van der Corput inequality for real line and Wiener-Wintner theorem for amenable groups.

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Abstract. We extend the classical van der Corput inequality to the real line. As a consequence, we obtain a simple proof of the Wiener-Wintner theorem for the $\mathbb{R}$-action which assert that for any family of maps $(T_t)_{t\in\mathbb{R}}$ acting on the Lebesgue measure space $(\Omega, \mathcal{A}, \mu)$ where μ is a probability measure and for any $t \in \mathbb{R}$, T_t is measure-preserving transformation on measure space $(\Omega, \mathcal{A}, \mu)$ with $T_t \circ T_s = T_{t+s}$, for any $t, s \in \mathbb{R}$. Then, for any $f \in L^1(\mu)$, there is a a single null set off which $\displaystyle \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(T_t\omega)e^{2i\pi\theta t}dt$ exists for all $\theta \in \mathbb{R}$. We further present the joining proof of the amenable group version of Wiener-Wintner theorem due to Weiss and Ornstein.

1 Introduction

In this paper, using our generalization of van der Corput inequality for the real line, we present a simple proof of Wiener-Wintner theorem for the flows. We further present the joining proof of the amenable groups version of it due to Ornstein and Weiss [O-W]. This accomplished by applying the Furstenberg joinings machinery. The classical Wiener-Wintner theorem [W-W] assert the following.

Theorem. *Let* $(\Omega, \mathcal{A}, \mu, T)$ *be a dynamical system where* μ *is a probability measure. Then, for any* f *in* $L^1(\mu)$*, There is a set* Ω' *of full measure such that for any* $\omega \in \Omega'$ *the sums*

$$\frac{1}{N}\sum_{0}^{N-1} f(T^n\omega)z^n$$

converge for all z *in the unit circle* $C = \{z \in \mathbb{C} \ : \ |z| = 1\}$.

The original proof can be found in [W-W]. Subsequently, Furstenberg in [F1] obtain a joining proof of Wiener-Wintner theorem. Later, I. Assani [A], A. Below & V. Losert [B-L] proved the stronger version of this theorem. This stronger version is due to Bourgain [B]. Theirs proofs is based on the Hellinger integral (Known also as affinity principale). In [L1], E. Lesigne generalize Wiener-Wintner theorem to the polynomial case. His proof is based on the Furstenberg's joinings technique. Afterwards, in [L2], using van der Corput inequality and the spectral theory of skew products, he extended the stronger version of polynomial Wiener-Wintner theorem to the case of weak-wixing dynamical systems[1].

In this paper, we extend van der Corput inequality to the continuous time and we give a simple proof of the flow version of Wiener-Wintner theorem. We further present the Ornstein-Weiss's joining of the amenable group version of this fundamental theorem in ergodic theory. The proof is based on Furstenberg's joinings machinery combined with the recent result of E. Lindenstrauss [Li].

The plan of the paper is as follows. In Section 2, we state and prove the continuous van der Corput inequality and the flow version of Wiener-Wintner theorem. In section 2, we state and prove the amenable group version of Wiener-Wintner theorem.

[1]Seven year after this note was written , M. Lacey and E. Terwilleger [L-T] produce an oscillation proof of the Hilbert version of Wiener-Wintner theorem.

2 van der Corput for real line

In this section, we state our first main result.

Theorem 2.1 (Theorem (van der Corput).) *Let $(u(t))_{t\in[0,T]}$ be an integrable complex valued function and let $S \in (0, T]$. Then*

$$\Big|\int_0^T u(t)dt\Big|^2 \leq \frac{S+T}{S^2}\int_0^S\int_0^S\int_0^T u(t+s'-s)\overline{u}(t)dsds'dt. \tag{1}$$

Proof: We start by noticing that we have

$$S\int_0^T u(t)dt = \int_0^{T+S}\int_0^S \tilde{u}(t-s)dsdt, \tag{2}$$

where $\tilde{u}$ stand for

$$\tilde{u}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ u(t) & \text{if } 0 \leq t \leq T, \\ 0 & \text{if not.} \end{cases}$$

Indeed, we have

$$\int_0^{T+S}\int_0^S \tilde{u}(t-s)dsdt = \int_0^S\int_{-s}^{T+S-s}\tilde{u}(t)dtds \tag{3}$$

$$= \int_0^S\int_0^T u(t)dtds \tag{4}$$

$$= S\int_0^T u(t)dt. \tag{5}$$

Whence,

$$S^2\Big|\int_0^T u(t)dt\Big|^2 = \Big|\int_0^{T+S}\int_0^S \tilde{u}(t-s)dsdt\Big|^2. \tag{6}$$

Now, applying Cauchy-Schwarz inequality, we obtain

$$S^2\Big|\int_0^T u(t)dt\Big|^2 \leq (T+S)\Big(\int_0^{T+S}\Big|\int_0^S \tilde{u}(t-s)ds\Big|^2dt\Big). \tag{7}$$

But

$$
\begin{aligned}
\Big|\int_0^S \tilde{u}(t-s)ds\Big|^2 &= \int_0^S \tilde{u}(t-s)\overline{\tilde{u}}(t-s')dsds' &(8)\\
&= \int_0^S \tilde{u}(t-s)\overline{\tilde{u}}(t-s')dsds' &(9)\\
&= \int_0^S \tilde{u}(t+s'-s)\overline{\tilde{u}}(t)dsds' &(10)
\end{aligned}
$$

Whence

$$
\Big|\int_0^T u(t)dt\Big|^2 \leq \frac{S+T}{S^2}\int_0^S\int_0^S\int_0^T u(t+s'-s)\overline{u}(t)dsds'dt. \quad (11)
$$

This achieve the proof of the theorem. ∎

Theorem 2.2 (Limit version of continuous van der Corput theorem.) *Let $(u(t))_{t\in\mathbb{R}}$ be a bounded complex valued function. Then*

$$
\begin{aligned}
&\limsup_{T\to\infty}\Big|\frac{1}{T}\int_0^T u(t)dt\Big|^2\\
&\leq \limsup_{S\to\infty}\frac{1}{S^2}\int_0^S\int_0^S \limsup_{T\to\infty}\frac{1}{T}\int_0^T u(t+s'-s)\overline{u}(t)dsds'dt. \quad (12)
\end{aligned}
$$

Proof: Straightforward from Theorem 2.1. ∎

Now, let us state the continuous version of Wiener-Wintner theorem.

Theorem 2.3 (Continuous version of Wiener-Wintner theorem.) *Let $(T_t)_{t\in\mathbb{R}}$ be a maps acting on the Lebesgue measure space $(\Omega,\mathcal{A},\mu)$ where μ is a probability measure and for any $t\in\mathbb{R}$, T_t is measure-preserving transformation on measure space $(\Omega,\mathcal{A},\mu)$ with $T_t \circ T_s = T_{t+s}$, for any $t,s\in\mathbb{R}$. Then, for any $f\in L^1(\mu)$, there is a a single null set off which $\displaystyle\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(T_t\omega)e^{2i\pi\theta t}dt$ exists for all $\theta\in\mathbb{R}$.*

We will assume without loss of generality that μ ergodic. Indeed, otherwise, on can use the ergodic decomposition of μ. So, it is sufficient to prove the following :

Theorem 2.4 *For any f in $L^2(\mu)$, there is a set Ω' of full measure such that the sums* $\displaystyle\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(T_t\omega)e^{2i\pi\theta t}dt$ *converge to 0 for all θ in $\mathbb{R}$, where $e^{2\pi i\theta}\notin e(T)$ and $\omega\in\Omega'$. $e(T)$ stand for the set of eigenvalue of the Koopman operator $U_T\ :\ g\mapsto g\circ T$.*

Before proceeding to the proof of Theorem 2.4, let us notice that it suffices to prove it for a dense class of functions (L^2 functions for instance). Indeed, Put

$$R(\omega,f)=\limsup_{T\longrightarrow+\infty}\Big|\int_0^T f(T_t(\omega))e^{2\pi it\theta}dt\Big|,$$

and assume that g in the dense class for which theorem holds. Then

$$R(\omega,f)=R(\omega,f-g).$$

and then

$$\mu\{\omega:R(\omega,f-g)>\epsilon\}\leq\frac{||f-g||_1}{\epsilon}.$$

We thus get by the density of $L^2(\mu)$ in $L^1(\mu)$, that there exist g in $L^2(\mu)$ such that : $||f-g||_1<\epsilon^2$. Then

$$\mu\{\omega:R(\omega,f-g)>\epsilon\}\leq\epsilon.$$

Since ϵ is arbitrary, we see $R(\omega,f)=0$ a.e., where the null set excluded is independent of z.

We start by recalling that by Bochner theorem, for any $f\in L^2(X)$, there exists a unique finite Borel measure σ_f on $\mathbb{R}$ such that

$$\widehat{\sigma_f}(t)=\int_{\mathbb{R}}e^{-it\xi}\ d\sigma_f(\xi)=\langle U_tf,f\rangle=\int_\Omega f\circ T_t(\omega)\cdot\overline{f}(\omega)\ d\mu(\omega).$$

σ_f is called the *spectral measure* of f. If f is eigenfunction with eigenfrequency λ then the spectral measure is the Dirac measure at λ.

We need also the following fundamental results.

Theorem 2.5 *Let* $(\Omega, \mathcal{A}, \mu, (T_t)_{t\in\mathbb{R}})$ *be an* ergodic *dynamical flow. Then, for any* $S > 0$ *and all* $f, g \in L^2(X)$*, for almost all* $\omega \in \Omega$*, we have*

$$\lim_{\tau\to+\infty} \frac{1}{\tau} \int_0^\tau f(T_{t+s}\omega) \cdot g(T_t\omega)\, dt = \int_\Omega f \circ T_s \cdot g\, d\mu$$

uniformly for s *in the interval* $[-S, S]$.

This yields the exact result need it.

Corollary 2.6 *Let* $f \in L^2(\mu)$*. There exist a full measure subset* Ω_f *of* Ω *such that, for any* $\omega \in \Omega_f$ *and any* $s \in \mathbb{R}$*, we have*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau f(T_{t+s}\omega) \cdot \overline{f}(T_t\omega)\, dt = \int_X f \circ T_s \cdot \overline{f}\, d\mu.$$

Proof: [**of Theorem** 2.4] Let f in $L^\infty(\mu)$ and $\omega \in \Omega_f$ as in Corollary 2.6, then we have

$$\begin{aligned}\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau f(T_{t+s}\omega) \cdot \overline{f}(T_t\omega)\, dt \;&= \int_X f \circ T_s \cdot \overline{f}\, d\mu &(13)\\ &\stackrel{\text{def}}{=} < f \circ T_s, f > . &(14)\end{aligned}$$

Put

$$u(t) = f(T_t\omega)e^{2\pi i t\theta},$$

and apply further van der Corput's inequality (Theorem 2.1) to get

$$\begin{aligned}&\Big|\tfrac{1}{\tau}\int_0^\tau f(T_t\omega)e^{2\pi i t\theta}dt\Big|^2\\ \le\; &\tfrac{S+\tau}{\tau S^2}\int_0^S\int_0^S e^{2\pi i(s-s')\theta}\frac{1}{\tau}\int_0^\tau f(T_{t+s-s'})\overline{f}(T_t\omega)dtdsds'. \qquad (15)\end{aligned}$$

We thus deduce that for almost all ω and all $\theta \in \mathbb{R}$, we have

$$\begin{aligned}&\limsup_{\tau\to\infty}\Big|\tfrac{1}{\tau}\int_0^\tau f(T_t\omega)e^{2\pi i t\theta}dt\Big|^2\\ \le\; &\tfrac{1}{S^2}\int_0^S\int_0^S e^{2\pi i(s-s')\theta}\Big(\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(T_{t+s-s'})\overline{f}(T_t\omega)dt\Big)dsds'. \qquad (16)\end{aligned}$$

This combined with Corollary 2.6 gives

$$\begin{aligned}&\limsup_{\tau\to\infty}\Big|\tfrac{1}{\tau}\int_0^\tau f(T_t\omega)e^{2\pi i t\theta}dt\Big|^2\\ \le\; &\tfrac{1}{S^2}\int_0^S\int_0^S \Big(\int_{\mathbb{R}} e^{2\pi i(s-s')(\theta-\gamma)}d\sigma_f(s-s')\Big)dsds', \qquad (17)\end{aligned}$$

where σ_f stand for the spectral measure of f. But, since

$$\frac{1}{S^2}\int_0^S\int_0^S e^{2\pi i(s-s')(\theta-\gamma)}dsds' = \Big|\frac{1}{S}\int_0^S e^{2\pi is(\theta-\gamma)}ds\Big|^2 \tag{18}$$

if $\theta \neq \gamma$, we have

$$\lim_{S\to\infty}\frac{1}{S^2}\int_0^S\int_0^S e^{2\pi i(s-s')(\theta-\gamma)}dsds' = 0. \tag{19}$$

Whence, if $e^{2\pi i\theta}$ is not a eigenvalue of (T_t), we have

$$\lim_{S\to\infty}\frac{1}{S^2}\int_0^S\int_0^S \Big(\int_{\mathbb{R}} e^{2\pi i(s-s')(\theta-\gamma)}d\sigma_f(s-s')\Big)dsds' = 0.$$

Since all the sums are bounded, we deduce from Lebesgue theorem that for almost all ω, and for all θ in $\mathbb{R}$, where $e^{2\pi i\theta} \notin e(T)$,

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(T_t\omega)e^{2\pi it\theta}dt = 0,$$

and this finish the proof of the theorem. ■

2. Joining's proof of Wiener-Wintner Theorem for action of amenable group

In this section, we deal with actions on Lebesgue spaces, that is, given a locally compact groupe G and the a Lebesgue space $(X, \mathcal{A}, \mu)$, a $G-$*action* is a measurable mapping $G \times X \to X$, $(g, x) \mapsto g.x$, such that for all $g, h \in G$, $g.(h.x) = (gh).x$ and $e.x = x$ for almost all $x \in X$ (where e is the identity in G). Furthermore, $T_g \ : \ x \mapsto g.x$ is measure -preserving for every $g \in G$. We will mainly concerned with G which is amenable group (locally compact second countable) or the subclass of locally compact abelian groups.
We recall that G is an amenable group if for any compact $K \subset G$ and $\delta > 0$ there is a compact set $F \subset G$ such that

$$\mathrm{h}_L(F\Delta KF) < \delta\mathrm{h}_L(F), \tag{20}$$

where h_L stand for the left Haar measure on G. It is well known that the amenability is equivalent to the existence of Følner sequence (F_n), that is,

(F_n) is a sequence of compact subsets of G for which for every compact K and $\delta > 0$, for all large enough n we have that F_n satisfy (20). Assume further that (F_n) satisfy the so-called Shulman Condition ,that is, for some $C > 0$ and all n

$$\mathrm{h}_L\Big(\bigcup_{k\leq n} F_k^{-1}F_n\Big) \leq C.\mathrm{h}_L\Big(F_n\Big). \tag{21}$$

Under this assumptions, E. Lindenstrauss proved that the Birkhoff pointwise ergodic theorem holds, that is, Then for any $f \in L^1(\mu)$, there is a G-invariant $f^* \in L^1(\mu)$ such that

$$\lim \frac{1}{\mathrm{h}_L(F_n)} \int_{F_n} f(g\omega) d\mathrm{h}_L(g) = f^*(\omega) \text{ a.e..}$$

To formulate the G-version of Wiener-Wintner theorem, we replace the group rotations by homomorphisms Θ from G to a finite dimensional unitary group U_d. The canonical action in this case is given by $g.u = \Theta(g).u$, $u \in U_d$ and $g \in G$. The invariant measure is the Haar measure on U_d. In this setting, we formulate the Wiener-Wintner theorem as follows:

Theorem 2.7 (Group version of Wiener-Wintner theorem) *Let G be an amenable group acting on a Lebesgue space $(\Omega, \mathcal{A}, \mu)$ and assume that G satisfy Shulman condition. Let $f \in L^\infty(\mu)$. Then, there is a set Ω_f of full measure such for any $\omega \in \Omega_f$*

$$\frac{1}{h_L(F_n)} \int_{F_n} f(g\omega)\phi(\Theta(a)u) dh_L(g)$$

converge for all finite dimensional unitary representation Θ of G into U_d (all d), all continuous function ϕ on U_d and all $u \in U_d$. We further have that the limit on the orthocomplement of the space spanned by the finite dimensional invariant subspaces is zero.

Before proceeding to the proof let us recall some important tools.

A *joining* of two actions of the same group $\mathcal{X} = (X, \mathcal{A}, \mu, G)$ and $\mathcal{Y} = (Y, \mathcal{B}, \nu, G)$ is the probability measure λ on $(X \times Y, \mathcal{A} \times \mathcal{B})$ which is invariant under the diagonal action of G $(g.(x, y) = (g.x, g.y))$ and whose marginals on $(\mathcal{A} \times Y)$ and $(X \times \mathcal{B})$ are μ and ν respectively (i.e. if $A \in \mathcal{A}$, $\lambda(A \times Y) = \mu(A)$;

and if $B \in \mathcal{B},\ \lambda(X \times B) = \nu(B))$. The set of joinings is never empty. (Take $\mu \times \nu$). As we deal with Lebesgue spaces, a joining λ of two ergodic G-actions $\mathcal{X}$ and $\mathcal{Y}$ has the property that there exists a Lebesgue space Ω and the probability $\mathbb{P}$ on Ω such that $\lambda = \int \lambda_\omega d\mathbb{P}(\omega)$, where λ_ω is ergodic. (This is just the ergodic decomposition of λ, and as the marginals of λ are ergodic a.e., λ_ω is joining). Therefore the set of ergodic joinings is never empty.[2]

Historically, joinings were introduced by H. Furstenberg in his paper [F2] on disjointness. In particular, he defined the important notion of disjointness for $\mathbb{Z}$-*action* in the following way : $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ is *disjoint* if the only joining between them is the product joining. In the case of *G-action* we have the following definition.

Definition 2.8 *Let $\mathcal{X}$ and $\mathcal{Y}$ be two actions of the same group G. $\mathcal{X}$ and $\mathcal{Y}$ are disjoint if the only joining between them is the product joining. We denote this disjointness by $\mathcal{X} \perp \mathcal{Y}$.*

In the case of $\mathbb{Z}$-actions, H. Hahn & W. Parry obtain in [H-P] that if two transformations have mutually singular maximal types, then they are disjoint. But, As for the joinings theory, the spectral theory of $\mathbb{Z}$-actions can be extended to the case of locally abelien $G-$actions. Therefore, we have the following group version of Hahn-Parry theorem.

Theorem 2.9 ((Hahn & Parry) *If two G-actions $\mathcal{X}$ and $\mathcal{Y}$ have mutually singular maximal spectral types, then they are disjoint.*

Proof: Let recall that the spectral measure of a function $f \in L^2(X)$ under the operators U_g (defined on $L^2(X)$ by $U_g(f) = f \circ T_g$) is the measure σ_f on $\hat{G}$ (dual group of G, i.e., the set of all continuous characters of G) where its Fourier transform $\overset{\wedge}{\sigma_f}$ is given by $\overset{\wedge}{\sigma_f}\ (g) =< U_g f, f)$. Now, we follows the proof given in [Th]. In $X \times Y$ endowed with a joining measure λ, consider $f_1 \in L^2(X)$ and $f_2 \in L^2(Y)$ and consider H_{f_1} the $L^2(\lambda)$ closure of the linear span of the functions $(U_g(f_1) - \int f_1 d\mu) \times 1_Y$, $g \in G$. The projection

[2] see [D-R], for instance.

of $1_X \times f_2$ on H_{f_1} will have a spectral measure absolutely continuous with respect to the spectral type of U_g on $L^2(X)$ and thus has to be 0. Therefore $1_X \times f_2 \perp (f_1 - \int f_1) \times 1_Y$, and $\int f_1(x) f_2(y) d\lambda(x,y) = \int f_1 d\mu \int f_2 d\nu$. ■

From this theorem we have the following.

Corollary 2.10 *Let χ_0 be a non trivial character and define the action of G on torus $\mathbb{T}$ by $(g, e^{ix}) \mapsto \chi_0(g)e^{ix}$. Assume that for any $n \in \mathbb{Z}$, the character χ_0^n define on G by $g \mapsto \chi_0(g^n)$ is not eigenvalue of the G-action on $\mathcal{X}$. The the G-action on $\mathbb{T}$ and the G-action on $\mathcal{X}$ are disjoint.*

Proof: Let recall that χ_0 is a eigenvalue of G- action if there exist a eigenfunction $f \in L^2(X,\mu)$ such that $f \circ T_g = \chi_0(g)\ f$. We deduce that the spectral measure of f is $||f||_2^2 \delta_{\chi_0}$ (δ_{χ_0} is the Dirac measure on χ_0). Since for any $n \in \mathbb{Z}$, χ_0^n is not eigenvalue of G-action on $\mathcal{X}$, we conclude that the maximal spectral types of this two G-actions are mutually singular. Now apply the Hahn-Parry theorem to complete the proof. ■

For the general case of amenable group which satisfy Shulman condition, we have the following lemma from [O-W].

Lemma 2.11 *Let U be the closure of $\Theta(G)$ in U_d. Then, if the product $(U, \Theta, G) \times (\Omega, \mathbf{A}, \mu, \mathbf{G})$ is ergodic then there is only on G-invariant measure on $U \times \Omega$ that projects onto μ on Ω.*

Proof: [**of Theorem 2.7**] We start by assuming without lost of generality that the action on $(\Omega, \mathbf{A}, \mu,)$ is ergodic and by presenting the proof for the case when G is locally abelien group. Let $f \in L^\infty(\mu)$ and ϕ continuous function. Then, by the pointwise theorem there is a set of full measure of ω. Assume that ω is in this subset and let $\chi_0 \in \hat{G}$ such χ_0 is not eigenvalue. Then, the product $(U, \Theta, G) \times (\Omega, \mathbf{A}, \mu, \mathbf{G})$ is ergodic. Moreover, by taking a subsequence (n_k), we can assume that

$$\lim_{k \longrightarrow +\infty} \frac{1}{\mathrm{h}_L(F_n)} \int_{F_{n_k}} f(g\omega)\phi(\Theta(a)u) d\mathrm{h}_L(g) = \lambda(f \otimes \phi).$$

It follows that λ is a joining and by Corollary 2.10 $\lambda = dh \times \mu$. We end the proof by noticing that there is a countable of eigenvalue. The general case follows in the same manner by taking

$$F(\omega) = \int \psi(u) I(u_1 \omega) du,$$

where I is a bounded invariant functions on $U \times \Omega$ and ψ is any positive continuous function on u. Therefore, transforming F by g is the same as transforming ψ by $\Theta(g)$. We thus have that a non-constant I will give rise to finite dimensional invariant subspaces for G on Ω. Moreover, by taking (U, Θ, G) not in in the list of countable representations (U_j, Θ_j, G), the condition of Lemma 2.11 is satisfied and therefore as before the only joining is the product measure, and we are done. ∎

Question 2.12 *We ask on the possible extension of van der Corput inequality to the locally compact group and its application to obtain produce a direct proof of the group version of Wiener-Wintner theorem.*

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