

AN IMPROVED UPPER BOUND ON THE NUMBER OF BILLIARD BALL COLLISIONS

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ABSTRACT. We give a new upper bound K_+ on the number of totally elastic collisions of n hard spheres with equal radii and equal masses in \mathbb{R}^d . Our bound satisfies $\log K_+ \leq c(d)n \log n$.

1. INTRODUCTION

Consider a family of n billiard balls in \mathbb{R}^d reflecting from each other in a totally elastic way. We assume that their masses and radii are identical. Note that the “billiard table” has no walls—it is the whole space \mathbb{R}^d . We will prove the following upper bound for the number of collisions.

Theorem 1.1. *The number of collisions is bounded above by*

$$(1.1) \quad 1600 \left(1000 \cdot 32^{5^d} \right)^n n^{((3/2)5^d + 9/2)n + 3/2}.$$

We will review the history of the problem in Section 2. Here we will discuss the question of optimality of our bound. Let K'_+ denote the best previously known upper bound for the number of collisions, stated below in (2.1). Let K_- denote the best known lower bound in dimensions $d \geq 3$, stated below in (2.3). Let K_+ be the new upper bound given in (1.1). For $d \geq 3$, some constants c_1, c_2 and c_3 depending on d , and large n ,

$$c_1 n \leq \log K_- < \log K_+ \leq c_2 n \log n < c_3 n^2 \log n \leq \log K'_+.$$

This shows that while there still remains a gap between the best lower and upper bounds K_- and K_+ , the gap is much smaller than between the previously known best upper bound K'_+ and the best known lower bound K_- .

The proof of Theorem 1.1 is based in an essential way on two results—one from [BFK98c] and another one from [BD20]. The latter one shows that the family of all balls “quickly” splits into two non-interacting families. An estimate from [BFK98c] can be used to give an upper bound for the number of collisions on the initial (“short”) time interval. The proof of Theorem 1.1 is an inductive construction of a branching collection of ball subfamilies. Subfamilies that are leaves in the branching structure have “lifetimes” short enough so that the estimate mentioned above can be applied. Finally, we estimate the number of subfamilies in the branching structure.

There are two sources of the factor of the form n^{cn} in (1.1). One of them is a bound adopted from [BFK98c]. The other one is a branching construction in the present

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article. Hence, there is little hope for a significant improvement of the upper bound by fine-tuning the argument given in this paper.

We will review the history of the problem in Section 2. The notation and assumptions will be presented in Section 3. The proof of Theorem 1.1 will be given in Section 4.

2. HARD BALL COLLISIONS—HISTORICAL REVIEW

The question of whether a finite system of hard balls can have an infinite number of elastic collisions was posed by Ya. Sinai. It was answered in negative in [Vas79]. For alternative proofs see [Il89, Il90, CI04, BD20]. It was proved in [BFK98c] that a system of n balls in the Euclidean space undergoing elastic collisions can experience at most

$$(2.1) \quad \left(32 \sqrt{\frac{m_{\max}}{m_{\min}}} \frac{r_{\max}}{r_{\min}} n^{3/2} \right)^{n^2}$$

collisions. Here m_{\max} and m_{\min} denote the maximum and the minimum masses of the balls. Likewise, r_{\max} and r_{\min} denote the maximum and the minimum radii of the balls. The following alternative upper bound for the maximum number of collisions appeared in [BFK98a]

$$(2.2) \quad \left(400 \frac{m_{\max}}{m_{\min}} n^2 \right)^{2n^4}.$$

The papers [BFK98c, BFK98b, BFK00, BFK02, BFK98a] were the first to present universal bounds (2.1)-(2.2) on the number of collisions of n hard balls in any dimension. No improved universal upper bounds were found since then, as far as we know.

It has been proved in [BD19] by example that the number of elastic collisions of n balls in d -dimensional space is greater than $n^3/27$ for $n \geq 3$ and $d \geq 2$, for some initial conditions. The previously known lower bound was of order n^2 (that bound was for balls in dimension 1 and was totally elementary). The lower bound estimate was improved in [BI18] to

$$(2.3) \quad 2^{\lfloor n/2 \rfloor}$$

in dimensions $d \geq 3$.

In a somewhat different direction, it has been shown in [Ser21] that no more than $O(n^2)$ collisions change the velocities of balls in a significant way.

3. ASSUMPTIONS AND NOTATION

We will consider $n \geq 3$ hard balls in \mathbb{R}^d , for $d \geq 2$, colliding elastically, on the time interval $(-\infty, \infty)$. We will assume that the balls have equal masses and their radii are 1.

The center and velocity of the k -th ball will be denoted $x^k(t)$ and $v^k(t)$, for $k = 1, 2, \dots, n$. We will say that the j -th and k -th balls collide at time t if $|x^j(t) - x^k(t)| = 2$ and their velocities change at this time. The velocities are constant between collision times. We will write $\mathbf{x}(t) = (x^1(t), \dots, x^n(t)) \in \mathbb{R}^{dn}$ and $\mathbf{v}(t) = (v^1(t), \dots, v^n(t)) \in \mathbb{R}^{dn}$.

Note that $\mathbf{v}(t)$ is well defined only when t is not a collision time, but both $\mathbf{v}(t-)$ and $\mathbf{v}(t+)$ are well defined for all times.

Recall that all balls have the same mass. This implies that the velocities change at the moment of collision as follows. Suppose that the j -th and k -th balls collide at time t . This implies that the velocities $v^j(t-)$ and $v^k(t-)$ (i.e., the velocities just before the collision) satisfy

$$(3.1) \quad (v^j(t-) - v^k(t-)) \cdot (x^j(t) - x^k(t)) < 0.$$

Let $x^{jk}(t) = (x^j(t) - x^k(t))/|x^j(t) - x^k(t)|$. Then the velocities just after the collision are given by

$$(3.2) \quad v^j(t+) = v^j(t-) + (v^k(t-) \cdot x^{jk}(t))x^{jk}(t) - (v^j(t-) \cdot x^{jk}(t))x^{jk}(t),$$

$$(3.3) \quad v^k(t+) = v^k(t-) + (v^j(t-) \cdot x^{jk}(t))x^{jk}(t) - (v^k(t-) \cdot x^{jk}(t))x^{jk}(t).$$

In other words, the balls exchange the components of their velocities that are parallel to the line through their centers at the moment of impact. The orthogonal components of velocities remain unchanged.

Consider the following assumptions.

(A1) The balls have equal masses and all radii are equal to 1.

(A2) We will assume that there are no simultaneous collisions. It is known that the set of vectors in the phase space of positions and velocities that lead to simultaneous collisions has measure zero (see [Ale76]). It has been proved in [CI04, Thm. 4] that there are no accumulation points for collision times.

(A3) We will assume that the momentum of the system is zero, i.e., $\sum_{j=1}^n v^j(t) = 0$ for all t . We can make this assumption because the number of collisions is the same in all inertial frames of reference. Since the total momentum is zero, the center of mass of all balls is constant, so it can be assumed to be at the origin. This, together with the fact that all masses are equal, implies that $\sum_{j=1}^n x^j(t) = 0$.

(A4) We will assume without loss of generality that the total “energy” is equal to 1, i.e., $|\mathbf{v}(t+)|^2 = 1$ for all t . If the initial energy is not zero then we can multiply all velocity vectors by the same scalar constant so that the energy is equal to 1. If all velocities are changed by the same multiplicative constant then the balls will follow the same trajectories at a different rate and hence there will be the same total number of collisions.

Remark 3.1. (i) The problem of the number of collisions is invariant under time shifts.

(ii) We recall [BD20, Rem. 4.3]. Let $\alpha(t) = \angle(\mathbf{x}(t), \mathbf{v}(t+))$. There is a unique $t_0 \in \mathbb{R}$ such that $\alpha(t) > \pi/2$ for $t < t_0$, and $\alpha(t) < \pi/2$ for $t > t_0$. Right continuity yields $\alpha(t_0) \leq \pi/2$. We have

$$(3.4) \quad |\mathbf{x}(t)| \geq |\mathbf{x}(t_0)|.$$

for all $t \in \mathbb{R}$.

Theorem 3.2. ([BD20, Thm. 5.1]) *Recall assumptions (A1)-(A4) and time t_0 defined in Remark 3.1. The family of n balls can be partitioned into two non-empty subfamilies such that no ball from the first family collides with a ball in the second family in the*

time interval $[t_0 + 100n^3|\mathbf{x}(t_0)|, \infty)$. By symmetry and time reversal, a similar claim applies to $(-\infty, t_0 - 100n^3|\mathbf{x}(t_0)|]$.

4. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. Step 1. Consider a family of n balls satisfying assumptions (A1)-(A4). Recall $t_0 \in \mathbb{R}$ defined in Remark 3.1.

Let $[s_1, s_2]$ be the smallest interval containing t_0 satisfying the following two conditions.

(i) The balls can be partitioned into two non-empty subfamilies such that no ball from the first family collides with a ball in the second family in the time interval $(-\infty, s_1)$.

(ii) The balls can be partitioned into two non-empty subfamilies such that no ball from the first family collides with a ball in the second family in the time interval (s_2, ∞) .

The division into subfamilies in (i) and (ii) is not unique. The subfamilies in (i) need not be the same as those in (ii).

By (3.4) and Theorem 3.2, for any $t \in \mathbb{R}$,

$$\begin{aligned} s_2 - t_0 &\leq 100n^3|\mathbf{x}(t_0)| \leq 100n^3|\mathbf{x}(t)|, \\ t_0 - s_1 &\leq 100n^3|\mathbf{x}(t_0)| \leq 100n^3|\mathbf{x}(t)|, \end{aligned}$$

so, for any $t \in \mathbb{R}$,

$$(4.1) \quad s_2 - s_1 \leq 200n^3|\mathbf{x}(t)|.$$

Step 2. Fix a time interval $[u, u + 1]$. Since $|\mathbf{v}(t+)| = 1$ for all t , we have $|v^i(t)| \leq 1$ for all i and t . Hence, a ball can travel at most distance 1 in the time interval $[u, u + 1]$. Suppose the center of a ball B_1 is at y at time u and B_1 collides with another ball B_2 at a time $u_1 \in [u, u + 1]$. The distance from the center of B_1 to y is at most 1 at time u_1 so the distance from the center of B_2 to y is at most 3 at the same time. It follows that distance from the center of B_2 to y is at most 4 at time u . This implies that B_2 is a subset of the ball \mathcal{B} (not a billiard ball) centered at y with radius 5 at time u . The volume of \mathcal{B} is 5^d times the volume of a ball with radius 1. Hence, B_1 might have collided with at most 5^d balls during the time interval $[u, u + 1]$. It follows that the number of pairs of balls that could have collided in $[u, u + 1]$ is bounded by $5^d n/2$ (the factor $1/2$ is present so as not to count pairs twice). Note that $5^d n/2$ is the number of pairs of balls that could have collided in $[u, u + 1]$, not the number of collisions, which could be much higher. We need an estimate of the number of pairs of balls because we want to use the results of [BFK98c]. In that paper, counting of collisions is based on the number of “walls B_i ” (in the notation of that paper), which is equal to the number of pairs of balls that could collide.

We will now apply an upper bound on the number of collisions given in [BFK98c]. In the notation of [BFK98c], $r_{\max} = 1$ and $m_{\max} = 1$ because (i) we have assumed that the balls have radii 1, and (ii) we have assumed that the masses of all balls are equal, so we can make them all equal to 1 without losing generality. In view of these remarks, the bound given on the left hand side of the last displayed formula on page 707 of [BFK98c] is $(8(2n\sqrt{n}))^{n(n-1)-2}$. There seems to be a mistake here, in view of Remark 5.3 in [BFK98c]. The correct version should be $(8(2n\sqrt{n} + 2))^{n(n-1)-2}$. Here

$n(n-1)$ is twice the number of pairs of balls. Since only $5^d n/2$ pairs of balls can collide in $[u, u+1]$, we replace $n(n-1)$ with $5^d n$ to see that the number of collisions of n balls during an interval $[u, u+1]$ is bounded by

$$(4.2) \quad (8(2n\sqrt{n} + 2))^{5^d n - 2} < (32n^{3/2})^{5^d n - 2}.$$

This bound agrees with Corollary 1.1 in [BFK98a] but our bound is more explicit.

We now offer a more formal justification of the bound in (4.2). Let \mathcal{A} be the family of all pairs of balls that collide in $[u, u+1]$. Consider a billiards evolution in which (i) pairs of balls in \mathcal{A} move along the same trajectories as in the original evolution in $[u, u+1]$, (ii) the trajectories of pairs of balls in \mathcal{A} are extended outside $[u, u+1]$ in the usual way, i.e., with elastic collisions, and (iii) pairs of balls that do not belong to \mathcal{A} do not collide, i.e., they pass through each other like ghosts. The results of [BFK98c] apply to this model with the number of “walls B_i ” (in the notation of that paper) equal to the number of pairs in \mathcal{A} .

Step 3. By (4.1) and (4.2), the number of collisions on the interval $[s_1, s_2]$ is bounded by

$$(4.3) \quad 200n^3 |\mathbf{x}(t)| (32n^{3/2})^{5^d n - 2},$$

for any $t \in \mathbb{R}$. This bound is based on Theorem 3.2 proved under the assumptions (A1)-(A4). We will argue that (4.3) holds even if (A3) and (A4) are not satisfied.

First, we will argue that (4.3) holds even if (A4) does not hold. Suppose that \mathbf{x} and \mathbf{v} do not satisfy (A4). Let $c_1 = 1/|\mathbf{v}(0+)|$ and $\tilde{\mathbf{v}}(t+) = c_1 \mathbf{v}(t+)$ for all t . Then $|\tilde{\mathbf{v}}(t+)| = 1$ for all t . We keep the same position at time t_0 , i.e., $\tilde{\mathbf{x}}(t_0) = \mathbf{x}(t_0)$. The balls will follow the same trajectories but at a different speed. Hence, $\inf_{t \in \mathbb{R}} |\mathbf{x}(t)| = \inf_{t \in \mathbb{R}} |\tilde{\mathbf{x}}(t)| = |\tilde{\mathbf{x}}(t_0)|$. Let $[\tilde{s}_1, \tilde{s}_2]$ be defined as in Step 1 relative to $\tilde{\mathbf{x}}$. Then the number of collisions in $[s_1, s_2]$ in the system characterized by \mathbf{x} and \mathbf{v} is the same as the number of collisions in $[\tilde{s}_1, \tilde{s}_2]$ in the system characterized by $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{v}}$. Since (4.3) holds for the latter evolution, it also holds for the former.

Next we will argue that we do not need to assume (A3) and (A4) for (4.3) to hold. Suppose that \mathbf{x} and \mathbf{v} do not necessarily satisfy (A3) and (A4), and recall $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{v}}$ defined in the previous paragraph. For some $\mathbf{z} \in \mathbb{R}^d$, we have $\sum_{j=1}^n \tilde{x}^j(t) = \sum_{j=1}^n \tilde{x}^j(0) + t\mathbf{z}$ for all t . Let $\mathbf{z}_1(t) = \frac{1}{n} \sum_{j=1}^n \tilde{x}^j(0) + t\mathbf{z}/n$, $\hat{x}^j(t) = \tilde{x}^j(t) - \mathbf{z}_1(t)$ and $\hat{v}^j(t) = \tilde{v}^j(t) - \mathbf{z}/n$ for all $j = 1, \dots, n$ and $t \in \mathbb{R}$. The pair $\hat{\mathbf{x}}$ and $\hat{\mathbf{v}}$ is a representation of the dynamical system defined by $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{v}}$ in a different inertial frame of reference. We have $\sum_{j=1}^n \hat{x}^j(t) = 0$ for all t , so the functions $\hat{\mathbf{x}}$ and $\hat{\mathbf{v}}$ characterize an evolution satisfying (A3). This and the previous paragraph show that (4.3) holds with $\hat{\mathbf{x}}$ in place of \mathbf{x} . A standard calculation shows that the function $a \rightarrow \sum_{j=1}^n |\hat{x}^j(t) - a\mathbf{z}_1(t)|^2$ achieves the maximum at $a = 0$ because $\sum_{j=1}^n \hat{x}^j(t) = 0$. This implies that $|\tilde{\mathbf{x}}(t)| \geq |\hat{\mathbf{x}}(t)|$ for every t , and, therefore, $\inf_{t \in \mathbb{R}} |\mathbf{x}(t)| = \inf_{t \in \mathbb{R}} |\tilde{\mathbf{x}}(t)| \geq \inf_{t \in \mathbb{R}} |\hat{\mathbf{x}}(t)|$. This completes the proof that (4.3) holds even if (A3) and (A4) are not satisfied.

Step 4. Consider any subfamily \mathcal{F} of the balls. Suppose that $T_1(\mathcal{F}) < T_2(\mathcal{F})$ are given and balls in \mathcal{F} do not collide with any balls outside of \mathcal{F} on the time interval $(T_1(\mathcal{F}), T_2(\mathcal{F}))$. We will define $r(\mathcal{F}), t_*(\mathcal{F}), U_1(\mathcal{F})$ and $U_2(\mathcal{F})$. These numbers depend

not only on \mathcal{F} , as indicated by the notation, but also on $T_1(\mathcal{F})$ and $T_2(\mathcal{F})$. Hopefully, our notation, chosen for typographical convenience, will not cause confusion.

Let $n_{\mathcal{F}}$ be the number of balls in \mathcal{F} and suppose that the indices of balls in \mathcal{F} are $i_1, \dots, i_{n_{\mathcal{F}}}$. Choose an inertial coordinate system $CS_{\mathcal{F}}$ such that if $\tilde{x}^{i_k}(t)$ is the position of the i_k -th ball $CS_{\mathcal{F}}$ then $\sum_{k=1}^{n_{\mathcal{F}}} \tilde{x}^{i_k}(t) = 0$ for all $t \in [T_1(\mathcal{F}), T_2(\mathcal{F})]$. Let

$$(4.4) \quad \begin{aligned} \mathbf{x}_{\mathcal{F}}(t) &= (\tilde{x}^{i_1}(t), \dots, \tilde{x}^{i_{n_{\mathcal{F}}}}(t)), \\ r(\mathcal{F}, t) &= \max_{1 \leq j, k \leq n_{\mathcal{F}}} |\tilde{x}^{i_j}(t) - \tilde{x}^{i_k}(t)|, \end{aligned}$$

$$(4.5) \quad \begin{aligned} r(\mathcal{F}) &= \inf_{t \in [T_1(\mathcal{F}), T_2(\mathcal{F})]} r(\mathcal{F}, t), \\ t_*(\mathcal{F}) &= \arg \min_{t \in [T_1(\mathcal{F}), T_2(\mathcal{F})]} r(\mathcal{F}, t), \end{aligned}$$

$$|\mathbf{v}_{\mathcal{F}}| = \left(\sum_{k=1}^{n_{\mathcal{F}}} (v^{i_k}(t))^2 \right)^{1/2}, \quad t \in [T_1(\mathcal{F}), T_2(\mathcal{F})].$$

Since $\sum_{k=1}^{n_{\mathcal{F}}} \tilde{x}^{i_k}(t) = 0$, the norm of the vector $\mathbf{x}_{\mathcal{F}}(t_*(\mathcal{F}))$ is smaller in $CS_{\mathcal{F}}$ than in any other coordinate system, for example, in a coordinate system with the origin at $\tilde{x}^{i_1}(t_*(\mathcal{F}))$. Hence,

$$|\mathbf{x}_{\mathcal{F}}(t_*(\mathcal{F}))|^2 \leq \sum_{k=2}^{n_{\mathcal{F}}} |\tilde{x}^{i_k}(t_*(\mathcal{F})) - \tilde{x}^{i_1}(t_*(\mathcal{F}))|^2 \leq (n_{\mathcal{F}} - 1)r(\mathcal{F})^2,$$

and, therefore,

$$(4.6) \quad |\mathbf{x}_{\mathcal{F}}(t_*(\mathcal{F}))| \leq n_{\mathcal{F}}^{1/2} r(\mathcal{F}).$$

Consider the following modified evolution of balls in \mathcal{F} . Let the evolution of balls in \mathcal{F} remain as in the original system in the time interval $[T_1(\mathcal{F}), T_2(\mathcal{F})]$. Let the evolution continue before $T_1(\mathcal{F})$ and after $T_2(\mathcal{F})$, with balls in \mathcal{F} colliding according to the usual laws of elastic collisions, but with no collisions between balls in \mathcal{F} with balls outside \mathcal{F} . According to Remark 3.1 and (3.4) there exists a unique $T_0(\mathcal{F}) \in \mathbb{R}$ such that for all $t \in \mathbb{R}$,

$$(4.7) \quad |\mathbf{x}_{\mathcal{F}}(t)| \geq |\mathbf{x}_{\mathcal{F}}(T_0(\mathcal{F}))|.$$

Let $[S_1(\mathcal{F}), S_2(\mathcal{F})]$ be the smallest interval containing $T_0(\mathcal{F})$ satisfying the following two conditions.

(i) The family \mathcal{F} can be partitioned into two non-empty subfamilies \mathcal{F}_1 and \mathcal{F}_2 such that no ball in \mathcal{F}_1 collides with a ball in \mathcal{F}_2 in the time interval $(-\infty, S_1(\mathcal{F}))$.

(ii) The family \mathcal{F} can be partitioned into two non-empty subfamilies \mathcal{F}_3 and \mathcal{F}_4 such that no ball in \mathcal{F}_3 collides with a ball in \mathcal{F}_4 in the time interval $(S_2(\mathcal{F}), \infty)$.

The division into subfamilies in (i) and (ii) is not unique. By Theorem 3.2, $S_1(\mathcal{F}) > -\infty$ and $S_2(\mathcal{F}) < \infty$. Let

$$(4.8) \quad U_1(\mathcal{F}) = \max(S_1(\mathcal{F}), T_1(\mathcal{F})), \quad U_2(\mathcal{F}) = \min(S_2(\mathcal{F}), T_2(\mathcal{F})).$$

It is possible that $U_1(\mathcal{F}) = U_2(\mathcal{F})$.

We have $[U_1(\mathcal{F}), U_2(\mathcal{F})] \subset [S_1(\mathcal{F}), S_2(\mathcal{F})]$ so by (4.1) and rescaling by the speed $|\mathbf{v}_{\mathcal{F}}|$,

$$(4.9) \quad U_2(\mathcal{F}) - U_1(\mathcal{F}) \leq 200n_{\mathcal{F}}^3 |\mathbf{x}_{\mathcal{F}}(t_*(\mathcal{F}))| / |\mathbf{v}_{\mathcal{F}}|.$$

By (4.3), the number of collisions between balls in \mathcal{F} in the time interval $[U_1(\mathcal{F}), U_2(\mathcal{F})]$ is bounded by

$$(4.10) \quad 200n_{\mathcal{F}}^3 |\mathbf{x}_{\mathcal{F}}(t_*(\mathcal{F}))| \left(32n_{\mathcal{F}}^{3/2} \right)^{5d_{n_{\mathcal{F}}}-2}.$$

Step 5. We will construct a branching family \mathcal{W} with elements of the form

$$\Lambda(\mathcal{F}) := (\mathcal{F}, r(\mathcal{F}), T_1(\mathcal{F}), T_2(\mathcal{F}), U_1(\mathcal{F}), U_2(\mathcal{F})),$$

where \mathcal{F} is a subfamily of the balls.

Let \mathcal{G}_1 be the set of all n balls. We initiate the construction of \mathcal{W} by declaring $(\mathcal{G}_1, r(\mathcal{G}_1), -\infty, \infty, U_1(\mathcal{G}_1), U_2(\mathcal{G}_1))$ to be the only “individual” in the first generation of the branching structure \mathcal{W} . In other words, $T_1(\mathcal{G}_1) = -\infty$, $T_2(\mathcal{G}_1) = \infty$, and $U_1(\mathcal{G}_1)$ and $U_2(\mathcal{G}_1)$ are defined as in (4.8) with \mathcal{G}_1 in place of \mathcal{F} .

We will now describe the branching mechanism. Suppose that

$$(4.11) \quad (\mathcal{F}, r(\mathcal{F}), T_1(\mathcal{F}), T_2(\mathcal{F}), U_1(\mathcal{F}), U_2(\mathcal{F})) \in \mathcal{W}.$$

We always have $[U_1(\mathcal{F}), U_2(\mathcal{F})] \subset [T_1(\mathcal{F}), T_2(\mathcal{F})]$. If $[U_1(\mathcal{F}), U_2(\mathcal{F})] = [T_1(\mathcal{F}), T_2(\mathcal{F})]$ then we declare the sextuplet in (4.11) to be a leaf of the branching tree, i.e., this sextuplet has no offspring. We also declare the sextuplet in (4.11) to be a leaf if $n_{\mathcal{F}} \leq 2$.

Suppose that $[U_1(\mathcal{F}), U_2(\mathcal{F})] \neq [T_1(\mathcal{F}), T_2(\mathcal{F})]$ and $n_{\mathcal{F}} \geq 3$. Recall families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and \mathcal{F}_4 defined in conditions (i) and (ii) below (4.7). Let

$$\begin{aligned} T_1(\mathcal{F}_1) &= T_1(\mathcal{F}_2) = T_1(\mathcal{F}), & T_2(\mathcal{F}_1) &= T_2(\mathcal{F}_2) = U_1(\mathcal{F}), \\ T_1(\mathcal{F}_3) &= T_1(\mathcal{F}_4) = U_2(\mathcal{F}), & T_2(\mathcal{F}_3) &= T_2(\mathcal{F}_4) = T_2(\mathcal{F}). \end{aligned}$$

We declare the following four sextuplets to be (some of the) offspring of the sextuplet in (4.11),

$$(4.12) \quad (\mathcal{F}_k, r(\mathcal{F}_k), T_1(\mathcal{F}_k), T_2(\mathcal{F}_k), U_1(\mathcal{F}_k), U_2(\mathcal{F}_k)), \quad k = 1, 2, 3, 4.$$

Here $U_1(\mathcal{F}_k)$ and $U_2(\mathcal{F}_k)$ are defined as in (4.8) with \mathcal{F}_k in place of \mathcal{F} .

It is easy to check that if $\mathcal{F}_5 = \mathcal{F}$, $T_1(\mathcal{F}_5) = U_1(\mathcal{F})$ and $T_2(\mathcal{F}_5) = U_2(\mathcal{F})$ then $r(\mathcal{F}_5) \leq r(\mathcal{F})$, $U_1(\mathcal{F}_5) = U_1(\mathcal{F})$ and $U_2(\mathcal{F}_5) = U_2(\mathcal{F})$.

If $r(\mathcal{F}) \leq 4n_{\mathcal{F}}$ then we declare that the sextuplet in (4.11) has five offspring—the four offspring listed in (4.12) and

$$(4.13) \quad (\mathcal{F}_5, r(\mathcal{F}_5), U_1(\mathcal{F}), U_2(\mathcal{F}), U_1(\mathcal{F}), U_2(\mathcal{F})).$$

This case is illustrated in Fig. 1.

Next we will discuss the case when $r(\mathcal{F}) > 4n_{\mathcal{F}}$. In this case, all sextuplets listed in (4.12) will be declared to be offspring of the sextuplet in (4.11) but there will be more offspring constructed as follows.

Let $t_1 = U_1(\mathcal{F})$ and

$$(4.14) \quad \beta = (r(\mathcal{F}) - 2n_{\mathcal{F}}) / (n_{\mathcal{F}} - 1).$$

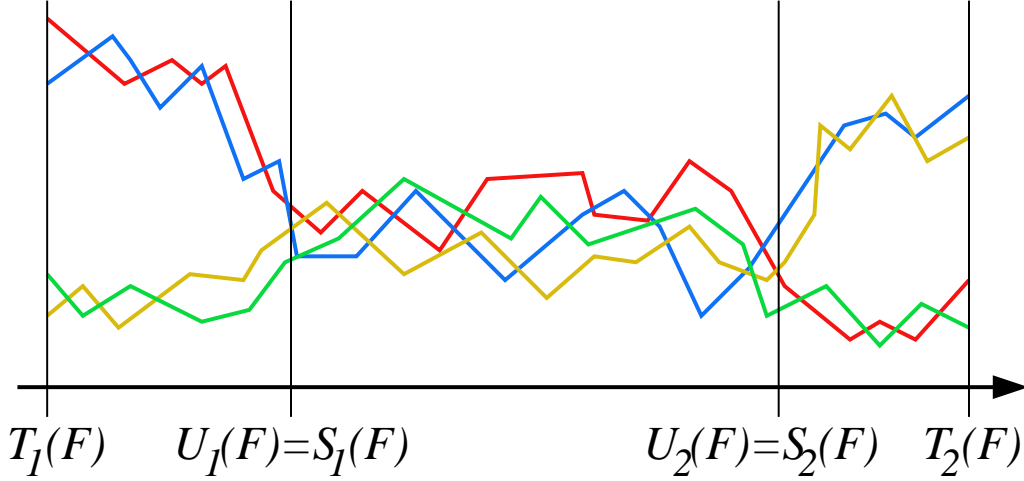


FIGURE 1. Schematic drawing of a branching event with five offspring. The parent family \mathcal{F} is represented by all trajectories on the interval $(T_1(\mathcal{F}), T_2(\mathcal{F}))$. In this generic case, $(U_1(\mathcal{F}), U_2(\mathcal{F}))$ lies strictly inside $(T_1(\mathcal{F}), T_2(\mathcal{F}))$. All trajectories are close to each other (at least at one time) in $(U_1(\mathcal{F}), U_2(\mathcal{F}))$. The five offspring consist of two non-interacting families to the left of $U_1(\mathcal{F}) = S_1(\mathcal{F})$, two non-interacting families to the right of $U_2(\mathcal{F}) = S_2(\mathcal{F})$, and the original family \mathcal{F} restricted to the interval $(U_1(\mathcal{F}), U_2(\mathcal{F}))$. The latter offspring will not have any descendants—it is a leaf in the branching structure.

We will argue that \mathcal{F} can be partitioned into nonempty disjoint families \mathcal{H}_1^1 and \mathcal{H}_2^1 such that the distance between any ball in \mathcal{H}_1^1 and any ball in \mathcal{H}_2^1 is greater than β at time t_1 . If this is not the case then every two balls in \mathcal{F} are connected by a chain of balls with distances between consecutive balls less than or equal to β . Hence, the distance between the centers of endpoint balls in the chain is bounded by

$$(n_{\mathcal{F}} - 1)(\beta + 2) = r(\mathcal{F}) - 2n_{\mathcal{F}} + 2(n_{\mathcal{F}} - 1) = r(\mathcal{F}) - 2.$$

This contradicts the definitions (4.4)-(4.5) of $r(\mathcal{F}, t)$ and $r(\mathcal{F})$ because according to these definitions, there must exist balls whose centers are at a distance equal to or greater than $r(\mathcal{F})$ for every $t \in [T_1(\mathcal{F}), T_2(\mathcal{F})]$. We conclude that families \mathcal{H}_1^1 and \mathcal{H}_2^1 exist.

The velocities of balls in \mathcal{F} are bounded by $|\mathbf{v}_{\mathcal{F}}|$ so no ball in \mathcal{H}_1^1 can collide with any ball in \mathcal{H}_2^1 in the time interval $[t_1, t_1 + \beta/|\mathbf{v}_{\mathcal{F}}|]$.

Let

$$(4.15) \quad \begin{aligned} k_* &= \lceil (U_2(\mathcal{F}) - U_1(\mathcal{F}))|\mathbf{v}_{\mathcal{F}}|/\beta \rceil, \\ t_k &= t_1 + (k - 1)\beta/|\mathbf{v}_{\mathcal{F}}|, \quad k = 2, \dots, k_*, \\ t_{k_*+1} &= U_2(\mathcal{F}). \end{aligned}$$

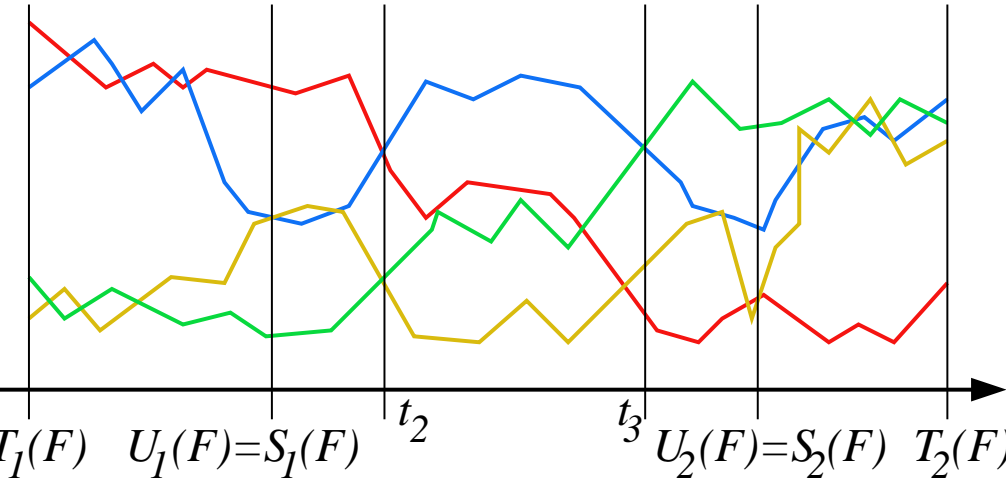


FIGURE 2. Schematic drawing of a branching event with more than five offspring. The parent family \mathcal{F} is represented by all trajectories on the interval $(T_1(\mathcal{F}), T_2(\mathcal{F}))$. In this generic case, $(U_1(\mathcal{F}), U_2(\mathcal{F}))$ lies strictly inside $(T_1(\mathcal{F}), T_2(\mathcal{F}))$. There is no time in $(U_1(\mathcal{F}), U_2(\mathcal{F}))$ such that all trajectories are close to each other. There are ten offspring. Four of these consist of two non-interacting families to the left of $U_1(\mathcal{F}) = S_1(\mathcal{F})$ and two non-interacting families to the right of $U_2(\mathcal{F}) = S_2(\mathcal{F})$. On each of the intervals $(U_1(\mathcal{F}), t_2)$, (t_2, t_3) and $(t_3, U_2(\mathcal{F}))$ there are at least two non-interacting families of trajectories. On each of these intervals, two non-interacting families are chosen and declared to be offspring of \mathcal{F} .

For every $k = 2, \dots, k_*$, we can find nonempty disjoint families \mathcal{H}_1^k and \mathcal{H}_2^k such that the distance between any ball in \mathcal{H}_1^k and any ball in \mathcal{H}_2^k is greater than β at time t_k . No ball in \mathcal{H}_1^k can collide with any ball in \mathcal{H}_2^k in the time interval $[t_k, t_{k+1}]$.

We declare the following sextuplets to be offspring of the sextuplet in (4.11),

$$(4.16) \quad (\mathcal{H}_i^k, r(\mathcal{H}_i^k), t_k, t_{k+1}, U_1(\mathcal{H}_i^k), U_2(\mathcal{H}_i^k)),$$

for $k = 1, \dots, k_*$ and $i = 1, 2$. Hence, in the case when $r(\mathcal{F}) > 4n_{\mathcal{F}}$, sextuplets listed in (4.12) and (4.16) are offspring of the sextuplet in (4.11). This case is illustrated in Fig. 2.

Step 6. We will estimate some quantities characterizing \mathcal{W} . We will write $\Lambda(\mathcal{G}) \prec \Lambda(\mathcal{F})$ to indicate that $\Lambda(\mathcal{G})$ is an offspring of $\Lambda(\mathcal{F})$. If $\Lambda(\mathcal{G}), \Lambda(\mathcal{F}) \in \mathcal{W}$ and $\Lambda(\mathcal{G}) \prec \Lambda(\mathcal{F})$ then either $\Lambda(\mathcal{G})$ is a leaf or $n_{\mathcal{G}} < n_{\mathcal{F}}$. It follows that the number of generations in \mathcal{W} is bounded by n .

An individual in \mathcal{W} has five offspring in the case $r(\mathcal{F}) \leq 4n_{\mathcal{F}}$.

For the next calculation, recall that $n_{\mathcal{F}} \geq 3$. If $r(\mathcal{F}) > 4n_{\mathcal{F}}$ then, in view of (4.6), (4.9), (4.14) and (4.15), the number of offspring is bounded above by

$$4 + 2k_* = 4 + 2[(U_2(\mathcal{F}) - U_1(\mathcal{F}))|\mathbf{v}_{\mathcal{F}}|/\beta] \leq 6 + 2(U_2(\mathcal{F}) - U_1(\mathcal{F}))|\mathbf{v}_{\mathcal{F}}|/\beta$$

$$\begin{aligned}
&= 6 + 2 \frac{(U_2(\mathcal{F}) - U_1(\mathcal{F}))|\mathbf{v}_{\mathcal{F}}|}{(r(\mathcal{F}) - 2n_{\mathcal{F}})/(n_{\mathcal{F}} - 1)} \leq 6 + 2 \frac{(U_2(\mathcal{F}) - U_1(\mathcal{F}))|\mathbf{v}_{\mathcal{F}}|}{(r(\mathcal{F})/2)/(n_{\mathcal{F}} - 1)} \\
&\leq 6 + 2 \frac{200n_{\mathcal{F}}^3 |\mathbf{x}_{\mathcal{F}}(t_*(\mathcal{F}))|}{(r(\mathcal{F})/2)/(n_{\mathcal{F}} - 1)} \leq 6 + 2 \frac{200n_{\mathcal{F}}^3 n_{\mathcal{F}}^{1/2} r(\mathcal{F})}{(r(\mathcal{F})/2)/(n_{\mathcal{F}} - 1)} \\
(4.17) \quad &\leq 6 + 800n_{\mathcal{F}}^{9/2} \leq 1000n_{\mathcal{F}}^{9/2} \leq 1000n^{9/2}.
\end{aligned}$$

This upper bound holds also in the case $r(\mathcal{F}) \leq 4n_{\mathcal{F}}$. Thus (4.17) implies that the number of individuals in the k -th generation is bounded by $(1300n^3)^{k-1}$. Since the number of generations is bounded by n , the total number of individuals in \mathcal{W} is bounded by

$$(4.18) \quad n(1000n^{9/2})^{n-1} \leq 1000^n n^{9n/2}.$$

Step 7. We will now bound the number of collisions. It follows from Step 5 that if two balls collide then there must exist a leaf

$$\Lambda(\mathcal{F}) = (\mathcal{F}, r(\mathcal{F}), T_1(\mathcal{F}), T_2(\mathcal{F}), U_1(\mathcal{F}), U_2(\mathcal{F})) \in \mathcal{W}$$

such that the two balls belong to \mathcal{F} and the collision takes place in the interval $[T_1(\mathcal{F}), T_2(\mathcal{F})]$.

First we will count collisions in open intervals of the form $(T_1(\mathcal{F}), T_2(\mathcal{F}))$.

If $n_{\mathcal{F}} \leq 2$ then the number of collisions in $(T_1(\mathcal{F}), T_2(\mathcal{F}))$ is bounded by 1.

The argument in Step 5 (see (4.13)) shows that if $n_{\mathcal{F}} \geq 3$ and $\Lambda(\mathcal{F})$ is a leaf then $(T_1(\mathcal{F}), T_2(\mathcal{F})) = (U_1(\mathcal{F}), U_2(\mathcal{F}))$ and $r(\mathcal{F}) \leq 4n_{\mathcal{F}} \leq 4n$. Hence, we can use (4.10) as an upper bound for the number of collisions in $(T_1(\mathcal{F}), T_2(\mathcal{F}))$. We combine (4.6), (4.10) and the estimate $r(\mathcal{F}) \leq 4n$ to obtain the following bound on the number of collisions in $(T_1(\mathcal{F}), T_2(\mathcal{F}))$ associated with $\Lambda(\mathcal{F})$,

$$\begin{aligned}
(4.19) \quad &200n_{\mathcal{F}}^3 |\mathbf{x}_{\mathcal{F}}(t_*(\mathcal{F}))| \left(32n_{\mathcal{F}}^{3/2}\right)^{5^d n_{\mathcal{F}} - 2} \leq 200n_{\mathcal{F}}^3 n_{\mathcal{F}}^{1/2} r(\mathcal{F}) \left(32n_{\mathcal{F}}^{3/2}\right)^{5^d n_{\mathcal{F}} - 2} \\
&\leq 200n^3 n^{1/2} \cdot 4n \left(32n^{3/2}\right)^{5^d n - 2} = 800n^{9/2} \left(32n^{3/2}\right)^{5^d n - 2}.
\end{aligned}$$

This upper bound applies also to leaves $\Lambda(\mathcal{F})$ with $n_{\mathcal{F}} \leq 2$.

The number of leaves in \mathcal{W} is bounded by the quantity in (4.18) so, in view of (4.19), the total number of collisions in open intervals of the form $(T_1(\mathcal{F}), T_2(\mathcal{F}))$ is bounded by

$$\begin{aligned}
(4.20) \quad &1000^n n^{9n/2} 800n^{9/2} \left(32n^{3/2}\right)^{5^d n - 2} = 800 \cdot 1000^n n^{(9/2)(n+1)} \left(32n^{3/2}\right)^{5^d n - 2} \\
&= 800 \left(1000 \cdot 32^{5^d}\right)^n n^{((3/2)5^d + 9/2)n + 3/2}.
\end{aligned}$$

The number of collisions at times $T_1(\mathcal{F})$ or $T_2(\mathcal{F})$ is bounded by the product of (i) the number of individuals in \mathcal{W} , (ii) number of endpoints of an interval, and (iii) one half of the number of balls, so, in view of (4.18), it is bounded by

$$1000^n n^{9n/2} \cdot 2 \cdot n/2 = 1000^n n^{9n/2+1}.$$

We combine this bound with (4.20) to conclude that the number of collisions is bounded by

$$\begin{aligned} & 800 \left(1000 \cdot 32^{5^d} \right)^n n^{((3/2)5^d + 9/2)n + 3/2} + 1000^n n^{9n/2 + 1} \\ & \leq 1600 \left(1000 \cdot 32^{5^d} \right)^n n^{((3/2)5^d + 9/2)n + 3/2}. \end{aligned}$$

□

Remark 4.1. (i) The estimates in Step 6 are crude and can be easily improved but we do not see a way to reduce the quantity in (4.18) so that its logarithm is $o(n \log n)$. Even if we could, the logarithm of the quantity in (4.3) is not $o(n \log n)$ so the bound in (1.1) would not change in a significant way.

(ii) Let τ_d denote the kissing number of a d -dimensional ball, i.e., the maximum number of mutually nonoverlapping translates of the ball that can be arranged so that they all touch the ball. According to [Bez10, Thm. 1.1.3],

$$2^{0.2075d(1+o(1))} \leq \tau_d \leq 2^{0.401d(1+o(1))}.$$

In Step 2, we derived the bound 5^d for the number of balls that could collide with a given ball on a time interval of length one. The lower bound for the kissing number shows that the bound 5^d cannot be improved to be less than exponential in d .

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