

QUANTITATIVE MARCINKIEWICZ'S THEOREM AND CENTRAL LIMIT THEOREMS: APPLICATIONS TO SPIN SYSTEMS AND POINT PROCESSES

TIEN-CUONG DINH, SUBHROSHEKHAR GHOSH, HOANG-SON TRAN, MANH-HUNG TRAN

ABSTRACT. The classical Marcinkiewicz theorem states that if an entire characteristic function $\Psi_X(u) := \mathbb{E}[e^{iuX}]$ of a non-degenerate real-valued random variable X is of the form $\exp(P(u))$ for some polynomial P , then X has to be a Gaussian. In this work, we obtain a broad, quantitative extension of this framework in several directions, establish central limit theorems (CLTs) with explicit rates of convergence, and demonstrate Gaussian fluctuations in continuous spin systems and general classes of point processes. Our work complements classical work of Ostrovskii, Linnik, Zimogljad and others, as well as recent advances by Michelen and Sahasrabudhe, and Eremenko and Fryntov. In particular, we obtain *quantitative decay estimates* on the Kolmogorov-Smirnov distance between X and a Gaussian under the condition that Ψ does not vanish *only on a bounded disk*. This leads to quantitative CLTs applicable to very general and possibly strongly dependent random systems. In spite of the general applicability, our rates for the CLT match the classic Berry-Esseen bounds for independent sums up to a log factor. We implement this programme for two important classes of models in probability and statistical physics. First, we extend to the setting of continuous spins a popular paradigm for obtaining CLTs for discrete spin systems that is based on the theory of Lee-Yang zeros, focussing in particular on the XY model, Heisenberg ferromagnets and generalised Ising models. Secondly, we establish Gaussian fluctuations for linear statistics of so-called α -determinantal processes for $\alpha \in \mathbb{R}$ (including the usual determinantal, Poisson and permanental processes) under very general conditions, including in particular higher dimensional settings where structural alternatives such as random matrix techniques are not available. Our applications demonstrate the significance of having to control the characteristic function only on a (small) disk, and lead to CLTs which, to the best of our knowledge, are not known in generality.

CONTENTS

1. Introduction	2
1.1. A quantitative Marcinkiewicz theory and quantitative CLTs	3
1.2. Spin systems and Lee-Yang theory	5
1.3. Linear statistics of α -determinantal processes	7
2. Proofs of Theorem 1.1, Corollaries 1.2 and 1.3	10
3. CLTs in general spin systems: generalised Ising, XY and Heisenberg models	18
3.1. Ferromagnetism and its generalisations	18
3.2. Lee-Yang theorem	19
3.3. The no-zeros condition for S_Λ	19
3.4. Variance growth for S_Λ	20
3.5. Proof of Theorem 1.5	24
4. Fluctuation theory for linear statistics of α -determinantal processes	24
4.1. The characteristic function of linear statistics	25
4.2. The no-zeros condition for linear statistics	27
4.3. Growth rate of the moment-generating function of $\Lambda(\varphi_L)$	27
4.4. Variance growth of $\Lambda(\varphi_L)$ for $\alpha > 0$	29

4.5. Variance growth of $\Lambda(\varphi_L)$ for $\alpha < 0$	30
4.6. Proof of Theorem 1.6	35
References	36

Classification AMS 2020: 60F05, 60G55

Keywords: Marcinkiewicz's theorem, central limit theorem, spin system, α -determinantal system.

1. INTRODUCTION

Let X be a real-valued random variable on a probability space. We define the associated centered random variable by

$$\bar{X} := X - \mathbb{E}[X].$$

Denote by $F_{\bar{X}} : \mathbb{R} \rightarrow [0, 1]$ the cumulative distribution function (c.d.f.) of \bar{X} , that is,

$$F_{\bar{X}}(x) := \mathbb{P}(\bar{X} \leq x) \text{ for } x \in \mathbb{R}.$$

The c.d.f. of the Gaussian $N(0, 1)$ is denoted by Φ . Denote also by σ^2 the variance of X with $\sigma \geq 0$. Define $\log^+ := \max(\log, 0)$. We will denote by $\Psi_X(u) := \mathbb{E}[e^{iuX}]$ the characteristic function of X . For brevity, we will suppress the X -dependency from Ψ_X and simply write Ψ from time to time, whenever the random variable is understood from context.

The function Ψ , although well defined for any $u \in \mathbb{R}$, admits a holomorphic extension to the entire complex plane only under appropriate decay of the tails of the distribution of X . Entire characteristic functions form a function class of independent interest, and their distinctive properties such as growth rates and zero distributions have received considerable attention in the literature [Pol, Ram, Luk72, LuSz].

A fundamental result on entire characteristic functions is the classic Marcinkiewicz Theorem. This entails that if Ψ is an entire characteristic function that is of the form $\exp(P(u))$ for some polynomial P , then X has to be a Gaussian (unless it is degenerate, i.e. purely atomic). The characteristic function Ψ being of the form $\exp(f)$ for some entire function f is equivalent to the assertion that Ψ has no zeros on the whole of \mathbb{C} . For such functions, the growth rate, or equivalently, the order of the entire function f is of considerable interest.

The simplest possible growth behaviour of f arises when X is degenerate, that is, a delta measure at a point; it is easy to see that in this case $f(u)$ grows at most linearly in $|u|$. It is well known that if X follows a standard Poisson distribution, then $f(u) = c \cdot (e^{iu} - 1)$ for some constant c . What growth rates for f are possible in between these two extremities of linear and exponential growth is a fundamental question.

To set notations, we define $M_f(r) := \max\{|f(z)| : |z| = r\}$. In 1960, Linnik conjectured that the regime of Gaussianity, i.e. the regime of growth rate for which a Marcinkiewicz-type theorem holds true, extends all the way till $\log^+ M_f(r) = o(r)$, as $r \rightarrow \infty$. The most significant result in this direction is the theorem of Ostrovskii [Os63, Os62, Os83], who demonstrates that Linnik's conjecture is indeed true as soon as $\limsup_{r \rightarrow \infty} r^{-1} \log^+ M_f(r) = 0$, using ideas from Wiman-Valiron theory. By the very setup of these results, it is required that the entire characteristic function be of the form $\Psi(u) = \exp(f(u))$ for some entire function f ; this entails in particular that the characteristic function Ψ does not vanish anywhere on \mathbb{C} .

1.1. A quantitative Marcinkiewicz theory and quantitative CLTs.

1.1.1. A quantitative Marcinkiewicz theory. In this work, we obtain a quantitative version of such Marcinkiewicz-type theorems, by obtaining upper bounds on the *Kolmogorov-Smirnov* distance (*abbrv* KS distance) between the cumulative distribution function (c.d.f.) F_X of X and the standard Gaussian c.d.f. Φ under *non-asymptotic* conditions on growth and bounded zero-free regions for the characteristic function. Under the much more relaxed hypothesis that Ψ does not vanish and satisfies a growth hypothesis *only* on a disk of finite radius r , we are able to provide an upper bound on the Kolmogorov-Smirnov distance $\sup_{x \in \mathbb{R}} |F_{\bar{X}}(x) - \Phi(x)|$ in terms of r . We elaborate this in the following.

Theorem 1.1. *Suppose there is a number $r > 0$ such that $\mathbb{E}[e^{r|X|}]$ is finite and the characteristic function $u \mapsto \mathbb{E}[e^{iuX}]$ doesn't vanish on the closed disk $\mathbb{D}(0, r)$ of center 0 and radius r . Then, we have for some universal constant $A > 0$*

$$\sup_{x \in \mathbb{R}} |F_{\bar{X}}(x) - \Phi(x)| \leq 2|\sigma - 1| + A(1 + \log^+ \log \max_{|u|=r} |\mathbb{E}[e^{iuX}]|)r^{-1}.$$

Theorem 1.1 easily implies the following extension by Zimogljad [Zim] of the classical Marcinkiewicz theorem:

Corollary 1.2. *Let X be a real-valued random variable such that $\mathbb{E}[e^{iuX}] = e^{f(u)}$ for some entire function $f(u)$. Assume that*

$$\liminf_{r \rightarrow \infty} r^{-1} \log^+ \sup_{|u|=r} \operatorname{Re} f(u) = 0.$$

Then, f is a polynomial of degree at most equal to 2.

In the classical Marcinkiewicz theorem, the function f is assumed to be a polynomial and therefore the hypothesis on the growth of f is automatically satisfied. Ostrovskii's theorem [Os63, Os62, Os83] entails a similar conclusion as Corollary 1.2 with the stronger growth assumption

$$\limsup_{r \rightarrow \infty} r^{-1} \log \sup_{|u|=r} |f(u)| = 0.$$

This condition may be shown to be equivalent to

$$\limsup_{r \rightarrow \infty} r^{-1} \log \sup_{|u|=r} \operatorname{Re} f(u) = 0.$$

On a related note, we also refer to the preprint due to Eremenko and Fryntov [ErFr], which has been announced shortly prior to the announcement of our results, and also addresses the question of stability in Marcinkiewicz theorem, using a very different approach than ours. We emphasise that in our main theorems we only assume the zero freeness on a disk of finite radius. This is crucial for important applications, such as spin systems and general α -determinantal processes examined in our paper (c.f. Remarks 3.4, 4.5 resp.), and is a main difference in comparison with the classical works by Marcinkiewicz, Ostrovskii, as well as [Zim] and [ErFr], where the zero freeness needs to hold for an infinite strip. In particular, a key technique using Phragmén-Lindelöf principle for \mathbb{C} or a strip in \mathbb{C} in the approach of Ostrovskii, Zimoglyad and Eremenko-Fryntov doesn't work in our setting. Note also that we don't assume any hypothesis on the global behaviour of the characteristic function on \mathbb{C} as needed in [ErFr]. In particular, our characteristic functions do not need to exist on the whole \mathbb{C} .

In this vein, we would like to mention a different flavour of stability results for the Marcinkiewicz theorem due to Golinskii ([Gol], see also [GoCh]). This answers a question of Sapogov ([Sap-I],[Sap-II]) which asks for conditions on the coefficients of a polynomial P under the assumption that $\exp(P(\cdot))$ is *close to* a characteristic function (but might not be a characteristic function itself).

It would be of interest to investigate if a combination of these approaches can lead to a more succinct and sharper quantitative Marcinkiewicz theory.

1.1.2. Quantitative central limit theorems. Our result can be used to obtain a quantitative version of the central limit theorem. Consider a sequence of real-valued random variables X_n of variances σ_n^2 with $\sigma_n > 0$. Define the associated normalized and centered random variables by

$$\hat{X}_n := \frac{\bar{X}_n}{\sigma_n} = \frac{X_n - \mathbb{E}[X_n]}{\sigma_n}.$$

Corollary 1.3. *Assume that there are positive real numbers r_n such that $\mathbb{E}[e^{r_n|X_n|}]$ is finite and the characteristic function $u \mapsto \mathbb{E}[e^{iuX_n}]$ doesn't vanish on the closed disk $\overline{\mathbb{D}(0, r_n)}$. Assume also that*

$$\lim_{n \rightarrow \infty} \frac{1 + \log^+ \log \sup_{|u|=r_n} |\mathbb{E}[e^{iuX_n}]|}{r_n \sigma_n} = 0.$$

Then, the sequence (X_n) satisfies the CLT, that is, \hat{X}_n converges in law to the Gaussian $N(0, 1)$ as n tends to infinity. Moreover, we have for some universal constant $A > 0$

$$(1.1) \quad \sup_{x \in \mathbb{R}} |F_{\hat{X}_n}(x) - \Phi(x)| \leq \frac{A(1 + \log^+ \log \sup_{|u|=r_n} |\mathbb{E}[e^{iuX_n}]|)}{r_n \sigma_n}.$$

At this point, we compare our rate estimates with the classical Berry-Esseen bounds for sums of random variables. For concreteness, we focus on the setting of sums of independent and identically distributed (i.i.d.) bounded random variables $S_n = \sum_{i=1}^n X_n$. Berry-Esseen theorem [Ber, Es42, Es56, Dur] gives a convergence rate of $n^{-1/2}$ to the standard normal for \hat{S}_n ; this rate is optimal in the situation under consideration. We now apply Corollary 1.3 to this setting.

Without loss of generality, we assume that the X_i -s are centered. We notice that if X is a random variable following the common distribution of X_i -s, then the characteristic functions satisfy $\Psi_{S_n}(u) = (\Psi_X(u))^n$. Since $\Psi_X(u)$ does not have zeros arbitrarily close to 0, we may conclude that Ψ_{S_n} does not have zeros in a neighbourhood of radius δ of 0, where δ does not depend on n . This shows that in Corollary 1.3, we may take $r_n = \delta$. On the other hand, a simple calculation shows that $\sigma_n = c \cdot \sqrt{n}$ for some constant $c > 0$. Finally, $\mathbb{E}[e^{r_n|S_n|}] \leq \Psi_{|X|}(\delta)^n$, implying that $\log^+ \log \mathbb{E}[e^{r_n|S_n|}] \lesssim \log n$.

Combining these ingredients, we obtain a CLT convergence rate of $\log n \cdot n^{-1/2}$ for \hat{S}_n , which differs from the classical Berry-Esseen rate by only a factor of $\log n$. On the other hand, the Berry-Esseen approach is well-suited for sums of random variables which are preferably independent or weakly dependent; whereas our techniques apply much more generally to strongly correlated settings, and does not assume any algebraic structure (such as sums of smaller ingredients) on the sequence of random variables under consideration.

In [MS-I] and [MS-II], Michelen and Sahasrabudhe investigate CLTs for random variables under zero-free conditions. Primarily, these works address the setting of non-negative integer-valued random variables under conditions on zero-free regions for their generating polynomials, in relation to the earlier works [LPRS], [GhLiPe] and a related variance growth question due to Pemantle. However, in [MS-II] Section 12, they demonstrate an extension of their approach to real-valued random variables; in particular see Theorem 12.2 therein. In the coordinate u given by $z = e^u$, this result has a similar flavour to Theorem 2 in [ErFr]; it obtains a CLT for real valued random variables under the assumptions that the characteristic function is entire, zero-free on an infinite vertical strip and satisfies global growth conditions.

A significant point to note about Theorem 1.1 and Corollary 1.1 is that these results provide effective versions of the quantitative CLT that are applicable to wide classes of random variables of interest in probability and statistical physics, including in situations that are not covered by existing results of similar flavour. In particular, we only need to assume that the characteristic function is finite on a (small, possibly shrinking) disk around the origin, zero-free on this disk, and a growth estimate that needs to hold only on this disk. The limited and local nature of these assumptions are crucial for applications, demonstrated in Remarks 3.4 and 4.5. This includes, in particular, the applications to continuous and multi-component spin systems and point processes considered in the present paper; for details see Sections 1.2 and 1.3 below.

1.1.3. Applications to strongly dependent random systems. We demonstrate below the generality of scope and the robustness of our approach by obtaining CLTs for observables of natural interest in general spin systems and in α -determinantal processes. A leitmotif of these stochastic systems is their strongly correlated nature, which renders ineffective most of the common approaches to CLT that involve exploiting independence or approximate independence in some form. We emphasize that, more than the specific CLTs in question, we use these as models to demonstrate the generality of scope and power of our technique, and its effectiveness with limited structural requirements. Nonetheless, we observe that, except for a few specialised settings, to the best of our knowledge these CLTs appear to be not available in the literature.

1.2. Spin systems and Lee-Yang theory. A significant approach in the statistical physics literature for obtaining CLTs for the total spin and similar observables in discrete spin systems such as the classical Ising model (and related discrete stochastic models such as monomer-dimer systems) connects to the famous Lee-Yang theorem on the zeros of the partition functions of such models; for a detailed and modern exposition to this approach, we refer the reader to [FrRo, LPRS]; for the original works of Lee and Yang see [LY-I, LY-II]. However, the approach is tailored to the situation where the observables are positive integer valued, and is based on understanding the partition function of a finite system as a polynomial (in the fugacity of the system). Further, other ingredients of the approach, such as the use of Ginibre's theorem [Gin] to demonstrate extensive fluctuations, is also addressed to the discrete setting.

However, in statistical physics, models with continuous spins are of fundamental importance. These include the important spin systems with continuous or vector valued spins; in particular, the XY model (i.e., the plane rotator model) with \mathbb{S}^1 -valued spins

and the classical Heisenberg spin system with \mathbb{S}^2 -valued spins. The approach of [LPRS], based on zeros of polynomials, do not apply to these settings.

In this work, we demonstrate a substantial extension of the partition function zero-based technique to obtain CLTs from the discrete setup of [LPRS] to the setting of more general, continuous and multi-component spin distributions. We exploit the fact that our approach, being based on characteristic functions, is independent of the discrete structure of the spins, and obtain CLTs for the first component of the total spin for the XY model as well as the classical Heisenberg model in the so-called ferromagnetic regime. In fact, we are able to obtain similar CLTs also for 1D continuous valued spins, thereby extending the partition function zero-based approach to generalised Ising models.

To be precise, let $\Lambda \subset \mathbb{Z}^d$ be a d -dimensional cube. For any two neighbouring vertices $x, y \in \Lambda$ we write $(x, y) \subset \Lambda$. A spin configuration on Λ is a map $\sigma_\Lambda : \Lambda \mapsto \mathbb{R}^N$, with $\sigma_x := \sigma_\Lambda(x) = (\sigma_x^1, \dots, \sigma_x^N) \forall x \in \Lambda$. The spins σ_x are distributed according to the *rotationally invariant* non-negative Borel measure μ_0 on \mathbb{R}^N . We consider the Hamiltonian $H_\Lambda(\cdot)$ given by

$$(1.2) \quad H_\Lambda(\sigma_\Lambda) = - \sum_{(x,y) \subset \Lambda} \sum_{i=1}^N J_{xy}^i \sigma_x^i \sigma_y^i - \sum_{x \in \Lambda} \sum_{i=1}^N h_x^i \sigma_x^i,$$

where $J_{xy}^i = J_{yx}^i$ are coupling constants for the edge (x, y) in the direction $i \in \{1, \dots, N\}$, and h_x is the external magnetic field at $x \in \Lambda$, whose component in the direction i is given by h_x^i . We are interested in the probability measure on the space of such σ_Λ -s given by

$$(1.3) \quad \mathbb{P}_\Lambda(\sigma_\Lambda) \propto \exp(-\beta H_\Lambda(\sigma_\Lambda)) \prod_{x \in \Lambda} d\mu_0(\sigma_x),$$

where $\beta > 0$ is the *inverse temperature*.

Remark 1.4. We observe that we denote spins (σ_x, σ_x^i) as well as standard deviation (σ_n) by the same symbol σ ; the connotation of σ will be understood from the context.

For the purposes of our CLT, we will work in the following modelling setup.

Model 1.1 (Model classes for CLTs in Ferromagnetic spin systems). *We have one among the following choices of μ_0 and interactions J_{xy} . The external field is uniform and always assumed to act purely along the x axis (i.e., along the first co-ordinate), and is given by $h > 0$.*

- (i) $N = 1$; μ_0 is a compactly supported even measure on \mathbb{R} (that is not degenerate at 0), and satisfies

$$\left| \int_{\mathbb{R}^N} e^{u\sigma} d\mu_0(\sigma) \right| < \infty \quad \forall u \in \mathbb{C} \text{ such that } \operatorname{Re}(u) \neq 0;$$

the ferromagnetic condition $J_{xy} \geq 0$ is satisfied $\forall (x, y) \subset \Lambda$.

- (ii) $N = 2$; (XY model) : μ_0 is the uniform measure on the unit circle \mathbb{S}^1 , and the ferromagnetic condition $J_{xy}^1 \geq |J_{xy}^2|$ is satisfied $\forall (x, y) \subset \Lambda$.

- (iii) $N = 3$; (Classical Heisenberg model) : μ_0 is the uniform measure on the unit sphere \mathbb{S}^2 , and the ferromagnetic conditions $J_{xy}^1 \geq \max\{|J_{xy}^2|, |J_{xy}^3|\}$ and $J_{xy}^3 \geq 0$ are satisfied $\forall (x, y) \subset \Lambda$.

- (iv) $N = 4$: μ_0 is the uniform measure on \mathbb{S}^3 , and the couplings satisfying the ferromagnetic conditions $|J_{uv}^2| \leq J_{uv}^1$ and $|J_{uv}^4| \leq J_{uv}^3 \forall (u, v) \in \Lambda$.

Then we may state:

Theorem 1.5. *Let σ_Λ be an N -component spin system defines on a cube $\Lambda \subset \mathbb{Z}^d$ satisfy the modelling hypotheses in Model 1.1. Then the first component of the total spin $S_\Lambda = \sum_{x \in \Lambda} \sigma_x^1$ satisfies a CLT upon centering and scaling as $|\Lambda| \uparrow \infty$, with its Kolmogorov-Smirnov distance from a standard Gaussian decaying at the rate $O(\log |\Lambda| \cdot |\Lambda|^{-1/2})$.*

A key feature of Theorem 1.5 is that it provides a unified approach that may be applied irrespective of the number of components N of the spins. This is particularly significant in the context of the fact that many effective tools to understand behaviour of spin systems, become unavailable with increasing number of components N , and rigorous analysis is often possible only in high temperature regimes via perturbative expansions. Furthermore, in the few cases where CLT is known (principally, the classical Ising model [Ell]), the literature on convergence rates is very limited. In the multi-component models and the generalised Ising models considered in this article, to our knowledge, CLT for the total spin is not known. In the one-component case ((i) in Model 1.1), our approach may be generalised to the setting where the spins are not compactly supported but only have a finite exponential moment; however, for the sake of brevity and unity of presentation, we adhere to the case of compactly supported spins in this article.

1.3. Linear statistics of α -determinantal processes. Determinantal point processes [Sos-I, Bor, HKPV] have emerged as a significant probabilistic model for capturing a wide class of phenomena in statistical physics, quantum theory, combinatorics, representation theory and the theory of integrable systems. These processes are characterised by their so-called correlation functions, which represent the probability (densities) of having particles at specified locations; the latter being given by a determinants of a certain kernel matrix with respect to a background measure.

Determinantal processes have been extended and generalized in multiple directions; these include in particular the so-called permanental processes where the determinantal structure of the correlation functions is replaced by a permanental one. A standard one-parameter family of point processes that interpolate between the determinantal and permanental ones, also including the classical Poisson point process, is the family of α -determinantal processes. To define these processes, we first define the notion of the α -determinant of a matrix $A \in \mathbb{C}^{n \times n}$. This is given by the relation

$$(1.4) \quad \text{Det}_\alpha[A] = \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^n A_{i\sigma(i)},$$

where S_n is the symmetric group on n symbols, and $\nu(\sigma)$ stands for the number of cycles in $\sigma \in S_n$. It is easy to see that for $\alpha = -1$, Det_α is the usual determinant; for $\alpha = +1$, it is the permanent and for $\alpha = 0$, we have $\text{Det}_0 = \prod_{i=1}^n A_{ii}$.

Let Ξ be a locally compact Polish space endowed with a non-negative Borel measure μ and a symmetric kernel $K : \Xi \times \Xi \rightarrow \mathbb{C}$. The α -determinantal point process on Ξ with kernel K and background measure μ is a random locally finite point set on Ξ such that for any finite subset $\{x_1, \dots, x_n\} \subset \Xi$, the probability (density, with respect to $\mu^{\otimes n}$ of

having points at these locations is given by the n -point correlation function

$$\rho_n(x_1, \dots, x_n) = \text{Det}_\alpha [(K(x_i, x_j))_{1 \leq i, j \leq n}].$$

It is known that for $\alpha > 0$, α -determinantal processes exist under the conditions that K is a bounded symmetric integral operator on $L^2(\mu)$ that is positive semi-definite and locally trace class, and $\alpha \in \{2/m : m \in \mathbb{N}_+\}$. For $\alpha < 0$, it is additionally necessary that $\text{Spec}(K) \subset [0, -1/\alpha] = [0, 1/|\alpha|]$. For a detailed account of α -determinantal processes, we refer the reader to [ShTa, CuMaOc, BaBlKa]. Clearly, $\alpha = -1$ corresponds to determinantal point processes, whereas $\alpha = +1$ corresponds to permanental processes.

Let X be a point process on a space Ξ . For a test function $\varphi : \Xi \rightarrow \mathbb{R}$ with compact support, the linear statistic $\Lambda(\varphi)$ is given by the sum $\Lambda(\varphi) = \sum_{x \in X} \varphi(x)$. Linear statistics are fundamental objects of interest in understanding point processes; indeed, under very general conditions, the statistical law of a point process is completely determined by the distribution of its linear statistics. In the present work, we will consider $\Xi = \mathbb{R}^d$ and consider functions $\varphi_L(\cdot) = \varphi(\cdot/L)$ for a parameter $L > 0$. We will investigate the family of random variables given by the linear statistics $\Lambda(\varphi_L)$ and, under very general conditions, obtain a CLT for them as $L \rightarrow \infty$.

CLTs for such families of linear statistics of α -determinantal processes are known in specialised situations [ShTa]. The fundamental problem in CLTs for linear statistics is that, although $\Lambda(\varphi_L)$ is expressible as a sum, the summands are highly correlated because of the correlation structure of the α -determinantal point process, and hence standard techniques for CLTs for sums of independent variates cannot be applied in this setting. Generally speaking, such CLTs require a delicate and elaborate analysis of cumulant expansions of $\Lambda(\varphi_L)$, often exploiting particular analytical structures accorded by the specific setting under consideration. For instance, [ShTa] deals with the scenario where K is a translation invariant convolutional operator and μ is the Lebesgue measure on \mathbb{R}^d . Such invariance assumption allows the application of Fourier analytic techniques, which are exploited to perform asymptotic analysis of cumulant expansions for $\Lambda(\varphi_L)$. Another special case where CLTs are better understood is that of determinantal processes, where, using a variety of specialised tools such as Fourier analysis and orthogonal polynomials, progress has been achieved in different settings [ShTa, Sos-II, BrDu, BaHa, RiVi]. Furthermore, in most cases, the literature appears to be limited regarding rates of convergence to normality.

We are able to invoke the techniques in this article to provide a succinct and self contained proof of CLTs for $(\Lambda(\varphi_L))_{L>0}$ for general α -determinantal processes, along with explicit rates of convergence.

We lay out the modelling setup below. In what follows, we will use the notation $C(x, r)$ to denote the cube with centre $x \in \mathbb{R}^d$ and side length r , and $A_n^x(r)$ to denote the annulus $A_n^x(r) = \{y \in \mathbb{R}^d : nr \leq \|x - y\| \leq (n+1)r\}$.

In addition to the existential criteria outlined above (K locally trace class and positive semi-definite on $L^2(\mu)$ and the additional spectral condition for $\alpha < 0$), we require the following conditions to hold for the kernel.

Model 1.2. *Let μ have a bounded density with respect to the Lebesgue measure on \mathbb{R}^d , i.e. $d\mu(x) = f(x)dx$ with $c_1 \leq f(x) \leq c_2 \forall x \in \mathbb{R}^d$ for constants $c_1, c_2 > 0$, and $K(x, x) \leq b$ for all $x \in \mathbb{R}^d$. We have either of:*

- (i) $\alpha > 0$, and for any $\delta > 0$, $\exists m_\delta > 0$ such that for some $r > 0$, it holds that $\forall x \in \mathbb{R}^d$ we have $\text{Vol}(\{y \in C(x, r) : |K(y, y)| \geq m_\delta\}) \geq (1 - \delta) \text{Vol}(C(x, r))$.
- (ii) $\alpha < 0$, and either of :
- (ii.a) $|K(x, y)| \geq a \forall \|x - y\| < \delta$ for some $a, \delta > 0$ and $d > 4$.
- (ii.b) $\exists d/2 < \beta < d; r, c_1, c_2 \in \mathbb{R}_+; n_0 \in \mathbb{N}_+$ such that for each $x \in \mathbb{R}^d$, the set

$$E_x = \{y : |K(x, y)| \geq c_1 \|x - y\|^{-\beta}\}$$

satisfies $\text{Vol}(E_x \cap A_n^x(r)) \geq c_2 \text{Vol}(A_n^x(r))$ for all $n \geq n_0$.

We note that these are very general conditions that are easy to verify for most kernel classes of interest; in Section 4, we demonstrate these conditions in some important cases. For instance, (i) above is a quantitative version of the statement that $K(x, x) > 0$ a.e. $x \in \mathbb{R}^d$; whereas (ii.a) entails positivity of the kernel near the diagonal, and the alternative (ii.b) covers the case where the kernel might vanish near the diagonal but does not decay too fast away from it. Taken together, these conditions cover nearly all translation invariant kernel classes of interest for $d > 1$; but significantly, they also cover perturbations thereof (e.g. via conjugation by bounded functions), whereas such perturbations may render ineffective Fourier analytic or other structure-based techniques. For instance, the intuition behind (ii.b) in Model 1.2 is that there is a polynomial lower bound on the decay of $K(x, y)$ in the separation $\|x - y\|$, but we do not assume it uniformly for all well-separated pairs x, y ; instead we require that it holds on a positive fraction of the space. This allows for zeros of the kernel $K(x, y)$, which is necessary for many applications, see Example 4.11.

We further observe that our approach has limited sensitivity to the ambient dimension d , and is particularly effective in dimensions $d > 2$, where connections of determinantal processes to random matrices are not available.

We are now ready to state the CLT for α -determinantal processes for general α .

Theorem 1.6. *Let X be an α -determinantal process with kernel K and background measure μ additionally satisfying the conditions in Model 1.2. Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ denote a bounded compactly supported function with $\|\varphi\|_2 > 0$. Then $\{\Lambda(\varphi_L)\}_{L>0}$ satisfies a CLT upon centering and scaling as $L \uparrow \infty$, with its Kolmogorov-Smirnov distance from a standard Gaussian decaying at the rate $O(\log L \cdot (\text{Var}(\Lambda(\varphi_L)))^{-1/2}) = O(\log L \cdot L^{-\eta})$, for a positive number η that may be explicitly specified depending on the model class in Model 1.2.*

Remark 1.7. *In particular, η takes the following values : for the model class Model 1.2 (i), $\eta = \frac{1}{2}d$; for the model class Model 1.2 (ii.a), $\eta = \frac{1}{2}d - 2$; for the model class Model 1.2 (ii.b), $\eta = d - \beta$.*

Thus, Model 1.2 implies power law lower bounds on $\text{Var}(\Lambda(\varphi_L))$, which translates into power law decay in Theorem 1.6. This is significant in the context of CLTs for linear statistics of α -determinantal processes, where literature on rates of convergence to Gaussianity is limited (even for the determinantal case $\alpha = -1$).

The case $\alpha = 0$ in α -determinantal processes corresponds to Poisson point processes, for which CLTs for linear statistics are understood to be simpler in nature due to spatial independence. This case can also be covered under the ambit of our technique; however we skip the details for reasons of brevity, and note in passing that our necessary estimates will follow directly from the well-known Campbell formula for the Poisson process [Ka].

Acknowledgements. T.C.D. is supported by the NUS and MOE grants R-146-000-319-114 and MOE-T2EP20120-0010. S.G. is supported in part by the MOE grants R-146-000-250-133 and R-146-000-312-114. We would like to thank Alexandre Eremenko for bringing to our notice the work of L.B. Golinskii. We would also like to thank Julian Sahasrabudhe and Marcus Michelen for illuminating discussions, and for pointing us to Theorem 12.2 in [MS-II].

2. PROOFS OF THEOREM 1.1, COROLLARIES 1.2 AND 1.3

We start by proving Theorem 1.1. In methodological terms, our proof strategy is similar spirit to [MS-II]; however we obtain a simpler and more succinct argument purely based on classical complex analytic techniques. In particular, the following lemma may be seen as a version of [MS-II, Lemma 4.1] and is a key ingredient in the proof. Our proof of the lemma only uses Poisson formula and Dirichlet problem.

Lemma 2.1. *Let h be a harmonic function on a neighbourhood of a rectangle $[-a, a] \times [0, b]$ such that $e^{\pi a/b} \geq 4 \max |h| + 1$ and $h(t) \geq h(t + is)$ for $t \in [-a, a]$ and $s \in [0, b]$. Then, we have $h(is) - h(is') + 1 \geq 0$ for $0 \leq s \leq s' \leq b - s$.*

Proof. By reducing b , we can assume that $0 \leq s \leq s' = b - s$. Let h_0, h_+, h_- be the solutions of the following Dirichlet problems on the rectangle $[-a, a] \times [0, b]$

$$\begin{cases} \Delta h_0 = 0 \\ h_0 = h & \text{on } (-a, a) \times \{0, b\} \\ h_0 = 0 & \text{on } \{-a, a\} \times (0, b) \end{cases} \quad \begin{cases} \Delta h_{\pm} = 0 \\ h_{\pm} = 0 & \text{on } (-a, a) \times \{0, b\} \text{ and } \{\mp a\} \times (0, b) \\ h_{\pm} = h & \text{on } \{\pm a\} \times (0, b). \end{cases}$$

We have $h = h_0 + h_+ + h_-$. So, it is enough to show that $h_0(is) \geq h_0(is')$ for s, s' as above and $|h_{\pm}(is)| \leq 1/2$ for all $s \in (0, b)$.

We prove the first inequality. Denote by \mathbb{D} the unit disk in \mathbb{C} and consider the unique conformal map

$$\Pi : (-a, a) \times (0, b) \rightarrow \mathbb{D} \quad \text{such that} \quad \Pi(ib/2) = 0 \quad \text{and} \quad \Pi'(ib/2) \in \mathbb{R}_+.$$

It can be extended continuously to a bijective map $\Pi : [-a, a] \times [0, b] \rightarrow \overline{\mathbb{D}}$, see [StSh, p.238]. Write $\Pi = \Pi_2 \circ \Pi_1$, where Π_1 is the translation $u \mapsto u - ib/2$ and Π_2 is the unique conformal map from $(-a, a) \times (-b/2, b/2)$ to \mathbb{D} with $\Pi_2(0) = 0$ and $\Pi_2'(0) \in \mathbb{R}_+$. One can also extend Π_2 continuously to a bijective map $\Pi_2 : [-a, a] \times [-b/2, b/2] \rightarrow \overline{\mathbb{D}}$.

Observe that the maps $z \mapsto \overline{\Pi_2(\bar{z})}, -\overline{\Pi_2(-\bar{z})}, -\Pi_2(-z)$ are conformal from $(-a, a) \times (-b/2, b/2)$ to \mathbb{D} and satisfy the same properties at 0 as Π_2 does. The uniqueness of Π_2 implies that all these maps are equal to Π_2 . We easily deduce that Π_2 sends points in each half-line $e^{ik\pi/2}\mathbb{R}_+$, $k = 0, 1, 2, 3$, to the same half-line. In particular, we have for some $y \in [0, 1]$

$$\Pi(is) = \Pi_2(is - ib/2) = -iy \quad \text{and} \quad \Pi(is') = \Pi_2(-is + ib/2) = iy,$$

where we used $is' - ib/2 = -(is - ib/2)$ and $\Pi_2(z) = -\Pi_2(-z)$. Define $g(z) := h_0(\Pi^{-1}(z))$. For $0 < \theta < \pi$, we have either

$$\Pi^{-1}(e^{-i\theta}) \in (-a, a) \quad \text{and} \quad \Pi^{-1}(e^{i\theta}) = \Pi^{-1}(e^{-i\theta}) + ib$$

or these two points belong to the vertical edges of the rectangle where h_0 vanishes. It follows from the definition of h_0 that $g(e^{-i\theta}) \geq g(e^{i\theta})$ for $0 < \theta < \pi$. We need to show that $g(-iy) \geq g(iy)$.

By Poisson's integral formula, we have

$$\begin{aligned} g(\pm iy) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-y^2}{|\pm iy - e^{i\theta}|^2} g(e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{\pi} \frac{1-y^2}{|\pm iy - e^{i\theta}|^2} g(e^{i\theta}) d\theta + \frac{1}{2\pi} \int_0^{\pi} \frac{1-y^2}{|\pm iy + e^{i\theta}|^2} g(e^{-i\theta}) d\theta. \end{aligned}$$

It follows that

$$g(-iy) - g(iy) = \frac{1}{2\pi} \int_0^{\pi} (1-y^2) \left[\frac{1}{|iy - e^{i\theta}|^2} - \frac{1}{|iy + e^{i\theta}|^2} \right] (g(e^{-i\theta}) - g(e^{i\theta})) d\theta.$$

It is clear that $g(-iy) - g(iy) \geq 0$ as each factor of the last integrand is non-negative.

It remains to prove the estimate for h_{\pm} . We only consider the case of h_+ as the case of h_- can be obtained in the same way. Using a dilation of coordinate, we can assume for simplicity that $b = \pi$. We can solve explicitly the Dirichlet problem (see [AgO, p.269]) and obtain

$$h_+(t+is) = \sum_{n=1}^{\infty} A_n \sinh(nt+na) \sin(ns) \quad \text{with} \quad A_n := \frac{2}{\pi \sinh(2na)} \int_0^{\pi} h(a+i\xi) \sin(n\xi) d\xi.$$

Observe that $\sinh(t)/\sinh(2t) = 1/(e^t + e^{-t}) \leq e^{-t}$ for $t \geq 0$. We then deduce that

$$|h_+(is)| \leq 2 \sum_{n=1}^{\infty} e^{-na} \max |h| = \frac{2}{e^a - 1} \max |h| \leq \frac{1}{2}.$$

The lemma follows. ■

We now prove a particular case of Theorem 1.1.

Proposition 2.2. *Let X be as in Theorem 1.1. Assume moreover that X is centered and normalized, i.e., $\mathbb{E}[X] = 0$ and $\sigma = 1$. Let $\kappa := \log^+ \log \mathbb{E}[e^{r|X|}]$. Then, we have for some universal constant $A > 0$*

$$\sup_{x \in \mathbb{R}} |F_X(x) - \Phi(x)| \leq \frac{A(\kappa + 1)}{r}.$$

Observe that the left hand side of the above estimate is always bounded by 1 and we can choose A large enough. Therefore, it is enough to consider r large enough. It follows that

$$\mathbb{E}[e^{r|X|}] \geq \mathbb{E}[r^2|X|^2/2] \geq r^2\sigma^2/2 = r^2/2$$

and hence κ is also large.

By hypothesis, there is a function $f(u)$ which is holomorphic on $\mathbb{D}(0, r)$ and continuous on $\overline{\mathbb{D}(0, r)}$ such that $f(0) = 0$ and $e^{f(u)} = \mathbb{E}[e^{uX}]$. Consider its Taylor's expansion

$$f(u) = \sum_{n \geq 2} a_n u^n = \frac{1}{2} u^2 + a_3 u^3 + \dots = \frac{1}{2} u^2 + R(u).$$

Since X is real-valued, we have $a_n \in \mathbb{R}$ for every n .

Define $h(u) := \operatorname{Re}(f(u)) = \log |\mathbb{E}[e^{uX}]|$. Since X is real-valued, we have $h(u) = h(\bar{u})$ for $|u| \leq r$. We then observe that h is harmonic satisfying $h(0) = 0$, $h \geq 0$ on $[-r, r]$ (by Jensen's inequality) and

$$h(u) = \log |\mathbb{E}[e^{uX}]| \leq \log \mathbb{E}[e^{r|X|}] \leq e^{\kappa} \quad \text{for} \quad |u| \leq r.$$

Thus, $\varphi(u) := e^\kappa - h(u)$ is a nonnegative harmonic function on $\mathbb{D}(0, r)$ and $\varphi(0) = e^\kappa$. By Harnack's inequality (see [Ah, p.243])

$$\varphi(u) \leq \frac{r + |u|}{r - |u|} \varphi(0) \leq 5\varphi(0) = 5e^\kappa \quad \text{for } |u| \leq 2r/3,$$

which implies

$$|h(u)| \leq 4e^\kappa \quad \text{for } |u| \leq 2r/3.$$

Moreover, we have

$$(2.1) \quad h(t) = \log |\mathbb{E}[e^{tX}]| \geq \log |\mathbb{E}[e^{tX} e^{isX}]| = h(t + is) \quad \text{for } |t + is| \leq r.$$

Define $r_1 := r/(2\kappa)$.

Lemma 2.3. *We have for $|t| \leq r_1$ and $0 \leq s \leq s' \leq r_1$*

$$h(t + is) - h(t + is') + 1 \geq 0.$$

Proof. Fix $t \in [-r_1, r_1]$ and define $h_t(u) := h(t + u)$ for all $u \in [-r/2, r/2] \times [0, 2r_1]$. Note that since κ is large, we have for $u \in [-r/2, r/2] \times [0, 2r_1]$

$$|t + u|^2 \leq (r_1 + r/2)^2 + 4r_1^2 < (2r/3)^2.$$

Thus, h_t is a harmonic function on a neighborhood of $[-r/2, r/2] \times [0, 2r_1]$ and

$$4 \max_{[-r/2, r/2] \times [0, 2r_1]} |h_t| + 1 \leq 4 \sup_{\mathbb{D}(0, 2r/3)} |h(u)| + 1 \leq 16e^\kappa + 1 \leq e^{\kappa+4} \leq e^{\pi(r/2)/(2r_1)}.$$

Moreover, by (2.1), we have for all $x \in [-r/2, r/2]$ and $y \in [0, 2r_1]$

$$h_t(x) = h(t + x) \geq h(t + x + iy) = h_t(x + iy).$$

Applying Lemma 2.1 gives

$$h_t(is) - h_t(is') + 1 \geq 0 \quad \text{for } 0 \leq s \leq s' \leq r_1,$$

or equivalently,

$$h(t + is) - h(t + is') + 1 \geq 0 \quad \text{for } 0 \leq s \leq s' \leq r_1.$$

The lemma follows. ■

Define $r_2 := r_1/3$.

Lemma 2.4. *There is a universal constant $c_0 > 0$ such that for every $0 \leq r' \leq r_2$ we have*

$$|h(u)| \leq c_0 \max(|h(ir')|, 1)$$

for $u = t + is$ with $|t| \leq \sqrt{2}r'$ and $|s| \leq \sqrt{2}r'$.

Proof. We will define a number $M > 0$ such that $|h(u)| \lesssim M$ and $|h(ir')| + 1 \gtrsim M$. The implicit constants we use in this lemma are universal.

Let $\Pi : (-3r'/2, 3r'/2)^2 \rightarrow \mathbb{D}$ be the unique conformal map such that $\Pi(0) = 0$ and $\Pi'(0) \in \mathbb{R}_+$. It can be extended to a continuous bijective map $\Pi : [-3r'/2, 3r'/2]^2 \rightarrow \overline{\mathbb{D}}$. Define $g := h \circ \Pi^{-1}$ and $g^\pm := \max(\pm g, 0)$. Let $P(z, \theta) \geq 0$ denote the Poisson kernel on the unit disk \mathbb{D} with $z \in \mathbb{D}$ and $\theta \in [0, 2\pi]$. By Poisson's formula, we have

$$(2.2) \quad \begin{aligned} h(u) &= g(\Pi(u)) = \int_0^{2\pi} P(\Pi(u), \theta) g(e^{i\theta}) d\theta \\ &= \int_0^{2\pi} P(\Pi(u), \theta) g^+(e^{i\theta}) d\theta - \int_0^{2\pi} P(\Pi(u), \theta) g^-(e^{i\theta}) d\theta. \end{aligned}$$

Since $h(0) = 0$ and $P(0, \theta) = 1/(2\pi)$, we obtain

$$\int_0^{2\pi} g^+(e^{i\theta})d\theta = \int_0^{2\pi} g^-(e^{i\theta})d\theta.$$

Define

$$M := \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|d\theta = \frac{1}{\pi} \int_0^{2\pi} g^-(e^{i\theta})d\theta \leq 2 \max_{[0, 2\pi]} g^-(e^{i\theta}).$$

By the last inequality, we have $g(e^{i\theta}) \leq -M/2$ for some $\theta \in [0, 2\pi]$. It follows that there is ζ_0 in the boundary of $[-3r'/2, 3r'/2]^2$ such that $h(\zeta_0) \leq -M/2$. Using the inequality in Lemma 2.3 and the fact that h is symmetric, we find $\zeta_1 \in [-3r'/2, 3r'/2] \times \{3r'/2\}$ such that $h(\zeta_1) \leq -M/2 + 1$.

Now, consider two functions

$$\phi_1(u) := h(u) - h(u + ir') + 1$$

and

$$\phi_2(u) := h(u) - h(\bar{u} + i3r'/2) + 1.$$

Lemma 2.3 and the fact that h is symmetric imply that the above functions are harmonic and non-negative on $(-3r', 3r') \times (-r'/2, 2r')$ and $(-3r', 3r') \times (-3r'/2, 3r'/4)$ respectively.

Observe that $\phi_2(\operatorname{Re}(\zeta_1)) \geq M/2$ because $h \geq 0$ on $[-r, r]$. By Harnack's inequality, we have $\phi_2(t + is) \gtrsim M$ for $|t| \leq 3r'/2$ and $-r' \leq s \leq r'/2$. It follows that $\phi_1(ir'/4) = \phi_2(ir'/4) \gtrsim M$. By Harnack's inequality again, we have $\phi_1(0) \gtrsim M$. Hence, $|h(ir')| + 1 \gtrsim M$ because $h(0) = 0$.

On another hand, observe that if K is a compact subset of \mathbb{D} , the above Poisson's formula (2.2) implies that $\max_K |g| \leq c_K M$ for some constant c_K depending only on K . We deduce from the definition of g that $|h(u)| \lesssim M$ for $u = t + is$ with $|t| \leq \sqrt{2}r'$ and $|s| \leq \sqrt{2}r'$. Now, the lemma follows easily. \blacksquare

The following is a version of [MS-II, Lemma 6.1].

Lemma 2.5. *There exists a universal integer $N \geq 3$ such that for $1 \leq r' \leq r_2$ we have*

$$\sum_{n \geq N} |a_n| r'^n \leq \frac{1}{300} \sum_{n=2}^{N-1} |a_n| r'^n.$$

Proof. By Lemma 2.4, the definition of h and $a_2 = 1/2$, we have for $1 \leq r' \leq r_2$

$$(2.3) \quad |h(u)| \leq c_0 \max(|h(ir')|, 1) \lesssim \sum_{n \geq 2} |a_n| r'^n$$

when $|u| \leq \sqrt{2}r'$. Using a Cauchy's type formula and that $f(0) = 0$, we also have

$$\begin{aligned} f(u) &= i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + (\sqrt{2}r')^{-1}u}{e^{i\theta} - (\sqrt{2}r')^{-1}u} \operatorname{Re} f(\sqrt{2}r' e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{2\sqrt{2}r' e^{i\theta}}{\sqrt{2}r' e^{i\theta} - u} - 1 \right] h(\sqrt{2}r' e^{i\theta}) d\theta. \end{aligned}$$

By taking derivatives at 0 and using (2.3), we obtain for $m \geq 2$

$$|a_m| = |f^{(m)}(0)/m!| \lesssim (\sqrt{2}r')^{-m} \max_{|u|=\sqrt{2}r'} |h(u)| \lesssim 2^{-m/2} r'^{-m} \sum_{n \geq 2} |a_n| r'^n.$$

Hence,

$$\sum_{n \geq N} |a_n| r^n \lesssim 2^{-N/2} \sum_{n \geq 2} |a_n| r^n = 2^{-N/2} \sum_{n \geq N} |a_n| r^n + 2^{-N/2} \sum_{n=2}^{N-1} |a_n| r^n.$$

We easily obtain the lemma by taking N large enough. \blacksquare

We continue the proof of Proposition 2.2 and study a_n for $n < N$ using an improvement of Marcinkiewicz's argument. Consider the points $P_n = (n, \log(|a_n| r_2^n))$ in \mathbb{R}^2 for $2 \leq n \leq N-1$. Define also

$$P_N := (N, \max_{2 \leq n \leq N-1} \log(|a_n| r_2^n)).$$

This sequence of points admits a unique subsequence

$$P_{n_1}, \dots, P_{n_{l-1}}, P_{n_l} \quad \text{with} \quad 2 = n_1 < \dots < n_{l-1} < n_l = N,$$

such that

- If Γ_j denotes the segment joining $P_{n_j}, P_{n_{j+1}}$ and Γ is the union of the Γ_j , then no point P_n , with $2 \leq n \leq N$, is above the polygon curve Γ ;
- If θ_j denotes the slope of Γ_j , then the sequence $(\theta_j)_{1 \leq j \leq l-1}$ is strictly decreasing and $\theta_{l-1} = 0$.

For example, if $\log(|a_n| r_2^n) \leq \log(|a_2| r_2^2)$ for $2 \leq n \leq N-1$, then $l = 2$ and Γ is just a segment parallel to the abscissa axis of \mathbb{R}^2 . Note that Γ can be seen as the graph of a concave non-decreasing function over the interval $[2, N]$ which is constant on $[n_{l-1}, N]$.

Lemma 2.6. *We have $\theta_{j-1} - \theta_j \leq 10$ for $2 \leq j \leq l-1$ and $\theta_1 \leq 10N$.*

Proof. Since $\theta_{l-1} = 0$, the second inequality is a direct consequence of the first one. We prove now the first inequality by contradiction. Recall that we have $\operatorname{Re} f(t + is) \leq |f(t)|$ for $t, s \in \mathbb{R}$. The goal is to find t, s such that $\operatorname{Re} f(t + is) > |f(t)|$, which contradicts the above inequality.

Let $2 \leq j \leq l-1$ be the largest integer for which the inequality in the lemma is wrong, that is

$$\theta_{j-1} - \theta_j > 10 \quad , \quad \theta_{k-1} - \theta_k \leq 10 \quad \text{for all} \quad j < k \leq l-1.$$

Since $\theta_{l-1} = 0$, the maximality of j implies that $\theta_j \leq 10(N-1)$. Define $\theta := \theta_j + 5$. We have

$$5 \leq \theta \leq 10N \quad \text{and} \quad \theta_{j-1} - \theta > 5.$$

Claim. There is a complex number $u = t + is$ such that $t = \pm e^{-\theta} r_2$, $a_{n_j} \operatorname{Re}(u^{n_j}) \geq \sqrt{3} |a_{n_j}| |u|^{n_j} / 2$ and $2|t|/\sqrt{3} \leq |u| \leq 8|t| < r_2$.

Proof of Claim. We will choose $t = \pm e^{-\theta} r_2$ and $\arg(u)$ in $[\pi/6, 11\pi/24] \cup [-11\pi/24, -\pi/6]$. Define $\alpha := \arg(u)$. Then $|u|/|t| = 1/|\cos \alpha|$ and it is easy to check that $2|t|/\sqrt{3} \leq |u| \leq 8|t| < r_2$. When α runs over the above set and $n_j \neq 3, 4, 6$, $\arg(u^{n_j}) = n_j \alpha$ takes all possible values modulo 2π . So, the claim is clear in this case. When $n_j = 3, 4, 6$, it is not difficult to check that there is a value of α in the considered set such that $\cos(n_j \alpha) \geq \sqrt{3}/2$ (resp. $\cos(n_j \alpha) \leq -\sqrt{3}/2$). Depending on the sign of a_{n_j} , the claim holds for one of these values of α . \blacksquare

We deduce from the properties of θ that $e^{-10N}r_2 \leq |t| \leq e^{-5}r_2$. Since we only consider big enough r , we can assume that $|t| \geq 1$. This allows us to apply Lemma 2.5 for t and u .

Denote by L^- (respectively, L^+) the line through P_{n_j} with slope θ_{j-1} (respectively, θ_j). For any $2 \leq n \leq n_j - 1$, P_n is under the line L^- . This implies

$$\log(|a_n|r_2^n) - \log(|a_{n_j}|r_2^{n_j}) \leq \theta_{j-1}(n - n_j)$$

hence

$$|a_n|r_2^n \leq |a_{n_j}|r_2^{n_j} e^{\theta_{j-1}(n-n_j)}.$$

Multiplying both sides by $e^{-n\theta}$, we have

$$|a_n|(e^{-\theta}r_2)^n \leq |a_{n_j}|(e^{-\theta}r_2)^{n_j} e^{(\theta_{j-1}-\theta)(n-n_j)} \leq |a_{n_j}|(e^{-\theta}r_2)^{n_j} e^{5(n-n_j)}.$$

Hence,

$$(2.4) \quad |a_n||t|^n \leq |a_{n_j}||t|^{n_j} e^{5(n-n_j)}, \quad 2 \leq n \leq n_j - 1.$$

It follows that

$$\sum_{n=2}^{n_j-1} |a_n||t|^n \leq |a_{n_j}||t|^{n_j} \sum_{k=1}^{\infty} e^{-5k} = |a_{n_j}||t|^{n_j} \frac{e^{-5}}{1 - e^{-5}} < \frac{1}{100} |a_{n_j}||t|^{n_j}.$$

Similarly, for any $n_j + 1 \leq n \leq N - 1$, we have that P_n is under L^+ . With the same argument, we obtain

$$(2.5) \quad |a_n||t|^n \leq |a_{n_j}||t|^{n_j} e^{-5(n-n_j)}, \quad n_j + 1 \leq n \leq N - 1.$$

Hence,

$$\sum_{n=n_j+1}^{N-1} |a_n||t|^n \leq |a_{n_j}||t|^{n_j} \sum_{k=1}^{\infty} e^{-5k} = |a_{n_j}||t|^{n_j} \frac{e^{-5}}{1 - e^{-5}} < \frac{1}{100} |a_{n_j}||t|^{n_j}.$$

Consider the following expansion of $f(t)$

$$f(t) = a_{n_j}t^{n_j} + \sum_{n=2}^{n_j-1} a_n t^n + \sum_{n=n_j+1}^{N-1} a_n t^n + \sum_{n=N}^{\infty} a_n t^n.$$

Denote by S_1, S_2 and S_3 the three summations in the above expansion of $f(t)$. From the above discussion, we deduce

$$|S_1| < \frac{1}{100} |a_{n_j}||t|^{n_j} \quad \text{and} \quad |S_2| < \frac{1}{100} |a_{n_j}||t|^{n_j}.$$

We apply Lemma 2.5 for t and obtain

$$|S_3| \leq \frac{1}{300} (|a_{n_j}||t|^{n_j} + |S_1| + |S_2|) \leq \frac{1}{100} |a_{n_j}||t|^{n_j}.$$

Hence,

$$(2.6) \quad |f(t)| \leq \frac{103}{100} |a_{n_j}||t|^{n_j}.$$

Now, consider the following expansion of $f(u)$

$$(2.7) \quad f(u) = a_{n_j}u^{n_j} + \sum_{n=2}^{n_j-1} a_n u^n + \sum_{n=n_j+1}^{N-1} a_n u^n + \sum_{n=N}^{\infty} a_n u^n.$$

Denote by S'_1, S'_2 and S'_3 the three summations in this expansion. Using (2.4) gives

$$\begin{aligned} |S'_1| &\leq \sum_{n=2}^{n_j-1} |a_n| |u|^n = |a_{n_j}| |u|^{n_j} \sum_{n=2}^{n_j-1} \frac{|a_n|}{|a_{n_j}|} |u|^{n-n_j} \\ &\leq |a_{n_j}| |u|^{n_j} \sum_{n=2}^{n_j-1} \frac{|a_n|}{|a_{n_j}|} |t|^{n-n_j} \leq |a_{n_j}| |u|^{n_j} \frac{e^{-5}}{1-e^{-5}} < \frac{1}{100} |a_{n_j}| |u|^{n_j}. \end{aligned}$$

Using (2.5) and the above Claim, we can bound S'_2 as follows

$$\begin{aligned} |S'_2| &\leq \sum_{n=n_j+1}^{N-1} |a_n| |u|^n = |a_{n_j}| |u|^{n_j} \sum_{n=n_j+1}^{N-1} \frac{|a_n|}{|a_{n_j}|} |u|^{n-n_j} \\ &\leq |a_{n_j}| |u|^{n_j} \sum_{n=n_j+1}^{N-1} \frac{|a_n|}{|a_{n_j}|} (8|t|)^{n-n_j} \leq |a_{n_j}| |u|^{n_j} \sum_{n=n_j+1}^{N-1} 8^{n-n_j} e^{-5(n-n_j)} \\ &\leq |a_{n_j}| |u|^{n_j} \sum_{k=1}^{\infty} (8e^{-5})^k = |a_{n_j}| |u|^{n_j} \frac{8e^{-5}}{1-8e^{-5}} \leq \frac{8}{100} |a_{n_j}| |u|^{n_j}. \end{aligned}$$

Applying Lemma 2.5 gives

$$|S'_3| \leq \frac{1}{300} \left(|a_{n_j}| |u|^{n_j} + |S'_1| + |S'_2| \right) \leq \frac{1}{100} |a_{n_j}| |u|^{n_j}.$$

Hence, using (2.7), the Claim, together with (2.6), we get

$$\begin{aligned} \operatorname{Re} f(u) &\geq \operatorname{Re} [a_{n_j} u^{n_j}] - \frac{1}{10} |a_{n_j}| |u|^{n_j} \geq \left(\frac{\sqrt{3}}{2} - \frac{1}{10} \right) |a_{n_j}| |u|^{n_j} \\ &\geq \left(\frac{\sqrt{3}}{2} - \frac{1}{10} \right) \left(\frac{2}{\sqrt{3}} \right)^3 |a_{n_j}| |t|^{n_j} > \frac{103}{100} |a_{n_j}| |t|^{n_j} \geq |f(t)|. \end{aligned}$$

This is the contradiction we are looking for. The proof is now completed. \blacksquare

Define $r_3 := e^{-10N} r_2$. Recall that $f(u) = u^2/2 + R(u)$.

Lemma 2.7. *We have $|a_n| \leq \frac{1}{2} r_3^{2-n}$ for $2 \leq n \leq N-1$. In particular, we have $|R(it)| \leq c_1 |t|^3 r_3^{-1}$ for $|t| \leq r_3$, where $c_1 > 0$ is a universal constant.*

Proof. By Lemma 2.6, we have $\theta_1 \leq 10N$. By the definition of θ_j and r_3 , we deduce that $|a_n| r_3^n \leq |a_2| r_3^2$. We obtain the first inequality using that $a_2 = 1/2$. We prove now the second inequality. By Lemma 2.5 and the first assertion, we have

$$\sum_N^{\infty} |a_n| |t|^n \leq |t/r_3|^N \sum_N^{\infty} |a_n| r_3^n \lesssim |t/r_3|^N \sum_2^{N-1} |a_n| r_3^n \lesssim |t/r_3|^N r_3^2 \leq |t|^3 r_3^{-1}.$$

On the other hand, we have

$$\sum_3^{N-1} |a_n| |t|^n = \sum_3^{N-1} (|t/r_3|^n |a_n| r_3^n) \lesssim \sum_3^{N-1} (|t/r_3|^n r_3^2) \lesssim (|t/r_3|^3 r_3^2) = |t|^3 r_3^{-1}.$$

The result follows. \blacksquare

End of the proof of Proposition 2.2. From the proof of the Berry-Esseen theorem [Fe, p.538], for every positive number T , we have

$$\sup_{x \in \mathbb{R}} |F_X(x) - \Phi(x)| \lesssim \int_{-T}^T \left| \frac{e^{f(it)} - e^{-t^2/2}}{t} \right| dt + \frac{1}{T} = \int_{-T}^T \left| \frac{e^{R(it)} - 1}{t} \right| e^{-t^2/2} dt + \frac{1}{T}.$$

Choose $T = \delta r_3$ for some constant $\delta > 0$ small enough. We have

$$\sup_{x \in \mathbb{R}} |F_X(x) - \Phi(x)| \lesssim \left\{ \int_{|t| \leq 2\sqrt{\log r_3}} + \int_{|t|=2\sqrt{\log r_3}}^{\delta r_3} \right\} \left| \frac{e^{R(it)} - 1}{t} \right| e^{-t^2/2} dt + r_3^{-1}.$$

For the first integral, by Lemma 2.7, we have $|R(it)| \leq 1$ for $|t| \leq 2\sqrt{\log r_3}$ because r_3 is large. Using the fact $|e^R - 1| \leq e|R|$ for $|R| \leq 1$ and Lemma 2.7 again, we deduce that

$$\int_{|t| \leq 2\sqrt{\log r_3}} \left| \frac{e^{R(it)} - 1}{t} \right| e^{-t^2/2} dt \lesssim \int_{|t| \leq 2\sqrt{\log r_3}} r_3^{-1} t^2 e^{-t^2/2} dt \lesssim r_3^{-1} \int_{\mathbb{R}} t^2 e^{-t^2/2} dt \lesssim r_3^{-1}.$$

For the second integral, observe that for $2\sqrt{\log r_3} \leq |t| \leq \delta r_3$ we have $|R(it)| \leq t^2/4$ because δ is small. Thus,

$$\int_{|t|=2\sqrt{\log r_3}}^{\delta r_3} \left| \frac{e^{R(it)} - 1}{t} \right| e^{-t^2/2} dt \lesssim \int_{|t|=2\sqrt{\log r_3}}^{\delta r_3} e^{t^2/4} e^{-t^2/2} dt \lesssim \int_{2\sqrt{\log r_3}}^{\infty} t e^{-t^2/4} dt \lesssim r_3^{-1}.$$

Combining everything, we have

$$\sup_{x \in \mathbb{R}} |F_X(x) - \Phi(x)| \lesssim r_3^{-1} \lesssim \frac{\kappa}{r}.$$

This ends the proof of the proposition. ■

End of the proof of Theorem 1.1. We can assume that $|\sigma - 1| < 1/2$ and hence $\sigma > 1/2$ because the theorem is clear otherwise. Define

$$\kappa := \log^+ \log \mathbb{E}[e^{r|X|}] \leq \log^+ \log (\mathbb{E}[e^{rX}] + \mathbb{E}[e^{-rX}]) \leq 1 + \log^+ \log \max_{|u|=r} |\mathbb{E}[e^{iuX}]|.$$

Define also

$$\hat{X} := \frac{X - \mathbb{E}[X]}{\sigma} = \frac{\bar{X}}{\sigma}, \quad \hat{r} := \frac{r}{2} < r\sigma, \quad \hat{\kappa} := \log^+ \log \mathbb{E}[e^{\hat{r}|\hat{X}|}].$$

By Jensen's inequality, we have

$$\mathbb{E}[r|X|] \leq \log \mathbb{E}[e^{r|X|}] \leq e^\kappa$$

and hence

$$\log \mathbb{E}[e^{\hat{r}|\hat{X}|}] \leq \log \mathbb{E}[e^{r\sigma|\hat{X}|}] = \log \mathbb{E}[e^{r|X - \mathbb{E}[X]|}] \leq \log \mathbb{E}[e^{r|X|}] + r\mathbb{E}[|X|] \leq 2e^\kappa < e^{\kappa+1}.$$

This implies $\hat{\kappa} < \kappa + 1$. Applying Proposition 2.2 for \hat{X} , \hat{r} , $\hat{\kappa}$ gives

$$\sup_{x \in \mathbb{R}} |F_{\hat{X}}(x) - \Phi(x)| \lesssim \frac{\hat{\kappa}}{\hat{r}} \lesssim \frac{\kappa + 1}{r} \lesssim \frac{1 + \log^+ \log \max_{|u|=r} |\mathbb{E}[e^{iuX}]|}{r}.$$

On another hand, we have

$$\begin{aligned} \sup_{x \in \mathbb{R}} |F_{\hat{X}}(x) - \Phi(x)| &= \sup_{x \in \mathbb{R}} |F_{\hat{X}}(x) - \Phi(\sigma x)| \\ &\leq \sup_{x \in \mathbb{R}} |F_{\hat{X}}(x) - \Phi(x)| + \sup_{x \in \mathbb{R}} |\Phi(\sigma x) - \Phi(x)|. \end{aligned}$$

So, in order to get the desired estimate, we only need to bound $|\Phi(\sigma x) - \Phi(x)|$ by $2|\sigma - 1|$ for $x \in \mathbb{R}$. Observe that since $|\sigma - 1| \leq 1/2$, we have

$$\begin{aligned} |\Phi(\sigma x) - \Phi(x)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{\sigma x}^x e^{-t^2/2} dt \right| = \frac{1}{\sqrt{2\pi}} \left| \int_{\sigma}^1 x e^{-x^2 s^2/2} ds \right| \\ &\leq \frac{1}{\sqrt{2\pi}} |\sigma - 1| |x| e^{-x^2/8} \leq 2|\sigma - 1| \end{aligned}$$

because $|x|e^{-x^2/8}$ attains its maximum when $|x| = 4$. The proof is finished. \blacksquare

Proof of Corollary 1.2. The case of zero variance is clear. By replacing X with X/σ , we can assume for simplicity that $\sigma = 1$. Choose a sequence $r_n \rightarrow \infty$ such that $r_n^{-1} \log \sup_{|u|=r_n} \operatorname{Re} f(u) \rightarrow 0$. Applying Theorem 1.1 to X , r_n and $\sigma = 1$ gives that $F_{\overline{X}} = \Phi$. Hence, \overline{X} is a Gaussian. The result follows. \blacksquare

Proof of Corollary 1.3. Without loss of generality, we can replace X_n, r_n by $X_n/\sigma_n, r_n\sigma_n$ so that we can assume that $\sigma_n = 1$. Now, it is enough to apply Theorem 1.1 to $X_n, r_n, 1$ instead of X, r, σ . \blacksquare

3. CLTs IN GENERAL SPIN SYSTEMS: GENERALISED ISING, XY AND HEISENBERG MODELS

We recall the definition of spin systems from Section 1.2; in particular, the Hamiltonian H_Λ (1.2) and the spin measure $\mathbb{P}_\Lambda(\sigma_\Lambda)$ (1.3). Recall that for each $x \in \Lambda$, the spin $\sigma_x \in \mathbb{R}^N$. We will consider one-component ($N = 1$) models with general real valued spins (essentially, generalised versions of the classical Ising model), and multi-component spin systems with $N \geq 2$ components.

We would be interested in the *partition function* for the spin system, which is given by the integral

$$(3.1) \quad \mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x\}_{x \in \Lambda}) := \int_{(\mathbb{R}^N)^{\otimes |\Lambda|}} \exp(-\beta H_\Lambda(\sigma_\Lambda)) \prod_{x \in \Lambda} d\mu_0(\sigma_x),$$

3.1. Ferromagnetism and its generalisations. When $N = 1$, the model is called *ferromagnetic* if $J_{xy} = J_{yx} \geq 0$ for all x, y . For ferromagnetic models with homogeneous nearest neighbour interaction J and uniform magnetic field h , it is known that, at positive temperature β^{-1} , the total spin σ exhibits a Gaussian central limit theorem as the d -dimensional cubic domain $\Lambda \uparrow \mathbb{Z}^d$ (see, e.g., [Ell]).

For multi-component spins (i.e., $N \geq 2$), the generalisation of the ferromagnetic condition is of considerable interest, but not straightforward. It is considered in the literature that for the XY model, the appropriate generalisation is $J_{xy}^1 \geq |J_{xy}^2|$, whereas for the classical Heisenberg model, the analogous condition is $J_{xy}^1 \geq \max\{|J_{xy}^2|, |J_{xy}^3|\}$; recall the spin Models 1.1.

For $N \geq 3$, to the best of our knowledge, entirely satisfactory generalisations are not known, even for special choices of the background spin measure μ_0 . A natural extension of ferromagnetism for the XY and classical Heisenberg models to higher component spins would be to posit that $J_{xy}^1 \geq \max_{2 \leq i \leq N} \{|J_{xy}^i|\}$, but unfortunately, crucial theorems on such spin models (such as the Lee-Yang theorem for the zeros of the partition function, see Section 3.2) are known to hold under the much more restrictive condition $J_{xy}^1 \geq \sum_{i=2}^N |J_{xy}^i|$.

3.2. Lee-Yang theorem. We will now discuss the celebrated Lee-Yang theorem for spin systems, which will be a key tool in verifying the zero-free condition with regard to CLT for total spin. To this end, we first introduce the following subset of \mathbb{C}^N

$$(3.2) \quad \Omega_N^+ := \left\{ \mathbf{h} = (h^1, \dots, h^N) \in \mathbb{C}^N : \operatorname{Re} h^1 > \sum_{i=2}^N |h^i| \right\}.$$

For the purposes of the Lee-Yang theorem, we will consider the following hypotheses on spin systems defined on a cube $\Lambda \subset \mathbb{Z}^d$; these may be easily seen to cover the Model class 1.1 for which we eventually want to establish CLTs.

Model 3.1 (Model classes for Lee-Yang theorem). *One of the following choices of measure μ_0 and ferromagnetic conditions on the interactions:*

- (a) $N = 1$; μ_0 is an even measure and satisfies (3.3) and (3.4) below, and the ferromagnetic condition $J_{xy} \geq 0$ is satisfied for all $(x, y) \subset \Lambda$.
- (b) $N = 2$; (XY model) μ_0 is the uniform measure on the unit circle \mathbb{S}^1 , and the ferromagnetic condition $J_{xy}^1 \geq |J_{xy}^2|$ is satisfied for all $(x, y) \subset \Lambda$.
- (c) $N = 3$; (classical Heisenberg model) μ_0 is the uniform measure on the unit sphere \mathbb{S}^2 , and the ferromagnetic condition $J_{xy}^1 \geq \max\{|J_{xy}^2|, |J_{xy}^3|\}$ is satisfied for all $(x, y) \subset \Lambda$.
- (d) $N \geq 2$; μ_0 is rotationally invariant, and satisfies

$$(3.3) \quad \int_{\mathbb{R}^N} e^{u\sigma^1} d\mu_0(\sigma) \neq 0, \quad \forall u \in \mathbb{C} \text{ such that } \operatorname{Re}(u) \neq 0,$$

and

$$(3.4) \quad \int_{\mathbb{R}^N} e^{b|\sigma|^2} d\mu_0(\sigma) < \infty, \quad \forall b > 0,$$

and the ferromagnetic condition

$$(3.5) \quad J_{xy}^1 \geq \sum_{i=2}^N |J_{xy}^i|$$

is satisfied for all $(x, y) \subset \Lambda$.

We have the following result.

Theorem 3.1 (Lee-Yang theorem). *Let the random spin configuration (1.3) with Hamiltonian (1.2) satisfy any of the conditions (a) – (d) in Model 3.1. Then for $\beta > 0$, the partition function $\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x\}_{x \in \Lambda})$ in (3.1) does not vanish whenever $\mathbf{h}_x \in \Omega_N^+$ for each $x \in \Lambda$.*

For detailed discussion on Lee-Yang theorem, we refer the reader to the articles [FrRo, LiSo, New], and the references therein.

3.3. The no-zeros condition for S_Λ . Herein we establish that the desired no-zeros condition for the characteristic function of the total spin for ferromagnetic systems at positive temperature β^{-1} . We state this as follows.

Lemma 3.2. *Let the random spin configuration (1.3) with Hamiltonian (1.2) satisfy any of the conditions (a) – (d) in Model 3.1. Let S_Λ denote the first component of the total spin on Λ , i.e.,*

$$S_\Lambda := \sum_{x \in \Lambda} \sigma_x^1,$$

and let $\Psi_{S_\Lambda}(u) := \mathbb{E}[\exp(iuS_\Lambda)]$. Then for $\beta > 0$ and $\mathbf{h}_x \in \Omega_N^+$ for all $x \in \Lambda$, the function Ψ_{S_Λ} does not vanish in a neighbourhood $U_\Lambda \subset \mathbb{C}$ around the origin. Furthermore, if we have a sequence $\Lambda \uparrow \mathbb{Z}^d$, then this U_Λ may be chosen to be uniform in Λ whenever

$$(3.6) \quad \inf_{x \in \mathbb{Z}^d} \left(\operatorname{Re} h_x^1 - \sum_{i=2}^N |h_x^i| \right) > 0.$$

Remark 3.3. It may be observed that (3.6) is satisfied in the crucial setting of a uniform external field, i.e., $\mathbf{h}_x = \mathbf{h} \in \Omega_N^+$ for all $x \in \mathbb{Z}^d$.

Proof of Lemma 3.2. To begin with, we set \mathbf{e}_1 to be the unit x -coordinate direction in \mathbb{C}^N . We may then write (the three integrals below are with respect to the measure $\prod_{x \in \Lambda} d\mu_0(\boldsymbol{\sigma}_x)$)

$$\begin{aligned} & \mathbb{E}[\exp(uS_\Lambda)] \\ &= \frac{\int_{(\mathbb{R}^N)^{\otimes |\Lambda|}} \exp(u \sum_{x \in \Lambda} \sigma_x^1) \exp(-\beta H_\Lambda(\boldsymbol{\sigma}_\Lambda))}{\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x\}_{x \in \Lambda})} \\ &= \frac{\int_{(\mathbb{R}^N)^{\otimes |\Lambda|}} \exp(u \sum_{x \in \Lambda} \sigma_x^1) \exp\left(\beta \sum_{(x,y) \subset \Lambda} \sum_{i=1}^N J_{xy}^i \sigma_x^i \sigma_y^i + \beta \sum_{x \in \Lambda} \sum_{i=1}^N h_x^i \sigma_x^i\right)}{\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x\}_{x \in \Lambda})} \\ &= \frac{\int_{(\mathbb{R}^N)^{\otimes |\Lambda|}} \exp\left(\beta \sum_{(x,y) \subset \Lambda} \sum_{i=1}^N J_{xy}^i \sigma_x^i \sigma_y^i + \beta \sum_{x \in \Lambda} \left((h_x^1 + \beta^{-1}u)\sigma_x^1 + \sum_{i=2}^N h_x^i \sigma_x^i\right)\right)}{\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x\}_{x \in \Lambda})} \\ (3.7) \quad &= \frac{\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x + \beta^{-1}u\mathbf{e}_1\}_{x \in \Lambda})}{\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x\}_{x \in \Lambda})}. \end{aligned}$$

It is apparent from (3.7) that the characteristic function Ψ_{S_Λ} satisfies

$$(3.8) \quad \Psi_{S_\Lambda}(u) = \mathbb{E}[\exp(iuS_\Lambda)] = \frac{\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x + \beta^{-1}iue_1\}_{x \in \Lambda})}{\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x\}_{x \in \Lambda})}.$$

Thus, $\Psi_{S_\Lambda}(u)$ vanishes if and only if $\mathcal{Z}_{\beta, \Lambda}(\{\mathbf{h}_x + \beta^{-1}iue_1\}_{x \in \Lambda})$ vanishes. Observe that $\mathbf{h}_x \in \Omega_N^+$ for each $x \in \Lambda$ and the condition (3.2) defining Ω_N^+ is an open condition. Therefore, for each $x \in \Lambda$, if u small enough (depending on x) then $\mathbf{h}_x + \beta^{-1}iue_1 \in \Omega_N^+$. Since the set Λ is finite, we may take the minimum over Λ of such allowable moduli for u , such that $\mathbf{h}_x + \beta^{-1}iue_1 \in \Omega_N^+$ for all $x \in \Lambda$. Theorem 3.1 implies the first assertion of the lemma.

The bound on $|u|$ above is a function of Λ . If the condition (3.6) holds, then (3.2) implies that this choice can be made uniformly in Λ , thereby completing the proof. \blacksquare

Remark 3.4. In the setting of a uniform magnetic field \mathbf{h} (for simplicity), observe in the proof of Lemma 3.2 that we need $\mathbf{h} + \beta^{-1}iue_1 \in \Omega_N^+$; equivalently $\operatorname{Re} h^1 - \beta^{-1} \operatorname{Im} u > \sum_{i=2}^N |h^i|$. Clearly, this requires us to be able to choose $|\operatorname{Im}(u)|$ to be small; in particular, u cannot be allowed to vary over an infinite vertical strip in \mathbb{C} .

3.4. Variance growth for S_Λ . In this section, we demonstrate variance growth conditions for the total spin (to be precise, the first component thereof), denoted S_Λ for the spin systems under our consideration, which ensure asymptotics normality of S_Λ .

3.4.1. Sufficiency of growth rates. Here, we lay out concrete growth rates for S_Λ and its standard deviation σ_Λ , calibrated against the system size (or volume) $|\Lambda|$, which ensure asymptotic normality. To this end, we have the following lemma.

Lemma 3.5. *Let the random spin configuration (1.3) with Hamiltonian (1.2) satisfy any of the conditions (a) – (d) in Model 3.1, with the external field gap condition (3.6) holding true. Suppose the following two bounds hold simultaneously for some positive numbers $c_1, c_2, \alpha, \epsilon$:*

$$(3.9) \quad (i) \quad |S_\Lambda| \leq c_1 |\Lambda|^\alpha \text{ almost surely} \quad ; \quad (ii) \quad \sigma_\Lambda \geq c_2 |\Lambda|^\epsilon.$$

Then the following growth condition from Corollary 1.3 is satisfied with any constant r_Λ (independent of Λ), as $|\Lambda| \uparrow \infty$:

$$(3.10) \quad \lim_{|\Lambda| \uparrow \infty} (r_\Lambda \sigma_\Lambda)^{-1} \log^+ \log |\mathbb{E}[e^{r_\Lambda S_\Lambda}]| = 0.$$

Proof. In order to verify (3.10), we notice that (3.9) (i) and the constancy of r_Λ ensures $\mathbb{E}[e^{r_\Lambda S_\Lambda}] \leq e^{c_3 |\Lambda|^\alpha}$, for some positive constant c_3 . As a result,

$$(3.11) \quad \log^+ \log |\mathbb{E}[e^{r_\Lambda S_\Lambda}]| \lesssim \log |\Lambda|.$$

On the other hand, r_Λ being constant and the variance growth condition (3.9) (ii) implies that

$$(3.12) \quad r_\Lambda \sigma_\Lambda \gtrsim |\Lambda|^\epsilon.$$

Putting together (3.11) and (3.12), we obtain (3.10). ■

3.4.2. Upper bound on S_Λ . For a compactly supported spin distribution μ_0 , there is a positive number M (depending only on μ_0) such that $|\sigma_x^1| \leq M$. In view of this, we easily bound

$$|S_\Lambda| = \left| \sum_{x \in \Lambda} \sigma_x^1 \right| \leq \sum_{x \in \Lambda} |\sigma_x^1| \leq M \cdot |\Lambda|,$$

thereby satisfying (3.9) (i). The hypothesis of a compactly supported spin distribution μ_0 is valid for most natural models, including the Ising model, the XY model and the classical Heisenberg model (where the spin measure μ_0 is supported on the sets $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^2$ respectively).

3.4.3. Lower bound on the variance of S_Λ . In this section, we discuss how to obtain lower bounds on the variance growth rate of S_Λ that scale as some power $|\Lambda|^\epsilon$ of the total system size (i.e. volume) $|\Lambda|$. Our approach will make use of positive association, which is generally believed to be true for spin systems in the ferromagnetic regime. Broadly speaking, we will make use of two ingredients: non-negativity of spin correlations and a variance lower bound for the spin at individual sites. These will suffice because of the following reason. We have

$$\text{Var}[S_\Lambda] = \sum_{x \in \Lambda} \text{Var}[\sigma_x^1] + \sum_{x \neq y \in \Lambda} \text{Cov}(\sigma_x^1, \sigma_y^1).$$

If $\text{Cov}(\sigma_x^1, \sigma_y^1) \geq 0$ for all $x, y \in \Lambda$, then we may lower bound

$$\text{Var}[S_\Lambda] \geq \sum_{x \in \Lambda} \text{Var}[\sigma_x^1].$$

If for some constant $c_1 > 0$ we further have $\text{Var}[\sigma_x^1] \geq c_1$ for $x \in A \subset \Lambda$ such that $|A| \geq c_2 \cdot |\Lambda|$ for some constant $c_2 > 0$, then we can further bound

$$\text{Var}[S_\Lambda] \geq \sum_{x \in A} \text{Var}[\sigma_x^1] \geq c_3 \cdot |\Lambda|$$

for a positive constant c_3 . This would imply that $\sigma_\Lambda \gtrsim |\Lambda|^{1/2}$, as desired.

3.4.4. Non-negativity of spin correlations. We will establish the non-negativity of spin correlations $\text{Cov}(\sigma_x^1, \sigma_y^1)$ via classic correlation inequalities for spin systems. Unfortunately, for higher dimensional spins ($N \geq 2$), such correlation inequalities are difficult to obtain, and are known only for special models of particular interest in statistical physics. These include the Ising model, the XY model and the classical Heisenberg ferromagnet for $N = 1, 2, 3$ respectively. The literature around such inequalities is quite substantive; for a relatively detailed account we refer to the lecture notes [Ba]. For the purposes of the present work, we will refer to the concrete statements from [MoPi]; the earlier work [KuPfvY] also addresses some of these models.

N=1: Proposition 2.2 in [Ba] gives us $\text{Cov}(\sigma_x^1, \sigma_y^1) \geq 0$ for symmetric invariant single spin measures μ_0 with ferromagnetic interactions $J_{xy} \geq 0 \forall (xy) \subset \Lambda$ and external magnetic field $h \geq 0$. This is obtained by setting $A = (x, 1)$ and $B = (y, 1)$ in this proposition. In particular, this covers the case of Ising models for any d .

N=2: The XY model, with external field $h \geq 0$ along the x axis, is addressed by [MoPi] Corollary 3.4. The non-negativity of $\text{Cov}(\sigma_x^1, \sigma_y^1)$ for XY model with $J_{uv}^i \geq 0, h_u^i \geq 0$ for all $i = 1, 2$ and $u, v \in \Lambda$ can be obtained from this corollary, by setting in Eq. (3.15) therein $A(x) = 1, A(z) = 0$ for all $z \neq x$ and $B(y) = 1, B(z) = 0$ for all $z \neq y$, where $x, y, z \in \Lambda$.

N=3: The classical Heisenberg ferromagnet, with external field $h \geq 0$ along the x axis is addressed by [MoPi] Corollary 4.3. In particular, the non-negativity of spin correlations in the classical Heisenberg model with $J_{uv}^3 \geq 0, |J_{uv}^2| \leq J_{uv}^1$ and $h_u^1 \geq 0, h_u^2 = 0, h_u^3 \geq 0$ for all $u, v \in \Lambda$ holds true. This follows by setting $A(x) = 1, A(z) = 0$ for all $z \neq x$, $B(y) = 1, B(z) = 0$ for all $z \neq y$, where $x, y, z \in \Lambda$ and $\alpha = 1$ in Eq. (4.16) therein.

N=4: For $N = 4$ with spins taking values in \mathbb{S}^3 , and the couplings satisfying the condition $|J_{uv}^2| \leq J_{uv}^1, |J_{uv}^4| \leq J_{uv}^3$ and $h_u^1 \geq 0, h_u^3 \geq 0, h_u^2 = h_u^4 = 0$ for all $u, v \in \Lambda$, we invoke Corollary 4.6 from [MoPi]. This is obtained in particular by setting $A(x) = 1, B(y) = 1$, otherwise 0 where $x, y \in \Lambda$ and $\alpha = 1$ in Eq. (4.28) therein.

3.4.5. Local lower bounds on single spin fluctuations. Our approach to lower bounding the single spin fluctuations $\text{Var}[S_\Lambda]$ involves considering conditional distributions. To set the stage, we recall that, for any pair of jointly distributed random variables (X, Y) we may write down the variance decomposition

$$\text{Var}[X] = \mathbb{E}[\text{Var}[X|Y]] + \text{Var}[\mathbb{E}[X|Y]] \geq \mathbb{E}[\text{Var}[X|Y]].$$

For our purposes, we will invoke this decomposition with $X = \sigma_x$ and $Y = \sigma_{\Lambda \setminus \{x\}}$.

Lemma 3.6. *Let the random spin configuration satisfy any of the conditions (a) – (d) in Model 3.1, and suppose that μ_0 is compactly supported. We assume that the coupling J_{xy}*

and the external field h are uniformly bounded (as the system size $|\Lambda|$ grows), i.e., there exist universal constants $H, J > 0$ such that

$$|h_x^i| \leq H \quad \text{and} \quad |J_{xy}^i| \leq J, \quad \forall i, x, y, \Lambda.$$

Then there exists a constant $c > 0$ (not depending on Λ) such that

$$\text{Var}[\sigma_x^1] \geq c, \quad \forall x \in \Lambda.$$

Proof. Denote by ∂x the set of all vertices in Λ that are connected by an edge to x . The conditional distribution for the spin at the site x given the rest of the spins has the following structure

$$\mathbb{P}_\Lambda[\sigma_x | \sigma_{\Lambda \setminus \{x\}}] \propto \exp\left(\beta \sum_{i=1}^N \sigma_x^i \left(h_x^i + \sum_{y \in \partial x} J_{xy}^i \sigma_y^i\right)\right) d\mu_0(\sigma_x).$$

Setting $\alpha = (\alpha_1(\partial x), \dots, \alpha_N(\partial x))$, where

$$\alpha_i(\partial x) := \beta \left(h_x^i + \sum_{y \in \partial x} J_{xy}^i \sigma_y^i\right), \quad i = 1, 2, \dots, N.$$

Then $|\alpha_i(\partial x)| \leq |\beta|H + |\partial x|J \leq |\beta|H + 2dJ =: M$ for all i (recall that Λ is a d -dimensional cube).

Since

$$\mathbb{P}_\Lambda[\sigma_x | \sigma_{\Lambda \setminus \{x\}}] \propto \exp\left(\sum_{i=1}^N \alpha_i(\partial x) \sigma_x^i\right) d\mu_0(\sigma_x)$$

then

$$\text{Var}[\sigma_x^1 | \sigma_{\Lambda \setminus \{x\}}] = \frac{f(\alpha)}{h(\alpha)} - \left(\frac{g(\alpha)}{h(\alpha)}\right)^2,$$

where the functions f, g and h from \mathbb{R}^N to \mathbb{R} are defined for $u = (u_1, \dots, u_N)$ as follows

$$f(u) := \int_{\mathbb{R}^N} (\sigma_x^1)^2 \exp\left(\sum_{i=1}^N u_i \sigma_x^i\right) d\mu_0(\sigma_x),$$

$$g(u) := \int_{\mathbb{R}^N} \sigma_x^1 \exp\left(\sum_{i=1}^N u_i \sigma_x^i\right) d\mu_0(\sigma_x),$$

$$h(u) := \int_{\mathbb{R}^N} \exp\left(\sum_{i=1}^N u_i \sigma_x^i\right) d\mu_0(\sigma_x).$$

Observe that $h > 0$. Since μ_0 is compactly supported, we can easily deduce that f, g, h are continuous functions (using Lebesgue's dominated convergence theorem).

Let

$$F(u) := \frac{f(u)}{h(u)} - \left(\frac{g(u)}{h(u)}\right)^2.$$

Since f, g, h are continuous, $F(u)$ is also continuous. Moreover, by Cauchy-Schwarz inequality we have $f(u)h(u) \geq g(u)^2$. In that inequality, when the equality is attained, there exist two constants $A, B \geq 0$, not all zero, such that

$$A(\sigma_x^1)^2 \exp\left(\sum_{i=1}^N u_i \sigma_x^i\right) = B \exp\left(\sum_{i=1}^N u_i \sigma_x^i\right) \quad \mu_0\text{-a.s.},$$

or equivalently,

$$|\sigma_x^1| = \text{constant } \mu_0\text{-a.s.}$$

This is not possible in our model. Thus, $f(u)h(u) - g(u)^2 > 0$ for all $u \in \mathbb{R}^N$. This implies $F(u) > 0$ for all $u \in \mathbb{R}^N$. Letting

$$c := \min_{|u_i| \leq M} F(u),$$

we have

$$\text{Var}[\sigma_x^1 | \sigma_{\Lambda \setminus \{x\}}] = F(\alpha) \geq c$$

and

$$\text{Var}[\sigma_x^1] \geq \mathbb{E}[\text{Var}[\sigma_x^1 | \sigma_{\Lambda \setminus \{x\}}]] \geq c > 0.$$

This ends the proof of the lemma. ■

3.5. Proof of Theorem 1.5.

Proof of Theorem 1.5. Herein, we put together the various ingredients developed in the earlier subsections, and complete the proof of Theorem 1.5. In view of Corollary 1.3, for every Λ large enough, we need to specify r_Λ such that the characteristic function of S_Λ , i.e. $\Psi_{S_\Lambda}(u)$, does not vanish on $\overline{\mathbb{D}(0, r_\Lambda)}$; furthermore, we need to verify the growth condition in that corollary.

We begin by recalling that the Model class 1.1 is a subset of the Model class 3.1; this implies in particular that Lemma 3.2 applies. The uniformity and positivity hypothesis on the external field in the Model class 1.1 implies that we may invoke Remark 3.3 in our case, implying in particular that we may choose a constant r_Λ such that $\Psi_{S_\Lambda}(u)$ does not vanish on $\overline{\mathbb{D}(0, r_\Lambda)}$.

It remains to verify the growth condition in Corollary 1.3. To this end, we invoke Lemma 3.5 and observe that it suffices to demonstrate the growth conditions (3.9). Growth condition (i) in (3.9) follows from Section 3.4.2. Growth condition (ii) in (3.9) follows from Section 3.4.3; the two main ingredients therein, namely non-negativity of correlations and lower bound on single spin fluctuations are demonstrated for the Model class 1.1 in Section 3.4.4 and Lemma 3.6 respectively.

This completes the verification of the conditions in Corollary 1.3, and thereby establishes a CLT for \hat{S}_Λ . By Section 3.4.2, we have $|S_\Lambda| \lesssim |\Lambda|$; on the other hand, by Section 3.4.3, we have $\sqrt{\text{Var}(S_\Lambda)} \gtrsim |\Lambda|^{1/2}$. Applying these bounds in the CLT convergence rate (1.1), we conclude that the Kolmogorov-Smirnov distance between \hat{S}_Λ and a standard Gaussian decays as $O(\log |\Lambda| / \sqrt{\text{Var}(S_\Lambda)}) = O(\log |\Lambda| \cdot |\Lambda|^{-1/2})$. ■

4. FLUCTUATION THEORY FOR LINEAR STATISTICS OF α -DETERMINANTAL PROCESSES

In this section, we will undertake a detailed study of α -determinantal processes in the context of our approach to CLTs based on zeros of characteristic functions. We will work in the setting of Section 1.3 in general, and Model 1.2 in particular.

4.1. The characteristic function of linear statistics. We begin with an expression for the Laplace transform of $\Lambda(\varphi_L)$ in terms of Fredholm determinants that is valid for bounded and compactly supported test functions φ and complex arguments u that are small enough. This is an extension of a related expression for non-negative u and non-negative φ that is available in the literature [ShTa, BaBlKa].

Proposition 4.1. *Let φ be a bounded, compactly supported function, and let X be the α -determinantal process as in Theorem 1.6. Denote by $M_{\varphi,u}$ the multiplication operator by the function $(1 - \exp(-u\varphi(x)))$. Further, set $B = \text{supp}(\varphi)$ and define K_B to be the operator compression of K given by $K_B(\cdot, \cdot) = \mathbb{1}_B(\cdot)K(\cdot, \cdot)\mathbb{1}_B(\cdot)$. Then the formula for the Laplace transform of $\Lambda(\varphi)$ given by*

$$(4.1) \quad \mathbb{E}[\exp(-u\Lambda(\varphi))] = \text{Det}[I + \alpha M_{\varphi,u}K]^{-\frac{1}{\alpha}} = \text{Det}[I + \alpha M_{\varphi,u}K_B]^{-\frac{1}{\alpha}},$$

where Det is the Fredholm determinant and I is the identity operator on $L^2(\mu)$, holds for $u \in U \subset \mathbb{C}$, where U is a small enough open set that depends on φ only via $\|\varphi\|_\infty$. Further, the Laplace transform is holomorphic in u on U .

Remark 4.2. *We observe that, while formulae for Laplace transforms of such linear statistics are available [ShTa], they are known to hold only for real argument $u \geq 0$ and non-negative test functions φ . For our purposes, we require a tractable expression for a general complex u (which can albeit be small in magnitude), and any bounded test function φ . This is the content of Proposition 4.1.*

Proof of Proposition 4.1. Set $g_u(x) = 1 - e^{-u\varphi(x)}$. We may write, using standard results on point process correlations (c.f. [DaVe, Ka]),

$$\begin{aligned} \mathbb{E}[\exp(-u\Lambda(\varphi))] &= \mathbb{E}\left[\prod_{x \in X} e^{-u\varphi(x)}\right] \\ &= \mathbb{E}\left[\prod_{x \in X} (1 - (1 - e^{-u\varphi(x)}))\right] = \sum_{k=0}^{\infty} (-1)^k \mathbb{E}\left[\sum_{\{x_1, \dots, x_k\} \subset X} \prod_{i=1}^k (1 - e^{-u\varphi(x_i)})\right] \\ &= \sum_{k=0}^{\infty} (-1)^k \mathbb{E}\left[\sum_{\{x_1, \dots, x_k\} \subset X} \prod_{i=1}^k g_u(x_i)\right] \\ (4.2) \quad &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\int^{\otimes k} g_u(x_1) \dots g_u(x_k) \rho_k(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k) \right) \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int^{\otimes k} \prod_{i=1}^k (-g_u(x_i)) \cdot \text{Det}_\alpha[(K(x_i, x_j))_{i,j=1}^k] d\mu(x_1) \dots d\mu(x_k) \right) \\ (4.3) \quad &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int^{\otimes k} \text{Det}_\alpha[(-g_u(x_i)K(x_i, x_j))_{i,j=1}^k] d\mu(x_1) \dots d\mu(x_k) \right) \end{aligned}$$

$$(4.4) \quad = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int^{\otimes k} \text{Det}_\alpha[(-g_u(x_i)K_B(x_i, x_j))_{i,j=1}^k] d\mu(x_1) \dots d\mu(x_k) \right),$$

where, in the last step, we have used the fact that the function $g_u(\cdot)$ is supported on $B = \text{supp}(\varphi)$, and therefore the integrand in (4.3) is non-zero only when $x_i \in B \forall i$.

Let M_{g_u} denote the multiplication operator by g_u acting on $L^2(\mu)$; notice that this is the same as the operator $M_{\varphi,u}$ considered earlier. By Theorem 2.4 in [ShTa] (for a concrete statement, see Theorem 4.3 below), we may deduce that as soon as the operator $\alpha M_{g_u} K = \alpha M_{\varphi,u} K$ satisfies $\|\alpha M_{g_u} K\|_{\text{op}} < 1$, we may write

$$(4.5) \quad \begin{aligned} & \text{Det}[I + \alpha M_{\varphi,u} K]^{-1/\alpha} = \text{Det}[I + \alpha M_{g_u} K]^{-1/\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int^{\otimes k} \text{Det}_{\alpha} [(-g_u(x_i) K(x_i, x_j))_{i,j=1}^k] d\mu(x_1) \dots d\mu(x_k) \right). \end{aligned}$$

Given any $\varepsilon > 0$, if $u \in \mathbb{C}$ is such that $|u|$ is small enough (depending to $\|\varphi\|_{\infty}^{-1}$ and ε), then $\|M_{\varphi,u}\|_{\text{op}} = \|1 - \exp(-u\varphi(\cdot))\|_{\infty} \leq \varepsilon |\alpha|^{-1} \|K\|_{\text{op}}^{-1}$. As a result, for $u \in \mathbb{C}$ such that $|u|$ is small enough (depending to $\|\varphi\|_{\infty}^{-1}$ and ε), we may write $\|\alpha M_{\varphi,u} K\|_{\text{op}} \leq |\alpha| \|M_{\varphi,u}\|_{\text{op}} \|K\|_{\text{op}} < \varepsilon$. Note that under such conditions, the operator $\alpha M_{\varphi,u} K$ is still trace class, which implies that $\text{Det}[I + \alpha M_{\varphi,u} K]$ is finite. On the other hand, the only way $\text{Det}[I + \alpha M_{\varphi,u} K]$ can be 0 is for one of its eigenvalues to be equal to 0, which cannot happen because all the eigenvalues of $\alpha M_{\varphi,u} K$ are of modulus bounded away from 1. This implies that for such u , the quantity $\text{Det}[I + \alpha M_{\varphi,u} K]^{-1/\alpha}$ is finite and has no singularities.

In such a situation, the right hand side of (4.1), and therefore the Laplace transform $\mathbb{E}[\exp(-u\Lambda(\varphi))]$, is finite and has no singularities. Notice that the choice of such u depends on φ only via $\|\varphi\|_{\infty}$.

For holomorphicity of the Laplace transform in a neighbourhood of 0, we may examine the series development in (4.2). Denoting

$$a_k(u) := \frac{(-1)^k}{k!} \left(\int^{\otimes k} g_u(x_1) \dots g_u(x_k) \rho_k(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k) \right).$$

Each $a_k(u)$ is differentiable in u , which follows from the fact that for a given x , $g_u(x)$ is differentiable in u . We will demonstrate holomorphicity of the Laplace transform, which equals $\sum_{k \geq 0} a_k(u)$, by showing that $\sum_{k \geq 0} |a'_k(u)|$ is bounded for $|u|$ small enough.

First, we notice that $g_u(x) = 0$ outside the support B of φ . As such, we may write $g_u(x) = g_u(x) \mathbb{1}_B(x)$. Since $|\varphi|$ is bounded (by M) and u is such that $\|g_u\|_{\infty}$ is small ($< \delta$), therefore

$$(4.6) \quad \left| \frac{d}{du} \left(\prod_{i=1}^k g_u(x_i) \right) \right| \lesssim k M \delta^{k-1} \prod_{i=1}^k \mathbb{1}_B(x_i).$$

Therefore, we have

$$(4.7) \quad \sum_{k \geq 0} |a'_k(u)| \leq \sum_{k \geq 1} \frac{1}{k!} \cdot k \delta^{k-1} \cdot \left(\int^{\otimes k} \text{Det}_{\alpha} [(\mathbb{1}_B(x_i) K(x_i, x_j))_{i,j=1}^k] d\mu(x_1) \dots d\mu(x_k) \right),$$

which we intend to show to be bounded for small enough δ .

Notice that the right hand side of (4.7) is the formal derivative of the power series

$$(4.8) \quad \sum_{k \geq 1} \frac{1}{k!} \cdot \delta^k \cdot \left(\int^{\otimes k} \text{Det}_{\alpha} [(\mathbb{1}_B(x_i) K(x_i, x_j))_{i,j=1}^k] d\mu(x_1) \dots d\mu(x_k) \right).$$

If a power series converges absolutely for $|z| < r$ for some r , then it is holomorphic and differentiable term by term for $|z| < r$. In view of this, to establish boundedness of

(4.7), we need to show that the power series in δ given by (4.8) is absolutely convergent for δ small enough. Since all the terms are non-negative, we merely need to show convergence. To this end, notice that this series can be written as

$$(4.9) \quad \sum_{k \geq 1} \frac{1}{k!} \left(\int^{\otimes k} \text{Det}_\alpha[(\delta \mathbb{1}_B(x_i) K(x_i, x_j))_{i,j=1}^k] d\mu(x_1) \dots d\mu(x_k) \right).$$

For δ small enough, we have $\|\delta \mathbb{1}_B(x) K(x, y)\|_{\text{op}} < 1$, whence we can invoke the determinantal identity [ShTa] Theorem 2.4 to conclude that (4.9) equals $\text{Det}[I - \alpha \delta \mathbb{1}_B(\cdot) K(\cdot, \cdot)]^{-1/\alpha}$, and therefore must be bounded.

Finally, the expression $\text{Det}[I + \alpha M_{\varphi, u} K_B]^{-\frac{1}{\alpha}}$ in (4.1) follows from (4.4) via similar considerations as above.

This completes the proof. ■

We complete this section with a statement of Theorem 2.4 in [ShTa] that was used in the above proof.

Theorem 4.3 (Theorem 2.4 in [ShTa]). *Let J be a trace class integral operator on $L^2(\mu)$. If $\|\alpha J\|_{\text{op}} < 1$, we have*

$$(\text{Det}[I - \alpha J])^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{1}{n!} \int^{\otimes n} \text{Det}_\alpha[(J(x_i, x_j))_{i,j=1}^n] d\mu(x_1) \dots d\mu(x_n).$$

4.2. The no-zeros condition for linear statistics. In this section, we demonstrate the no-zeros condition for the moment-generating function of the linear statistics $\Lambda(\varphi_L)$. We will work in the setting where φ is a bounded compactly supported function on \mathbb{R}^d .

Proposition 4.4. *Let X be an α -determinantal process and let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded compactly supported test function. Then there exists $r_0 > 0$ such that the characteristic function $\Psi_{\Lambda(\varphi_L)}(u) = \mathbb{E}[\exp(iu\Lambda(\varphi_L))]$ does not vanish whenever $|u| \leq r_0$ for all $L > 0$.*

Proof. We begin with (4.1) and notice that for $\alpha \neq 0$ we have

$$\Psi_{\Lambda(\varphi_L)}(u) = \mathbb{E}[\exp(iu\Lambda(\varphi_L))] = \text{Det}[I + \alpha M_{\varphi_L, -iu} K]^{-\frac{1}{\alpha}}.$$

We now use the same argument as in Section 4.1 to conclude that $\Psi_{\Lambda(\varphi_L)}(u)$ does not have any zeros for $|u| \leq r_0(L)$ small enough. As in Section 4.1, the choice of u will only depend on $\|\varphi_L\|_\infty$. However, since $\|\varphi_L\|_\infty = \|\varphi\|_\infty$ which is independent of L , our choice of $r_0(L)$ is also independent of L . ■

Remark 4.5. *We observe that the above proof entails $|u|$ small implying $\|1 - e^{iu\varphi(\cdot)}\|_\infty$ small, in turn implying $\|\alpha M_{\varphi, -iu} K\|_{\text{op}} < 1$; this would imply non-vanishing of the Fredholm determinant. However, if $|\text{Im}(u)|$ was large, $\|1 - e^{iu\varphi(\cdot)}\|_\infty$ would be large, and the above argument would break down. In particular, if u is allowed to vary in an infinite vertical strip in \mathbb{C} , then it might be possible to choose u such that -1 is an eigenvalue of $\alpha M_{\varphi, -iu} K$, and hence such u would be a zero of the Fredholm determinant. This would lead to the vanishing of the characteristic function $\Psi_{\Lambda(\varphi_L)}$ for $\alpha < 0$, and to its blow up for $\alpha > 0$.*

4.3. Growth rate of the moment-generating function of $\Lambda(\varphi_L)$. In this section, we provide a result about the growth rate of the moment-generating function of $\Lambda(\varphi_L)$ which is necessary in our approach.

Proposition 4.6. *Let X be an α -determinantal process as in Model 1.2. Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded compactly supported test function. There exist constants $r, c > 0$ such that for $L \geq 1$ we have*

$$\mathbb{E}[\exp(r|\Lambda(\varphi_L)|)] \leq \exp(cL^d).$$

Proof. We recall our hypothesis that φ is compactly supported, and therefore $\text{supp}\varphi_L$ is contained in a ball of radius $c'L$ for some positive constant c' that depends on φ . If $N(R)$ denotes the number of points of the process in the ball of radius R , then

$$|\Lambda(\varphi_L)| \leq N(c'L) \cdot \|\varphi\|_\infty.$$

Thus, we are reduced to bounding from above $\mathbb{E}[\exp(rN(c'L))]$. Observe that $N(c'L) = \Lambda(\mathbb{1}_{c'L})$, where $\mathbb{1}_R$ is the indicator function of the ball of radius R in \mathbb{R}^d .

We primarily focus on the case $\alpha \neq 0$. By the formula for characteristic functions of linear statistics, we have

$$\mathbb{E}[\exp(rN(c'L))] = \text{Det}[I + \alpha M_{h_L} K]^{-1/\alpha},$$

where $h_L(x) := 1 - \exp(r\mathbb{1}_{c'L}) = (1 - \exp(r))\mathbb{1}_{c'L}$ and M_{h_L} is the multiplication operator by h_L acting on $L^2(\mu)$.

Let $\{\lambda_i\}_{i \geq 1}$ be the eigenvalues of the operator $\alpha K_{h_L} := \alpha M_{h_L} K$. By the argument in Section 4.1, by choosing u small enough compared to $\|h_L\|_\infty = \|h\|_\infty = 1$, we may ensure that the λ_i -s are smaller in modulus than $1/2$ (implying in particular that they are bounded away in modulus from 1).

Using this, we may proceed as

$$|\mathbb{E}[\exp(rN(c'L))]| = \exp\left(-\alpha^{-1} \log \text{Det}[I + \alpha K_{h_L}]\right) = \exp\left(-\alpha^{-1} \sum_{i=1}^{\infty} \log(1 + \lambda_i)\right).$$

Then we may write

$$|\mathbb{E}[\exp(rN(c'L))]| = \exp\left(-\alpha^{-1} \sum_{i=1}^{\infty} \log(1 + \lambda_i)\right) \leq \exp\left(2|\alpha|^{-1} \sum_{i=1}^{\infty} |\lambda_i|\right)$$

where we have used the fact that $0 \leq |\lambda_i| < 1/2$ and used small parameter expansion for $\log(1 + \lambda)$ for small $|\lambda|$.

We now investigate the λ_i -s, which are the eigenvalues of the operator αK_{h_L} . For r small enough, $h_L(x)$ is non-negative, which, coupled with the non-negativity of K , implies that the operator K_{h_L} is positive semi-definite. This implies in particular that all λ_i are of the same sign, hence

$$\sum_{i=1}^{\infty} |\lambda_i| = |\text{Tr}[\alpha K_{h_L}]|.$$

The operator αK_{h_L} equals $\alpha(1 - \exp(r))\mathbb{1}_{c'L}K$. Thus, we have

$$\text{Tr}[\alpha K_{h_L}] = \int \alpha K_{h_L}(x, x) d\mu(x) = \alpha(1 - e^r) \int \mathbb{1}_{c'L}(x) K(x, x) f(x) dx.$$

Recall that the one point intensity $K(x, x)$ and the density f are bounded from above on \mathbb{R}^d , which gives us

$$|\text{Tr}[\alpha K_{h_L}]| \lesssim \int \mathbb{1}_{c'L}(x) dx = O(L^d).$$

Combining this with the argument above, we obtain $|\mathbb{E}[\exp(rN(c'L))]| \lesssim \exp(cL^d)$ for some positive constant c as desired. \blacksquare

4.4. Variance growth of $\Lambda(\varphi_L)$ for $\alpha > 0$. Now we establish the variance growth condition for $\Lambda(\varphi_L)$, which is a necessary ingredient in our approach. To this end, we recall that for any point process with one and two point correlation functions ρ_1 and ρ_2 respectively with respect to background measure μ and any real valued test function ϕ we have

$$(4.10) \quad \mathbb{E}[\Lambda(\phi)] = \int \phi(x)\rho_1(x)d\mu(x)$$

and

$$(4.11) \quad \mathbb{E}[\Lambda(\phi)^2] = \int \phi(x)^2\rho_1(x)d\mu(x) + \iint \phi(x)\phi(y)\rho_2(x,y)d\mu(x)d\mu(y).$$

Proposition 4.7. *For α -determinantal processes in Model 1.2 with $\alpha > 0$, we have*

$$\text{Var}[\Lambda(\varphi_L)] \gtrsim L^d.$$

Proof. For the α -determinantal process, we have

$$\rho_1(x) = K(x, x) \quad , \quad \rho_2(x, y) = K(x, x)K(y, y) + \alpha K(x, y)^2.$$

By direct computation, we obtain

$$\text{Var}[\Lambda(\varphi_L)] = \int \varphi_L(x)^2 K(x, x)d\mu(x) + \alpha \iint \varphi_L(x)K(x, y)^2\varphi_L(y)d\mu(x)d\mu(y).$$

Notice that, since the kernel $K(\cdot, \cdot)$ is positive semi-definite on $L^2(\mu)$, so is the kernel $K(\cdot, \cdot)^2$. This follows from the so-called Schur product theorem for operators [HoJo].

This implies that

$$\alpha \iint \varphi_L(x)K(x, y)^2\varphi_L(y)d\mu(x)d\mu(y) \geq 0.$$

The upshot of this is that, for $\alpha > 0$ we have

$$\text{Var}[\Lambda(\varphi_L)] \geq \int \varphi_L(x)^2 K(x, x)d\mu(x).$$

Recall that μ has a density f with respect to the Lebesgue measure that is bounded from below by a positive constant. Then we may write

$$\text{Var}[\Lambda(\varphi_L)] \gtrsim \int \varphi_L(x)^2 K(x, x)dx.$$

Recall that the kernel K satisfies the condition (i) in Model 1.2, which is the following condition: for any $0 < \delta < 1$, there exist $m_\delta > 0$ and $r > 0$ such that

$$\text{Vol}(\mathcal{F}_{m_\delta} \cap C(x, r)) \geq (1 - \delta) \text{Vol}(C(x, r)) \quad \forall x \in \mathbb{R}^d,$$

where $\mathcal{F}_{m_\delta} := \{y \in \mathbb{R}^d : K(y, y) \geq m_\delta\}$ and $C(x, r)$ is the cube centered at x with side length r . From this hypothesis, we can easily deduce that

$$\text{Vol}(\mathcal{F}_{m_\delta} \cap C(0, Nr)) \geq (1 - \delta) \text{Vol}(C(0, Nr)) \quad \forall N \in \mathbb{N}_+.$$

Let $n_0 \in \mathbb{N}_+$ be such that $\text{supp}(\varphi) \subset C(0, n_0r)$. For each $\varepsilon > 0$, denote by Ω_ε the set $\{x \in \mathbb{R}^d : |\varphi(x)| \geq \varepsilon\}$. Since φ is bounded and $\|\varphi\|_2 > 0$, there must exist $\varepsilon > 0$ such that $\text{Vol}(\Omega_\varepsilon) > 0$.

Let $\nu := \text{Vol}(\Omega_\varepsilon) / \text{Vol}(C(0, n_0r)) > 0$. Choose $\delta < \nu/2$, then for all L large enough we will have

$$\begin{aligned}
 \text{Vol}(\mathcal{F}_{m_\delta} \cap L \cdot \Omega_\varepsilon) &= \text{Vol}(\mathcal{F}_{m_\delta} \cap C(0, Ln_0r) \cap L \cdot \Omega_\varepsilon) \\
 &\geq \text{Vol}(\mathcal{F}_{m_\delta} \cap C(0, Ln_0r)) + \text{Vol}(L \cdot \Omega_\varepsilon) - \text{Vol}(C(0, Ln_0r)) \\
 &\geq \text{Vol}(\mathcal{F}_{m_\delta} \cap C(0, \lfloor L \rfloor n_0r)) + \text{Vol}(L \cdot \Omega_\varepsilon) - \text{Vol}(C(0, Ln_0r)) \\
 &\geq (1 - \delta) \text{Vol}(C(0, \lfloor L \rfloor n_0r)) + \nu \text{Vol}(C(0, Ln_0r)) - \text{Vol}(C(0, Ln_0r)) \\
 &\geq \left((1 - \delta) \left(\frac{L-1}{L} \right)^d + \nu - 1 \right) \text{Vol}(C(0, Ln_0r)) \\
 &\geq \left(\left(\frac{L-1}{L} \right)^d - 1 + \frac{\nu}{2} \right) \text{Vol}(C(0, Ln_0r)) \\
 &\geq \frac{\nu}{4} \text{Vol}(C(0, n_0r)) L^d.
 \end{aligned}$$

Then we can further bound for all L large enough

$$\begin{aligned}
 \int \varphi_L(x)^2 K(x, x) dx &= \int \varphi(x/L)^2 K(x, x) dx \geq \int_{\mathcal{F}_{m_\delta} \cap L \cdot \Omega_\varepsilon} \varphi(x/L)^2 K(x, x) dx \\
 &\geq \varepsilon^2 m_\delta \text{Vol}(\mathcal{F}_{m_\delta} \cap L \cdot \Omega_\varepsilon) \geq \varepsilon^2 m_\delta \frac{\nu}{4} \text{Vol}(C(0, n_0r)) L^d.
 \end{aligned}$$

This completes the proof. ■

Remark 4.8. While, for the sake of brevity, we establish Proposition 4.7 as stated, we observe that the condition in Model 1.2 (i) can be further weakened. In particular, . For that purpose, we can replace m_δ therein by $m_\delta \|x\|^{-L} \log^2 \|x\| c(\|x\|)$ for any function $c(\|x\|)$ going to infinity. To see this, we may use the same argument as in the proof of Proposition 4.7, but remove from $C(0, Ln_0r)$ a suitable cube of center 0 and size $O(L)$. The estimate would then give $\log^2 L$ times a function $\rightarrow \infty$. It remains to observe that, in the context of the mgf growth bound in Proposition 4.6, we require a variance growth rate of only $\text{Var}[\Lambda(\varphi_L)] / \log^2 L \rightarrow \infty$ to deduce CLT via Theorem 1.1.

4.5. Variance growth of $\Lambda(\varphi_L)$ for $\alpha < 0$.

Proposition 4.9. For α -determinantal processes in Model 1.2 with $\alpha < 0$, there exists a positive constant γ such that for all L large enough

$$\text{Var}[\Lambda(\varphi_L)] \gtrsim L^\gamma.$$

Proof. As in the case of $\alpha > 0$, we may obtain

$$(4.12) \quad \text{Var}[\Lambda(\varphi_L)] = \int \varphi_L(x)^2 K(x, x) d\mu(x) + \alpha \iint \varphi_L(x) K(x, y)^2 \varphi_L(y) d\mu(x) d\mu(y).$$

Since $\alpha < 0$, we cannot simply neglect the second term, which is a source of difficulties in this setting.

We consider the operator composition $K^{\circ 2} := K \circ K$, given by the integral kernel $K^{\circ 2}(x, y) = \int K(x, z) K(z, y) d\mu(z)$. In particular, this entails, using the symmetry of K , that $K^{\circ 2}(x, x) = \int K(x, y)^2 d\mu(y)$, a fact that will be useful shortly. Recall that for $\alpha < 0$, a requirement on the operator K is that $\|K\|_{\text{op}} \leq 1/|\alpha|$. As a result, we have

$$(4.13) \quad 0 \preceq K^{\circ 2} \preceq \|K\|_{\text{op}} K \preceq \frac{1}{|\alpha|} \cdot K,$$

where \preceq denotes domination of operators in a positive semi definite sense.

For a bounded Borel set E , we denote by χ_E the characteristic function (i.e., the indicator function) of this set. Now we consider the inequality of local traces (that is a consequence of (4.13)):

$$(4.14) \quad 0 \leq \text{Tr}[\chi_E K^{\circ 2} \chi_E] \leq \frac{1}{|\alpha|} \cdot \text{Tr}[\chi_E K \chi_E]$$

Writing out (4.14) as integrals, we obtain the following inequality, valid for any bounded Borel set E :

$$(4.15) \quad \int_E \left(\int K(x, y)^2 d\mu(y) \right) d\mu(x) \leq \int_E \frac{1}{|\alpha|} \cdot K(x, x) d\mu(x).$$

Since (4.15) is valid for every bounded Borel set E , we may deduce that

$$(4.16) \quad \int K(x, y)^2 d\mu(y) \leq \frac{1}{|\alpha|} \cdot K(x, x)$$

holds for μ -a.e. x .

Combining (4.16) with (4.12), we obtain a crucial lower bound on the variance for the range $\alpha < 0$

$$(4.17) \quad \text{Var}[\Lambda(\varphi_L)] \geq \frac{1}{2} |\alpha| \iint |\varphi_L(x) - \varphi_L(y)|^2 K(x, y)^2 d\mu(x) d\mu(y).$$

Henceforth, we focus on lower bounding the double integral on the right hand side.

By making a change of variables $u = x/L, v = y/L$, we may lower bound

$$(4.18) \quad \begin{aligned} & \iint |\varphi_L(x) - \varphi_L(y)|^2 K(x, y)^2 f(x) f(y) dx dy \\ &= L^{2d} \iint |\varphi(u) - \varphi(v)|^2 K(Lu, Lv)^2 f(Lu) f(Lv) du dv \\ &\gtrsim L^{2d} \iint |\varphi(u) - \varphi(v)|^2 K(Lu, Lv)^2 du dv, \end{aligned}$$

where in the last step we use the hypothesis on the lower bound on the density f .

It now remains to lower bound the integral expression in (4.18). This will exploit the conditions (ii.a) or (ii.b) in Model 1.2. The precise implementation of this programme will be taken up in Sections 4.5.1 and 4.5.2 respectively. \blacksquare

4.5.1. Using hypothesis (ii.a) in Model 1.2. Recall the condition (ii.a) in Model 1.2 is that there exist constants $a, \delta > 0$ such that

$$|K(x, y)| \geq a \quad \forall \|x - y\| < \delta.$$

Thus, we can further bound

$$(4.19) \quad L^{2d} \iint |\varphi(u) - \varphi(v)|^2 K(Lu, Lv)^2 du dv \gtrsim L^{2d} \iint_{\|u-v\| < \frac{\delta}{L}} |\varphi(u) - \varphi(v)|^2 du dv.$$

Recall that we denote by $B(x, r)$ the ball with centre x and radius r in \mathbb{R}^d ; for brevity we set $B = B(0, 1)$. We denote by ν_L the volume of $B(0, \delta/L)$. Further, we consider the local average function $\bar{\varphi}$, defined by $\bar{\varphi}(x) = \frac{1}{\nu_L} \int_{B(0, \delta/L)} \varphi(x - u) du$. Set χ_B to the indicator function of B and χ_B^L to be that of $B(0, \delta/L)$. Then we have the convolutional identity $\bar{\varphi} = \frac{1}{\nu_L} \cdot \varphi * \chi_B^L$. Also observe the scaling relationship between χ_B^L and χ_B given

by $\chi_B^L = (\chi_B)_{\delta/L}$, leading to the relationship of Fourier transforms (4.22) below. Also observe that $\nu_L = \left(\frac{\delta}{L}\right)^d \nu$, where $\nu = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$ is the volume of B . We then have

$$\begin{aligned}
 & \iint_{\|u-v\| < \frac{\delta}{L}} |\varphi(u) - \varphi(v)|^2 du dv \\
 &= \int_{\mathbb{R}^d} \left(\int_{\|u-v\| < \frac{\delta}{L}} |\varphi(u) - \varphi(v)|^2 dv \right) du \\
 &= \nu_L \int_{\mathbb{R}^d} \left(\frac{1}{\nu_L} \int_{\|u-v\| < \frac{\delta}{L}} |\varphi(u) - \varphi(v)|^2 dv \right) du \\
 &\geq \nu_L \int_{\mathbb{R}^d} \left| \varphi(u) - \frac{1}{\nu_L} \int_{\|v-u\| < \frac{\delta}{L}} \varphi(v) dv \right|^2 du \quad (\text{by Jensen's inequality}) \\
 (4.20) \quad &= \nu_L \int_{\mathbb{R}^d} |\varphi(u) - \bar{\varphi}(u)|^2 du.
 \end{aligned}$$

We may continue from here as

$$(4.21) \quad \int_{\mathbb{R}^d} |\varphi(u) - \bar{\varphi}(u)|^2 du = \|\varphi - \bar{\varphi}\|_2^2 = \|\hat{\varphi} - \hat{\bar{\varphi}}\|_2^2 = \|\hat{\varphi} - \nu_L^{-1} \cdot \hat{\varphi} \cdot \widehat{\chi_B^L}\|_2^2 = \|\hat{\varphi} \cdot (1 - \nu_L^{-1} \cdot \widehat{\chi_B^L})\|_2^2,$$

where we have used the Parseval-Plancherel Theorem.

Thus, we are reduced to examining the function $(1 - \nu_L^{-1} \cdot \widehat{\chi_B^L})$, and in particular showing that it is $\gtrsim \frac{1}{L}$ on a set that has a substantial intersection with $\text{supp}(\hat{\varphi})$.

To this end, we observe, via the behaviour of Fourier transforms under scaling (see [Fi] for a complete derivation of the Fourier transform of the unit ball), that

$$(4.22) \quad \widehat{\chi_B^L}(\xi) = \left(\frac{\delta}{L}\right)^d \widehat{\chi_B}\left(\frac{\delta}{L} \cdot \xi\right) = \left(\frac{\delta}{L}\right)^d \cdot (2\pi)^{d/2} \left\| \frac{\delta}{L} \cdot \xi \right\|^{-d/2} J_{d/2}\left(\frac{\delta}{L} \|\xi\|\right),$$

where J_β is the Bessel function of the first kind with order $\beta > 0$.

We also consider the following well-known power series expansion of such Bessel functions around 0 (for details, see e.g. [AbStRo])

$$(4.23) \quad J_\beta(x) = \left(\frac{x}{2}\right)^\beta \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k k! \Gamma(k + \beta + 1)}.$$

As a consequence, for small x , J_β can be further approximated as

$$(4.24) \quad J_\beta(x) = \left(\frac{x}{2}\right)^\beta \cdot \left(\frac{1}{\Gamma(\beta + 1)} - \frac{x^2}{4\Gamma(\beta + 2)} + o(x^2) \right).$$

With the ingredients from (4.22), (4.24) in hand, we are ready to examine $\nu_L^{-1} \cdot \widehat{\chi}_B^L$ for ξ in a fixed compact set (to be specified later). We proceed as

$$\begin{aligned}
 & \nu_L^{-1} \cdot \widehat{\chi}_B^L(\xi) \\
 &= \nu_L^{-1} \cdot \left(\frac{\delta}{L}\right)^d \cdot (2\pi)^{d/2} \left\| \frac{\delta}{L} \cdot \xi \right\|^{-d/2} J_{d/2} \left(\frac{\delta}{L} \|\xi\| \right) \\
 &= \left(\left(\frac{\delta}{L}\right)^d \nu \right)^{-1} \cdot \left(\frac{\delta}{L}\right)^{d/2} \cdot (2\pi)^{d/2} \|\xi\|^{-d/2} \left(\frac{\delta \|\xi\|}{2L} \right)^{d/2} \cdot \left(\frac{1}{\Gamma(\frac{d}{2} + 1)} - \frac{\delta^2 \|\xi\|^2}{4L^2 \Gamma(\frac{d}{2} + 2)} + o(L^{-2}) \right) \\
 &= \pi^{d/2} \nu^{-1} \cdot \left(\frac{1}{\Gamma(\frac{d}{2} + 1)} - \frac{\delta^2 \|\xi\|^2}{4L^2 \Gamma(\frac{d}{2} + 2)} + o(L^{-2}) \right) \\
 &= \pi^{d/2} \left(\frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} \right)^{-1} \cdot \left(\frac{1}{\Gamma(\frac{d}{2} + 1)} - \frac{\delta^2 \|\xi\|^2}{4L^2 \Gamma(\frac{d}{2} + 2)} + o(L^{-2}) \right) \\
 &= 1 - \frac{\delta^2}{2(d+2)} L^{-2} \|\xi\|^2 + o(L^{-2}) \\
 (4.25) \quad &= 1 - \theta L^{-2} \|\xi\|^2 + o(L^{-2}), \quad \text{where } \theta := \frac{\delta^2}{2(d+2)}.
 \end{aligned}$$

Since $\|\widehat{\varphi}\|_2 = \|\varphi\|_2 > 0$, there exist $0 < p < q < \infty$ such that

$$\int_{\|\xi\| \in [p, q]} |\widehat{\varphi}(\xi)|^2 d\xi > 0.$$

Note that the set $\{p \leq \|\xi\| \leq q\}$ is compact, we now continue from (4.21) as

$$\begin{aligned}
 & \|\widehat{\varphi} \cdot (1 - \nu_L^{-1} \cdot \widehat{\chi}_B^L)\|_2^2 \\
 & \geq \int_{\|\xi\| \in [p, q]} |\widehat{\varphi}(\xi)|^2 \cdot |1 - \nu_L^{-1} \cdot \widehat{\chi}_B^L|^2 d\xi \\
 & = \int_{\|\xi\| \in [p, q]} |\widehat{\varphi}(\xi)|^2 \cdot |1 - (1 - \theta L^{-2} \|\xi\|^2 + o(L^{-2}))|^2 d\xi \quad (\text{using (4.25) on } \{p \leq \|\xi\| \leq q\}) \\
 (4.26) \quad & \gtrsim L^{-4}.
 \end{aligned}$$

Combining (4.19), (4.20), (4.21) and (4.26), we may deduce that

$$(4.27) \quad \text{Var}[\Lambda(\varphi_L)] \gtrsim L^{2d} \cdot \nu_L \cdot L^{-4} \gtrsim L^{d-4},$$

as desired.

4.5.2. Using hypothesis (ii.b) in Model 1.2. If B denotes the support of the function φ , and B_1 denotes the set of points of distance ≤ 1 to B (with $B \subset B_1$) then we may further lower bound

$$L^{2d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(u) - \varphi(v)|^2 K(Lu, Lv)^2 dudv \geq L^{2d} \int_B \int_{B_1^c} |\varphi(u) - \varphi(v)|^2 K(Lu, Lv)^2 dudv.$$

Since φ vanishes on B_1^c , we have

$$(4.28) \quad L^{2d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(u) - \varphi(v)|^2 K(Lu, Lv)^2 dudv \geq L^{2d} \int_B \int_{B_1^c} |\varphi(u)|^2 K(Lu, Lv)^2 dudv.$$

Recall the condition (ii.b) in Model 1.2, which says that there exist constants $\beta, r, c_1, c_2 \in \mathbb{R}_+, n_0 \in \mathbb{N}_+$ with $d/2 < \beta < d$ such that for each $x \in \mathbb{R}^d$, the set

$$E_x := \{y : |K(x, y)| \geq c_1 \|x - y\|^{-\beta}\}$$

satisfies $\text{Vol}(E_x \cap A_n^x(r)) \geq c_2 \text{Vol}(A_n^x(r))$ for all $n \geq n_0$.

Proposition 4.10. *Let K satisfy the above condition. Then for $R \geq R_0$ we have*

$$\int_{\|x-y\|>R} K(x, y)^2 dy \geq c_3 R^{d-2\beta}$$

for some positive constants c_3, R_0 that are independent of $x \in \mathbb{R}^d$.

Proof. For simplicity of notations, we only prove for the case $r = 1$. The argument for general case is similar. Here we will denote $A_n^x := A_n^x(1)$.

In this case, choose $R_0 = n_0$. For any $R \geq R_0$ and $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \int_{\|x-y\|>R} K(x, y)^2 dy &\geq \sum_{n=[R]+1}^{\infty} \int_{A_n^x} K(x, y)^2 dy \geq \sum_{n=[R]+1}^{\infty} \int_{A_n^x \cap E_x} K(x, y)^2 dy \\ &\geq \sum_{n=[R]+1}^{\infty} \int_{A_n^x \cap E_x} c_1^2 \|x - y\|^{-2\beta} dy \geq c_1^2 \sum_{n=[R]+1}^{\infty} (n+1)^{-2\beta} \text{Vol}(E_x \cap A_n^x) \\ &\geq c_1^2 c_2 \sum_{n=[R]+1}^{\infty} (n+1)^{-2\beta} \text{Vol}(A_n^x) \geq c_1^2 c_2 \sum_{n=[R]+1}^{\infty} (n+1)^{-2\beta} n^{2\beta} \int_{A_n^x} \|x - y\|^{-2\beta} dy \\ &\geq c_1^2 c_2 2^{-2\beta} \sum_{n=[R]+1}^{\infty} \int_{A_n^x} \|x - y\|^{-2\beta} dy \geq c_1^2 c_2 2^{-2\beta} \int_{\|x-y\| \geq 2R} \|x - y\|^{-2\beta} dy \\ &= c_1^2 c_2 2^{-2\beta} R^{d-2\beta} \int_{\|y\| \geq 2} \|y\|^{-2\beta} dy. \end{aligned}$$

Let $c_3 := c_1^2 c_2 2^{-2\beta} \int_{\|y\| \geq 2} \|y\|^{-2\beta} dy$. This completes the proof. ■

We now continue from (4.28) with $L \geq L_0$ for some large enough L_0 as

$$\begin{aligned} &L^{2d} \int_B \int_{B_1^c} |\varphi(u)|^2 K(Lu, Lv)^2 du dv \\ &= L^d \int_B |\varphi(u)|^2 \left(L^d \int_{B_1^c} K(Lu, Lv)^2 dv \right) du \\ &= L^d \int_B |\varphi(u)|^2 \left(\int_{L \cdot B_1^c} K(Lu, y)^2 dy \right) du \quad (\text{changing variables to } y = Lv) \\ &\geq c_3 L^d \int_B |\varphi(u)|^2 L^{d-2\beta} du \\ &\quad (\text{using Proposition 4.10 with } x = Lu; \text{ note that } \|Lu - Lv\| \geq L) \\ (4.29) \quad &= c_3 \|\varphi\|_2^2 L^{2(d-\beta)}. \end{aligned}$$

Setting $\gamma := 2(d-\beta) > 0$, we have the desired polynomial lower bound on the variance growth of $\Lambda(\varphi_L)$.

Example 4.11. The kernel $K(x, y) = \hat{\chi}_{B(0,1)}(x-y)$, where $\hat{\chi}_{B(0,1)}$ is the Fourier transform of the characteristic function of the unit ball $B(0, 1)$ in \mathbb{R}^d , $d \geq 2$, satisfies the decay condition in Model 1.2 with

$$\beta = \frac{d+1}{2}, \quad r = 2\pi, \quad c_1 = \frac{(2\pi)^{d/2}}{4\pi^{1/2}}, \quad c_2 = \frac{b-a}{2^d\pi},$$

for some constants $a < b \in [0, 2\pi]$.

Indeed, we have

$$K(x, y) = \hat{\chi}_{B(0,1)}(x-y) = (2\pi)^{d/2} \|x-y\|^{-d/2} J_{d/2}(\|x-y\|),$$

where J_n is the Bessel function of the first kind with order n [Fi]. Use well-known asymptotics of Bessel functions [AbStRo], we have

$$J_{d/2}(t) \sim \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{d+1}{4}\pi\right) + O(t^{-3/2}) \quad \text{as } t \rightarrow +\infty.$$

Hence there exists a positive integer n_0 such that for all $t > n_0$, we have

$$\sqrt{t} J_{d/2}(t) \geq \sqrt{2}\pi^{-1/2} \cos\left(t - \frac{d+1}{4}\pi\right) - \frac{1}{4}\pi^{-1/2}.$$

Let $[a, b] \subset [0, 2\pi]$ be a subinterval such that $\cos\left(t - \frac{d+1}{4}\pi\right) \geq 1/2$ for all $t \in [a, b]$. Then for all positive integer $n > n_0$, we will have

$$\sqrt{t} J_{d/2}(t) \geq \frac{\sqrt{2}}{2}\pi^{-1/2} - \frac{1}{4}\pi^{-1/2} > \frac{1}{4}\pi^{-1/2}, \quad \forall t \in [2\pi n + a, 2\pi n + b].$$

For each $x \in \mathbb{R}^d$, consider the set

$$\begin{aligned} E_x &:= \left\{ y \in \mathbb{R}^d : |K(x, y)| \geq \frac{(2\pi)^{d/2}}{4\pi^{1/2}} \|x-y\|^{-(d+1)/2} \right\} \\ &= \left\{ y \in \mathbb{R}^d : \|x-y\|^{1/2} |J_{d/2}(\|x-y\|)| \geq \frac{1}{4}\pi^{-1/2} \right\}. \end{aligned}$$

By the argument above, we see that for every integer $n > n_0$

$$E_x \cap A_n^x(2\pi) \supset \{y \in \mathbb{R}^d : 2\pi n + a \leq \|x-y\| \leq 2\pi n + b\}.$$

Thus for all $n > n_0$

$$\frac{\text{Vol}(E_x \cap A_n^x(2\pi))}{\text{Vol}(A_n^x(2\pi))} \geq \frac{(2\pi n + b)^d - (2\pi n + a)^d}{(2\pi(n+1))^d - (2\pi n)^d} \geq \frac{b-a}{2^d\pi}$$

where we use the identity $x^d - y^d = (x-y)(x^{d-1} + x^{d-2}y + \dots + y^{d-1})$.

4.6. Proof of Theorem 1.6.

Proof of Theorem 1.6. Let r_0 be the constant in Proposition 4.4 and let r, c be the constants in Proposition 4.6. By choosing $r_L = \min\{r_0, r\}$ for all $L > 0$, we deduce that $\mathbb{E}[\exp(iu\Lambda(\varphi_L))]$ does not vanish in $\mathbb{D}(0, r_L)$ and

$$\mathbb{E}[\exp(r_L|\Lambda(\varphi_L)|)] \leq \exp(cL^d) \text{ for all } L > 0.$$

Let γ be the constant in Proposition 4.9 and define $\eta := d$ if $\alpha > 0$ and $\eta := \gamma$ if $\alpha < 0$. Then for all L large enough

$$\text{Var}(\Lambda(\varphi_L)) \gtrsim L^\eta.$$

It remains to verify the growth condition in Corollary 1.3. From the constancy of r_L and the inequality above, we easily deduce that

$$\log^+ \log \mathbb{E}[e^{r_L |\Lambda(\varphi_L)|}] \leq \log^+ \log \exp(cL^d) \leq \log^+ c + d \log^+ L.$$

Denote by σ_L the standard deviation of $\Lambda(\varphi_L)$, then for all L big enough we will have

$$(r_L \sigma_L)^{-1} (1 + \log^+ \log |\mathbb{E}[e^{r_L |\Lambda(\varphi_L)|}]|) \lesssim \log L \cdot L^{-\eta/2} \rightarrow 0$$

as $L \rightarrow +\infty$. Using Corollary 1.3, the theorem follows. ■

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE - 10, LOWER KENT RIDGE ROAD
- SINGAPORE 119076

Email address: matdte@nus.edu.sg; subhrowork@gmail.com; e0546172@u.nus.edu; e0511873@u.nus.edu