# A note on derivatives, expansions and $\Pi_{1}^{1}$-ranks 

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#### Abstract

$\Pi_{1}^{1}$-ranks are a natural tool for studying coanalytic sets in descriptive set theory. In [5], Kechris provided a technique to build $\Pi_{1}^{1}$-ranks using derivatives. In this note we will prove a variant of this result that is applicable to the $\Gamma$-rank. Some dynamical ranks, like the entropy rank can be stated in terms of the $\Gamma$-rank.


## 1 Introduction

At the dawn of descriptive set theory, Lebesgue made an infamous mistake by assuming that the continuous projection of a Borel set was Borel. Suslin spotted this mistake 10 years later and began the study of analytic or $\Sigma_{1}^{1}$ sets. One of tools which was developed later to understand the complexity of Borel sets and the difference between Borel and co-analytic sets were $\Pi_{1}^{1}$-ranks. One of the most well known $\Pi_{1}^{1}$-ranks is the Cantor-Bendixson rank. Kechris developed a very general set up were, using the concept of derivatives (or the dual version of expansions), one can prove that the Cantor-Bendixson rank, as well as several other natural ranks, are $\Pi_{1}^{1}$ (see Theorem 2.7) [5].

A natural rank that appears in dynamics as well as other areas of mathematics is the $\Gamma$-rank on product spaces. Though the $\Gamma$-rank can be stated in terms of expansions it does not fit exactly in the context of the dual of Theorem 2.7 In this note, we adapt the proof given in [5] in order to show that the $\Gamma$-rank is a $\Pi_{1}^{1}$-rank. A concrete example of the $\Gamma$-rank is the entropy rank for topological dynamical systems that was introduced by Barbieri and the second author [1] to classify dynamical systems with completely positive entropy.

We note that, using effective descriptive set theory, Westrick recently proved that the entropy (or TCPE) rank is an effective $\Pi_{1}^{1}$-rank [9, Corollary 2]. It is possible to transfer this result to the classic descriptive theory setting, nonetheless, the approach of this paper gives a direct proof using only classical descriptive set theory.

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## $2 \Pi_{1}^{1}$-ranks

In this section we present the necessary basic definitions and background results concerning $\Pi_{1}^{1}$-ranks. We also prove a new result concerning Borel expansions.

Recall that a subset of a Polish space is analytic or $\Sigma_{1}^{1}$ if it is the continuous image of a Borel set of a Polish space. Complements of $\Sigma_{1}^{1}$ sets are $\Pi_{1}^{1}$ or coanalytic.
Definition 2.1 Let C be a set. A rank on $C$ is a function $\varphi: C \rightarrow \omega_{1}$, where $\omega_{1}$ is the set of countable ordinals. Associated with $\varphi$ we have the relations $<_{\varphi}$ and $\leq_{\varphi}$ defined as follows:

$$
\begin{aligned}
x<_{\varphi} y & \Longleftrightarrow \varphi(x)<\varphi(y) \\
x \leq_{\varphi} y & \Longleftrightarrow \varphi(x) \leq \varphi(y) .
\end{aligned}
$$

Definition 2.2 Let $X$ be a Polish space, $C \subseteq X$ and $\varphi: C \rightarrow \omega_{1}$ a rank on $C$. We say that $\varphi$ is $a \Pi_{1}^{1}$-rank if $C$ is $\Pi_{1}^{1}$ and there are relations $P, Q \subseteq X^{2}$, with one of them $\Sigma_{1}^{1}$ and the other $\Pi_{1}^{1}$, such that for all $y \in C$ we have that

$$
\{x \in C: \varphi(x) \leq \varphi(y)\}=\{x \in X:(x, y) \in P\}=\{x \in X:(x, y) \in Q\} .
$$

Loosely speaking, $\varphi$ is a $\Pi_{1}^{1}$-rank if $\{x: \varphi(x) \leq \varphi(y)\}$ is "uniformly Borel in $y$ ".
We will use the following reformulation of $\Pi_{1}^{1}$-rank in our proof.
Proposition 2.3 [5] Exercise 34.3] Let X, C, $\varphi$ as in Definition 2.2] Then, $\varphi$ is a $\Pi_{1}^{1}$-rank if and only if there are $\Sigma_{1}^{1}$ relations $P, Q \subseteq X^{2}$ such that for all $y \in C$ we have that

$$
\begin{aligned}
& \{x \in C: \varphi(x) \leq \varphi(y)\}=\{x \in X:(x, y) \in P\}, \text { and } \\
& \{x \in C: \varphi(x)<\varphi(y)\}=\{x \in X:(x, y) \in Q\} .
\end{aligned}
$$

The following are fundamental results on $\Pi_{1}^{1}$-ranks [5].
Theorem 2.4 Every $\Pi_{1}^{1}$ set admits a $\Pi_{1}^{1}$-rank.
Theorem 2.5 Let $C$ be a $\Pi_{1}^{1}$ set and $\varphi$ be a $\Pi_{1}^{1}$-rank on $C$. If $A \subseteq C$ is $\Sigma_{1}^{1}$, then $\varphi$ is bounded on $A$, i.e., there exists $\alpha<\omega_{1}$ such that $\varphi(x)<\alpha$ for all $x \in A$. In particular,

$$
C \text { is Borel } \Longleftrightarrow \varphi \text { is bounded on } C .
$$

We next recall the notion of derivatives and how it induces $\Pi_{1}^{1}$-ranks in a natural way [5, Section 34.D]. Let $\mathcal{K}(X)$ denote the space of all compact subsets of $X$ endowed with the Hausdorff metric.

Definition 2.6 A map $\mathbf{D}: \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is a derivative if the following holds:

$$
\mathbf{D}(A) \subseteq A \quad \& \quad A \subseteq B \Longrightarrow \mathbf{D}(A) \subseteq \mathbf{D}(B)
$$

Derivatives appear in a variety of contexts and they induce $\Pi_{1}^{1}$-ranks in a natural way. For a derivative D, let

$$
\begin{aligned}
& \mathbf{D}^{0}(A)=A \\
& \mathbf{D}^{\alpha+1}=\mathbf{D}\left(\mathbf{D}^{\alpha}(A)\right) \\
& \mathbf{D}^{\lambda}(A)=\cap_{\beta<\lambda} \mathbf{D}^{\beta}(A) \text { if } \lambda \text { is a limit ordinal. }
\end{aligned}
$$

Let $A \in \mathcal{K}(X)$. Then, there exists a countable ordinal $\alpha$ such that $\mathbf{D}^{\alpha}=\mathbf{D}^{\alpha+1}$. Such an ordinal exists since in a separable metric space a chain of strictly decreasing sequence of closed sets must be countable. We let $|A|_{\mathbf{D}}$ be the least such $\alpha$. Moreover, we let $\mathbf{D}^{\infty}(A)=\mathbf{D}^{|A|_{\mathbf{D}}}$, i.e., the stable part of $A$.

A useful classical Borel derivative is the Cantor-Bendixson derivative given by

$$
A \rightarrow A^{\prime}
$$

where $A^{\prime}$ is the set of limit-points of $A$ [5] Theorem 6.11]. The $\alpha^{\text {th }}$ Cantor-Bendixson derivative of $A$ is denoted by $A^{\alpha},|A|_{C B}$ denotes least ordinal $\alpha$ such that $A^{\alpha+1}=A^{\alpha}$, and $A^{\infty}=A^{|A|_{C B}}$, i.e., the stable part of $A$.

The following is an important theorem which relates derivatives to $\Pi_{1}^{1}$-ranks.
Theorem 2.7 [5] Theorem 34.10] Let $\mathbf{D}: \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ be a Borel derivative and

$$
C=\left\{A \in \mathcal{K}(X): D^{\infty}(A)=\emptyset\right\} .
$$

Then, $C$ is $\Pi_{1}^{1}$ and $\varphi: C \rightarrow \omega_{1}$ defined by $\varphi(A)=|A|_{\mathbf{D}}$ is a $\Pi_{1}^{1}$-rank on $C$.
A dual notion of derivatives is the concept of expansion.
Definition 2.8 A map $\mathbf{E}: \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is an expansion means that

$$
A \subseteq \mathbf{E}(A) \quad \& \quad A \subseteq B \Longrightarrow \mathbf{E}(A) \subseteq \mathbf{E}(B)
$$

For an expansion $E$, as earlier, we let

$$
\begin{aligned}
& \mathbf{E}^{0}(A)=A \\
& \mathbf{E}^{\alpha+1}=\mathbf{E}\left(\mathbf{E}^{\alpha}(A)\right) \\
& \mathbf{E}^{\lambda}(A)=\overline{\cup_{\beta<\lambda} \mathbf{E}^{\beta}(A)} \text { if } \lambda \text { is a limit ordinal. }
\end{aligned}
$$

We let $|A|_{\mathbf{E}}$ be the least such $\alpha$ such that $\mathbf{E}^{\alpha+1}=\mathbf{E}^{\alpha}(A)$. Moreover, we let $\mathbf{E}^{\infty}(A)=$ $\mathbf{E}^{|A| \mathbf{E}}$, i.e., the stable part of A.

For every expansion one can define a derivative (and vice-versa). Furthermore, one can formulate the above Theorem 2.7in terms of expansions. We will prove a variant of this dual.

Theorem 2.9 Let $X$ be a compact metric space and $\mathbf{E}$ be a Borel expansion on $\mathcal{K}(X)$ and let

$$
C=\left\{A \in K(X): \mathbf{E}^{\alpha}(A)=X \text { for some } \alpha\right\} .
$$

Then, $C$ is $\Pi_{1}^{1}$ and $\varphi: C \rightarrow \omega_{1}$ defined by $\varphi(A)=|A|_{\mathbf{E}}$ is a $\Pi_{1}^{1}$-rank on $C$.
Before proving the theorem, let us show a specific instance of a Borel expansion, the $\Gamma$ map.

Definition 2.10 Let $X$ be a compact metric space and $E \subseteq X^{2}$ a closed set. We define $E^{+}$as the smallest equivalence relation that contains $E$ and $\Gamma(E)=\overline{E^{+}}$. For an ordinal $\alpha, \Gamma^{\alpha}(E)$ is defined by

$$
\Gamma^{\alpha}(E)=\Gamma\left(\Gamma^{\alpha-1}(E)\right)
$$

if $\alpha$ is the successor ordinal and

$$
\Gamma^{\alpha}(E)=\overline{U_{\beta<\alpha} \Gamma^{\beta}(E)}
$$

if $\alpha$ is a limit ordinal.
Recall that in a topological space with countable basis, a chain of strictly increasing sequence of closed sets must be countable. From this we have the following.

Proposition 2.11 Let $X$ be a compact metrizable space and $E \subset X$. There exists a countable ordinal $\alpha$ such that $\Gamma^{\alpha}(E)=\Gamma^{\alpha+1}(E)$.
The smallest ordinal that satisfies the statement in the previous proposition is called the $\Gamma$-rank of $E$.

Before proving that $\Gamma: K(X \times X) \rightarrow K(X \times X)$ we will prove a lemma.
Lemma 2.12 Let $X$ be a compact metrizable space, $\varphi_{n}: K(X) \rightarrow K(X)$ be a Borel map, $n \in \mathbb{N}$, and $\varphi: K(X) \rightarrow K(X)$ defined by

$$
\varphi(A):=\overline{\bigcup_{n=1}^{\infty} \varphi_{n}(A)}
$$

Then, $\varphi$ is Borel.
Proof. Define $\psi_{n}: K(X) \rightarrow K(X)^{n}$ by

$$
\psi_{n}(A):=\left(\varphi_{1}(A), \ldots, \varphi_{n}(A)\right) .
$$

Then, $\psi_{n}$ is Borel. Moreover, as the union map is continuous, we have that, for each $n \in \mathbb{N}, A \rightarrow \cup_{i=1}^{n} \varphi_{i}(A)$ is Borel. Now $\varphi$ is simply the pointwise limit of these maps and hence itself Borel.
Proposition 2.13 Let $X$ be a compact metric space. Then,

$$
\Gamma: K(X \times X) \rightarrow K(X \times X)
$$

is a Borel map.
Proof. We first note that $A \rightarrow A^{+}$is a continuous map. Define $\sim_{n}: K(X \times X) \rightarrow$ $K(X \times X)$ by $\sim_{n}(A):=\left\{(x, y): \exists x=x_{0}, \ldots x_{n}=y\right.$ such that $\left(x_{i}, x_{i+1}\right) \in A \forall 0 \leq$ $i<n\}$. We note that $\sim_{n}$ is a continuous map. Hence, the map $A \rightarrow \sim_{n}\left(A^{+}\right)$is continuous. Now we have that

$$
\Gamma(A)=\overline{\cup_{n=1}^{\infty} \sim_{n}\left(A^{+}\right)}
$$

is Borel by Lemma 2.12
We now proceed to prove Theorem 2.9 We follow the general outline of [5] Theorem 34.10] adapted to this set up.
Proof of Theorem 2.9. It suffices to show that $|\cdot|_{\mathbf{E}}$ is a $\Pi_{1}^{1}$-rank on $C \backslash\{X\}$.
We first show that $C$ is $\Pi_{1}^{1}$. As $\mathbf{E}$ is Borel, $g r(\mathbf{E})$, the graph of $E$, is also Borel. Let $\Delta$ be the diagonal of $K(X) \backslash\{X\}$. Then, $g r(\mathbf{E}) \cap \Delta$ is Borel. As

$$
\mathcal{F}=\{A \in K(X): \mathbf{E}(A)=A \& A \neq X\}
$$

is the 1-1 projection of the Borel set $\operatorname{gr}(\mathbf{E}) \cap \Delta$, we have that $\mathcal{F}$ is Borel. Hence,

$$
\mathcal{G}=\{(A, B) \in K(X) \times K(X): B \in \mathcal{F} \& A \subseteq B\}
$$

is Borel as it is the intersection of two sets, one closed and the other Borel, namely,

$$
\{(A, B) \in K(X) \times K(X): A \subseteq B\} \quad K(X) \times \mathcal{F} .
$$

As the projection of Borel sets are $\Sigma_{1}^{1}$, and

$$
A \notin C \Longleftrightarrow(A, B) \in \mathcal{G} \text { for some } B,
$$

we have that $K(X) \backslash C$ is $\Sigma_{1}^{1}$, or, equivalently, $C$ is $\Pi_{1}^{1}$.

We proceed to construct required $\Sigma_{1}^{1}$ sets as in Proposition 2.3 In order to do this we need a a variant of a standard combinatorial $\Pi_{1}^{1}$ set as in the proof of [5] Theorem 34.10]. We recall the basic terminology.

For $x \in 2^{\mathbb{N} \times \mathbb{N}}$ we let

$$
D^{*}(x)=\{m \in \mathbb{N}: x(m, m)=1\}
$$

and we define

$$
m \leq_{x}^{*} n \Leftrightarrow\left[m, n \in D^{*}(x) \& x(m, n)=1\right] .
$$

We let $\mathbf{L O} \mathbf{O}^{*}$ be the set of all $x \in 2^{\mathbb{N} \times \mathbb{N}}$ such that $\leq_{x}^{*}$ is a linear order, $0 \in D^{*}(x)$ and $0 \leq_{x}^{*} m$ for all $m \in D^{*}(x)$ and let $\mathbf{W} \mathbf{F}^{*}$ be the set of all $x \in \mathbf{L} \mathbf{O}^{*}$ such that $\leq_{x}^{*}$ is a wellordering. It is known that $\mathbf{L O} \mathbf{O}^{*}$ is a closed subset of $2^{\mathbb{N} \times \mathbb{N}}$ and $\mathbf{W F}^{*}$ is a $\Pi_{1}^{1}$-complete subset of $2^{\mathbb{N} \times \mathbb{N}}$ and $x \mapsto|x|^{*}$ is a $\Pi_{1}^{1}$ rank on $\mathbf{W F}^{*}$ where $|x|^{*}$ is the order type of $x \in \mathbf{W F}^{*}$. Moreover, the range of $\mathbf{W F}^{*}$ is $\omega_{1} \backslash\{0\}$.

We next show that it suffices to construct $\Sigma_{1}^{1}$ subsets $\mathcal{R}$ and $\mathcal{S}$ of $\mathbf{L} \mathbf{O}^{*} \times K(X)$ which satisfy the following properties:

$$
\begin{array}{lc}
\forall A \in C \backslash\{X\}, & \left\{x \in \mathbf{L O}^{*}:(x, A) \in \mathcal{R}\right\}=\left\{x \in \mathbf{W F}^{*}:|x|^{*} \leq|A|_{\mathbf{E}}\right\} \\
\forall x \in \mathbf{W F}^{*}, & \{A \in K(X):(x, A) \in \mathcal{S}\}=\left\{A \in C:|x|^{*}=|A|_{\mathbf{E}}\right\} . \tag{S}
\end{array}
$$

Indeed, let

$$
\mathcal{P}=\left\{(A, B) \in K(X)^{2}: \exists x \in \mathbf{L} \mathbf{O}^{*} \text { such that }(x, B) \in \mathcal{R} \&(x, A) \in \mathcal{S}\right\} .
$$

Then, $\mathcal{P}$ is $\Sigma_{1}^{1}$ and for all $B \in C \backslash\{X\}$ we have that

$$
\left\{A \in C \backslash\{X\}:|A|_{\mathbf{E}} \leq|B|_{\mathbf{E}}\right\}=\{A \in \mathcal{K}(X):(A, B) \in \mathcal{P}\} .
$$

Indeed, the containment $\subseteq$ of the above equality is clear. To see the containment $\supseteq$, let $(A, B) \in \mathcal{P}$ and $x \in \mathbf{L O}^{*}$ be such that $(x, B) \in \mathcal{R}$ and $(x, A) \in \mathcal{S}$. Applying Condition (R) to our set $B \in C \backslash\{X\}$, we have that $x \in \mathbf{W F}^{*}$ and $|x|^{*} \leq|B|_{\mathbf{E}}$. As $x \in \mathbf{W F}^{*}$ and $(x, A) \in \mathcal{S}$, by Condition (S) we have that $A \in C$ and $|x|^{*}=|A|_{\mathbf{E}}$. As $|x|^{*}>0$, we have that $A \neq X$. Hence, we have that $A \in C \backslash\{X\}$ with $|A|_{\mathbf{E}} \leq|B|_{\mathbf{E}}$.

In order to obtain $Q$, we choose a Borel function $x \mapsto x^{\prime}$ from $\mathbf{L O}^{*}$ to $\mathbf{L O}^{*}$ such that $\left|x^{\prime}\right|^{*}=|x|^{*}+1$ and $x \in \mathbf{W F}^{*}$ iff $x^{\prime} \in \mathbf{W F}^{*}$. We let

$$
Q=\left\{(A, B) \in K(X)^{2}: \exists x \in \mathbf{L} \mathbf{O}^{*} \text { such that }\left(x^{\prime}, B\right) \in \mathcal{R} \&(x, A) \in \mathcal{S}\right\} .
$$

As $x \mapsto x^{\prime}$ is Borel, we have that $Q$ is $\Sigma_{1}^{1}$. Moreover for all $B \in C \backslash\{X\}$ we have that

$$
\left\{A \in C \backslash\{X\}:|A|_{\mathbf{E}}<|B|_{\mathbf{E}}\right\}=\{A \in \mathcal{K}(X):(A, B) \in Q\} .
$$

Indeed, the containment $\subseteq$ of the above equality is clear. To see the containment $\supseteq$, let $(A, B) \in Q$ and $x \in \mathbf{L} \mathbf{O}^{*}$ be such that $\left(x^{\prime}, B\right) \in \mathcal{R}$ and $(x, A) \in \mathcal{S}$. Applying Condition (R) to our set $B \in C \backslash\{X\}$, we have that $x^{\prime} \in \mathbf{W F}^{*}$ and $\left|x^{\prime}\right|^{*} \leq|B|_{\mathbf{E}}$. As $x^{\prime} \in \mathbf{W F}^{*}$, we have that $x \in \mathbf{W F}^{*}$. Now, as $x \in \mathbf{W F}^{*}$ and $(x, A) \in \mathcal{S}$, by Condition (S) we have that $A \in C$ and $|x|^{*}=|A|_{\mathbf{E}}$. As $|x|^{*}>0$, we have that $A \neq X$. Hence, we have that $A \in C \backslash\{X\}$ with $|A|_{\mathbf{E}}=|x|^{*}<\left|x^{\prime}\right|^{*} \leq|B|_{\mathbf{E}}$, i.e., $|A|_{\mathbf{E}}<|B|_{\mathbf{E}}$

By Proposition 2.3, we have that $|\cdot|_{\mathbf{E}}$ is a $\Pi_{1}^{1}$-rank on $C \backslash\{K\}$, provided that we construct $\Sigma_{1}^{1}$ sets $\mathcal{R}, \mathcal{S}$ with the required properties.

We define

$$
\begin{array}{r}
\mathcal{R}=\left\{(x, A) \in \mathbf{L} \mathbf{O}^{*} \times K(X): \exists h \in K(X)^{\mathbb{N}} \text { s.t. } h(0)=A,\right. \\
\left.\& m \in D^{*}(x) \backslash\{0\} \Rightarrow \overline{U_{n<_{x}^{*} m} \mathbf{E}(h(n))} \subseteq h(m)\right\} . \\
\mathcal{S}=\left\{(x, A) \in \mathbf{L} \mathbf{O}^{*} \times K(X): \exists h \in K(X)^{\mathbb{N}} \text { s.t. } h(0)=A,\right. \\
\& \forall m \in D^{*}(x), h(m) \neq X \\
\& m \in D^{*}(x) \backslash\{0\} \Rightarrow \overline{U_{n<_{x}^{*} m} \mathbf{E}(h(n))} \subseteq h(m) \\
\left.\& \frac{U_{m \in D^{*}(x)} \mathbf{E}(h(m))}{}=X\right\}
\end{array}
$$

By Proposition 2.12we have that $\overline{\mathrm{U}_{n}}: K(X)^{\mathbb{N}} \rightarrow K(X)$ defined by $\overline{\mathrm{U}_{n}}\left(A_{n}\right)=\overline{\mathrm{U}_{n}\left(A_{n}\right)}$ is Borel. This fact together with the standard quantifier counting technique imply that $\mathcal{R}$ and $\mathcal{S}$ are $\Sigma_{1}^{1}$.

Let us now observe that $\mathcal{R}$ and $\mathcal{S}$ satisfies Conditions (R) and (S), respectively.
To see that $\mathcal{R}$ satisfies Condition (R), let $A \in C \backslash\{X\}$. That containment $\supseteq$ holds in Condition (R) follows directly from the definition of $\mathcal{R}$. For the containment $\subseteq$, we note that if $m>0$ is in $D^{*}(x)$, then for some $\alpha<|A|_{\mathbf{E}}$ we have that $\overline{\cup_{n<{ }_{x}^{*} m} \mathbf{E}(h(n))} \nsupseteq \mathbf{E}^{\alpha+1}(A)$. This is so, for otherwise, for all $\alpha<|A|_{\mathbf{E}}$ we would have that $\mathbf{E}^{\alpha+1}(A) \subseteq \overline{U_{n<{ }_{x}^{*} m} \mathbf{E}(h(n))} \subseteq h(m) \neq X$, implying that $\mathbf{E}^{\infty}(A)=\overline{U_{\alpha<\left|| |_{\mathbf{E}}\right.} \mathbf{E}^{\alpha+1}(A)} \neq$ $X$ and that $A \notin C$. Now we define $f$ from $<_{x}^{*}$ into $|A|_{\mathbf{E}}$. Define $f(0)=0$ and for $m \in D^{*}(x) \backslash\{0\}$, let

$$
f(m)=\text { the least } \alpha<|A|_{\mathbf{E}} \text { such that } \overline{\mathrm{U}_{n<_{x}^{*} m} \mathbf{E}(h(n))} \nsupseteq \mathbf{E}^{\alpha+1}(A) .
$$

It suffices to show that $f$ is order preserving, i.e., $m<{ }_{x}^{*} p$ implies $f(m)<f(p)$. Indeed, this would imply that $x \in \mathbf{W F}^{*}$ and $|x|^{*} \leq|A|_{\mathbf{E}}$. Finally, to see that $f$ is order preserving, let $m<{ }_{x}^{*} p, m \neq 0$. Then,

$$
\mathbf{E}^{f(m)}(A)=\overline{U_{\alpha<f(m)} \mathbf{E}^{\alpha+1}(A)} \subseteq \overline{U_{n<{ }_{x}^{*} m} \mathbf{E}(h(n))} .
$$

Hence, $\mathbf{E}^{f(m)}(A) \subseteq h(m)$, implying that $\mathbf{E}^{f(m)+1}(A) \subseteq \mathbf{E}(h(m))$. As $m<_{x}^{*} p$, we have that $\mathbf{E}^{f(m)+1}(A) \subseteq \mathbf{E}(h(m)) \subseteq \overline{U_{q<{ }_{x}^{*} p} \mathbf{E}(h(q))}$, implying that $f(q) \geq f(m)+1$ and exhibiting that $f$ is order preserving.

To see that $\mathcal{S}$ satisfies Condition (S), let $x \in \mathbf{W F}^{*}$. That containment $\supseteq$ holds in Condition (S) follows directly from the definition of $\mathcal{S}$. That containment $\subseteq$ holds follows from our order preserving function $f$ from $<_{x}^{*}$ into $|A|_{\mathbf{E}}$ and the fact that $\overline{\mathrm{U}_{m \in D^{*}(x)} \mathbf{E}(h(m))}=X$.

## 3 Application: entropy rank

A set $I \subseteq \mathbb{N}$ has positive density if $\liminf _{n} \frac{|I n[1, n]|}{n>0}>0$.
We say $(X, T)$ is a topological dynamical system (TDS) if $X$ is a compact metrizable space and $T: X \rightarrow X$ is a continuous function. We say $\left(X_{2}, T_{2}\right)$ is a factor of $\left(X_{1}, T_{1}\right)$ if there exists a surjective continuous function $\phi: X_{2} \rightarrow X_{1}$ (called a factor map) such that $\varphi \circ T_{1}=T_{2} \circ \varphi$.

Given a TDS $(X, T)$ and $\{U, V\} \subset X$, we say $I \subset \mathbb{N}$ is an independence set for $\{U, V\}$ if for all finite $J \subseteq I$, and for all $\left(Y_{j}\right) \in \prod_{j \in J} \mathcal{A}$, we have that

$$
\cap_{j \in J} T^{-j}\left(Y_{j}\right) \neq \emptyset .
$$

Let $(X, T)$ be a TDS and $\mathcal{U}, \mathcal{V}$ open covers of $X$. We denote the smallest cardinality of a subcover of $\mathcal{U}$ with $N(\mathcal{U})$, and

$$
\mathcal{U} \vee \mathcal{V}=\{U \cap V: U \in \mathcal{U} \text { and } V \in \mathcal{V}\} .
$$

We define the entropy of $(X, T)$ with respect to $\mathcal{U}$ as

$$
h_{\mathrm{top}}(X, T, \mathcal{U})=\lim _{n \rightarrow \infty} \frac{1}{n} \log N\left(\mathrm{~V}_{m=1}^{n} T^{-m}(\mathcal{U})\right)
$$

The (topological) entropy of $(X, T)$ is defined as

$$
h_{\mathrm{top}}(X, T)=\sup _{\mathcal{U}} h_{\mathrm{top}}(T, \mathcal{U}) .
$$

Definition 3.1 A TDS has complete positive entropy (CPE) if every non-trivial factor has positive entropy.
Let $X$ be a compact metrizable space and $C(X, X)$ be the set of all continuous functions from $X$ into $X$ endowed with the uniform topology. We will now define a subspace of $C(X, X)$.
Definition 3.2 Given a compact metrizable space $X$ we define

$$
C P E(X)=\{T \in C(X, X):(X, T) \text { has } C P E\} .
$$

Local entropy theory was initiated in [2]. For more information see the survey [4] or the book [7].

Definition 3.3 Let $(X, T)$ be a TDS. We say that $\left[x_{1}, x_{2}\right] \in X \times X$ is an independence entropy pair (IE-pair) of $(X, T)$ if for every pair of open sets $A_{1}, A_{2}$, with $x_{1} \in A_{1}$ and $x_{2} \in A_{2}$, there exists an independence set for $\left\{A_{1}, A_{2}\right\}$ with positive density. The set of IE-pairs of $(X, T)$ will be denoted by $E(X, T)$.

We are particularly interested in studying the $\Gamma$-rank in the case when $E=$ $E(X, T)$ is the set of independence entropy pairs.

Definition 3.4 Let $(X, T)$ be a TDS. The $\Gamma$-rank of the set of entropy pairs is called the entropy rank of $(X, T)$.

An equivalent statement of the following result was proved in [2] (also see [7] Theorem 12.30]).
Theorem 3.5 A TDS has CPE if and only if $\Gamma^{\alpha}(E(X, T))=X^{2}$ where $\alpha$ is the entropy rank of $(X, T)$.

The following proposition was proved in [3]. We give a proof for completeness.
Proposition 3.6 Consider the mapping $E: C(X, X) \rightarrow K(X \times X)$ given by $E(T)=$ $E(X, T)$. Then, $E$ is a Borel map.

Proof. Let $U, V$ be open in $X$. We first observe that

$$
\{T \in C(X, X): E(T) \cap(U \times V) \neq \emptyset\}
$$

is Borel. Indeed, using an equivalent definition of independence given in 6 Lemma 3.2] we have that $\dagger$ is satisfied by $T$ if and only if there is a rational number $r>0$ such that for all $l \in \mathbb{N}$ there is an interval $I \subseteq \mathbb{N}$ with $|I| \geq l$ and a finite set $F \subseteq I$ with $|F| \geq r|I|$ such that $F$ is an independent set for $(U, V)$. It is easy to verify that for fixed $U, V, r, l, I, F$ set

$$
\{T \in C(X, X): F \text { is an independent set for }(U, V) \text { for } T\}
$$

is open. Now the set in $\dagger$ is the result of a sequence of countable union and countable intersections of sets of type $\ddagger$. Hence, $\dagger$ is Borel. Since $X$ has a countable basis, by taking unions, we have that $\dagger$ is Borel when $U \times V$ is replaced by any open set $W \subseteq X \times X$. Every closed set in $X \times X$ is the monotonic intersection of a sequence of open sets in $X \times X$. This and the fact that $E(T)$ is closed imply that $\dagger$ is Borel when $U \times V$ is replaced by a closed set $C \subseteq X \times X$. Reformulating the last statement, we have that for all open $W \in X \times X$, the set

$$
\{T \in C(X, X): E(T) \subseteq W\}
$$

is Borel. Putting $\dagger$ and $\diamond$ together, we have that

$$
\left\{T \in C(X, X): E(T) \subseteq \cup_{i=1}^{n}\left(U_{i} \times V_{i}\right) \& E(T) \cap\left(U_{i} \times V_{i}\right) \neq \emptyset, 1 \leq i \leq n\right\}
$$

is Borel whenever $U_{1}, \ldots, U_{n}, V_{1}, \ldots V_{n}$ are open in $X$, completing proof.

Theorem 3.7 Let $X$ be a compact metrizable space. Then, $\varphi: C(X, X) \rightarrow \omega_{1}$ defined by $\varphi(T)=|E(X, T)|_{\Gamma}$ is a $\Pi_{1}^{1}-$ rank on $C(X, X)$.

Proof. By Proposition 2.13 we have that $\Gamma$ is Borel. Setting $\mathbf{E}=\Gamma$ in Theorem 2.9 we have that the map that takes $A \in K(X \times X)$ to $|A|_{\Gamma}$ is a $\Pi_{1}^{1}$-rank on the set $C^{\prime}=\left\{A \in K(X \times X): \Gamma^{|A|_{\Gamma}}(A)=X\right\}$. By the definition of $\Pi_{1}^{1}$ rank, there are sets $P, Q \in K(X \times X)^{2}$, as in the Definition [2.2, which verify that $|\cdot|_{\Gamma}$ is a $\Pi_{1}^{1}$ rank on $C^{\prime}$. By Lemma 3.6, we have that $E$ is Borel and hence $(E \times E)^{-1}(P)$ and $(E \times E)^{-1}(Q)$, are $\Sigma_{1}^{1}$, and $\Pi_{1}^{1}$ in $C(X, X) \times C(X, X)$, respectively. This and the fact that $E^{-1}\left(C^{\prime}\right)=C$, we have that $(E \times E)^{-1}(P)$ and $(E \times E)^{-1}(Q)$ exhibit that $\varphi$ is a $\Pi_{1}^{1}$-rank on $C$.

Examples of TDSs with CPE and arbitrarily high entropy rank have been constructed in [1] 8, 3].

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