

Emergence of Hopf bifurcation in an extended SIR dynamic

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1 Abstract

In this paper, the SIR dynamics is extended by considering another compartmental which represents hospitalization of the critical cases. So a system of differential equations with four blocks is considered when there is intensive care unit (ICU) to cure critical cases. Outgoing rate of survived infected individuals is divided into nI and $\frac{bI}{I+b}$. The second term represents the rate of critical cases who enter ICUs. It is proved that there are forward, backward and Hopf bifurcations in different regimes of parameters.

2 Introduction

In the article by Kermack and Mckendrick [1], the first susceptible-infected-recovered(SIR) model was introduced in order to simulate and predict a diseases spreading phenomena and its epidemic. In this standard model, there are three compartments labeled as S , I and R which indicate number of (or percentage of) susceptible, infectious and recovered individuals and the transitions between these blocks happen with the constant parameters β , the infection rate, and γ , the recovery rate.

After publishing of this article, the parameters β and γ have been changed in various works and also some new blocks have been introduced and simulated in order to achieve more accurate models for better prediction of the behavior of different epidemics. For instance, β has been considered as $kI^{p-1}S^{q-1}$, $\beta a^{S+bI+cR}$ and μe^{-mI} , in the articles [2],[5],[6], respectively. In these articles, it has been tried to model behavior of people and strategies of government with considering β as a function of other blocks.

In order to consider capacity of the heath-care system in a country, γ has been changed in articles too. For instance, γI was been considered as:

$$\gamma I = \begin{cases} 0 & I = 0 \\ rI & I > 0 \end{cases} \quad (1)$$

$$\gamma I = \begin{cases} rI & 0 \leq I \leq I_0 \\ k & I_0 < I \end{cases} \quad (2)$$

$$\gamma I = (\mu_0 + (\mu_1 - \mu_0) \frac{b}{b+I}) I \quad (3)$$

in the articles [3],[4] and [8] respectively.

In our study, an improved model, which considers the effect of capacity ICU (Intensive Care Unit) in a hospital, will be discussed. To model this effect, a new variable H will be defined which expresses the number of individuals in ICU. So there is a new block in differential equations of SIR model. Fig. 2

represents a schematic of this model and the dynamics' equations are expressed as follow:

$$\left\{ \begin{array}{l} \frac{dS}{dt} = A - dS - \beta IS \\ \frac{dI}{dt} = \beta IS - dI - \alpha I - nI - \mu(I) \\ \frac{dH}{dt} = \mu(I) - dH - n'H \\ \frac{dR}{dt} = nI + n'H - dR \end{array} \right. \quad (4)$$

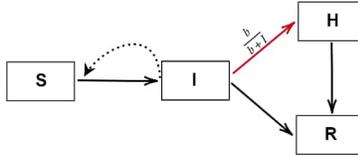


Figure 1: A schematic of our model. Please note that the natural birth and death are not shown here and all the quantities and parameters of the model are listed in Table 1 and 2.

In the equations (4), A is birth rate, d natural death rate and α is death rate caused by disease and β is the incidence rate. In this new model, the individuals of I , that will be survived, will be divided in two groups. The individuals of first group will be cured directly and they will not need to use special medical treatments of health system such as ICU so they will be transported in block R with a rate n . In other hand, the individuals of second group need to get special medical treatments and they must be cured in hospitals. These individuals will be transported in a new block H with a rate $\mu(I)$. The individuals of H will be recovered with the rate n' .

remark 2.1. Obviously, the parameters n and n' must be functions of I and H and there must be a connection between I and H , for example α can be a function of H , but we will ignore these effects to simplify of equations. So n, n' and α will be considered as constants.

A function, that can model $\mu(I)$ in real situations, can be expressed as follow, where b indicates the number of beds in ICU:

$$\mu_1(I) = \begin{cases} I & I \leq b \\ b & b \leq I \end{cases}$$

At first, when the number of infected individuals I is less than the number of beds b , the rate of transport to ICU will be I , for example, there are 100 beds and 50 infected people, so exactly 50 individuals will be transported in ICU. On the other hand, when the number of infected individuals I is more than the number of beds b , the rate of transport to ICU will be independent of the number of infected individuals and will be b , for example, there are 150 infected people and 100 beds, so 100 infected people will be transported in ICU.

But there is a big problem about this function. The function is obviously undifferentiable at the point $I = b$. So to simplify the solving of differential equations, we will choose another function, which is similar to this function and will be smooth. A good choice can be expressed as follows:

$$\mu_2(I) = \frac{bI}{I+b}$$

Obviously, in the limits $I \rightarrow 0$ and $I \rightarrow \infty$, we have $\mu_2(I) \rightarrow \mu_1(I)$ and $\frac{d\mu_2}{dI} \rightarrow \frac{d\mu_1}{dI}$ will be defined as follow:

$$\mu(I) := \frac{bI}{I+b}$$

And the differential equations will be expressed as follow:

$$\begin{cases} \frac{dS}{dt} = A - dS - \beta IS \\ \frac{dI}{dt} = \beta IS - dI - \alpha I - nI - \frac{bI}{I+b} \\ \frac{dH}{dt} = \frac{bI}{I+b} - dH - n'H \\ \frac{dR}{dt} = nI + n'H - dR \end{cases} \quad (5)$$

quantities	definition
$S(t)$	susceptible individuals
$I(t)$	infected individuals
$H(t)$	individuals of ICU
$R(t)$	recovered individuals

parameter	definition
β	infectious rate
b	number of beds
A	birth rate
d	natural death rate
α	death rate caused by infectious
n	natural recovery rate
n'	recovery rate of individuals of ICU

Table 1,2: List of all quantities and parameters of the model. Dimension of all parameters except b is T^{-1} . b and quantities are dimensionless.

In the system (5), the first two equations are independent of the other equations, so to analyse the system (5), such as finding the fixed points and bifurcations, just these equations will be considered. Thus the essential equations are:

$$\begin{cases} \frac{dS}{dt} = A - dS - \beta IS \\ \frac{dI}{dt} = \beta IS - \delta I - \frac{bI}{I+b} \end{cases} \quad (6)$$

where δ is defined as follow:

$$\delta := d + \alpha + n$$

3 Fixed Points

If one presents the system (6) as $\dot{X} = f(X)$, to find fixed points, $f(X^*) = 0$ must be solved, so:

$$\begin{cases} A - dS - \beta IS = 0 \\ \beta IS - \delta I - \frac{bI}{I+b} = 0 \end{cases} \quad (7)$$

With solving second equation, one can find:

$$\begin{cases} I = 0 \\ or \\ S = \frac{1}{\beta}(\delta + \frac{b}{I+b}) \end{cases}$$

Now with combining the first case and the first equation, one fixed point can be found as follow:

$$E_0 = (\frac{A}{d}, 0)$$

Which is the *disease free equilibrium(DFE)*.

With combining the second case and the first equation, the following quadratic equation can be found:

$$\delta I^2 + (b(\delta + 1) + \frac{\delta d}{\beta} - A)I + b(\frac{d}{\beta}(\delta + 1) - A) = 0 \quad (8)$$

As we see in the next sections, it will be useful to define R_0 , *basic reproduction number*, as follow:

$$R_0 := \frac{\beta A}{d(\delta + 1)}$$

Now with this definition, the equation (8) can be rewritten as follow:

$$P(I) = c_2 I^2 + c_1 I + c_0 = 0 \quad (9)$$

With following coefficients:

$$\begin{cases} c_2 = \delta \\ c_1 = b(\delta + 1) + \frac{\delta d}{\beta} - A \\ c_0 = \frac{bd(\delta+1)}{\beta}(1 - R_0) \end{cases}$$

So there will be following theorems about other fixed points $X^* \neq E_0$ of system (6).

Theorem 3.1. *There is exactly one fixed point X^* if $R_0 > 1$.*

Proof. It is obvious:

$$R_0 > 1 \Rightarrow \frac{bd(\delta + 1)}{\beta}(1 - R_0) < 0$$

so the quadratic equation (9) have two solutions I_1 and I_2 with the condition $I_1 I_2 < 0$. But in the system (6), I must be non-negative, so there will be just one permissible solution for the quadratic equation (9) and this proves this theorem. \square

Theorem 3.2. *If $R_0 < \frac{\delta}{(\delta+1)}$, there is no fixed point X^* .*

Proof. it is obvious that $R_0 < 1$ so $c_0 > 0$ and we can drive from the assumption that:

$$\begin{aligned} R_0 < \frac{\delta}{(\delta + 1)} &\Rightarrow \frac{A\beta}{d(\delta + 1)} < \frac{\delta}{(\delta + 1)} \\ &\Rightarrow \frac{A\beta}{d} < \delta \\ &\Rightarrow 0 < \frac{d\delta}{\beta} - A \quad \& \quad 0 < b(\delta + 1) \\ &\Rightarrow 0 < b(\delta + 1) + \frac{\delta d}{\beta} - A \\ &\Rightarrow 0 < c_1 \end{aligned}$$

So firstly, if there are real solutions for quadratic equation (9), both of them must be negative ($c_1 > 0$ & $c_0 > 0$). On the other hand there are no real solutions for this equation. So the theorem has been proved. \square

Theorem 3.3. *If $\frac{\delta}{(\delta+1)} < R_0 < 1$, there will be following cases:*

$b > \frac{d}{\beta}(\frac{1}{\delta+1})$: there is no fixed point X^* .

$b < \frac{d}{\beta}(\frac{1}{\delta+1})$: there are two fixed points X^* for R_0 close enough to 1 .

Proof. If $R_0 < 1$, it is obvious for two cases that $0 < c_0$. Now we can drive following result for first case:

$$\begin{aligned}
 & \frac{d}{\beta}(\frac{1}{\delta+1}) < b \Rightarrow \frac{d}{\beta} < b(1+\delta) \\
 \Rightarrow & \frac{(1+\delta)d}{\beta} < b(1+\delta) + \delta\frac{d}{\beta} \\
 \Rightarrow & \frac{(1+\delta)d}{\beta} - A < b(1+\delta) + \delta\frac{d}{\beta} - A \\
 \Rightarrow & \frac{(1+\delta)d}{\beta}(1 - \frac{\beta A}{d(\delta+1)}) < c_1 \\
 \Rightarrow & \frac{(1+\delta)d}{\beta}(1 - R_0) < c_1 \ \& \ 0 < 1 - R_0 \\
 \Rightarrow & 0 < c_1
 \end{aligned}$$

Because $c_1 > 0$ and $c_0 > 0$, the proof of this case will be similar to the previous theorem. So there is no fixed point X^* in this case.

The second case is a bit more complicated. At first, the new parameter ϵ will be defined as follow:

$$\epsilon := \frac{d(\delta+1)}{\beta} - A$$

ϵ will be positive, because:

$$\begin{aligned}
 R_0 < 1 \Rightarrow \frac{\beta A}{d(\delta+1)} < 1 \Rightarrow A < \frac{d(\delta+1)}{\beta} \\
 & 0 < \epsilon
 \end{aligned}$$

and the following properties are obvious:

$$\begin{aligned}
 A &= \frac{d(\delta+1)}{\beta} - \epsilon \\
 \epsilon \rightarrow 0^+ &\Leftrightarrow R_0 \rightarrow 1^-
 \end{aligned}$$

Now we claim, if ϵ is small enough, $c_1 < 0$:

$$\begin{aligned}
c_1 < 0 &\Leftrightarrow b(\delta + 1) + \frac{\delta d}{\beta} - A < 0 \\
&\Leftrightarrow 0 < A - b(\delta + 1) - \frac{\delta d}{\beta} \\
&\Leftrightarrow 0 < \frac{d(\delta + 1)}{\beta} - \epsilon - b(\delta + 1) - \frac{\delta d}{\beta} \\
&\Leftrightarrow \epsilon < \frac{d}{\beta} - b(\delta + 1) \\
&\Leftrightarrow \epsilon < (\delta + 1)\left(\frac{d}{\beta(\delta + 1)} - b\right)
\end{aligned}$$

But because in the second case $b < \frac{d}{\beta(\delta + 1)}$, the right side of the last inequality is positive, so ϵ can be found. So in this case:

$$c_1 < 0 \Leftrightarrow \epsilon < (\delta + 1)\left(\frac{d}{\beta(\delta + 1)} - b\right)$$

Now with considering the properties of ϵ , which is mentioned above, we can find, that the right inequality is equal to the condition, R_0 close enough to 1.

Now it is obvious, when $c_1 < 0$ and $c_2 > 0$, $I_{min} := \operatorname{argmin} P(I)$ is positive and when $R_0 = 1$, there are two solutions for quadratic equation (9) $I_1 = 0$ and $I_2 > 0$. So when $R_0 = 1$, $P(I_{min}) < 0$. Now with decreasing R_0 , when R_0 is close enough to 1 and the conditions $c_1 < 0$ and $P(I_{min}) < 0$ are hold, there must be two positive solutions for the quadratic equation (9). (the condition $P(I_{min}) < 0$ can be hold because of continuously). So the second case of theorem can be proved. \square

Corollary 3.1. *When $R_0 < 1$, E_0 will be a unique fixed point if $b > \frac{d}{(\delta + 1)\beta}$ and there will be two other fixed points X^* if $b < \frac{d}{(\delta + 1)\beta}$.*

4 Stability of Fixed Points

Firstly, to analyze the stability of fixed points, Df must be determined. Df will be expressed as follow:

$$Df(X) = \begin{pmatrix} -d - \beta I & -\beta S \\ \beta I & \beta S - \delta - \frac{b^2}{(I+b)^2} \end{pmatrix} \quad (10)$$

Now Df at the point E_0 can be expressed as follow:

$$Df(E_0) = \begin{pmatrix} -d & -\frac{\beta A}{d} \\ 0 & \frac{\beta A}{d} - (\delta + 1) \end{pmatrix} = \begin{pmatrix} -d & -\frac{\beta A}{d} \\ 0 & (\delta + 1)(R_0 - 1) \end{pmatrix} \quad (11)$$

Now there will be following theorem:

Theorem 4.1. E_0 will be a stable node if $R_0 < 1$ or will be a saddle point if $R_0 > 1$.

Proof. Because $Df(E_0)$ is an upper triangular matrix, its eigenvalues will be $-d$ and $(\delta + 1)(R_0 - 1)$. Now if $R_0 < 1$, there are two negative eigenvalues and E_0 will be a stable node and if $R_0 > 1$, there is a negative and a positive eigenvalue and E_0 will be a saddle point. \square

Df at other fixed points X^* , when they exist, can be expressed as follow:

$$Df(X^*) = \begin{pmatrix} -d - \beta I & -(\delta + \frac{b}{I+b}) \\ \beta I & \frac{b}{I+b} - \frac{b^2}{(I+b)^2} \end{pmatrix} = \begin{pmatrix} -d - \beta I & -(\delta + \frac{b}{I+b}) \\ \beta I & \frac{bI}{(I+b)^2} \end{pmatrix} \quad (12)$$

Now, the trace of $Df(X^*)$, which is the sum of eigenvalues of $Df(X^*)$, will be expressed as follow:

$$\lambda_1 + \lambda_2(I) = \text{tr}(Df(X^*)) = \frac{Ib}{(I+b)^2} - d - \beta I$$

For determinant of $Df(X^*)$, which is the product of eigenvalues of $Df(X^*)$, one can drive the following equations:

$$\begin{aligned} \lambda_1 \lambda_2(I) &= \det(Df(X^*)) = \beta I \left(\delta + \frac{b}{I+b} \right) - \frac{dbI}{(I+b)^2} - \frac{b\beta I^2}{(I+b)^2} \\ &= \frac{I}{(I+b)^2} (\beta(\delta(I+b)^2 + b(I+b)) - db - \beta bI) \\ &= \frac{\beta I}{(I+b)^2} (\delta I^2 + 2b\delta I + b^2((\delta + 1) - \frac{d}{b\beta})) \end{aligned}$$

Now, there will be following theorems about stability of fixed points X^* :

Theorem 4.2. When $R_0 > 1$, the unique fixed point X^* will be stable if $\frac{1}{4} < d$ and $\frac{d}{(\delta+1)\beta} < b$.

Proof. Firstly, the necessary condition, which must be satisfied in both of real and complex eigenvalues cases, is that, $\lambda_1 + \lambda_2 < 0$, i.e:

$$\frac{Ib}{(I+b)^2} - d - \beta I < 0$$

Now, it is easy to see, that the function $\frac{Ib}{(I+b)^2}$ is less than or equal to $\frac{1}{4}$ for nonnegative I . And as we saw in the theorem, there will be a fixed point with $I \in (0, +\infty)$ when $R_0 > 1$. So the above inequality must be satisfied for

$\forall I \in (0, +\infty)$. So:

$$\begin{aligned} \frac{1}{4} < d &\Rightarrow -d < -\frac{1}{4} \\ \Rightarrow \frac{Ib}{(I+b)^2} - d &< \frac{Ib}{(I+b)^2} - \frac{1}{4} < 0 \\ &\Rightarrow \frac{Ib}{(I+b)^2} - d - \beta I < 0 \end{aligned}$$

But when the eigenvalues of $Df(X^*)$ are real, an other condition, which must be satisfied, is that, $\lambda_1 \lambda_2 > 0$, i.e:

$$\frac{\beta I}{(I+b)^2} (\delta I^2 + 2b\delta I + b^2((\delta+1) - \frac{d}{b\beta})) > 0 \quad \forall I \in (0, +\infty)$$

It is easy to see, that the above inequality will be satisfied in this interval, if and only if:

$$\begin{aligned} b^2((\delta+1) - \frac{d}{b\beta}) &> 0 \Rightarrow \\ (\delta+1) - \frac{d}{b\beta} &> 0 \Rightarrow \\ \frac{d}{(\delta+1)\beta} &< b \end{aligned}$$

And this completes the proof of this theorem. \square

5 Bifurcations

In this subsection will be shown that the system (6) undergoes three kinds of bifurcation, forward and backward bifurcation and Hopf bifurcation.

5.1 forward & backward bifurcation

Theorem 5.1 (forward bifurcation). *When $R_0 = 1$, the system (6) undergoes a forward bifurcation if $\frac{1}{4} < d$ and $\frac{d}{(\delta+1)\beta} < b$.*

Proof. According to the theorems of the section 3 and 4, when $\frac{d}{(\delta+1)\beta} < b$, E_0 is a unique stable fixed point for $R_0 < 1$ and when $\frac{1}{4} < d$ and $\frac{d}{(\delta+1)\beta} < b$, there will be a unique stable fixed point X^* for $R_0 > 1$ and E_0 will be an unstable fixed point. So when $\frac{d}{(\delta+1)\beta} < b$ and $\frac{1}{4} < d$, both of conditions will be satisfied and there will be a forward bifurcation in the system (6). \square

As an example, this theorem can be seen in the following plots:

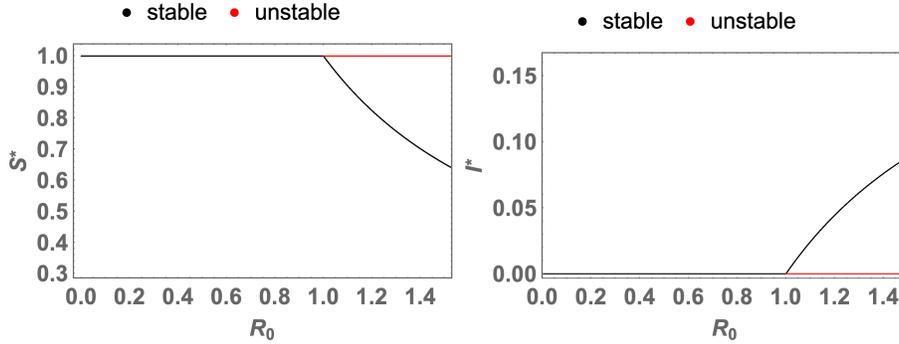


Figure 2: Simulation of theorem 4.1: plots of S^* and I^* , coordinates of fixed points, as a function of R_0 where black and red curves indicate stable and unstable fixed points respectively and when the parameters are $b = 1, A = 1, d = 1, \alpha = 1, n = 1, n' = 1$ and β changes. The quantities and parameters have been defined in the table 1 and 2 and R_0 has been defined in the section 2.

Theorem 5.2 (backward bifurcation). *When $R_0 = 1$, the system (6) undergoes a backward bifurcation if $\frac{1}{4} < d$ and $b < \frac{d}{(\delta+1)\beta}$.*

Proof. According to the theorems in the section 3 and 4, when $R_0 < 1$, E_0 is always a stable fixed point and there will be two other fixed points X^* if $b < \frac{d}{(\delta+1)\beta}$ and R_0 close enough to 1. Now we claim that one of these fixed points X^* will be a stable node and the other one will be a saddle point. We prove this with *Index Theory*. Firstly, we choose one closed curve C as follow:

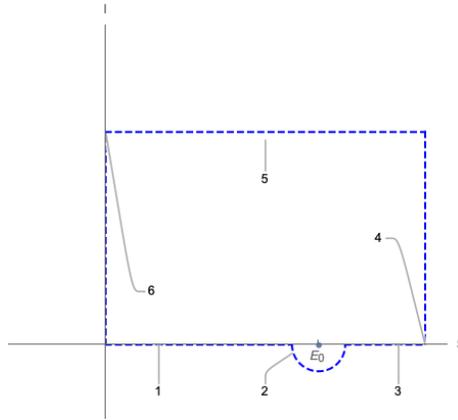


Figure 3: closed curve C

This curve C has been chosen large enough so when the fixed points X^* exist, this curve C will enclose them. (The circle in the fourth quadrant is too small so it encloses just E_0 for $R_0 < 1$).

Now, when $R_0 < 1$ and small, there is just E_0 in the curve C , so the index of closed curve C will be as follow:

$$I_C = I_0$$

where I_0 is the index of E_0 . E_0 is a stable node so $I_0 = 1$ and:

$$I_C = 1$$

Now assume that R_0 is close enough to 1 and two other fixed points X^* exist in the C . So I_C will be expressed as follow:

$$I_C = I_0 + I_1 + I_2 = 1 + I_1 + I_2$$

where I_1 and I_2 are the indexes of other fixed points X^* . So I_C can be considered as a function of R_0 .

Now we claim that the $I_C(R_0)$ is a continuous function. $I_C(R_0)$ can be rewritten as an integral in the complex plane as follow:

$$I_C(R_0) = \frac{1}{2\pi i} \oint_{\tilde{C}} \frac{1}{\xi} d\xi$$

where \tilde{C} is $f(C)$ (C has been considered as a closed curve in \mathbb{C} and f as a function $\mathbb{C} \rightarrow \mathbb{C}$). So we can obtain:

$$I_C(R_0) = \frac{1}{2\pi i} \sum_{i=1}^6 \int_0^1 \frac{1}{f(g_i(t))} f'(g_i(t)) g_i'(t) dt$$

where $g_i(t)$ are the parameterization of lines and circle of closed curve C in complex plane as it is shown in figure 3.

Now f and f' are continuous function of R_0 , so integrands of above integrals are continuous function of R_0 and $I_C(R_0)$ will be a continuous function of R_0 .

Now we know $I_{1,2} = \pm 1$ and $I_C(R_0)$ is continuous, so:

$$\begin{cases} I_C(R_0) = 1 & R_0 \in (0, R_0^{(1)}) \\ I_C(R_0) = 1 + I_1 + I_2 & R_0 \in (R_0^{(1)}, 1) \end{cases}$$

$$\Rightarrow I_1 + I_2 = 0$$

$$\Rightarrow \begin{cases} I_1 = 1 \ \& \ I_2 = -1 \\ or \\ I_2 = 1 \ \& \ I_1 = -1 \end{cases}$$

So one of the fixed points X^* must be a saddle point ($I = -1$) and the other one must be a stable or an unstable node ($I = 1$) but $d > \frac{1}{4}$ and $\lambda_1 + \lambda_2 < 0$, so it must be a stable node.

Now when $R_0 > 1$, E_0 is unstable fixed point(saddle point) and there is always an other fixed point X_1^* in the first quadrant and a fixed point X_2^* near E_0 in the circle(X_2^* is not a permissible fixed point because it exists in fourth quadrant). So in this case $I_0 = -1$ and $I_C(R_0)$ is continuous:

$$\begin{cases} I_C(R_0) = 1 & R_0 \in (0, 1) \\ I_C(R_0) = -1 + I_1 + I_2 & R_0 \in (1, +\infty) \end{cases}$$

$$\Rightarrow I_1 + I_2 = 2$$

$$\Rightarrow I_{1,2} = 1$$

So $X_{1,2}^*$ are nodes. But similar to the case $R_0 < 1$, permissible node must be stable, so X_1^* will be a stable fixed point.

So there is just a stable fixed point (E_0) for $R_0 < R_0^{(1)}$ and two stable fixed point(E_0, X_1^*) and an unstable fixed point (X_2^*) for $R_0^{(1)} < R_0 < 1$ and a stable fixed point X_1^* and an unstable fixed point E_0 for $R_0 > 1$. So there must be a backward bifurcation in $R_0 = 1$. \square

As an example, this theorem can be seen in the following plots:

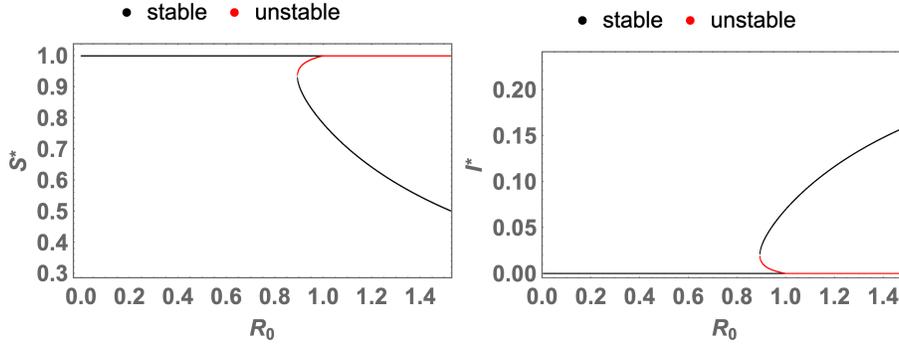


Figure 4: Simulation of theorem 4.2: plots of S^* and I^* , coordinates of fixed points, as a function of R_0 where the black and red curves indicate the stable and unstable fixed point respectively and when the parameters are $b = .01$, $A = 1$, $d = 1$, $\alpha = 1$, $n = 1$, $n' = 1$ and β changes. The quantities and parameters have been defined in the table 1 and 2 and R_0 has been defined in the section 2.

remark 5.1. The saddle node bifurcation in backward bifurcation occurs when the quadratic equation (9) has just one root, a.e:

$$\Delta = 0 \Leftrightarrow c_1^2 - 4c_2c_0 = 0 \quad (13)$$

Now when β is changing and other parameters are constant, β can be obtained from the above equation, for example in the simulation of previous theorem, $\beta = 0.892857$.

5.2 Hopf bifurcation

remark 5.2. In the following theorem, R_0 will be as a function of β and other parameters will be constant, in other words because $R_0 \propto \beta$, there will be no difference between changing of R_0 and β . It will be useful to define β_0 as follow:

$$\beta_0 := \frac{(\delta + 1)d}{A}$$

$$\Rightarrow R_0 = 1 \Leftrightarrow \beta = \beta_0$$

Theorem 5.3 (Hopf bifurcation). Suppose $\frac{b(2\delta+1)}{2} < A$ and $\frac{d}{(\delta+1)\beta} < b$. Define Δ and β_{max} as follow:

$$\Delta := A - \frac{b(2\delta + 1)}{2} \quad \beta_{max} := \frac{(\delta + 1)d}{\Delta}$$

With above assumptions, the system (6) undergoes a Hopf bifurcation for some $\beta \in (\beta_0, \beta_{max})$ or equally for some $R_0 \in (1, \frac{A}{\Delta})$ if $d(1 + \frac{(\delta+1)b}{\Delta}) < \frac{1}{4}$.

Proof. Firstly, we will consider the $\lambda_1 + \lambda_2(I)$. We have shown:

$$\lambda_1 + \lambda_2(I) = \frac{Ib}{(I+b)^2} - d - \beta I$$

First we fix $\beta \in [\beta_0, \beta_{max}]$. Now when $I = 0$, $\lambda_1 + \lambda_2 = -d < 0$ and when $I = b$

$$\lambda_1 + \lambda_2(b) = \frac{1}{4} - d - \beta b$$

Now we have

$$\begin{aligned} & \beta \leq \beta_{max} \\ \Rightarrow & \frac{1}{4} - d - \beta b \geq \frac{1}{4} - d - b\beta_{max} \\ \Rightarrow & \frac{1}{4} - d - \beta b \geq \frac{1}{4} - d - \frac{(\delta + 1)db}{\Delta} \\ \Rightarrow & \frac{1}{4} - d - \beta b \geq \frac{1}{4} - d(1 + \frac{(\delta + 1)b}{\Delta}) > 0 \\ \Rightarrow & \lambda_1 + \lambda_2(b) = \frac{1}{4} - d - \beta b > 0 \end{aligned}$$

So under above assumptions, for each $\beta \in [\beta_0, \beta_{max}]$, we have $\lambda_1 + \lambda_2(0) < 0$ and $\lambda_1 + \lambda_2(b) > 0$. So according to the *intermediate value theorem*, for each $\beta \in [\beta_0, \beta_{max}]$ there is some $I_1 \in (0, b)$ such that $\lambda_1 + \lambda_2(I_1) = 0$. But if we consider

I_1 as the intersection point of function $\frac{Ib}{(I+b)^2}$ and line $\beta I + d$, considering the behavior of this function and line, there is just one I_1 for each $\beta \in [\beta_0, \beta_{max}]$ and $I_1(\beta)$ is a continuous and $0 < I_1(\beta) < b$ and $\lambda_1 + \lambda_2(I_1(\beta)) = 0$ when $\beta \in [\beta_0, \beta_{max}]$.

Now we consider the quadratic equation (9). When $\frac{d}{(\delta+1)\beta} < b$ and $R_0 = 1$ or equally $\beta = \beta_0$, there is just one permissible solution, $I^* = 0$, and when $R_0 > 1$, there is always just one permissible root, which is a continuous function of R_0 or other parameters like $\beta(I^*(\beta))$. Now we consider the root of quadratic equation (9) for $\beta = \beta_{max}$. In this case, the quadratic equation (9) will be expressed as follow:

$$P_M(I) = \delta I^2 + (b(\delta + 1) - A)I - bA + \frac{\delta d}{\beta_{max}}I + \frac{bd}{\beta_{max}}(\delta + 1)$$

Now we will consider the $P_M(b)$ and the condition $\frac{b(2\delta+1)}{2} < A$:

$$\begin{aligned} P_M(b) &= \delta b^2 + (b(\delta + 1) - A)b - bA + \frac{\delta db}{\beta_{max}} + \frac{bd}{\beta_{max}}(\delta + 1) \\ &= b(\delta b + b(\delta + 1) - 2A) + b\left(\frac{\delta d}{\beta_{max}} + \frac{d}{\beta_{max}}(\delta + 1)\right) \\ &= -2b\left(A - \frac{b(2\delta + 1)}{2}\right) + \frac{b(2\delta + 1)d}{\beta_{max}} \\ &= b\left(-2\Delta + \frac{(2\delta + 1)\Delta}{\delta + 1}\right) \\ &= b\Delta\left(-2 + \frac{2\delta + 1}{\delta + 1}\right) \\ &= b\Delta\left(\frac{-1}{\delta + 1}\right) < 0 \end{aligned}$$

So obviously, $P_M(I)$ has a root I^* in the interval $(b, +\infty)$. In other words, with assumption of the conditions $\frac{d}{(\delta+1)\beta} < b$ and $\frac{b(2\delta+1)}{2} < A$, the quadratic equation (9) has a root $I^* = 0$ when $\beta = \beta_0$ and a root $I^* > b$ when $\beta = \beta_{max}$ and the $I^*(\beta)$ is a continuous function of $\beta \in [\beta_0, \beta_{max}]$. Now we use *intermediate value theorem* for the following function $f(\beta) = I_1 - I^*(\beta)$ and $\beta \in [\beta_0, \beta_{max}]$. This function is obviously continuous and

$$\begin{aligned} f(\beta_0) &= I_1(\beta_0) - I^*(\beta_0) = I_1(\beta_0) > 0 \\ f(\beta_{max}) &= I_1(\beta_{max}) - I^*(\beta_{max}) < 0 \end{aligned}$$

So there is a $\beta_1 \in (\beta_0, \beta_{max})$ such that $f(\beta_1) = 0$ or equally $I_1(\beta_1) = I^*(\beta_1)$ and

$$\lambda_1 + \lambda_2(I^*(\beta_1)) = \lambda_1 + \lambda_2(I_1(\beta_1)) = 0$$

Now with considering the differentiability of functions $I^*(\beta)$ and $\lambda_1 + \lambda_2(I)$ and *intermediate value theorem*, it will be possible to find a β_1 with the following

properties:

$$(\exists \epsilon > 0) I^*(\beta_1) = I_1(\beta_1), \begin{cases} \lambda_1 + \lambda_2(I^*(\beta)) < 0 & \beta \in (\beta_1 - \epsilon, \beta_1) \\ \lambda_1 + \lambda_2(I^*(\beta)) = 0 & \beta = \beta_1 \\ \lambda_1 + \lambda_2(I^*(\beta)) > 0 & \beta \in (\beta_1, \beta_1 + \epsilon) \end{cases}$$

Now we will consider $\lambda_1 \lambda_2(I)$. We have shown that $\lambda_1 \lambda_2(I)$ will be positive when $I \in (0, +\infty)$ and $\frac{d}{(\delta+1)\beta} < b$. So:

$$\begin{cases} \lambda_1 + \lambda_2(I^*(\beta_1)) = 0 \\ \lambda_1 \lambda_2(I^*(\beta_1)) > 0 \end{cases} \Rightarrow -\lambda_1^2 > 0 \Rightarrow \begin{cases} \lambda_1(I^*(\beta_1)) = i\omega_0 \\ \lambda_2(I^*(\beta_1)) = -i\omega_0 \end{cases}, \omega_0 > 0$$

Now obviously, $\lambda_{1,2}$ are continuous function of I^* and β and I^* is a continuous function of β , so $\lambda_{1,2}(\beta)$ will be continuous with values in complex plane \mathbb{C} . So we can choose a ϵ , which has been defined in the above arguments, with following properties:

$$\text{Im}(\lambda_1(I^*(\beta))) > 0, \beta \in (\beta_1 - \epsilon, \beta_1 + \epsilon)$$

Now because of $\lambda_2 = \bar{\lambda}_1$, the above results can be summarized and rewritten as follow:

$$[(\exists \epsilon > 0) \forall \beta \in (\beta_1 - \epsilon, \beta_1 + \epsilon)] \lambda_{1,2}(\beta) = r \pm i\omega, \omega > 0$$

$$, \begin{cases} r < 0 & \beta \in (\beta_1 - \epsilon, \beta_1) \\ r = 0 & \beta = \beta_1 \\ r > 0 & \beta \in (\beta_1, \beta_1 + \epsilon) \end{cases}$$

So there must be a Hopf bifurcation when $\beta = \beta_1 \in (\beta_0, \beta_{max})$ or equally $R_0 = \frac{\beta_1 A}{(\delta+1)\beta} \in (1, \frac{A}{\Delta})$. \square

As an example, this theorem can be seen in the following plot:

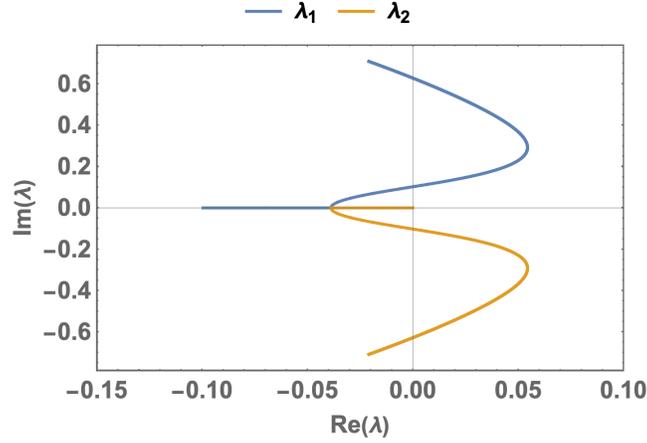


Figure 5: Simulation of theorem 4.3: curves of eigenvalues in complex plane, where the blue and yellow curves indicate the different eigenvalues, and when the parameters are $A = 10, d = .1, \alpha = 1.9, n = 1, b = 1$ and β changes in the interval $(.04, .09)$ or equally $R_0 \in (1, 2.25)$. The quantities and parameters have been defined in the tables 1 and 2 and R_0 has been defined in the section 2.

It is easy to check that these parameters satisfy the conditions of this theorem and $\beta_0 = .04$ and $\beta_{max} \approx .06154$. Obviously, there are two Hopf bifurcations in this case. These bifurcations occur $\beta_1 \approx 0.040968$ and $\beta_2 \approx 0.078668$ or equally $R_0^{(1)} \approx 1.0242$ and $R_0^{(2)} \approx 1.9667$. The case β_1 was predicted in the our theorem.

6 Numerical Analysis

In this section, we will concentrate on the regime of parameters where there is a Hopf bifurcation and analyze the time evolution of the dynamics. Thus for instance, we can consider the case in the Figure 5, where β changes in the interval $(.04, .09)$ and obviously, there is a limit cycle in this situation for $\beta \in (0.040968, 0.078668)$. Now for a better observation of behavior of system, when β changes, we will choose two different β , before (figures 6) and after (figures 7) $\beta_1 \approx 0.040968$, with two different initial conditions for each β and sketch stream plot $S - I$ and S, I, H, R and fluxes as function of t , where the fluxes are defined as follow:

$$\begin{cases} \phi_{in} := \beta IS \\ \phi_{out,1} := (d + \alpha + n)I \\ \phi_{out,2} := \frac{bI}{I+b} \end{cases} \quad (14)$$

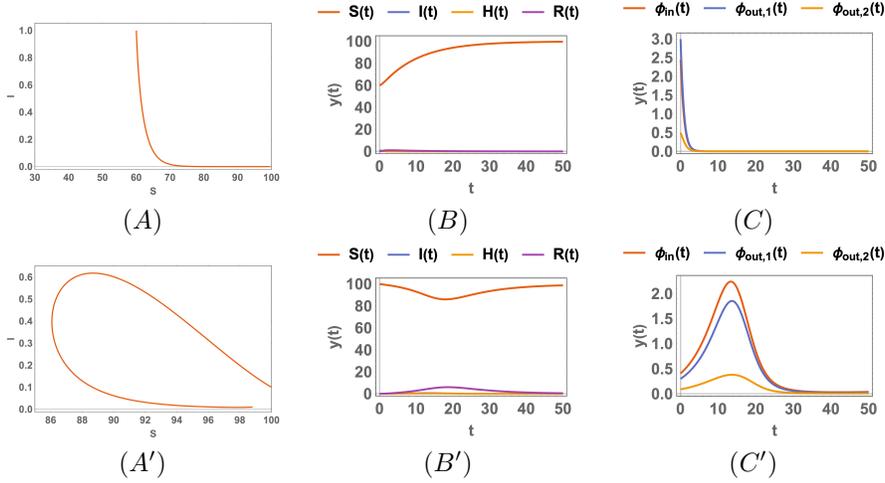


Figure 6: stream plot $S - I$ (A, A') and plots of S, I, H, R (B, B') and fluxes (C, C') as a function of t when the parameters are $b = 1, A = 10, d = .1, \alpha = 1.9, n = 1, n' = 1$ and $\beta = .0408$ and initial conditions are $X_0 = (60, 1, 0, 0)$ For panels (A, B, C) and $X_0 = (100, .1, 0, 0)$ (A', B', C'). The quantities and parameters have been defined in the table 1 and 2 and the fluxes have been defined in the equations 14.

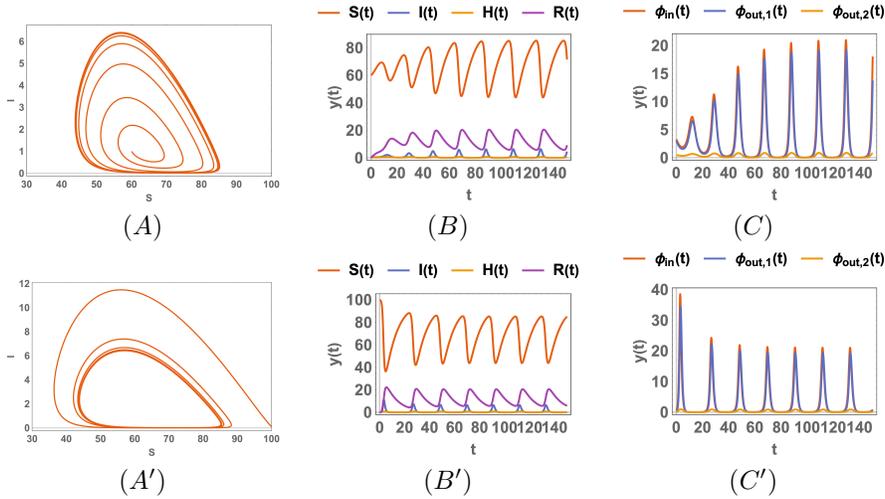


Figure 7: stream plot $S - I$ (A) and plots of S, I, H, R (B) and fluxes (C) as a function of t when the parameters are $b = 1, A = 10, d = .1, \alpha = 1.9, n = 1, n' = 1$ and $\beta = .055$ and initial conditions are $X_0 = (60, 1, 0, 0)$ for panels (A, B, C) and $X_0 = (100, .1, 0, 0)$ for panels (A', B', C'). The quantities and parameters have been defined in the table 1 and 2 and the fluxes have been defined in the

equations 14.

In the figure 6, β is less than 0.040968, so there is no limit cycle in this condition and curves in the stream plots (panels (A, A')) approach to a fixed point. Because of that, plots of S, I, H, R and fluxes have limit when $t \rightarrow \infty$. In contrast, in the figure 7, $\beta > 0.040968$ and there is a limit cycle and curves in the stream plots (panels (A, A')) approach to this limit cycle. Because of that, plots of S, I, H, R and fluxes behave periodic when t is large enough.

7 Discussion

In our study, we changed the standard SIR model and divided survived infected individuals into block R and a new introduced block H in order to model number of individuals of ICU, who need special and intensive treatment, where hospitalization rates were defined as nI and $\frac{bI}{I+b}$ respectively. Our model can be considered as a special case of model by Zang et al. [7] but in our study we have concentrated on existence and type of bifurcations, which is a new aspect of previous model, and we could observe forward, backward and Hopf bifurcation.

The most important result of our study, which makes our study different from previous works, is existence of Hopf bifurcation and so existence of limit cycle. Existence of limit cycle shows that a disease can survive in the society and this is a difficult and critic situation for governments in order to control this disease.

As an application of our model, we will consider following case, which has been based on existence of Hopf bifurcation. Suppose that we are in a regime of parameters where a supercritical Hopf bifurcation can occur when $\beta = \beta_0$ like the case, which has been considered in numerical analysis section 6. After spread of a special disease, governments usually use interventions in order to reduce contact between people for example quarantine. So β is small and we can suppose that $\beta < \beta_0$, where there is no limit cycle. But after a while, it is possible that β increases and $\beta > \beta_0$. This event can happen for example when people ignore the interventions of governments and quarantine. And in this situation, there is a limit cycle and disease can survive.

As a limitation of our study, we can mention that we could not analyze time evolution of fluxes, which have defined in the equations (14), although we have simulated them numerically in the section 6. Please note that the fluxes are important since they are the practical observable in the hospitals and can be used as a sort of warning signal, also can help us to estimate the other parameters of the dynamics. Therefore further theoretical works are needed in order to figure out how to detect such behaviours in practice.

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