

Topological axion electrodynamics and 4-group symmetry

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Abstract

We study higher-form symmetries and a higher group in the low energy limit of a $(3+1)$ -dimensional axion electrodynamics with a massive axion and a massive photon. A topological field theory describing topological excitations with the axion-photon coupling, which we call a topological axion electrodynamics, is obtained in the low energy limit. Higher-form symmetries of the topological axion electrodynamics are specified by equations of motion and Bianchi identities. We find that there are induced anyons on the intersections of symmetry generators. By a link of worldlines of the anyons, we show that the worldvolume of an axionic domain wall is topologically ordered. We further specify the underlying mathematical structure elegantly describing all salient features of the theory to be a 4-group.

1. Introduction

Axions are hypothetical particles originally proposed as a solution to the strong CP problem [1–7] in particle physics, but now play crucial roles in several contexts in modern physics; they are not only a dark matter candidate in cosmology [8–12] (see, e.g., Refs. [13–16] as a review) but also appear in string theories [17–22] and even in topological insulators [23–25] and topological superconductors [26–28] in condensed matter physics. (see, e.g., Refs. [29, 30] as a review). One of the salient features of the axions is a topological coupling to the photon due to a chiral anomaly. In particular, the simplest system, the axion electrodynamics [23], has been studied extensively to capture magneto-electric responses due to the topological coupling [23–25, 31–39]. When both the axion and photon have mass gaps, the axion electrodynamics is applicable to topological superconductors [26–28] and admits topological solitons such as axionic domain walls [40, 41] and quantized magnetic vortex strings [42, 43].

Topological solitons are necessary ingredients to determine the dynamics and phases of systems. In the absence of the axion coupling, quantized magnetic vortices can give fractional Aharonov-Bohm (AB) effects. Gapped phases exhibiting these effects in the low energy limit are called topologically ordered phases [44–47], which are realized by anyons in fractional quantum Hall (FQH) states and s-wave superconductors in $(2+1)$ and $(3+1)$ dimensions, respectively. Once we take into account the axion cou-

pling, electromagnetic properties become richer: for instance, electric charges are induced by penetrating magnetic fluxes to the axionic domain wall [23, 24, 32, 48]. Effects on the domain wall can be described by an Abelian Chern-Simons (CS) term on the domain wall, induced by the axion-photon coupling [49, 50]. Since FQH states can be described by this CS term in the low energy limit [44–46], one can expect that the axionic domain wall is topologically ordered. However, topological objects giving the AB phases, which we will call topological order parameters in this Letter, have not been identified to the best of our knowledge.

On the other hand, recently physics of topological solitons and extended objects has been extensively studied in the language of global higher-form symmetries; symmetries under actions on p -dimensional extended objects, called p -form symmetries [51–62], were found as natural extensions of ordinary symmetries acting on local 0-dimensional operators. Higher-form symmetries give us new understandings for the classification of phases of matter and physical effects discussed by extended objects, e.g., topologically ordered phases as spontaneous symmetry breaking of 1-form symmetries associated to the AB effect [53, 60, 63], the magneto-electric responses for the axion electrodynamics in gapless phase [64, 65] as correlations of 0- and 1-form symmetries. One of the most elegant notions of symmetries in modern quantum field theory can be formulated as so-called higher groups; an n -group symmetry denotes a set of 0-, ..., $(n-1)$ -form symmetries with nontrivial correlations between them. 2- and 3-groups have been extensively studied (see e.g., Refs. [66–85]), and in particular, a 3-group structure has been found in gapless axion electrodynamics [64, 65].

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In this Letter, we investigate higher-form global symmetries and a higher group in the low energy limit of the $(3+1)$ -dimensional axion electrodynamics in the gapped phase, to show a topological order on the axionic domain wall. The higher-form symmetries can be specified by employing a dual topological quantum field theory for the massive photon [86–90] and massive axion [91–95], which we will call a *topological axion electrodynamics*. We find 0-, 1-, 2-, and 3-form symmetries and show the topological order on the axionic domain wall in terms of the higher-form symmetries, by identifying the topological order parameters as intersections of 0- and 1-form symmetry generators on which anyons are induced. On the axionic domain wall, the intersections can have a fractional phase determined by the groups of the higher-form symmetries. The fractional AB effect for the intersected symmetry generators implies nontrivial correlations between higher-form symmetries. We further find that the symmetries organize a 4-group symmetry. While a 4-group symmetry in a simpler case has been studied in Ref. [96], our 4-group is the first example where all the 0-, 1-, 2-, and 3-form symmetries are nontrivial.

2. Topological axion electrodynamics

Here, we give an action of the low energy limit of the axion electrodynamics with a massive axion and a massive photon. Using dual transformations, we derive a topological field theory describing this limit of the system. We begin with the effective action,

$$S = - \int_{M_4} \left(\frac{v^2}{2} |d\phi|^2 + \frac{1}{2e^2} |da|^2 + \frac{v'^2}{2} |d\chi - qa|^2 + V(\phi) \star 1 - \frac{N}{8\pi^2} \phi da \wedge da \right), \quad (1)$$

where we use notations of differential forms: $|\omega_p|^2 = \omega_p \wedge \star \omega_p$ for a p -form ω_p , and $\star$ is the Hodge star operator. M_4 denotes the $(3+1)$ -dimensional spacetime spin manifold [97, 98]. The quantities v, v' are mass dimension 1 parameters, e is a coupling constant, and N is an integer. The axion ϕ is given by a 2π periodic pseudo-scalar field, $\phi(\mathcal{P}) + 2\pi \sim \phi(\mathcal{P})$, for a point $\mathcal{P}$ in the spacetime. By this periodicity, the axion can have a winding number along a loop $\mathcal{C}$: $\int_{\mathcal{C}} d\phi \in 2\pi\mathbb{Z}$. In addition, we have introduced a potential term $V(\phi)$ for the axion, which has a global symmetry under the shift $V(\phi + 2\pi/k) = V(\phi)$. The potential has k of degenerated minima, $V(2\pi n/k) = V'(2\pi n/k) = 0$ and $V''(2\pi n/k) > 0$ for $n \in \mathbb{Z} \bmod k$. The photon is described by a $U(1)$ 1-form gauge field a with a gauge transformation, $a \rightarrow a + d\lambda$. Here, λ is a $U(1)$ 0-form gauge parameter normalized as $\int_{\mathcal{C}} d\lambda \in 2\pi\mathbb{Z}$. The photon is subject to the Dirac quantization on a closed 2-dimensional subspace $\mathcal{S}$ (e.g., a sphere S^2), $\int_{\mathcal{S}} da \in 2\pi\mathbb{Z}$. The 2π periodic scalar field χ is introduced as a phase component of a charge $q(\in \mathbb{Z})$ Higgs field. The field χ is shifted as $\chi \rightarrow \chi + q\lambda$ under the gauge transformation $a \rightarrow a + d\lambda$.

To investigate topological properties of the system, we dualize the theory to a topological theory after neglecting the kinetic terms of the axion and photon at low energy. We dualize $|d\chi - qa|^2$ and $V(\phi)$ to topological term given by 2- and 3-form gauge fields, respectively [86–95]. The dual topological action has the simple form,

$$S_{\text{TAE}} = \int_{M_4} \left(\frac{k}{2\pi} c \wedge d\phi + \frac{q}{2\pi} b \wedge da + \frac{N}{8\pi^2} \phi da \wedge da \right). \quad (2)$$

Here, b and c are $U(1)$ 2- and 3-form gauge fields, whose gauge transformation laws are given by 1- and 2-form gauge parameters λ_1 and λ_2 as $b \rightarrow b + d\lambda_1$ and $c \rightarrow c + d\lambda_2$, respectively. They are normalized by the flux quantization conditions as $\int_{\mathcal{V}} db, \int_{\Omega} dc, \int_{\mathcal{S}} d\lambda_1, \int_{\mathcal{V}} d\lambda_2 \in 2\pi\mathbb{Z}$ where $\mathcal{V}$ and Ω are closed 3- and 4-dimensional subspaces, respectively. We will call the dual theory the “topological axion electrodynamics,” since it can be described by only topological terms.

3. Higher-form global symmetries in topological axion electrodynamics

We show higher-form global symmetries in this system found by the equations of motion for the dynamical fields ϕ, a, b , and c :

$$\begin{aligned} \frac{k}{2\pi} dc + \frac{N}{8\pi^2} da \wedge da &= 0, & \frac{q}{2\pi} db + \frac{N}{4\pi^2} d\phi \wedge da &= 0, \\ \frac{q}{2\pi} da &= 0, & \frac{k}{2\pi} d\phi &= 0, \end{aligned} \quad (3)$$

respectively. The corresponding symmetry generators with groups parametrizing them are

$$\begin{aligned} U_0(e^{2\pi i n_0/m}, \mathcal{V}) &= e^{-i \frac{n_0}{m} \int_{\mathcal{V}} (kc + \frac{N}{4\pi} a \wedge da)}, \\ U_1(e^{2\pi i n_1/p}, \mathcal{S}) &= e^{-i \frac{n_1}{p} \int_{\mathcal{S}} (qb + \frac{N}{2\pi} \phi da)}, \\ U_2(e^{2\pi i n_2/q}, \mathcal{C}) &= e^{-i n_2 \int_{\mathcal{C}} a}, \\ U_3(e^{2\pi i n_3/k}, (\mathcal{P}, \mathcal{P}')) &= e^{-i n_3 (\phi(\mathcal{P}) - \phi(\mathcal{P}'))}. \end{aligned} \quad (4)$$

Here, we have introduced $n_0, \dots, n_3 \in \mathbb{Z}$, $p := \gcd(N, q)$, and $m := \gcd(N, k)$, where “gcd” represents the greatest common divisor. Hereafter, we assume that the subspaces $\mathcal{V}, \mathcal{S}$, and $\mathcal{C}$ do not have any self-intersection for simplicity. All symmetry generators have the standard form of generators $U_n = e^{i\theta_n Q_n}$ with $Q_n = \int j_n$, where the $(3-n)$ -form current j_n is closed under the equations of motion (4), i.e., $dj_n = 0$. Note that the phase θ_n is constrained by the large gauge invariance of the dynamical fields [64, 65, 99–102]. Therefore, there are the electric $\mathbb{Z}_m$ 0-form, $\mathbb{Z}_p$ 1-form, $\mathbb{Z}_q$ 2-form, and $\mathbb{Z}_k$ 3-form global symmetries. Their physical interpretations are as follows: $U_0(e^{2\pi i n_0/m}, \mathcal{V})$ is a worldvolume of the axionic domain wall: the equation of motion of c in the presence of U_0 is $d\phi = (2\pi n_0/m) \delta_1(\mathcal{V})$. Here, $\delta_{4-n}(\Sigma_n)$ is a delta function $(4-n)$ -form satisfying $\int_{\Sigma_n} \omega_n = \int_{M_4} \omega_n \wedge \delta_{4-n}(\Sigma_n)$ for a n -dimensional subspace

Σ_n [63, 103–106]. Similarly, $U_1(e^{2\pi i n_1/p}, \mathcal{S})$ represents a worldsheet of a quantized magnetic flux or an impulse of a quantized electric field, because the equation of motion of b in the presence of U_1 is $da = (2\pi n_1/p)\delta_2(\mathcal{S})$. U_2 and U_3 are a Wilson loop and a 2-point local operator of the axion, respectively. All of them are topological, and the vacuum expectation values (VEVs) are trivial, $\langle U_0 \rangle = \dots = \langle U_3 \rangle = 1$ thanks to the equations of motion in Eq. (3). Here, the symbol ‘ $\langle \dots \rangle$ ’ means the VEV. We remark that the 2- and 3-form global symmetries are emergent symmetries in the low energy limit, where the magnetic vortices and axionic domain walls can be treated as objects with infinitely large tensions. We summarize corresponding charged objects, which are not necessary in the following discussion in Appendix A.

In addition to these electric symmetries, there are four magnetic $U(1)$ (–1)-, 0-, 1-, 2-form symmetries, which are given as the form $U_n^M(e^{i\theta_n^M}, \Sigma_{3-n}) = e^{i\theta_n^M \int_{\Sigma_n} j_n^M}$ with closed currents $j_{-1}^M = dc$, $j_0^M = db$, $j_1^M = da$, $j_2^M = d\phi$. The conservation laws are given by the Bianchi identities. In this Letter, we mainly focus on the electric symmetries.

4. Topological order in bulk

Before discussing the topological order on the axionic domain wall, we show the topological order in the bulk for $p = \gcd(N, q) \neq 1$ with a fractional AB phase given by $\mathbb{Z}_p$. Topological order parameters can be identified as the symmetry generators U_1 and U_2 , since they are topological and have nontrivial fractional phases. First, we have seen that they are topological, developing nonzero VEVs $\langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) \rangle = \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) \rangle = 1$. Second, they have a fractional AB phase characterized by $\mathbb{Z}_p$:

$$\langle U_1(e^{\frac{2\pi i n_1}{p}}, \mathcal{S}) U_2(e^{\frac{2\pi i n_2}{q}}, \mathcal{C}) \rangle = e^{i\theta_{12}}. \quad (5)$$

Here, we have defined $\theta_{12} = -(2\pi n_1 n_2/p) \text{Link}(\mathcal{S}, \mathcal{C})$ with the linking number between n - and $(3-n)$ -dimensional subspaces Σ_n and Σ'_{3-n} as $\text{Link}(\Sigma_n, \Sigma'_{3-n}) = \int_{\Omega_{\Sigma_n}} \delta_{n+1}(\Sigma'_{3-n})$ by using $(n+1)$ -dimensional subspace Ω_{Σ_n} satisfying $\partial\Omega_{\Sigma_n} = \Sigma_n$ (see Appendix B). By the linking phase, the topological axion electrodynamics is topologically ordered in the bulk. We remark that this fractional AB phase is different from a $\mathbb{Z}_q$ AB phase of an Abelian Higgs model without an axion [47, 53], due to the presence of the axion-photon coupling deforming the 1-form symmetry.

5. Topological order on axionic domain walls

Now, we show that the axionic domain walls are also topologically ordered with a nontrivial fractional linking phase in a different manner from the bulk. The topological order parameter can be identified as an intersection of the 0- and 1-form symmetry generators. The fractional linking

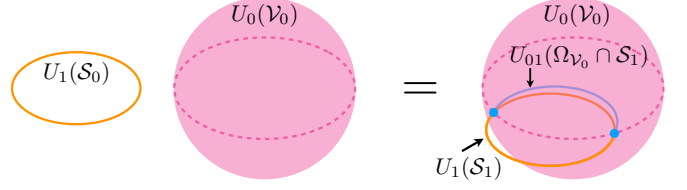


Figure 1: Intersection of temporally and spatially extended symmetry generators U_0 (pink sphere) and U_1 (orange line). This figure is a time slice of the configuration. The blue line in the right panel is a time slice of an induced static surface U_{01} . The blue dots denote the boundary of $\Omega_{V_0} \cap \mathcal{S}_1$ on the time slice, which corresponds to an anyon on the domain wall.

phase is given by the group $\mathbb{Z}_{mp^2/\gcd(N, mp^2)}$, different from the one in the bulk.

A rough description is as follows. The worldvolume of the axionic domain wall can be understood as a FQH state, since U_0 in Eq. (4) has a level $N/m \in \mathbb{Z}$ CS term, $\frac{N}{4\pi m} \int_{\mathcal{V}} a \wedge da$. By intersecting U_1 to U_0 , we can have an anyon, whose worldline is a topological order parameter of FQH states. Linked anyons on the domain wall can be obtained by two U_1 's intersected in the bulk.

A more precise proof can be given as follows. First, we intersect U_0 to U_1 to create a worldline of an anyon on the domain wall. We begin with the correlation function $\langle U_0(e^{2\pi i n_0/m}, \mathcal{V}_0) U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) \rangle$ assuming $\mathcal{V}_0 \cap \mathcal{S}_0 = \emptyset$ (the left panel of Fig. 1). We deform $\mathcal{S}_0$ to $\mathcal{S}_1$ that is intersected to $\mathcal{V}_0$ (the right panel of Fig. 1). Here, we choose $\mathcal{S}_1$ so that $\mathcal{S}_1 \cap \mathcal{V}_0$ is a 1-dimensional closed subspace, and $\mathcal{S}_0 \cap \mathcal{S}_1 = \emptyset$. We interpolate between $\mathcal{S}_0$ and $\mathcal{S}_1$ using a 3-dimensional subspace $\mathcal{V}_{01}$ satisfying $\partial\mathcal{V}_{01} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that $\mathcal{V}_{01}$ does not intersect with any singularity such as an 't Hooft line. Using $U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) = U_1(e^{2\pi i n_1/p}, \partial\mathcal{V}_{01}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_1)$, and absorbing $U_1(e^{2\pi i n_1/p}, \partial\mathcal{V}_{01})$ into the action by a redefinition $a + (2\pi n_1/p)\delta_1(\mathcal{V}_{01}) \rightarrow a$, the correlation function can be written as

$$\begin{aligned} & \langle U_0(e^{2\pi i n_0/m}, \mathcal{V}_0) U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) \rangle \\ &= \langle U_{01}(e^{\frac{2\pi i N n_0 n_1}{mp}}, \Omega_{\mathcal{V}_0} \cap \mathcal{S}_1) \\ & \quad \times U_0(e^{2\pi i n_0/m}, \mathcal{V}_0) U_1(e^{2\pi i n_1/p}, \mathcal{S}_1) \rangle. \end{aligned} \quad (6)$$

Here, we have introduced an object on a 2-dimensional subspace, $\Omega_{\mathcal{V}_0} \cap \mathcal{S}_1$, with a boundary,

$$U_{01}(e^{\frac{2\pi i N n_0 n_1}{mp}}, \Omega_{\mathcal{V}_0} \cap \mathcal{S}_1) = e^{\frac{2\pi i N n_0 n_1}{mp} \int_{\Omega_{\mathcal{V}_0}} \frac{da}{2\pi} \wedge \delta_2(\mathcal{S}_1)}, \quad (7)$$

where $\Omega_{\mathcal{V}_0}$ is a 4-dimensional subspace whose boundary is $\mathcal{V}_0$. It is straightforward to show that U_{01} is topological under the deformation of $\mathcal{S}_1$ or $\mathcal{V}_0$ by the redefinition of a or ϕ , respectively. Therefore, we find that, when U_0 and U_1 are intersected, there should be an additional magnetic 1-form symmetry generator with the boundary U_{01} . The boundary object expresses a worldline of an anyon on the domain wall. This anyon cannot solely exist in the bulk if $N/mp \notin \mathbb{Z}$; it is always trapped on the domain wall. From

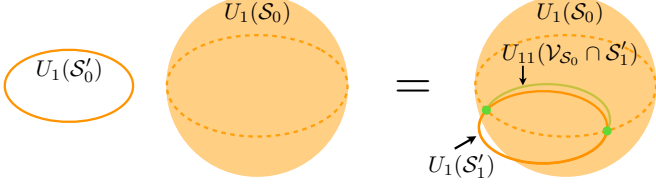


Figure 2: Intersection of symmetry generators U_1 . This figure is a time slice of the configuration: $U_1(\mathcal{S}_0)$ expressed by orange spheres are extended to only spatial directions, while $U_1(\mathcal{S}'_0)$ and $U_1(\mathcal{S}'_1)$ expressed by the orange lines are temporally and spatially extended. The green line in the right panel is an induced instantaneous line object U_{11} . The green dots denote boundaries of $\mathcal{V}_{\mathcal{S}_0} \cap \mathcal{S}'_1$ on the time slice.

the phase factor in Eq. (7), the anyon line has a fractional electric charge $Nn_0n_1/(mp)$ [32]. The appearance of anyons trapped on the domain wall is one of the main results of this Letter. The anyons also have a fractional linking phase, which we will show in the following.

We next consider intersections of 1-form symmetry generators, which are necessary to derive the fractional linking phase of anyons. We begin with the correlation function (left panel of Fig. 2), $\langle U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) U_1(e^{2\pi i n'_1/p}, \mathcal{S}'_0) \rangle$, where 2-dimensional subspaces $\mathcal{S}_0$ and $\mathcal{S}'_0$ satisfy $\mathcal{S}_0 \cap \mathcal{S}'_0 = \emptyset$. We deform $\mathcal{S}'_0$ to $\mathcal{S}'_1$ that intersects with $\mathcal{S}_0$, where the deformation is characterized by a 3-dimensional subspace $\mathcal{V}_{01}$ satisfying $\partial \mathcal{V}_{01} = \mathcal{S}'_0 \cup \mathcal{S}'_1$. By the same procedure as Eq. (6), the correlation function becomes

$$\begin{aligned} & \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) U_1(e^{2\pi i n'_1/p}, \mathcal{S}'_0) \rangle \\ &= \langle U_{11}(e^{\frac{2\pi i N n_1 n'_1}{p^2}}, \mathcal{V}_{\mathcal{S}_0} \cap \mathcal{S}'_1) \\ & \quad \times U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) U_1(e^{2\pi i n'_1/p}, \mathcal{S}'_1) \rangle, \end{aligned} \quad (8)$$

where we have introduced an object on a 1-dimensional subspace, $\mathcal{V}_{\mathcal{S}_0} \cap \mathcal{S}'_1$, with boundaries,

$$U_{11}(e^{\frac{2\pi i N n_1 n'_1}{p^2}}, \mathcal{V}_{\mathcal{S}_0} \cap \mathcal{S}'_1) = e^{\frac{2\pi i N n_1 n'_1}{p^2} \int_{\mathcal{V}_{\mathcal{S}_0}} \frac{d\phi}{2\pi} \wedge \delta_2(\mathcal{S}'_1)}, \quad (9)$$

and $\mathcal{V}_{\mathcal{S}_0}$ is a 3-dimensional subspace whose boundary is $\mathcal{S}_0$. Thus, we should add the magnetic 2-form symmetry generator with boundaries U_{11} if we intersect two U_1 's. Again, the intersected configuration is topological under deformations of $\mathcal{S}_0$ or $\mathcal{S}'_1$. Using the interpretation of U_1 as external electric or magnetic fluxes, the presence of U_{11} can be physically understood as the fact that the $\mathbf{E} \cdot \mathbf{B}$ is a source of the axion, since the boundary objects of U_{11} can be identified as local objects of the axion.

Finally, we consider the intersection of three symmetry generators to show the fractional linking phase of anyons on the domain wall. We begin with the following cubic but trivial correlation function,

$$\langle U_1(e^{\frac{2\pi i n_1}{p}}, \mathcal{S}_0) U_1(e^{\frac{2\pi i n'_1}{p}}, \mathcal{S}'_0) U_0(e^{\frac{2\pi i n_0}{m}}, \mathcal{V}_0) \rangle = 1, \quad (10)$$

where we choose the subspaces such that $\mathcal{S}_0 \cap \mathcal{S}'_0 = \mathcal{S}_0 \cap \mathcal{V}_0 = \mathcal{S}'_0 \cap \mathcal{V}_0 = \emptyset$. We deform $\mathcal{S}_0$ to $\mathcal{S}_1$ using $\mathcal{V}_{01}$, where $\mathcal{S}_1$

intersects with $\mathcal{V}_0$, but $\mathcal{S}_1 \cap \mathcal{S}'_0 = \emptyset$. As shown in Eq. (6), we have $U_{01}(e^{\frac{2\pi i N n_0 n_1}{mp}}, \Omega_{\mathcal{V}_0} \cap \mathcal{S}_1)$ in the correlation function. We then deform $\mathcal{S}'_0$ to $\mathcal{S}'_1$ interpolated by $\mathcal{V}'_{01}$, where $\mathcal{S}'_1$ intersects with $\mathcal{V}_0$, and also with $\mathcal{S}_1$ transversally. We thus obtain

$$\begin{aligned} & C_{011}(e^{2\pi i n_0/m}, \mathcal{V}_0; e^{2\pi i n_1/p}, \mathcal{S}_1; e^{2\pi i n'_1/p}, \mathcal{S}'_1) \\ &:= \langle U_{01}(e^{\frac{2\pi i N n_0 n_1}{mp}}, \Omega_{\mathcal{V}_0} \cap \mathcal{S}_1) U_{01}(e^{\frac{2\pi i N n_0 n'_1}{mp}}, \Omega_{\mathcal{V}_0} \cap \mathcal{S}'_1) \\ & \quad \times U_{11}(e^{\frac{2\pi i N n_1 n'_1}{p^2}}, \mathcal{V}_{\mathcal{S}_1} \cap \mathcal{S}'_1) U_0(e^{2\pi i n_0/m}, \mathcal{V}_0) \\ & \quad \times U_1(e^{2\pi i n_1/p}, \mathcal{S}_1) U_1(e^{2\pi i n'_1/p}, \mathcal{S}'_1) \rangle = e^{i\theta_{011}}. \end{aligned} \quad (11)$$

Here, $\theta_{011} := -2\pi N n_0 n_1 n'_1 / (mp^2) \text{Link}(\mathcal{S}_1, \mathcal{S}'_1)|_{\mathcal{V}_0}$, and we have used the linking number of $\mathcal{S}_1$ and $\mathcal{S}'_1$ on $\mathcal{V}_0$ $\text{Link}(\mathcal{S}_1, \mathcal{S}'_1)|_{\mathcal{V}_0} := \int_{\mathcal{V}_0} \delta_1(\mathcal{V}_{\mathcal{S}_1}) \wedge \delta_2(\mathcal{S}'_1)$, and $\mathcal{V}_{\mathcal{S}_1}$ is a 3-dimensional subspace whose boundary is $\mathcal{S}_1$. The above relation means that the boundary line objects of U_{01} are topological order parameters with the fractional linking phase. Thus, the axionic domain wall is topologically ordered, and the fractional phase, $e^{i\theta_{011}} \in \mathbb{Z}_{mp^2/\text{gcd}(N, mp^2)}$, is different from that of the bulk. The linking phase comes from the fractionally quantized magnetic fluxes $(1/p)^2$ on the level N/m Abelian CS action.

6. Global 4-group symmetry in topological axion electrodynamics

The topological order on the axionic domain wall implies a so-called higher-group structure; as shown below, there is a 4-group symmetry which is a set of 0-, 1-, 2-, and 3-form symmetries with nontrivial correlations between them, where the 0- and 1-form symmetry generators lead to a 2-form symmetry generator, and two 1-form symmetry generators lead to a 3-form symmetry generator.

As shown in Eq. (6), the correlation of the 0- and 1-form symmetry generators induces a magnetic 1-form symmetry generator, whose boundary is the intersection of the 0- and 1-form symmetry generators. The intersection is a closed 1-dimensional object, which generate a 2-form symmetry. To see this, we again focus on Eq. (11), and evaluate it as follows:

$$\begin{aligned} & C_{011}(e^{2\pi i n_0/m}, \mathcal{V}_0; e^{2\pi i n_1/p}, \mathcal{S}_1; e^{2\pi i n'_1/p}, \mathcal{S}'_1) \\ &= e^{i\theta_{011}} \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}_1) \rangle. \end{aligned} \quad (12)$$

The 2-form symmetry and its symmetry group can be identified as follows. We remark that U_1 is charged under the 2-form symmetry with the charge $-n_1 q/p$, $\langle U_2(e^{\frac{2\pi i n_2}{q}}, \mathcal{C}) U_1(e^{\frac{2\pi i n_1}{p}}, \mathcal{S}) \rangle = e^{i\theta_{12}} \langle U_1(e^{\frac{2\pi i n_1}{p}}, \mathcal{S}) \rangle$, as in Eq. (5). Thus, the intersection of U_0 and U_1 can be regarded as a 2-form symmetry generator. By comparing θ_{011} to the charge $-n_1 q/p$, we find that the intersection of U_0 and U_1 is parametrized by $\mathbb{Z}_Q$ with $Q = q \cdot mp / \text{gcd}(N, mp)$ being product of q and the denominator of N/mp . Thus, the 2-form symmetry group is identified as $\mathbb{Z}_Q$, which is transmuted from $\mathbb{Z}_q$.

Similarly, the intersection of two 1-form symmetry generators becomes the boundary of a magnetic 2-form symmetry (8), which behaves as a 3-form symmetry generator. We again focus on the correlation function in Eq. (11), and evaluate it as follows:

$$C_{011}(e^{2\pi i n_0/m}, \mathcal{V}_0; e^{2\pi i n_1/p}, \mathcal{S}_1; e^{2\pi i n'_1/p}, \mathcal{S}'_1) = e^{i\theta_{011}} \langle U_0(e^{2\pi i n_0/m}, \mathcal{V}_0) \rangle. \quad (13)$$

Since the correlation function $\langle U_3(e^{2\pi i n_3/k}, (\mathcal{P}, \mathcal{P}')) U_0(e^{2\pi i n_0/m}, \mathcal{V}) \rangle = e^{-2\pi i \frac{n_0 n_3}{k} \cdot \frac{k}{m} \text{Link}(\mathcal{V}, (\mathcal{P}, \mathcal{P}'))} \langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) \rangle$ implies the charge of U_0 under the 3-form symmetry is $-n_0 k/m$, we find that the intersection of two U_1 's becomes a symmetry generator of the 3-form symmetry with the symmetry group $\mathbb{Z}_K$, where $K := k \cdot p^2 / \gcd(N, p^2)$ with the nontrivial denominator of N/p^2 .

The appearance of topological objects at intersections of symmetry generators is a signal of higher groups. Our discussion in this Letter can have potential applications to both of physics and mathematics. For the physics side, we may apply this discussion to, e.g., topological superconductors. To discuss 't Hooft anomalies would be an important issue to determine the ground state structures. For the mathematics side, clarifying the precise definition of the 4-group would lead to a construction of a new class of higher-groups. We will address these issues in the forthcoming paper [107].

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Appendix A. Charged objects for higher-form symmetries

Here, we summarize charged objects for the higher-form symmetries. The corresponding charged objects are the Wilson loop and its analogues. For 0-, 1-, 2-, and 3-form symmetries, the charged objects are explicitly written as

$$\begin{aligned} L(q_0, \mathcal{P}) &= e^{iq_0 \phi(\mathcal{P})}, \\ W(q_1, \mathcal{C}) &= e^{iq_1 \int_{\mathcal{C}} a} = U_2(e^{-2\pi i q_1/q}, \mathcal{C}), \\ V(q_2, \mathcal{S}) &= e^{iq_2 \int_{\mathcal{S}} b}, \quad D(q_3, \mathcal{V}) = e^{iq_3 \int_{\mathcal{V}} c}, \end{aligned} \quad (A.1)$$

respectively. Here, the charges are quantized $q_0 \in \mathbb{Z}_m$, $q_1 \in \mathbb{Z}_p$, $q_2 \in \mathbb{Z}_q$, and $q_3 \in \mathbb{Z}_k$ because of the large gauge invariance for the gauge fields. We remark that $W(q_1, \mathcal{C})$ is identical to the symmetry generator $U_2(e^{-2\pi i q_1/q}, \mathcal{C})$. The symmetry transformations are induced by the link of a

charged operator and the symmetry generator (see Appendix B):

$$\begin{aligned} \langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) L(q_0, \mathcal{P}) \rangle &= e^{2\pi i q_0 n_0 \text{Link}(\mathcal{V}, \mathcal{P})/m} \langle L(q_0, \mathcal{P}) \rangle, \\ \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) W(q_1, \mathcal{C}) \rangle &= e^{2\pi i q_1 n_1 \text{Link}(\mathcal{S}, \mathcal{C})/p} \langle W(q_1, \mathcal{C}) \rangle, \\ \langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) V(q_2, \mathcal{S}) \rangle &= e^{2\pi i q_2 n_2 \text{Link}(\mathcal{C}, \mathcal{S})/q} \langle V(q_2, \mathcal{S}) \rangle, \\ \langle U_3(e^{2\pi i n_3/k}, (\mathcal{P}, \mathcal{P}')) D(q_3, \mathcal{V}) \rangle &= e^{2\pi i q_3 n_3 \text{Link}((\mathcal{P}, \mathcal{P}'), \mathcal{V})/k} \langle D(q_3, \mathcal{V}) \rangle. \end{aligned} \quad (A.2)$$

Appendix B. Derivations of correlation functions

Here, we briefly explain derivations of the correlation functions. The derivations are based on the reparametrizations of dynamical fields in the path-integral formalism, which can be understood as finite versions of the Ward-Takahashi identity or Schwinger-Dyson equation.

As an example, we explain the derivation of Eq. (5). In the path-integral formalism, the correlation function can be written as

$$\begin{aligned} \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) U_2(e^{2\pi i n_2/q}, \mathcal{C}) \rangle &= \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{iS_{\text{TAE}}} U_1(e^{2\pi i n_1/p}, \mathcal{S}) U_2(e^{2\pi i n_2/q}, \mathcal{C}), \end{aligned} \quad (B.1)$$

where $\mathcal{N}$ is the normalization factor such that $\langle 1 \rangle = 1$. We can absorb the symmetry generator U_1 into the action by the reparametrization of a as follows. We take a 3-dimensional subspace $\mathcal{V}_S$ whose boundary is $\mathcal{S} = \partial \mathcal{V}_S$, and express the integral in the symmetry generator by the Stokes theorem as

$$\int_{\mathcal{S}} \left(\frac{q}{2\pi} b + \frac{N}{4\pi^2} \phi da \right) = \int_{\mathcal{V}_S} \left(\frac{q}{2\pi} db + \frac{N}{4\pi^2} d\phi \wedge da \right). \quad (B.2)$$

By using this relation, one can absorb the symmetry generator to the action,

$$\begin{aligned} e^{iS_{\text{TAE}}[\phi, a, b, c]} e^{-2\pi i \frac{n_1}{p} \int_{\mathcal{V}_S} \left(\frac{q}{2\pi} db + \frac{N}{4\pi^2} d\phi \wedge da \right)} &= e^{iS_{\text{TAE}}[\phi, a + \frac{2\pi n_1}{p} \delta_1(\mathcal{V}_S), b, c]}. \end{aligned} \quad (B.3)$$

By the reparametrization $a + (2\pi n_1/p) \delta_1(\mathcal{V}_S) \rightarrow a$ in the path integral, we obtain

$$\begin{aligned} \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) U_2(e^{2\pi i n_2/q}, \mathcal{C}) \rangle &= e^{-\frac{2\pi i n_1 n_2}{p} \text{Link}(\mathcal{S}, \mathcal{C})} \langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) \rangle. \end{aligned} \quad (B.4)$$

Here, $\text{Link}(\mathcal{S}, \mathcal{C}) = \int_{\mathcal{V}_S} \delta_3(\mathcal{C})$ is the intersection number of $\mathcal{V}_S$ and $\mathcal{C}$, which is equal to the linking number of $\mathcal{S}$ and $\mathcal{C}$.

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