Deviations from the Majority: A Local Flip Model

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Abstract

We study the effect of probabilistic distortions to the local majority rules used in the Galam model of opinion dynamics and bottom-up hierarchical voting. A different probability for a flip against the local majority within the discussion group is associated with each ratio of majority / minority. The cases of groups of sizes 3 and 5 are investigated in detail. For hierarchical voting, the local flip corresponds to a 'faithless elector', a representative who decides to vote against the choice of their electing group. Depending on the flip probabilities, the model exhibits a rich variety of patterns for the dynamics, which include novel features in the topology of the landscape. In particular, for size 5, we uncover for the first time an interplay between five fixed points, which split into either three attractors and two tipping points or two attractors and three tipping points, depending on the flip probabilities. Larger groups are also analysed. These features were absent in the former versions of the Galam model, which has at maximum three fixed points for any group size. The results shed a new light on a series of social phenomena triggered by one single individual who acts against the local majority.

Keywords: opinion dynamics, hierarchical voting, contrarianism, minority spreading

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1 Introduction

The study of opinion dynamics has been and still is one of the main topics of interest within the field of sociophysics [9, 8, 35]. Several recent surveys of the opinion dynamics literature are available [33, 7, 32]. Models of opinion dynamics can be classified according to the opinion space, which is the set of possible opinions, and by whether time is discrete or continuous (the former is more common; a recent example of the latter is found in [34]). In the 1980s and 90s, most models featured discrete opinion spaces. The scenario where there are two choices has been especially thoroughly studied (an influential article is [36]). One of the main contributions to this field is the Galam model [15], in which individuals hold a certain binary opinion. Much later, continuous opinion spaces were introduced such as the so called bounded confidence models around the year 2000 (e.g. [11]), the Friedkin-Johnsen model [13, 14], and the Hegselmann-Krause model [26].

Those continuous models connect to earlier modelling of opinion dynamics by social psychologists and mathematicians starting in the 1950s [3, 1, 12], an approach which remained disconnected from sociophysics and physicists for a few decades due to the compartmentalised nature of the different disciplines and the lack of access to literature from other disciplines pre internet. An increasing number of mathematicians have been joining the field recently [28, 2, 5].

The basic Galam model describes the dynamics of binary opinions. We call the options A and B, with an initial proportion of $p_0 \in [0, 1]$ in favour of A. The model can be interpreted in two different ways:

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1. As a model of *voting in a hierarchy*. The hierarchy is a homogeneous rooted tree (with branching factor $r \in \mathbb{N}$) with the leaves being the bottom level individuals. Each group within the bottom level is comprised of r individuals who vote for one of the options by the majority rule. This decision is carried to the next level of the hierarchy, where r representatives form a group that in turn votes on A versus B. Their decision is passed to the next level, etc. With this setup, the number of voters on the bottom level of an n-hierarchy with group size r is r^n and the total number of individuals comprising the hierarchy is $N := \left(r^{n+1} - 1\right) / (r - 1)$. This model was first introduced in [15]. See also [18] for an in-depth treatment.

2. As a model of *opinion dynamics*. Rather than voting within fixed groups as part of a hierarchy, the entire population randomly meets in groups of fixed size r. A discussion takes place in each group and the majority opinion is adopted by all members of the group. Next, the groups are broken up and the individuals randomly form new groups of size r to resume discussion. This is repeated n times. Contrary to the model of voting in a hierarchy, the total number of individuals is not a function of the number of discussion rounds n. However, when studying the dynamics of the model, the number of rounds n plays the same role as the number of levels in the hierarchy in the voting model to determine the updated distribution of opinions. As a rule of thumb, the total number of individuals N should be at a minimum 100 when dealing with proportions of opinions, since this allows us to round to two digits. The influence of contrarians on opinion dynamics was first studied using this model in [16, 17]. There is also a version of 'global' contrarianism [6], which takes into account the global rather than the local (or group) majority. Other authors have also studied the phenomenon of contrarianism, e.g. [38, 4, 24, 29, 25]. The idea of using contrarianism to explain social phenomena is not exclusive to the study of opinion dynamics, however: [10] is a study of contrarianism in finance.

In both cases, the interest lies in determining the dynamics of distribution of preferences as n increases. Aside from the basic model, there have been several extensions [16, 18, 22, 23]. The authors of [31] studied a model with group size 3 and probabilistic adoption of the group majority. In this paper, we compare the dynamics of the basic opinion dynamics model, the contrarian model, and a new local flip model, where the likelihood of an individual flipping against the majority is a function of the magnitude of this majority. We will assume exclusively that the size of the group $r \in \mathbb{N}$ is odd. This is motivated by the observation that if we introduce a tiebreaker in case of a draw for even r, where each option is then chosen with probability $1/2$, then the dynamics of the model correspond to that of the model with $r - 1$ odd. See equation (2.8) in [18].

A number of problems have been studied over the course of the last decades using the general setup of the Galam model with suitable modifications. Versions of this model have been used to predict surprising election results, such as the French rejection of the European constitution in 2005 [30], the Brexit vote [20], and Trump's election for president in 2016 [19]. However, the prediction of a second Trump victory in 2020 failed by a short margin [21]. From a mathematical point of view, these scenarios can be modelled by the repeated application of an update equation that takes as input the current distribution of opinions, given by the proportion of individuals with opinion A, and outputs the new proportion of individuals with opinion A. Thus, this is a dynamic model where, for any initial distribution of opinions, we have a trajectory of distributions over time. The properties of the update equation determine whether the dynamics are convergent and, if so, where they converge to. This is how the model explains the emergence of stable majorities in favour of one of the options, or to the contrary, under what circumstances the dynamics tend to an even split between the two options.

In this article, we introduce a new model based on the Galam model, called the local flip model. It is in a similar vein as the contrarian model first introduced in [17], which deals with opinion dynamics under the presence of some individuals who reject the majority opinion. However, while the contrarian model uses the same probability for all local configurations for a shift against the majority including the case of unanimity, here we assume a different probability for each configuration as a function of the ratio of majority to minority. We investigate the local flip model for two settings, which are:

1. In the hierarchical voting scenario (the ‘vertical frame’), the group representative faces incentives to deviate from the majority opinion for personal gain. Suppose there is some mechanism in place to detect deviations from the majority, e.g. a statistical sampling of opinions across all groups, which would allow detection of deviations some of the time, and make detection more likely when the majority is large. Then it is reasonable to assume that flip probabilities are decreasing in the magnitude of the majority. The local flip corresponds to a ‘faithless elector’, a representative who decides to vote against the choice of their electing group.

2. The other scenario is the opinion dynamics interpretation. In this ‘horizontal frame’, there are contrarian tendencies which hamper the adoption of the local majority opinion. Here, there is no reason to assume that these tendencies are stronger for smaller majorities. In fact, contrarianism may be stronger when facing a larger majority. Hence, different parameter ranges may be suited to the vertical and to the horizontal frame.

We study in detail the cases of groups of sizes 3 and 5 as a function of the various flip probabilities. The associated dynamics exhibit a rich variety of patterns including novel features which were absent in the previous works. In particular, contrary to the Galam model which yields at maximum three fixed points (two attractors and one tipping point) for any size r, the flip model is found to exhibit five fixed points for $r = 5$. Moreover, modifying the flip probabilities produces an unexpected interplay between the stabilities of the five fixed points with a transformation of a set of three attractors and two tipping points into a set of two attractors and three tipping points. These new features persist for larger group sizes. The previous models referred to as the ‘basic model’ and the ‘contrarian model’ in this article are recovered as sub-cases of the local flip model. The results shed a new light on a series of social phenomena, which could not be explained by a flat probability of contrarian behaviour. Counterintuitive strategies can be designed to optimise outcomes for competing groups.

This paper is organised in six sections and an appendix. In Sections 2 and 3, we provide a review of the basic Galam model and the contrarian model, respectively. Thereafter, in Section 4, we introduce the local flip model. We first study the model when the groups are very small, specifically group size 3, and then we turn to the larger group size 5. We consider versions of the model where flips only occur when the majority has some specific magnitude, as well as versions where there are flips for different majority sizes with distinct probabilities. Section 5 discusses the three models, their similarities and differences. The novelties yielded by the local flip model are reviewed. Section 6 concludes the paper. Finally, the Appendix contains some technical details as well as the case of arbitrarily large group sizes.

2 Basic Model

Mathematically speaking, all the models are completely determined by their update equation. This equation describes what the distribution of opinions is for the next level / round given a current distribution p. Given a current probability $p \in [0, 1]$ of a preference for A, the updated probability of a preference for A is given by the polynomial

$$P_r(p) := \sum_{i=\frac{r+1}{2}}^{r} \binom{r}{i} p^i (1-p)^{r-i} . \tag{1}$$

As mentioned in the Introduction, we are interested in the dynamics of the probability that a preference for option A exists: $p_0 \mapsto p_1 = P(p_0) \mapsto \cdots$. The key to understanding these dynamics is the analysis of the fixed points of the update function P_r. It is well known that for the basic model, the three fixed points are $0, 1/2, 1$ for any group size r. The points 0 and 1 are attractors. The fixed point $1/2$ is unstable even for $r = 3$ and it becomes ‘more unstable’ the larger r is. This can be seen by calculating

r	p_0	p_1	p_2	p_3	p_4	p_5	p_6
3	0.4	0.35	0.28	0.2	0.1	0.03	0
5	0.4	0.32	0.19	0.05	0	0	0
7	0.4	0.29	0.11	0	0	0	0
9	0.4	0.27	0.06	0	0	0	0

Table 1: Dynamics of the Basic Model ($p_0 = 0.4$)

r	p_0	p_1	p_2	p_3	p_4	p_5	p_7	p_8	p_9	p_{10}	p_{11}
3	0.48	0.47	0.46	0.43	0.4	0.35	0.28	0.2	0.1	0.03	0
5	0.48	0.46	0.43	0.37	0.27	0.12	0.02	0	0	0	0
7	0.48	0.46	0.41	0.3	0.13	0.01	0	0	0	0	0
9	0.48	0.45	0.38	0.23	0.03	0	0	0	0	0	0

Table 2: Dynamics of the Basic Model ($p_0 = 0.48$)

the derivative

$$P_r'(1/2) = \left(\frac{1}{2}\right)^{r-1} \sum_{i=\frac{r+1}{2}}^{r} \left(\begin{array}{c} r \\ i \end{array}\right) (2i - r).$$

This derivative is strictly larger than 1 and it is asymptotically equivalent to $\sqrt{2r/\pi}$ as $r \to \infty$. Thus, for any initial distribution of preferences for A given by the probability $p_0 \neq 1/2$, we have a tendency to move towards one of the two attractors: If $p_0 < 1/2$, then $p_n \searrow 0$, and if $p_0 > 1/2$, then $p_n \nearrow 1$ as $n \to \infty$. This means that the convergence to the stable fixed points becomes faster the larger the group size r becomes. In the limit $r \to N$ where there is a single large group, the convergence becomes immediate ($n = 1$).

Generally, the number of iterations n required to converge to the fixed points is quite small as long as p_0 is not too close to the repeller $1/2$. We present numerical convergence data, i.e. the orbits $p_0, p_1 = P(p_0), p_2 = P(P(p_0)), \ldots$ rounded to two decimal digits for initial probabilities $p_0 = 0.4$ and $p_0 = 0.48$ in Table 1 and 2. The rounding to two digits has a probabilistic interpretation: If k is the smallest value such that $p_k = 0$, i.e. $p_k = 0.00$, it means that either $p_k = 0.001, 0.002, 0.003, 0.004$, which in turn implies that for $n = k$, on average, out of 1000 trials, we will find that A wins – instead of B as expected – 1, 2, 3, or 4 times, respectively. We observe that – even for small group sizes – the convergence toward the fixed point 0 is very fast. This is due to the superstability ($P_r'(0) = P_r'(1) = 0$) of the fixed points $0, 1$. So even though formally we only have convergence as $n \to \infty$, in reality, small values of n suffice to approach the fixed point.

3 Contrarian Model

This model is more adapted to the study of opinion dynamics. As in the basic model above, the population meets repeatedly in randomly formed groups of fixed size r. Discussion takes place and all r members adopt the majority opinion. However, there are some individuals in the population who have a tendency to adopt the opposite opinion. This happens independently of their own initial opinion with a probability of a. The behaviour of the model is summed up for $r = 3$ in Table 3. For arbitrary r, the update equation of the contrarian model is given by

$$Q_{r,a}(p) := (1 - 2a) \sum_{i=\frac{r+1}{2}}^{r} \left(\begin{array}{c} r \\ i \end{array}\right) p^i (1 - p)^{r-i} + a. \tag{2}$$

Pre discussion	Probability	Post discussion	Post contrarian	Probability
AAA	p^3	AAA	AAA	$(1-a)^3$
			$AAB \cdot 3$	$3a\,(1-a)^2$
$AAB \cdot 3$	$3p^2\,(1-p)$		$ABB \cdot 3$	$3a^2\,(1-a)$
			BBB	a^3
$ABB \cdot 3$	$3p\,(1-p)^2$	BBB	AAA	a^3
			$AAB \cdot 3$	$3a^2\,(1-a)$
BBB	$(1-p)^3$		$ABB \cdot 3$	$3a\,(1-a)^2$
			BBB	$(1-a)^3$

Table 3: Contrarian Model, $r = 3$

r	p_0	p_1	p_2	p_3	p_4	p_5	p_6
3	0.02	0.6	0.47	0.51	0.5	0.5	0.5
5	0.02	0.6	0.46	0.51	0.49	0.5	0.5
7	0.02	0.6	0.46	0.52	0.49	0.5	0.5
9	0.02	0.6	0.45	0.52	0.49	0.51	0.5

Table 4: Dynamics of the Contrarian Model ($a = 3/5, p_0 = 0.02$)

Depending on the contrarian probability a, the dynamics differ from those of the basic model. While 1/2 is still a fixed point, as in the basic model, 0 and 1 are no longer fixed points. We consider group size $r = 3$ first. Instead of 0 and 1, for $a \in (0, 1/6)$, there are two stable fixed points $p_\pm$: $p_- \in (0, 1/2)$ and, symmetrically, $p_+ = 1 - p_- \in (1/2, 1)$. Specifically, these fixed points are located at

$$p_\pm = \frac{1 - 2a \pm \sqrt{12a^2 - 8a + 1}}{2\,(1 - 2a)}. \tag{3}$$

For higher values of $a \in [1/6, 1/2)$, 1/2 is the unique fixed point and it is stable. Qualitatively similar results hold for $r \geq 5$. The critical value of the contrarian probability a_c, at which the phase transition occurs, equals 1/6 for $r = 3$. a_c is increasing in r and $a_c \nearrow 1/2$ as $r \to \infty$. For each group size, the pattern we observe is a repeller at 1/2 and two stable fixed points $p_\pm$ if $a < a_c$ and the probability of contrarianism is low. For probabilities of contrarianism in the interval $[a_c, 1/2)$, we have a unique stable fixed point at 1/2. For $a = 1/2$, the convergence to the attractor 1/2 is immediate. If the contrarian probability is even higher, we obtain an interesting alternating pattern. For $a \in (1/2, 1 - a_c]$, 1/2 is still an attractor, just as for $a \in [a_c, 1/2)$. For $a > 1 - a_c$, there is an alternating pattern of converging subsequences towards p_- and $p_+ = 1 - p_-$, respectively.

Convergence toward the stable fixed points is fast. We illustrate this in Table 4 which is representative of the convergence behaviour of the model in a wide range of parameter values. The values of p_n given are rounded to two digits once again. For the chosen parameter value $a = 3/5$, we are in the alternating regime with 1/2 being an attractor. As we see, even starting at $p_0 = 0.02$ very far away from the attractor 1/2, convergence is very fast.

Configuration	Group vote	Probability
AAA	A	$(1-b)\,p^3$
	B	bp^3
$AAB \cdot 3$	A	$(1-a)\cdot 3p^2\,(1-p)$
	B	$a\cdot 3p^2\,(1-p)$
$ABB \cdot 3$	A	$a\cdot 3p\,(1-p)^2$
	B	$(1-a)\cdot 3p\,(1-p)^2$
BBB	A	$b\,(1-p)^3$
	B	$(1-b)\,(1-p)^3$

Table 5: Local Flip Model, $r=3$

4 Local Flip Model

Now we turn to the new model. One of the problems we want to study is the behaviour of voting in hierarchies also referred to as 'bottom-up voting'. We imagine a hierarchy with group size r. Each group votes and the majority decision is passed up the hierarchy. However, the group's representative on the next higher level may not respect the vote. We will consider a scenario where the deviation probability depends on the size of the majority. If $(r+1)/2$ voters within the group vote for A and $(r-1)/2$ vote for B, and we count each vote for A as a $+1$ and each vote for B as a -1, then there is a $+1$ voting margin. We will write S for the voting margin. Similarly, an $(r-1)/2$-$(r+1)/2$ majority in favour of B corresponds to a voting margin of $S=-1$. Therefore, a minimal majority corresponds to an absolute voting margin $|S|$ of 1. Similarly, any number of votes $i=0,1,\ldots,r$ in favour of A is equivalent to a voting margin of $S=i-(r-i)=2i-r$.

Before we define and analyse the local flip model in full generality, we take a look at small groups of size 3 and, subsequently, of size 5. These models can be analysed exhaustively with rigorous results concerning almost all of their properties. In the Appendix, we will discuss the general model with any group size r.

4.1 Group Size $r=3$

For simplicity's sake, we first consider the case of groups of size $r=3$. Then there are only two tiers of voting margins: $\pm1,\pm3$, and hence the model has only two parameters which we will call a and b. Let a be the probability of deviating from the majority when the absolute voting margin equals 1 and b the corresponding probability if $|S|=3$. The behaviour of the model is summarised in Table 5. The update equation of the model is

$$R_{3,a,b}\,(p)=(-2+6a-2b)\,p^3+(3-9a+3b)\,p^2+(3a-3b)\,p+b.$$

We mention here that the local flip model can also be understood to represent opinion dynamics / hierarchical voting under exogenous shocks: Imagine a group decision process where the group decides according to the majority rule, and then there is an exogenous shock that potentially modifies the decision and which is stochastically independent of the individual opinions. Provided that the shock probabilities are symmetric and only depend on the size of the majority but not its sign, i.e. which of the option A or B is favoured, we obtain precisely the local flip model. The symmetry of the flip probabilities implies that the model is 'neutral' in the sense that there is no distinction between the two options A and B. If we flipped all initial opinions $A \longleftrightarrow B$, then the winning opinion would flip as well.

Due to the small group size, we can completely solve the model and analyse its dynamics. If the flip probability b is non-zero, then 0 and 1 are not fixed points of the dynamics. Due to the symmetry of the model, however, $1/2$ is always a fixed point. To determine the regimes of the model, we partition the parameter space $[0,1]^2 = \{(a,b) \mid a,b \in [0,1]\}$ first according to the stability of the fixed point $1/2$ and then according to the number of fixed points the model has.

As far as the stability of $1/2$ is concerned, we distinguish four regions:

1. The unstable region $\mathbf{L}$ with monotonic dynamics, given by the inequality $b < 1/3 - a$.
2. The stable region $\mathbf{M}_1$ with monotonic dynamics, given by the inequalities $1/3 - a < b < 1 - a$.
3. The stable region $\mathbf{M}_2$ with alternating dynamics, given by the inequalities $1 - a < b < 5/3 - a$.
4. The unstable region $\mathbf{H}$ with alternating dynamics which lies above the line $b = 5/3 - a$.

We can also determine what happens at the boundaries between these regions. See the Appendix for the details.

Now we partition the parameter space into regions which exhibit either a single fixed point or three fixed points, as well as a single point where every value $p \in [0,1]$ is a fixed point.

1. The region in which every value $p \in [0,1]$ is a fixed point is $\mathbf{F}_\infty := \{(1/3, 0)\}$.
2. The region $\mathbf{F}_1$ with only a single fixed point which is $1/2$ is given by the inequalities $1/3 - a \leq b$ and $b > 0$.
3. The region with three different fixed points is the complement $\mathbf{F}_3 := [0,1]^2 \setminus (\mathbf{F}_\infty \cup \mathbf{F}_1)$, i.e. the corner region around the origin and the a-axis excluding $(1/3, 0)$.

The regimes of the model can now be determined by forming intersections of the regions given above.

1. The region $\mathbf{F}_\infty$ is a regime of its own. Here the dynamics of the model are stationary.
2. The region $\mathbf{L} \subset \mathbf{F}_3$ is a regime characterised by having three fixed points. $1/2$ is unstable and there are two additional fixed points located at

   $$p_\pm := \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1-3a-3b}{1-3a+b}}.$$

   Note that in the fraction both the numerator and the denominator are positive. The points $p_\pm$ are attractors and their values are not $0, 1$ if and only if $b > 0$. The location not being at $0, 1$ is a new feature when compared to $b = 0$ and similar to the contrarian model where $a = b > 0$.
3. The region $\mathbf{M}_1 \cap \mathbf{F}_1$ exhibits a single fixed point $1/2$ that is a global attractor. The dynamics of the model starting at any p_0 is monotonic convergence towards $1/2$.
4. The region $\mathbf{M}_1 \cap \mathbf{F}_3$, which is the section of the a-axis with $a > 1/3$, has three fixed points, $0, 1/2, 1$, with $1/2$ being an attractor and $0, 1$ repellers.
5. The region $\mathbf{M}_2 \subset \mathbf{F}_1$ also has a single fixed point $1/2$ that is a global attractor. However, the dynamics of the model are dampened oscillations: starting at any p_0, the orbits alternate around the limit $1/2$.

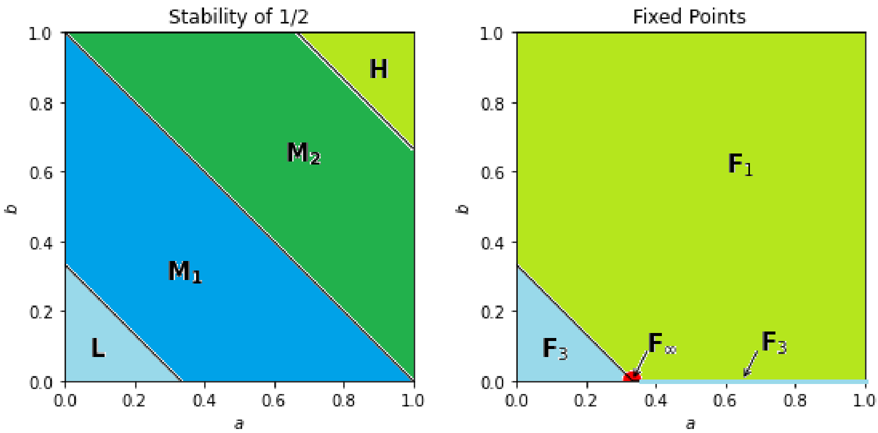

Figure 1: Regimes of the Local Flip Model, $r = 3$

6. The region $\mathbf{H} \subset \mathbf{F}_1$ is characterised by having a single fixed point $1/2$ which acts as a repeller. The dynamics are oscillatory, so there are wild swings from one round to the next from large majorities for A to large majorities of B and back. In the extreme case $(a, b) = (1, 1)$, we obtain a model that is the inverse of the basic model: $R_{3,1,1}(p) = 1 - P_3(p)$. The subsequences p_{2n} and p_{2n-1} converge and their limits can be calculated explicitly:

$$\omega_{\pm} := \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{-10 + 6a + 6b}{-2 + 6a - 2b}}.$$

These accumulation points are created at the centre, $p = 1/2$, when we cross over from $\mathbf{M}_2$ to $\mathbf{H}$ and they wander towards the corners $0, 1$ as the parameter values (a, b) go toward the corner $(1, 1)$. There is a period-doubling bifurcation when passing from $\mathbf{M}_2$ to $\mathbf{H}$.

A graph of the regimes can be found in Figure 1. In the Introduction, we mentioned two scenarios: the vertical frame is the study of hierarchical voting, and we consider that a representative who decides whether to deviate from the group majority has a stronger incentive to do so when the majority is of size 1, than when it is of size 3. Thus, in this scenario, a reasonable assumption would be $a > b$. In the horizontal frame, the group discussions may lead to the phenomenon of contrarianism, where we consider that a larger majority may strengthen the resolve of the contrarians. This interpretation of the model leads to the assumption $a < b$. As we see in Figure 1, these two assumptions neatly bisect the two dimensional parameter space, with the vertical frame being the lower right triangle and the horizontal frame, the upper left triangle. Thus, with the exceptions of $\mathbf{F}_\infty$ and $\mathbf{M}_1 \cap \mathbf{F}_3$, all other regions, both concerning the stability of $1/2$ and the number of fixed points overlap with both assumptions.

4.2 Group Size $r = 5$

We will analyse various aspects of the model when the groups are of size $r = 5$. This model has three flip parameters: in addition to a and b which are flip probabilities for $|S|$ equals 1 and 3, respectively, we have the flip parameter c for $|S| = 5$. First, we will talk about a linear regime of the model which allows us to determine the model's behaviour completely. Then, we will take a look at the three one-parameter models where only one of a, b, c is different than 0. We will subsequently compare these models to the contrarian model which belongs to the category of one-parameter models, too. To conclude this section, we will analyse the two-parameter model with $c = 0$ similarly to the discussion in Section 4.1.

c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
a	0.4	0.42	0.44	0.46	0.48	0.5	0.52	0.54	0.56	0.58	0.6
b	0.2	0.26	0.32	0.38	0.44	0.5	0.56	0.62	0.68	0.74	0.8

Table 6: Parameter Values of the Linear Regime ($r = 5$)

For groups of size 5, the update equation is

$$\begin{aligned} R_{5,a,b,c}(p) = {} & (6 - 20a + 10b - 2c)\, p^5 + (-15 + 50a - 25b + 5c)\, p^4 \\ & + (10 - 40a + 30b - 10c)\, p^3 + (10a - 20b + 10c)\, p^2 \\ & + (5b - 5c)\, p + c. \end{aligned} \tag{4}$$

By varying all three flip parameters simultaneously, we obtain a 'linear' regime, i.e. for certain values of the flip parameters, the update function $R_{5,a,b,c}$ becomes an affine function of the form

$$R_{5,a,b,c} = \alpha + \beta p. \tag{5}$$

For $r = 5$, we can determine the region in the parameter space $[0,1]^3$ explicitly for which $R_{5,a,b,c}$ has the form (5). From the update equation (4), we obtain the coefficients of each power $p^k, k = 2, \ldots, 5$, and equate them to 0. Thus, we obtain a linear equation system, which we solve to obtain the solutions given by

$$5a - c = 2 \quad \text{and} \quad 5b - 3c = 1.$$

This reduced equation system describes a line through parameter space, along which the update function has the form (5), with the coefficients $\alpha = c$ and $\beta = 1 - 2c$. We can parametrise this line through parameter space as $\gamma : [0,1] \to [0,1]^3$ as a function of c. The first coordinate of γ is $\gamma_1(c) = 2/5 + c/5 = a$, the second $\gamma_2(c) = 1/5 + 3c/5 = b$, and the last coordinate is the value $\gamma_3(c) = c$. One way of looking at this is to say that for each possible value of flip parameter c, we have a unique configuration of all parameters that yields the linear regime. Some values of γ are given in Table 6. We will call this region of the parameter space $\mathbf{Lin} = \gamma([0,1])$. Except for the two extremes, $(a,b,c) = (0.4, 0.2, 0)$ and $(a,b,c) = (0.6, 0.8, 1)$, there is a unique fixed point $1/2$ which is a global attractor. For $c \leq 1/2$, the dynamics are monotonic, whereas for $c > 1/2$, the dynamics are alternating.

The existence of this linear regime allows us to find a result concerning the fixed points of the update function $R_{5,a,b,c}$ given in (4): fix a point $(a_0, b_0, c_0) \in \mathbf{Lin}$. Then for all $c > c_0$, the update function $R_{5,a_0,b_0,c}$ is strictly convex on the interval $[0, 1/2]$ and strictly concave on $[1/2, 1]$. Similarly, if $c < c_0$, then $R_{5,a_0,b_0,c}$ is strictly concave on the interval $[0, 1/2]$ and strictly convex on $[1/2, 1]$. In both cases, there are no fixed points aside from $1/2$ and no 2-cycles (with the single exception of the 2-cycle $(0,1)$ if $c = 1$). Therefore, if $(a_0, b_0, c_0) \in \mathbf{Lin}$, then, for all $c \in (0, 1]$, $R_{5,a_0,b_0,c}$ has no fixed points aside from $1/2$. See Figure 2 for an illustration of this behaviour. It should be noted that – generally speaking – when all three parameters are distinct, $R_{5,a,b,c}$ will be neither concave nor convex over the intervals $[0, 1/2]$ and $[1/2, 1]$.

Rigorous results concerning the existence and number of fixed points aside from $1/2$ in this style are generally difficult to come by for $r \geq 5$ when all parameters assume distinct (non-zero) values.

In the right graph in Figure 2, any value $0 < c < 0.4$ would serve as a model for a scenario that involves an initially dominant opinion, and then, after one person deviates, others follow suit, eventually leading to an even split.

Next, we consider models in which flips can only happen in the case of a certain absolute voting margin.

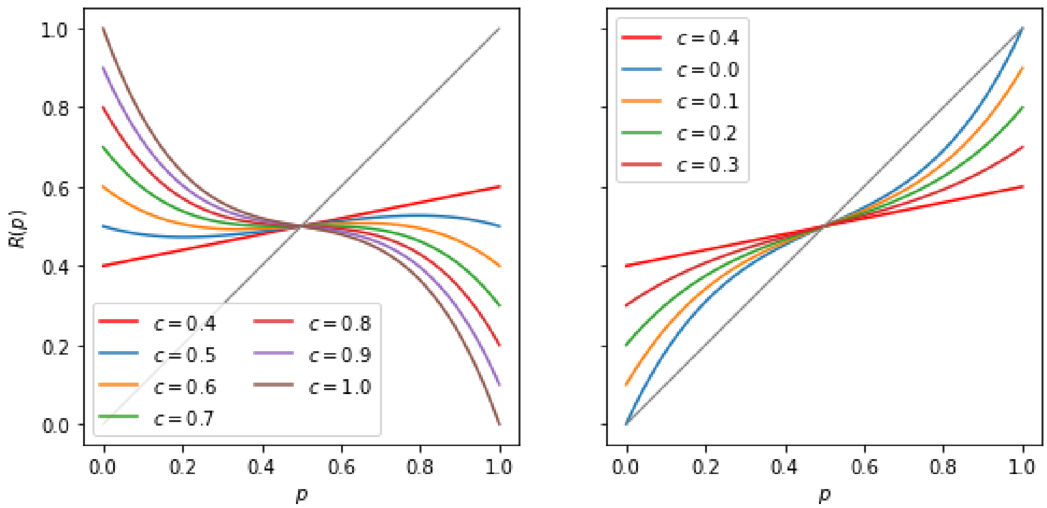

Figure 2: Linear Regime Separates Concave and Convex $R_{5,a_0,b_0,c}$ with $a_0 = 0.48, b_0 = 0.44, c_0 = 0.4$

4.2.1 $|S| = 1$

Flips are now only possible for minimal majorities, i.e. when $|S| = 1$, and $b = c = 0$. The update equation (4) thus becomes

$$R_{5,a}(p) = (6 - 20a)\,p^5 + (-15 + 50a)\,p^4 + (10 - 40a)\,p^3 + 10ap^2 \tag{6}$$

The fixed points vary in number and stability depending on a. Here is an overview:

1. For small $a \in [0, 7/10]$, the model behaves similarly to $a = 0$, i.e. the basic model. So $0, 1/2, 1$ are the only fixed points and they have their usual stability properties.
2. If $a = 3/10$, the update equation reduces to $R_{5,3/10}(p) = R_{3,0}(p)$.
3. At $a = 7/10$, there is a phase transition.
4. For $a \in (7/10, 1]$, there are five fixed points. The new pair of fixed points is created at $1/2$, it is symmetric around $1/2$, and located at

$$p_\pm := \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{10a - 7}{10a - 3}}. \tag{7}$$

   Thus, we have a subcritical pitchfork bifurcation at $a = 7/10$.
5. The fixed points 0 and 1 are always stable – independently of a. The fixed point $1/2$ is unstable in the regime $a \in [0, 7/10]$ and stable for $a \in (7/10, 1]$. The additional fixed points $p_\pm$ are unstable.

The behaviour with five fixed points only surfaces when there is a large probability of flipping the vote.

This version of the model fits into the vertical frame interpretation mentioned in the Introduction. We next investigate how the model changes if we only consider flips at an absolute voting margin of 3.

4.2.2 $|S| = 3$

Now only flips at absolute voting margins equal to 3 are considered. So the model has a single flip parameter b. The update equation (4) under this assumption is

$$R_{5,b}(p) = (6 + 10b)\, p^5 + (-15 - 25b)\, p^3 + (10 + 30b)\, p^3 - 20bp^2 + 5bp. \tag{8}$$

The values $0, 1/2, 1$ are fixed points of $R_{r,b}$ for all values of b. However, the number of fixed points and their stability depends on b. Here is an overview:

1. For small $b \in [0, 1/5]$ and for large $b \in [7/15, 1]$, the model has the three fixed points $0, 1/2, 1$. On the interval $(1/5, 7/15)$, there are five fixed points. The two additional fixed points are

$$p_\pm := \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{7 - 15b}{3 + 5b}}.$$

2. For $b \leq 1/5$, 0 and 1 are stable and $1/2$ is unstable. For all $b > 1/5$, the fixed points $0, 1$ are unstable. However, the stability of $1/2$ still varies for $b > 1/5$.
3. On the interval $(1/5, 7/15)$, $1/2$ is unstable and $p_\pm$ are stable. For $b \geq 7/15$, $1/2$ is stable.

It should be mentioned that this version of the model with $a = c = 0$ does not fit into either the vertical or the horizontal frame interpretation mentioned in the Introduction, because the finite sequence a, b, c is neither increasing nor decreasing here (unless we disregard the possibility of flips for $|S| = 5$, in which case it fits into the horizontal frame).

4.2.3 $|S| = 5$

Now we consider a model with the possibility of a flip in case of absolute voting margins equal to 5. So the flip parameters a and b are equal to 0. The update equation is

$$R_{5,c}(p) = (6 - 2c)\, p^5 + (-15 + 5c)\, p^4 + (10 - 10c)\, p^3 + 10cp^2 - 5cp + c. \tag{9}$$

It should be noted that for $r = 5$ the parameter c is the flip parameter of unanimous majorities. We observe that $0, 1$ are fixed points if and only if $c = 0$.

Previously we saw that a and b could induce a phase transition. The flip parameter c, on the other hand, cannot change the stability of $1/2$ on its own. However, c does change the location of the other two fixed points. In fact, we observe that the fixed points $p_\pm$, which lie between 0 and 1, wander towards $1/2$ as we increase c. The range $[0, 1]$ is not sufficient to arrive at $1/2$. So, even for $c = 1$, there are three fixed points. It seems that $p_\pm$ are both stable throughout the entire range of c as we see in Figure 3. This is related to the fact that $p_\pm$ are close to local extrema of $R_{5,c}$, where the derivative is 0.

This version of the model fits into the horizontal frame interpretation mentioned in the Introduction.

To conclude this part of the article, we compare the behaviour of four one-parameter models: the contrarian model from Section 3 and the $|S| = 1, 3, 5$ models from this section. These graphs are found in Figure 4. We see that both a and b can change the stability of the fixed point $1/2$. For small values of these parameters, $1/2$ is unstable and $0, 1/2, 1$ are the only fixed points. As the respective parameter increases, $1/2$ becomes stable and an additional pair of fixed points $p_\pm$ appears. Interestingly, the mechanism by which the parameters create these fixed points is different: the parameter a leaves $0, 1$ superstable and only affects the stability of $1/2$. Thus, the unstable fixed points $p_\pm$ appear at the point where $1/2$ switches to an attractor. The parameter b on the other hand affects the stability of both $0, 1$ and $1/2$. For small values of b, the fixed points $0, 1$ are still stable (but no longer superstable!) and $1/2$ is unstable. At a critical value $b = 1/5$, the fixed points $0, 1$ become unstable while $1/2$ is still

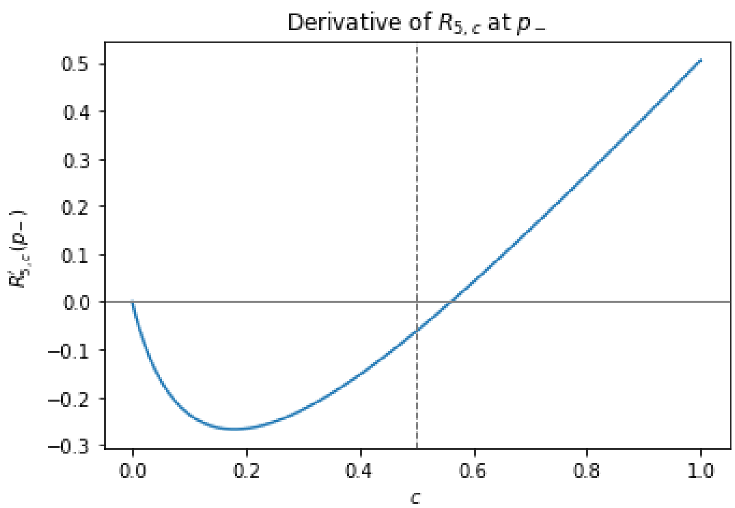

Figure 3: Stability of $p_\pm$

unstable. This is only possible due to the creation of two additional fixed points $p_\pm$ which are stable. As b further increases beyond $b = 1/2$, the fixed point $1/2$ becomes an attractor and $p_\pm$ disappear.

The parameter c, however, cannot on its own switch $1/2$ from a repeller to an attractor. In the Appendix, we analyse the effect of flip parameters for the general r model. In equation (15), we define the stability parameter λ_r which is the derivative of the update function R_r at $1/2$. The finding that c cannot affect the stability of $1/2$ is in line with the derivatives of λ_r with respect to the flip parameter a_r found in Table 8. (In the notation a_i, i stands for the number of votes in favour of A; thus, for $r = 5$, $a = a_3, b = a_4, c = a_5$.) Note that even for the case considered here where $r = 5$, the partial derivative $\partial\lambda_5/\partial c$ is -0.62. Since this partial derivative does not depend on the value of c according to (17), this explains why c cannot affect the stability of $1/2$: even going from $c = 0$ to $c = 1$ does not change λ_5 enough.

In Figure 5, we compare the contrarian model to three different two-parameter models, in which we set one of the three flip parameters a, b, c equal to 0 and allow the other two to vary. We note that the model with $a = b = 0$ is qualitatively similar for any value of c to the contrarian model with low contrarian probability. In Figure 6, we present a scenario where all three parameters are non-zero and the model is very close to $R_{5,a,b,c}(p) = p$ but there are three fixed points. The additional fixed points $p_\pm$ are half-stable: they are stable from the outside, i.e. from the left in case of $p_- = 0.13$ and from the right for $p_+ = 0.87$, and unstable from the inside, i.e. from the direction of $1/2$. This gives rise to a pattern where $p_0 \in (0, p_-)$ implies convergence to p_-, $p_0 \in (p_-, p_+)$ leads to convergence to $1/2$, and $p_0 \in (p_+, 1)$ towards p_+. The dynamics are monotonic in all cases. Both the models with $b = 0$ and $a = 0$ can explain phenomena where initially one of the opinions dominates completely, and then, after one person deviates, others follow suit, with convergence either to some intermediate point $p_\pm$ or $1/2$.

4.2.4 $|S| \leq 3$

Now we analyse a two-parameter model that includes flips for absolute voting margins equal to 1 and 3. That is a and b lie in $[0, 1]$ and $c = 0$. For this case, we can solve the model in similar depth for

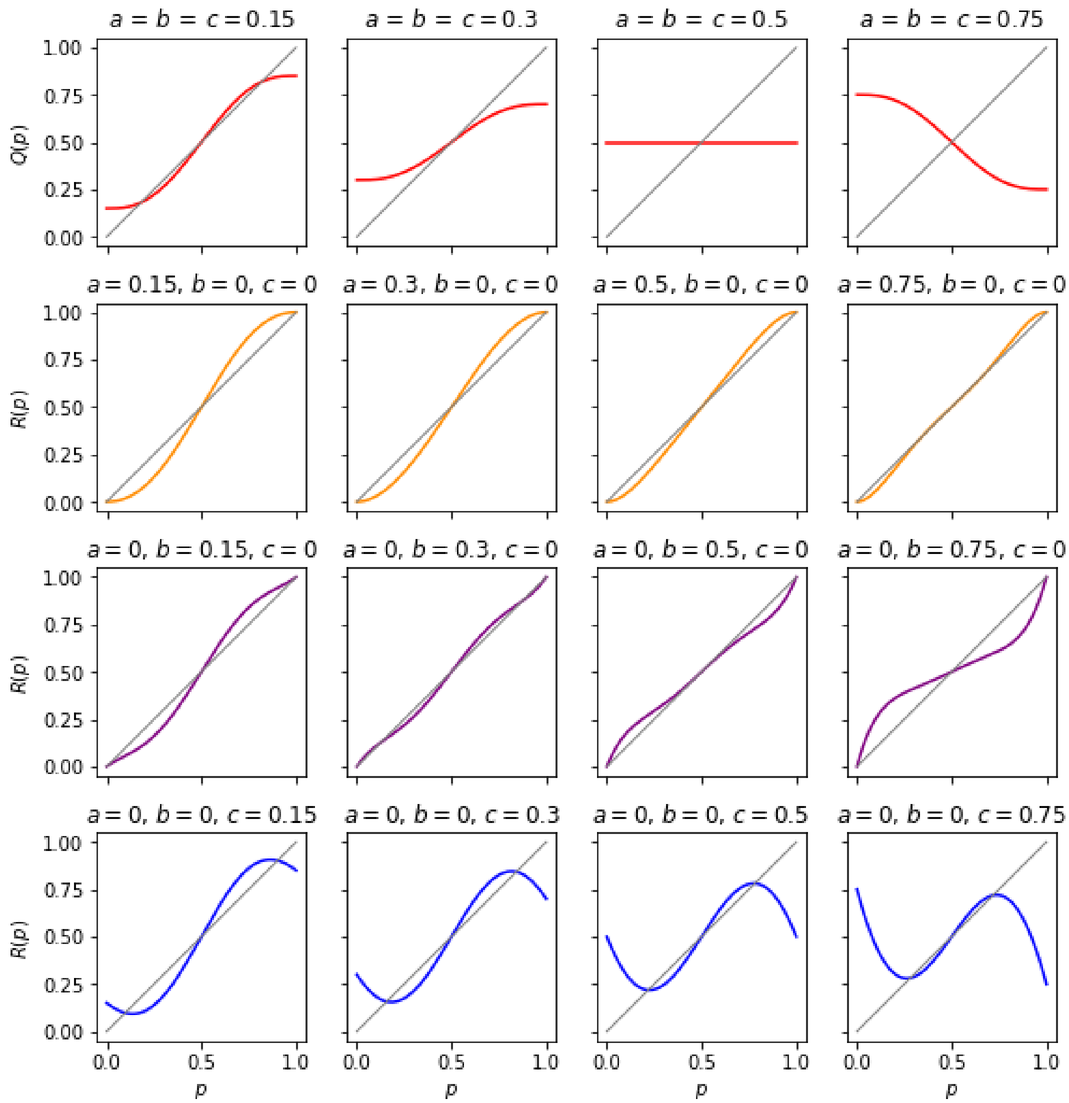

Figure 4: Comparison of the Contrarian and One-Parameter Local Flip Models

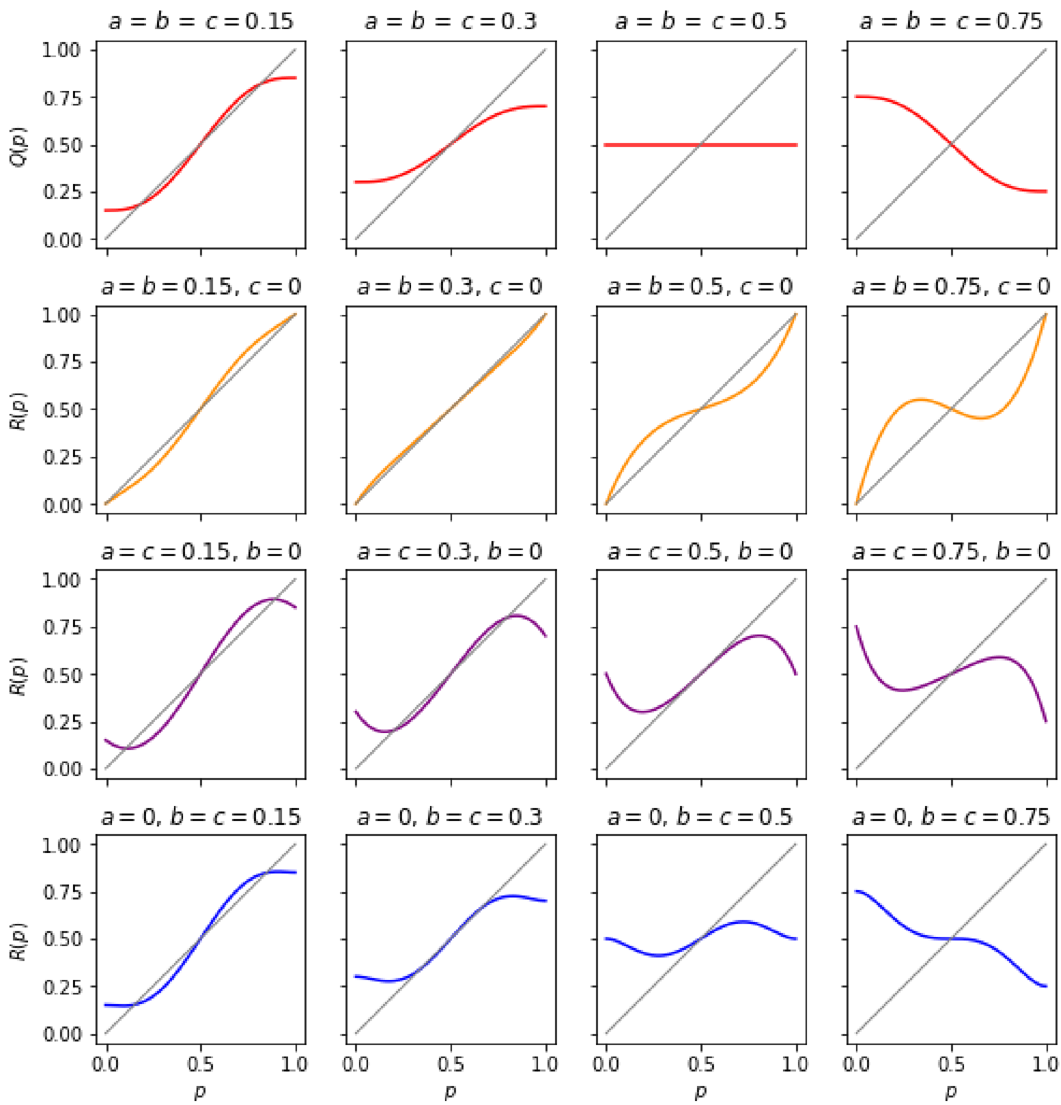

Figure 5: Comparison of the Contrarian and Two-Parameter Local Flip Models

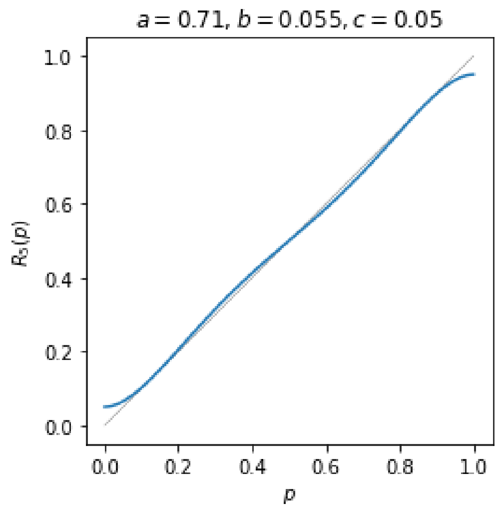

Figure 6: Half-Stable Fixed Points in Local Flip Models

$r = 5$ as we did for $r = 3$ in Section 4.1. The update equation is

$$R_{5,a,b,c}(p) = (6 - 20a + 10b)\, p^5 + (-15 + 50a - 25b)\, p^3 \\ + (10 - 40a + 30b)\, p^3 + (10a - 20b)\, p^2 + 5bp. \tag{10}$$

Contrary to $r = 3$, $0, 1/2, 1$ are always fixed points for $r \geq 5$. We proceed as in Section 4.1, and partition the parameter space $[0,1]^2$ first according to the stability of the fixed point $1/2$ and then according to the number of fixed points the model has.

As far as the stability of $1/2$ is concerned, we distinguish four regions:

1. The unstable region $\mathbf{L}$ with monotonic dynamics, given by the inequality $b < 7/15 - 2a/3$.
2. The stable region $\mathbf{M}_1$ with monotonic dynamics, given by the inequalities $7/15 - 2a/3 < b < 1 - 2a/3$.
3. The stable region $\mathbf{M}_2$ with alternating dynamics, given by the inequalities $1 - 2a/3 < b < 23/15 - 2a/3$.
4. The unstable region $\mathbf{H}$ with alternating dynamics which lies above $b = 23/15 - 2a/3$.

A similar remark as for $r = 3$ regarding the inclusion of the boundaries applies. The analysis is found in the Appendix.

Now we partition the parameter space into regions according to the number of fixed points.

1. The region in which every value $p \in [0,1]$ is a fixed point is $\mathbf{F}_\infty := \{(2/5, 1/5)\}$.
2. The region $\mathbf{F}_5$ which has five fixed points consists of the union of the two areas $1/5 < b < 7/15 - 2a/3$ and $7/15 - 2a/3 < b < 1/5$.
3. The region $\mathbf{F}_3$ with only the fixed points $0, 1/2, 1$ is the complement of $\mathbf{F}_\infty \cup \mathbf{F}_5$.

The regimes of the model can now be determined by forming intersections of the regions given above.

1. The region $\mathbf{F}_\infty$ is a regime of its own. Here the dynamics of the model are stationary.

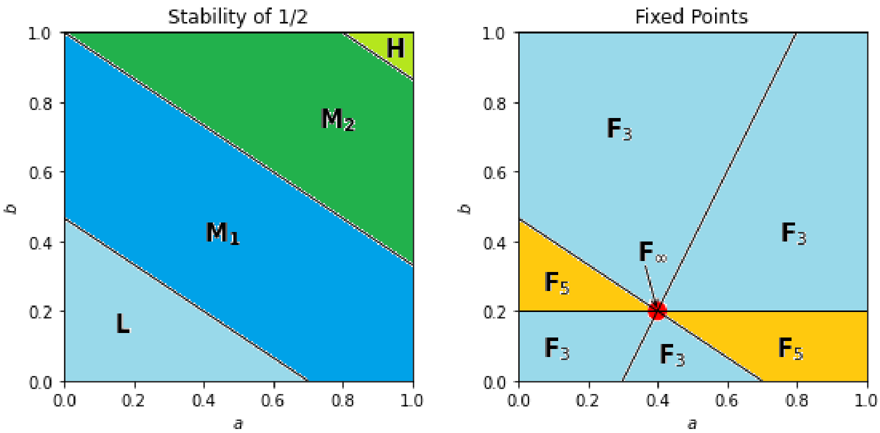

Figure 7: Regimes of the Local Flip Model ($|S| \leq 3$), $r = 5$

2. The region $\mathbf{L} \cap \mathbf{F}_3$ is a regime characterised by having the three fixed points $0, 1/2, 1$. $1/2$ is unstable and $0, 1$ are stable. This is the regime most similar in behaviour to the basic model since here the flip probabilities are low.

3. The region $\mathbf{L} \cap \mathbf{F}_5$ has five fixed points: there are two additional fixed points located at

$$p_{\pm} := \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{7 - 10a - 15b}{3 - 10a + 5b}}. \tag{11}$$

   Note that in the fraction both the numerator and the denominator are positive. As $1/2$ is unstable, the points $p_{\pm}$ are attractors and $0, 1$ are repellers. This is a feature the one-parameter model with $b = c = 0$ does not exhibit: if $b = 0$, then the points $p_{\pm}$ are unstable.

4. The region $\mathbf{M}_1 \cap \mathbf{F}_3$ has the fixed points $0, 1/2, 1$. Here, $1/2$ is stable and the dynamics are monotonic.

5. The region $\mathbf{M}_1 \cap \mathbf{F}_5$ has five fixed points, $0, 1/2, 1$, and $p_{\pm}$ given by the formula above. $0, 1/2, 1$ act as attractors and $p_{\pm}$ as repellers.

6. The region $\mathbf{M}_2 \subset \mathbf{F}_3$ also has the fixed points $0, 1/2, 1$ with the same stability properties as $\mathbf{M}_1 \cap \mathbf{F}_3$. However, the dynamics of the model starting at any p_0 are now alternating with limit $1/2$.

7. The region $\mathbf{H} \subset \mathbf{F}_3$ is characterised by having the fixed points $0, 1/2, 1$ which acts as a repeller. The dynamics are alternating, so there are wild swings from one round to the next from large majorities for A to large majorities of B and back. $0, 1$ are fixed points contrary to the corresponding regime for $r = 3$, so if p_0 is even slightly different than 0 or 1, the dynamics tend to produce large swings. There are accumulation points of the dynamics in this regime provided $p_0 \in (0, 1)$. Similarly to $r = 3$, these are symmetric $\omega_- \in (0, 1/2)$ and $\omega_+ = 1 - \omega_-$. However, as $0, 1$ are fixed points, not even in the extreme case $(a, b) = (1, 1)$ do we see the complete unanimity alternating between A and B. Instead, if the initial $p_0 > 0$ is very close to the origin, then there is some lead time before the oscillation between majorities for A and B starts. The discrepancy between the group sizes 3 and 5 is because for the latter we are not allowing flips for unanimous configurations.

A graph of the regimes can be found in Figure 7.

The regime $\mathbf{L} \cap \mathbf{F}_5$ exhibits five fixed points: two attractors $p_\pm$ given by (11) which lie inside $(0, 1/2)$ and $(1/2, 1)$, respectively. So the dynamics tend to majorities for one of the options. If only a is allowed to vary (as in Section 4.2.1), then there is no such regime: the only regime which exhibits five fixed points has different stability properties. The additional fixed points given by (7) act as tipping points that push the dynamics of the model either to a fifty-fifty split (if $p_- < p_0 < p_+$) or towards unanimous opinions (if $|p_0 - 1/2| > p_+ - 1/2$). Also, the regime $\mathbf{L} \cap \mathbf{F}_5$ appears at lower flip probabilities than the five fixed point regime in the model with $b = c = 0$.

In the Introduction, we mentioned two scenarios for the local flip models: in the vertical frame, a reasonable assumption would be $a > b$. The horizontal frame interpretation leads to the assumption $a < b$ (if we disregard the possibility of flips for $|S| = 5$). As we see in Figure 7, the situation is similar to $r = 3$, as these two assumptions bisect the two dimensional parameter space, with the vertical frame being the lower right triangle and the horizontal frame, the upper left triangle. Once again, with the exception of $\mathbf{F}_\infty$, all other regions, both concerning the stability of $1/2$ and the number of fixed points overlap with both assumptions. However, contrary to $r = 3$, here we have two additional fixed points, and as mentioned in the last paragraph, their stability properties differ in the left hand side $\mathbf{L} \cap \mathbf{F}_5$, where $p_\pm$ are attractors. This region belongs mostly to the horizontal frame interpretation, although a small part of it lies in the lower right triangle of the vertical frame. However, the right hand side region $\mathbf{M}_1 \cap \mathbf{F}_5$, where $p_\pm$ are repellers, belongs entirely to the vertical frame interpretation of the model. Thus, contrary to $r = 3$, the two interpretations of the model, and their accompanying assumptions on the flip parameters, do lead to different behaviours of the model. In particular, in the two regimes with five fixed points each, the vertical frame induces dynamics that tend to sizeable (but not unanimous) majorities in favour of one of the two options. In the horizontal frame, the dynamics for the most part either tend to an even split if p_0 starts between $p_\pm$ and $1/2$, or toward unanimous majorities if p_0 starts between 0 and p_- or between p_+ and 1.

5 Novel Behaviour of the Local Flip Model – A Comparison of the Three Models

We have defined three models in the previous three sections of this article. In this section, we first take a look at the behaviour of each of the three models, and then we describe the specific novel aspects of the local flip model which do not occur in the other two models. Each model is completely determined by its respective update equation given in (1), (2), and (12), and the model's properties can be deduced by analysing these update equations.

5.1 Basic Model

We have seen that the basic model does not exhibit a phase transition. For any update group size $r \in \mathbb{N}$, the model exhibits three fixed points. These are $0, 1/2, 1$. The stability properties of these fixed points do not depend on r. Instead, 0 and 1 are always stable, and $1/2$ is always unstable. For any initial probability $p_0 \neq 1/2$ of a preference for A, the distribution of opinions tends to the closer of the two attractors, either 0 or 1. The third fixed point acts as a repeller. This behaviour is stable over all group sizes r. In fact, as r increases, the dynamics towards the attractors become faster.

We can sum up the basic model by stating that there are stable unanimous fixed points and a tipping point $1/2$ which separates the two basins of attraction.

For the other two models, we have observed that there are phase transitions depending on the parameters, which in both models measure a probability that some kind of deviation from the majority occurs.

5.2 Contrarian Model

In the contrarian model, each individual decides whether to go along with the majority opinion or oppose it, regardless of their own initial opinion. The contrarian model is a one-parameter model. It exhibits a phase transition for all group sizes r although the critical value a_c increases with the group size. In the low contrarian probability regime, i.e. $a < a_c$, the model behaves similarly to the basic model with the main difference being the shift of the stable fixed points from 0 and 1 to the points $p_\pm$ given in equation (3), which lie in the interior of the interval $[0, 1]$. The change of $p_\pm$ is continuous with respect to the parameter value a on the interval $[0, a_c]$. In the intermediate contrarian regime $a \in [a_c, 1 - a_c]$, the stable fixed points $p_\pm$ merge with the repeller $1/2$, and this fixed point becomes globally stable. For high contrarian probabilities, $a > 1 - a_c$, $1/2$ is once again a repeller, and it is the only fixed point of the model. However, as opposed to the low regime, p_n does not converge to either p_- or p_+; instead, there is an alternating pattern with subsequences converging to $p_\pm$. Similarly to the basic model, the contrarian model does not exhibit more than three fixed points for any group size. Also, we note that the contrarian model does not feature $p_\pm$ as unstable fixed points. As we saw, the local flip model does have this distinction.

5.3 Local Flip Model

We considered two different scenarios to be explained by the local flip model. One is hierarchical voting, where the representative, whose task it is to pass along the majority decision of their group, sometimes decides to flip to the contrary opinion but does so taking into consideration the size of the majority. The other scenario is opinion dynamics, where each person decides on their own whether to adopt the majority opinion.

For group size r, the local flip model spans a $(r + 1)/2$-dimensional parameter space, in contrast to the basic and the contrarian model, which are restricted to zero- and one-dimensional parameter spaces, respectively. Exploring the multidimensional parameter space of the local flip model reveals a wide variety of different and novel behaviours. In particular, depending on the region in the parameter space, we found dynamics which are driven by respectively one, three, and – for the first time – five fixed points for groups of size $r = 5$. The finding of five fixed points is new and significant in that, for both the basic and the contrarian models, all sizes r for the discussion groups always yield only one or three valid fixed points. The others roots of the polynomial of degree r that is the update function given by (1) and (2), respectively, lie outside the interval $[0, 1]$.

While $1/2$ is always a fixed point, its stability is contingent on the parameter values. Moreover, in the local flip model, the fixed points 0 and 1 can become unstable, contrary to what happens in the other two models where they are always attractors. Likewise the additional fixed points $p_\pm$, which lie respectively in the interior of the intervals $(0, 1/2)$ and $(1/2, 1)$, can be stable or unstable. These changes of stability create a series of counterintuitive dynamics with flow toward or away from $1/2$ and $p_\pm$. The phenomena of convergence to large majorities, ties, or alternating majorities are obtained within one single framework. Within the local flip model, both the basic and the contrarian model appear as special cases. In addition, new dynamics are found, which do not occur in those two models. For instance, unlike in the contrarian model, where $p_\pm$ are always attractors, in the local flip model $p_\pm$ can be tipping points directing the dynamics towards either $1/2$ or one of the unanimous fixed points 0 and 1.

This new feature highlights a novel strategy for the competing alternatives A and B. For A to avoid a loss, an initial support $p_0 > p_-$ is required; otherwise, the dynamics tend to 0 and a unanimous majority in favour of B. When $1/2 > p_0 > p_-$, the support for A increases to reach a tie at $1/2$. However, getting a high initial support $p_+ > p_0 > 1/2$ is futile and even a waste since then the dynamics tend down to $1/2$. To trigger an increase in support, A must reach an initial support $p_0 > p_+$. These results suggest a twofold strategy to either reach a tie or to win, with an interval of support $(1/2, p_+)$ which is illusionary in the sense that it does not suffice to win in the end.

	Basic	Contrarian	Local flip $b=c=0$	Local flip $a=c=0$	Local flip $a=b=0$	Local flip $c=0$
$0,1$ fixed points	✓	○	✓	✓	○	✓
$0,1$ stable	✓	✓[1]	✓	○	✓[1]	○
$1/2$ fixed point	✓	✓	✓	✓	✓	✓
$1/2$ unstable	✓	○	○	○	✓	○
Phase transition	×	✓	✓	✓	×	✓
Min # fixed points	3	1	3	3	3	1
Max # fixed points	3	3	5	5	3	5
Alternating pattern	×	○	×	×	×	○

Table 7: Summary of the Models for $r = 5$ (✓ 'yes', ○ 'sometimes', × 'no')

We summarise the properties of the three models in Table 7.

6 Conclusion

In this article, starting from the Galam model of opinion dynamics which uses local majority rules to update individual opinions in small discussion groups, we have included probabilistic deviations from the local majority as a function of the ratio majority / minority, which resulted in the local flip model of opinion dynamics and hierarchical voting. Investigating groups of size 3 and 5 has revealed a series of novel features which were absent in the previous works. In particular, for group size 5, the dynamics are driven by five fixed points, which split into three attractors and two tipping points or two attractors and three tipping points, depending on the parameter values. Such an unexpected and new interplay of fixed point stabilities creates a rich diversity of non-linear dynamics, which could shed some new light on several social phenomena triggered by one or a few individuals acting against larger local majorities, such as:

- At a party, initially people are standing around, having conversations. Then at some point in time, the first person starts dancing and others follow. After a short time, some fixed proportion of all people in attendance are dancing. This illustration was considered using a different model [37, 27]. Note that the iteration can consist of the people moving around the room and thus reshuffling the update groups in which conversations take place. Also, every new song which is played can be considered a new round of discussions as it changes the dynamics.
- During a demonstration, one person starts throwing objects at the police and others close by follow suit. Rioters move among demonstrators and drag more and more people into violence. Also, in this scenario, the movement in the crowd leads to iterations with different configurations of small groups that influence each other.
- Sharing some confidential or sensational piece of information, be it political, societal, or financial in nature, can cause a few people to shift their opinion. They will keep their new stance after the meeting and subsequently even use the new information to convince others in subsequent meetings. This includes the spreading of fake news.

[1] Provided $0, 1$ are fixed points.

In future work, we intend to introduce asymmetric flip probabilities along the same lines as the asymmetric contrarianism probabilities in [22]. Another direction we will consider is the case of large size update groups with correlated initial opinions, which could be more appropriate to describe some social situations involving crowd behaviour.

Appendix

Boundary Analysis between Regimes

We analysed the two-parameter model for $r = 3$ in Section 4.1. As mentioned there, we can determine the behaviour of the model on the boundary between the stability regions of the universal fixed point 1/2. On the boundary between the regions $\mathbf{L}, \mathbf{M}_1, \mathbf{M}_2$, and $\mathbf{H}$, the first derivative of the update function $R'_{3,a,b}(1/2)$ has some critical value: $-1, 0$, or 1. Therefore, it is not possible to determine the stability just by looking at the first derivative. The second derivative at 1/2 is 0, so we go to the third derivative. The line $b = -1 + 3a$ represents the location of all combinations of a and b such that the third derivative $R'''_{3,a,b}(1/2)$ equals 0. To the left of this line, the third derivative is negative; to the right, it is positive. Thus, to the left of $b = -1 + 3a$, the boundaries between $\mathbf{L}$ and $\mathbf{M}_1$, $\mathbf{M}_1$ and $\mathbf{M}_2$ belong to the higher one of the regions. To the right of the line, the pattern is opposite and the boundaries belong to the lower one of the neighbouring regions.

Similarly, for the two-parameter model with $r = 5$ and $c = 0$ analysed in Section 4.2.4, we can sort out the behaviour on the boundaries. The line $b = 1 - 2a$ represents the location of all combinations of a and b such that the third derivative $R'''_{5,a,b}(1/2)$ equals 0. To the left of this line, the boundary between $\mathbf{L}$ and $\mathbf{M}_1$ belongs to $\mathbf{M}_1$. To the right of the line, the pattern is opposite and the boundaries belong to the lower one of the neighbouring regions.

Larger Group Sizes

We discuss the general model of group size r. This model has a flip probability a_i for any number of votes belonging to the majority, i.e. $i = (r+1)/2, \ldots, r$, and symmetrically for a majority of the same respective magnitude for B. The update equation is given by

$$R_{r,\mathbf{a}}(p) = \sum_{i=\frac{r+1}{2}}^{r} \binom{r}{i} \left[(1-a_i)\, p^i (1-p)^{r-i} + a_i p^{r-i} (1-p)^i \right]. \tag{12}$$

The parameters a_i are the flip probabilities when there are i votes for A (or, symmetrically, i votes for B). Just as in the case of the contrarian model, the local flip model also reduces to the basic model if all parameters are 0: $\mathbf{a} = \left(a_{(r+1)/2}, a_{(r+3)/2}, \ldots, a_r\right) = (0, \ldots, 0)$. Due to the symmetry of the model, 1/2 is a universal fixed point for any group size. Furthermore, the unanimous points $0, 1$ are fixed if and only if $a_r = 0$, i.e. the probability that the group representative deviates from a unanimous group vote is 0. What is more, the values are

$$R_{r,\mathbf{a}}(0) = a_r \quad \text{and} \quad R_{r,\mathbf{a}}(1) = 1 - a_r. \tag{13}$$

Since disregarding a unanimous vote might easily lead to widespread discontent, assuming $a_r = 0$ may be reasonable. In fact, this is called the Pareto criterion in the voting theory literature; see e.g. Chapter 1 of [39]. The derivatives of $R_{r,\mathbf{a}}$ at $0, 1$ are given by

$$R'_{r,\mathbf{a}}(0) = R'_{r,\mathbf{a}}(1) = r\,(a_{r-1} + a_r). \tag{14}$$

Thus, only the two parameters of the unanimous and unanimous-but-one flip affect the stability of $0, 1$.

r	$\partial\lambda_r/\partial a_i$, where $i =$						
	$(r+1)/2$	$(r+3)/2$	$(r+5)/2$	$(r+7)/2$	$(r+9)/2$	$(r+11)/2$	$(r+13)/2$
3	-1.5	-1.5	–	–	–	–	–
5	-1.25	-1.88	-0.62	–	–	–	–
7	-1.09	-1.97	-1.09	-0.22	–	–	–
9	-0.98	-1.97	-1.41	-0.49	-0.07	–	–
11	-0.9	-1.93	-1.61	-0.75	-0.19	-0.02	–
13	-0.84	-1.89	-1.75	-0.98	-0.34	-0.07	-0.01
15	-0.79	-1.83	-1.83	-1.17	-0.5	-0.14	-0.02
17	-0.74	-1.78	-1.89	-1.32	-0.65	-0.23	-0.05
19	-0.7	-1.73	-1.92	-1.45	-0.8	-0.33	-0.1

Table 8: Partial Derivatives of λ_r (Rounded to Two Digits)

If $a_{r-1} = a_r = 0$, then $0, 1$ are superstable fixed points. Thus, there is some neighbourhood of 0 or 1 in which convergence towards 0 or 1 is very fast.

As we saw in Section 4.1, even varying a single flip parameter can switch the model into another regime when $r = 3$. Now we turn to the question of how much a single flip parameter can influence the dynamics of the model when the group size is larger. Specifically, we analyse the stability of $1/2$. We define

$$\lambda_r := R'_{r,\mathbf{a}}(1/2) = \frac{1}{2^{r-1}} \sum_{i=\frac{r+1}{2}}^{r} \binom{r}{i} (1-2a_i)(2i-r). \tag{15}$$

We observe that λ_r can be expressed as the sum of the derivative of the update function of the basic model, $P'_r(1/2)$, and another term

$$-\frac{1}{2^{r-2}} \sum_{i=\frac{r+1}{2}}^{r} \binom{r}{i} a_i (2i-r) \tag{16}$$

which is negative. The stability parameter λ_r of the fixed point $1/2$ can be regarded as a differentiable function of the flip parameters a_i: $\lambda_r : [0,1]^{\frac{r+1}{2}} \to \mathbb{R}$ Thus, we will also write $\lambda_r(\mathbf{a})$. The partial derivatives

$$\frac{\partial\lambda_r}{\partial a_i} = -\frac{1}{2^{r-2}} \binom{r}{i} (2i-r) \tag{17}$$

are all negative, whereas λ_r can be positive or negative. This means that in regimes where λ_r is positive, increasing a_i contributes to making $1/2$ more stable. In regimes with alternating dynamics around $1/2$, increasing a_i makes $1/2$ less stable. We present some of the values in Table 8. As we see, the flip parameter with the largest impact is not $a_{(r+1)/2}$ but rather some a_i with i close to $(r+1)/2$. So, to change the dynamics, flips of small – but not minimal – majorities are most effective. We also observe that the flip parameter a_r, which by (13) completely determines the values $R_{r,\mathbf{a}}(0) = a_r$ and $R_{r,\mathbf{a}}(1) = 1 - a_r$, only marginally affect the stability of the universal fixed point $1/2$. Lastly, note that $\partial\lambda_r/\partial a_i$ is constant with respect to any of the flip parameters. In fact, we can characterise the behaviour of the index i with the most negative partial derivative. Asymptotically, i.e. for large r, the index i with the most negative partial derivative $\partial\lambda_r/\partial a_i$ is $i \approx r/2 + \sqrt{r}/4$. Here, we used the symbol '$\approx$' in the sense that two sequences $f(n), g(n)$ are asymptotically equivalent, i.e. $f(n) \approx g(n)$, if $\lim_{n\to\infty} f(n)/g(n) = 1$.

However, each flip parameter individually does not have a significant effect on the stability of $1/2$ as r grows large. As we will see, for r as small as 13, no single parameter on its own can turn $1/2$ stable.

r	5	7	9	11
$b_c(r)$	0.47	0.6	0.74	0.88

Table 9: Critical Values for b (Rounded to two Digits)

Hence, the statement concerning the most impactful flip parameter should be interpreted in the sense that a set of flip parameters with indices close to $r/2 + \sqrt{r}/4$ jointly have the most impact.

The situation differs with respect to the fixed points $0, 1$. We already know from (14) that only the two parameters a_{r-1} and a_r can affect the stability of these fixed points. Since $a_r > 0$ implies that $0, 1$ are not fixed points in the first place, we see that a_{r-1} determines on its own whether $0, 1$ are stable or unstable. Given $a_r = 0$, the points $0, 1$ are stable if and only if $a_{r-1} \leq 1/r$. Interestingly, as the group size becomes larger, even small values of the flip parameter a_{r-1} can make $0, 1$ unstable and induce a tendency toward smaller majorities when starting at some p_0 arbitrarily close to either 0 or 1.

How come a single parameter can affect the stability of $0, 1$ but the same is not possible for $1/2$? The slope parameter λ_r at $1/2$ is positive for $\mathbf{a} = 0$ which corresponds to the basic model. Here, the term (16) is 0. As r becomes larger, this slope $\lambda_r(0)$ becomes larger as well, behaving like $\lambda_r(0) = P_r'(1/2) \approx \sqrt{2r/\pi}$ as noted in Section 2. So even though the largest derivatives in absolute value found in Table 8 and given by the formula (17) do *not* decay as r increases, their magnitude compared to $\lambda_r(0)$ becomes insignificant. The same does not apply to the fixed points $0, 1$: the slope of $R_{r,\mathbf{a}}$ at these points is 0 for $\mathbf{a} = 0$. That, in combination with the multiplicative constant r in (14), allows a single parameter to affect the stability of these fixed points even (and especially) when r is large.

What are the ranges of r such that each of three parameters $a = a_{(r+1)/2}, b = a_{(r+3)/2}, c = a_{(r+5)/2}$ can induce a phase transition on their own?

- The flip parameter a can induce a phase transition if and only if $r \leq 5$. For larger values of r, even $a = 1$ does not make the fixed point $1/2$ stable.
- The flip parameter b is more effective at producing a phase transition in the model: it can turn $1/2$ stable for group sizes up to and including $r = 11$. The critical values are given in Table 9. For $r \geq 13$, there is no phase transition. Here, for all values $b \in [0, 1]$, the dynamics of the model resemble those of the basic model, i.e. the case where $b = 0$.
- The flip parameter c cannot induce a phase transition even for $r = 5$. (For $r = 3$, there is no flip parameter c, only a and b.)

References

[1] Abelson, R. P.: Mathematical models of the distribution of attitudes under controversy, In Frederiksen N and Gulliksen H (Eds.), Contributions to Mathematical Psychology, New York, NY: Holt, Rinehart, and Winston. (1964)

[2] Ajmone Marsan, G.; Bellomo N.; and Gibelli, L.: Stochastic evolutionary differential games toward a systems theory of behavioral social dynamics, Math. Models Methods, Appl. Sci. 26 1051–1093. (2016)

[3] Asch, S. E.: Studies of independence and conformity: a minority of one against a unanimous majority. Psychological Monographs 70, no. 9, whole no. 416. (1956)

[4] Bagnoli, F. and Rechtman, R.: Bifurcations in models of a society of reasonable contrarians and conformists, Phys. Rev. E 92 042913 (2015)

[5] Bellomo, N.; Brezzi, F.; and Pulvirenti, M.: Modeling behavioral social systems, Mathematical Models and Methods in Applied Sciences Vol. 27, No. 01, pp. 1-11 (2017)

[6] Borghesi, C. and Galam, S.: Chaotic, staggered, and polarized dynamics in opinion forming: The contrarian effect, Phys. Rev. E 73, 066118 (2006)

[7] Brazil, R.: The physics of public opinion, Physics World, January issue (2020)

[8] Castellano, C.; Fortunato, S.; and Loreto, V.: Statistical physics of social dynamics, Rev. Mod. Phys. 81, 591-646 (2009)

[9] Chakrabarti, B. K.; Chakraborti, A.; and Chatterjee, A. (Eds.): Econophysics and Sociophysics: Trends and Perspectives, Wiley-VCH Verlag (2006)

[10] Corcos, A.; Eckman, J. P.; Malaspinas, A.; Malvergne, Y.; and Sornette, D.: Imitation and Contrarian Behaviour, Research Paper Quantitative Finance, Vol. 2 264--281 (2002)

[11] Deffuant, G.; Neau, D.; Amblard, F.; and Weisbuch, G.: Mixing beliefs among interacting agents. Advances in Complex Systems, 03(01n04):87–98, 2000.

[12] DeGroot, M.: Reaching a consensus. Journal of the American Statistical Association, 69(345):118–121, 1974.

[13] Friedkin, N. and Johnsen, E.: Social influence and opinions. Journal of Mathematical Sociology, J Math Sociol, 15:193–206, Jan. 1990.

[14] Friedkin, N. and Johnsen, E.: Social Influence Networks and Opinion Change Models of opinion formation, Advances in Group Processes, 16:1–29, Jan. 1999.

[15] Galam, S.: Majority Rule, Hierarchical Structure, and Democratic Totalitarianism: A Statistical Approach, Journal of Mathematical Psychology 30, 426-434 (1986)

[16] Galam, S.: Minority Opinion Spreading in Random Geometry, European Physical Journal B 25 Rapid Note, 403-406 (2002)

[17] Galam, S.: Contrarian Deterministic Effects on Opinion Dynamics: "The Hung Elections Scenario", Physica A 333, 453-460 (2004)

[18] Galam, S.: Geometric Vulnerability of Democratic Institutions against Lobbying: A Sociophysics Approach, Mathematical Models and Methods in Applied Sciences, Vol. 27, No. 1, 13-44 (2017)

[19] Galam, S. The Trump phenomenon, an explanation from sociophysics. Int J Mod Phys B (2017) 31:1742015. doi:10.1142/s0217979217420152.

[20] Galam, S. Are referendum a machinery to turn our prejudices into rational choices? An unfortunate answer from sociophysics, in The Routlrdge Handbook to Referendums and Direct Democracy, Eds. L Morel M Qvortrup, Routledge 19, 334–47 (2018)

[21] Galam, S.: Will Trump win again in the 2020 election? An answer from a sociophysics model, Physica A 570 125835 (2021)

[22] Galam, S. and Cheon, T.: Asymmetric Contrarians in Opinion Dynamics, Entropy 22, 25 (2020)

[23] Galam, S. and Cheon, T.: Tipping Points in Opinion Dynamics: A Universal Formula in Five Dimensions. Front. Phys. 8:566580. doi: 10.3389/fphy.2020.566580 (2020)

[24] Gambaro, J. P. and Crokidakis, N.: The influence of contrarians in the dynamics of opinion formation, Physica A 486, 465–472 (2017)

[25] Hamann, H.: Opinion Dynamics With Mobile Agents: Contrarian Effects by Spatial Correlations, Front. Robot. AI, 06 June 2018 | https://doi.org/10.3389/frobt.2018.00063

[26] Hegselmann, R. and Krause, U.: Opinion dynamics and bounded confidence: Models, analysis and simulation, Journal of Artificial Societies and Social Simulation, 5(3), 2002.

[27] Jędrzejewski, A.; Marcjasz, G.; Nail, P. R.; Sznajd-Weron, K.: Think then act or act then think?, PLoS ONE 13(11): e0206166. https://doi.org/10.1371/journal.pone.0206166 (2018)

[28] Lanchier, N. and Taylor, N.: Galam's bottom-up hierarchical system and public debate model revisited, Adv. Appl. Probab. 47 668–692 (2013)

[29] Lee, E.; Holme, P.; and Lee, S. H.: Modeling the dynamics of dissent, Physica A, 262-272 (2017)

[30] Lehir, P.: Les mathï¿œmatiques s'invitent dans le dï¿œbat europï¿œen, (interview of S. Galam). Monde (2005) 26/02:23.

[31] Mobilia, M.; Redner, S.: Majority versus minority dynamics: phase transition in an interacting two-state spin system, Phys Rev E Stat Nonlin Soft Matter Phys. 2003 Oct;68(4 Pt 2):046106. doi: 10.1103/PhysRevE.68.046106. Epub 2003 Oct 9. PMID: 14683001.

[32] Noorazar, H.: Recent advances in opinion propagation dynamics: a 2020 survey. Eur. Phys. J. Plus 135, 521 (2020). https://doi.org/10.1140/epjp/s13360-020-00541-2

[33] Redner, S.: Reality-inspired voter models: A mini-review, C. R. Physique 20 275–292 (2019)

[34] Sabin-Miller, D.; Abrams, D. M.: When pull turns to shove: A continuous-time model for opinion dynamics, Phys. Rev. Res. 2 (2020) 043001

[35] Schweitzer, F.: Sociophysics, Physics Today 71, 40–47 (2018)

[36] Sznajd-Weron, K. and Sznajd, J.: Opinion evolution in closed community, Int. J. Mod. Phys. C 11, 1157–1165 (2000)

[37] Sznajd-Weron, K.: private communication (2021)

[38] Tanabe, S. and Masuda, N.: Complex dynamics of a nonlinear voter model with contrarian agents, Chaos 23, 043136(6pp) (2013)

[39] Taylor, A.; Pacelli, A.: Mathematics and Politics. Strategy, Voting, Power and Proof, 2nd edition, Springer (2008)