# Exact solutions to homogeneous and quasi-homogeneous systems of nonlinear ODEs 

Andrei D. Polyanin ${ }^{\text {a }}$, Alexei I. Zhurov ${ }^{\text {a,* }}$<br>${ }^{a}$ Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadsky Avenue, bldg 1, 119526 Moscow, Russia


#### Abstract

This note considers fairly general quasi-homogeneous systems of first-order nonlinear ODEs and homogeneous systems of second-order nonlinear ODEs that contain arbitrary functions of several arguments. It presents several exact solutions to these systems in terms of elementary functions.


Keywords: system of nonlinear equations, system of first-order equations, homogeneous system, quasi-homogeneous system, exact solutions

## 1. Brief introduction

Systems of ordinary differential equations are a common object of study in various scientific disciplines. Many exact solutions to such systems can be found in the handbooks [1, 2]. The article [3] presented partial solutions to a class of systems of first-order nonlinear ordinary differential equations with homogeneous polynomial right-hand sides. The current note deals with more general (than in [3]), quasi-homogeneous systems of first-order nonlinear ODEs as well as homogeneous systems of second-order nonlinear ODEs. It presents several exact solutions to these systems in terms of elementary functions.

## 2. Quasi-homogeneous systems of nonlinear first-order ODEs

Consider the following quasi-homogeneous systems of $N$ first-order ordinary differential equations for the unknowns $x_{1}=x_{1}(t), \ldots, x_{N}=x_{N}(t)$ :

$$
\begin{equation*}
x_{n}^{\prime}=x_{n}^{m_{n}+1} F_{n}\left(\frac{x_{2}^{m_{2} / m_{1}}}{x_{1}}, \frac{x_{3}^{m_{3} / m_{1}}}{x_{1}}, \ldots, \frac{x_{N}^{m_{N} / m_{1}}}{x_{1}}\right), \quad n=1, \ldots, N, \tag{1}
\end{equation*}
$$

where $F_{n}(\ldots)$ are given arbitrary functions, $m_{n}$ are arbitrary constants, and $N$ is an arbitrary positive integer.

System (11) preserves its form under the transformation

$$
t=\lambda^{-m_{1}} \bar{t}, \quad x_{n}=\lambda^{m_{1} / m_{n}} \bar{x}_{n}, \quad n=1, \ldots, N
$$

where $\lambda>0$ is an arbitrary constant.

[^0]Assuming that $m_{n} \neq 0(n=1, \ldots, N)$, we look for an exact solution to system (1) in the form

$$
\begin{equation*}
x_{n}(t)=a_{n}(1+C t)^{-1 / m_{n}}, \quad n=1, \ldots, N, \tag{2}
\end{equation*}
$$

where $C$ is an arbitrary constant and $a_{n}=x_{n}(0)$ are constants (initial values of the unknowns) to be determined.

The system of equations (1) admits an exact solution of the form (2) with the constants $a_{n}$ related by the algebraic (or transcendental) constraints

$$
\begin{equation*}
a_{n}^{m_{n}} m_{n} F_{n}\left(\frac{a_{2}^{m_{2} / m_{1}}}{a_{1}}, \frac{a_{3}^{m_{3} / m_{1}}}{a_{1}}, \ldots, \frac{a_{N}^{m_{N} / m_{1}}}{a_{1}}\right)+C=0, \quad n=1, \ldots, N . \tag{3}
\end{equation*}
$$

Example 1. For $m_{1}=\cdots=m_{N}=m$, the quasi-homogeneous system (1) simplifies to become a homogeneous system that can be represented as

$$
\begin{equation*}
x_{n}^{\prime}=x_{n}^{m+1} F_{n}\left(\frac{x_{2}}{x_{1}}, \frac{x_{3}}{x_{1}}, \ldots, \frac{x_{N}}{x_{1}}\right), \quad n=1, \ldots, N, \tag{4}
\end{equation*}
$$

where $F_{n}(\ldots)$ are arbitrary functions. The system of ODEs (4) admits an exact solution of the form (2) with $m_{n}=m$, where the constants $a_{n}$ are related by the constraints

$$
a_{n}^{m} m F_{n}\left(\frac{a_{2}}{a_{1}}, \frac{a_{3}}{a_{1}}, \ldots, \frac{a_{N}}{a_{1}}\right)+C=0, \quad n=1, \ldots, N .
$$

The note [4] presents system (4) and its solution for positive integer $m$.
In the degenerate case $m=0$, exact solutions to system (4) can be sought in the exponential form

$$
x_{n}(t)=a_{n} \exp (-C t), \quad n=1, \ldots, N,
$$

where $C$ is an arbitrary constant. The constants $a_{n}$ and $C$ are related by the constraints

$$
F_{n}\left(\frac{a_{2}}{a_{1}}, \frac{a_{3}}{a_{1}}, \ldots, \frac{a_{N}}{a_{1}}\right)+C=0, \quad n=1, \ldots, N .
$$

Example 2. The homogeneous system of ODEs (4) can be represented in the equivalent form

$$
\begin{equation*}
x_{n}^{\prime}=x_{1}^{m+1} G_{n}\left(\frac{x_{2}}{x_{1}}, \frac{x_{3}}{x_{1}}, \ldots, \frac{x_{N}}{x_{1}}\right), \quad n=1, \ldots, N \tag{5}
\end{equation*}
$$

where $G_{n}(\ldots)=\left(x_{n} / x_{1}\right)^{m+1} F_{n}(\ldots)$. Suppose that the functions $G_{n}$ are all multivariate polynomials of degree $M=m+1$ such that

$$
G_{n}=\sum \alpha_{n \mu_{2} \ldots \mu_{N}}\left(\frac{x_{2}}{x_{1}}\right)^{\mu_{2}} \ldots\left(\frac{x_{N}}{x_{1}}\right)^{\mu_{N}}, \quad n=1, \ldots, N
$$

where $\alpha_{n \mu_{2} \ldots \mu_{N}}$ are some constants and $\mu_{2}, \ldots, \mu_{N}$ are some nonnegative integers. Then system (5) becomes

$$
x_{n}^{\prime}=\sum c_{n \mu_{1} \ldots \mu_{N}} x_{1}^{\mu_{1}} \ldots x_{N}^{\mu_{N}}, \quad n=1, \ldots, N
$$

where $c_{n \mu_{1} \ldots \mu_{N}}$ are some constants and $\mu_{1}=M-\mu_{2}-\cdots-\mu_{N}$. We see that $\mu_{1}+\cdots+\mu_{N}=M$. If $\mu_{1}$ is also a nonnegative integer, we obtain system (1) from [3]. It admits the exact solution

$$
x_{n}(t)=a_{n}(1+C t)^{1 /(1-M)}, \quad n=1, \ldots, N,
$$

where $C$ is an arbitrary parameter and the constants $a_{n}$ satisfy the algebraic constraints

$$
C a_{n}=(1-M) \sum_{\mu_{1}+\cdots+\mu_{N}=M} c_{n \mu_{1} \ldots \mu_{N}} a_{1}^{m_{1}} \ldots a_{N}^{m_{N}}, \quad n=1, \ldots, N
$$

## 3. Homogeneous systems of nonlinear second-order ODEs

Now we look at homogeneous systems of second-order ODEs of the form

$$
\begin{equation*}
x_{n}^{\prime \prime}=x_{n}^{m}\left(x_{n}^{\prime}\right)^{k} F_{n}\left(\frac{x_{2}}{x_{1}}, \frac{x_{3}}{x_{1}}, \ldots, \frac{x_{N}}{x_{1}}, \frac{x_{2}^{\prime}}{x_{1}^{\prime}}, \frac{x_{3}^{\prime}}{x_{1}^{\prime}}, \ldots, \frac{x_{N}^{\prime}}{x_{1}^{\prime}}\right), \quad n=1, \ldots, N \tag{6}
\end{equation*}
$$

where $F_{n}(\ldots)$ are arbitrary functions of their arguments, $m$ and $k$ are arbitrary constants, and $N$ is an arbitrary positive integer.

Assuming all unknowns to be proportional, we look for a particular solution to system (6) in the special form

$$
\begin{equation*}
x_{n}=a_{n} y, \quad y=y(t), \quad n=1, \ldots, N, \tag{7}
\end{equation*}
$$

where $a_{n}$ are constants to be determined. As a results, we arrive at the following second-order ODE for $y$ :

$$
\begin{equation*}
y^{\prime \prime}=\lambda y^{m}\left(y^{\prime}\right)^{k} \tag{8}
\end{equation*}
$$

where $\lambda$ is an arbitrary constant, while $a_{n}$ satisfy the relations

$$
a_{n}^{m+k-1} F_{n}\left(\frac{a_{2}}{a_{1}}, \frac{a_{3}}{a_{1}}, \ldots, \frac{a_{N}}{a_{1}}, \frac{a_{2}}{a_{1}}, \frac{a_{3}}{a_{1}}, \ldots, \frac{a_{N}}{a_{1}}\right)=\lambda, \quad n=1, \ldots, N
$$

It is noteworthy that equation (8) is solvable and its general solution can be represented in implicit form.

For arbitrary $m$ and $k$ such that $m+k-1 \neq 0$, equation (8) admits the simple power-law particular solution

$$
y=A(1+C t)^{\sigma}, \quad \sigma=\frac{k-2}{m+k-1}, \quad A=\left[\frac{C^{2-k}(\sigma-1)}{\lambda \sigma^{k-1}}\right]^{\frac{1}{m+k-1}}
$$

where $C$ is an arbitrary constant.
For $m+k-1=0$, equation (8) admits the exponential partial solution

$$
y=B \exp (C t), \quad \lambda=C^{2-k}
$$

where $B$ and $C$ are arbitrary constants.
Below are two examples of more complicated solutions to equation (8) and system (6).
Example 3. In the special case of $m=1$ and $k=0$, system (6) becomes

$$
x_{n}^{\prime \prime}=x_{n} F_{n}\left(\frac{x_{2}}{x_{1}}, \frac{x_{3}}{x_{1}}, \ldots, \frac{x_{N}}{x_{1}}, \frac{x_{2}^{\prime}}{x_{1}^{\prime}}, \frac{x_{3}^{\prime}}{x_{1}^{\prime}}, \ldots, \frac{x_{N}^{\prime}}{x_{1}^{\prime}}\right), \quad n=1, \ldots, N .
$$

Its has two different exact solutions depending on the sign of $\lambda$. These are obtained from the second-order linear equation (8) with $m=1$ and $k=0$ :

$$
\begin{array}{ll}
x_{n}=a_{n}\left[C_{1} \exp (-\beta t)+C_{2} \exp (\beta t)\right] & \text { if } \lambda=\beta^{2}>0 \\
x_{n}=a_{n}\left[C_{1} \cos (\beta t)+C_{2} \sin (\beta t)\right] & \text { if } \lambda=-\beta^{2}<0 \tag{9}
\end{array}
$$

where $C_{1}$ and $C_{2}$ are arbitrary constants. The constants $a_{n}$ satisfy the constraints

$$
F_{n}\left(\frac{a_{2}}{a_{1}}, \frac{a_{3}}{a_{1}}, \ldots, \frac{a_{N}}{a_{1}}, \frac{a_{2}}{a_{1}}, \frac{a_{3}}{a_{1}}, \ldots, \frac{a_{N}}{a_{1}}\right)= \pm \beta^{2}, \quad n=1, \ldots, N
$$

where the upper sign refers to the first group of solutions in (9), while the lower sign refers to the second group of solutions.

Example 4. In the special case of $m=-3$ and $k=0$, system (6) becomes

$$
x_{n}^{\prime \prime}=x_{n}^{-3} F_{n}\left(\frac{x_{2}}{x_{1}}, \frac{x_{3}}{x_{1}}, \ldots, \frac{x_{N}}{x_{1}}, \frac{x_{2}^{\prime}}{x_{1}^{\prime}}, \frac{x_{3}^{\prime}}{x_{1}^{\prime}}, \ldots, \frac{x_{N}^{\prime}}{x_{1}^{\prime}}\right), \quad n=1, \ldots, N
$$

An exact solution to this system can be found from equation (8) with $m=-3$ and $k=0$. It is given by

$$
x_{n}=a_{n}\left(C_{1} t^{2}+C_{2} t+C_{3}\right)^{1 / 2}, \quad n=1, \ldots, N,
$$

where $C_{1}, C_{2}$, and $C_{3}$ are constants of integration, which are related by $C_{1} C_{3}-\frac{1}{4} C_{2}^{2}=\lambda$. The constants $a_{n}$ satisfy the constraints

$$
a_{n}^{-4} F_{n}\left(\frac{a_{2}}{a_{1}}, \frac{a_{3}}{a_{1}}, \ldots, \frac{a_{N}}{a_{1}}, \frac{a_{2}}{a_{1}}, \frac{a_{3}}{a_{1}}, \ldots, \frac{a_{N}}{a_{1}}\right)=C_{1} C_{3}-\frac{1}{4} C_{2}^{2}, \quad n=1, \ldots, N
$$

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[^0]:    *Corresponding author
    Email addresses: polyanin@ipmnet.ru (Andrei D. Polyanin), zhurovai@cardiff.ac.uk (Alexei I. Zhurov)

