

HISTORIC AND PHYSICAL WANDERING DOMAINS FOR WILD BLENDER-HORSESHOES

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ABSTRACT. We present diffeomorphisms of wild blender-horseshoes which belong to C^r ($1 \leq r < \infty$) closures of two types of diffeomorphisms, one of which has a historic contracting wandering domain, and the other has a non-trivial Dirac physical measure supported by saddle periodic orbit. It is a non-trivial extension of Colli-Vargas' model [CV01] to the higher dimensional dynamics with the use of wild blender-horseshoes.

1. INTRODUCTION

1.1. Historicity and physicality. Ruelle and Takens focused on dynamics which is irregular under Lebesgue measure by introducing the concept of “historic behaviour” in [Rue01, Tak08]. Here, for a map f on a Riemannian manifold M , we say that f or some orbit of f has *historic behaviour* if there is $x \in M$ such that the forward orbit $\{f^i(x) : i \geq 0\}$ has non-converging Birkhoff averages. That is,

$$\frac{1}{n+1} \sum_{i=0}^n \delta_{f^i(x)}$$

dose not converge as $n \rightarrow \infty$ in the weak*-topology, where $\delta_{f^i(x)}$ is the Dirac measure on M supported at $f^i(x)$. There have been sporadic studies of examples of the non-existence of Birkhoff average, while the following open questions have been proposed in order to study dynamical historicity from a unified point of view.

Takens' Last Problem ([Tak08]). *Are there persistent classes of smooth dynamical systems such that the set of points whose orbits have historic behaviour has positive Lebesgue measure?*

Affirmative answers to the problem are already provided for non-hyperbolic situations. To explain it, recall the non-hyperbolic phenomenon given by Newhouse [New79] that, for any C^2 diffeomorphism on a smooth manifold M of $\dim M = 2$ with a homoclinic tangency of a saddle periodic point, there is an open set $\mathcal{N} \subset \text{Diff}^2(M)$ whose closure contains f and such that any $g \in \mathcal{N}$ has a C^2 -robust homoclinic tangency of some hyperbolic sets Λ_g which is homoclinically related to the continuation of the saddle periodic point. We say that such a C^2 -open set of nonhyperbolic diffeomorphisms is a (C^2 -) *Newhouse open set* or *domain*. An affirmative answer was detected in any C^2 -Newhouse open set by the use of non-trivial

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wandering domains as follows. Here the *non-trivial wandering domain* for f means a non-empty connected open set $\mathbb{D} \subset M$ with the following conditions:

- $f^i(\mathbb{D}) \cap f^j(\mathbb{D}) = \emptyset$ for any integers $i, j \geq 0$ with $i \neq j$;
- the union of ω -limit sets of all $x \in \mathbb{D}$, $\omega(\mathbb{D}, f) = \bigcup_{x \in \mathbb{D}} \omega(x, f)$, is not equal to a single periodic orbit.

A wandering domain \mathbb{D} is called *contracting* if the diameter of $f^i(\mathbb{D})$ converges to zero as $i \rightarrow +\infty$. Existence of contracting non-trivial wandering domains was first brought to light by Colli-Vargas [CV01] for a prototype of wild hyperbolic set, that is, an affine thick horseshoe with homoclinic tangencies. Kiriki and Soma [KS17] not only generalised their results in any C^2 -Newhouse open set, but also identified the presence of historic behaviour as follows: any Newhouse open set in $\text{Diff}^r(M)$, $\dim M = 2$ and $2 \leq r < \infty$, contains a dense subset every element of which has a *historic wandering domain*, i.e. a non-trivial wandering domain of points whose orbits have historic behaviour. Note that arguments in [KS17] are not extendable to C^∞ -diffeomorphisms, but Berger and Biebler [BBar] overcome this difficulty with completely different methods.

The arguments of [New79] would not be applicable to C^1 -diffeomorphisms on 2-dimensional manifold M . In fact, $\text{Diff}^1(M)$ with $\dim M = 2$ contains a generic subset where every diffeomorphism has no homoclinic tangency [Mor11]. Thus any method similar to that in [KS17] might be irrelevant for $\text{Diff}^1(M)$ if $\dim M = 2$. On the other hand, if $\dim M \geq 3$, Bonatti and Díaz [BD12] presented an open set of $\text{Diff}^1(M)$ such that every diffeomorphism in the open set has a C^1 -robust homoclinic tangency. Nowadays, it is called a *C^1 -Newhouse domain*.

Before stating our result, we recall the notion of classical regularity which is quite opposite to that of historic behaviour. We say that f has a *Dirac physical measure* ν associated with a wandering domain \mathbb{D} if for every $x \in \mathbb{D}$

$$\lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{i=0}^n \delta_{f^i(x)} = \nu$$

and the support of ν is equal to a periodic orbit of f . Moreover such a ν is *non-trivial* if the periodic orbit is of saddle type. It implies that \mathbb{D} is contained in the basin of ν , which has positive Lebesgue measure because \mathbb{D} is a non-empty open set. In this sense, it is traditional usage to refer to ν as *physical* or *SRB*, see [CTV19]. Dirac physical measures were studied for some transitive flows such that the supports of measures are non-attracting orbits in [SSV10, SV13]. On the other hand, for diffeomorphisms, using the prototype of wild hyperbolic set, Colli and Vargas presented a non-trivial Dirac physical measure associated with a wandering domain in [CV01], which can be extended in some dense subset of the C^r ($2 \leq r < \infty$)-Newhouse domain in [KS17]. In a C^1 -generic standpoint, several negative observations about the existence of Dirac physical measures supported on non-attracting periodic orbits and examples are provided in [San18, GGS20].

We are now ready to state the main theorem.

Theorem A. *There is a 3-dimensional diffeomorphism f in the C^1 Newhouse domain such that every C^r ($1 \leq r < \infty$) neighbourhood of f contains two types of diffeomorphisms, one of which has a historic contracting wandering domain, and the others have non-trivial Dirac physical measures supported by saddle periodic orbits.*

Remark 1.1. The following scenario may come to mind to detect historic wandering domains using several known facts in C^1 dynamics of dimension at least 3. In fact, Barrientos [Bar] essentially implemented the scenario. In [BDP03] Bonatti, Díaz and Pujals gave C^1 open sets of diffeomorphisms with a C^1 dense subset of diffeomorphisms admitting a periodic disk, that is, there is a large integer n such that f^n is equal to the identity map on some disk. Then perturb f^n in order to get a C^2 -homoclinic tangency in a smooth normally contracting surface Σ by [Rom95], and apply [KS17] to get a historic wandering domain by perturbation of the restriction $f^n|_{\Sigma}$. However, the proofs in [KS17] are founded on some *unbalanced weight assumption* to control dynamics, see [KS17, Remark 2.1]. Thus, one can use the scenario to confirm the existence of non-trivial Dirac physical measure supported on the saddle fixed point, but one cannot use it to confirm that on the 2-periodic orbit. On the other hand, the direct way provided in this paper is not only useful for both, but may be applied to non-trivial Dirac physical measure supported by saddle periodic orbit of every period. See the final Remark 5.6.

We believe that there exist a locally dense C^1 diffeomorphisms which satisfy the same properties as in Theorem A and Remark 5.6. In particular, there will be Dirac physical measures for periodic orbits of every period. While this paper is limited to specific models, it will shed light on “pluripotentiality” of wandering domains, that is, the existence of them whose orbits can approximate statistically to every dynamics on given invariant sets.

Note that the residual subset of blender-horseshoe causing historic behaviour in [BKN⁺20] has zero Lebesgue measure, while the historic wandering domain given in Theorem A has positive Lebesgue measure. We mention another novelty that this paper contains. To prove Theorem A, we borrow the key idea called “critical chain” from [CV01]. However Colli and Vargas proved their main technical results (Linking and Linear Growth Lemmas) which support the proof of their Critical Chain Lemma by using C^2 -robust homoclinic tangencies, and hence they did not employ contexts of C^1 -robust ones. So, instead of their technical results, we present an innovation (Lemma 2.1 with Proposition 2.3 in Section 2), which takes advantage of the distinctive property of inverse dynamics of the cs-blender horseshoe. It should also be emphasised that our proof might be considerably simpler than that of Colli-Vargas.

1.2. Centre stable blender-horseshoes. In this subsection and the next, we give a concrete construction of f in Theorem A. All this can be extended to structures called blender-horseshoes, which can be defined for diffeomorphisms of all dimensions greater than or equal to three. However, a 3-dimensional case contains all essential properties on blender-horseshoes, and therefore we discuss them only in this case.

Let $\lambda_{ss}, \lambda_{cs0}, \lambda_{cs1}$ and λ_u be real positive constants with

$$(1.1) \quad \lambda_{ss} < \lambda_{cs0} < 1/2 < \lambda_{cs1} < 1 < \lambda_{cs0} + \lambda_{cs1}, \quad 2 < \lambda_u.$$

Furthermore, we suppose that λ_{cs0} is relatively small compared to λ_{cs1} and λ_u so that

$$(1.2) \quad \lambda_{cs0}\lambda_{cs1}\lambda_u^2 < 1,$$

which corresponds to the partially dissipative condition for 3-dimensional diffeomorphisms given below. We first consider the 2-dimensional affine horseshoe map

F with

$$F(x, y) = \begin{cases} (\lambda_u x, \lambda_{ss} y) & \text{if } (x, y) \in [0, \lambda_u^{-1}] \times [0, 1], \\ (\lambda_u(1-x), 1 - \lambda_{ss} y) & \text{if } (x, y) \in [1 - \lambda_u^{-1}, 1] \times [0, 1], \end{cases}$$

and the iterated function system consisting of the pair of contracting 1-dimensional maps defined as, for $z \in [0, 1]$,

$$(1.3a) \quad \zeta_0(z) = \lambda_{cs0} z, \quad \zeta_1(z) = \lambda_{cs1} z + \beta,$$

where $\beta = 1 - \lambda_{cs1}$. Let \mathbb{B} be the unit cube $[0, 1]^3$ and let $f : \mathbb{B} \rightarrow \mathbb{R}^3$ be a local diffeomorphism satisfying the following conditions:

- $f|_{\mathbb{V}_0 \cup \mathbb{V}_1}$ is the skew product $F \ltimes (\zeta_0, \zeta_1)$ given by

$$(1.3b) \quad f(x, y, z) = \begin{cases} (\lambda_u x, \lambda_{ss} y, \lambda_{cs0} z) & \text{if } (x, y, z) \in \mathbb{V}_0, \\ (\lambda_u(1-x), 1 - \lambda_{ss} y, \lambda_{cs1} z + \beta) & \text{if } (x, y, z) \in \mathbb{V}_1, \end{cases}$$

where $\mathbb{V}_0 = [0, \lambda_u^{-1}] \times [0, 1]^2$ and $\mathbb{V}_1 = [1 - \lambda_u^{-1}, 1] \times [0, 1]^2$.

- For $\mathbb{G} = \mathbb{B} \setminus (\mathbb{V}_0 \cup \mathbb{V}_1)$, $f(\mathbb{G})$ is contained in $\mathbb{R}^3 \setminus \mathbb{B}$.

We now consider the hyperbolic set $\Lambda = \bigcap_{i \in \mathbb{Z}} f^i(\mathbb{V}_0 \cup \mathbb{V}_1)$ of f in \mathbb{B} on which f is conjugate to the full shift of two symbols, and besides Λ contains the saddle fixed points

$$P = (0, 0, 0) \in \mathbb{V}_0 \cap f(\mathbb{V}_0), \quad Q = (\lambda_u(1 + \lambda_u)^{-1}, (1 + \lambda_{ss})^{-1}, 1) \in \mathbb{V}_1 \cap f(\mathbb{V}_1).$$

Remark 1.2 (Asymmetry). The inequalities (1.1) and (1.2) imply a partially dissipative situation for $f|_\Lambda$, which gives asymmetrical contractions along the centre-stable direction for the cs-blender horseshoe, see Figure 1.1. We will see that these conditions are essential to show Lemma 4.5 and Theorem 4.4. Note that these conditions cannot be fulfilled if both ζ_0 and ζ_1 are close to the identity. Therefore, the cs-blender horseshoe with partially dissipative situation might be C^1 -away from usual one which can be derived from a heterodimensional cycle via some strongly homoclinic intersection by an arbitrarily small perturbation in [BD08, Section 4]. See also Remark 5.6.

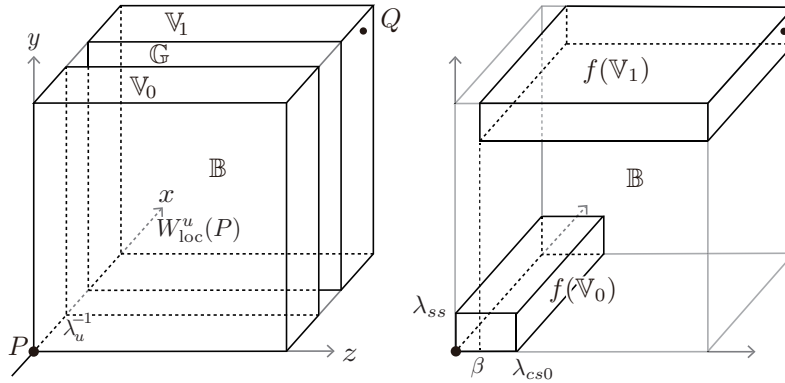


Figure 1.1

Despite its asymmetric structure, Λ still satisfies all the standard properties of blender. In fact, Λ is an example of *cs-blender horseshoe*, refer to [BD12, Definition

3.9] for the precise definition. Note that there is a C^1 -neighbourhood \mathcal{N}_f of f such that every $\tilde{f} \in \mathcal{N}_f$ has the continuation $\Lambda_{\tilde{f}}$ of Λ which is a cs-blender horseshoe containing the continuations $P_{\tilde{f}}$ and $Q_{\tilde{f}}$ of P and Q , respectively. Moreover it follows from (1.1) that $\beta < \lambda_{cs0}$, that is, \tilde{f} still has a superposition region associated with $\Lambda_{\tilde{f}}$ and lying between $W_{\text{loc}}^u(P_{\tilde{f}})$ and $W_{\text{loc}}^u(Q_{\tilde{f}})$. See [BD96, Lemma 1.11] and [BD12, Lemma 3.10] for details.

1.3. Configurations of tangency. Next, to investigate homoclinic tangencies under the setting of cs-blender horseshoe, we assume the following conditions on the second iterate of $f|_{\mathbb{G}}$: for a given $0 < \delta < \frac{1}{2} - \lambda_u^{-1}$, the restriction of f^2 to the δ -neighbourhood U_δ of the 2-dimensional disc $\{x = 1/2\} \cap \mathbb{G}$ is given by

$$(1.4a) \quad f^2(x, y, z) = \left(-a_1 \left(x - \frac{1}{2} \right)^2 + a_2 z, \ a_3 \left(y - \frac{1}{2} \right) + \frac{1}{2}, \ a_4 \left(x - \frac{1}{2} \right) + \frac{1}{2} \right)$$

for $(x, y, z) \in U_\delta$, where the coefficients a_1, a_2, a_3 and a_4 are nonzero real constants with

$$(1.4b) \quad a_1 > (1 - 2\lambda_u^{-1})^{-1}, \ a_2, a_4 > 0, \ |a_3| < 1 - 2\lambda_{ss}.$$

The first condition is used to show Proposition 4.3 which leads to wandering domains disjoint from Λ , and the second one is necessary to show Lemma 1.3. The last one assures that $f^2(\mathbb{G}) \cap \mathbb{B}$ has no intersection with $f(\mathbb{V}_0) \cup f(\mathbb{V}_1)$, see Figure 1.2.

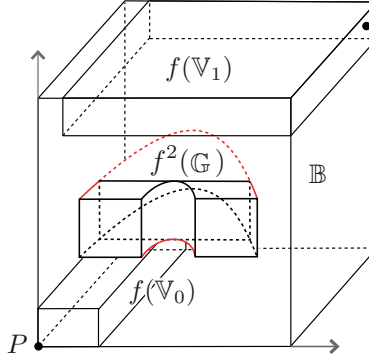


Figure 1.2

We say that a blender horseshoe Λ is *wild* if there are points $x, x' \in \Lambda$ such that $W^u(x)$ and $W^s(x')$ have a non-transverse intersection. Let the f^2 -image of $(1/2, 0, 0) \in \mathbb{G}$ be written as X , which satisfies

$$X = (0, 1/2 - a_3/2, 1/2) \in W^u(P) \cap W_{\text{loc}}^s(P), \ T_X W^u(P) \subsetneq T_X W_{\text{loc}}^s(P).$$

That is, f has a homoclinic tangency of $P \in \Lambda$. See Figure 1.3-(a). Therefore the cs-blender horseshoe Λ for (1.3b) is wild. Furthermore, this homoclinic tangency is robust for small C^1 perturbation:

Lemma 1.3. *Let f be a diffeomorphism with (1.3b) and (1.4a). Then f has a C^1 -robust homoclinic tangency of the cs-blender horseshoe Λ .*

The proof of this lemma is given in the Appendix A as it is just confirmed that [BD12, Theorem 4.8] can be applicable to our setting.

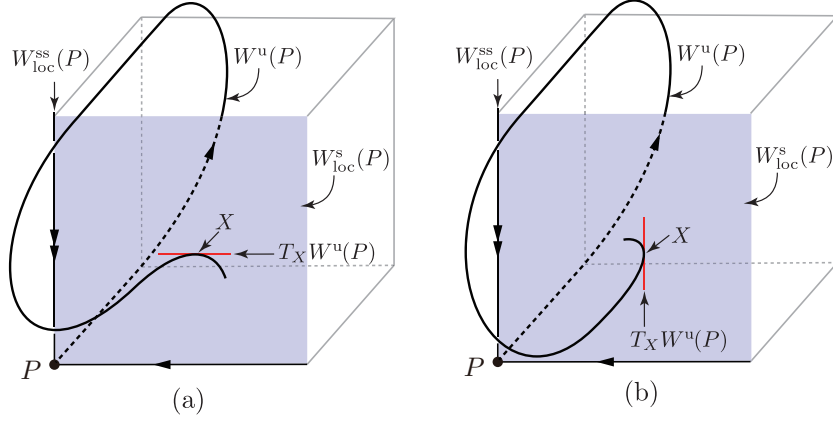


Figure 1.3

Remark 1.4 (Generality of configuration). We here explain why the above setting is a non-trivial extension of [CV01], which is also the second novelty of the present paper. For a certain map instead of (1.4a), we have another situation such that the direction of $T_X W^u(P)$ is parallel to the ss-direction of Λ , as in Figure 1.3-(b). This is an actually trivial extension of Colli-Vargas' 2-dimensional model with one more dimension, and hence one might obtain similar results as in [CV01] with the help of several techniques in [PV94]. However, such a strategy will not get us out of the C^2 -category. Furthermore, with a little perturbation, it is possible to maintain the tangency but make a direction different from the ss-direction. Thus, the tangent directions at forward images of such a perturbed tangency are gradually close to the cs-direction, since it is pressed strongly along the ss-direction. As a consequence, the situation will be essentially the same as defined by (1.4a) as in Figure 1.3-(a). In this sense, the configuration of tangency in this paper is general. Combining this situation with Lemma 1.3, we now obtain a diffeomorphism having both cs-blender horseshoe and C^1 -robust homoclinic tangency simultaneously.

It follows from the above definitions and facts that Theorem A is a consequence of the next theorem.

Theorem A'. *Every C^r ($1 \leq r < \infty$)-neighbourhood of the above diffeomorphism f with a wild blender-horseshoe contains a diffeomorphism which has a historic contracting wandering domain. Moreover, it contains another diffeomorphism having non-trivial Dirac physical measures supported by saddle periodic orbits associated with a contracting wandering domain.*

For the proof of Theorem A', we need to prepare some tools associated with the blender-horseshoes, and give key results (Lemma 2.1, Proposition 2.3) for projected dynamics in Section 2 and some infinite sequence of perturbations in Section 3. Using the results, the existence of the wandering domain is proved by several geometric steps in Theorem 4.4 of Section 4. Finally, the existences of historic behaviour (Theorem 5.1) and Dirac physical measure (Theorem 5.5) are shown by probabilistic approaches in Section 5.

2. CRITICAL CHAINS OF BRIDGES

The results given in this section are keys to this paper, which is associated with several subsequences of u-bridges and its copies in the cs-direction.

2.1. Unstable bridges and gaps. Let f be a diffeomorphism with the cs-blender horseshoe Λ by (1.3b). We first extend the notations of bridges and gaps given in [CV01] as follows. For any integer $n \geq 1$, let \underline{w} be an n -tuple of binary codes, that is, $\underline{w} = w_1 \cdots w_n$ with $w_i \in \{0, 1\}$. Define dynamically defined rectangular solids as

$$\begin{aligned}\mathbb{B}^u(n; \underline{w}) &= \{x \in \mathbb{B} : f^{i-1}(x) \in \mathbb{V}_{w_i}, i = 1, \dots, n\}, \\ \mathbb{G}^u(n; \underline{w}) &= \mathbb{B}^u(n; \underline{w}) \setminus (\mathbb{B}^u(n+1; \underline{w}0) \cup \mathbb{B}^u(n+1; \underline{w}1)).\end{aligned}$$

The former set is called an *unstable bridge* or a *u-bridge*, while the latter one an *unstable gap* or a *u-gap*. Sometimes n and \underline{w} of $\mathbb{B}^u(n; \underline{w})$ are called the *generation* and *itinerary* for the u-bridge, respectively. If there is no confusion, the number of generation may be omitted and $\mathbb{B}^u(n; \underline{w})$ and $\mathbb{G}^u(n; \underline{w})$ may be written as $\mathbb{B}^u(\underline{w})$ and $\mathbb{G}^u(\underline{w})$, respectively. Observe that if n is fixed, the family $\{\mathbb{B}^u(\underline{w}) : \underline{w} \in \{0, 1\}^n\}$ consists of 2^n mutually disjoint rectangular solids, which consequently contains 2^n mutually disjoint arcs of $W_{\text{loc}}^u(P)$. For every $n \geq 1$ and $\underline{w} \in \{0, 1\}^n$, we denote by $B^u(n; \underline{w})$ (or $B^u(\underline{w})$ for short) the arc $\mathbb{B}^u(n; \underline{w}) \cap W_{\text{loc}}^u(P)$, which can be regarded as a subinterval in $[0, 1]$, that is,

$$\mathbb{B}^u(\underline{w}) \cap W_{\text{loc}}^u(P) = B^u(\underline{w}) \times \{(0, 0)\}.$$

Since $\mathbb{G}^u(\underline{w}) \subset \mathbb{B}^u(\underline{w})$, one can obtain the open interval $G^u(n; \underline{w})$ (or $G^u(\underline{w})$ for short) on $[0, 1]$ satisfying

$$\mathbb{G}^u(\underline{w}) \cap W_{\text{loc}}^u(P) = G^u(\underline{w}) \times \{(0, 0)\}.$$

The closed interval $B^u(\underline{w})$ is called a *u-bridge*, while the open interval $G^u(\underline{w})$ is called a *u-gap* of the *u-Cantor set* $\Lambda_u = \Lambda \cap W_{\text{loc}}^u(P)$. Finally, we write $G_0^u = [0, 1] \setminus (B^u(0) \cup B^u(1))$, and hence $\mathbb{G} \cap W_{\text{loc}}^u(P) = G_0^u \times \{(0, 0)\}$.

2.2. Projected dynamics. The following simple projection can be used, since our model consists of the affine forms by (1.3b) with a tangency without any distortion given in (1.4a). For any $(x, y, z) \in \mathbb{B}$ and integer $n > 0$, we write

$$(2.1) \quad \varphi^n(x, z) = \hat{\pi}(f^n(x, y, z))$$

if the value of the right-hand side of the equation does not depend on y , where $\hat{\pi} : \mathbb{B} \rightarrow \mathbb{R}^2$ is the projection defined by $\hat{\pi}(x, y, z) = (x, z)$. By (1.4a), we have

$$(2.2) \quad \varphi^2(1/2, z) = (a_2 z, 1/2).$$

First, we define sequences $\{B_k^u\}_{k \geq 1}$ and $\{\tilde{B}_k^u\}_{k \geq 0}$ of unstable bridges as follows. For any integer $n_0 \geq 0$ and any code $\tilde{w}^{(0)} \in \{0, 1\}^{n_0}$, let us define

$$\tilde{B}_0^u = B^u(n_0, \tilde{w}^{(0)}),$$

which is a *u-bridge* contained in $\pi_1 \circ \varphi^2(\{1/2\} \times [0, 1])$. One can take n_0 so that $\tilde{B}_0^u \subset (0, a_2)$, see Figure 2.1. Let B_1^u, \tilde{B}_1^u be the pair of maximal sub-bridges of \tilde{B}_0^u such that \tilde{B}_1^u lies in the left side of B_1^u , that is, $\max \tilde{B}_1^u < \min B_1^u$. Then they are

represented as $B_1^u = B^u(n_0 + 1, \underline{w}^{(0)}\alpha_1)$ and $\tilde{B}_1^u = B^u(n_0 + 1, \underline{w}^{(0)}\tilde{\alpha}_1)$, where α_1 is either 0 or 1 and $\tilde{\alpha}_1 = 1 - \alpha_1$. If we write $\underline{w}^{(0)}\alpha_1 = \underline{w}^{(1)}$ and $\underline{w}^{(0)}\tilde{\alpha}_1 = \underline{w}^{(1)}$, then

$$B_1^u = B^u(n_0 + 1, \underline{w}^{(1)}), \quad \tilde{B}_1^u = B^u(n_0 + 1, \underline{w}^{(1)}).$$

For integer $k > 1$, we inductively define the sub-bridges B_k^u and \tilde{B}_k^u of \tilde{B}_{k-1}^u satisfying

$$B_k^u = B^u(n_0 + k, \underline{w}^{(k)}), \quad \tilde{B}_k^u = B^u(n_0 + k, \underline{w}^{(k)}),$$

where $\underline{w}^{(k)} = \underline{w}^{(k-1)}\alpha_k$ and $\underline{w}^{(k)} = \underline{w}^{(k-1)}\tilde{\alpha}_k$ for some $\alpha_k, \tilde{\alpha}_k$ with $\{\alpha_k, \tilde{\alpha}_k\} = \{0, 1\}$. See Figure 2.1.

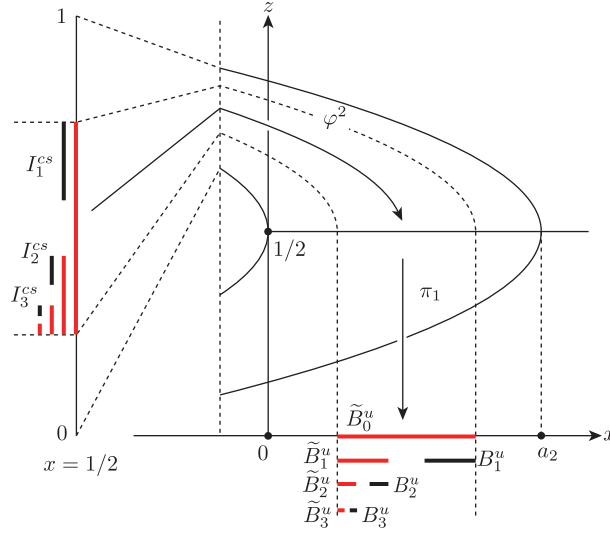


Figure 2.1

Next, we define a sequence $\{J_k^{cs}\}_{k \geq 1}$ of φ^2 -inverse images of B_k^u as follows. For every integer $k \geq 1$ and sub-bridge B_k^u of \tilde{B}_0^u , let I_k^{cs} be the arc in $\{1/2\} \times [0, 1]$ with

$$\pi_1 \circ \varphi^2(I_k^{cs}) = B_k^u,$$

where $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection with $\pi_1(x, z) = x$. Define J_k^{cs} as the sub-interval of $[0, 1]$ such that

$$\{1/2\} \times J_k^{cs} = I_k^{cs},$$

and call it the *cs-interval* associated with B_k^u . By (2.2),

$$(2.3) \quad J_k^{cs} = a_2^{-1} B_k^u.$$

For any code $\underline{\gamma} = \gamma_1 \gamma_2 \cdots \gamma_n \in \{0, 1\}^n$, the map $\zeta_{\underline{\gamma}}^n$ (or $\zeta_{\underline{\gamma}}$ for short) is defined by

$$\zeta_{\underline{\gamma}}^n = \zeta_{\gamma_n} \circ \cdots \circ \zeta_{\gamma_2} \circ \zeta_{\gamma_1},$$

where each ζ_{γ_i} is the function given in (1.3a). Moreover, we define the *length* $|\underline{\gamma}|$ of $\underline{\gamma}$ as the total number of symbols in $\underline{\gamma}$, that is, $|\underline{\gamma}| = |\gamma_1 \gamma_2 \cdots \gamma_n| = n$.

Lemma 2.1. *For any integer $L > 0$, any u -bridge \bar{B}_k^u ($k = 1, 2, \dots$) with $\bar{B}_k^u = B^u(n_0 + k + Lk, \underline{w}^{(k+Lk)}) \subset B_k^u$ and any code $\underline{u}^{(k)}$ with $|\underline{u}^{(k)}| \geq 0$, there exist codes $\underline{w}^{(k)}$ and sub-bridges $\hat{B}_k^u = B^u(\hat{n}_k, \underline{w}^{(k)})$ of \bar{B}_k^u satisfying the following conditions.*

- (1) $\underline{w}^{(k)} = \underline{w}^{(k+Lk)} \underline{v}^{(k)} \underline{\gamma}^{(k)}$, where
 - $\underline{\gamma}^{(k)} = \gamma_{m_k} \gamma_{m_k-1} \cdots \gamma_2 \gamma_1$, $\gamma_i \in \{0, 1\}$ ($i = 1, 2, \dots, m_k$), for some integer m_k satisfying $0 < m_k \leq N_0 + N_1 k$, where N_0, N_1 are positive integers independent of k ,
 - either $\underline{v}^{(k)} = \underline{u}^{(k)}$ or $\underline{v}^{(k)} = \underline{u}^{(k)} \alpha$ for some $\alpha \in \{0, 1\}$.
- (2) $\zeta_{\underline{w}^{(k)}}(1/2) \in \bar{J}_{k+1}^{\text{cs}}$ and $\hat{J}_{k+1}^{\text{cs}} \subset \bar{J}_{k+1}^{\text{cs}}$, where $\bar{J}_{k+1}^{\text{cs}}$ is the cs-interval associated with \bar{B}_{k+1}^u .

Remark 2.2. The freedom of the choice of $\underline{u}^{(k)}$ in Lemma 2.1 is crucial in the study of historic behaviour of wandering domains for diffeomorphisms C^1 -close to f .

Proof of Lemma 2.1. Take $\gamma_1 \in \{0, 1\}$ with $\bar{J}_{k+1}^{\text{cs}} \subsetneq \text{Image}(\zeta_{\gamma_1})$, where $\text{Image}(\zeta_{\gamma_1})$ is the image of ζ_{γ_1} . Define

$$\bar{J}_{k+1}^{\text{cs}(1)} = \zeta_{\gamma_1}^{-1}(\bar{J}_{k+1}^{\text{cs}}).$$

For any integer $i \geq 1$, define the interval $\bar{J}_{k+1}^{\text{cs}(i)}$ in $[0, 1]$ inductively if $\bar{J}_{k+1}^{\text{cs}(i-1)} \subset \text{Image}(\zeta_{\gamma_i})$ holds for at least one γ_i of 0, 1. Suppose that this process finishes m_k times. Accordingly, $\bar{J}_{k+1}^{\text{cs}(m_k-1)}$ is contained in $\text{Image}(\zeta_{\gamma_{m_k}})$. We here note that

$$|\bar{B}_{k+1}^u| = (\lambda_u^{-1})^{n_0+k+1+L(k+1)}$$

and

$$\pi_1 \circ \varphi^2(\{1/2\} \times \bar{J}_{k+1}^{\text{cs}}) = \bar{B}_{k+1}^u.$$

Then, by (2.2),

$$|\bar{J}_{k+1}^{\text{cs}}| = |a_2^{-1}|(\lambda_u^{-1})^{n_0+k+1+L(k+1)}.$$

Thus, there are integers $m_{k-1}^{(0)}, m_{k-1}^{(1)}$ with $m_k - 1 = m_{k-1}^{(0)} + m_{k-1}^{(1)}$ such that

$$|\bar{J}_{k+1}^{\text{cs}(m_k-1)}| = \lambda_{cs0}^{-m_{k-1}^{(0)}} \lambda_{cs1}^{-m_{k-1}^{(1)}} (|a_2^{-1}|(\lambda_u^{-1})^{n_0+k+1+L(k+1)}) \leq 1 - \beta,$$

where λ_{cs0} and λ_{cs1} are derivatives of ζ_0 and ζ_1 , respectively, see (1.3a). Since $\lambda_{cs0} < \lambda_{cs1} < 1$, we have

$$m_k \leq \frac{\log |a_2| (1 - \beta) \lambda_u^{n_0+1+L}}{\log \lambda_{cs0}^{-1}} + 1 + \frac{\log \lambda_u^{1+L}}{\log \lambda_{cs0}^{-1}} k.$$

Thus the smallest integers N_0 and N_1 with

$$N_0 \geq \frac{\log |a_2| (1 - \beta) \lambda_u^{n_0+1+L}}{\log \lambda_{cs0}^{-1}} + 1, \quad N_1 \geq \frac{\log \lambda_u^{1+L}}{\log \lambda_{cs0}^{-1}}$$

fulfill the required condition on m_k .

From the definition of m_k , neither $\bar{J}_{k+1}^{\text{cs}(m_k)} \subsetneq \text{Image}(\zeta_0) = [0, \lambda_{cs0}]$ nor $\bar{J}_{k+1}^{\text{cs}(m_k)} \subsetneq \text{Image}(\zeta_1) = [\beta, 1]$ occurs. It follows that

$$\max \left\{ \bar{J}_{k+1}^{\text{cs}(m_k)} \right\} \geq \lambda_{cs0}, \quad \min \left\{ \bar{J}_{k+1}^{\text{cs}(m_k)} \right\} \leq \beta.$$

So $\bar{J}_{k+1}^{\text{cs}(m_k)}$ contains the interval $[\beta, \lambda_{cs0}]$. In the case when $\zeta_{\underline{w}^{(k+Lk)} \underline{u}^{(k)}}(1/2) \in [\beta, \lambda_{cs0}]$, we set $\underline{v}^{(k)} = \underline{u}^{(k)}$. When $\zeta_{\underline{w}^{(k+Lk)} \underline{u}^{(k)}}(1/2) \in [0, \beta)$ (resp. $\zeta_{\underline{w}^{(k+Lk)} \underline{u}^{(k)}}(1/2) \in$

$(\lambda_{cs0}, 1]$), we set $\underline{v}^{(k)} = \underline{u}^{(k)}1$ (resp. $\underline{v}^{(k)} = \underline{u}^{(k)}0$). Since $1/2 < \lambda_{cs} < 1$, we have in either case $\zeta_{\underline{w}^{(k+Lk)}\underline{v}^{(k)}}(1/2) \in [\beta, \lambda_{cs0}]$. Hence the code

$$\widehat{\underline{w}}^{(k)} = \underline{w}^{(k+Lk)}\underline{v}^{(k)}\gamma_{m_k}\gamma_{m_k-1}\cdots\gamma_2\gamma_1$$

satisfies our desired conditions. \square

Proposition 2.3. *For any integer $L > 0$, let $\widehat{B}_k^u = B^u(\widehat{n}_k, \widehat{\underline{w}}^{(k)})$ ($k = 1, 2, \dots$) be the sub-bridges of $\bar{B}_k^u = B^u(n_0 + k + Lk, \underline{w}^{(k+Lk)})$ and \widehat{J}_{k+1}^{cs} the cs-interval associated with \widehat{B}_{k+1}^u given in Lemma 2.1. Then there exists a $t_{k+1} \in \mathbb{R}$ such that*

- $\varphi^{\widehat{n}_k}(\widehat{x}_k^u, 1/2) = (1/2, \widehat{z}_{k+1}^{cs}) - (0, a_2^{-1}t_{k+1})$, where \widehat{x}_k^u and \widehat{z}_{k+1}^{cs} are the centres of \widehat{B}_k^u and \widehat{J}_{k+1}^{cs} , respectively,
- $|t_{k+1}| < \lambda_u^{-(n_0+k+1+L(k+1))}$.

Remark 2.4. Note that $\widehat{B}_k^u \subset \bar{B}_k^u \subset B_k^u$, and B_k^u will be used to specify the domain of a perturbation. On the other hand, \widehat{B}_k^u determined from \bar{B}_k^u controls the size of the perturbation, and it will be important in the proof of Proposition 3.1 that its size, which is exactly $|t_{k+1}|$ above, can be much smaller than the size of B_k^u by taking a sufficiently large L .

Proof of Proposition 2.3. Since $|\widehat{\underline{w}}^{(k)}| = \widehat{n}_k$, we have $\pi_1 \circ \varphi^{\widehat{n}_k}(\widehat{x}_k^u, 1/2) = 1/2$ and hence, by Lemma 2.1, there is the cs-interval $\bar{J}_k^{cs} \subset J_k^{cs}$ and

$$\varphi^{\widehat{n}_k}(\widehat{x}_k^u, 1/2) = \left(1/2, \zeta_{\widehat{\underline{w}}^{(k)}}(1/2)\right) \in \{1/2\} \times \bar{J}_{k+1}^{cs} = \bar{I}_{k+1}^{cs}.$$

See Figure 2.2. Furthermore,

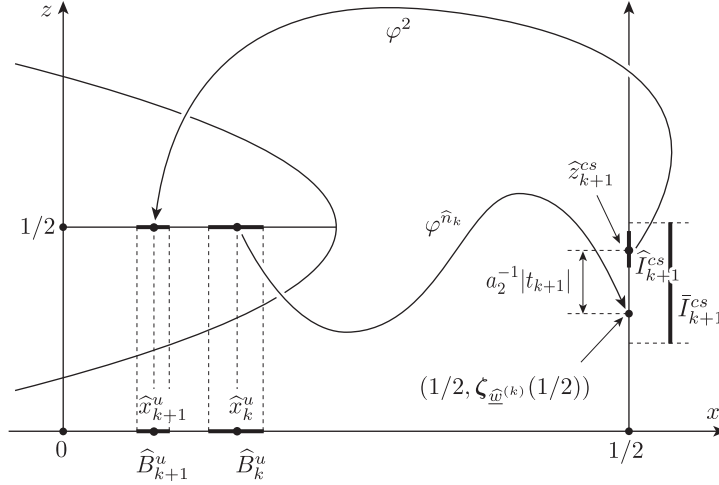


Figure 2.2

$$\widehat{z}_{k+1}^{cs} \in \text{Int} \widehat{J}_{k+1}^{cs} \subset \bar{J}_{k+1}^{cs}.$$

We here set

$$(2.4) \quad t_{k+1} = a_2(\widehat{z}_{k+1}^{cs} - \zeta_{\widehat{\underline{w}}^{(k)}}(1/2)).$$

Since both $\zeta_{\underline{w}^{(k)}}(1/2)$ and $\widehat{z}_{k+1}^{\text{cs}}$ belong to $\bar{J}_{k+1}^{\text{cs}}$, it follows from (2.3) that

$$|t_{k+1}| \leq a_2 |\bar{J}_{k+1}^{\text{cs}}| = |\bar{B}_{k+1}^u| = \lambda_u^{-(n_0+k+1+L(k+1))}. \quad \square$$

3. PERTURBATIONS

We construct some map arbitrarily C^r -close to f by countably many small perturbations near homoclinic tangencies.

For any integer $k \geq 1$ let $B_k^u = B^u(n_0 + k, \underline{w}^{(k)})$ be the u -bridge and J_k^{cs} the cs -interval associated with B_k^u defined in the previous section.

Proposition 3.1. *For any $\varepsilon > 0$, there is a diffeomorphism g which is contained in the ε -neighbourhood of f in the C^r -topology ($1 \leq r < \infty$) and satisfies the following conditions:*

- $g|_{U_{3\delta/2}^c} = f$ for any $0 < \delta < \frac{1}{3}(1 - 2\lambda_u^{-1})$, where $U_{3\delta/2}$ is the $3\delta/2$ -neighbourhood of the 2-dimensional disc $\{x = 1/2\} \cap \mathbb{B}$ and $U_{3\delta/2}^c$ is the complement of $U_{3\delta/2}$ in \mathbb{B} .
- For every $k \geq 1$ and $(x, y, z) \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta] \times [0, 1] \times J_{k+1}^{\text{cs}}$,

$$g^2(x, y, z) = (t_{k+1}, 0, 0) + f^2(x, y, z),$$

where t_{k+1} is the number given in Proposition 2.3.

Proof. The idea of the proof is already described in Remark 2.4. Here we prove it in practice.

Let $\mathbf{b} : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative, non-decreasing C^r function such that $\mathbf{b}(x) = 0$ if $x \leq -1$ while $\mathbf{b}(x) = 1$ if $x \geq 0$. Using \mathbf{b} , we consider the bump function $\mathbf{b}_{\rho, I}$ with $\mathbf{b}_{\rho, I} = 1$ on I as follows:

$$\mathbf{b}_{\rho, I}(x) = \mathbf{b}\left(\frac{x-a}{\rho|I|}\right) + \mathbf{b}\left(-\frac{x-b}{\rho|I|}\right) - 1,$$

where ρ is a positive constant and I is the interval $[a, b]$ with $a < b$. The function satisfies

$$\|\mathbf{b}_{\rho, I}\|_{C^r} \leq \frac{1}{(\rho|I|)^r} \|\mathbf{b}\|_{C^r},$$

where $\|\cdot\|_{C^r}$ is the supremum norm of the derivatives of corresponding maps. Next we set

$$b_u = \mathbf{b}_{\frac{1}{4}, [\frac{1}{2}-\delta, \frac{1}{2}+\delta]}, \quad b_{ss} = \mathbf{b}_{\frac{1}{4}, [0, 1]}, \quad b_{cs, k} = \mathbf{b}_{\frac{1}{3\tau_{cs}}, J_k^{\text{cs}}},$$

where $\tau_{cs} = \lambda_u^{-1}/(1 - 2\lambda_u^{-1})$, which is independent of k .

For every $k \geq 1$, let t_{k+1} be the constant given in Proposition 2.3, the absolute value of which has an upper bound depending on a given $L > 0$. We write

$$\underline{t}(L) = (t_2, \dots, t_{k+1}, \dots),$$

and define the perturbation map $h_{\underline{t}(L)} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as

$$(3.1) \quad h_{\underline{t}(L)}(x, y, z) = \left(x, y, z + a_2^{-1} b_u(x) \sum_{k=1}^{\infty} t_{k+1} b_{ss}(y) b_{cs, k+1}(z) \right).$$

Then we have

$$\begin{aligned} \|h_{\underline{t}(L)} - id\|_{C^r} &= \left\| a_2^{-1} b_u(x) \sum_{k=1}^{\infty} t_{k+1} b_{ss}(y) b_{cs,k+1}(z) \right\|_{C^r} \\ &< |a_2|^{-1} \left(\frac{18\tau_{ss}\tau_{cs}}{\delta} \right)^r \|b\|_{C^r} \sum_{k=1}^{\infty} \frac{|t_{k+1}|}{|J_{k+1}^{cs}|^r}. \end{aligned}$$

Moreover, it follows from (2.3) and Proposition 2.3 that

$$\begin{aligned} (3.2) \quad \sum_{k=1}^{\infty} \frac{|t_{k+1}|}{|J_{k+1}^{cs}|^r} &\leq \sum_{k=1}^{\infty} \frac{\lambda_u^{-(n_0+k+1+L(k+1))}}{(|a_2|^{-1} \lambda_u^{-n_0-k-1})^r} \\ &= \frac{|a_2|^r \lambda_u^{(n_0+1)(r-1)}}{\lambda_u^L} \sum_{k=1}^{\infty} \left(\frac{\lambda_u^r}{\lambda_u^{1+L}} \right)^k = \frac{|a_2|^r \lambda_u^{(n_0+1)(r-1)+r}}{\lambda_u^L (\lambda_u^{1+L} - \lambda_u^r)}. \end{aligned}$$

Consequently, $h_{\underline{t}(L)}$ can be taken arbitrarily C^r -close to the identity map if L is sufficiently large as long as r is fixed.

By using the perturbation map, we define

$$(3.3) \quad g = f \circ h_{\underline{t}(L)},$$

which is arbitrarily C^1 -close to f if L is large.

We first note that $[\frac{1}{2} - \delta, \frac{1}{2} + \delta] \times [0, 1] \times J_{k+1}^{cs} \subset U_{3\delta/2}$ for every $k \geq 1$ and $h_{\underline{t}(L)}|_{U_{3\delta/2}^c}$ is equal to the identity. That is, $g|_{U_{3\delta/2}^c} = f$. On the other hand, for any $(x, y, z) \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta] \times [0, 1] \times J_{k+1}^{cs}$, we have

$$g(x, y, z) = f \circ h_{\underline{t}(L)}(x, y, z) = f(x, y, z + a_2^{-1} t_{k+1}) \in U_{3\delta/2}^c.$$

Since $h_{\underline{t}(L)}|_{U_{3\delta/2}^c} = id$, it follows from (1.4a) that

$$g^2(x, y, z) = f^2(x, y, z + a_2^{-1} t_{k+1}) = (t_{k+1}, 0, 0) + f^2(x, y, z).$$

This ends the proof. \square

Remark 3.2. As $r = \infty$, the evaluation (3.2) is useless. Hence the regularity condition in Proposition 3.1 does not reach infinity.

4. CONTRACTING WANDERING DOMAINS

4.1. Two conditions in freedom term. From the results for $\widehat{B}_k^u = B^u(\widehat{n}_k, \widehat{w}^{(k)})$ obtained in Lemma 2.1, one can make some further conditions. Since $\widehat{w}^{(k)} = \underline{w}^{(k+Lk)} \underline{v}^{(k)} \underline{\gamma}^{(k)}$, we have

$$\left| \underline{w}^{(k+Lk)} \right| = n_0 + k + Lk = O(k), \quad \left| \underline{\gamma}^{(k)} \right| = m_k = O(k).$$

Also, as in Remark 2.2, the sub-code $\underline{u}^{(k)}$ of $\underline{v}^{(k)}$ can be chosen freely. Thus, we may assume that the length of $\underline{v}^{(k)}$ is quadratic for k such that

$$(4.1a) \quad \left| \underline{v}^{(k)} \right| = k^2,$$

which is called the *quadratic condition*. Note that the same condition was already used in [CV01]. It follows from (4.1a) that

$$\frac{\widehat{n}_{k+1}}{\widehat{n}_k} = \frac{(k+1)^2 + O(k+1)}{k^2 + O(k)} \rightarrow 1 \text{ as } k \rightarrow +\infty.$$

That is, we have the subexponential growth in generations of \widehat{B}_k^u ($k = 1, 2, \dots$) as follows:

Lemma 4.1. *For any $\eta > 0$, there is an integer $k_0 > 0$ such that for any integer $k \geq k_0$,*

$$\widehat{n}_k < \widehat{n}_{k+1} < (1 + \eta)\widehat{n}_k. \quad \square$$

In addition to (4.1a), we have to make another condition on $\widehat{B}_k^u = B^u(\widehat{n}_k, \widehat{w}^{(k)})$. From the freedom of the choice of $\underline{u}^{(k)}$ again, one can assume that the total number $\widehat{n}_{k(0)}$ of zeros in $\widehat{w}^{(k)}$ is greater than or equal to the number $\widehat{n}_{k(1)}$ of ones, that is,

$$(4.1b) \quad \widehat{n}_{k(1)} \leq \widehat{n}_{k(0)},$$

which is called the *majority condition*.

Remark 4.2. Both (4.1a) and (4.1b) are indispensable to show Lemma 4.5 which is a key to Theorem 4.4. On the other hand, (4.1b) may be an obstacle to realise some type of dynamics of wandering domain. See also Remark 5.4.

In order to see the region in the code occupied by each symbol, we sometimes denote $\widehat{n}_{k(0)}$ and $\widehat{n}_{k(1)}$ by $\left| \widehat{w}^{(k)} \right|_{(0)}$ and $\left| \widehat{w}^{(k)} \right|_{(1)}$, respectively. So we have

$$(4.2) \quad \widehat{n}_k = \widehat{n}_{k(0)} + \widehat{n}_{k(1)} = \left| \widehat{w}^{(k)} \right|_{(0)} + \left| \widehat{w}^{(k)} \right|_{(1)} = \left| \widehat{w}^{(k)} \right|.$$

4.2. Identifying of wandering domains. In the same way as in (2.1), the following similar notations are useful here. For any $(x, y, z) \in \mathbb{B}$ and integer $n > 0$, we write

$$\psi^n(x, z) = \widehat{\pi}(g^n(x, y, z)), \quad \widetilde{\psi}^n(y) = \pi_2(g^n(x, y, z))$$

if the value of the right-hand side of the former (resp. latter) equation does not depend on y (resp. on x or z), where $\widehat{\pi}$ is the projection as in (2.1) and $\pi_2 : \mathbb{B} \rightarrow \mathbb{R}$ is the projection defined by $\pi_2(x, y, z) = y$.

To show the existence of our desired wandering domain, we have to prepare some notations. The first one is the following. For every integer $k \geq k_0$, we set

$$b_k = a_1^{-1} \lambda_u^{-\sum_{i=0}^{\infty} \widehat{n}_{k+i}/2^i}.$$

It implies that

$$(4.3) \quad a_1 \lambda_u^{2\widehat{n}_k} b_k^2 = b_{k+1},$$

which will be useful for some evaluations later. The next one is the following. Let $Y_{k_0} = [\lambda_{ss}, 1 - \lambda_{ss}]$ and, for each integer $k > k_0$,

$$(4.4) \quad Y_k = \widetilde{\psi}^{\widehat{n}_{k-1}+2} \circ \widetilde{\psi}^{\widehat{n}_{k-2}+2} \circ \dots \circ \widetilde{\psi}^{\widehat{n}_{k_0}+2}(Y_{k_0}).$$

Using these items, for each integer $k \geq k_0$, we define

$$\mathbb{W}_k = \left[\widehat{x}_k^u - \frac{b_k}{2}, \widehat{x}_k^u + \frac{b_k}{2} \right] \times Y_k \times \left[\frac{1}{2} - z_k^*, \frac{1}{2} + z_k^* \right],$$

where \widehat{x}_k^u be the centre point of $\widehat{B}_k^u = B^u(\widehat{n}_k, \widehat{w}^{(k)})$ ($k = 1, 2, \dots$) given in Proposition 2.3, and

$$(4.5) \quad z_k^* = 20a_1^{-1/2} a_4 b_k^{1/2}.$$

From the above definition of \mathbb{W}_k , we can immediately see that

Proposition 4.3. *For every $k \geq k_0$,*

$$\pi_1 \circ \widehat{\pi}(\mathbb{W}_k) \subset \widehat{G}_k^u = G^u(\widehat{n}_k, \widehat{w}^{(k)}).$$

Proof. By (1.1), (1.4b) and $n_k < n_{k+1}$, we have

$$(\lambda_u^{-1})^{\sum_{i=1}^{\infty} \widehat{n}_{k+i}/2^i} < \lambda_u^{-2n_k} < a_1(1 - 2\lambda_u^{-1})\lambda_u^{-n_k},$$

and hence

$$b_k = a_1^{-1}\lambda_u^{-\sum_{i=0}^{\infty} \widehat{n}_{k+i}/2^i} < (1 - 2\lambda_u^{-1})(\lambda_u^{-1})^{\widehat{n}_k} = |\widehat{G}_k^u|.$$

Moreover, since the centre point \widehat{x}_k^u of \widehat{B}_k^u is identical to that of \widehat{G}_k^u , the claim of this proposition has been shown. \square

Moreover for the diffeomorphism g given in Proposition 3.1, the following result is obtained:

Theorem 4.4. *There is an integer $k_1 \geq k_0$ such that, for every integer $k \geq k_1$, $\mathbb{D}_k = \text{Int}(\mathbb{W}_k)$ is a contracting wandering domain for g satisfying*

$$g^{\widehat{n}_k+2}(\mathbb{D}_k) \subset \mathbb{D}_{k+1}.$$

The proof of this theorem can be obtained immediately from Propositions 4.7, 4.8 and 4.9. To show them, we need two technical lemmas as follows. As mentioned in Remark 1.2, this is the place where the partially dissipative condition (1.2) comes into play.

Lemma 4.5.

$$\lim_{k \rightarrow +\infty} \left| \frac{a_2 \lambda_{cs0}^{\widehat{n}_{k(0)}} \lambda_{cs1}^{\widehat{n}_{k(1)}} z_k^*}{b_{k+1}/2} \right| = 0.$$

Proof. By (4.3) and (4.5), we have

$$\begin{aligned} \frac{a_2 \lambda_{cs0}^{\widehat{n}_{k(0)}} \lambda_{cs1}^{\widehat{n}_{k(1)}} z_k^*}{2^{-1} b_{k+1}} &= 40a_1^{1/2} a_2 a_4 \lambda_{cs0}^{\widehat{n}_{k(0)}} \lambda_{cs1}^{\widehat{n}_{k(1)}} \lambda_u^{-2\widehat{n}_k} b_k^{-3/2} \\ &= 40a_1^{1/2} a_2 a_4 \lambda_{cs0}^{\widehat{n}_{k(0)}} \lambda_{cs1}^{\widehat{n}_{k(1)}} \lambda_u^{-2\widehat{n}_k} (a_1 \lambda_u^{\sum_{i=0}^{\infty} \widehat{n}_{k+i}/2^i})^{3/2}. \end{aligned}$$

Let η be any positive integer. By Lemma 4.1 based on the quadratic condition (4.1a), there exists $k_0 > 0$ such that, for any integers $k \geq k_0$ and $i \geq 0$, $\widehat{n}_{k+i} < (1 + \eta)^i \widehat{n}_k$. Thus we have the following evaluation.

$$\frac{3}{2} \sum_{i=0}^{\infty} \frac{\widehat{n}_{k+i}}{2^i} \leq \frac{3\widehat{n}_k}{2} \sum_{i=0}^{\infty} \left(\frac{1 + \eta}{2} \right)^i = \frac{3\widehat{n}_k}{1 - \eta} = (3 + \eta_1)\widehat{n}_k,$$

where $\eta_1 = 3\eta/(1 - \eta)$. Recall that $\lambda_{cs0}\lambda_{cs1}\lambda_u^2 < 1$ by (1.2). One can take $\eta > 0$ sufficiently small so that η_1 satisfies $\lambda_{cs0}\lambda_{cs1}\lambda_u^{2(1+\eta_1)} < 1$. Since $\lambda_{cs1}\lambda_u > 1$ by (1.1) and $\widehat{n}_{k(1)} \leq \widehat{n}_{k(0)}$ by (4.1b), $(\lambda_{cs1}\lambda_u)^{\widehat{n}_{k(1)}} \leq (\lambda_{cs1}\lambda_u)^{\widehat{n}_{k(0)}}$. It follows that

$$\begin{aligned} \left| \frac{a_2 \lambda_{cs0}^{\widehat{n}_{k(0)}} \lambda_{cs1}^{\widehat{n}_{k(1)}} z_k^*}{2^{-1} b_{k+1}} \right| &\leq 40a_1^2 |a_2 a_4| \lambda_{cs0}^{\widehat{n}_{k(0)}} \lambda_{cs1}^{\widehat{n}_{k(1)}} \lambda_u^{(1+\eta_1)\widehat{n}_k} \\ &= 40a_1^2 |a_2 a_4| (\lambda_{cs0}\lambda_u^{(1+\eta_1)})^{\widehat{n}_{k(0)}} (\lambda_{cs1}\lambda_u^{(1+\eta_1)})^{\widehat{n}_{k(1)}}, \\ &\leq 40a_1^2 |a_2 a_4| (\lambda_{cs0}\lambda_{cs1}\lambda_u^{2(1+\eta_1)})^{\widehat{n}_{k(0)}} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Thus the proof is now completed. \square

We denote by V_δ the δ -neighbourhoods of $\{x = 1/2\} \cap [0, 1]^2$ in the xz -plane. From (2.1) and (3.3), we have

$$(4.6) \quad \begin{cases} \psi(x, z) = \varphi(x, z) & \text{if } (x, z) \in [0, 1]^2 \setminus V_\delta, \\ \psi^2(x, z) = \varphi^2(x, z + a_2^{-1}t_{k+1}) & \text{if } (x, z) \in V_\delta. \end{cases}$$

Let \hat{x}_k^u be the centre point of $\hat{B}_k^u = B^u(\hat{n}_k, \hat{w}^{(k)})$ ($k = 1, 2, \dots$) given in Proposition 2.3.

Lemma 4.6. *For any $(\hat{x}_k^u + x, 1/2 + z) \in \hat{B}_k^u \times [0, 1]$,*

$$\psi^{\hat{n}_k+2}(\hat{x}_k^u + x, 1/2 + z) = (\hat{x}_{k+1}^u, 1/2) + \left(-a_1 \lambda_u^{2\hat{n}_k} x^2 + a_2 \lambda_{cs0}^{\hat{n}_k(0)} \lambda_{cs1}^{\hat{n}_k(1)} z, \right. \\ \left. a_4 (-1)^{\hat{n}_k(1)} \lambda_u^{\hat{n}_k} x \right).$$

Proof. For simplicity, let us here write $\hat{w}^{(k)} = w_1 w_2 \dots w_i \dots w_{\hat{n}_k}$. Let ξ_0, ξ_1 be the functions on \mathbb{R} defined by $\xi_0(x) = \lambda_u x$ and $\xi_1(x) = \lambda_u(1 - x)$. Then, for any $\hat{x}, x \in \mathbb{R}$,

$$\xi_0(\hat{x} + x) = \lambda_u(\hat{x} + x) = \xi_0(\hat{x}) + \lambda_u x, \quad \xi_1(\hat{x} + x) = \lambda_u(1 - \hat{x} - x) = \xi_1(\hat{x}) - \lambda_u x.$$

Similarly, by (1.3a), for any α, z such that $\alpha + z$ and z are in the domains of the corresponding functions,

$$\zeta_0(\alpha + z) = \zeta_0(\alpha) + \lambda_{cs0} z, \quad \zeta_1(\alpha + z) = \zeta_1(\alpha) + \lambda_{cs1} z.$$

Hence, by the first equation of (4.6) together with (1.3b) and (2.1), for each $i \in \{1, \dots, \hat{n}_k\}$,

$$(4.7) \quad \psi^i(\hat{x}_k^u + x, 1/2 + z) = \left(\xi_{w_i} \circ \dots \circ \xi_{w_2} \circ \xi_{w_1}(\hat{x}_k^u) + \lambda_u^{\hat{n}_i(0)} (-\lambda_u)^{\hat{n}_i(1)} x, \right. \\ \left. \zeta_{w_i} \circ \dots \circ \zeta_{w_2} \circ \zeta_{w_1}(1/2) + \lambda_{cs0}^{\hat{n}_i(0)} \lambda_{cs1}^{\hat{n}_i(1)} z \right).$$

Since \hat{x}_k^u is the centre point of \hat{B}_k^u ,

$$\xi_{w_{\hat{n}_k}} \circ \dots \circ \xi_{w_2} \circ \xi_{w_1}(\hat{x}_k^u) = 1/2.$$

Moreover, by (2.4),

$$\zeta_{w_{\hat{n}_k}} \circ \dots \circ \zeta_{w_2} \circ \zeta_{w_1}(1/2) = \zeta_{\hat{w}^{(k)}}(1/2) = \hat{z}_{k+1}^{cs} - a_2^{-1} t_{k+1}.$$

Since $\hat{n}_k = \left| \hat{w}^{(k)} \right|_{(0)} + \left| \hat{w}^{(k)} \right|_{(1)} = \hat{n}_{k(0)} + \hat{n}_{k(1)}$ by (4.2), the equation (4.7) shows that

$$\psi^{\hat{n}_k}(\hat{x}_k^u + x, 1/2 + z) = \left(1/2 + \lambda_u^{\hat{n}_k(0)} (-\lambda_u)^{\hat{n}_k(1)} x, \hat{z}_{k+1}^{cs} - a_2^{-1} t_{k+1} + \lambda_{cs0}^{\hat{n}_k(0)} \lambda_{cs1}^{\hat{n}_k(1)} z \right) \in V_\delta.$$

By the second equation of (4.6),

$$\begin{aligned} \psi^2 \circ \psi^{\hat{n}_k}(\hat{x}_k^u + x, 1/2 + z) &= \varphi^2 \left(\psi^{\hat{n}_k}(\hat{x}_k^u + x, 1/2 + z) + (0, a_2^{-1} t_{k+1}) \right) \\ &= \varphi^2 \left(1/2 + \lambda_u^{\hat{n}_k(0)} (-\lambda_u)^{\hat{n}_k(1)} x, \hat{z}_{k+1}^{cs} - a_2^{-1} t_{k+1} + \lambda_{cs0}^{\hat{n}_k(0)} \lambda_{cs1}^{\hat{n}_k(1)} z + a_2^{-1} t_{k+1} \right) \\ &= \varphi^2 \left(1/2 + (-1)^{\hat{n}_k(1)} \lambda_u^{\hat{n}_k} x, \hat{z}_{k+1}^{cs} + \lambda_{cs0}^{\hat{n}_k(0)} \lambda_{cs1}^{\hat{n}_k(1)} z \right), \end{aligned}$$

by (1.4a), (2.1) and (4.2),

$$= \left(-a_1 \lambda_u^{2\hat{n}_k} x^2 + a_2 \lambda_{cs0}^{\hat{n}_k(0)} \lambda_{cs1}^{\hat{n}_k(1)} z + a_2 \hat{z}_{k+1}^{cs}, a_4 (-1)^{\hat{n}_k(1)} \lambda_u^{\hat{n}_k} x + 1/2 \right).$$

Since $a_2 \widehat{z}_{k+1}^{cs} = \widehat{x}_{k+1}^u$ from (2.3), we have obtained the equation required in this lemma. \square

For each $k > 0$, we have the rectangle $W_k = \widehat{\pi}(\mathbb{W}_k)$ with the sides $\partial_z W_k = \widehat{\pi}(\mathbb{W}_k \cap \{z = 1/2 \pm z_k^*\})$, $\partial_x W_k = \widehat{\pi}(\mathbb{W}_k \cap \{x = \widehat{x}_k^u \pm b_k/2\})$ and the central line $\mathbf{c}(W_k) = \widehat{\pi}(\mathbb{W}_k \cap \{x = \widehat{x}_k^u\})$. See Figure 4.1.

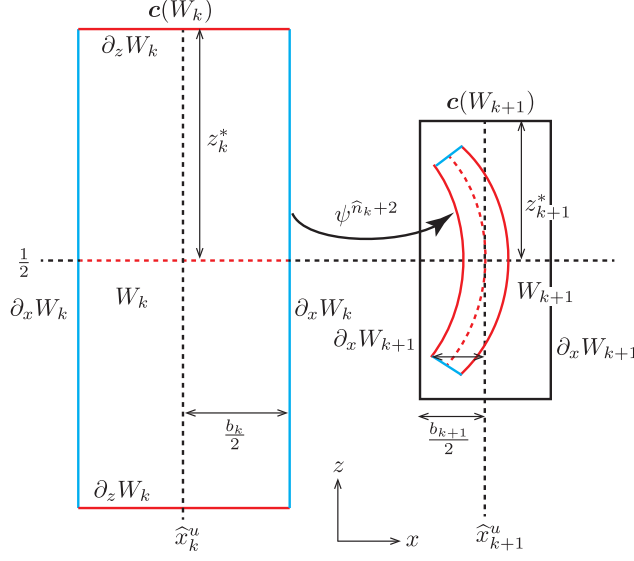


Figure 4.1

Proposition 4.7. *There is an integer $k'_0 \geq k_0$ such that, for any integer $k > k'_0$,*

$$\pi_1(\psi^{\widehat{n}_k+2}(W_k)) \subset \pi_1(W_{k+1}),$$

where π_1 is the projection with $\pi_1(x, z) = x$.

Proof. From the form (1.4a), $\psi^{\widehat{n}_k+2}(\partial_z W_k)$ consists of two quadratic curves. See Figure 4.1. Points of $\psi^{\widehat{n}_k+2}(W_k)$ furthest from $\mathbf{c}(W_{k+1})$ are endpoints of one of the quadratic curves. By Lemma 4.6 and (4.3), we have

$$\begin{aligned} d_h(\mathbf{c}(W_{k+1}), \psi^{\widehat{n}_k+2}(W_k)) &= a_1(\lambda_{cs0}^{\widehat{n}_k} b_k/2)^2 + \left| a_2 \lambda_{cs0}^{\widehat{n}_k(0)} \lambda_{cs1}^{\widehat{n}_k(1)} z_k^* \right| \\ &= 4^{-1} b_{k+1} + \left| a_2 \lambda_{cs0}^{\widehat{n}_k(0)} \lambda_{cs1}^{\widehat{n}_k(1)} z_k^* \right|, \end{aligned}$$

where d_h is the Hausdorff distance of the two subsets. It follows from (4.3) and Lemma 4.5 that the width comparison along the x -direction is the following:

$$\frac{d_h(\mathbf{c}(W_{k+1}), \psi^{\widehat{n}_k+2}(W_k))}{d_h(\mathbf{c}(W_{k+1}), \partial_x W_{k+1})} = \frac{1}{2} + \left| \frac{a_2 \lambda_{cs0}^{\widehat{n}_k(0)} \lambda_{cs1}^{\widehat{n}_k(1)} z_k^*}{b_{k+1}/2} \right|.$$

Note that, from Lemma 4.5, the right-hand side of the inequality is less than 1 if one takes k sufficiently large. This proves the desired assertion and completes the proof of the proposition. \square

Proposition 4.8. *There is an integer $k_0'' \geq k_0$ such that, for any integer $k > k_0''$,*

$$\pi_3(\psi^{\widehat{n}_k+2}(W_k)) \subset \pi_3(W_{k+1}),$$

where π_3 is the projection with $\pi_3(x, z) = z$.

Proof. By the same reason stated in the beginning of the proof of Proposition 4.7, it is sufficient to evaluate how the endpoints of components of $\psi^{\widehat{n}_k+2}(\partial_z W_k)$ are far from $\{z = 1/2\}$. More concretely, it follows from Lemma 4.6 that it suffices to prove that the following inequality:

$$|a_4 \lambda_u^{n_k} b_k / 2| < z_{k+1}^* / 2.$$

By (4.5), it is equivalent to

$$a_1 \lambda_u^{2n_k} b_k^2 < 400 b_{k+1}.$$

This is established from (4.3) and the proof is accomplished. \square

Now let us turn our attention to Y_k defined in (4.4).

Proposition 4.9. *For every integer $k > k_0$, Y_k is contained in $(\frac{1}{2} - \frac{a_3}{2}, \frac{1}{2} + \frac{a_3}{2})$ and*

$$\lim_{k \rightarrow +\infty} |Y_k| = 0.$$

Proof. For the generation \widehat{n}_{k_0} of $\widehat{B}_{k_0}^u = B^u(\widehat{n}_{k_0}, \underline{\widehat{w}}^{(k_0)})$, we have

$$\left| \widetilde{\psi}^{\widehat{n}_{k_0}}(Y_{k_0}) \right| = \lambda_{ss}^{\widehat{n}_{k_0}} |Y_{k_0}| = \lambda_{ss}^{\widehat{n}_{k_0}} (1 - 2\lambda_{ss}).$$

From (1.4a) together with (1.4b), $Y_{k_0+1} = \widetilde{\psi}^{\widehat{n}_{k_0}+2}(Y_{k_0}) \subset (\frac{1}{2} - \frac{a_3}{2}, \frac{1}{2} + \frac{a_3}{2})$ and

$$|Y_{k_0+1}| = |a_3| \lambda_{ss}^{\widehat{n}_{k_0}} |Y_{k_0}| = |a_3| \lambda_{ss}^{\widehat{n}_{k_0}} (1 - 2\lambda_{ss}).$$

By inductive steps, one can show that, for every integer $k > k_0$, $Y_k \subset (\frac{1}{2} - \frac{a_3}{2}, \frac{1}{2} + \frac{a_3}{2})$ and

$$|Y_k| = |a_3|^{k-k_0} \lambda_{ss}^{\sum_{i=0}^{k-k_0} \widehat{n}_{k+i}} (1 - 2\lambda_{ss}).$$

Hence, it converges to 0 as $k \rightarrow +\infty$. \square

Proof of Theorem 4.4. From Propositions 4.7 and 4.8, there is an integer $k_1 \geq k_0$ such that, for any integer $k \geq k_1$,

$$\psi^{\widehat{n}_k+2}(W_k) \subset \text{Int}(W_{k+1}),$$

and moreover

$$\lim_{k \rightarrow +\infty} \text{diam}(W_{k+1}) = 0.$$

On the other hand, Proposition 4.9 implies that, for any $k > k_1$, diameter of Y_k converges to zero as $k \rightarrow +\infty$. Since $W_k \times Y_k$ is equal to \mathbb{W}_k , the proof is complete. \square

5. PROBABILISTIC REPRESENTATIONS

Let g be the diffeomorphism defined in (3.3), $\mathbb{D}_k = \text{Int}(\mathbb{W}_k)$ the contracting wandering domain of g and k_1 the integer obtained in Theorem 4.4. Also with Proposition 4.3, for each $k > k_1$,

$$(5.1a) \quad g^{\widehat{n}_k+2}(\mathbb{D}_k) \subset \mathbb{D}_{k+1}, \quad \pi_1 \circ \widehat{\pi}(\mathbb{D}_k) \subset \widehat{B}_k^u = B^u(\widehat{n}_k, \widehat{w}^{(k)}),$$

where $\widehat{\pi}$ and π_1 are the projections given as in (2.1) and Proposition 4.7, respectively, and $\widehat{n}_k = |\widehat{w}^{(k)}|$. The itinerary $\widehat{w}^{(k)}$ consists of three parts as

$$(5.1b) \quad \widehat{w}^{(k)} = \underline{w}^{(k+Lk)} \underline{v}^{(k)} \underline{\gamma}^{(k)},$$

where the sub-code $\underline{v}^{(k)}$ is a k^2 -tuple $\underline{v}^{(k)} = (v_1 v_2 \dots v_{k^2})$, at least $k^2 - 1$ elements of which can be chosen freely, and the other parts satisfy

$$(5.1c) \quad |\underline{w}^{(k+Lk)}| = n_0 + k + Lk, \quad |\underline{\gamma}^{(k)}| = m_k,$$

where $n_0 + k + Lk$ and m_k are integers given in Lemma 2.1. Let us now take advantage of this freedom of $\underline{v}^{(k)}$ to realise historicity and physicality.

5.1. Historicity. The following theorem guarantees half of the claim in Theorem A', the part about historicity.

Theorem 5.1. *There exists a sequence $\mathbf{v} = (\underline{v}^{(k)})_{k > k_1}$ of codes such that \mathbb{D}_k is a historic contracting wandering domain for $g = g_{\mathbf{v}}$.*

To show this claim we need the following two conditions:

Era condition: We consider an increasing sequence $(k_s)_{s \in \mathbb{N}}$ of integers, which satisfies the following condition: for every $s \in \mathbb{N}$,

$$(5.2) \quad \sum_{k=k_s}^{k_{s+1}-1} k^2 > s \sum_{k=k_1}^{k_s-1} k^2.$$

Note that this setting provides us with a situation that the new era from k_s until $k_{s+1} - 1$ to be so dominant as to ignore the old one from k_1 until $k_s - 1$, see Claim 5.2.

Code condition (for historic behaviour): On the era condition, for any integer $k = k(s)$ with $k_s < k \leq k_{s+1}$, we consider each entry of $\underline{v}^{(k)} = (v_1 v_2 \dots v_{k^2})$ satisfying (4.1b) and the following rules:

- if s is even,

$$(5.3a) \quad v_i = \begin{cases} 0 & \text{for } i = 1, \dots, \lfloor 3k(s)^2/4 \rfloor \\ 1 & \text{for } i = \lceil 3k(s)^2/4 \rceil, \dots, k(s)^2, \end{cases}$$

$$\text{that is, } \underline{v}^{(k)} = \overbrace{000 \dots 0}^{\lfloor 3k^2/4 \rfloor} \overbrace{1 \dots 1}^{\lceil k^2/4 \rceil},$$

- if s is odd,

$$(5.3b) \quad v_i = \begin{cases} 0 & \text{for } i = 1, \dots, \lfloor 7k(s)^2/8 \rfloor \\ 1 & \text{for } i = \lceil 7k(s)^2/8 \rceil, \dots, k(s)^2, \end{cases}$$

$$\text{that is, } \underline{v}^{(k)} = \overbrace{000 \dots 0}^{\lfloor 7k^2/8 \rfloor} \overbrace{1 \dots 1}^{\lceil k^2/8 \rceil},$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ indicate the floor and ceiling functions, respectively.

Note that both (5.3a) and (5.3b) satisfy the quadratic condition (4.1a) and the majority condition (4.1b).

It may be obvious that historic behaviour appears under (5.3a) and (5.3b). Indeed, in [CV01, KS17] they gave constructions similar to ours, but did not provide detailed proofs. However, we here describe a proof in detail for the convenience of readers.

Proof of Theorem 5.1. The proof is carried out under the era and code conditions, which are required to show Claims 5.2 and 5.3.

For given non-negative integers n, m with $n < m$ and $x \in \mathbb{D}_{k_1}$, the *empirical probability measure* is defined by

$$\nu_x^{(n,m)} = \frac{1}{m-n} \sum_{i=n}^{m-1} \delta_{g^i(x)}.$$

For any integer $k_s > 0$, we write

$$\hat{N}_{k_s} = \sum_{k=k_1}^{k_s-1} (\hat{n}_k + 2).$$

Let $\hat{\mathbb{B}}$ be a compact subset of \mathbb{R}^3 containing $\bigcup_{i=0}^2 g^i(\mathbb{B})$. For any $\Phi \in C^0(\hat{\mathbb{B}}, \mathbb{R})$, we have

$$(5.4) \quad \int \Phi d\nu_x^{(\hat{N}_{k_s}, \hat{N}_{k_{s+1}})} = \frac{1}{\hat{N}_{k_{s+1}} - \hat{N}_{k_s}} \sum_{i=\hat{N}_{k_s}}^{\hat{N}_{k_{s+1}}-1} \Phi \circ g^i(x).$$

Claim 5.2. *For any $x \in \mathbb{D}_{k_1}$,*

$$\lim_{s \rightarrow +\infty} \left| \int \Phi d\nu_x^{(\hat{N}_{k_s}, \hat{N}_{k_{s+1}})} - \int \Phi d\nu_x^{(0, \hat{N}_{k_{s+1}})} \right| = 0.$$

Here we show the claim. Consider

$$|A_s + B_s| = \left| \int \Phi d\nu_x^{(\hat{N}_{k_s}, \hat{N}_{k_{s+1}})} - \int \Phi d\nu_x^{(0, \hat{N}_{k_{s+1}})} \right|,$$

where

$$\begin{aligned} A_s &= \frac{1}{\hat{N}_{k_{s+1}} - \hat{N}_{k_s}} \sum_{i=\hat{N}_{k_s}}^{\hat{N}_{k_{s+1}}-1} \Phi \circ g^i(x) - \frac{1}{\hat{N}_{k_{s+1}}} \sum_{i=\hat{N}_{k_s}}^{\hat{N}_{k_{s+1}}-1} \Phi \circ g^i(x), \\ B_s &= \frac{1}{\hat{N}_{k_{s+1}}} \sum_{i=\hat{N}_{k_s}}^{\hat{N}_{k_{s+1}}-1} \Phi \circ g^i(x) - \frac{1}{\hat{N}_{k_{s+1}}} \sum_{i=0}^{\hat{N}_{k_{s+1}}-1} \Phi \circ g^i(x). \end{aligned}$$

Thus, the proof will be complete if $|A_s|$ and $|B_s|$ converge to 0 as $s \rightarrow 0$. In fact,

$$\begin{aligned} |A_s| &\leq \left| \frac{(\hat{N}_{k_{s+1}} - \hat{N}_{k_s})(\hat{N}_{k_{s+1}} - (\hat{N}_{k_{s+1}} - \hat{N}_{k_s})) \|\Phi\|_{C^0}}{(\hat{N}_{k_{s+1}} - \hat{N}_{k_s}) \hat{N}_{k_{s+1}}} \right| \\ &= \frac{\hat{N}_{k_s}}{\hat{N}_{k_{s+1}}} \|\Phi\|_{C^0} < \frac{1}{1+s} \|\Phi\|_{C^0}, \end{aligned}$$

where the last inequality follows from (5.2). On the other hand,

$$|B_s| \leq \frac{|(\widehat{N}_{k_{s+1}} - \widehat{N}_{k_s}) - \widehat{N}_{k_{s+1}}| \|\Phi\|_{C^0}}{\widehat{N}_{k_{s+1}}} = \frac{\widehat{N}_{k_s}}{\widehat{N}_{k_{s+1}}} \|\Phi\|_{C^0} < \frac{1}{1+s} \|\Phi\|_{C^0}.$$

Hence, $|A_s|, |B_s| \rightarrow 0$ as $s \rightarrow +\infty$. This ends the proof of Claim 5.2.

Based on the result of Claim 5.2, we focus only on (5.4). So we divide it into three parts as follows:

$$\frac{1}{\widehat{N}_{k_{s+1}} - \widehat{N}_{k_s}} \sum_{i=\widehat{N}_{k_s}}^{\widehat{N}_{k_{s+1}}-1} \Phi(g^i(x)) = \frac{1}{\widehat{N}_{k_{s+1}} - \widehat{N}_{k_s}} (S_1 + S_2 + S_3),$$

where

$$\begin{aligned} S_1 &= \sum_{i=\widehat{N}_{k_s}}^{\widehat{N}_{k_s}+(n_0+k_s+Lk_s)-1} \Phi(g^i(x)), & S_2 &= \sum_{i=\widehat{N}_{k_s}+(n_0+k_s+Lk_s)}^{\widehat{N}_{k_s}+(n_0+k_s+Lk_s)+k_s^2-1} \Phi(g^i(x)), \\ S_3 &= \sum_{i=\widehat{N}_{k_s}+(n_0+k_s+Lk_s)+k_s^2}^{\widehat{N}_{k_{s+1}}-1} \Phi(g^i(x)). \end{aligned}$$

Note that the number of terms in the sum of S_1 and S_3 is $O(k_s)$, while that of S_2 is k_s^2 . Since

$$\widehat{N}_{k_{s+1}} - \widehat{N}_{k_s} = \widehat{n}_{k_s} + 2 = (n_0 + k_s + Lk_s) + k_s^2 + m_{k_s} + 2 = k_s^2 + O(k_s),$$

we have

$$\begin{aligned} (5.5) \quad \lim_{s \rightarrow +\infty} \left| \frac{1}{\widehat{N}_{k_{s+1}} - \widehat{N}_{k_s}} \sum_{i=\widehat{N}_{k_s}}^{\widehat{N}_{k_{s+1}}-1} \Phi(g^i(x)) - \frac{S_2}{k_s^2} \right| \\ = \lim_{s \rightarrow +\infty} \left| \left(\frac{S_1}{\widehat{n}_{k_s} + 2} + \frac{k_s^2}{\widehat{n}_{k_s} + 2} \left(\frac{S_2}{k_s^2} \right) + \frac{S_3}{\widehat{n}_{k_s} + 2} \right) - \frac{S_2}{k_s^2} \right| = 0. \end{aligned}$$

For simplicity, write $\widehat{x} := g^{\widehat{N}_k+(n_0+k_s+Lk_s)}(x)$ and hence

$$S_2 = \sum_{j=0}^{k_s^2-1} \Phi(g^j(\widehat{x})).$$

It is sufficient to prove the following claim for Theorem 5.1:

Claim 5.3.

$$\lim_{s \rightarrow +\infty} \frac{S_2}{k_s^2} = \begin{cases} (3\Phi(P_g) + \Phi(Q_g))/4 & \text{if } s \text{ is even,} \\ (7\Phi(P_g) + \Phi(Q_g))/8 & \text{if } s \text{ is odd,} \end{cases}$$

where P_g and Q_g are the continuations of the fixed points P and Q , respectively.

To prove this claim, define

$$\begin{aligned} N_s^{(P_g)} &= N_s^{(P_g)}(\rho) = \max \{ N > 0 : g^i(\hat{x}) \in U_\rho(P_g) \text{ for } 0 \leq \forall i \leq N \}, \\ \tilde{N}_s^{(P_g)} &= \max \{ N > 0 : g^i(\hat{x}) \in \mathbb{V}_0 \text{ for } 0 \leq \forall i \leq N \}, \\ N_s^{(Q_g)} &= N_s^{(Q_g)}(\rho) = \max \{ N > 0 : g^i(\hat{x}) \in U_\rho(Q_g) \text{ for } N_s^{(P_g)} \leq \forall i < N \}, \end{aligned}$$

where $U_\rho(P_g)$ and $U_\rho(Q_g)$ are the ρ -neighbourhoods of P_g and Q_g , respectively, for a given constant $\rho > 0$, and \mathbb{V}_0 is the component of $g^{-1}(\mathbb{B}) \cap \mathbb{B}$ containing P_g . Using them, we have

$$\begin{aligned} \sum_{i=0}^{k_s^2-1} \Phi(g^i(\hat{x})) &= \sum_{i=0}^{N_s^{(P_g)}-1} \Phi(g^i(\hat{x})) + \sum_{i=N_s^{(P_g)}}^{\tilde{N}_s^{(P_g)}-1} \Phi(g^i(\hat{x})) \\ &\quad + \sum_{i=\tilde{N}_s^{(P_g)}}^{N_s^{(Q_g)}-1} \Phi(g^i(\hat{x})) + \sum_{i=N_s^{(Q_g)}}^{k_s^2-1} \Phi(g^i(\hat{x})). \end{aligned}$$

For any small $\varepsilon > 0$, there is a $\rho > 0$ such that

$$\left| \frac{\sum_{j=0}^{N_s^{(P_g)}-1} \Phi(g^j(\hat{x}))}{N_s^{(P_g)}} - \Phi(P_g) \right| < \varepsilon, \quad \left| \frac{\sum_{i=\tilde{N}_s^{(P_g)}}^{N_s^{(Q_g)}-1} \Phi(g^i(\hat{x}))}{N_s^{(Q_g)} - \tilde{N}_s^{(P_g)}} - \Phi(Q_g) \right| < \varepsilon.$$

It implies that

$$\begin{aligned} \frac{1}{k_s^2} \sum_{i=0}^{k_s^2-1} \Phi(g^i(\hat{x})) &< \frac{N_s^{(P_g)}}{\tilde{N}_s^{(P_g)}} \frac{\tilde{N}_s^{(P_g)}}{k_s^2} (\Phi(P_g) + \varepsilon) + \frac{1}{k_s^2} \sum_{i=N_s^{(P_g)}}^{\tilde{N}_s^{(P_g)}-1} \Phi(g^i(x)) \\ &\quad + \frac{1}{k_s^2} \sum_{i=N_s^{(Q_g)}}^{k_s^2-1} \Phi(g^i(x)) + \frac{N_s^{(Q_g)} - \tilde{N}_s^{(P_g)}}{k_s^2 - \tilde{N}_s^{(P_g)}} \frac{k_s^2 - \tilde{N}_s^{(P_g)}}{k_s^2} (\Phi(Q_g) + \varepsilon), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{k_s^2} \sum_{i=0}^{k_s^2-1} \Phi(g^i(\hat{x})) &> \frac{N_s^{(P_g)}}{\tilde{N}_s^{(P_g)}} \frac{\tilde{N}_s^{(P_g)}}{k_s^2} (\Phi(P_g) - \varepsilon) + \frac{1}{k_s^2} \sum_{i=N_s^{(P_g)}}^{\tilde{N}_s^{(P_g)}-1} \Phi(g^i(x)) \\ &\quad + \frac{1}{k_s^2} \sum_{i=N_s^{(Q_g)}}^{k_s^2-1} \Phi(g^i(x)) + \frac{N_s^{(Q_g)} - \tilde{N}_s^{(P_g)}}{k_s^2 - \tilde{N}_s^{(P_g)}} \frac{k_s^2 - \tilde{N}_s^{(P_g)}}{k_s^2} (\Phi(Q_g) - \varepsilon). \end{aligned}$$

Since ρ is already fixed, it follows from code conditions (5.3a) and (5.3b) that

$$\lim_{s \rightarrow +\infty} \frac{N_s^{(P_g)}}{\tilde{N}_s^{(P_g)}} = \lim_{s \rightarrow +\infty} \frac{N_s^{(Q_g)} - \tilde{N}_s^{(P_g)}}{k_s^2 - \tilde{N}_s^{(P_g)}} = 1,$$

and

$$\lim_{s \rightarrow +\infty} \frac{\tilde{N}_s^{(P_g)}}{k_s^2} = \begin{cases} 3/4 & \text{if } s \text{ is even,} \\ 7/8 & \text{if } s \text{ is odd,} \end{cases} \quad \lim_{s \rightarrow +\infty} \frac{k_s^2 - \tilde{N}_s^{(P_g)}}{k_s^2} = \begin{cases} 1/4 & \text{if } s \text{ is even,} \\ 1/8 & \text{if } s \text{ is odd.} \end{cases}$$

Moreover,

$$\lim_{s \rightarrow +\infty} \frac{1}{k_s^2} \left(\sum_{i=N_s^{(P_g)}}^{\tilde{N}_s^{(P_g)}-1} \Phi(g^i(x)) + \sum_{i=N_s^{(Q_g)}}^{k_s^2-1} \Phi(g^i(x)) \right) = 0.$$

This finishes the proof of Claim 5.3.

Finally, by Claims 5.2 and 5.3 together with (5.5),

$$\lim_{s \rightarrow +\infty} \int \Phi d\nu_x^{(0, \tilde{N}_{k_{s+1}})} = \begin{cases} (3\Phi(P_g) + \Phi(Q_g))/4 & \text{if } s \text{ is even,} \\ (7\Phi(P_g) + \Phi(Q_g))/8 & \text{if } s \text{ is odd.} \end{cases}$$

We complete the proof of Theorem 5.1. \square

5.2. Physicality. The final discussion in this paper concerns the existence of non-trivial Dirac physical measure associated with a contracting wandering domain in Theorem A'. To show it we need not to take any era condition as (5.2) into account. Moreover instead of (5.3a) and (5.3b) we adopt simpler code condition, which is the same as that in [CV01, Section 9], as follows:

Code condition (for Dirac physical measure supported on P_g): For a given integer $k > 0$, we suppose that the freedom part $\underline{v}^{(k)}$ of the itinerary $\underline{\hat{w}}^{(k)}$ in (5.1b) consists of k^2 zeros, that is,

$$(5.6) \quad \underline{v}^{(k)} = \underline{0}^{k^2} = \overbrace{0000 \dots 00}^{k^2}.$$

Note that (5.6) is not contradict to the quadratic condition (4.1a) and the majority condition (4.1b).

Remark 5.4. On the other hand, since the itinerary $\underline{1}^{k^2}$ does not meet (4.1b), the other saddle fixed point Q_g may not be a support of the Dirac physical measure. See also Remark 5.6.

Theorem 5.5. *There exist an integer $k_2 > 0$ and a sequence $\mathbf{v} = (\underline{v}^{(k)})_{k > \max\{k_1, k_2\}}$ of codes such that $g = g_{\mathbf{v}}$ has the non-trivial Dirac physical measure supported on P_g associated with the contracting wandering domain \mathbb{D}_k .*

Proof. For any given integers $u, s > 0$, we take an integer k_2 which satisfies

$$k_2^2 > u + s.$$

Moreover, we write

$$\mathbb{B}^{\text{ss}}(s; \underline{w}) = \{x \in \mathbb{B} : g^{-i}(x) \in \mathbb{V}_{w_i}, i = 1, \dots, s\}.$$

Hereafter, we suppose that the code condition (5.6) holds for every

$$k \geq \max\{k_1, k_2\},$$

where k_1 is the integer given in Theorem 4.4. Consider the wandering domain \mathbb{D}_k with (5.1a). First, observe that, for any integers $u, s > 0$,

$$P_g \in \mathbb{B}^u(u; \underline{0}^{(u)}) \cap \mathbb{B}^{\text{ss}}(s; \underline{0}^{(s)}),$$

where P_g is the saddle fixed point of the cs-blender horseshoe Λ_g and $\mathbb{B}^u(u; \underline{0}^{(u)})$ is the u-bridge given in Subsection 2.1.

Next, verify the following facts:

- It follows from Proposition 4.3 that

$$\mathbb{D}_k \subset \mathbb{G}^u(\hat{n}_k, \hat{w}^{(k)}),$$

where $\mathbb{G}^u(\hat{n}_k, \hat{w}^{(k)})$ is the u -gap given in Subsection 2.1.

- Thus, under the code condition (5.6), at least $k^2 - (u + s)$ of the first $\hat{n}_k - 1$ iterates of \mathbb{D}_k are contained in the interior of $\mathbb{B}^u(u; \underline{Q}^u) \cap \mathbb{B}^{ss}(s; \underline{Q}^s)$.

The corresponding fact for a 2-dimensional horseshoe is stated by Colli-Vargas [CV01, Section 9]. The cs-blender horseshoe, on the other hand, is a 3-dimensional object, but this fact is actually the same as for the 2-dimensional horseshoe. Based on this, the following evaluations will be conducted.

For any integer $\hat{k} > 0$, we here define

$$\hat{N}_{\hat{k}} = \sum_{i=0}^{\hat{k}} (\hat{n}_{k+i} + 2).$$

Then for any $x \in \mathbb{D}_k$, the empirical probability measure from $\hat{N}_{\hat{k}-1}$ to $\hat{N}_{\hat{k}}$ is

$$\nu_x^{(\hat{N}_{\hat{k}-1}, \hat{N}_{\hat{k}})} = \frac{1}{\hat{N}_{\hat{k}} - \hat{N}_{\hat{k}-1}} \sum_{i=\hat{N}_{\hat{k}-1}}^{\hat{N}_{\hat{k}}-1} \delta_{g^i(x)} = \frac{1}{\hat{n}_{k+\hat{k}} + 2} \sum_{i=\hat{N}_{\hat{k}-1}}^{\hat{N}_{\hat{k}}-1} \delta_{g^i(x)}.$$

Hence, for any $\Phi \in C^0(\mathbb{B}, \mathbb{R})$,

$$\int \Phi d\nu_x^{(\hat{N}_{\hat{k}-1}, \hat{N}_{\hat{k}})} = \frac{1}{\hat{n}_{k+\hat{k}} + 2} \sum_{i=\hat{N}_{\hat{k}-1}}^{\hat{N}_{\hat{k}}-1} \Phi \circ g^i(x) = \frac{1}{\hat{n}_{k+\hat{k}} + 2} \sum_{j=0}^{\hat{n}_{k+\hat{k}}+1} \Phi \circ g^j(x_{\hat{N}_{\hat{k}-1}}),$$

where $x_{\hat{N}_{\hat{k}-1}} = g^{\hat{N}_{\hat{k}-1}}(x)$. Taking (5.1c) into account, let us divide the sum from 0 to $\hat{n}_{k+\hat{k}} + 1$ into the following three parts:

$$\begin{aligned} \sum_{j=0}^{\hat{n}_{k+\hat{k}}+1} \Phi \circ g^j(x_{\hat{N}_{\hat{k}-1}}) &= \sum_{j=0}^{\hat{K}+u-1} \Phi \circ g^j(x_{\hat{N}_{\hat{k}-1}}) \\ &\quad + \sum_{j=\hat{K}+u}^{\hat{K}+(k+\hat{k})^2-s} \Phi \circ g^j(x_{\hat{N}_{\hat{k}-1}}) + \sum_{j=\hat{K}+(k+\hat{k})^2-s+1}^{\hat{n}_{k+\hat{k}}+1} \Phi \circ g^j(x_{\hat{N}_{\hat{k}-1}}), \end{aligned}$$

where $\hat{K} = n_0 + (k + \hat{k}) + L(k + \hat{k})$. Note that

$$\hat{n}_{k+\hat{k}} = \hat{K} + (k + \hat{k})^2 + m_{k+\hat{k}} = (k + \hat{k})^2 + O(k + \hat{k}).$$

Therefore, for any $\varepsilon > 0$, there exist integers $u_0, s_0 > 0$ such that, for any $u \geq u_0$ and $s \geq s_0$,

$$\begin{aligned} \sum_{j=0}^{\hat{n}_{k+\hat{k}}+1} \Phi \circ g^j(x_{\hat{N}_{\hat{k}-1}}) &\leq (\hat{K} + u) \|\Phi\|_{C^0} \\ &\quad + \left((k + \hat{k})^2 - s - u + 1 \right) (\Phi(P_g) + \varepsilon) + (m_{k+\hat{k}} + s) \|\Phi\|_{C^0}, \end{aligned}$$

and

$$\sum_{j=0}^{\widehat{n}_{k+\widehat{k}}+1} \Phi \circ g^j(x_{\widehat{N}_{\widehat{k}-1}}) \geq \left((k+\widehat{k})^2 - s - u + 1\right) (\Phi(P_g) - \varepsilon).$$

In consequence, it follows from the code condition (5.6) that for any sufficiently large \widehat{k} ,

$$\Phi(P_g) - 2\varepsilon \leq \frac{1}{\widehat{n}_{k+\widehat{k}} + 2} \sum_{j=0}^{\widehat{n}_{k+\widehat{k}}+1} \Phi \circ g^j(x_{\widehat{N}_{\widehat{k}-1}}) \leq \Phi(P_g) + 2\varepsilon,$$

and hence

$$\left| \int \Phi d\nu_x^{(\widehat{N}_{\widehat{k}-1}, \widehat{N}_{\widehat{k}})} - \int \Phi d\delta_{P_g} \right| < 2\varepsilon.$$

That is, $\nu_x^{(\widehat{N}_{\widehat{k}-1}, \widehat{N}_{\widehat{k}})}$ converges to δ_{P_g} as $\widehat{k} \rightarrow +\infty$ in the weak*-topology. \square

Remark 5.6. Instead of (5.6), for any positive integer k and $n \geq 2$, consider a n -periodic itinerary such that

$$\underline{v}^{(k)} = \overbrace{\underbrace{000 \dots 01}_n \underbrace{000 \dots 01}_n \dots \underbrace{000 \dots 01}_n \underbrace{000 \dots 00}_{k^2 - \lfloor k^2/n \rfloor n}}^{k^2},$$

where $\lfloor \cdot \rfloor$ stands for the floor function. Since

$$\lim_{k \rightarrow +\infty} \frac{k^2 - \lfloor k^2/n \rfloor n}{k^2} = 0,$$

$\underline{v}^{(k)}$ still satisfies both (4.1a) and (4.1b). Then, for such a $\underline{v}^{(k)}$, by the same procedure as in proof of Theorem 5.5, one can obtain the non-trivial Dirac physical measure associated with the wandering domain supported by the n -periodic orbit.

APPENDIX A.

To show Lemma 1.3, we only need to verify the existence of a folding manifold which is contained in $W^s(\Lambda)$, because it implies that f has a C^1 -robust homoclinic tangency of Λ from [BD12, Theorem 4.8]. Here the *folding manifold* of Λ is a 2-dimensional manifold with the following conditions:

- $\mathcal{S} = \bigcup_{t \in [t_-, t_+]} \mathcal{S}_t$, where $t_{\pm} \in \mathbb{R}$ with $t_- < t_+$ and \mathcal{S}_t is a (1-dimensional) ss-disc of \mathbb{B} ;
- both \mathcal{S}_{t_-} and \mathcal{S}_{t_+} intersect $W_{\text{loc}}^u(P)$;
- for any $t \in (t_-, t_+)$, \mathcal{S}_t lies between $W_{\text{loc}}^u(P)$ and $W_{\text{loc}}^u(Q)$.

Proof of Lemma 1.3. Let ℓ_0 and ℓ_1 be the parallel edges of \mathbb{B} given as

$$\ell_0 = [1/2 - \delta, 1/2 + \delta] \times (0, 0), \quad \ell_1 = [1/2 - \delta, 1/2 + \delta] \times (1, 1).$$

By (1.4a), $\tilde{\ell}_0 = f^2(\ell_0)$ and $\tilde{\ell}_1 = f^2(\ell_1)$ are contained in the quadratic curves, respectively, as

$$\left\{ x = -a_1 a_4^{-2} \left(z - \frac{1}{2} \right)^2, y = \frac{1}{2} - \frac{a_3}{2} \right\}, \left\{ x = -a_1 a_4^{-2} \left(z - \frac{1}{2} \right)^2 + a_2, y = \frac{1}{2} + \frac{a_3}{2} \right\}.$$

See Figure A.1.

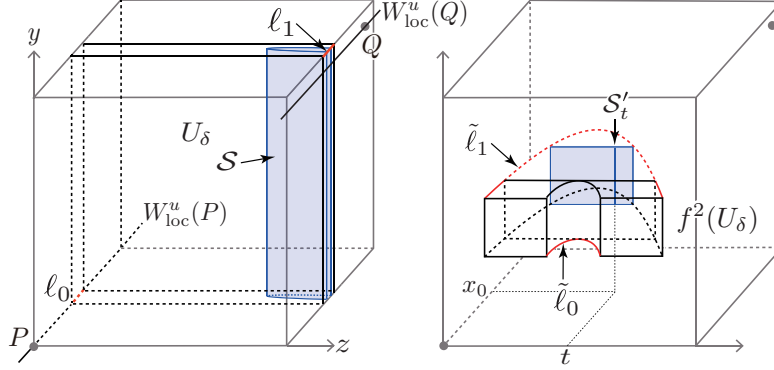


Figure A.1

For a given $0 < x_0 < a_2$, we write

$$t_{\pm}(x_0) = \frac{1}{2} \pm \sqrt{a_1^{-1}a_4(a_2 - x_0)}.$$

By the second condition in (1.4b), the value inside the root is positive. For any $t \in [t_-(x_0), t_+(x_0)]$, consider the vertical ss-disc defined as

$$\mathcal{S}'_t = \mathcal{S}'_t(x_0) = \{x_0\} \times \left[\frac{1}{2} - \frac{a_3}{2}, \frac{1}{2} + \frac{a_3}{2}\right] \times \{t\}.$$

Note that $\mathcal{S}'_t(x_0)$ can be contained in $W_{\text{loc}}^s(\Lambda)$ if one chooses x_0 appropriately. Let \mathcal{S}' be the collection of all of \mathcal{S}'_t with $t \in [t_-(x_0), t_+(x_0)]$. Observe that the intersection of \mathcal{S}' and $\tilde{\ell}_0$ consists of two transverse points. It implies that $\mathcal{S}_{t_-(x_0)}$ and $\mathcal{S}_{t_+(x_0)}$ intersect $W_{\text{loc}}^u(Q)$. Moreover, it follows from (1.3b) and (1.4a) that f preserves the y -direction. Thus $\mathcal{S}_t = f^{-1}(\mathcal{S}'_t)$ is an ss-disc. In consequence, $\mathcal{S} = f^{-1}(\mathcal{S}')$ is a folding stable manifold of Λ . \square

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