Invariant cones for semigroups and controllability of bilinear control systems

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Abstract

In this paper we present necessary and sufficient conditions to guarantee the existence of invariant cones, for semigroup actions, in the space of the k-fold exterior product. As consequence we establish a necessary and sufficient condition for controllability of a class of bilinear control systems.

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1 Introduction

In this paper we deal with invariant cones for semigroup actions and applications to study controllability of control systems. In our context this question is related with the flag type of the semigroup (in particular semigroup of the control system) and hence with the control sets of the semigroup (or of the control system). Note that it is far from achieving global results on controllability of bilinear control systems, that is, to find sufficient conditions for controllability is a long term and still incomplete area of research (see e.g. Elliot [6]). But, in the last few decades, several papers have been published showing that the Lie theory, especially the theory of semigroups of semisimple Lie groups, provides important tools to study controllability (see e.g. Do Rocio, San Martin and Santana [3], Do Rocio, Santana and Verdi [4], Dos Santos and San Martin [5] and San Martin [10]). The semigroup appears naturally in the context of control systems, in fact, given a bilinear control system

$$\dot{x} = Ax + uBx, x \in \mathbb{R}^d \setminus \{0\}, u \in \mathbb{R},\tag{1}$$

where A and B are $d \times d$ -matrices, we have that the semigroup S of the system is given by the concatenations of solutions:

 $S = \{e^{t_k(A+u_kB)}e^{t_{k-1}(A+u_{k-1}B)} \dots e^{t_1(A+u_1B)}, t_i \ge 0, k \in \mathbb{N}\}$

and the group system has a similar definition just changing the positive times t_i by real times (see e.g Colonius and Kliemann [2] and Elliot [6]). And if we consider A and B generating a semisimple Lie algebra \mathfrak{g} we have the possibility to use the semisimple Lie theory to study controllability of the system, for example in case of $\mathfrak{g} = \mathfrak{sl}(d,\mathbb{R})$ we have that this system is controllable in $\mathbb{R}^d \setminus \{0\}$ ($Sx = \mathbb{R}^d \setminus \{0\}$ for all $x \in \mathbb{R}^d \setminus \{0\}$) if and only if $S = \mathrm{Sl}(d,\mathbb{R})$ (see [3] and [14]).

One of the most interesting ways to prove that the above system is not controllable is to show the existence of some S-invariant proper subset of \mathbb{R}^d , a trap of the system. This problem was addressed in [8], by Sachkov, but in [3] the authors searched these invariant sets among the convex cones, since if a set C is invariant by the system then the convex closure of C is also invariant. In our work we follow similar approach to improve and generalize the results contained in [3] and in particular to give a necessary and sufficient condition for controllability of the above system when $A, B \in \mathfrak{sl}(d, \mathbb{R})$. More specifically, we prove that the system is controllable if and only if it does not have an invariant proper cone in the k-fold exterior product of \mathbb{R}^d , $\bigwedge^k \mathbb{R}^d$, for all $k \in \{1, \ldots, d-1\}$. In fact, this is a consequence of our following transitivity result: Let $S \subset Sl(d, \mathbb{R})$ be a connected semigroup with nonempty interior. Then $S = Sl(d, \mathbb{R})$ if and only if there are no S-invariant and proper cones in $\bigwedge^k \mathbb{R}^d$, for all $k \in \{1, \cdots, d-1\}$. These two results are built from the theory of flag type of a semigroup.

We briefly recall the main concept or tool of this paper. Consider $S \subset$ $\mathrm{Sl}(d,\mathbb{R})$ a semigroup with nonempty interior. Denote by \mathbb{F}_{Θ} the flag manifold of all flags $(V_1 \subset \cdots \subset V_k)$ of subspaces $V_i \subset \mathbb{R}^d$ with dim $V_i = r_i$, i = $1,\ldots,k$ and $\Theta = \{r_1,\ldots,r_k\}$. Take the canonical projection $\pi_{\Theta_1}^{\Theta} : \mathbb{F}_{\Theta} \to \mathbb{F}_{\Theta}$ \mathbb{F}_{Θ_1} with $\Theta_1 \subset \Theta$ and denote by \mathbb{F} the full flag manifold with the sequence $\Theta_M = \{1, 2, \dots, d-1\}$. There is a natural (transitive) action of $Sl(d, \mathbb{R})$ in these flag manifolds, then an invariant control set, in \mathbb{F}_{Θ} , for the S-action is a subset $C \subset \mathbb{F}_{\Theta}$ such that $\operatorname{cl}(Sx) = C$, for all $x \in C$, and C is maximal with this property. Recall that an invariant control set is closed and its interior is dense on it. One important result is that in each flag manifold \mathbb{F}_{Θ} there exists just one S-invariant control set. Moreover, there exist $\Theta \subset \Theta_M$ such that $\pi_{\Theta}^{-1}(C_{\Theta}) = C$ where $\pi_{\Theta} : \mathbb{F} \to \mathbb{F}_{\Theta}$ is the canonical projection, and C_{Θ}, C are the invariant control sets in $\mathbb{F},\mathbb{F}_\Theta$ respectively. In addition, among these flag manifolds there is exactly one, denoted by $\mathbb{F}_{\Theta(S)}$, which is minimal (see [10]). The flag manifold $\mathbb{F}_{\Theta(S)}$ (or $\Theta(S)$) is called the flag (or parabolic) type of S (for details see San Martin [9] and San Martin and Tonelli [14]). We note that once we know the invariant control set $C_{\Theta(S)}$ in the flag type $\mathbb{F}_{\Theta(S)}$ then every invariant control set is described because for any Θ we have $C_{\Theta} = \pi_{\Theta}(C)$ and $C = \pi_{\Theta(S)}^{-1} (C_{\Theta(S)})$. Given $\Theta = \{r_1, \ldots, r_n\}$ with $0 < r_1 < \cdots < r_n < d$ define $\Theta^* = \{d - r_n, \dots, d - r_1\}$. The flag manifold \mathbb{F}_{Θ^*} is said to be dual of \mathbb{F}_{Θ} . With this we have that the flag type of S^{-1} is given by the flag manifold $\mathbb{F}_{\Theta(S)^*}$ dual to the flag type of S (see [11]).

From this semigroup theoretical development, considering S a connected semigroup with nonempty interior and taking $\Theta(S)$ its flag type, we prove our main result: there exists a non-trivial S-invariant cone $W \subset \bigwedge^k \mathbb{R}^d$ if and only if $k \in \Theta(S)$. Hence, as a consequence we show the controllability and transitivity results mentioned above.

About the structure of this paper, in Section 2 we establish the main concepts necessary for the next sections. In the third section we study the invariance of cones in $\bigwedge^k \mathbb{R}^d$. In Section 4 we have the main results of this paper, in special we present necessary and sufficient conditions for the existence of these invariant cones. In Section 5 we extend the previous results to convex set instead of cones. And finally, in the last section we present some examples where the fundamental conditions are verified using a computational implementation created by the authors.

2 Preliminaries

In this section we collect some concepts and facts about flag manifolds related with the $\mathrm{Sl}(d,\mathbb{R})$ actions. In particular we are interested in some dynamical properties originating from the action of semigroups $S \subset \mathrm{Sl}(d,\mathbb{R})$ on flag manifolds (for details see San Martin [12] and [13]).

Now we complement the previous introduction about flag manifolds and control sets. Recall that the flag manifolds \mathbb{F}_{Θ} are compact and the **minimal flag manifolds** are the Grassmannians $\mathbb{F}_{\Theta} = \mathbb{G}_k(d)$, where $\Theta = \{k\}$. A particular case, when k = 1, is the projective space $\mathbb{P}^{d-1} = \mathbb{G}_1(d)$.

From now on, in this section we discuss the special case $\mathbb{G}_k(d)$, $1 \leq k \leq d-1$. In this work it is convenient represent $\mathbb{G}_k(d)$ in the following algebraic way. Let $B_k(d)$ be the set of $d \times k$ matrices of rank k. Define in $B_k(d)$ the following equivalence relation: $p \sim q$ if exists $a \in \mathrm{Gl}(k, \mathbb{R})$ with q = pa. In other words, $p \sim q$ if, and only if, the columns of p and q generate the same subspace of \mathbb{R}^d . Then we can see $\mathbb{G}_k(d)$ as $B_k(d)/\sim$. Denote the elements of $\mathbb{G}_k(d)$ by [p]. There is a natural action ρ_k of the Lie group $\mathrm{Sl}(d, \mathbb{R})$ on $\mathbb{G}_k(d)$, which is given by $\rho_k(g, [p]) = [gp]$.

Now take an arbitrary basis \mathcal{B} of \mathbb{R}^d and $N_{\mathcal{B}}$ the nilpotent group of lower triangular matrices (with respect to \mathcal{B}) with ones on the main diagonal. The decomposition of $\mathbb{G}_k(d)$ into $N_{\mathcal{B}}$ -orbits is called **Bruhat decomposition** of $\mathbb{G}_k(d)$, moreover if we change the basis the decomposition also changes. There is just a finite number of these orbits, $N_{\mathcal{B}}[p]$ with $[p] \in \mathbb{G}_k(d)$. It is well known that exists only one open and dense orbit, $N_{\mathcal{B}}[p_0]$, where $[p_0]$ is the subspace spanned by the first k basic vectors (see [12]). We have that $N_{\mathcal{B}}[p_0]$ can be written as

$$\left[\begin{array}{c}I_k\\X\end{array}\right]$$

with I_k the $k \times k$ identity and X an arbitrary $(d-k) \times k$ matrix. Taking

$$\eta = \left[\begin{array}{cc} A_1 & 0\\ Y & A_2 \end{array} \right] \in N_{\mathcal{B}}$$

with A_1 and A_2 invertible, it follows that

$$\rho_k(\eta, [p_0]) = [\eta p_0] = \begin{bmatrix} A_1 & 0 \\ Y & A_2 \end{bmatrix} \begin{bmatrix} I_k \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ Y \end{bmatrix} = \begin{bmatrix} I_k \\ YA_1^{-1} \end{bmatrix}.$$

Note that this orbit is diffeomorphic to euclidean spaces.

Another important concept here is the **split regular** or just **regular** element, that is the $h \in Sl(d, \mathbb{R})$ with positive and distinct eigenvalues, where in some basis (denoted by $\mathcal{B}(h)$), $h = \text{diag}\{\lambda_1, \ldots, \lambda_d\}$ with $\lambda_1 > \cdots > \lambda_d > 0$. Considering the action on $\mathbb{G}_k(d)$, the fixed points for h are the subspaces spanned by k basic vectors. Moreover, these fixed points are hyperbolic and with respect to $\mathcal{B}(h)$, the stable manifolds are the $N_{\mathcal{B}}$ -orbits. One interesting dynamical property is that the stable manifold of the subspace $[p_0]$ is open and dense so that it is the unique attractor for h and $h^m[q] \to [p_0]$ for generic [q]. Now taking h^{-1} instead of h and reverting the order of the basis, it follows that h has also just one repeller, and it is the subspace spanned by the last k basic vectors $\{e_{d-k+1}, \ldots, e_d\}$.

We recall other dynamical facts. Let $S \subset \text{Sl}(d, \mathbb{R})$ be a semigroup with nonempty interior and denote by reg(S) the set of regular elements in intS. As before take C_k the S-invariant control set in $\mathbb{G}_k(d)$, its uniqueness implies that

$$C_k = \bigcap_{[p] \in \mathbb{G}_k(d)} \operatorname{cl}(S[p]).$$

According to the above comments, for $h \in \operatorname{reg}(S)$ we have that $b_{\{k\}}(h) = [p_0]$ and $C_k \subset N_{\mathcal{B}(h)}[p_0]$ if $k \in \Theta(S)$. The **set of transitivity** of an invariant control set C_k is the set C_k^0 of the fixed points which are the attractors for elements in $\operatorname{reg}(S)$ (see [10]). Specifically, we have that for any $[p] \in C_k^0$, there exists a basis $\mathcal{B}(h) = \{e_1, \ldots, e_d\}$ of \mathbb{R}^d and $h = \operatorname{diag}\{\lambda_1, \ldots, \lambda_d\}$ with $\lambda_1 > \cdots > \lambda_d > 0$ (in this basis), such that $h \in \operatorname{int} S$ and $[p] = \langle e_1, \ldots, e_k \rangle$, i.e., [p] is the attractor of h. From this fact it follows that the set of attractors of elements in $\operatorname{reg}(S)$ coincides with C_k^0 and this set is dense in C_k . Hence $\operatorname{reg}(S)$ is dense in $\operatorname{int} S$ and C_k is formed, in some sense, by attractors for these regular elements. This is a kind of converse to the fact that $[p] \in C_k$ if [p] is the attractor of a element $h \in \operatorname{reg}(S)$. Therefore C_k is contained in the open Bruhat component corresponding to $\mathcal{B}(h)$. Another interesting result in this context is that $C_k = \mathbb{G}_k(d)$ for some k if and only if S is transitive on $\mathbb{G}_k(d)$. On the other hand, we have that if S is a proper semigroup of $\mathrm{Sl}(d, \mathbb{R})$, then $C_k \neq \mathbb{G}_k(d)$ for any $k \in \{1, \ldots, d-1\}$ and S is not transitive on $\mathbb{G}_k(d)$ (see [14]).

We finish this section recalling some necessary facts about tensorial product and Grassmanianns.

For $k \in \{1, \ldots, d\}$, denote by $\bigwedge^k \mathbb{R}^d$ the k-fold exterior product of \mathbb{R}^d and let $\mathcal{F}_k(d)$ be the set of all k multi-index $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}$ with $1 \leq i_1 < \cdots < i_k \leq d$. It is well known that if we fix a basis $\mathcal{B} = \{e_1, \ldots, e_d\}$, then $\{e_I := e_{i_1} \land \cdots \land e_{i_k}; I = \{i_1, \ldots, i_k\} \in \mathcal{F}_k(d)\}$ is a basis of $\bigwedge^k \mathbb{R}^d$. Along the text, we use the notation \mathcal{D} to designate the set of all decomposable elements of $\bigwedge^k \mathbb{R}^d$, that is, the set of elements that can be written as $u_1 \wedge \cdots \wedge u_k$ with $u_i \in \mathbb{R}^d$.

The manifold $\mathbb{G}_k(d), k \in \{1, \ldots, d-1\}$, can be seen as a compact submanifold of the projective space $\mathbb{P}\left(\bigwedge^k \mathbb{R}^d\right)$ of $\bigwedge^k \mathbb{R}^d$ via Plücker embedding $\varphi : \mathbb{G}_k(d) \to$ $\mathbb{P}\left(\bigwedge^k \mathbb{R}^d\right), \varphi([p]) = [u_1 \land \cdots \land u_k]$, where $p = [u_1 \ldots u_k]$ is a $d \times k$ matrix and $[u_1 \land \cdots \land u_k] \in \mathbb{P}\left(\bigwedge^k \mathbb{R}^d\right)$ denotes the class of all non-zero multiples of $u_1 \land \cdots \land u_k \in \bigwedge^k \mathbb{R}^d$.

Identifying the Grassmaniann $\mathbb{G}_k(d)$ as a subset of $\mathbb{P}\left(\bigwedge^k \mathbb{R}^d\right)$, we can write the action ρ_k of $\mathrm{Sl}(d,\mathbb{R})$ on $\mathbb{G}_k(d)$ as

$$\rho_k(g, [u_1 \wedge \dots \wedge u_k]) = [gu_1 \wedge \dots \wedge gu_k]$$

and denote $\rho_k(g, [p])$ simply by g[p].

In the next sections $\pi : \left(\bigwedge^k \mathbb{R}^d\right) \setminus \{0\} \to \mathbb{P}\left(\bigwedge^k \mathbb{R}^d\right)$ represents the canonical projection.

3 Cones in *k*-fold exterior product

From now on we consider a connected semigroup $S \subset \text{Sl}(d, \mathbb{R})$ with nonempty interior. In this work a **cone** means a closed convex cone in a finite dimensional vector space V and if not otherwise specified the cones are proper and nontrivial. Remember that a cone W is **pointed** if $W \cap -W = \{0\}$ and **generating** if $\operatorname{int} W \neq \emptyset$. Our main interest is to study the S-invariance of this kind of cones in $\bigwedge^k \mathbb{R}^d$, with $1 \leq k \leq d-1$.

In this section we present some technical and useful results about cones. In particular, we show that any S-invariant cone $W \subset \bigwedge^k \mathbb{R}^d$ contains a decomposable element and it is pointed and generating (see the following Propositions 4 and 5). To obtain these results we need some lemmas.

Lemma 1 Let $F : V_1 \to V_2$ be an analytic map where V_1 and V_2 are finite dimensional vector spaces. Assume that for a nonempty open set $U \subset V_1$ there is a subspace $V \subset V_2$ such that $F(U) \subset V$. Then $F(V_1) \subset V$.

Proof. The canonical projection $p: V_2 \to V_2/V$ is linear and then analytic. Therefore, $p \circ F$ is an analytic function, and $p \circ F(U) = 0 + V$, since $p(x) \in V$ for all $x \in U$. Therefore, $p \circ F(x) = 0 + V$, for every $x \in V_1$, because the unique analytic map between finite dimensional vector spaces which vanishes on an open subset of the domain is the null map. Hence, $F(x) \in V$, for all $x \in V_1$.

Lemma 2 Let V be a d-dimensional vector space and take the cone $W \subset V$. Then W is generating if, and only if, W is not contained in any proper subspace of V. **Proof.** If W has nonempty interior, then W is not contained in a proper subspace of V. For the converse, observe that convex cones spanned by any basis of V have nonempty interior. In fact, let $\{e_1, \ldots, e_d\}$ be a basis of V. The convex cone spanned by this basis is the set

$$\left\{\sum_{i=1}^d \alpha_i e_i; \ \alpha_1, \dots, \alpha_d \ge 0\right\}.$$

Then the interior of this set is nonempty. Now, assuming that W is not contained in a proper subspace of V, we have that W contains a basis \mathcal{B} . Since W is a convex cone it follows that W also contains the convex cone spanned by \mathcal{B} . Therefore, $\operatorname{int} W \neq \emptyset$.

Lemma 3 If U is open in the set of decomposable elements \mathcal{D} (in the relative topology), then U contains a basis of $\bigwedge^k \mathbb{R}^d$.

Proof. First note that we can write the set of decomposable elements as the image of a polynomial function. In fact, let $F : (\mathbb{R}^d)^k \to \bigwedge^k \mathbb{R}^d$ be the map given by $F(u_1, u_2, \dots, u_k) = u_1 \land u_2 \land \dots \land u_k$. Clearly, $F((\mathbb{R}^d)^k) = \mathcal{D}$. On the other hand, F is polynomial due to the multi-linearity of the wedges, hence it is analytic.

Since F is continuous, the set $F^{-1}(U)$ is open in $(\mathbb{R}^d)^k$. In fact, since U is open in \mathcal{D} , there is an open $U' \subset \bigwedge^k \mathbb{R}^d$ with $U = U' \cap \mathcal{D}$. So $F^{-1}(U) = F^{-1}(U') \cap (\mathbb{R}^d)^k = F^{-1}(U')$.

Now, suppose that U does not contain a basis of $\bigwedge^k \mathbb{R}^d$, then S is contained in a proper subspace Z, so $F(F^{-1}(U)) \subset U \subset Z$, and by Lemma 1, $F((\mathbb{R}^d)^k) \subset Z$. Hence \mathcal{D} is contained in a proper subspace of $\bigwedge^k \mathbb{R}^d$. This is a contradiction, because \mathcal{D} spans $\bigwedge^k \mathbb{R}^d$. Therefore U contains a basis of $\bigwedge^k \mathbb{R}^d$.

In the next two propositions we consider the representation δ : $\mathrm{Sl}(d,\mathbb{R}) \to \mathrm{Gl}(\bigwedge^k \mathbb{R}^d)$ where

$$\delta(g)(u_1 \wedge \cdots \wedge u_k) := gu_1 \wedge \cdots \wedge gu_k.$$

To abbreviate, we denote $\delta(g)$ simply by g.

Now we can prove that an invariant cone contains a decomposable element.

Proposition 4 Take $S \subset Sl(d, \mathbb{R})$ a semigroup with nonempty interior. Let $\{0\} \neq W \subset \bigwedge^k \mathbb{R}^d$ be an S-invariant cone. Then W intercepts a non-null decomposable element of $\bigwedge^k \mathbb{R}^d$.

Proof. Take $h \in \operatorname{reg}(S)$ and consider as before the basis $\mathcal{B} = \{e_1, \dots, e_d\}$ of \mathbb{R}^d such that $h = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ with $\lambda_1 > \dots > \lambda_d > 0$. Note that for $I = \{i_1, \dots, i_k\} \in \mathcal{F}_k(d)$, the vectors $e_I = e_{i_1} \wedge \dots \wedge e_{i_k} \in \bigwedge^k \mathbb{R}^d$ are eigenvectors of h, with eigenvalues $\lambda_{i_1} \cdots \lambda_{i_k}$. Moreover, they form a basis of $\bigwedge^k \mathbb{R}^d$.

Define the following order relation on $\mathcal{F}_k(d)$: given $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$ in $\mathcal{F}_k(d)$,

$$I \prec J$$
 if $\lambda_{i_1} \cdots \lambda_{i_k} < \lambda_{j_1} \cdots \lambda_{j_k}$.

If necessary, we take a perturbation of $h = \text{diag}(\lambda_1, \ldots, \lambda_d) \in \text{int}S$ such that \prec become a total order. Consider $0 \neq v \in W$ with $v = \sum_{I \in \mathcal{F}_k(d)} \alpha_I e_I$ and define

$$J_0 = \{j_1, \ldots, j_k\} = \max\{I \in \mathcal{F}_k(d); \ \alpha_I \neq 0\}.$$

As W is S-invariant, we have that W is invariant under h. Hence,

$$\left(\frac{h^m(v)}{(\lambda_{j_1}\cdots\lambda_{j_k})^m}\right)_{m\in\mathbb{N}}$$

is a sequence in W and

$$\frac{h^m(v)}{(\lambda_{j_1}\cdots\lambda_{j_k})^m} = \sum_{I\in\mathcal{F}_k(d)} \alpha_I \frac{h^m(e_I)}{(\lambda_{j_1}\cdots\lambda_{j_k})^m} = \sum_{I\in\mathcal{F}_k(d)} \alpha_I \frac{(\lambda_{i_1}\cdots\lambda_{i_k})^m}{(\lambda_{j_1}\cdots\lambda_{j_k})^m} e_I,$$

for all $m \in \mathbb{N}$. Note that if $I = \{i_1, \dots, i_k\} \notin \{I \in \mathcal{F}_k(d); \alpha_I \neq 0\}$, then $\alpha_I \frac{(\lambda_{i_1} \cdots \lambda_{i_k})^m}{(\lambda_{j_1} \cdots \lambda_{j_k})^m} = 0$, for all $m \in \mathbb{N}$. Moreover $\lambda_{j_1} \cdots \lambda_{j_k} > \lambda_{i_1} \cdots \lambda_{i_k}$ for all $\{i_1, \dots, i_k\}$ in $\{I \in \mathcal{F}_k(d); \alpha_I \neq 0\} \setminus \{J_0\}$. Hence,

$$\lim_{m \to \infty} \frac{(\lambda_{i_1} \cdots \lambda_{i_k})^m}{(\lambda_{j_1} \cdots \lambda_{j_k})^m} = 0.$$

Therefore

$$\lim_{m \to \infty} \frac{h^m(v)}{(\lambda_{j_1} \cdots \lambda_{j_k})^m} = \lim_{m \to \infty} \sum_{I \in \mathcal{F}_k(d)} \alpha_I \frac{(\lambda_{i_1} \cdots \lambda_{i_k})^m}{(\lambda_{j_1} \cdots \lambda_{j_k})^m} e_I = \alpha_{J_0} e_{J_0}$$

The closeness of W implies that the decomposable element $\alpha_{J_0} e_{J_0}$ belongs to W, and moreover, this element is non-null.

Hence we have the main result of this section.

Proposition 5 Let $S \subset \text{Sl}(d, \mathbb{R})$ be a semigroup with non empty interior. If $\{0\} \neq W \subset \bigwedge^k \mathbb{R}^d$ is a S-invariant cone, then W is pointed and generating.

Proof. First recall that the representation of $\mathrm{Sl}(d, \mathbb{R})$ on $\bigwedge^k \mathbb{R}^d$ is irreducible. Now, define $H = W \cap -W$. Then H is an S-invariant vector subspace.

We have also that H is S^{-1} -invariant, because if $g \in S$, then $gH \subset H$. Since g is invertible, gH is a subspace of H with $\dim gH = \dim H$, i.e., gH = H. Consequently, $H = g^{-1}H$. The fact that $\operatorname{int} S \neq \emptyset$ implies that $\operatorname{Sl}(d, \mathbb{R})$ is generated by $S \cup S^{-1}$. Hence H is $\operatorname{Sl}(d, \mathbb{R})$ -invariant, now knowing that W is proper and $\operatorname{Sl}(d, \mathbb{R})$ is irreducible we have that $H = \{0\}$. Hence W is pointed.

Finally, assume that $\operatorname{int} W = \emptyset$. By Lemma 2, $W \cup -W$ is contained in a proper subspace V of $\bigwedge^k \mathbb{R}^d$. Consider a decomposable element $x \in W$ and take $\rho_k^q : \operatorname{Sl}(d, \mathbb{R}) \to \mathbb{G}_k(d)$ the open map $\rho_k^q(g) = [gq]$ where $[q] := \varphi^{-1}(\pi(x))$ and φ is the Plücker embedding defined in the second section. Then $\varphi(\rho_k^q(\operatorname{int} S))$ is open in $\varphi(\mathbb{G}_k(d))$, that is, there exists an open set $B \subset \mathbb{P}(\bigwedge^k \mathbb{R}^d)$ such that

$$\varphi(\rho_k^q(\text{int}S)) = B \cap \varphi(\mathbb{G}_k(d)).$$

Knowing that

$$\pi^{-1}(\varphi(\phi(\operatorname{int} S))) = \pi^{-1}(B \cap \varphi(\mathbb{G}_k(d))) = \pi^{-1}(B) \cap \mathcal{D},$$

we have that $\pi^{-1}(\varphi(\phi(\text{int}S)))$ is open in \mathcal{D} . By Lemma 3, $\pi^{-1}(\varphi(\phi(\text{int}S)))$ contains a basis of $\bigwedge^k \mathbb{R}^d$. Note also that

$$\pi^{-1}(\varphi(\phi(\text{int}S))) = \pi^{-1}(\varphi((\text{int}S)[q])) = \pi^{-1}(\pi((\text{int}S)x)) = \pi^{-1}((\text{int}S)\pi(x)).$$

So, if $y \in \pi^{-1}(\varphi(\phi(\operatorname{int} S)))$, then $\pi(y) \in (\operatorname{int} S)\pi(x)$, hence there is $g \in \operatorname{int} S$ with $\pi(y) = g\pi(x) = \pi(gx)$, that is, $y = \alpha gx$ for some $\alpha \neq 0$. If $\alpha > 0$, then $y \in \alpha Sx \subset \alpha W = W$ and if $\alpha < 0$, then $y \in \alpha Sx \subset \alpha W = -W$. Anyway $y \in W \cup -W$ and we conclude that $\pi^{-1}(\varphi(\phi(\operatorname{int} S))) \subset W \cup -W$. But it is a contradiction, because $\pi^{-1}(\varphi(\phi(\operatorname{int} S)))$ is contained in the proper subspace Vof $\bigwedge^k \mathbb{R}^d$ and contains a basis of $\bigwedge^k \mathbb{R}^d$. Therefore, $\operatorname{int} W \neq \emptyset$.

4 Cones, flag type and controllability

In this section we prove that there exists an S-invariant cone in $\bigwedge^k \mathbb{R}^d$ if and only if the flag type of S contains k. Consequently we have the main result of this section, Theorem 10, that gives a necessary and sufficient condition for the equality $S = Sl(d, \mathbb{R})$ in terms of the existence of S-invariant cones in the spaces $\bigwedge^k \mathbb{R}^d$, $k \in \{1, \ldots, d-1\}$. As application, we determine necessary and sufficient conditions for the controllability of a bilinear control system.

Theorem 6 Let $S \subset \text{Sl}(d, \mathbb{R})$ be a connected semigroup with flag type given by $\Theta(S)$. If $k \in \Theta(S)$, then there exists an S-invariant cone $\{0\} \neq W \subset \bigwedge^k \mathbb{R}^d$.

Proof. Take $h \in \operatorname{reg}(S)$ and consider $\mathcal{B}(h) = \{e_1, \ldots, e_d\}$ the special basis of \mathbb{R}^d . We saw in the second section that $b_{\{k\}}(h) = (\operatorname{span}\{e_1, \ldots, e_k\})$ and the orbit

$$N_{\mathcal{B}(h)}b_{\{k\}}(h) = \left\{ \left[\begin{array}{c} I_k \\ X \end{array} \right]; \ X \in \mathbb{R}^{(d-k) \times k} \right\}$$

contains C_k . Note that $\varphi(N_{\mathcal{B}(h)}b_{\{k\}}(h)) \subset \pi(M)$, where M is the affine subspace

$$M = \left\{ (1, x_2, \cdots, x_{\binom{d}{k}}); \ x_2, \cdots, x_{\binom{d}{k}} \in \mathbb{R} \right\} \subset \bigwedge^k \mathbb{R}^d$$

in the basis $\{e_I; I \in \mathcal{F}_k(d)\}$. Since the invariant control set $C_k \subset \mathbb{G}_k(d)$ is contained in $N_{\mathcal{B}(h)}b_{\{k\}}(h)$, we have

$$\varphi(C_k) \subset \varphi(N_{\mathcal{B}(h)}b_{\{k\}}(h)) \subset \pi(M).$$

Define $M_1 := \pi^{-1}(\varphi(C_k)) \cap M$. Let W be the cone generated by M_1 , W is clearly non-null.

Now, we show that W is S-invariant. Since C_k is S-invariant, it follows that $\varphi(C_k)$ is S-invariant. We claim that $(\mathbb{R}\setminus\{0\})M_1$ is S-invariant. In fact, given

 $\alpha \in \mathbb{R} \setminus \{0\}, u_1 \wedge \cdots \wedge u_k \in M_1 \text{ and } g \in S$, we have that $\pi(g(\alpha \ u_1 \wedge \cdots \wedge u_k)) = \pi(g(u_1 \wedge \cdots \wedge u_k))$ is contained in $\pi(gM_1) = g\pi(M_1) = g\varphi(C_k) \subset \varphi(C_k)$, due to the equality $\pi(M_1) = \varphi(C_k)$ and the S-invariance of $\varphi(C_k)$. Hence knowing that $\pi|_M$ is injective, we conclude the claim. As S is connected this implies that $C_k, \varphi(C_k)$ and M_1 are connected.

Furthermore, since for every $x \in \left(\bigwedge^k \mathbb{R}^d\right) \setminus \{0\}$ the mapping $g \in S \mapsto gx \in \bigwedge^k \mathbb{R}^d$ is continuous, we conclude that S leaves invariant the connected com-

ponents of $(\mathbb{R}\setminus\{0\})M_1$. As $(\mathbb{R}^+)M_1$ is one of these components, $(\mathbb{R}^+)M_1$ is invariant, implying that its convex closure W is S-invariant.

Remark 7 Our result generalizes Theorem 4.2 in [3] and also improves its hypotheses in the sense that we do not need to have the identity in clS. In [3] the authors assume $1 \in S$ to guarantee that S leaves invariant the connected components of $(\mathbb{R}\setminus\{0\})M_1$, but we can show that this is not necessary. In fact, let $g \in S$, then g leaves $(\mathbb{R}\setminus\{0\})M_1$ invariant. So g is a bijection between the connected components of $(\mathbb{R}\setminus\{0\})M_1$. Denote by $M_1^+ = (\mathbb{R}^+)M_1$ and $M_1^- =$ $(\mathbb{R}^-)M_1$ these connected components. Suppose that there is an element $g \in S$ which does not leave M_1^+ invariant. Then $g(M_1^+) = M_1^-$ and $g(M_1^-) = M_1^+$. Hence we have another element in S, g^2 , that leaves invariant the components, but this contradicts the connectedness of S.

We also note that by Proposition 5, the cone W, in the above theorem is pointed and generating.

The following results prove that the existence of a pointed invariant cone in $\bigwedge^k \mathbb{R}^d$ implies that the flag type of the semigroup contains k.

Lemma 8 Assume that $k \notin \Theta(S)$. Let C_k be the invariant control set for the action of S on $\mathbb{G}_k(d)$. Then there is a two-dimensional subspace $V \subset \bigwedge^k \mathbb{R}^d$ such that $\pi(V) \subset \varphi(C_k)$.

Proof. Denote by $\pi_k : \mathbb{F} \to \mathbb{G}_k(d)$ the natural projection and consider $[p] \in C_k$. Let f be an element of the invariant control set C of the full flag \mathbb{F} with $\pi_k(f) = [p]$. Such element exists because $C_k = \pi_k(C)$. Let $\Theta(S) = \{r_1, \ldots, r_n\}$ be the flag type of S and observe that $\pi_{\Theta(S)}^{-1}(\pi_{\Theta(S)}(f))$ is a subset of C, where $\pi_{\Theta(S)} : \mathbb{F} \to \mathbb{F}_{\Theta(S)}$. Therefore, $\pi_k(\pi_{\Theta(S)}^{-1}(\pi_{\Theta(S)}(f))) \subset C_k$. Since $k \notin \Theta(S)$ then $\pi_{\Theta(S)}(f) = (V_1 \subset \cdots \subset V_n)$ with $\dim V_i = r_i, 1 \leq i \leq n$. We have the following cases:

Case 1: Assume that $r_1 < k < r_n$. In this case, there exists $l \in \{1, \ldots, n-1\}$ such that the elements of $\pi_k(\pi_{\Theta(S)}^{-1}(\pi_{\Theta(S)}(f)))$ are the k-subspaces that contain V_l and are contained in V_{l+1} . Let $\{v_1, \cdots, v_{r_l}\}$ be a basis of V_l , and complete it to an ordered basis $\{v_1, \cdots, v_{r_l}, v_{r_l+1}, \cdots, v_{r_{l+1}}\}$ of V_{l+1} . Since $r_l < k$ and $r_{l+1} > k$, consider the element v_k in this basis of V_{l+1} and, moreover, there is a basic element v_j with $k < j \le r_{l+1}$. In this way, define the subspace

$$V = \{v_1 \wedge \cdots \wedge v_{r_k} \wedge \cdots \wedge v_{k-1} \wedge (\alpha v_k + \beta v_j); \ \alpha, \beta \in \mathbb{R}\}.$$

Case 2: Now, suppose that $k < r_1$. Here, the elements of $\pi_k(\pi_{\Theta(S)}^{-1}(\pi_{\Theta(S)}(f)))$ are the k-subspaces contained in V_1 . Since $k \ge 1$, then $r_1 \ge 2$. Hence, given an ordered basis $\{v_1, \dots, v_{r_1}\}$ of V_1 , we can find $v_k, v_j \in \{v_1, \dots, v_{r_1}\}$ where j satisfies $k < j \le r_1$. Consider the subspace

$$V = \{ v_1 \wedge \dots \wedge v_{k-1} \wedge (\alpha v_k + \beta v_j); \ \alpha, \beta \in \mathbb{R} \}.$$
⁽²⁾

Case 3: Finally, assume $k > r_n$. Hence, $\pi_k(\pi_{\Theta(S)}^{-1}(\pi_{\Theta(S)}(f)))$ is the set formed by the k-subspaces which contains V_n . Since $k \leq d-1$, we can consider a basis $\{v_1, \ldots, v_{r_n}\}$ of V_{r_n} and complete it to obtain the ordered basis $\{v_1, \ldots, v_{r_n}, v_{r_n+1}, \ldots, v_d\}$ of \mathbb{R}^d . In this case, we can also take v_k and v_j in this basis, with $k < j \leq d$ and consider the subspace defined as in (2).

In the three cases, the subspace $V \subset \bigwedge^k \mathbb{R}^d$ is two-dimensional and satisfies

$$\pi(V) \subset \varphi(\pi_k(\pi_{\Theta(S)}^{-1}(\pi_{\Theta(S)}(f)))) \subset \varphi(C_k).$$

The following theorem is a reciprocal of Theorem 6.

Theorem 9 If $\{0\} \neq W \subset \bigwedge^k \mathbb{R}^d$ is an S-invariant cone, then $k \in \Theta(S)$.

Proof. Assume that $k \notin \Theta(S)$ and denote by L the intersection of W with the set \mathcal{D} of the decomposable elements of $\bigwedge^k \mathbb{R}^d$. By Proposition 4 we have that L is nonempty. Moreover, L is S-invariant, since the set of decomposable elements is also S-invariant. Therefore, $\varphi^{-1}(\pi(L))$ is also invariant. As W is a closed set then L is closed in \mathcal{D} and hence $\varphi^{-1}(\pi(L))$ is a closed set in $\mathbb{G}_k(d)$. Since $\mathbb{G}_k(d)$ is compact, $\varphi^{-1}(\pi(L))$ is also compact, then there is an invariant control set contained in $\varphi^{-1}(\pi(L))$. But there is only one invariant control set $C_k \subset \mathbb{G}_k(d)$ implying that $C_k \subset \varphi^{-1}(\pi(L))$ and hence $\pi^{-1}(\varphi(C_k)) \subset L \subset W$. As proved in Lemma 8, there is a two-dimensional subspace V such that $\pi(V) \subset \varphi(C_k)$. But this means that $V \subset \pi^{-1}(\varphi(C_k)) \subset W$, which is a contradiction because W is pointed (see Proposition 5).

Recall that if $S \subset \operatorname{Sl}(d, \mathbb{R})$ is a nonempty semigroup, then S is transitive on $\mathbb{R}^d \setminus \{0\}$ if and only if $S = \operatorname{Sl}(d, \mathbb{R})$ (see [3]). In this context, the next theorem gives a necessary and sufficient condition in terms of the existence of invariant cones.

Theorem 10 Let $S \subset Sl(d, \mathbb{R})$ be a semigroup with nonempty interior. Then $S = Sl(d, \mathbb{R})$ if and only if there are no S-invariant cones in $\bigwedge^k \mathbb{R}^d$, for all $k \in \{1, \ldots, d-1\}$.

Proof. Let $W \subset \bigwedge^k \mathbb{R}^d$ be a proper S-invariant cone, for some $k \in \{1, \ldots, d-1\}$. Note that W does not contain \mathcal{D} , otherwise the convexity of W would imply that the convex closure of \mathcal{D} , $\bigwedge^k \mathbb{R}^d$, would be contained in W, which would contradicts the fact that W is proper.

By Proposition 4 we can consider an element $v_1 \in W \cap \mathcal{D}$. Take $v_2 \in \mathcal{D} \setminus W$. If $S = \mathrm{Sl}(d, \mathbb{R})$ and knowing that \mathcal{D} is S-invariant then there exists $g \in S$ such that $gv_1 = v_2 \notin W$, but this contradicts the S-invariance of W. Hence $S \neq \mathrm{Sl}(d, \mathbb{R})$.

On the other hand, assume that $S \subset \text{Sl}(d, \mathbb{R})$ is proper. Then $\Theta(S) \neq \emptyset$, hence there exists $k \in \Theta(S)$, for some $k \in \{1, \ldots, d-1\}$. Therefore, Theorem 6 implies the existence of a such cone.

Remark 11 This theorem complement and improve Section 7 of [3].

The next example shows that, as we commented before, the connectedness of S is fundamental in the previous results.

Example 12 Let $S^+ \subset Sl(2, \mathbb{R})$ be the set of matrices with positive entries. It is not difficult to show that S^+ is a proper semigroup with nonempty interior in $Sl(2, \mathbb{R})$, the positive orthant $Q^+ = \{(a, b) \in \mathbb{R}; a, b \ge 0\}$ is S^+ -invariant and S^+ is a open set. Now take the following proper semigroup

$$S = S^+ \cup (-S^+) = (-1)^{\mathbb{Z}} S^+ = \{(-1)^k A; k \in \mathbb{Z}, A \in S^+\}.$$

Note that S has nonempty interior. Moreover, S is not transitive on \mathbb{R}^2 because it leaves invariant the double cone $Q^+ \cup -Q^+ = (-1)^{\mathbb{N}}Q^+$:

$$S((-1)^{\mathbb{N}}Q^{+}) = (-1)^{\mathbb{N}}S^{+}(-1)^{\mathbb{N}}Q^{+} = (-1)^{\mathbb{N}+\mathbb{N}}S^{+}Q^{+} = (-1)^{\mathbb{N}}Q^{+}.$$

However, S does not leave invariant proper cones in $\mathbb{R}^2 = \bigwedge^1 \mathbb{R}^2$. In fact, we have that $-I \in S$, therefore, if C is a proper S invariant cone then $-I(C) = -C \subset C$. This implies that C is a subspace, which is a contradiction.

As a consequence of the above results, we get a necessary and sufficient condition for controllability of

$$\dot{x} = Ax + uBx, x \in \mathbb{R}^d \setminus \{0\}, u \in \mathbb{R},$$

with $A, B \in \mathfrak{sl}(d, \mathbb{R})$.

Recall that the system semigroup

$$S = \{ e^{t_1(A+u_1B)} \cdots e^{t_n(A+u_nB)}; \ t_1, \dots, t_n \ge 0, \ u_1, \dots, u_n \in \mathbb{R}, \ n \in \mathbb{N} \}$$

is a semigroup of $\mathrm{Sl}(d, \mathbb{R})$. Moreover, if the Lie algebra, generated by A and B, coincides with $\mathfrak{sl}(d, \mathbb{R})$, then $\mathrm{int} S \neq \emptyset$. Furthermore, S is path connected. It is well know that this system is controllable if, and only if, $S = \mathrm{Sl}(d, \mathbb{R})$ (see e.g. [3]). Hence, as a result of Theorem 10 we have the necessary and sufficient condition for controllability of this bilinear system.

Theorem 13 The above system is controllable if and only if it does not leave invariant a cone in $\bigwedge^k \mathbb{R}^d$, for all $k \in \{1, \ldots, d-1\}$.

5 Flag type and invariance of convex sets

In this section, we generalize the previous one. Or rather, instead of proper cones, we study the existence of proper convex sets in $\bigwedge^k \mathbb{R}$ which are invariant by the action of a semigroup $S \subset \text{Sl}(d, \mathbb{R})$. We also relate the existence of this convex sets with the flag type $\Theta(S)$ of S.

Initially, given $h \in \operatorname{reg}(S)$, take as before the basis $\mathcal{B}(h) = \{e_1, \ldots, e_d\}$ of \mathbb{R}^d . Since $1 = \det(h) = \lambda_1 \cdots \lambda_d$, then for all $k \in \{1, \ldots, d-1\}$, we can prove that $\lambda_1 \cdots \lambda_k > 1$.

The following lemma gives an expression for the closed convex cone generated by a convex set in $\bigwedge^k \mathbb{R}^d$.

Lemma 14 If the set $K \subset \bigwedge^k \mathbb{R}^d$ is convex, then the closed convex cone W generated by K is

$$W := \operatorname{cl}(\bigcup_{\alpha > 0} \alpha K).$$

Proof. Let $\{W_l\}_{l \in \Lambda}$ be the family of all closed cones that contains K and consider $V := \bigcap W_l$ the closed convex cone generated by K.

Note that W is a closed cone which contains K. To show that W is convex, take $x, y \in W$. There are sequences $(\gamma_n x_n), (\delta_n y_n)$ in $\bigcup_{\alpha>0} \alpha K$ with $\gamma_n, \delta_n > 0$ and $x_n, y_n \in K$ (for all $n \in \mathbb{N}$) converging to x and y respectively. Take $t \in [0, 1]$ and define

$$z_n = \left(\frac{(1-t)\gamma_n}{(1-t)\gamma_n + t\delta_n}\right) x_n + \left(\frac{t\delta_n}{(1-t)\gamma_n + t\delta_n}\right) y_n, \ n \in \mathbb{N}.$$

Note that (z_n) is a sequence in K, then $(((1-t)\gamma_n + t\delta_n) z_n)$ is a sequence in $\bigcup_{\alpha>0} \alpha K$, since $(1-t)\gamma_n + t\delta_n > 0$. But $((1-t)\gamma_n + t\delta_n) z_n = (1-t)\gamma_n x_n + t\delta_n y_n$ converges to (1-t)x + ty hence $(1-t)x + ty \in W$. Therefore $V \subset W$.

converges to (1-t)x + ty, hence $(1-t)x + ty \in W$. Therefore $V \subset W$.

On the other hand, for each $\gamma > 0$ we have $\gamma K \subset W_l$, for all $l \in \Lambda$, then $\bigcup_{\gamma > 0} \gamma K \subset W_l$, for all $l \in \Lambda$. Hence the closeness of each W_l implies that $W \subset W_l$, for all $l \in \Lambda$, so $W \subset V$.

Proposition 15 Let $K \subset \bigwedge^k \mathbb{R}^d$ be a proper S-invariant convex set. Then the closed cone generated by K is S-invariant.

Proof. Denote by W the closed cone generated by K. Since K is S-invariant, for each $g \in S$ it holds that $gK \subset K$. Hence

$$gW = g\left(\operatorname{cl}(\bigcup_{\alpha>0} \alpha K)\right) \subset \operatorname{cl}(g\left(\bigcup_{\alpha>0} \alpha K\right)) = \operatorname{cl}(\bigcup_{\alpha>0} \alpha gK) \subset \operatorname{cl}(\bigcup_{\alpha>0} \alpha K) = W,$$

that is, W is S-invariant.

Proposition 16 If $K \subset \bigwedge^k \mathbb{R}^d$ is a proper S-invariant convex set, then $0 \notin intK$.

Proof. For each $h \in \operatorname{reg}(S)$ denote by $b_k(h)$ the attractor of h in G_k . The set of transitivity of C_k , C_k^0 , satisfies

$$C_k^0 = \{b_k(h); h \in \operatorname{reg}(S)\},\$$

and has nonempty interior. In particular, there is an open set $V \,\subset\, C_k^0 = \{b_k(h); h \in \operatorname{reg}(S)\}$. As a consequence, $\phi(V)$ is an open set in \mathcal{D} , and therefore, by Lemma 3, $\pi^{-1}(\phi(V))$ contains a basis $\{b_1, b_2, \ldots, b_n\}$ of the exterior space. Since $b_i \in \pi^{-1}(\phi(V))$ and V is a subset of $C_k^0 = \{b_k(h); h \in \operatorname{reg}(S)\}$, then, for each b_i exists $h_i \in \operatorname{reg}(S)$ such that $b_i \in \pi^{-1}(\phi(b_k(h_i)))$ or, equivalently, there is a basis $\{e_1(h_i), e_2(h_i), \ldots, e_d(h_i)\}$ of \mathbb{R}^d where h_i is written as diag $(\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{di})$ and $b_i = e_1(h_i) \wedge e_2(h_i) \wedge \cdots \wedge e_k(h_i) = e_I(h_i)$ with $I = \{1, \ldots, k\}$. So, if we suppose that $0 \in \operatorname{int} K$, then there are $\alpha \neq 0$ and $h_1, \ldots, h_r \in \operatorname{reg}(S)$ with $r = \binom{d}{k}$, such that $\alpha e_I(h_i)$ is a basis of $\bigwedge^k \mathbb{R}^d$ with $\pm \alpha e_I(h_i)$ contained in $\operatorname{int} K$, $i = 1, \ldots, r$.

But for all $m \in \mathbb{N}$ and $i \in \{1, \ldots, r\}$ we have $h_i^m(\pm \alpha e_I(h_i)) \in K$ due to S-invariance of K. Moreover,

$$\|h_i^m(\pm \alpha e_I(h_i))\| = |\alpha|(\lambda_{1i}\cdots\lambda_{ki})^m\|e_I(h_i)\| \to +\infty,$$

then the convexity of K implies that $K = \bigwedge^k \mathbb{R}^d$.

The above proposition has the following consequence.

Corollary 17 Let $K \subset \bigwedge^k \mathbb{R}^d$ be an S-invariant convex set and denote by W the closed cone generated by K. The following statements are equivalents:

- *i)* W *is proper;*
- ii) K is proper.
- *iii*) $0 \notin \text{int}K$.

Proof. The implication $(i) \Rightarrow (ii)$ holds because $K \subset W$. Moreover, $(ii) \Rightarrow (iii)$ follows by Proposition 16. Finally, to prove that $(iii) \Rightarrow (i)$ we first note that if $W = \bigwedge^k \mathbb{R}^d$ then $\operatorname{int} K \neq \emptyset$. In fact, if $\operatorname{int} K = \emptyset$ then K is contained in a proper affine subspace $V + u_0$, where $V \subset \bigwedge^k \mathbb{R}^d$ is a proper vector subspace and $u_0 \in \bigwedge^k \mathbb{R}^d$. Hence

$$\bigwedge^{k} \mathbb{R}^{d} = W = \operatorname{cl}(\bigcup_{\alpha > 0} \alpha K) \subset \operatorname{cl}(\bigcup_{\alpha > 0} \alpha (V + u_{0})) = \operatorname{cl}(\bigcup_{\alpha > 0} (V + \alpha u_{0}))$$
$$= V + [0, +\infty)u_{0} \not\subseteq \bigwedge^{k} \mathbb{R}^{d}.$$

which is a contradiction. Hence, given the open set -intK, there are $\alpha > 0$ and $k \in K$ with $\alpha k \in -intK$, that is, $-\alpha k \in intK$. Since K is convex, the line $[-\alpha k, k) := \{(t-1)\alpha k + tk; t \in [0,1)\}$ is contained in int*K*, therefore $0 \in \text{int}K$.

The next result presents a synthesis of this section, the relation among invariant convex set, invariant cone and flag type.

Theorem 18 Let $S \subset Sl(d, \mathbb{R})$ a semigroup with nonempty interior. Then the following statements are equivalents:

- i) There exists an S-invariant proper convex set in $\bigwedge^k \mathbb{R}^d$;
- ii) There exists an S-invariant proper closed cone in $\bigwedge^k \mathbb{R}^d$;
- *iii*) $k \in \Theta(S)$.

Proof. By Proposition 15 and Corollary 17 we have that $(i) \Rightarrow (ii)$. By Theorem 9 it follows that (ii) implies (iii). Moreover, since a cone is a convex set, the implication $(iii) \Rightarrow (i)$ follows by Theorem 6.

6 Examples

In order to present examples to illustrate our results, we create a computational implementation in Julia Language [1] called LieAlgebraRankCondition.jl¹. The basic idea of this implementation is the following: given the bilinear control system

$$\dot{x} = Ax + uBx, x \in \mathbb{R}^4 \setminus \{0\}, u \in \mathbb{R} \text{ and } A, B \in \mathfrak{sl}(4, \mathbb{R})$$

put the Lie brackets in a convenient way and analyse all the possibilities until get, if possible, a linearly independent (L.I.) set for $\mathfrak{sl}(4,\mathbb{R})$. In the following we

¹Available in https://github.com/evcastelani/LieAlgebraRankCondition.jl

describe a conceptual algorithm.

Algorithm 1: Lie Algebra Rank Condition Algorithm. **Data:** A: Array, B: Array, dim: dimension of $\mathfrak{sl}(4,\mathbb{R})$ Result: True: a set of L. I. arrays were found; False: Does not exists an L. I. set of arrays. $C \leftarrow \{A, B, [A, B]\};$ if C is L. I. then $k \leftarrow 3;$ else return False; \mathbf{end} while $k \leq \dim \operatorname{do}$ $j \leftarrow k-1;$ $C_{trial} \leftarrow C_j;$ while $(C \cup [C_{trial}, C_k]$ is not L.I) and (j > 3) do $j \leftarrow j - 1$; $C_{trial} \leftarrow C_j;$ $\quad \text{end} \quad$ if j=3 then remove C_k from C; $k \leftarrow k - 1;$ else add $[C_{trial}, C_k]$ to C; $k \leftarrow k+1;$ end if k=3 then return False; end end return True;

Remark 19 The parameter dim can be changed in order to find solutions for higher order spaces.

Example 20 Consider the bilinear system

 $(\Sigma) \quad \dot{x} = Ax + uBx, \text{ with } x \in \mathbb{R}^4 \setminus \{0\}, u \in \mathbb{R},$ $A = \begin{bmatrix} 0 & 2 & 0 & -1\\ 2 & 0 & 2 & 0\\ 0 & 2 & 0 & 2\\ -1 & 0 & 2 & 0 \end{bmatrix} \text{ and } B = \text{diag}(4, 1, -2, -3) \in \mathfrak{sl}(4, \mathbb{R}).$

The matrix A has the distinct eigenvalues, 3, 2, -2, -3, with the following eigenvectors $v_1 = (1, 2, 2, 1)$, $v_2 = (-2, -1, 1, 2)$, $v_3 = (2, -1, -1, 2)$ and $v_4 = (-1, 2, -2, 1)$, respectively. Let S be the semigroup of (Σ) , that is,

$$S = \{ e^{t_1(A+u_1B)} \cdots e^{t_k(A+u_nB)}; t_1, \dots, t_n \ge 0, \ n \in \mathbb{N} \}.$$

Using the implementation of Algorithm 1, we can show that this system satisfies the Lie algebra rank condition, hence S has nonempty interior in $Sl(4, \mathbb{R})$. Moreover, S is a proper semigroup. In fact, by [13, Proposition 2], we have

$$A + uB \in \mathcal{L}(S_2) = \{ X \in \mathfrak{sl}(4, \mathbb{R}); \exp(X) \in S_2 \},\$$

where $S_2 = \{g \in Sl(4,\mathbb{R}); g\mathcal{O}_2 \subset \mathcal{O}_2\}$ is the the compression semigroup of

the positive orthant
$$\mathcal{O}_2 = \left\{ \sum_{I = \{i_1 < i_2\} \subset \{1, 2, 3, 4\}} \alpha_I e_I; \ \alpha_I \ge 0 \right\} \subset \bigwedge^2 \mathbb{R}^4.$$
 This

semigroup coincides with the set of all matrix in $Sl(4,\mathbb{R})$ such that the minors of order 2 have non-negative determinant. Note that $S \subset S_2$. Since S_2 leaves invariant the cone \mathcal{O}_2 , then $S\mathcal{O}_2 \subset \mathcal{O}_2$. Hence (Σ) is not controllable and therefore S is proper, in particular S leaves invariant the positive orthant of $\bigwedge^2 \mathbb{R}^4$.

On the other hand, neither $\pm A$ nor $\pm (A + uB)$ leave invariant an orthant of \mathbb{R}^4 . In fact, by [8, Lemma 1], a matrix $X = (x_{ij})$ leaves invariant the orthant with signs $(\sigma_1, \ldots, \sigma_d)$ if and only if $\sigma_i \sigma_j x_{ij} > 0$. Applying this condition to $\pm A, \pm (A + uB)$, we get the contradictory fact that $\sigma_1 \sigma_4$ must be simultaneously 1 and -1, so that there are no invariant orthants in $\mathbb{R}^4 = \bigwedge^1 \mathbb{R}^4$. The system (Σ) is a counter-example for the following conjecture proposed by Sachkov in [8]. Consider a bilinear control system with A symmetric and $B = \text{diag}(b_1, \ldots, b_n)$ where $b_i \neq b_j$ for $i \neq j$. Is it true that if this system has no invariant orthants and everywhere satisfies the necessary Lie algebra rank controllability condition, then it is controllable in $\mathbb{R}^d \setminus \{0\}$?

Now we prove that, although (Σ) is not controllable, there are no S-invariant cones in $\mathbb{R}^4 = \bigwedge^1 \mathbb{R}^4$ neither in $\bigwedge^3 \mathbb{R}^4$. Suppose that $W \subset \mathbb{R}^4$ is an S-invariant cone. Then W has nonempty interior and it is not contained in the plane generated by $\{e_2, e_3, e_4\}$. Therefore there is a vector $w = (w_1, w_2, w_3, w_4) \in W$ such that $w_1 \neq 0$. Since $e^{tB} \in cl(S)$ for all $t \in \mathbb{R}$, then if $w_1 > 0$ we have that

$$\lim_{t \to +\infty} \frac{e^{tB}w}{\|e^{tB}w\|} = e_1 \in W.$$

If $w_1 < 0$ then

$$\lim_{t \to +\infty} \frac{e^{tB}w}{\|e^{tB}w\|} = -e_1 \in W.$$

Without loss of generality, assume that $e_1 \in W$. Knowing that v_1 is the attractor eigenvalue of A and considering the basis $\{v_1, v_2, v_3, v_4\}$, a similar argument assures that either $v_1 \in V$ or $-v_1 \in V$.

Let $H := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 < 0\}$, then, for all $x \in H$,

$$\lim_{t \to +\infty} \frac{e^{t(-B)}x}{||e^{t(-B)}x||} = -e_4$$

In particular, note that if $W \cap H \neq \emptyset$, then $-e_4 \in W$. Now we show that $W \cap H \neq \emptyset$. Since the inner product between Ae_1 and e_4 is negative, then the

curve $t \mapsto e^{tA}e_1$ intersects H for t > 0. By S-invariance and knowing that $e_1 \in W$, we have $e^{\mathbb{R}_+A}e_1 \subset W$, then $W \cap H \neq \emptyset$.

As stated early, either $v_1 \in W$ or $-v_1 \in W$. As v_1 has a positive fourth coordinate, then

$$\lim_{t \to +\infty} \frac{e^{t(-B)}v_1}{||e^{t(-B)}v_1||} = e_4,$$

and as $-v_1$ has a negative first coordinate, we have

$$\lim_{t \to +\infty} \frac{e^{tB}(-v_1)}{||e^{tB}(-v_1)||} = -e_1.$$

Hence if $v_1 \in W$ then $e_4 \in W$. But $-e_4$ is also in W, then W is not pointed. On the other hand, if $-v_1 \in W$, then $-e_1 \in W$. Analogously, since e_1 is also in W, then W is not pointed also in his case. Anyway W is not pointed, but this contradicts Proposition 5.

Since W is arbitrary, we conclude that (Σ) does not have invariant cones in $\mathbb{R}^4 = \bigwedge^1 \mathbb{R}$.

Now in the case of $\bigwedge^3 \mathbb{R}^4$, we recall that S has invariant cones in $\bigwedge^3 \mathbb{R}^4$ if, and only if, S^{-1} has invariant cones in \mathbb{R}^4 , and the linear isomorphism from \mathbb{R}^4 to $\bigwedge^3 \mathbb{R}^4$ (that preserves basis) is also a one to one correspondence between the respective invariant cones (see e.g. [13]).

Therefore, it is enough to prove that S^{-1} does not leave invariant cones in \mathbb{R}^4 . Since S is generated by the exponential of the elements of $\{A + uB; u \in \mathbb{R}\} \subset \mathfrak{sl}(4,\mathbb{R})$, then S^{-1} is generated by the exponential of the elements -A+uB with $u \in \mathbb{R}\}$. Then S^{-1} is also the semigroup of the bilinear control system $\dot{x} = -Ax + uBx$ with $x \in \mathbb{R}^4 \setminus \{0\}$ and $u \in \mathbb{R}$.

Let $W \neq \{0\}$ be an S^{-1} -invariant cone. Note that S^{-1} has nonempty interior in $Sl(4, \mathbb{R})$. Therefore, $e_1 \in W$ or $-e_1 \in W$. Without loss of generality, we assume $e_1 \in W$. Since the highest eigenvalue of -A is 3 and the corresponding eigenvector is v_4 , then $v_4 \in W$ or $-v_4 \in W$. Furthermore, the inner product between $-Ae_1$ and e_4 is positive, and, therefore, $e_4 \in W$. If $v_4 \in W$, then $\lim_{t \to +\infty} \frac{e^{tB}v_4}{||e^{tB}v_4||} = -e_1 \in W$ and W is not pointed, because $e_1, -e_1 \in W$. Other-

wise, if $-v_4 \in W$, then $\lim_{t \to +\infty} \frac{e^{t(-B)}(-v_4)}{||e^{t(-B)}(-v_4)||} = -e_4$ and W is still not pointed, because $e_4, -e_4 \in W$. Since W is not pointed in both cases, by Proposition 5

because $e_4, -e_4 \in W$. Since W is not pointed in both cases, by Proposition 5 we have a contradiction. We conclude that the proper semigroup S does not leave invariant a proper cone in $\bigwedge^1 \mathbb{R}^4$ neither in $\bigwedge^3 \mathbb{R}^4$ but S has an invariant cone in $\bigwedge^2 \mathbb{R}^4$ (in fact, we showed that it leaves invariant the positive orthant of that space). Then by Theorem 10, the system (Σ) is not controllable. Moreover, Theorem 9 implies that S has parabolic type $\Theta(S) = \{2\}$, in other words, $\mathbb{F}_{\Theta(S)} = \mathbb{G}_2(4)$. Example 21 Consider the above bilinear control system, but with

$$A := \left[\begin{array}{rrrr} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -1 \end{array} \right], \ B := \left[\begin{array}{rrrr} 2 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{10} & 0 \\ 0 & \frac{1}{10} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

and denote the system semigroup by S. Using again the implementation of Algorithm 1, we can see that S satisfies the Lie algebra rank condition, so $intS \neq$ \emptyset . Now we show that S does not have invariant cones in $\bigwedge^1 \mathbb{R}^4$, $\bigwedge^2 \mathbb{R}^4$ or $\bigwedge^3 \mathbb{R}^4$ and therefore $S = Sl(4, \mathbb{R})$.

First note that

$$e^{\frac{\pi}{2}A} = \begin{bmatrix} 0 & d & 0 & 0 \\ -d & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d}\frac{\sqrt{2}}{2} & \frac{1}{d}\frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{1}{d}\frac{\sqrt{2}}{2} & \frac{1}{d}\frac{\sqrt{2}}{2} \end{bmatrix}$$

with $e^{\frac{\pi}{2}} = d$.

th $e^{\frac{\pi}{2}} = d$. Now we compute $e^{\frac{\pi}{2}A}$ in the canonical basis of $\bigwedge^3 \mathbb{R}^4$.

$$e^{\frac{\pi}{2}A}(e_1 \wedge e_2 \wedge e_3) = d\frac{\sqrt{2}}{2}e_1 \wedge e_2 \wedge e_3 - d\frac{\sqrt{2}}{2}e_1 \wedge e_2 \wedge e_4,$$
$$e^{\frac{\pi}{2}A}(e_1 \wedge e_2 \wedge e_4) = d\frac{\sqrt{2}}{2}e_1 \wedge e_2 \wedge e_3 + d\frac{\sqrt{2}}{2}e_1 \wedge e_2 \wedge e_4,$$
$$e^{\frac{\pi}{2}A}(e_1 \wedge e_3 \wedge e_4) = -\frac{1}{d}e_2 \wedge e_3 \wedge e_4$$

and

$$e^{\frac{\pi}{2}A}(e_2 \wedge e_3 \wedge e_4) = \frac{1}{d}e_1 \wedge e_3 \wedge e_4.$$

Then $e^{\frac{\pi}{2}A}$ can be written, with respect to the canonical basis of $\bigwedge^3 \mathbb{R}^4$, as

$$\begin{bmatrix} d\frac{\sqrt{2}}{2} & d\frac{\sqrt{2}}{2} & 0 & 0\\ -d\frac{\sqrt{2}}{2} & d\frac{\sqrt{2}}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{1}{d}\\ 0 & 0 & -\frac{1}{d} & 0 \end{bmatrix} = \begin{bmatrix} dI & 0\\ 0 & \frac{1}{d}I \end{bmatrix} \begin{bmatrix} R_1 & 0\\ 0 & R_2 \end{bmatrix}$$

with R_1, R_2 rotations by angles different from 0 and π .

In the next lemma we prove that the cones in \mathbb{R}^4 , which are invariant by above matrix, are subspaces.

Lemma 22 Let $T \in Sl(4, \mathbb{R})$ be the matrix

$$T = \begin{bmatrix} dI & 0\\ 0 & \frac{1}{d}I \end{bmatrix} \begin{bmatrix} R_1 & 0\\ 0 & R_2 \end{bmatrix}$$

where $R_1, R_2 \in SO(2, \mathbb{R}) \setminus \{I, -I\}, I$ is (2×2) -identity matrix and $d \in \mathbb{R} \setminus \{0\}$. If W is a T-invariant cone in \mathbb{R}^4 then W is a subspace.

Proof. Note that $\langle e_1, e_2 \rangle$ and $\langle e_3, e_4 \rangle$ are *T*-invariant spaces, and the restrictions of *T* to these spaces are αR where $\alpha > 0$ and *R* is the rotation different from *I* and -I. The only cones in a two-dimensional space that are invariant by these maps are (0,0) or the whole space, hence if $W \subset \langle e_1, e_2 \rangle$ then $W = \{0\}$ or $W = \langle e_1, e_2 \rangle$. If $W \subset \langle e_3, e_4 \rangle$ then $W = \{0\}$ or $W = \langle e_3, e_4 \rangle$. Suppose that *W* is not contained in these spaces. Then there exists $v \in W$ such that $v \neq \langle e_1, e_2 \rangle$ and $v \neq \langle e_3, e_4 \rangle$. As \mathbb{R}^4 is a direct sum of these two spaces, then *v* has the unique decomposition v = u + w, with $0 \neq u \in \langle e_1, e_2 \rangle$ and $0 \neq w \in \langle e_3, e_4 \rangle$. Knowing the eigenvalues of the restriction of *A* to $\langle e_1, e_2 \rangle$ we can show that

$$|T^n u|| = ||dR_1^n u|| \to +\infty \text{ and } ||T^n w| = ||(1/d)R_2^n w|| \to 0.$$

In particular, the distance of $\frac{T^n v}{\|T^n v\|}$ to $\langle e_1, e_2 \rangle$ converges to zero, this sequence is contained in a compact set and has a subsequence that converges to p. Note that $p \in \langle e_1, e_2 \rangle$ and $\|p\| = 1$. As W is a T-invariant cone then $p \in cl(W) = W$.

We have also that $W \cap \langle e_1, e_2 \rangle$ is a *T*-invariant cone which contains *p*. Then $W \cap \langle e_1, e_2 \rangle = \langle e_1, e_2 \rangle$ and so $\langle e_1, e_2 \rangle \subset W$. It implies that $-u \in W$, then $w = v + (-u) \in W$ and therefore *W* has a non-null element of $\langle e_3, e_4 \rangle$. In a similar way we can see that $\langle e_3, e_4 \rangle \subset W$. Hence *W* contains $\langle e_1, e_2 \rangle$ and $\langle e_3, e_4 \rangle$, that is, $W = \mathbb{R}^4$. In all cases, *W* is a subspace of \mathbb{R}^4 .

By the above lemma, any $e^{\frac{\pi}{2}A}$ -invariant cone in $\bigwedge^1 \mathbb{R}^4$ or in $\bigwedge^3 \mathbb{R}^4$, is a subspace. Therefore there are no S-invariant cones in $\bigwedge^1 \mathbb{R}^4$ neither in $\bigwedge^3 \mathbb{R}^4$.

Now it remains to prove that in $\bigwedge^2 \mathbb{R}^4$ there are no S-invariant cones. First note that the following submatrix of B,

$$B_2 = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{3}{2} \end{bmatrix}$$

satisfies $\lim_{t \to +\infty} e^{tB_2} = 0$ implying that $\lim_{t \to +\infty} e^{tB}v = 0$ for all $v \in \langle e_2, e_3 \rangle$. Moreover $e^{tB}e_1 = e^{2t}e_1$ and $e^{tB}e_4 = e^te_4$.

Note that when $t \to +\infty$ we have that

$$\frac{e^{tB}(e_1 \wedge e_4)}{e^{2t}e^t} = \frac{e^{2t}e_1 \wedge e^t e_4}{e^{2t}e^t} = e_1 \wedge e_4 \to e_1 \wedge e_4$$

and moreover

$$\frac{e^{tB}(e_i \wedge e_j)}{e^{2t}e^t} \to 0 \text{ for } (i,j) \neq (1,4).$$

Hence, for any vector $v \in \bigwedge^2 \mathbb{R}^4$ we have

$$v = \alpha_1 e_1 \wedge e_4 + \alpha_2 e_1 \wedge e_2 + \alpha_3 e_1 \wedge e_3 + \alpha_4 e_4 \wedge e_2 + \alpha_5 e_4 \wedge e_3 + \alpha_6 e_2 \wedge e_3, \quad (3)$$

for some $v_1, \ldots, v_4 \in \mathbb{R}$ and we have $\lim_{t \to +\infty} \frac{e^{tB}(v)}{e^{2t}e^t} = \alpha_1 e_1 \wedge e_4$. Now, suppose that exists an S-invariant cone W. Then, there is $v \in W$ of the form (3) such that

$$\lim_{t \to +\infty} \frac{e^{tB}(v)}{e^{2t}e^t} = \alpha e_1 \wedge e_4$$

with $\alpha \neq 0$, because $\operatorname{int} W \neq \emptyset$.

As $\alpha e_1 \wedge e_4 \in W$ and

$$e^{2\pi A} = \begin{bmatrix} d^4 & 0 & 0 & 0\\ 0 & d^4 & 0 & 0\\ 0 & 0 & -\frac{1}{d^4} & 0\\ 0 & 0 & 0 & -\frac{1}{d^4} \end{bmatrix}$$

we have that $e^{2\pi A}(\alpha e_1 \wedge e_4) = \alpha e_1 \wedge -e_4 = -\alpha e_1 \wedge e_4$. As W is invariant by the $e^{2\pi A}$ -action, then $-\alpha e_1 \wedge e_4 \in W$, hence any straight line generated by $\alpha e_1 \wedge e_4$ is contained in W, that is, W is not pointed. Consequently, $\bigwedge^2 \mathbb{R}^4$ does not have S-invariant cones. Therefore, by Theorem 10, $S = Sl(4, \mathbb{R})$, that is, the system is controllable.

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