THE GHS AND OTHER CORRELATION INEQUALITIES FOR THE TWO-STAR MODEL

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ABSTRACT. We consider the two-star model, a family of exponential random graphs indexed by two real parameters, h and α, that rule respectively the total number of edges and the mutual dependence between them. Borrowing tools from statistical mechanics, we study different classes of correlation inequalities for edges, that naturally emerge while taking the partial derivatives of the (finite size) free energy. In particular, if $\alpha, h \geq 0$, we derive first and second order correlation inequalities and then prove the so-called GHS inequality. As a consequence, under the above conditions on the parameters, the average edge density turns out to be an increasing and concave function of the parameter h, at any fixed size of the graph. Some of our results can be extended to more general classes of exponential random graphs.

1. INTRODUCTION

Correlation inequalities are an important tool in equilibrium statistical mechanics. They are used to estimate moments and correlations in ferromagnetic systems, allowing in turn to obtain analyticity properties of some physical observables (such as magnetization and susceptibility) and to prove/disprove the presence of a phase transition. Among these inequalities, we find the Griffiths, Hurst and Sherman (GHS) inequality, that rules the three-particle interactions and is mainly known for providing convexity properties of relevant functionals. As the Griffiths, Kelley and Sherman (GKS) inequality [12, 15], it was firstly proved for the classical Ising model, to show that the average magnetization is a concave function of the positive external field [14], and then extended to general classes of even ferromagnets that can be derived out of the Ising model [8, 13, 5, 20].

However, the aforementioned result is only one of the different implications entailed by the GHS inequality. For example, it has been used to characterize possible phase transitions, to prove monotonicity of correlation length, and to derive critical exponent inequalities for the Ising model on $\mathbb{Z}^d$; to obtain monotonicity of mass gap and to estimate coupling constants in φ^4 field theory; or also to show convexity-preserving properties of certain differential equations and diffusion processes. For further details we refer the reader to [9] and references therein.

In the present paper we consider a family of exponential random graphs known as two-star model [22]. Specifically, we consider a Gibbs probability measure on the

set of all simple graphs on n vertices, whose Hamiltonian depends on the densities of edges and two-star graphs. Our goal is to study some correlation inequalities for such a model, with a particular focus on the GHS inequality.

In comparison with ferromagnetic systems, the major difference is that the Gibbs measure of our system, being supported on $\{0,1\}^{\binom{n}{2}}$, does not enjoy $\mathbb{Z}_2$-simmetry. As a consequence, although the positivity of the support of the measure allows to easily deduce positivity of the moments and derive the Fortuin, Kasteleyn and Ginibre (FKG) inequality [10], higher order correlations are non-trivial to analyze, and generally depend on the choice of the parameters.

The manuscript is organized as follows. In Section 2 we introduce the two-star model and we define the corresponding free energy function. Moreover, we briefly recall some recent results about its asymptotic behavior, including the characterization of the phase diagram and some limit theorems for the edge density. Section 3 is devoted to correlation inequalities and it collects our main results. We first provide the formal definition of the aforementioned FKG, GKS and GHS inequalities in the context of a two-star model with generalized parameters (see Eq. (3.1)). In Subsection 3.1 we show that the FKG and GKS inequalities hold for this model whenever $\alpha \geq 0$, and then we derive some preliminary results used afterwards in the proof of the GHS inequality, that is the core of the present work (see Theorem 3.10). The statement of this result, that holds under the additional hypothesis $h \geq 0$, is given in Subsection 3.2 together with its proof. This is mainly based on ideas from [16], where an alternative and simplified strategy of the original proof has been devised. We then bring back the results to the classical two-star model, and make a few comments about some immediate consequences of the derived correlation inequalities. In Section 4 we discuss which of our techniques can be extended to prove the FKG and GKS inequalities for general exponential random graphs and which are the issues in adapting the proofs to obtain the GHS inequality in this setting.

2. Model and background

2.1. **Two-star model.** Let us consider the set $\mathcal{G}_n$ of all simple graphs on n labeled vertices that are identified with the elements of the set $[n] = \{1, 2, 3, \ldots, n\}$. We define a probability distribution on $\mathcal{G}_n$ by means of the *homomorphism densities* of the subgraphs of the graph. Specifically, if $G \in \mathcal{G}_n$ and H is a given simple subgraph, we define

$$(2.1) \qquad t(H, G) := \frac{|\mathrm{hom}(H, G)|}{|V(G)|^{|V(H)|}},$$

i.e. the probability that a random mapping $V(H) \mapsto V(G)$ from the vertex set of H to the vertex set of G is edge-preserving.

For any $k \in \mathbb{N}$, let $H_1, H_2, \ldots, H_k$ be pre-chosen finite simple graphs (edges, stars, triangles, cycles, …) and let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)$ be a collection of real parameters. For any choice of $\boldsymbol{\beta}$, an exponential random graph is identified by the Gibbs probability

density

$$\mu_{n;\boldsymbol{\beta}}(G) = \frac{\exp\left(H_{n,\boldsymbol{\beta}}(G)\right)}{Z_{n;\boldsymbol{\beta}}} \quad \text{for } G \in \mathcal{G}_n. \tag{2.2}$$

The function $H_{n,\boldsymbol{\beta}}$, called Hamiltonian, is given by

$$H_{n;\boldsymbol{\beta}}(G) = n^2 \sum_{j=1}^{k} \beta_j t(H_j, G) \tag{2.3}$$

and the normalizing factor

$$Z_{n;\boldsymbol{\beta}} = \sum_{G \in \mathcal{G}_n} \exp\left(H_{n;\boldsymbol{\beta}}(G)\right) \tag{2.4}$$

is the *partition function*.

In the present setting we focus on the *two-star model*, characterized by a Gibbs measure that depends only on the densities of edges and two-star graphs. Recall that a two-star graph is an undirected graph with one *root* vertex and two other vertices connected with the root, and otherwise disconnected. Under this assumption, the measure can be conveniently expressed as follows.

Let $\mathcal{E}_n$ denote the edge set of the complete graph on n vertices, with elements labeled from 1 to $\binom{n}{2}$. If $i, j \in \mathcal{E}_n$ are neighboring edges, we write $i \sim j$ and we identify the unordered pair $\{i,j\}$ with the resulting two-star graph, that will be called *wedge* $\{i,j\}$ in short. Let $\mathcal{W}_n := \{\{i,j\} : i, j \in \mathcal{E}_n, i \sim j\}$ be the set of wedges of $\mathcal{E}_n$, and set $\mathcal{A}_n := \{0,1\}^{|\mathcal{E}_n|}$, $|\cdot|$ being the cardinality of a set.

Notice that there is a one-to-one correspondence between graphs $G \in \mathcal{G}_n$ and elements $x = (x_i)_{i \in \mathcal{E}_n} \in \mathcal{A}_n$ so that, if G corresponds to x, it holds that

$$t(H_1, G) = \frac{2}{n^2} \sum_{i \in \mathcal{E}_n} x_i \qquad t(H_2, G) = \frac{2}{n^3} \sum_{\{i,j\} \in \mathcal{W}_n} x_i x_j + \frac{2}{n^3} \sum_{i \in \mathcal{E}_n} x_i, \tag{2.5}$$

with H_1 an edge and H_2 a wedge. Hence, we may look at the Hamiltonian of the two-star model as a function on $\mathcal{A}_n$ defined by

$$H_{n;\beta_1,\beta_2}(x) = \frac{2\beta_2}{n} \sum_{\{i,j\} \in \mathcal{W}_n} x_i x_j + 2\left(\beta_1 + \frac{\beta_2}{n}\right) \sum_{i \in \mathcal{E}_n} x_i. \tag{2.6}$$

Notice that this Hamiltonian is asymptotically equivalent (see also [17]) to

$$H_{n;\alpha,h}(x) = \frac{\alpha}{n} \sum_{\{i,j\} \in \mathcal{W}_n} x_i x_j + h \sum_{i \in \mathcal{E}_n} x_i, \tag{2.7}$$

where, for convenience, we have set $h = 2\beta_1$ and $\alpha = 2\beta_2$. In the following, we will focus on the corresponding two-star model, having Gibbs density on $\mathcal{A}_n$ given by

$$\mu_{n;\alpha,h}(x) = \frac{\exp\left(H_{n;\alpha,h}(x)\right)}{Z_{n;\alpha,h}} \quad \text{with} \quad Z_{n;\alpha,h} = \sum_{x \in \mathcal{A}_n} \exp\left(H_{n;\alpha,h}(x)\right). \tag{2.8}$$

Accordingly, we will denote the related measure and expectation by $\mathbb{P}_{n;\alpha,h}$ and $\mathbb{E}_{n;\alpha,h}$, respectively.

2.2. **Free energy.** The *free energy* is a key function in the context of statistical mechanics, as it encodes most of the asymptotic properties of the system. Specifically, the finite and infinite size free energies associated with (2.7) are

$$(2.9) \qquad f_{n;\alpha,h} := \frac{1}{n^2} \ln Z_{n;\alpha,h} \quad \text{and} \quad f_{\alpha,h} := \lim_{n \to +\infty} f_{n;\alpha,h} \,.$$

To understand the important role of the free energy, we first observe that its partial derivatives w.r.t. α and h, respectively give the average edge and wedge densities of the model. More precisely, if we denote by E_n the number of edges of the graph G, and by W_n the number of wedges of G, we get

$$(2.10) \qquad \partial_h f_{n;\alpha,h} = \frac{\mathbb{E}_{n;\alpha,h}(E_n)}{n^2} \quad \text{and} \quad \partial_\alpha f_{n;\alpha,h} = \frac{\mathbb{E}_{n;\alpha,h}(W_n)}{n^3} \,.$$

The characterization of the infinite size free energy, together with its analytical properties, then provides a relevant tool to infer some structural properties of the graph.

As an application of Theorems 4.1 and 6.4 in [6], for any $(\alpha, h) \in \mathbb{R}^2$ we have that

$$(2.11) \qquad f_{\alpha,h} = \sup_{0 \le u \le 1} \left(\frac{\alpha u^2}{2} + \frac{hu}{2} - \frac{1}{2}I(u) \right) = \frac{\alpha(u^*)^2}{2} + \frac{hu^*}{2} - \frac{1}{2}I(u^*),$$

where $I(u) = u \ln u + (1-u)\ln(1-u)$ and $u^* = u^*(\alpha, h)$ is a maximizer that solves the fixed-point equation

$$(2.12) \qquad \frac{e^{2\alpha\,u+h}}{1 + e^{2\alpha\,u+h}} = u.$$

Depending on the parameters, Eq (2.12) can have more than one solution at which the supremum in (2.11) is attained. Having multiplicity of optimizers translates in the possibility of having limiting graphs with very different edge densities.

2.3. **Edge-occurrence probability.** As already observed by Park and Newman for the edge-triangle model (see [21], Eq. (4)), the probability that the edge x_i is present can be also written as the expectation of a function of the Hamiltonian where $x_i = 1$. Explicitly, in our context we obtain

$$(2.13) \qquad \mathbb{E}_{n;\alpha,h}(x_i) = \mathbb{E}_{n;\alpha,h}\left[\left(1 + \exp\left(-\frac{\alpha}{n} \sum_{j \in \mathcal{E}_n : j \sim i} x_j - h \right) \right)^{-1} \right].$$

Since the model enjoys a symmetry in the edge structure, in the sense that each edge in the complete graph has precisely the same neighborhood, the aforementioned expectation turns out to be the same for all i. This leads to

$$(2.14) \qquad \mathbb{E}_{n;\alpha,h}(E_n) = \sum_{i \in \mathcal{E}_n} \mathbb{E}_{n;\alpha,h}(x_i) = \binom{n}{2} \mathbb{E}_{n;\alpha,h}(x_i) \,.$$

Hence, the average edge density corresponds asymptotically to the edge-occurrence probability.

Remark 2.1. *At this point, the following remark is in order. The symmetry in the edge structure is intrinsic to the edge set $\mathcal{E}_n$, and does not depend on the specific exponential random graph taken into account. Hence, the analog of the identity (2.14) holds true for general Hamiltonians of the form (2.3). To our knowledge, this property, which is evident from the interacting particle system perspective, has not been pointed out before.*

Taking into account identity (2.11) and the aforementioned results, it holds that

$$(2.15) \qquad \lim_{n\to\infty} \mathbb{E}_{n;\alpha,h}(x_i) = 2 \lim_{n\to\infty} \partial_h f_{n;\alpha,h} = 2\partial_h f_{\alpha,h} = u^*(\alpha,h)\,.$$

While an explicit expression of the edge-occurrence probability as function of (α, h) is missing even in the infinite size limit, it is easy to verify from (2.13) that $\mathbb{E}_{n;\alpha,h}(x_i) \geq 1/2$ for all $n \in \mathbb{N}$, whenever $\alpha, h \geq 0$. However, simulations suggest that the region of parameters where the average edge density is bigger than $1/2$ is larger, and it also includes negative values of h. For large enough n, this region can be approximately characterized by the analysis of the asymptotic behavior of the model. The study of equations (2.11) and (2.12) leads to the phase diagram that we are going to summarize.

2.4. **Phase diagram.** We collect here the relevant features of the phase diagram of the two-star model, that can be obtained as a special case of some of the results in [23].

The infinite size free energy $f_{\alpha,h}$ is well-defined in $\mathbb{R}^2$. Moreover, it is analytic in the whole plane except for a continuous critical curve

$$\mathcal{M} := \{(\alpha, h) \in (\alpha_c, +\infty) \times (-\infty, h_c) : h = q(\alpha)\}\,,$$

starting at the critical point $(\alpha_c, h_c) = (2, -2)$ and contained in the cone $\alpha > 2$, $h < -2$. In particular, the system undergoes a first order phase transition across the curve and a second order phase transition at the critical point (see [23], Thms. 2.1 & 2.2). The scalar problem (2.11) admits one solution in the uniqueness region $\mathcal{U} := \mathbb{R}^2 \setminus \mathcal{M}$ while it has two solutions along the curve $\mathcal{M}$ (see [23], Prop. 3.2). A qualitative graphical representation of the phase diagram is provided in Fig. 2.1.

An analogous analysis has been performed in a sparse regime in [1], in the directed graph case in [2], and for a mean-field version of the model in [3].

2.5. **Limiting distribution for the edge density.** We summarize some results on the asymptotic behavior of the edge density of the two-star model. By retracing the proofs in [4], we can obtain the following strong law of large numbers and standard central limit theorem:

$$(2.16) \qquad \frac{2E_n}{n^2} \xrightarrow[n\to\infty]{\text{a.s.}} u^*(\alpha,h) \qquad \text{w.r.t. } \mathbb{P}_{n;\alpha,h}, \text{ for } (\alpha,h) \in \mathcal{U}$$

and
$$(2.17)$$
$$\sqrt{2}\,\frac{E_n - \mathbb{E}_{n;\alpha,h}(E_n)}{n} \xrightarrow[n\to\infty]{\text{d}} \mathcal{N}(0, v(\alpha,h)) \qquad \text{w.r.t. } \mathbb{P}_{n;\alpha,h}, \text{ for } (\alpha,h) \in \mathcal{U}\setminus\{(\alpha_c, h_c)\},$$
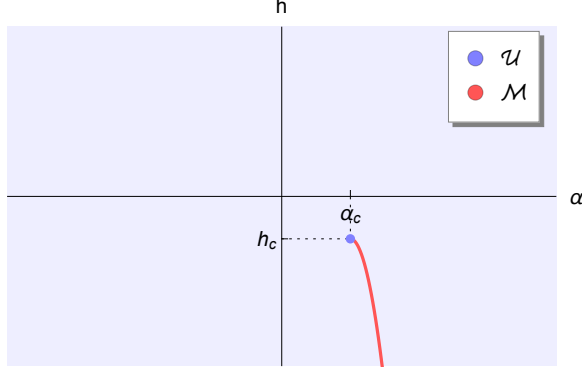
FIGURE 2.1. Phase space (α, h) for the two-star model (2.7). The blue region, that includes the critical point, is the uniqueness region $\mathcal{U}$ for the maximization problem (2.11); whereas, the red curve corresponds to the critical curve $\mathcal{M}$ along which (2.11) admits two solutions.

where $\mathcal{N}(0, v(\alpha, h))$ is a centered Gaussian distribution with variance $v(\alpha, h) := \partial_h u^*(\alpha, h)$, being u^* the unique maximizer of (2.11). A further result can be also given in the multiplicity region; for all $(\alpha, h) \in \mathcal{M}$, it holds

$$\frac{2E_n}{n^2} \xrightarrow[n \to \infty]{\mathrm{d}} \kappa \delta_{u_1^*(\alpha,h)} + (1 - \kappa) \delta_{u_2^*(\alpha,h)} \qquad \text{w.r.t. } \mathbb{P}_{n;\alpha,h},$$

where u_1^*, u_2^* solve the maximization problem in (2.11) and $0 < \kappa < 1$ is a suitable (unknown) constant.

Similar limit theorems are obtained, with different techniques, in [19], where also results on the partial sum of the degrees can be found.

3. CORRELATION INEQUALITIES

In statistical mechanics the study of correlations between particles, so as the analysis of local functions, is often performed with the help of two important inequalities, both related to the sign of the derivatives of the free energy; the **GKS inequality** and the **GHS inequality** (see [11, 14, 15, 16] and references therein for further details). We aim at deriving the analogs of these two inequalities for our reference measure $\mu_{n;\alpha,h}$, given in (2.8).

To understand the connection between the GKS inequality and the sign of the derivatives of the free energy, we introduce a slightly more general setting.

Let $\boldsymbol{\alpha} = (\alpha_{ij})_{i,j \in \mathcal{E}_n}$ and $\boldsymbol{h} = (h_i)_{i \in \mathcal{E}_n}$ be two collections of real numbers (we write $\boldsymbol{\alpha} \geq 0$ (resp. $\boldsymbol{h} \geq 0$) as a shortcut for $\alpha_{ij} \geq 0$ (resp. $h_i \geq 0$) for all $i, j \in \mathcal{E}_n$). For $x \in \mathcal{A}_n$, we define the Hamiltonian

$$(3.1) \qquad H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x) = \frac{1}{n} \sum_{\{i,j\} \in \mathcal{W}_n} \alpha_{ij} x_i x_j + \sum_{i \in \mathcal{E}_n} h_i x_i \,.$$

In analogy with (2.8) and (2.9), we denote by $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ the Gibbs measure obtained from (3.1), by $\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ the corresponding expectation, and we set $f_{n;\boldsymbol{\alpha},\boldsymbol{h}} := \frac{1}{n^2} \ln Z_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ to be the finite size free energy. Observe that we recover the Hamiltonian (2.7) and

the related Gibbs measure $\mu_{n;\alpha,h}$ by setting $\alpha_{ij} \equiv \alpha$, for all $i, j \in \mathcal{E}_n$, and $h_i \equiv h$, for all $i \in \mathcal{E}_n$.

Let $A \subseteq \mathcal{E}_n$ be a given subset of edges. The GKS inequality deals with expectations and covariances of random variables of the type $x_A := \prod_{i \in A} x_i$, with the convention that $x_\emptyset = 1$.

Definition 3.1 (GKS inequality). *The Gibbs measure $\mu_{n;\alpha,h}$ on $\mathcal{A}_n$ satisfies the GKS inequality if, for all $A, B \subseteq \mathcal{E}_n$,*

$$(3.2) \qquad \mathbb{E}_{n;\alpha,h}(x_A x_B) \geq \mathbb{E}_{n;\alpha,h}(x_A) \cdot \mathbb{E}_{n;\alpha,h}(x_B).$$

Remark 3.2. *Notice that, by choosing $A = \{i\}$ and $B = \{j\}$, with $i \neq j$, from the GKS inequality it follows that x_i and x_j are positively correlated under $\mu_{n;\alpha,h}$.*

A useful link between the correlations of the system and the partial derivatives of the free energy w.r.t. the parameters h_i's is provided by the MacLaurin expansion of the log moment generating function of $x \in \mathcal{A}_n$.

The coefficients of this expansion are the so-called *Ursell functions*, that are formally defined, for $\ell \in [n]$ and any choice of $i_1, \ldots, i_\ell \in \mathcal{E}_n$, by

$$(3.3) \qquad u_\ell(i_1, \ldots, i_\ell) := n^2 \partial_{h_{i_1} \ldots h_{i_\ell}} f_{n;\alpha,h}.$$

For instance, this yields

$$(3.4) \qquad u_1(i) = \mathbb{E}_{n;\alpha,h}(x_i),$$

$$(3.5) \quad u_2(i,j) = \mathbb{E}_{n;\alpha,h}(x_i x_j) - \mathbb{E}_{n;\alpha,h}(x_i)\mathbb{E}_{n;\alpha,h}(x_j),$$

$$u_3(i,j,k) = \mathbb{E}_{n;\alpha,h}(x_i x_j x_k) - \mathbb{E}_{n;\alpha,h}(x_i)\,\mathbb{E}_{n;\alpha,h}(x_j x_k) - \mathbb{E}_{n;\alpha,h}(x_j)\,\mathbb{E}_{n;\alpha,h}(x_i x_k)$$

$$(3.6) \qquad - \mathbb{E}_{n;\alpha,h}(x_k)\,\mathbb{E}_{n;\alpha,h}(x_i x_j) + 2\mathbb{E}_{n;\alpha,h}(x_i)\mathbb{E}_{n;\alpha,h}(x_j)\mathbb{E}_{n;\alpha,h}(x_k).$$

Remark 3.3. *Notice that the definition of the Ursell functions (3.3) necessarily passes through the generalized setting with vector parameters α, h, of which they are functions. However, when computed along the constant vectors α and h, with $\alpha_{ij} \equiv \alpha$ for all $i, j \in \mathcal{E}_n$, and $h_i \equiv h$ for all $i \in \mathcal{E}_n$, they are also useful to characterize the derivatives of the classical free energy $f_{n;\alpha,h}$ through the identities*

$$(3.7) \qquad n^2 \partial_{\underbrace{h \ldots h}_{\ell \ times}} f_{n;\alpha,h} = \sum_{i_1, \ldots, i_\ell \in \mathcal{E}_n} u_\ell(i_1, \ldots, i_\ell), \qquad \forall \ell \in [n].$$

While the GKS inequality implies $u_2(i,j) \geq 0$, giving positive correlation between the random variables x_i and x_j, the GHS inequality concerns the sign of the Ursell function $u_3(i,j,k)$.

Definition 3.4 (GHS inequality). *The Gibbs measure $\mu_{n;\alpha,h}$ on $\mathcal{A}_n$ satisfies the GHS inequality if, for all $i, j, k \in \mathcal{E}_n$, $u_3(i,j,k) \leq 0$ or, equivalently, if*

$$(3.8) \quad \mathbb{E}_{n;\alpha,h}(x_i x_j x_k) - \mathbb{E}_{n;\alpha,h}(x_i)\,\mathbb{E}_{n;\alpha,h}(x_j x_k) - \mathbb{E}_{n;\alpha,h}(x_j)\,\mathbb{E}_{n;\alpha,h}(x_i x_k)$$

$$- \mathbb{E}_{n;\alpha,h}(x_k)\,\mathbb{E}_{n;\alpha,h}(x_i x_j) + 2\mathbb{E}_{n;\alpha,h}(x_i)\mathbb{E}_{n;\alpha,h}(x_j)\mathbb{E}_{n;\alpha,h}(x_k) \leq 0.$$

Observe that, in our case, $u_1(i) \geq 0$ trivially, due to the fact that $x_i \in \{0, 1\}$. The rest of the section is devoted to proving $u_2(i,j) \geq 0$ and $u_3(i,j,k) \leq 0$.

3.1. **The FKG and GKS inequalities.** We start with a preliminary result, the FKG inequality, that will help us in deriving the more advanced inequalities (3.2) and (3.8).

We first show that the measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ on $\mathcal{A}_n$ satisfies a proper lattice condition. Recall that $\mathcal{A}_n$ is partially ordered by

$$(3.9) \qquad\qquad x \leq y \quad \text{if} \quad x_i \leq y_i \quad \text{for all } i \in \mathcal{E}_n.$$

Moreover, given two configurations $x, y \in \mathcal{A}_n$, the (pointwise) maximum and minimum configurations are defined as

$$(x \vee y)(i) := \max\{x_i, y_i\} \quad \text{and} \quad (x \wedge y)(i) := \min\{x_i, y_i\},$$

for all $i \in \mathcal{E}_n$. The following property holds true.

Lemma 3.5. *If $\boldsymbol{\alpha} \geq 0$, then the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ fulfills the FKG lattice condition*

$$(3.10) \qquad \mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x \vee y)\, \mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x \wedge y) \geq \mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x)\, \mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}(y) \quad \text{for } x, y \in \mathcal{A}_n.$$

Proof. For a configuration $z \in \mathcal{A}_n$, let $E_z := \{i \in \mathcal{E}_n : z_i = 1\}$, namely the set of edges present in z. We have $E_{x \vee y} = E_x \cup E_y$ and $E_{x \wedge y} = E_x \cap E_y$. Observe that

- the edges in the configuration $x \vee y$ are the edges the configurations x and y have in common, the edges present in configuration x only and those present in configuration y only;
- the edges in the configuration $x \wedge y$ are the edges the configurations x and y have in common;
- the wedges in the configuration $x \vee y$ are the wedges the configurations x and y have in common, the wedges present in configuration x (resp. configuration y) only and the wedges you may create by superimposing the edges of the two configurations;
- the wedges in the configuration $x \wedge y$ are the wedges the configurations x and y have in common.

Therefore, verifying that (3.10) is satisfied reduces to show the validity of the inequality

$$(3.11) \qquad\qquad \exp\left\{ \frac{1}{n} \sum_{\{i,j\} \in E} \alpha_{ij} x_i x_j \right\} \geq 1,$$

where

$$E = \left\{ \{i,j\} : \{i,j\} \subset E_{x \vee y} \text{ is a wedge and } \{i,j\} \begin{bmatrix} \not\subset E_x \\ \not\subset E_y \end{bmatrix} \right\}.$$

The conclusion follows as $\boldsymbol{\alpha} \geq 0$ by assumption. ∎

An immediate consequence of Lemma 3.5 is the positive correlation of increasing random variables. Specifically, if f and g are increasing functions on $\mathcal{A}_n$ (i.e., $f(x) \leq f(y)$ if $x \leq y$), then we obtain the **FKG inequality**

$$(3.12) \qquad\qquad \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(fg) \geq \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(f) \cdot \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(g).$$

Corollary 3.6. *If $\boldsymbol{\alpha} \geq 0$, then the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ satisfies the GKS inequality.*

Proof. Notice that for all $A \subseteq \mathcal{E}_n$, the function $x_A = \prod_{i \in A} x_i$ is increasing in $x \in \mathcal{A}_n$. Hence, by applying the FKG inequality (3.12) to the functions $f(x) = x_A$ and $g(x) = x_B$, we immediately derive (3.2). ∎

Remark 3.7. *A straightforward adaptation of the arguments of Lemma 3.5 applies to general exponential random graphs. We refer the reader to Section 4 for more details.*

We now provide two useful consequences of the GKS inequality. To state properly the results we need to introduce a few more notation; we need a suitable "restriction" of the system on a subset.

For $A \subseteq \mathcal{E}_n$, set $\mathcal{W}_A := \{\{i,j\} : i, j \in A\, ,\, i \sim j\}$ and define the Hamiltonian

$$(3.13) \qquad H_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x) = \frac{1}{n} \sum_{\{i,j\} \in \mathcal{W}_A} \alpha_{ij} x_i x_j + \sum_{i \in A} h_i x_i\,.$$

Let $\mu_{A;\boldsymbol{\alpha},\boldsymbol{h}}$ be the associated Gibbs measure, with normalizing constant $Z_{A;\boldsymbol{\alpha},\boldsymbol{h}}$ (partition function), and let $\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}$ denote the corresponding expectation.

A first consequence of the GKS inequality is a form of monotonicity, with respect to the volume, that can be established for the averages of x_Λ, with $\Lambda \subseteq \mathcal{E}_n$.

Lemma 3.8. *If the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ satisfies the GKS inequality then, for any $\Lambda \subseteq A \subseteq B \subseteq \mathcal{E}_n$,*

$$(3.14) \qquad \mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda) \leq \mathbb{E}_{B;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)\,.$$

Proof. Observe first that, for all $\Lambda \subseteq A \subseteq \mathcal{E}_n$, the function $\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)$ is non-decreasing in $\boldsymbol{\alpha}$. Indeed, by differentiating $\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}$ w.r.t α_{ij}, we get

$$(3.15) \qquad \partial_{\alpha_{ij}} \mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda) = \frac{1}{n} \left(\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda x_i x_j) - \mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j)\right) \geq 0\,,$$

where the last inequality follows from the GKS inequality.

Now let $\mathcal{W}_{A,B} := \{\{i,j\} : i \in A\, ,\, j \in B \setminus A\, ,\, i \sim j\}$ and, for $s \in [0,1]$, consider the Hamiltonian

$$H_{B;\boldsymbol{\alpha}(s),\boldsymbol{h}}(x) := \frac{1}{n} \sum_{\{i,j\} \in \mathcal{W}_B \setminus \mathcal{W}_{A,B}} \alpha_{ij} x_i x_j + \frac{s}{n} \sum_{\{i,j\} \in \mathcal{W}_{A,B}} \alpha_{ij} x_i x_j + \sum_{i \in B} h_i x_i\,,$$

with corresponding Gibbs measure $\mu_{B;\boldsymbol{\alpha}(s),\boldsymbol{h}}$ and relative average $\mathbb{E}_{B;\boldsymbol{\alpha}(s),\boldsymbol{h}}$. Notice that, if $s = 1$, we obtain the system on the set B, so that $\mathbb{E}_{B;\boldsymbol{\alpha}(1),\boldsymbol{h}}(x_\Lambda) = \mathbb{E}_{B;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)$. Moreover, since $\mathcal{W}_B = \mathcal{W}_A \sqcup \mathcal{W}_{B \setminus A} \sqcup \mathcal{W}_{A,B}$, when $s = 0$, we get

$$H_{B;\boldsymbol{\alpha}(0),\boldsymbol{h}}(x) = H_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x) + H_{B \setminus A;\boldsymbol{\alpha},\boldsymbol{h}}(x)\,.$$

Then $\mu_{B;\boldsymbol{\alpha}(0),\boldsymbol{h}} = \mu_{A;\boldsymbol{\alpha},\boldsymbol{h}} \cdot \mu_{B \setminus A;\boldsymbol{\alpha},\boldsymbol{h}}$ and, as a consequence, being $\Lambda \subseteq A$, we have $\mathbb{E}_{B;\boldsymbol{\alpha}(0),\boldsymbol{h}}(x_\Lambda) = \mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)$. Finally, since $\boldsymbol{\alpha} \mapsto \mathbb{E}_{B;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)$ is a non-decreasing mapping and $\boldsymbol{\alpha}(0) < \boldsymbol{\alpha}(1)$, we conclude

$$\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda) = \mathbb{E}_{B;\boldsymbol{\alpha}(0),\boldsymbol{h}}(x_\Lambda) \leq \mathbb{E}_{B;\boldsymbol{\alpha}(1),\boldsymbol{h}}(x_\Lambda) = \mathbb{E}_{B;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)\,,$$

as claimed. ∎

A second consequence of the GKS inequality is a comparison between partition functions.

Lemma 3.9. *If the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ satisfies the GKS inequality then, for any $E, F \subseteq \mathcal{E}_n$,*

$$(3.16) \qquad Z_{E;\boldsymbol{\alpha},\boldsymbol{h}} Z_{F;\boldsymbol{\alpha},\boldsymbol{h}} \leq Z_{E \cup F;\boldsymbol{\alpha},\boldsymbol{h}} Z_{E \cap F;\boldsymbol{\alpha},\boldsymbol{h}} \,.$$

Proof. We follow some ideas developed in [16] to prove an analogous result for Ising spin systems. We set $K_1 := E \cap F$, $K_2 := E \setminus K_1$ and $K_3 := F \setminus K_1$, so that we can express the sets E, F, $E \cup F$ and $E \cap F$ as proper disjoint unions of the subsets K_i's. With this notation, the inequality (3.16) becomes equivalent to

$$(3.17) \qquad L(\boldsymbol{\alpha}, \boldsymbol{h}) := \ln Z_{K_1 \cup K_2 \cup K_3;\boldsymbol{\alpha},\boldsymbol{h}} - \ln \frac{Z_{K_1 \cup K_2;\boldsymbol{\alpha},\boldsymbol{h}} Z_{K_1 \cup K_3;\boldsymbol{\alpha},\boldsymbol{h}}}{Z_{K_1;\boldsymbol{\alpha},\boldsymbol{h}}} \geq 0 \,.$$

Notice that, if there is no interaction between the edges in K_1 and those in K_3, then

$$Z_{K_1 \cup K_2 \cup K_3;\boldsymbol{\alpha},\boldsymbol{h}} = Z_{K_1 \cup K_2;\boldsymbol{\alpha},\boldsymbol{h}} Z_{K_3;\boldsymbol{\alpha},\boldsymbol{h}} \quad \text{and} \quad Z_{K_1 \cup K_3;\boldsymbol{\alpha},\boldsymbol{h}} = Z_{K_1;\boldsymbol{\alpha},\boldsymbol{h}} Z_{K_3;\boldsymbol{\alpha},\boldsymbol{h}},$$

that yields $L(\boldsymbol{\alpha}, \boldsymbol{h}) = 0$. To conclude, it suffices to show that the function $L(\boldsymbol{\alpha}, \boldsymbol{h})$ is non-decreasing with respect to the interaction parameter $\boldsymbol{\alpha}$. To this purpose, we consider the change in $L(\boldsymbol{\alpha}, \boldsymbol{h})$, when an interaction of strength α_{ij}, between the edges $i \in K_1$ and $j \in K_3$, is added to the system. By differentiating w.r.t. α_{ij} we get

$$(3.18) \qquad \partial_{\alpha_{ij}} L(\boldsymbol{\alpha}, \boldsymbol{h}) = \frac{1}{n} \left(\mathbb{E}_{K_1 \cup K_2 \cup K_3;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j) - \mathbb{E}_{K_1 \cup K_3;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j) \right) \geq 0 \,,$$

where the last inequality follows from Lemma 3.8. All together this implies that $L(\boldsymbol{\alpha}, \boldsymbol{h}) \geq 0$. ∎

3.2. **The GHS inequality.** We are now ready to derive our main result: the GHS inequality for the model associated with the Hamiltonian (3.1).

Theorem 3.10. *If $\boldsymbol{\alpha}, \boldsymbol{h} \geq 0$, then the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ satisfies the GHS inequality. In particular, for any choice of $i, j, k \in \mathcal{E}_n$, we have*

$$(3.19) \qquad \partial_{h_i h_j h_k} f_{n;\boldsymbol{\alpha},\boldsymbol{h}} \leq 0 \,.$$

Remark 3.11. *The above theorem provides sufficient conditions for the validity of the GHS inequality, and it is then natural to wonder whether they are also necessary. A hint on this question is given when considering the statement for indices $i = j = k$. Inequality (3.19) reduces to (see also (3.8))*

$$(3.20) \qquad \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i) \left(1 - \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i) \right) \left(1 - 2\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i) \right) \leq 0 \,,$$

that is verified if and only if

$$\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i) \geq 1/2 \,.$$

Recall that by (2.13) the above condition is fulfilled whenever $\boldsymbol{\alpha}, \boldsymbol{h} \geq 0$. This assumption is indeed the only strict requirement on the parameter $\boldsymbol{h}$ that we will use along the proof, and precisely in (3.38) below, though in a modified setting that requires the validity of this condition uniformly in n. However, as mentioned in Subsection

2.3, the edge-occurence probabilities are implicit functions of the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{h}$, and are also dependent on n. For these reasons, we believe that the derivation of explicit necessary conditions could be in general a hard task.

The strategy of the proof is based on the trick of introducing a duplicate set of variables. Let $y \in \mathcal{A}_n$ be an independent copy of $x \in \mathcal{A}_n$, with the same Hamiltonian as in (3.1), and let $\mathbb{E}$ denote the expectation with respect to the joint measure

$$(3.21) \qquad \mu(x,y) := \frac{\exp\{H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x) + H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(y)\}}{Z^2_{n;\boldsymbol{\alpha},\boldsymbol{h}}}.$$

For any $i \in \mathcal{E}_n$, define the variables $z_i = x_i - y_i$ and $v_i = \frac{1}{2}(x_i + y_i)$. Notice that z_i takes value on $\{-1, 0, +1\}$, while v_i takes value on $\{0, \frac{1}{2}, 1\}$, and that the following equivalence of events holds for all $i \in \mathcal{E}_n$:

$$(3.22) \qquad \{z_i \in \{-1, +1\}\} = \left\{v_i = \frac{1}{2}\right\} \quad \text{and} \quad \{v_i \in \{0,1\}\} = \{z_i = 0\}.$$

With standard notation we set $z := (z_i)_{i \in \mathcal{E}_n}$ and $v := (v_i)_{i \in \mathcal{E}_n}$. Moreover, for any given $A \subseteq \mathcal{E}_n$, we define the functions $z_A := \prod_{i \in A} z_i$ and $v_A := \prod_{i \in A} v_i$.

Proposition 3.12. *Let $\boldsymbol{\alpha}, \boldsymbol{h} \geq 0$. Then, for any $C, D \subseteq \mathcal{E}_n$, it holds that*

$$(3.23) \qquad \mathbb{E}(z_C z_D) \geq \mathbb{E}(z_C)\mathbb{E}(z_D),$$

$$(3.24) \qquad \mathbb{E}(z_C v_D) \leq \mathbb{E}(z_C)\mathbb{E}(v_D).$$

Remark 3.13. *It is easy to check that the Ursell function $u_3(i,j,k)$, given explicitly in (3.6), can be written as a function of the random variables z_i's and v_i's as*

$$(3.25) \qquad u_3(i,j,k) = \mathbb{E}(z_i z_j v_k) - \mathbb{E}(z_i z_j)\mathbb{E}(v_k).$$

The statement of Theorem 3.10 is then a consequence of the inequality (3.24). Similarly, it can be shown that Eq. (3.23) implies the GKS inequality for the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$.

Proof of Proposition 3.12. We first consider two general functions $\Phi(z)$ and $\Psi(v)$, with $z = (z_i)_{i \in \mathcal{E}_n}$ and $v = (v_i)_{i \in \mathcal{E}_n}$, and we try to express the average $\mathbb{E}(\Phi(z)\Psi(v))$ in a convenient form. Later we will focus on the functions $\Phi(z) = z_C$ and $\Psi(v) = v_D$.

Observe that, due to the identity $x_i x_j + y_i y_j = \frac{1}{2}z_i z_j + 2v_i v_j$, the exponent of the joint measure (3.21) can be phrased in terms of the variables z and v. It yields

$$(3.26) \qquad H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x) + H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(y) = \widehat{H^1}_{n;\boldsymbol{\alpha}}(z) + \widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v),$$

where

$$(3.27) \qquad \begin{aligned} \widehat{H^1}_{n;\boldsymbol{\alpha}}(z) &= \frac{1}{2n} \sum_{\{i,j\} \in \mathcal{W}_n} \alpha_{ij} z_i z_j, \\ \widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v) &= \frac{2}{n} \sum_{\{i,j\} \in \mathcal{W}_n} \alpha_{ij} v_i v_j + 2 \sum_{i \in \mathcal{E}_n} h_i v_i. \end{aligned}$$

Moreover, by exploiting the constraints (3.22), we can partition the state space of the couple (z, v) in a disjoint union, over subsets $A \subseteq \mathcal{E}_n$, of the sets

(3.28)
$$\mathcal{S}_A := \left\{ (z, v) : z_i = 0, v_i \in \{0, 1\} \, \forall i \in A \text{ and } v_i = \frac{1}{2}, z_i \in \{-1, 1\} \, \forall i \in A^c \right\}.$$

Hence, we can write

(3.29) $\qquad \mathbb{E}(\Phi(z)\Psi(v)) = \sum_{A \subseteq \mathcal{E}_n} \sum_{(z,v) \in \mathcal{S}_A} \Phi(z)\Psi(v) \dfrac{\exp\left\{ \widehat{H^1}_{n;\boldsymbol{\alpha}}(z) + \widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v) \right\}}{Z^2_{n;\boldsymbol{\alpha},\boldsymbol{h}}}.$

It is easy to see that if $(z, v) \in \mathcal{S}_A$, and with the same notation introduced in (3.13), we obtain

(3.30) $\qquad \widehat{H^1}_{n;\boldsymbol{\alpha}}(z) = \dfrac{1}{2n} \sum_{\{i,j\} \in \mathcal{W}_{A^c}} \alpha_{ij} z_i z_j, \quad \text{with } z_i \in \{-1, 1\}, \, \forall i \in A^c$

and

(3.31) $\quad \widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v) = \dfrac{2}{n} \sum_{\{i,j\} \in \mathcal{W}_A} \alpha_{ij} v_i v_j + \sum_{i \in A} \left(2h_i + \dfrac{1}{n} \sum_{j \in A^c : j \sim i} \alpha_{ij} \right) v_i$

$$+ \dfrac{1}{2n} \sum_{\{i,j\} \in \mathcal{W}_{A^c}} \alpha_{ij} + \sum_{i \in A^c} h_i, \qquad \text{with } v_i \in \{0, 1\}, \, \forall i \in A.$$

In particular, on the set $\mathcal{S}_A$,

- the Hamiltonian $\widehat{H^1}_{n;\boldsymbol{\alpha}}(z)$ corresponds to the Hamiltonian of an Ising spin system on the set A^c, with inverse temperature $\boldsymbol{\beta} := \boldsymbol{\alpha}/2n$, magnetic field $\boldsymbol{h} = \boldsymbol{0}$, and associated Gibbs measure

$$\mu^{\text{Is}}_{A^c;\boldsymbol{\beta},\boldsymbol{0}}(z) := \dfrac{e^{H^{\text{Is}}_{A^c;\boldsymbol{\beta},\boldsymbol{0}}(z)}}{Z^{\text{Is}}_{A^c;\boldsymbol{\beta},\boldsymbol{0}}}.$$

- the Hamiltonian $\widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v)$ corresponds to the two-star Hamiltonian on A given in (3.13), but with parameters $\boldsymbol{\alpha'} := 2\boldsymbol{\alpha}$ and $\boldsymbol{h'} := (h'_i)_{i \in \mathcal{E}_n}$, where $h'_i := 2h_i + \frac{1}{n} \sum_{j \in A^c : j \sim i} \alpha_{ij}$. Indeed, the two Hamiltonians differ only for the constant term $\frac{1}{2n} \sum_{\{i,j\} \in \mathcal{W}_{A^c}} \alpha_{ij} + \sum_{i \in A^c} h_i$ that, being irrelevant in the Gibbs measure, will be neglected. As before, we write $\mu_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}$ for the Gibbs measure related to the Hamiltonian (3.31).

Going back to Eq. (3.29), in view of the previous considerations, it turns out that

(3.32) $\qquad\qquad \mathbb{E}(\Phi(z)\Psi(v)) = \sum_{A \subseteq \mathcal{E}_n} P(A) f^\Phi(A) g^\Psi(A),$

where, with self-explanatory notation, we set

(3.33) $\quad f^\Phi(A) := \mathbb{E}^{\text{Is}}_{A^c,\boldsymbol{\beta},\boldsymbol{0}}(\Phi(z)|_{z_i=0, \forall i \in A}), \qquad g^\Psi(A) := \mathbb{E}_{A,\boldsymbol{\alpha'},\boldsymbol{h'}}(\Psi(v)|_{v_i=\frac{1}{2}, \forall i \in A^c})$

and

$$(3.34) \qquad P(A) := \frac{Z^{\text{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}} \cdot Z_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}}{Z^2_{n;\boldsymbol{\alpha},\boldsymbol{h}}} \, .$$

Notice that P is a probability on $\mathcal{E}_n$ by construction. Specializing the identity (3.32) to the functions $\Phi(z) = z_C$ and $\Psi(v) = v_D$, with $C, D \subset \mathcal{E}_n$, we get

$$(3.35) \qquad \mathbb{E}(z_C v_D) = \sum_{A \subseteq \mathcal{E}_n} P(A) \mathbb{E}^{\text{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z_C|_{z_i=0,\forall i\in A}) \mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_D|_{v_i=\frac{1}{2},\forall i \in A^c}) \, .$$

The proof of the two inequalities (3.24) and (3.23) is an immediate application of the FKG inequality relatively to P. Indeed, if $\boldsymbol{\alpha}$, $\boldsymbol{h} \geq 0$, then also $\boldsymbol{\beta}$, $\boldsymbol{\alpha}', \boldsymbol{h}' \geq 0$, and the conditions for the application of the FKG inequality are fulfilled:

- If $\boldsymbol{\beta} \geq 0$, the GKS inequality for ferromagnetic Ising systems [11] guarantees that the function $\mathbb{E}^{\text{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z_C|_{z_i=0,\forall i\in A})$ is non-increasing in A, for any choice of $C \subseteq \mathcal{E}_n$.
- If $\boldsymbol{\alpha}', \boldsymbol{h}' \geq 0$ the function $\mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_D|_{v_i=\frac{1}{2},\forall i \in A^c})$ is non-decreasing in A, for any choice of $D \subseteq \mathcal{E}_n$. This is a consequence of the GKS inequality together with Lemma 3.8. Indeed, let $A \subseteq B$ and observe that

$$(3.36) \qquad \begin{aligned} \mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_D|_{v_i=\frac{1}{2},\forall i \in A^c}) &= \frac{1}{2^{|D \cap A^c|}} \mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_{D\cap A}) \\ \mathbb{E}_{B;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_D|_{v_i=\frac{1}{2},\forall i \in B^c}) &= \frac{1}{2^{|D \cap B^c|}} \mathbb{E}_{B;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_{D\cap B}) \, . \end{aligned}$$

Since $D \cap A \subseteq D \cap B$ by hypothesis, we can write $v_{D\cap B} = v_{D\cap A} v_{D\cap(B\setminus A)}$ and hence, from the GKS inequality and Lemma 3.8,

$$(3.37) \qquad \begin{aligned} \mathbb{E}_{B;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_{D\cap B}) &\geq \mathbb{E}_{B;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_{D\cap A}) \mathbb{E}_{B;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_{D\cap(B\setminus A)}) \\ &\geq \mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_{D\cap A}) \mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_{D\cap(B\setminus A)}) \, . \end{aligned}$$

We now recall that for $\boldsymbol{\alpha}' \geq 0$ and $\boldsymbol{h}' \geq 0$, it holds that $\mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_i) \geq 1/2$, for all $i \in A$ and $A \subseteq \mathcal{E}_n$. Applying the GKS inequality to the second factor of the r.h.s of the above equation, and using this bound, we then get

$$(3.38) \qquad \mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_{D\cap(B\setminus A)}) \geq \prod_{i \in D \cap (B\setminus A)} \mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_i) \geq \frac{1}{2^{|D\cap(B\setminus A)|}} \, .$$

Putting together (3.36)-(3.38), we conclude that

$$(3.39) \qquad \mathbb{E}_{B;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_D|_{v_i=\frac{1}{2},\forall i \in B^c}) \geq \mathbb{E}_{A;\boldsymbol{\alpha}',\boldsymbol{h}'}(v_D|_{v_i=\frac{1}{2},\forall i \in A^c}) \, .$$

- If $\boldsymbol{\alpha} \geq 0$, the probability P, defined in (3.34) and acting on subsets of $\mathcal{E}_n$, satisfies the FKG lattice condition, namely

$$(3.40) \qquad P(E)P(F) \leq P(E \cup F)P(E \cap F) \, , \qquad \forall E, F \subseteq \mathcal{E}_n \, .$$

According to the definition of P, the inequality (3.40) follows if the two inequalities

$$Z^{\text{Is}}_{E;\boldsymbol{\beta},\mathbf{0}} Z^{\text{Is}}_{F;\boldsymbol{\beta},\mathbf{0}} \leq Z^{\text{Is}}_{E\cup F;\boldsymbol{\beta},\mathbf{0}} Z^{\text{Is}}_{E\cap F;\boldsymbol{\beta},\mathbf{0}}$$

and

$$Z_{E;\boldsymbol{\alpha}',\boldsymbol{h}'} Z_{F;\boldsymbol{\alpha}',\boldsymbol{h}'} \leq Z_{E\cup F;\boldsymbol{\alpha}',\boldsymbol{h}'} Z_{E\cap F;\boldsymbol{\alpha}',\boldsymbol{h}'}$$

are simultaneously satisfied. The first inequality holds true as a consequence of the GKS inequality for Ising spin systems with $\boldsymbol{\beta} \geq 0$ and magnetic field $\boldsymbol{h} \geq 0$ (see [16], Lemma on p. 90). The second inequality is instead verified thanks to Lemma 3.9.

Thus, as P obeys the FKG lattice condition, and the functions $\mathbb{E}^{\mathrm{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z_C|_{z_i=0,\forall i \in A})$ and $\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_i=\frac{1}{2},\forall i \in A^c})$ are respectively non-increasing and non-decreasing in A, from Eq. (3.35) we get

$$
\begin{aligned}
(3.41) \quad \mathbb{E}(z_C v_D) &\leq \sum_{A \subseteq \mathcal{E}_n} P(A)\mathbb{E}^{\mathrm{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z_C|_{z_i=0,\forall i \in A}) \sum_{A \subseteq \mathcal{E}_n} P(A)\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_i=\frac{1}{2},\forall i \in A^c}) \\
&= \mathbb{E}(z_C)\mathbb{E}(v_D)\,,
\end{aligned}
$$

providing inequality (3.24). The inequality (3.23) can be obtained in the same way by setting $\phi(z) = z_C z_D$, so that $g^\psi(A) = 1$, and by observing that $f^\phi(A)$ is non-decreasing in A, hence giving the reverse inequality.   ∎

Proof of Theorem 3.10. The statement follows readily from Remark 3.13 and Proposition 3.12.   ∎

3.3. **The GHS inequality for the two-star model.** Let $\alpha, h \in \mathbb{R}$. Recall that the two-star model is obtained, as a particular case, by setting $\alpha_{ij} \equiv \alpha$, for all $i, j \in \mathcal{E}_n$, and $h_i \equiv h$, for all $i \in \mathcal{E}_n$, in the Hamiltonian (3.1). This means that, whenever $\alpha, h \geq 0$, the GHS inequality holds true for the Gibbs measure $\mu_{n;\alpha,h}$, given in (2.2).

Observe that, by differentiating the free energy $f_{n;\alpha,h}$ w.r.t. h, we get the following identities in terms of the Ursell functions (see also Remark 3.3)

$$
n^2\partial_h f_{n;\alpha,h} = \sum_{i \in \mathcal{E}_n} u_1(i)\,, \quad n^2\partial_{hh} f_{n;\alpha,h} = \sum_{i,j \in \mathcal{E}_n} u_2(i,j)\,,
$$

$$
n^2\partial_{hhh} f_{n;\alpha,h} = \sum_{i,j,k \in \mathcal{E}_n} u_3(i,j,k)\,,
$$

and so on. Thus, not only the sign of each Ursell function provides a specific correlation inequality between the random variables x_i's, but also it gives a definite sign to a derivative of the free energy.

A direct computation easily shows that, being a variance, the second order partial derivative of $f_{n;\alpha,h}$ w.r.t. h is always non-negative. Thus, proving that $u_2(i,j) \geq 0$ (GKS inequality) is useful to know the covariance between x_i and x_j, but it is somehow irrelevant to the purpose of showing that the free energy is a convex function of h. On the contrary, the GHS inequality ($u_3(i,j,k) \leq 0$) is of particular importance, as it implies that the average edge density (2.10) is a concave function of the parameter h at any fixed size of the graph.

Explicitly, setting $m_n(\alpha, h) := \frac{\mathbb{E}_{n;\alpha,h}(E_n)}{n^2}$ and assuming that $\alpha, h \geq 0$, from the GKS and GHS inequalities we readily get

$$
(3.42) \qquad \partial_h m_n(\alpha, h) = \partial_{hh} f_{n;\alpha,h} \geq 0\,, \qquad \partial_{hh} m_n(\alpha, h) = \partial_{hhh} f_{n;\alpha,h} \leq 0\,.
$$

Understanding the limiting behavior of the above derivatives has then a twofold purpose. On the one hand, it allows to infer properties regarding the edge density and its limiting behavior; for example, the existence of $\lim_{n\to+\infty} \partial_h m_n(\alpha, h)$ is fundamental for proving the standard central limit theorem in (2.17) (see [4]). On the other hand, it is crucial for detecting the occurence of phase transitions, that are generally associated with the emergence of singularities in the infinite size free energy. In particular, one can exploit convergence results on the derivatives of convex functions to guarantee that the limits and the derivatives w.r.t. the external field commute (see [7, Lemma V.7.5]), and then obtain proper regularity conditions. Notice that this procedure can be seen as an alternative approach to the investigation of the hypotheses that allow for the application of the Lee-Yang theorem [18]. However, in this respect, the convexity property (3.42) provides a more specific information that may enter in the characterization of further features of the model.

4. Discussion on possible extensions

The results presented in Lemmas 3.5–3.9 can be extended to the general case where the Hamiltonian is a function of the homomorphism densities of an arbitrary collection of subgraphs of the graph G. In this Section we will briefly elaborate on this.

In the sequel, we will be dealing with the general Hamiltonian (2.3) and the corresponding Gibbs probability density (2.2). We will denote by $\mathbb{E}_{n;\boldsymbol{\beta}}$ the relative expectation. Moreover, as a standard choice in the literature, we will set the subgraph H_1 to be an edge.

Going through the proof of Lemma 3.5, it is easy to understand that the crucial condition for the validity of the FKG lattice condition is inequality (3.11). When moving to the general setting we are adopting, the analogous condition reads as

$$(4.1) \qquad \exp\left\{n^2 \sum_{j=2}^{k} \beta_j t(H_j, G)\right\} \geq 1\,.$$

As consequence, since the homomorphism densities are non-negative, the FKG lattice condition is in force whenever the parameters $\beta_2, \ldots, \beta_k$ are non-negative. Thus, we obtain the following result.

Lemma 4.1. *For all $\beta_1 \in \mathbb{R}$ and $\beta_2, \ldots, \beta_k \geq 0$, the Gibbs measure $\mu_{n;\boldsymbol{\beta}}$ fulfills the FKG lattice condition*

$$(4.2) \qquad \mu_{n;\boldsymbol{\beta}}(x \vee y)\, \mu_{n;\boldsymbol{\beta}}(x \wedge y) \geq \mu_{n;\boldsymbol{\beta}}(x)\, \mu_{n;\boldsymbol{\beta}}(y) \quad \text{for } x, y \in \mathcal{A}_n.$$

Two immediate consequences of Lemma 4.1 are the positive correlation between increasing functions of the configuration and, in turn, the GKS inequality for the Gibbs measure $\mu_{n;\boldsymbol{\beta}}$. Specifically, for all $\beta_1 \in \mathbb{R}$ and $\beta_2, \ldots, \beta_k \geq 0$, all increasing functions f and g, and all $A, B \subseteq \mathcal{E}_n$, it holds

$$(4.3) \qquad \mathbb{E}_{n;\boldsymbol{\beta}}(fg) \geq \mathbb{E}_{n;\boldsymbol{\beta}}(f) \cdot \mathbb{E}_{n;\boldsymbol{\beta}}(g) \qquad\qquad \text{(FKG inequality)}$$

$$(4.4) \qquad \mathbb{E}_{n;\boldsymbol{\beta}}(x_A x_B) \geq \mathbb{E}_{n;\boldsymbol{\beta}}(x_A) \cdot \mathbb{E}_{n;\boldsymbol{\beta}}(x_B), \qquad\qquad \text{(GKS inequality)}$$

where $x_C = \prod_{i \in C} x_i$, for $C \subseteq \mathcal{E}_n$.

An extension of Lemmas 3.8 and 3.9 to this general context is also straightforward. However, while they were crucial to prove the GHS inequality for the two-star model, they are quite irrelevant in the present setting, as the techniques used in Subsection 3.2 can not be replicated.

Indeed, when dealing with a generic exponential random graph, the trick of variable duplication (3.21) does not work. The problem is twofold. On the one hand, in general the decomposition (3.26) fails to exist, as mixed terms remain. Thus, it is not possible to factorize the joint measure of the doubled model and then characterize correlations exploiting averages over an Ising and an ERG subsystem (see (3.29)). On the other hand, even if the joint measure factored out and the analog of (3.29) were available, to conclude we would need FKG and GKS inequalities for Ising models with multi-body interactions, that are not known.

However, if we specify the Ursell function $u_3(i, j, k)$, given in (3.6), in the special cases when $i = j = k$ and $i = j \neq k$, we obtain respectively

$$\mathbb{E}_{n;\boldsymbol{\beta}}(x_i)\left(1 - \mathbb{E}_{n;\boldsymbol{\beta}}(x_i)\right)\left(1 - 2\mathbb{E}_{n;\boldsymbol{\beta}}(x_i)\right)$$

and

$$\mathrm{Cov}_{n;\boldsymbol{\beta}}(x_i, x_j)(1 - 2\mathbb{E}_{n;\boldsymbol{\beta}}(x_i)).$$

Since $\mathrm{Cov}_{n;\boldsymbol{\beta}}(x_i, x_j) \geq 0$, due to the GKS inequality (4.4), we can conjecture that the necessary and sufficient condition for the GHS inequality to hold in the present general setting is again only the requirement $\mathbb{E}_{n;\boldsymbol{\beta}}(x_i) \geq 1/2$.

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