Regular Foliations and Trace Divisors

Paulo Sad

December 1, 2021

Abstract

A example is given of a divisor of a curve which is not a trace divisor of a foliation.[1]

1 Introduction

We deal in this paper with a very concrete question about existence of foliations with prescribed singularities.

Let $F = 0$ be the equation of a plane smooth curve Q of degree 4; we fix the line at infinity $L_\infty \subset \mathbb{C}P^2$, transverse to Q, and put $D_\infty = Q \cap L_\infty$. We select 16 points $Q_1, \ldots, Q_{16}$ outside L_∞ and blow up $\mathbb{C}P^2$ at these points; the curve Q becomes a curve $\hat{Q} \subset \widehat{\mathbb{C}P^2}$.

Given the quartic Q and a foliation $\mathcal{F}$ which leaves Q invariant we say that the configuration of points $\{Q_1, \ldots, Q_{16}\}$ is *pre-regular* for $\mathcal{F}$ when $\hat{Q}$ is a leaf of the foliation $\hat{\mathcal{F}}$ obtained after blowing up at each of these points (in particular, $\hat{Q}$ has no singularities of $\hat{\mathcal{F}}$). A simple example is given by the pencil generated by Q and another transversal quartic; if $\{Q_1, \ldots, Q_{16}\}$ is the intersection of the two quadrics, then the divisor $[\sum_{j=1}^{16} Q_j - 4D_\infty]$ is a principal divisor (for simplicity we write $D = 0$ for a principal divisor D).

In ([1]) it is observed that if the points $Q_1, \ldots, Q_{16}$ are singularities of $\mathcal{F}$ then the divisor $[\sum_{j=1}^{16} m_j Q_j - LD_\infty]$ (its *trace divisor along* Q) defined putting m_j as the tangent multiplicity of $\mathcal{F}$ along Q at Q_j and $4L = \sum_{j=1}^{16} m_j$ is a principal divisor. A generic choice of the configuration of points satisfies none of these "resonance" conditions no matter what the choice of the

[1]MSC Number: 32S65

$m_1, \ldots, m_{16}$ is; in particular such a configuration will not be pre-regular. We may then allow resonances for the configuration and ask whether it is pre-regular for some foliation. This seems to be a difficult problem. Here in this paper we give an example of a configuration satisfying $[\sum_{j=1}^{16} Q_j - 4D_\infty] \neq 0$ and $2[\sum_{j=1}^{16} Q_j - 4D_\infty] = 0$ which is pre-regular for no foliation having $2[\sum_{j=1}^{16} Q_j - 4D_\infty]$ as its trace divisor.

This is related to the so called *Ueda type* of $\hat{Q}$ ([4]), as studied (in great generality) in ([2]). When $D = \sum_{j=1}^{16} Q_j - 4D_\infty$ satisfies $mD = 0$ for some integer smaller or equal to the Ueda type of $\hat{Q}$, then the configuration is pre-regular for some foliaton. It is interesting to look for cases where $mD = 0$ but the configuration is not pre-regular for any foliation (the simplest case being $2D = 0$ and Ueda type of $\hat{Q}$ equal to 1).

I would like to thank J.V.Pereira and M. Falla Luza for very helpful conversations.

2 The example

The set $\{Q_1, \ldots, Q_{16}\}$ is to be chosen for now as the zero set in Q of the function $G = l_1 l_2 C^2$, where C is a curve of degree 3 transverse to Q and l_1 and l_2 are bitangent lines to Q; we assume that the points of $l_1 = 0$ and $l_2 = 0$ lying in Q are not contained in C. It follows that $D \neq 0$ and $2D = 0$, for $D = \sum_{j=1}^{16} Q_j$, This example was introduced by Neeman in ([3]). We intend to prove the

Theorem Let Q be a generic smooth quartic. There exists a Neeman example such that the configuration of points is not pre-regular for any foliation which has $2D$ as its trace divisor along Q.

We will fix $C = F_X$ and assume: 1) the curve C is transverse to Q; 2) $F_Y \neq 0$ at the points of $l_1 l_2 = 0$ (there is a third condition that will appear in the proof of the Theorem). The Neeman example mentioned in the statement is the set $\{l_1 l_2 C^2 = 0\} \cap Q$.

We intend to study a foliation $\mathcal{F}$ which has Q as an invariant set and which is singular along Q exactly at the points $Q_1, \ldots, Q_{16}$; it is assumed that the singularities have multiplicities (along Q) equal to 2. Then $\mathcal{F}$ has degree 10 and it is defined by the polynomial 1-form

$$\Omega = \tilde{G}dF + F\tilde{\eta} \tag{1}$$

with $\tilde{\eta} = BdX - AdY$; the polynomials A and B have both degree 7.

Since the quotient $\dfrac{\tilde{G}}{G}$ is holomorphic along Q, we have that $\tilde{G} = G + \lambda F$ for some $\lambda \in \mathbb{C}$; therefore

$$\Omega = (G + \lambda F)dF + F\tilde{\eta} = GdF + F(\tilde{\eta} + \lambda F dF) = GdF + F\eta$$

where $\eta = \tilde{\eta} + \lambda F dF$. This means that we may use G in the expression of Ω.

Let us remark that more generally we may state

Proposition 1 Assume H and $\tilde{H}$ are polynomials such that the curves $H = 0$ and $\tilde{H} = 0$ intersect Q (in $\mathbb{C}P^2$) exactly at the points $Q_1, \ldots, Q_{16}$ with the same multiplicities. If $\mathcal{H}$ is a foliation defined by $HdF + F\xi = 0$ which has all its singularities along Q at the points $Q_1, \ldots, Q_{16}$, then $\mathcal{H}$ is also defined by $\tilde{H}dF + F\tilde{\xi}$ for some polynomial 1-form $\tilde{\xi}$.

From now on we consider a foliation $\mathcal{F}$ defined by a polynomial 1-form $\Omega = GdF + F(BdX - AdY)$ which has singularities along Q exactly at the points $Q_1, \ldots, Q_{16}$. We assume that $\mathcal{F}$ has multiplicity greater or equal to 2 along Q at any of these points. Let us now take local coordinates (x, y) in a neighborhood of some Q_j

$$x = X, \;\; y = F(X, Y)$$

or, equivalently

$$X = x, \;\; Y = l(x, y)$$

The relations

$$F_X(x, l(x, y)) + l_x(x, y).F_Y(x, l(x, y)) = 0, \quad F_Y(x, l(x, y)).l_y(x, y) = 1$$

follow from $F(x, l(x, y)) \equiv y$.

We remark that for each small $c \in \mathbb{C}$ the map $x \mapsto (x, l(x, c))$ parametrizes the curve $F(X, Y) = c$.

In these local coordinates the 1-form Ω becomes

$$g(x, y)dy + y[B(x, l)dx - A(x, l)(l_x dx + l_y dy)] = \tag{2}$$

$$[g - yl_y A(x,l)]dy + y[B(x,l) - l_x A(x,l)]dx,$$

where $g = g(x,y) = G(x, l(x,y))$ and $l = l(x,y), l_x = l_x(x,y), l_y = l_y(x,y)$.

$$\dot{x} = -g + y\alpha(x,y)), \ \ \dot{y} = y\beta(x,y)$$

The associated vector field is

$$[-g(x,y) + yl_y A(x, l(x,y))\frac{\partial}{\partial x} + y[B(x, l(x,y)) - l_x A(x, l(x,y))]\frac{\partial}{\partial y}$$

or simply

$$\dot{x} = -g(x,y) + y\alpha(x,y), \ \ \dot{y} = \beta(x,y)$$

for $\alpha(x,y) = l_y A(x, l(x,y))$ and $\beta(x,y) = B(x, l(x,y)) - l_x A(x, l(x,y))$.

Let us write $g(x,y) = \lambda x^m + y\tilde{g}(x,y)$ and $\tilde{\alpha}(x,y) = \alpha(x,y) - \tilde{g}(x,y)$; we have then

$$\dot{x} = -\lambda x^m + y\tilde{\alpha}(x,y), \ \ \dot{y} = y\beta(x,y) \tag{3}$$

3 Blowing-up the singularities

In this section we analyse necessary conditions a singularity as in (7) must satisfy in order to be a *pre-regular* singularity of the foliation $\mathcal{F}$.

Definition 1 A singularity in Q is *pre-regular* when after one blow-up the intersection of the exceptional divisor with the strict transform of Q is a regular point of the strict transform of $\mathcal{F}$.

After one blow-up of (7) for the case $m = 2$ (putting $y = tx$) we get

$$\dot{x} = -\lambda x^2 + tx\tilde{\alpha}(x, tx), \ \ \ \dot{t} = t\beta(x, tx) - t[-\lambda x + t\tilde{\alpha}(x, tx)];$$

let us write $\tilde{\alpha}(x,y) = \tilde{\alpha} + \tilde{\alpha}_x x + \tilde{\alpha}_y y + \ldots$ and $\beta(x,y) = \beta + \beta_x x + \beta_y y + \ldots$.

Since the Camacho-Sad index of a pre-regular singularity is equal to 1, we have $\beta_x = -\lambda$. The necessary conditions for pre-regularity are then

$$\tilde{\alpha} = 0, \ \ \beta = 0, \ \ \beta_y - \tilde{\alpha}_x = 0, \ \ \tilde{\alpha}_y = 0. \tag{4}$$

Let us remark that if the singularity was

$$\dot{x} = -\lambda x^m + y\tilde{\alpha}(x, y), \;\; \dot{y} = y\beta(x, y)$$

for $m > 2$, then the conditions in (8) of pre-regularity

$$\tilde{\alpha} = 0, \;\beta = 0, \;\beta_y - \tilde{\alpha}_x = 0, \;\tilde{\alpha}_y = 0, \;\beta_x = 0$$

are also, but not all, necessary conditions; the fact that the Camacho-Sad index is 1 corresponds to $\beta_x^{m-1} = -\lambda$, where $\beta_x^{m-1} x^{m-1}$ is the $(m-1)$-term in the Taylor development of $\beta(x, y)$.

We proceed to write the conditions in terms of G, A and B; for simplicity we assume the singularity to be the point $(0, 0) \in \mathbb{C}^2$.

1) $\tilde{\alpha} = 0$.

Since $\tilde{\alpha}(x, y) = \alpha(x, y) - \tilde{g}(x, y)$ we have $\alpha(0, 0) = \tilde{g}(0, 0)$. But $g(x, y) = \lambda x^2 + y\tilde{g}(x, y)$; therefore $\dfrac{\partial g}{\partial y}(x, y) = \tilde{g}(x, y) + y\dfrac{\partial \tilde{g}}{\partial y}(x, y)$ and $\dfrac{\partial g}{\partial y}(0, 0) = \tilde{g}(0, 0)$ Now we have $\alpha(0, 0) = \dfrac{\partial g}{\partial y}(0, 0)$ so $l_y(0, 0)A(0, l(0, 0)) = \dfrac{\partial G}{\partial Y}(0, l(0, 0)).l_y(0, 0)$.

We conclude that

$$A(0, 0) = \frac{\partial G}{\partial Y}(0, 0) \tag{5}$$

2) $\beta = 0$

We have $B(0, 0) - l_x(0, 0)A(0, 0) = 0$ and $l_x(0, 0) = -\dfrac{F_X}{F_Y}(0, 0)$ so that

$$A(0, 0)F_X(0, 0) + B(0, 0)F_Y(0, 0) = 0. \tag{6}$$

3) $\tilde{\alpha}_y = 0$

As before, $\tilde{\alpha}(x, y) = \alpha(x, y) - \tilde{g}(x, y)$ and $\tilde{\alpha}_y(x, y) = \alpha_y(x, y) - \tilde{g}_y(x, y)$. Since $\alpha(x, y) = l_y(x, y)A(x, l(x, y))$ we get

$$\alpha_y(x, y) = l_{yy}(x, y)A(x, l(x, y)) + l_y^2(x, y)A_Y(x, l(x, y))$$

As before, $\dfrac{\partial g}{\partial y}(x,y) = \tilde{g}(x,y) + y\dfrac{\partial \tilde{g}}{\partial y}(x,y)$; this implies

$$g_{yy}(x,y) = 2\tilde{g}_y(x,y) + y\tilde{g}_{yy}(x,y)$$

and

$$l_{yy}(0,0)G_Y(0,0) + l_y^2(0,0)A_Y(0,0) - \frac{g_{yy}(0,0)}{2} = 0$$

We have now to compute $g_{yy}(0,0)$, $l_y(0,0)$ and $l_{yy}(0,0)$.

- $g(x,y) = G(x,l(x,y))$ and $g_y(x,y) = G_Y(x,l(x,y))l_y(x,y)$ so $g_{yy}(x,y) = l_{yy}(x,y)G_Y(x,l(x,y)) + l_y^2(x,y)G_{YY}(x,l(x,y))$ and $l_y^2(0,0)A_Y(0,0) = \dfrac{l_y^2(0,0)G_{YY}(0,0) - l_{yy}(0,0)G_Y(0,0)}{2}$.
- $F(x,l(x,y)) = y$ implies $F_Y(x,l(x,y))l_y(x,y) = 1$ and $F_{YY}(x,l(x,y))l_y^2(x,y) + F_Y(x,l(x,y)l_{yy}(x,y) = 0$ so that $l_{yy}(x,y) = -\dfrac{F_{YY}(x,l(x,y))l_y^2(x,y)}{F_Y}$.

Finally we get

$$A_Y(0,0) = \frac{G_{YY}(0,0)}{2} + \frac{F_{YY}(0,0)G_Y(0,0)}{2F_Y(0,0)} \tag{7}$$

4) $\beta_y - \tilde{\alpha}_x = 0$

From $\beta(x,y) = B(x,l(x,y)) - l_x(x,y)A(x,l(x,y))$ we get

$$\beta_y(x,y) = B_Y(x,l(x,y))l_y(x,y) - l_{xy}(x,y)A(x,l(x,y)) - l_x(x,y)l_y(x,y)A_Y(x,l(x,y))$$

In order to compute $\tilde{\alpha}_x = \alpha_x - \tilde{g}_x$ we use:

- $\alpha(x,y) = l_y(x,y)A(x,l(x,y))$ implies

  $$\alpha_x(x,y) = l_{xy}(x,y)A(x,l(x,y)) + l_y(x,y[(A_X(x,l(x,y)) + l_x(x,y)A_Y(x,l(x,y))]$$

- $g_y(x,y) = \tilde{g}(x,y) + y\tilde{g}(x,y)$ implies

$$\tilde{g}_x(x,y) = g_{xy}(x,y) - y\tilde{g}_x(x,y) =$$
$$(G_Y.l_y)_x(x,y) - y\tilde{g}_x(x,y) =$$
$$[G_{YX}(x,l(x,y)) + G_{YY}(x,l(x,y))l_x(x,y)]l_y(x,y)+$$
$$G_Y(x,l(x,y)).l_{xy}(x,y) - y\tilde{g}_x(x,y)$$

It follows that

$$\tilde{\alpha}_x(0,0) = l_y(0,0)[A_X(0,0) + l_x(0,0)A_Y(0,0) - G_{YX}(0,0) - G_{YY}l_x(0,0)]$$

Using (9) and (11) we have $[B_Y(0,0) - A_X(0,0)]l_y(0,0) =$

$$l_{xy}(0,0)G_Y(0,0) + \frac{F_{YY}(0,0)G_Y(0,0)}{F_Y(0,0)}l_{xy}(0,0) - G_{XY}(0,0)l_y(0,0)$$

To finish the computation we notice that from $l_y(x,y)F_Y(x,l(x,y)) = 1$ it follows that

$$l_{xy}(x,y)F_Y(x,l(x,y)) + l_y(x,y)[F_{YX}(x,l(x,y)) + F_{YY}(x,l(x,l(x,y))l_x(x,y) = 0$$

and finally

$$B_Y(0,0) - A_X(0,0) = -\frac{F_{XY}(0,0)G_Y(0,0)}{F_Y(0,0)} - G_{XY}(0,0) \tag{8}$$

5) $\beta_x = -\lambda$

We have $\lambda = -\dfrac{g_{xx}(0,0)}{2}$.
From $\beta(x,y) = B(x,l(x,y)) - l_x(x,y)A(x,l(x,y))$ it follows

$$\beta_x = B_X(0,0) + [B_Y(0,0) - A_X(0,0)]l_x(0,0) - l_{xx}(0,0)A(0,0) - l_x^2(0,0)A_Y(0,0)$$

On the other hand $g(x,y) = G(x,l(x,y))$ implies

$$g_{xx}(0,0) = G_{XX}(0,0) + 2G_{XY}(0,0) + l_{xx}(0,0)G_Y(0,0) + G_{YY}(0,0)l_x^2(0,0)$$

and $F(x,l(x,y)) + l_x(x,y)F_Y(x,l(x,y)) = 0$ implies

$$l_{xx}(0,0)F_Y(0,0) = -2F_{XY}(0,0)l_x(0,0) - F_{XX}(0,0) - l_x^2(0,0)F_{YY}(0,0)$$

Using (11) and (12) we finally get

$$B_X(0,0) = -\frac{G_{XX}(0,0)}{2} - \frac{F_{XX}(0,0)G_Y(0,0)}{2F_Y(0,0)} \tag{9}$$

4 Consequences and Proof of the Theorem

Let us consider again the foliation $\mathcal{F}_0$ of degree 10 defined by the 1-form $\Omega = GdF + F(BdX - AdY)$; here $G = l_1 l_2 C^2$ and A, B are polynomials of degree 7. The question we address now is: is it possible that all the singularities $Q_1, \ldots, Q_{16}$ are pre-regular?

We denote by Q_{13}, Q_{14} the points of $\{l_1(X,Y) = 0\} \cap Q$ and by Q_{15}, Q_{16} the points of $\{l_2(X,Y) = 0\} \cap Q$; we have $\dfrac{(l_1 l_2)_X}{(l_1 l_2)_Y} = \dfrac{F_X}{F_Y}$ at these points.

We have the following list of identities

- $G = (l_1 l_2) C^2$
- $G_X = (l_1 l_2)_X C^2 + 2(l_1 l_2) C C_X$
- $G_Y = (l_1 l_2)_Y C^2 + 2(l_1 l_2) C C_Y$
- $G_{XX} = (l_1 l_2)_{XX} C^2 + 4(l_1 l_2)_X C C_X + 2(l_1 l_2) C C_{XX} + 2(l_1 l_2) C_X^2$
- $G_{XY} = (l_1 l_2)_{XY} C^2 + 2[(l_1 l_2)_X C C_Y + (l_1 l_2)_Y C C_X + l_1 l_2 C_Y C_X + l_1 l_2 C C_{XY}]$
- $G_{YY} = (l_1 l_2)_{YY} C^2 + 4(l_1 l_2)_Y C C_Y + 2(l_1 l_2) C_Y^2 + 2(l_1 l_2) CC)_{YY}$

Let us remind that $C = F_X$ and $F_{XX} \neq 0$ at the points of $\{F_X = 0\} \cap Q$.

Lemma 1 (i) $A = B = 0$ at the points of $Q \cap \{C = 0\}$; (ii) $B_X = -(l_1 l_2) C_X^2$ at $Q_1, \ldots, Q_{12}$; (iii) $B_X = -\dfrac{1}{2}(l_1 l_2)_{XX} C^2 - 2(l_1 l_2)_X C C_X - \frac{1}{2}(l_1 l_2)_X F_X F_{XX}$ at the point Q_{13}.

Proof. It is enough to use (9) and (10) and the identities above. □

Proposition 2 There exist polynomials h, h' of degree 4 and k, k' polynomials of degree 3 such that

$$A = hC + kF, \quad B = h'C + k'F \tag{10}$$

Proof. Since the meromorphic function $(\dfrac{A}{C})|_Q$ has its polar divisor supported in L_∞, there exists a polynomial h such that $(\dfrac{A}{C})|_Q = h|_Q$. As $(A - hC)|_Q = 0$, it follows that there exists a polynomial k satisfying $A - hC = kF$. The same argument works for B. □

We observe that (h', k') can be changed by $(h' + \mu F, k' - \mu C)$ for any $\mu \in \mathbb{C}$. Consequently we may assume that at Q_{13} we have $h' \neq 0$ and $k' \neq 0$.

Lemma 2 $h' + (l_1 l_2)F_{XX} + (l_1 l_2)_X F_X = 0$ at the points $Q_1, \ldots, Q_{16}$.

Proof. Let us look first at the points of $\{F_X\} \cap Q = 0$. Since $B_X = h' F_{XX}$ and $B_X = -(l_1 l_2)C_X^2$, we have $h' = -(l_1 l_2)F_{XX}$. As for $Q_{13}, \ldots . Q_{16}$, $A = (l_1 l_2)_Y F_X^2$ (because of (5)). But $B = h' F_X$ and $AF_X + BF_Y = 0$ (because of (6)), so we get $h' = -(l_1 l_2)_X F_X$ (using $\dfrac{(l_1 l_2)_X}{(l_1 l_2)_Y} = \dfrac{F_X}{F_Y}$). $\square$

Let us put $u = h' + (l_1 l_2)F_{XX} + (l_1 l_2)_X F_X$; we wish to show that u is not identically zero along Q. In order to do that, we parametrize a neighborhood of $Q_{13} = (0, 0)$ with the variable X and compute the derivative of $u|_Q$. We observe that $(u|_Q)' = u_X - \frac{F_X}{F_Y} u_Y$ at this point.

Without any loss of generality we may assume that $l_1(X, Y) = Y + aX$ and $l_2(X, Y) = Y + aX + b$, for some $a \neq 0$ and $b \neq 0$. Using Lemma 1 we get after a straight computation that at Q_{13}:

$$(u|Q)' = -a^2 F_X - \frac{1}{2} ab F_{XX} - a^2 b F_{XY} - (k' + a h'_Y)$$

In order to get rid of the term $k' + a h'_Y$, we replace the 1-form $G dF - F\eta$ that defines the foliation by $G d\hat{F} - \hat{F}\eta$ where $\hat{F} = cF$ for some $c \in \mathbb{C}$. Since, according to Proposition 2, $B = \hat{h}' C + \hat{k}' \hat{F}$ we see that $\hat{h}' = h'$ and $\hat{k}' = c^{-1} k'$. Consequently $\hat{k}' + a\hat{h}'_Y = c^{-1} k' + a h'_Y$; we can therefore choose $c \in \mathbb{C}$ in order that $\hat{k}' + a\hat{h}'_Y = 0$. The corresponding function $\hat{u}$ that replaces u satisfies

$$(\hat{u}|Q)' = c[-a^2 F_X - \frac{1}{2} ab F_{XX} - a^2 b F_{XY}]$$

We may now complete the meaning of the term "generic" which appears in the statement of the Theorem. We ask that the following inequality holds in Q_{13}:

$$a F_X + \frac{1}{2} b F_{XX} + ab F_{XY} \neq 0$$

This ensures that $\hat{u}|Q$ in not identically zero along Q.

In order to finish the proof of the Theorem we use the degree 4 polynomial $\hat{u}$. Since this polynomial vanishes at $Q_1, \ldots, Q_{16}$, we get that the divisor $\sum_{j=1}^{16} Q_j - 4D_\infty$ is a principal divisor, contradiction.

References

[1] P. Sad. *Regular Foliations along Curves*, Annales de la Faculte des Sciences de Toulouse : Mathematiques, Serie 6, Tome 8 (1999) no. 4, pp. 661-675.

[2] B. Claudon, F. Loray,J.V. Pereira, F. Touzet. *Compact Leaves of Codimension one Holomorphic Foliations on Projective Manifolds*, Annales Scientifiques de l'Ecole Normale Superieure (4) 51 (2018), 1457-1506.

[3] A. Neeman. Memoirs of the American Mathematical Society, 1989.

[4] T. Ueda. *On the neighborhood of a compact complex curve with topologically trivial normal bundle*, J. Math. Kyoto Univ. 22 (1982/83), no. 4, 583-607.

Paulo Sad
IMPA- Instituto de Matematica Pura e Aplicada
sad@impa.br