

GALOIS CONJUGATES FOR SOME FAMILY OF GENERALIZED BETA-MAPS

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ABSTRACT. A real number $\beta > 1$ is called an Yrrap (or Ito-Sadahiro) number if the corresponding negative β -transformation defined by $x \mapsto 1 - \{\beta x\}$ for $x \in [0, 1]$, where $\{y\}$ denotes the fraction part of $y \in \mathbb{R}$, has a finite orbit at 1. Yrrap numbers are an analogy of Parry numbers for positive β -transformations given by $x \mapsto \{\beta x\}$ for $x \in [0, 1]$, $\beta > 1$. In this paper, we determine the closure of the set of Galois conjugates of Yrrap numbers. In addition, we show an analogy of the result to the family of piecewise linear continuous maps each of which is obtained by changing the odd-numbered branches (left-most one is regarded as 0-th) of the β -transformation to negative ones for $\beta > 1$. As an application, we see that both the set of Yrrap numbers which are non-Parry numbers and that of Parry numbers which are non-Yrrap numbers are countable.

1. INTRODUCTION

Let β be a real number greater than 1. The so-called β -transformation $T_\beta : [0, 1] \rightarrow [0, 1]$ is defined by

$$T_\beta(x) = \{\beta x\}$$

for $x \in [0, 1]$, where $\{y\}$ denotes the fraction part of $y \in \mathbb{R}$. In case the map has a finite orbit at 1, it is easily seen that β is an algebraic integer, which relates the dynamical properties of the map to the algebraic properties of β . In that case, β is called a Parry number and such numbers have been studied from viewpoints of number theory and ergodic theory ([5, 8, 9, 10, 12]). One of the natural questions for Parry numbers is to ask which algebraic integers are actually Parry numbers. It is well-known that Pisot numbers are Parry numbers (see [11]), although it has not been known if the same is true for all Salem numbers (see [1, 2]). As a direction to provide non-Parry numbers, Solomyak [12] determined the closure of the set of Galois conjugates of Parry numbers using zeros of power series whose coefficients are in $[0, 1]$, which yields that real algebraic integers greater than 1 whose Galois conjugates do not belong to that set are non-Parry numbers. In [15], Thompson extended the result to the case of the family of all generalized β -maps in the sense of Góra [4], each of whose element is obtained by changing some branches of the β -transformation by negative ones.

In this paper, we investigate two subfamilies of generalized β -maps in the sense of Góra. One is the family of negative β -transformations defined by $x \mapsto 1 - \{\beta x\}$, $\beta > 1$, $x \in [0, 1]$. The other is the family of piecewise linear continuous maps each of which is obtained by changing the odd-numbered branches (left-most one is regarded 0-th) of the β -transformation to negative ones for $\beta > 1$. The first main result in this paper is an analogy of the result by Solomyak to the family of negative β -transformations. We recall that $\beta > 1$ is called an Yrrap number if

the corresponding negative β -transformation has a finite orbit at 1. In Theorem 2.3, we determine the closure of the set of Galois conjugates of all Yrrap numbers, which gives the answer to an open question in [7]. Together with the result by Solomyak [12], our result yields that the closure of the set of Galois conjugates of Parry numbers and that of Yrrap numbers are symmetric with respect to the imaginary axis in the complex plane. As its application, we see that the modulus of Galois conjugates of Yrrap numbers is less than or equal to the golden ratio (Corollary 2.4). In addition, we show that the set of Yrrap (resp. Parry) numbers which are non-Parry (resp. non-Yrrap) numbers is countable in Theorem 2.7.

The second main result is also an analogy of the result by Solomyak to the other family of piecewise linear continuous maps stated above (Theorem 2.5). Comparing with the result for all generalized β -maps by Thompson [15], we see that the closure of the set of Galois conjugates corresponding to that family coincides with that to the family of all generalized β -maps, although the family of maps we consider is a special class of generalized β -maps.

This paper is organized as follows. In Section 2, we introduce basic notions used throughout this paper and state the main results precisely. Section 3 is devoted to giving all the proofs of the main results.

2. PRELIMINARIES AND MAIN RESULTS

In the following, we denote by \mathbb{N} the set of all non-negative integers and assume that $\beta > 1$ is a non-integer.

Let $E = (E(0), E(1), \dots) \in \{0, 1\}^{\mathbb{N}}$ be an infinite 0-1 sequence. Set $I_\beta(x) = [\beta x]$ for $x \in [0, 1]$, where $[y]$ denotes the integer part of $y \in \mathbb{R}$. We define the piecewise linear map on the unit interval $\tau_{\beta, E} : [0, 1] \rightarrow [0, 1]$ by

$$\tau_{\beta, E}(x) = E(I_\beta(x)) + (-1)^{E(I_\beta(x))} T_\beta(x)$$

for $x \in [0, 1]$. Note that the definition of the map $\tau_{\beta, E}$ is slightly different from the original one in that a 0-1 sequence in [4] is defined as a finite sequence with $([\beta] + 1)$ -terms. In our definition, actually, the map $\tau_{\beta, E}$ depends only on the first $([\beta] + 1)$ -terms in E for $\beta > 1$. The reason why we define a 0-1 sequence as an infinite sequence is to consider a family of generalized β -maps $\{\tau_{\beta, E}\}_{\beta > 1}$, in which E is defined independently of the integer part of $\beta > 1$.

As in the case of β -transformations, we can consider an expansion of numbers in $[0, 1]$ using the map $\tau_{\beta, E}$ as follows. For $x \in [0, 1]$ and every non-negative integer $n \geq 0$, we define the digit function $d_n(\beta, E, x)$ and the sign function $e_n(\beta, E, x)$ by

$$\begin{aligned} d_n(\beta, E, x) &= E(I_\beta(\tau_{\beta, E}^n(x))) + I_\beta(\tau_{\beta, E}^n(x)), \\ e_n(\beta, E, x) &= (-1)^{E(I_\beta(\tau_{\beta, E}^n(x)))}, \end{aligned}$$

and the cumulative sign function $s_n(\beta, E, x)$ by

$$s_n(\beta, E, x) = \begin{cases} 1, & n = 0, \\ \prod_{i=0}^{n-1} e_i(\beta, E, x), & n \geq 1. \end{cases}$$

By definition we can represent $x \in [0, 1]$ by

$$\begin{aligned} x &= \frac{s_0(\beta, E, x)d_0(\beta, E, x)}{\beta} + \frac{s_1(\beta, E, x)\tau_{\beta, E}(x)}{\beta} \\ &= \frac{s_0(\beta, E, x)d_0(\beta, E, x)}{\beta} + \frac{s_1(\beta, E, x)d_1(\beta, E, x)}{\beta^2} + \frac{s_2(\beta, E, x)\tau_{\beta, E}^2(x)}{\beta^2} \\ &= \dots \\ &= \sum_{i=0}^{n-1} \frac{s_i(\beta, E, x)d_i(\beta, E, x)}{\beta^{i+1}} + \frac{s_n(\beta, E, x)\tau_{\beta, E}^n(x)}{\beta^n} \end{aligned}$$

for every positive integer $n \geq 1$ (see Proposition 1 in [4]). Since $\tau_{\beta, E}^n(x) \in [0, 1]$ and $s_n(\beta, E, x) \in \{-1, 1\}$ for $n \geq 0$, taking $n \rightarrow \infty$ in the right side of the above equation yields

$$x = \sum_{n=0}^{\infty} \frac{s_n(\beta, E, x)d_n(\beta, E, x)}{\beta^{n+1}}.$$

We call the above series the $\tau_{\beta, E}$ -expansion of x and the sequence of integers $\{s_n(\beta, E, x)d_n(\beta, E, x)\}_{n=0}^{\infty}$ the coefficient sequence of the $\tau_{\beta, E}$ -expansion of x . Set the monotone pieces of the map $\tau_{\beta, E}$ as

$$J_i = \begin{cases} [\frac{i}{\beta}, \frac{i+1}{\beta}) & (0 \leq i \leq [\beta] - 1) \\ [\frac{[\beta]}{\beta}, 1] & (i = [\beta]) \end{cases}$$

and denote by EP the set of all end points of J_i 's except 0 and 1, that is, set

$$EP = \left\{ \frac{1}{\beta}, \dots, \frac{[\beta]}{\beta} \right\}.$$

If there exists a positive integer $n \geq 1$ such that $\tau_{\beta, E}^n(x) \in EP$, we have

$$x = \sum_{i=0}^{n-1} \frac{s_i(\beta, E, x)d_i(x)}{\beta^{i+1}} + \frac{s_n(\beta, E, x)n_0}{\beta^{n+1}},$$

where $n_0 = \beta\tau_{\beta, E}^n(x) \in \{1, \dots, [\beta]\}$. In this case, we call the $\tau_{\beta, E}$ -expansion of x finite.

In the case of β -transformations (resp. negative β -transformations), recall that β is called a Parry (resp. Yrrap) number if the set of the orbit of 1 by the corresponding map is a finite set. In particular, β is called simple if the orbit of 1 by the map eventually falls to EP (see [15], [6]). Similarly, for $E \in \{0, 1\}^{\mathbb{N}}$, we call β a generalized Parry number with respect to (w.r.t) E if the set of the orbit of 1 by the map $\tau_{\beta, E}$ is a finite set. In particular, we call β is simple w.r.t E if the orbit of 1 by the corresponding map eventually falls to EP .

Let $I \subset \mathbb{R}$ be a bounded interval and let \mathbb{D} be the open unit disk in \mathbb{C} . Set

$$\mathcal{F}_I = \left\{ 1 + \sum_{n=1}^{\infty} a_n z^n ; a_i \in I \text{ for } i \geq 1 \right\}$$

and

$$\mathcal{G}_I = \{z \in \mathbb{D} ; \exists f \in \mathcal{F}_I \text{ such that } f(z) = 0\}.$$

For an infinite 0-1 sequence $E \in \{0, 1\}^{\mathbb{N}}$, denote by $\Phi(E)$ the closure of the set of Galois conjugates of all simple generalized Parry numbers w.r.t E and by $\hat{\Phi}(E)$ the closure of the set of Galois conjugates of all generalized Parry numbers w.r.t E .

We define the formal power series $\phi_{\beta,E}(z)$ by

$$\phi_{\tau_{\beta,E}}(z) = \sum_{n=0}^{N-1} s_n(\beta, E, 1) d_n(\beta, E, 1) z^{n+1} + k_0 s_N(\beta, E, 1) z^{N+1}$$

if β is simple, where $N = \min\{n \geq 1; \tau_{\beta,E}^n(1) \in EP\}$ and $k_0 = \beta \tau_{\beta,E}^N(1)$, and by

$$\phi_{\tau_{\beta,E}}(z) = \sum_{n=0}^{\infty} s_n(\beta, E, 1) d_n(\beta, E, 1) z^{n+1}$$

if β is non-simple. Note that the convergence radius of this power series is greater than or equal to 1 since the coefficient sequences $\{s_n(\beta, E, 1) d_n(\beta, E, 1)\}_{n=0}^{\infty}$ is bounded. In case that β is a generalized Parry number, the power series $\phi_{\tau_{\beta,E}}$ can be extended to a rational function, and otherwise has the unit circle as its natural boundary by Szegő's theorem, which states that if the coefficients $\{b_n\}_{n=0}^{\infty}$ of the power series $\sum_{n=0}^{\infty} b_n z^n$ line in some finite set in \mathbb{C} , then either it has the unit circle as its natural boundary or b_n is eventually periodic (see [14]).

Set

$$\Phi^*(E) = \text{clos}(\{z^{-1} \in \mathbb{C}; \exists \beta > 1 \text{ such that } 1 - \phi_{\tau_{\beta,E}}(z) = 0\}),$$

where $\text{clos}(A)$ denotes the closure of $A \subset \mathbb{C}$. Since $1 - \phi_{\tau_{\beta,E}}(\beta^{-1}) = 0$ for $\beta > 1$, it is easily seen that $\Phi(E) \subset \hat{\Phi}(E) \subset \Phi^*(E)$.

In the following, we set

$$E_0 = (0, 0, 0, \dots), \quad E_1 = (1, 1, 1, \dots) \text{ and } E_{alt} = (0, 1, 0, 1, 0, 1, 0, \dots).$$

Notice that $\{\tau_{\beta,E_0}\}_{\beta>1}$ is the family of all β -transformations and $\{\tau_{\beta,E_1}\}_{\beta>1}$ is that of all negative β -transformations. For $\beta > 1$ the map $\tau_{\beta,E_{alt}}$ is the continuous piecewise linear map obtained by changing the odd-numbered branches (left-most one is regarded as 0-th) of the β -transformation by negative ones. For a set of complex numbers $A \subset \mathbb{C}$, let $A^{-1} = \{z^{-1} \in \mathbb{C}; z \in A\}$ and $-A = \{-z \in \mathbb{C}; z \in A\}$. In [12], Solomyak determine the closure of the set of all Galois conjugates of (simple) Parry numbers:

Theorem 2.1 (Theorem 2.1 in [12]).

$$\Phi(E_0) = \hat{\Phi}(E_0) = \Phi^*(E_0) = \text{clos}(\mathbb{D}) \cup \mathcal{G}_{[0,1]}^{-1}.$$

As its generalization, Thompson [15] determined the closure of the set of Galois conjugates of all generalized Parry numbers by extending Parry's criteria (Theorem 3 in [8]) to the case of generalized β -maps under some conditions.

Theorem 2.2 (Theorem 5.7 in [15]).

$$\bigcup_{E \in \{0,1\}^{\mathbb{N}}} \Phi(E) = \bigcup_{E \in \{0,1\}^{\mathbb{N}}} \hat{\Phi}(E) = \bigcup_{E \in \{0,1\}^{\mathbb{N}}} \Phi^*(E) = \text{clos}(\mathbb{D}) \cup \mathcal{G}_{[-1,1]}^{-1}.$$

The first main theorem in this paper is an analogy of the result by Solomyak to negative β -transformations:

Theorem 2.3.

$$\Phi(E_1) = \hat{\Phi}(E_1) = \Phi^*(E_1) = \text{clos}(\mathbb{D}) \cup -\mathcal{G}_{[0,1]}^{-1}.$$

Together with Theorem 2.1, we can see that $\Phi(E_1) = -\Phi(E_0)$, which means that these sets are symmetric with respect to the imaginary axis in the complex plane.

In [12] and [3], it was shown that $\mathcal{G}_{[0,1]}$ includes the open disk whose radius is $(\sqrt{5} - 1)/2$ and the value of its radius is best possible. This immediately implies the following corollary:

Corollary 2.4. *If $z \in \mathbb{C}$ is a Galois conjugate of an Yrrap number, then its modulus is less than or equal to the golden ratio $(1 + \sqrt{5})/2$.*

The second main theorem in this paper is also an analogy of the result by Solomyak to the maps $\{\tau_{\beta, E_{alt}}\}_{\beta > 1}$.

Theorem 2.5.

$$\Phi(E_{alt}) = \hat{\Phi}(E_{alt}) = \Phi^*(E_{alt}) = \text{clos}(\mathbb{D}) \cup \mathcal{G}_{[-1,1]}^{-1}.$$

Together with Theorem 2.2, this theorem shows that $\Phi(E_{alt}) = \cup_{E \in \{0,1\}^{\mathbb{N}}} \Phi(E)$, although E_{alt} is just an element in $\{0,1\}^{\mathbb{N}}$.

In [4], Góra defined a generalization of Chebyshev maps $F_{\beta} : [-1, 1] \rightarrow [-1, 1]$ by

$$F_{\beta}(x) = \cos(\beta \arccos x)$$

for $x \in [-1, 1]$, where $\beta > 1$. If β is a positive integer, the map is the well-known Chebyshev polynomial of β -th order. He showed that it is topologically conjugate to the map $\tau_{\beta, E_{alt}}$ (see Proposition 17 in [4]) and the T_{β} -invariant density is expressed as a function associated with the orbit $\{F_{\beta}^n(-1)\}_{n=0}^{\infty}$. As an application of Theorem 2.5, we have the following result:

Corollary 2.6. *The set of Galois conjugates of β 's such that the orbit of -1 by F_{β} is a finite set is equal to $\text{clos}(\mathbb{D}) \cup \mathcal{G}_{[-1,1]}^{-1}$.*

Finally, as a relevant work, we investigate the set of all Parry numbers and that of all Yrrap numbers. In [6], Liao and Steiner gave two examples of β 's such that one is a Parry number but a non-Yrrap number and the other is a Parry number but a non-Yrrap number (see Proposition 6.1 and 6.2 in [6]). As an application of Theorem 2.1 and 2.3, we construct countably many such β 's.

Theorem 2.7. *Let P be the set of all Parry numbers and let Y be the set of all Yrrap numbers. Then $P \cap Y^c$ and $P^c \cap Y$ are countable sets.*

3. PROOFS OF MAIN RESULTS

This section is devoted to giving the proofs of the main results stated in the previous section. First, we provide the proof of Theorem 2.3 in the same analogy of the result for β -transformations by Solomyak via the following two lemmas. The first lemma, proved by Thompson in [15], provides a sufficient condition for β to be a simple generalized Parry number.

Lemma 3.1 (Theorem 5.5 in [15]). *For a positive integer $N \geq 2$, suppose that $M(0), \dots, M(N)$ are distinct non-zero integers such that $|M(i)| + 1 < M(0)$ for all $i \geq 1$ and $|M(j)| \neq |M(k)| - 1$ for j, k with $j \neq k$. Then the equation*

$$1 = \frac{M(0)}{x} + \dots + \frac{M(N)}{x^{N+1}}$$

has a solution $\beta > 1$ which is a simple generalized Parry number.

Remark 3.2. Note that the original theorem in [15] states that the solution β above is only a generalized Parry number. By Lemma 5.4 in [15], this solution is in fact a simple generalized Parry number, i.e., $\tau_{\beta,E}^N(1) \in EP$. Furthermore, by Lemma 5.3 in [15], we know that $|M(n)| = d_n(\beta, E, 1)$ and $\text{sign}(M(n)) = s_n(\beta, E, 1)$ for $0 \leq n \leq N$, where $\text{sign}(x)$ denotes the function defined by $\text{sign}(x) = 1$ if $x > 0$ and $\text{sign}(x) = -1$ if $x < 0$ for $x \in \mathbb{R}$.

The next lemma relates the generating function for the coefficient sequence of the $\tau_{\beta,E}$ -expansion of 1 to that for the orbit $\{\tau_{\beta,E}^n(1)\}_{n=0}^\infty$.

Lemma 3.3 (Proposition 4.1 in [13]). *For $z \in \mathbb{C}$ with $|z| < 1$, we have*

$$1 - \sum_{n=0}^N s_n(\beta, E, 1) d_n(\beta, E, 1) z^{n+1} = (1 - \beta z) \sum_{n=0}^N s_n(\beta, E, 1) \tau_{\beta,E}^n(1) z^n,$$

where N is the minimal number such that $\tau_{\beta,E}^N \in EP$, regarded as $+\infty$ if no such N exists.

Proof of Theorem 2.3 We first see that $\Phi^*(E_1) \setminus \text{clos}(\mathbb{D}) \subset -\mathcal{G}_{[0,1]}^{-1}$. By Lemma 3.3, we have that

$$1 - \phi_{\beta,E_1}(z) = (1 - \beta z) \left(1 + \sum_{n=1}^N (-1)^n \tau_{\beta,E_1}^n(1) z^n \right).$$

By the definition of $\Phi^*(E_1)$, if $z_0 \in \Phi^*(E_1) \setminus \text{clos}(\mathbb{D})$, then z_0^{-1} is a convergent of zeros of $1 - \phi_{\beta,E_1}(z)$ for $\beta > 1$ except $1/\beta$. This ensures that z_0^{-1} is also a convergent of zeros of $1 + \sum_{n=1}^N \tau_{\beta,E_1}^n(1)(-z)^n$, which means that $-z_0^{-1} \in \mathcal{G}_{[0,1]}$, together with $\tau_{\beta,E_1}^i(1) \in [0, 1]$ for $i \geq 1$. That is, we have $z_0 \in -\mathcal{G}_{[0,1]}^{-1}$.

We next see that $-\mathcal{G}_{[0,1]}^{-1} \subset \Phi(E_1) \setminus \text{clos}(\mathbb{D})$. Take a polynomial $f(z)$ of the form $f(z) = 1 + \sum_{n=1}^N a_n z^n$, where $N \geq 1$ and $a_n \in (0, 1)$ for $1 \leq n \leq N$ with $a_i \neq a_j$ if $i \neq j$. Note that it is sufficient to show that for every positive number $\varepsilon > 0$, there exists a simple $\beta > 1$ such that

$$(3.1) \quad |\tau_{\beta,E_1}^n(1) - a_n| < \varepsilon$$

for $1 \leq n \leq N$ and

$$(3.2) \quad \begin{aligned} x^N - d_0(\beta, E_1)x^{N-1} + \cdots + (-1)^{N-1}d_N(\beta, 1) \\ = z^N(1 - \beta/z) \left(1 + \sum_{n=1}^N \tau_{\beta,E_1}^n(1) \left(\frac{-1}{z} \right)^n \right) \end{aligned}$$

is an irreducible polynomial. In fact, the irreducibility of the polynomial (3.2) ensures that the reciprocal of every zero of $1 + \sum_{n=1}^N \tau_{\beta,E_1}^n(1)(-z)^n$ is a Galois conjugate of β . Furthermore, the inequality (3.1) ensures that any coefficient a_n of the power series $f(z)$ can be approximated by $\tau_{\beta,E_1}^n(1)$ for simple β , which implies that there is a sequence $\{\beta_m\}_{m=1}^\infty$ of simple numbers such that $1 + \sum_{n=1}^N \tau_{\beta_m,E_1}^n(1)z^n$ converges to $f(z)$ uniformly on compact subsets of \mathbb{D} as $m \rightarrow \infty$. By Hurwitz's theorem, we have that every zero of $f(z)$ in \mathbb{D} is obtained as the limit of zeros of

$1 + \sum_{n=1}^N \tau_{\beta_m, E_1}(1)z^n$ as $m \rightarrow \infty$. Since the set of all polynomials of the form $f(z)$ is dense in the set $\mathcal{F}_{[0,1]}$ with uniform topology, we get the conclusion.

Set $a_0 = 0$, $a_{N+1} = 1$ and $\delta = \min_{0 \leq i, j \leq N+1} |a_i - a_j|$. For a positive integer $M \geq 1$, we can see that $I_\beta(a_n) \in \{I_M(a_n), I_M(a_n) + 1\}$ for $\beta \in (M, M+1)$ and $1 \leq n \leq N$. Take an odd integer M so large as $M > \max\{7/\delta, 5/\varepsilon\}$. Let $D(0) = M+1$. For $1 \leq n \leq N-1$, set $D(n) = I_M(a_n)$ if $I_M(a_n)$ is even and $D(n) = I_M(a_n) + 1$ if $I_M(a_n)$ is odd. Note that each $D(n)$ is an even integer with $0 \leq D(n) - I_M(a_n) \leq 1$. Define the even integer $D(N)$ by $D(N) = I_M(a_N) + k$, where

$$k = \begin{cases} 2, & (I_M(a_N) \equiv 0 \pmod{4}) \\ 1, & (I_M(a_N) \equiv 1 \pmod{4}) \\ 0, & (I_M(a_N) \equiv 2 \pmod{4}) \\ -1, & (I_M(a_N) \equiv 3 \pmod{4}) \end{cases}$$

By definition, $D(N) \equiv 2 \pmod{4}$ and $|D(N) - I_M(a_N)| \leq 2$. Note that by $M > 7/\delta$, we have

$$|D(i) - D(j)| \geq |I_M(a_i) - I_M(a_j)| - 3 \geq M|a_i - a_j| - 4 \geq 3$$

for $0 \leq i, j \leq N$ with $i \neq j$. By Lemma 3.1 and 3.2, the equation

$$1 = \frac{D(0)}{x} - \frac{D(1)}{x^2} + \cdots + (-1)^N \frac{D(N)}{x^{N+1}}$$

has a unique positive solution $\beta_0 \in (M, M+1)$, which is a simple Yrrap number with

$$d_n(\beta_0, E_1, 1) = D(n)$$

for $0 \leq n \leq N$.

Since every $D(n)$ is even for $0 \leq n \leq N$ and $D(N) \equiv 2 \pmod{4}$, the polynomial (3.2) for β_0 is irreducible by Eisenstein's criterion. By definition,

$$\frac{I_{\beta_0}(\tau_{\beta_0, E_1}^n(1))}{\beta_0} \leq \tau_{\beta_0, E_1}^n(1) < \frac{I_{\beta_0}(\tau_{\beta_0, E_1}^n(1)) + 1}{\beta_0}$$

and

$$\frac{I_{\beta_0}(a_n)}{\beta_0} \leq a_n < \frac{I_{\beta_0}(a_n) + 1}{\beta_0}$$

for $1 \leq n \leq N$. This shows

$$|\tau_{\beta_0, E_1}^n(1) - a_n| \leq \frac{|I_{\beta_0}(\tau_{\beta_0, E_1}^n(1)) - I_{\beta_0}(a_n)| + 1}{\beta_0}.$$

Since $D(n) = d_n(\beta_0, E_1, 1) = I_{\beta_0}(\tau_{\beta_0, E_1}^n(1)) + 1$, $|D(n) - I_M(a_n)| \leq 2$ and $I_{\beta_0}(a_n) \in \{I_M(a_n), I_M(a_n) + 1\}$ for $0 \leq n \leq N$, we have

$$|I_{\beta_0}(\tau_{\beta_0, E_1}^n(1)) - I_{\beta_0}(a_n)| \leq |D(n) - I_M(a_n)| + 2 \leq 4,$$

which yields

$$|\tau_{\beta_0, E_1}^n(1) - a_n| \leq \frac{5}{\beta_0} < \frac{5}{M} < \varepsilon,$$

as desired.

Finally, we see that $\text{clos}(\mathbb{D}) \subset \Phi(E_1)$. Note that each zero of a polynomial of the form $f_r(z) = 1 + z/r + \cdots + z^N/r^N$, where $N \geq 2$ and $r \in (1, +\infty)$ can be approximated by Galois conjugates of elements in $\Phi(E_1)$ as in the above argument. Since the set of all zeros of $f_r(z)$ is equal to the set $\{re^{2\pi i n/N}\}_{n=0}^{N-1}$, we can conclude

that $\text{clos}(\mathbb{D}) \subset \Phi(E_1)$ by the fact that the set $\cup_{r>1} \cup_{N \geq 1} \{re^{2\pi i n/N}\}_{n=0}^{N-1}$ is dense in $\{z \in \mathbb{C} ; |z| \geq 1\}$, which finishes the proof. \square

Remark 3.4. In the second part of the above proof, we take a simple Yrrap number satisfying (3.1) and (3.2) so that its integer part is odd. Then the proof in fact yields the stronger result: Denoting by $\Phi_{\text{odd}}(E_1)$ the closure of the set of all Galois conjugates of simple Yrrap numbers each of whose integer part is odd, we have $\Phi_{\text{odd}}(E_1) = \text{clos}(\mathbb{D}) \cup -\mathcal{G}_{[0,1]}^{-1}$.

Next we show Theorem 2.5 with some modifications of the above proof.

Proof of Theorem 2.5 As in the first part of the proof of Theorem 2.3, the equation by Lemma 3.3

$$1 - \phi_{\beta, E_{\text{alt}}}(z) = (1 - \beta z) \left(1 + \sum_{n=1}^N s_n(\beta, E_{\text{alt}}, 1) \tau_{\beta, E_{\text{alt}}}^n(1) z^n \right),$$

where N is the minimal number with $\tau_{\beta, E_{\text{alt}}}^N(1) \in EP$ regarded as $+\infty$ if no such N exists, yields that $\Phi^*(E_{\text{alt}}) \setminus \text{clos}(\mathbb{D}) \subset \mathcal{G}_{[-1,1]}^{-1}$.

We shall show that $\mathcal{G}_{[-1,1]}^{-1} \subset \Phi(E_{\text{alt}}) \setminus \text{clos}(\mathbb{D})$. Let $f(z)$ be a polynomial of the form

$$f(z) = 1 + \sum_{n=1}^N c_n a_n z^n,$$

where $N \geq 2$, $c_n \in \{-1, 1\}$ and $a_n \in (0, 1)$ for $1 \leq n \leq N$ with $a_i \neq a_j$ for $1 \leq i < j \leq N$. As in the same reason for the proof of Theorem 2.3, it is sufficient to show that for every positive number $\varepsilon > 0$, there exists a simple $\beta_0 > 1$ such that

$$(3.3) \quad s_n(\beta_0, E_{\text{alt}}, 1) = c_n, \quad |\tau_{\beta_0, E_1}^n(1) - a_n| < \varepsilon$$

for $1 \leq n \leq N$ and

$$(3.4) \quad x^N - s_0(\beta_0, E_{\text{alt}}, 1) d_0(\beta_0, E_{\text{alt}}) x^{N-1} - \cdots - s_N(\beta_0, E_{\text{alt}}, 1) d_N(\beta_0, E_{\text{alt}}, 1) \\ = x^N (1 - \beta/x) \left(1 + \sum_{n=1}^N s_n(\beta_0, E_{\text{alt}}, 1) \tau_{\beta_0, E_{\text{alt}}}^n(1) (1/x)^n \right)$$

is an irreducible polynomial. Set $a_0 = 0$, $a_{N+1} = 1$ and $\delta = \min_{0 \leq i, j \leq N+1} |a_i - a_j|$. Take an even integer M so large as $M > \max\{7/\delta, 5/\varepsilon\}$. Define the sequence of even integers $\{D(n)\}_{n=0}^N$ by $D(0) = M$,

$$D(n) = \begin{cases} I_M(a_n) & \text{if } I_M(a_n) \text{ is even,} \\ I_M(a_n) + 1 & \text{if } I_M(a_n) \text{ is odd,} \end{cases}$$

for $1 \leq n \leq N-1$ and $D(N)$ as the even positive integer with $D(N) \equiv 2 \pmod{4}$ with $|D(N) - I_M(a_N)| \leq 2$. Then we can see in the same way of the proof of Theorem 2.3 that the sequence $\{c_n D(n)\}_{n=0}^N$ satisfies the assumptions of Lemma 3.1. Hence there are a simple $\beta_0 > 1$ and some 0-1 sequence $E' \in \{0, 1\}^{\mathbb{N}}$ such that the inequality (3.3) holds for $1 \leq n \leq N$ and the polynomial (3) for (β, E') is irreducible with $d_n(\beta_0, E', 1) = D(n)$ and $s_n(\beta_0, E', 1) = c_n$ for $1 \leq n \leq N$.

In fact, we can replace E' by E_{alt} from the following argument. Since every $D(n)$ is even and $e_n(\beta_0, E', 1) = s_n(\beta_0, E', 1)s_{n+1}(\beta_0, E', 1) = c_n c_{n+1}$ for $1 \leq n \leq N$ by definition, we have

$$I_\beta(\tau_{\beta_0, E'}^n(1)) = D(n) \text{ is an even integer if } e_n(\beta_0, E', 1) = +1$$

and

$$I_\beta(\tau_{\beta_0, E'}^n(1)) = D(n) - 1 \text{ is an odd integer if } e_n(\beta_0, E', 1) = -1$$

for $0 \leq n \leq N$. This means that $E(\tau_{\beta_0, E'}^n(1)) = 0$ if $I_\beta(\tau_{\beta_0, E'}^n(1))$ is even and $E(\tau_{\beta_0, E'}^n(1)) = 1$ if $I_\beta(\tau_{\beta_0, E'}^n(1))$ is odd. Therefore, we conclude that $\tau_{\beta_0, E'}^n(1) = \tau_{\beta_0, E_{alt}}^n(1)$ and $s_n(\beta_0, E', 1) = s_n(\beta_0, E_{alt}, 1)$ for $1 \leq n \leq N$, which allows us to replace E' by E_{alt} .

We can show $\text{clos}(\mathbb{D}) \subset \Phi(E_{alt})$ by the same argument as the last part of the proof of Theorem 2.3. \square

Finally, we prove Theorem 2.7 as an application of Theorem 2.3.

Proof of Theorem 2.7 Let $f \in \mathcal{F}_{[0,1]}$. By the definition of $\mathcal{F}_{[0,1]}$, the power series f has the form $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, where $a_n \in [0, 1]$ for $n \geq 1$. Since f has the convergence radius greater than or equal to 1 and $a_n \in [0, 1]$ for $n \geq 1$, we have that $f(t) = 1 + \sum_{n=1}^{\infty} a_n t^n \geq 1 > 0$ for $0 < t < 1$, which means that $\mathcal{G}_{[0,1]} \cap (0, +\infty) = \emptyset$. Together with Theorem 2.1, we have that every Parry number has no positive Galois conjugate greater than 1. In addition, we note that the set $\mathcal{G}_{[0,1]}$ has a ‘spike’ on the negative real axis, which means that $\mathcal{G}_{[0,1]} \cap (-1, 0] = (-1, -(\sqrt{5} - 1)/2]$ and there is $0 < C < (\sqrt{5} - 1)/2$ such that every $z \in \mathcal{G}_{[0,1]} \setminus (-1, 0]$ satisfies $|z| < C$ (see Corollary 2.3 and Lemma 4.1 in [12]). Since $\Phi(E_1) \supset -\mathcal{G}_{[0,1]}^{-1}$ by Theorem 2.3, there are countably many distinct Yrrap numbers one of whose Galois conjugates lies in $(1, (\sqrt{5} + 1)/2)$. Together with $\mathcal{G}_{[0,1]} \cap (0, \infty) = \emptyset$, we have that they are non-Parry numbers, ensuring the set $P^c \cap Y$ is countable. Similarly, $\Phi(E_0) \supset \mathcal{G}_{[0,1]}^{-1}$ by Theorem 2.1, there are countably many distinct Parry numbers one of whose Galois conjugates lies in $(-(\sqrt{5} + 1)/2, -1)$. Together with $-\mathcal{G}_{[0,1]} \cap (-\infty, 0) = \emptyset$, we have that they are non-Yrrap numbers, ensuring the set $P \cap Y^c$ is countable. \square

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