f)$ , let  $\sigma_f : E_f \to E_f$  be the involution induced by  $X \mapsto X^{-1}$ , and let  $(E_f)_0$  be the fixed field of  $\sigma$  in  $E_f$ .

**Definition 16.4.** Let  $f \in \mathbb{Z}[X]$  be an irreducible, symmetric polynomial of even degree, and let p be a prime number. We say that f is *ramified at* p if there exists a place w of  $(E_f)_0$  above p that is ramified in  $E_f$ .

**Proposition 16.5.** Suppose that  $d \leq 18$  and that one of the following holds :

- (i) |S(1)| and S(-1) are not both squares.
- (ii) S is ramified at the prime 2.

Then there exists a complemented Salem polynomial F with Salem factor S such that  $\Lambda_{3,19}$  has a realizable isometry with Milnor index  $\tau_{\delta}$ .

**Proof.** (a) Suppose first that there exists a prime number p such that  $v_p(S(-1)) \equiv 1 \pmod{2}$ . Set  $F(X) = S(X)(X-1)^{20-d}(X+1)^2$ . We have  $D_- = (-1)^{s_-}S(-1) = S(-1)$ ; this implies that  $D_- \neq -1$  in  $\mathbf{Q}_p^{\times}/\mathbf{Q}_p^{\times 2}$ , since  $v_p(D_-) = v_p(S(-1)) \equiv 1 \pmod{2}$ . Therefore  $p \in \prod_{S,X+1}$ , and  $\coprod_F(D_-) = 0$ .

(b) Suppose now that no such prime number exists; this implies that S(-1) is a square. Then either |S(1)| is not a square, or S is ramified at 2.

Set  $F(X) = S(X)(X-1)^{22-d}$ . We have  $d \leq 18$ , hence  $22 - d \neq 2$ ; this implies that  $p \in \prod_{S,X-1}$  and hence  $\coprod_F = 0$ .

Let (L,q) be an even, unimodular lattice of signature (3,19), and let  $t : L \to L$  be a semi-simple isometry with characteristic polynomial F and with Milnor index  $\tau_{\delta}$ ; such an isometry exists  $\text{III}_F(D_-) = 0$  in case (a) and  $\text{III}_F = 0$  in case (b).

Set  $L_S = \text{Ker}(S(t))$ , and let  $S_C$  be the orthogonal of  $L_S$  in L. If we are in case (b), then the restriction of t to  $L_C$  is the identity, hence it is a positive isometry in the terminology of McMullen; this implies that  $t : L \to L$  is realizable.

Suppose now that we are in case (a). Let  $L_1 = L_S = \text{Ker}(S(t)), L_2 = \text{Ker}(t+1)$  and  $L_3 = \text{Ker}(t-1)$ ; let  $t_i : L_i \to L_i$  be the restriction of t to  $L_i$ .

A root of  $(L_2, q)$  is by definition an element  $x \in L_2$  such that q(x, x) = -2. If  $(L_2, q)$  has no roots, then  $t_2$  is a positive isometry of  $(L_2, -q)$  by [McM 11], Theorem 2.1, and hence [McM 11], Theorem 6.2 (see also [McM 16], Theorem 6.1) implies that  $(F_+, \delta)$  is realizable, hence (b) holds.

Suppose that  $(L_2, q)$  has at least one root. By hypothesis, S(-1) is not a square, therefore  $det(L_2, q)$  is not a square. Since  $(L_2, q)$  is of rank 2, even and negative definite, there exist integers  $D \ge 1$  and  $f \ge 1$  such that  $det(L_2,q) = f^2 D$ , where -D is the discriminant of an imaginary quadratic field. The lattice  $(L_2, q)$  is isomorphic to a quadratic form q' on an order O of the imaginary quadratic field  $\mathbf{Q}(\sqrt{-D})$  (see for instance [Co 89], Theorem 7.7). Complex conjugation induces an isometry of the quadratic form (O, q')with characteristic polynomial  $X^2 - 1$ . If D = 3 and f = 1, then (O, q')is isomorphic to the root lattice  $A_2$ , and complex conjugation is a positive isometry of (O, -q') (see [McM 11], §5, Example); otherwise, (O, q') contains only two roots, fixed by complex conjugation, hence we obtain a positive isometry of (O, -q') in this case as well. Let  $t'_2 : L_2 \to L_2$  be the isometry of  $(L_2,q)$  obtained via the isomorphism  $(O,q') \simeq (L_2,q)$ . Then  $t'_2$  is a positive isometry of  $(L_2, -q)$ . Let  $G(L_2) = (L_2)^{\sharp}/L_2$ , and note that  $t_2$  and  $t'_2$  both induce -id on  $G(L_2)$ . This implies that (L,q) has a semi-simple isometry  $t': L \to L$  inducing the positive isometry  $t_2$  or  $t'_2$  on  $L_2$  and  $t_i$  on  $L_i$  for i = 1, 3. By [McM 11], Theorem 6.2 (see also [McM 16], Theorem 6.1) this implies that the isometry is realizable.

**Proof of Theorem 16.2.** If |S(1)| and S(-1) are not both squares or if if d is divisible by 4, then the result follows from the proposition; if  $d \equiv 6 \pmod{8}$ , then it follows from Proposition 15.5.

**Definition 16.6.** Let  $\alpha$  be a Salem number and let  $\delta$  be a conjugate of  $\alpha$  such that  $|\delta| = 1$ . We say that  $(\alpha, \delta)$  is *realizable* (resp. projectively realizable) if there exists an automorphism of a K3 surface (resp. a projective K3 surface) having an automorphism of dynamical degree  $\alpha$  and and determinant  $\delta$ .

**Corollary 16.7.** Let  $\alpha$  be a Salem number of degree d with  $4 \leq d \leq 16$ , let S be the minimal polynomial of  $\alpha$ , and let  $\delta$  be a root of S with  $|\delta| = 1$ . If d = 4, 6, 8, 12, 14 or 16, then  $(\alpha, \delta)$  is realizable.

**Remark 16.8.** If d = 20 with |S(1)| is a square and S(-1) is not a square, then the method of Proposition 16.5 still works, and therefore Corollary 16.7 holds in this case as well.

**Example 16.9.** McMullen proved that the Salem numbers  $\lambda_{14}$ ,  $\lambda_{16}$  and  $\lambda_{20}$  are not realized as dynamical degrees of automorphisms of *projective* K3 surfaces (cf. [McM 16], §9). Corollary 16.7 and Remark 16.8 show that they are realized by automorphisms of *non-projective* K3 surfaces.

## 17. A nonrealizable Salem number

McMullen proved that the Salem number  $\lambda_{18} = 1.1883681475...$  (the second smallest known Salem number) is the dynamical degree of an automorphism of a *projective* K3 surface (cf. [McM 16], Theorem 8.1). The aim of this section is to show that this is not possible for *non-projective* K3 surfaces.

Let S be a Salem polynomial of degree 18, and let  $\delta$  be a root of S with  $|\delta| = 1$ . Let  $\sigma_{\delta} \in \operatorname{Mil}_{3,15}(S)$  be such that  $\sigma_{\delta}(\mathcal{P}) = 2$  for  $\mathcal{P}(x) = (x-\delta)(x-\delta^{-1})$  and that  $\sigma_{\delta}(\mathcal{Q}) = -2$  for all  $\mathcal{Q} \in \operatorname{Irr}_{\mathbf{R}}(S)$  with  $\mathcal{Q} \neq \mathcal{P}$ .

If  $f \in \mathbf{Z}[X]$  is a monic polynomial, we denote by  $\operatorname{Res}(S, f)$  the resultant of the polynomials S and f.

**Proposition 17.1.** Assume that  $|\operatorname{Res}(S, f)| = 1$  for all  $f \in \{\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_6\}$ . Let C be a product of cyclotomic polynomials such that  $\deg(C) = 4$ , and set F = SC. Let  $\tau_{\delta} \in \operatorname{Mil}_{3,19}(F)$  be such that the restriction of  $\tau_{\delta}$  to  $\operatorname{Mil}_{3,15}(S)$  is  $\sigma_{\delta}$ , and that  $\tau_{\delta}(\mathcal{Q}) < 0$  for all  $\mathcal{Q} \in \operatorname{Irr}_{\mathbf{R}}(C)$ .

If  $\Lambda_{3,19}$  has a semi-simple isometry with characteristic polynomial F = SCand Milnor index  $\tau_{\delta}$ , then  $C = \Phi_{12}$ .

**Proof.** If  $C = \Phi_5$ ,  $\Phi_8$  or  $\Phi_{10}$ , then *FC* does not satisfy Condition (C 1), hence  $\Lambda_{3,19}$  does not have any isometry with characteristic polynomial *F* for these choices of *C*.

Assume that all the factors of C belong to the set  $\{\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_6\}$ . Then S and C are relatively prime over  $\mathbb{Z}$ . If  $\Lambda_{3,19}$  has an isometry with characteristic polynomial F, then  $\Lambda_{3,19} = L_1 \oplus L_2$ , where  $L_1$  and  $L_2$  are even, unimodular lattices such that  $L_1$  has an isometry with characteristic polynomial S and Milnor index  $\sigma_{\delta}$ , and  $L_2$  has an isometry with characteristic polynomial C. This implies that the signature of  $L_1$  is (3, 15) and that the signature of  $L_2$  is (0, 4), and this is impossible.

Therefore the only possiblity is  $C = \Phi_{12}$ , as claimed.

**Notation 17.2.** Let  $C = \Phi_{12}$ , and set F = SC. Let  $\zeta$  be a primitive 12th root of unity. Let  $\tau_{\delta}, \tau_{\zeta} \in \text{Mil}_{3,19}(F)$  be such that

 $\tau_{\delta}(\mathcal{P}) = 2$  for  $\mathcal{P}(x) = (x - \delta)(x - \delta^{-1})$  and that  $\tau_{\delta}(\mathcal{Q}) = -2$  for all  $\mathcal{Q} \in \operatorname{Irr}_{\mathbf{R}}(F)$  with  $\mathcal{Q} \neq \mathcal{P}$ ;

 $\tau_{\zeta}(\mathcal{P}) = 2$  for  $\mathcal{P}(x) = (x - \zeta)(x - \zeta^{-1})$  and that  $\tau_{\zeta}(\mathcal{Q}) = -2$  for all  $\mathcal{Q} \in \operatorname{Irr}_{\mathbf{R}}(F)$  with  $\mathcal{Q} \neq \mathcal{P}$ .

**Theorem 17.3.** Let S be an Salem polynomial of degree 18 such that

$$|S(1)S(-1)| = 1,$$

let  $C = \Phi_{12}$ , and set F = SC. Let  $\delta$  be a root of S with  $|\delta| = 1$ , and let  $\zeta$  be a primitive 12th root of unity. With the above notation, we have

(a) The lattice  $\Lambda_{3,19}$  has an isometry with characteristic polynomial F and Milnor index  $\tau_{\zeta}$ .

(b) The lattice  $\Lambda_{3,19}$  has an isometry with characteristic polynomial F and Milnor index  $\tau_{\delta}$  if and only if  $\coprod_F = 0$ .

**Proof.** The polynomial F satisfies Condition (C 1), since F(1) = -1 and F(-1) = 1.

Let us prove (a). Let  $\sigma_1 \in \text{Mil}_{1,17}(S)$  and  $\sigma_2 \in \text{Mil}_{2,2}(C)$  be the restrictions of of  $\tau_{\zeta} \in \text{Mil}_{3,19}(F)$ . Since S and C are both irreducible, we have  $\text{III}_S = 0$  and  $\operatorname{III}_C = 0$ . Therefore by Corollary 13.2,  $\Lambda_{1,17}$  has an isometry with characteristic polynomial S and Milnor index  $\sigma_1$  and  $\Lambda_{2,2}$  has an isometry with characteristic polynomial C and Milnor index  $\sigma_2$ . This implies (a).

Let us prove (b). If  $III_F = 0$ , then Corollary 13.2 implies that  $\Lambda_{3,19}$  has an isometry with characteristic polynomial F and any Milnor index.

Assume that  $\coprod_F \neq 0$ ; since F has two irreducible factors, this implies that  $\coprod_F \simeq \mathbf{Z}/2\mathbf{Z}$ . Recall from §12 that  $\epsilon_{\tau_{\delta}} = \epsilon^{\text{finite}} + \epsilon_{\tau_{\delta}}^{\infty}$  and  $\epsilon_{\tau_{\zeta}} = \epsilon^{\text{finite}} + \epsilon_{\tau_{\zeta}}^{\infty}$ .

By (a) we know that  $\Lambda_{3,19}$  has an isometry with characteristic polynomial Fand Milnor index  $\tau_{\zeta}$ ; this implies that  $\epsilon_{\tau_{\zeta}} = 0$ . Note that  $\epsilon_{\tau_{\delta}}^{\infty} \neq \epsilon_{\tau_{\zeta}}^{\infty}$ . Therefore  $\epsilon_{\tau_{\delta}} \neq 0$ , and by Theorem 13.1 this implies that  $\Lambda_{3,19}$  does not have an isometry with characteristic polynomial F and Milnor index  $\tau_{\delta}$ . This completes the proof of (b).

**Example 17.4.** Let S be the Salem polynomial corresponding to the Salem number  $\lambda_{18}$ . This polynomial satisfies the conditions of Proposition 17.1 : we have |Res(S, f)| = 1 for all  $f \in \{\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_6\}$ . Therefore by Proposition 17.1, if  $\Lambda_{3,19}$  has an isometry with characteristic polynomial SC and Milnor index  $\tau_{\delta}$  for some product C of cyclotomic polynomials, then we have  $C = \Phi_{12}$ .

Let  $F = S\Phi_{12}$ . We have  $III_F \neq 0$ . Indeed, |Res(S, f)| = 169, and the common factors modulo 13 of S and  $\Phi_{12}$  in  $\mathbf{F}_{13}[X]$  are  $X + 6, X + 11 \in \mathbf{F}_{13}[X]$ . These polynomials are not symmetric. Therefore  $\Pi_{S,\Phi_{12}} = \emptyset$ , and hence  $III_F \simeq \mathbf{Z}/2\mathbf{Z}$ . Theorem 17.3 implies that  $\Lambda_{3,19}$  does not have any isometry with characteristic polynomial  $S\Phi_{12}$  and Milnor index  $\tau_{\delta}$ .

Since this holds for all roots  $\delta$  of S with  $|\delta| = 1$ , the Salem number  $\lambda_{18}$  is not realized by an automorphism of a non-projective K3 surface.

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#### EVA BAYER-FLUCKIGER

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