

Ergodic problems for contact Hamilton-Jacobi equations

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Abstract

This paper deals with the generalized ergodic problem

$$H(x, u(x), Du(x)) = c, \quad x \in M,$$

where the unknown is a pair (c, u) of a constant $c \in \mathbb{R}$ and a function u on M for which u is a viscosity solution. We assume $H = H(x, u, p)$ satisfies Tonelli conditions in the argument $p \in T_x^*M$ and the Lipschitz condition in the argument $u \in \mathbb{R}$.

For a given $c \in \mathbb{R}$, we first discuss necessary and sufficient conditions for the existence of viscosity solutions. Let \mathfrak{C} denote the set of all real numbers c 's for which the above equation admits viscosity solutions. Then we show \mathfrak{C} is an interval, whose endpoints c_l, c_r with $c_l \leq c_r$ can be characterized by a min-max formula and a max-min formula, respectively.

The most significant finding is that we figure out the structure of \mathfrak{C} without monotonicity assumptions on u .

Keywords. Hamilton-Jacobi equations, generalized ergodic problem, contact Hamiltonian systems

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1 Introduction

1.1 Assumptions and main results

Let M be a closed (compact, without boundary), connected and smooth manifold. Denote by TM its tangent bundle and T^*M the cotangent one. \mathbb{R} stands for 1-dimensional real Euclidean space and $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. Let $H = H(x, u, p)$ be a C^3 function on $T^*M \times \mathbb{R}$ with $(x, p) \in T^*M$ and $u \in \mathbb{R}$, satisfying

- (H1) Strict convexity: the second partial derivative $\frac{\partial^2 H}{\partial p^2}(x, u, p)$ is positive definite as a quadratic form for all $(x, u, p) \in T^*M \times \mathbb{R}$;
- (H2) Superlinearity: $H(x, u, p)$ is superlinear in p for all $(x, u) \in M \times \mathbb{R}$;
- (H3) Lipschitz continuity: there exists $\lambda > 0$ such that $|\frac{\partial H}{\partial u}(x, u, p)| \leq \lambda$ for all $(x, u, p) \in T^*M \times \mathbb{R}$.

Consider the contact Hamilton-Jacobi equation

$$H(x, u(x), Du(x)) = c, \quad x \in M. \quad (E_c)$$

The symbol D in equation (E_c) denotes the spatial gradient.

Let

$$\mathfrak{C} := \left\{ c \in \mathbb{R} : \text{equation } (E_c) \text{ has viscosity solutions} \right\}.$$

We get two main results in this paper:

- Theorem A provides a set of necessary and sufficient conditions for the existence of viscosity solutions of (E_c) for any given $c \in \mathbb{R}$.
- Theorem B shows that \mathfrak{C} is an interval with the left endpoint \mathfrak{c}_l and the right endpoint \mathfrak{c}_r , where $\mathfrak{c}_l \in [-\infty, +\infty)$ and $\mathfrak{c}_r \in (-\infty, +\infty]$. Moreover, \mathfrak{C} may be an open interval, a closed interval or a half-open interval. Furthermore, we give a min-max formula for \mathfrak{c}_l and a max-min formula for \mathfrak{c}_r .

Before stating the main results, we recall the key tools used in this paper—solution semigroups first. The contact Lagrangian $L(x, u, \dot{x})$ associated with $H(x, u, p)$ is defined by

$$L(x, u, \dot{x}) := \sup_{p \in T_x^* M} \{ \langle \dot{x}, p \rangle_x - H(x, u, p) \}, \quad (x, \dot{x}) \in TM, u \in \mathbb{R}.$$

Under assumptions (H1)-(H3) the authors of [43] introduced two semigroups of operators associated with the contact Lagrangian L , denoted by $\{T_t^-\}_{t \geq 0}$ and $\{T_t^+\}_{t \geq 0}$. For each $\varphi \in C(M, \mathbb{R})$, denote by $(x, t) \mapsto T_t^-\varphi(x)$ the unique continuous function on $(x, t) \in M \times [0, +\infty)$ such that

$$T_t^-\varphi(x) = \inf_{\gamma} \left\{ \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_{\tau}^-\varphi(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau \right\},$$

where the infimum is taken among curves $\gamma \in C^{ac}([0, t], M)$ with $\gamma(t) = x$. We call $\{T_t^-\}_{t \geq 0}$ the backward solution semigroup for equation

$$w_t(x, t) + H(x, w(x, t), Dw(x, t)) = 0. \quad (1.1)$$

The function $(x, t) \mapsto T_t^-\varphi(x)$ is the unique viscosity solution of equation (1.1) with the initial condition $w(x, 0) = \varphi(x)$. Similarly, one can define another semigroup of operators $\{T_t^+\}_{t \geq 0}$, called the forward solution semigroup by

$$T_t^+\varphi(x) = \sup_{\gamma} \left\{ \varphi(\gamma(t)) - \int_0^t L(\gamma(\tau), T_{t-\tau}^+\varphi(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau \right\},$$

where the supremum is taken among curves $\gamma \in C^{ac}([0, t], M)$ with $\gamma(0) = x$. We use $\{T_t^{-,c}\}_{t \geq 0}$ (resp. $\{T_t^{+,c}\}_{t \geq 0}$) to denote the backward (resp. forward) solution semigroup associated with $L + c$, where $c \in \mathbb{R}$.

• **Existence of viscosity solutions of (E_c) .**

Let us recall the additive eigenvalue problem (or ergodic problem): let G be a Hamiltonian defined on T^*M . Finding solutions (c, u) of equation $G(x, Du(x)) = c$ is a well-known problem, called the cell (or, corrector) problem. Under a set of standard assumptions, the real number c is unique, for which the equation has viscosity solutions. Let \mathfrak{G}_t be the solution operator of the corresponding evolutionary equation $w_t(x, t) + G(x, Dw(x, t)) = 0$. Then u is a viscosity solution of $G(x, Du(x)) = c$ if and only if $\mathfrak{G}_t u = u - ct$ for all $t \geq 0$. Since the relation $\mathfrak{G}_t u = u - ct$ looks like a nonlinear eigenvalue problem, finding solutions (c, u) of $G(x, Du(x)) = c$ is also called an additive eigenvalue problem. The additive eigenvalue c determines the effective Hamiltonian in the homogenization of Hamilton-Jacobi equations [17, 31]. An interesting dynamical feature of c was discovered by weak KAM theory for Tonelli Lagrangians [18, 19, 20, 21], where c is called Mañé critical value of G , and a link between viscosity solutions of $G(x, Du(x)) = c$ and Aubry sets, Mather sets of Hamiltonian systems generated by G was established. Under Tonelli conditions, Contreras et al. [9] provided a representation formula for c :

$$c = \inf_{f \in C^\infty(M)} \sup_{x \in M} G(x, Df(x)).$$

The above infimum is not a minimum. This formula still holds true when $C^\infty(M)$ is replaced by $C^{1,1}(M)$, $C^1(M)$, or $\text{Lip}(M)$ and \inf is replaced by \min , see [4, 22].

Now come back to our problem (E_c) , which we call it *generalized additive eigenvalue problem (or generalized ergodic problem)*. When H satisfies (H1), (H2) and $0 < \delta \leq \frac{\partial H}{\partial u} \leq \lambda$, it is well-known that (E_c) has viscosity solutions for each real number c . When H satisfies (H1), (H2) and $0 \leq \frac{\partial H}{\partial u} \leq \lambda$, (E_c) has viscosity solutions if and only if there is $a \in \mathbb{R}$ such that Mañé critical value of $H(x, a, p)$ is c . See [44] for an example where the range of the function $a \mapsto \text{Mañé critical value of } H(x, a, p)$ is a proper subset of \mathbb{R} , which means that there exists $c \in \mathbb{R}$ such that (E_c) has no viscosity solutions. Seen in this light, studying the generalized additive eigenvalue problem (E_c) under (H1)-(H3) is not a straightforward task at all.

Theorem A. *Let $c \in \mathbb{R}$. The following statements are equivalent.*

- (1) *Equation (E_c) has viscosity solutions;*
- (2) *There exist $\varphi, \psi \in C(M, \mathbb{R})$ and $t_1, t_2 \in \mathbb{R}_+$ such that $T_{t_1}^{+,c} \varphi \leq \varphi$, $T_{t_2}^{+,c} \psi \geq \psi$;*
- (3) *There exist $\varphi, \psi \in C(M, \mathbb{R})$ and $t_1, t_2 \in \mathbb{R}_+$ such that $T_{t_1}^{-,c} \varphi \geq \varphi$, $T_{t_2}^{-,c} \psi \leq \psi$;*

(4) There exist $\varphi, \psi \in C(M, \mathbb{R})$ such that $T_t^{-,c}\varphi$ is bounded from below and $T_t^{-,c}\psi$ is bounded from above on $M \times [0, +\infty)$.

• **Structure of the set \mathfrak{C} .**

We call \mathfrak{C} the admissible set for the generalized ergodic problem (E_c) . Under the same assumptions imposed in this paper, the existence of solutions (c, u) of (E_c) was proven in [43], i.e, $\mathfrak{C} \neq \emptyset$. But, the structure of the set \mathfrak{C} was not discussed there.

$\text{SCL}^-(M)$ (resp. $\text{SCL}^+(M)$) stands for the set of all functions which are semiconcave (resp. semiconvex) on M with a linear modulus. $\text{Lip}(M)$ stands for the space of Lipschitz continuous functions on M . Since $\text{SCL}^\pm(M) \subset \text{Lip}(M)$, then by Rademacher's theorem $Du(x)$ exists almost everywhere for each $u \in \text{SCL}^\pm(M)$. See for example, [7] for more about semiconcave and semiconvex functions. Let $\text{Dom}(Du)$ denote the domain of definition of Du . We attempt to characterize the set \mathfrak{C} using the following two constants (may be $\pm\infty$) determined by H . Define

$$\mathfrak{c}_l := \inf_{u \in \text{SCL}^+(M)} \sup_{x \in \text{Dom}(Du)} H(x, u(x), Du(x)),$$

$$\mathfrak{c}_r := \sup_{u \in \text{SCL}^+(M)} \inf_{x \in \text{Dom}(Du)} H(x, u(x), Du(x)).$$

The following result gives a complete answer to the structure problem for \mathfrak{C} .

Theorem B.

$$(\mathfrak{c}_l, \mathfrak{c}_r) \subset \mathfrak{C} \subset [\mathfrak{c}_l, \mathfrak{c}_r].$$

Remark 1. Let us take a closer look at \mathfrak{C} .

★ The interval \mathfrak{C} will be one of the following: $(\mathfrak{c}_l, \mathfrak{c}_r)$, $[\mathfrak{c}_l, \mathfrak{c}_r]$, $(\mathfrak{c}_l, \mathfrak{c}_r]$, $[\mathfrak{c}_l, \mathfrak{c}_r)$. More precisely, each case can happen. Let $H(x, u, p) := \|p\|_x^2 + f(u)$, where $f(u)$ is a smooth function on \mathbb{R} with $|f'(u)| \leq \lambda$. If $\text{Ran}(f) = (a, b)$, then $\mathfrak{C} = (a, b)$. If $\text{Ran}(f) = [a, b]$, then $\mathfrak{C} = [a, b]$. If $\text{Ran}(f) = (a, b]$, then $\mathfrak{C} = (a, b]$. If $\text{Ran}(f) = [a, b)$, then $\mathfrak{C} = [a, b)$. Here, $a, b \in \mathbb{R}$ with $a \leq b$, and $\text{Ran}(f)$ denotes the range of f .

★ We will prove in Section 4 that there is no viscosity solutions of (E_c) either when $c < \mathfrak{c}_l$, or when $c > \mathfrak{c}_r$. So, in view of the non-emptiness of \mathfrak{C} , one can deduce that $\mathfrak{c}_l \leq \mathfrak{c}_r$.

★ If H does not depend on u , then by classical results we deduce that

$$\mathfrak{C} = \{\text{Mañé critical value of } H\}$$

is a singleton and thus $\mathfrak{c}_l = \mathfrak{c}_r$.

★ We will show what $\mathfrak{c}_l, \mathfrak{c}_r$ are in several examples in Section 4.

1.2 Historical remarks

Hamilton-Jacobi equations have been first introduced in classical mechanics, but find applications in many other fields of mathematics. The theory of viscosity solutions of Hamilton-Jacobi equations was introduced in the early 80's by Crandall and Lions [11], Crandall, Evans and Lions [12]. It provides a suitable PDE framework for studying Hamilton-Jacobi equations which does not have classical solutions. For a good introductory book on viscosity solutions, we refer readers to [2]. The theory of viscosity solutions has been extensively studied and refined by many authors, and, among the numerous contributions in the literature, we would like to point out that the weak KAM theory opened a way to study viscosity solutions of Hamilton-Jacobi equations with Tonelli Hamiltonians using the dynamical information of action minimizing orbits of Hamiltonian systems. We refer readers to [1, 5, 10, 14, 22, 23, 24, 25, 27, 29, 32, 38, 40, 41] and the references therein for more details on this topic. Along this line, it is natural to consider whether one can use weak KAM type results for contact Hamiltonian systems to study viscosity solutions of contact Hamilton-Jacobi equations (E_c). For weak KAM aspects for contact Hamiltonian systems, we refer readers to [15, 33, 35, 44, 45]. Under assumptions imposed in this paper, it was shown in [43] the existence of solutions (c, u) of equation (E_c). See [28] for a similar result using traditional PDE methods. We aim to refine and deepen the results in [43] in the present paper. We still use dynamical tools from the weak KAM theory for contact Hamiltonian systems satisfying (H1)-(H3). These assumptions will be weakened in a forthcoming paper.

Our method is dynamical in nature and inspired by the deep connection between contact Hamilton-Jacobi equations (E_c) and contact Hamiltonian system

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x, u, p), \\ \dot{p} = -\frac{\partial H}{\partial x}(x, u, p) - \frac{\partial H}{\partial u}(x, u, p)p, \\ \dot{u} = \frac{\partial H}{\partial p}(x, u, p) \cdot p - H(x, u, p). \end{cases} \quad (1.2)$$

The authors of [42, 43, 44, 45] discussed the weak KAM [21] and Aubry-Mather [34, 39] aspects of contact Hamiltonian systems from variational principles, dynamical properties of action minimizing orbits, to weak KAM solutions, viscosity solutions of stationary and evolutionary contact Hamilton-Jacobi equations. Contact Hamiltonian systems have deep connection with contact topology and non-equilibrium thermodynamics, see for example, [16].

1.3 Notations

We write as follows a list of symbols used throughout this paper.

- We choose, once and for all, a C^∞ Riemannian metric on M . It is classical that there is a canonical way to associate to it a Riemannian metric on TM . We use the same

symbol d to denote the distance function defined by the Riemannian metric on M and the distance function defined by the Riemannian metric on TM . We use the same symbol $\|\cdot\|_x$ to denote the norms induced by the Riemannian metrics on T_xM and T_x^*M for $x \in M$, and by $\langle \cdot, \cdot \rangle_x$ the canonical pairing between the tangent space T_xM and the cotangent space T_x^*M .

- $C^k(M, \mathbb{R})$ ($k \in \mathbb{N}$) stands for the function space of k -times continuously differentiable functions on M , and $C^\infty(M, \mathbb{R}) := \bigcap_{k=0}^{\infty} C^k(M, \mathbb{R})$. And $\|\cdot\|_\infty$ denotes the supremum norm on these spaces.
- $C^{ac}([a, b], M)$ stands for the space of absolutely continuous curves $[a, b] \rightarrow M$.
- $\text{Lip}(M)$ stands for the space of Lipschitz continuous functions on M .
- Denote by $\text{SCL}^-(M)$ (resp. $\text{SCL}^+(M)$) the set of all functions which are semiconcave (resp. semiconvex) on M with a linear modulus.
- $Du(x) = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$ and $Dw(x, t) = (\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_n})$.
- $D^+u(x)$ denotes the Fréchet superdifferential of u at x .
- Let $u \in \text{Lip}(M)$. Denote by $\text{Dom}(Du)$ the set of all points $x \in M$ where $Du(x)$ exists.
- \mathcal{S}_- (resp. \mathcal{S}_+) denotes the set of all backward (resp. forward) weak KAM solutions of equation $H(x, u(x), Du(x)) = 0$.
- Let Φ_t denote the local flow of contact Hamiltonian system (1.2).
- $\text{cl}(A)$ denotes the closure of a set $A \subset T^*M \times \mathbb{R}$.
- $\text{co}(A)$ denotes the convex hull of a set $A \subset T^*M \times \mathbb{R}$.
- $h_{x_0, u_0}(x, t)$ (resp. $h^{x_0, u_0}(x, t)$) denotes the forward (resp. backward) implicit action function associated with L .
- $h_{x_0, u_0}^c(x, t)$ (resp. $h_c^{x_0, u_0}(x, t)$) denotes the forward (resp. backward) implicit action function associated with $L + c$, where $c \in \mathbb{R}$.
- $\{T_t^-\}_{t \geq 0}$ (resp. $\{T_t^+\}_{t \geq 0}$) denotes the backward (resp. forward) solution semigroup associated with L .
- $\{T_t^{-,c}\}_{t \geq 0}$ (resp. $\{T_t^{+,c}\}_{t \geq 0}$) denotes the backward (resp. forward) solution semigroup associated with $L + c$, where $c \in \mathbb{R}$.

- We use $\varphi, \psi, \varphi', \psi'$ to denote generic functions in $C(M, \mathbb{R})$ not necessarily the same in any two proofs, and use $\varphi_\infty, \psi_\infty, \varphi'_\infty, \psi'_\infty$ to denote limit functions of $T_t^{\pm, c} f$ as $t \rightarrow +\infty$, with $f = \varphi, \psi, \varphi', \psi'$.

The rest of this paper is organized as follows. In Section 2, we first recall some known results on the weak KAM theory for contact Hamiltonian systems, and then prove several new results on the solution semigroups which will be used later. In Section 3 we show Proposition 14 first. Then we prove Proposition 15. Using the results obtained in Proposition 15 we prove Proposition 16. Theorem A is a direct consequence of Proposition 14 and Proposition 16. Section 4 is devoted to the proof of Theorem B. The proofs of some preliminary results are given in the Appendix.

2 Preliminaries

2.1 Weak KAM theory for contact Hamiltonian systems

We recall some definitions and basic results in the weak KAM theory for contact Hamiltonian system (1.2), where an implicit variational principle plays an essential role. Most of the results in this section can be found in [42, 43, 44, 45].

Since H satisfies (H1)-(H3), it is direct to check that L satisfies: (L1) Strict convexity: the second partial derivative $\frac{\partial^2 L}{\partial \dot{x}^2}(x, u, \dot{x})$ is positive definite as a quadratic form for all $(x, u, \dot{x}) \in TM \times \mathbb{R}$; (L2) Superlinearity: $L(x, u, \dot{x})$ is superlinear in \dot{x} for all $(x, u) \in M \times \mathbb{R}$; (L3) Lipschitz continuity: there exists $\lambda > 0$ such that $|\frac{\partial L}{\partial u}(x, u, \dot{x})| \leq \lambda$ for all $(x, u, \dot{x}) \in TM \times \mathbb{R}$.

• **Variational principles.** First recall implicit variational principles for contact Hamiltonian system (1.2), which connects contact Hamilton-Jacobi equations and contact Hamiltonian systems.

Proposition 1. *For any given $x_0 \in M$, $u_0 \in \mathbb{R}$, there exist two continuous functions $h_{x_0, u_0}(x, t)$ and $h^{x_0, u_0}(x, t)$ defined on $M \times (0, +\infty)$ satisfying*

$$h_{x_0, u_0}(x, t) = u_0 + \inf_{\substack{\gamma(0)=x_0 \\ \gamma(t)=x}} \int_0^t L(\gamma(\tau), h_{x_0, u_0}(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau, \quad (2.1)$$

$$h^{x_0, u_0}(x, t) = u_0 - \inf_{\substack{\gamma(t)=x_0 \\ \gamma(0)=x}} \int_0^t L(\gamma(\tau), h^{x_0, u_0}(\gamma(\tau), t - \tau), \dot{\gamma}(\tau)) d\tau, \quad (2.2)$$

where the infimums are taken among the Lipschitz continuous curves $\gamma : [0, t] \rightarrow M$. Moreover, the infimums in (2.1) and (2.2) can be achieved. If γ_1 and γ_2 are curves achieving the

infinimums (2.1) and (2.2) respectively, then γ_1 and γ_2 are of class C^1 . Let

$$\begin{aligned} x_1(s) &:= \gamma_1(s), & u_1(s) &:= h_{x_0, u_0}(\gamma_1(s), s), & p_1(s) &:= \frac{\partial L}{\partial \dot{x}}(\gamma_1(s), u_1(s), \dot{\gamma}_1(s)), \\ x_2(s) &:= \gamma_2(s), & u_2(s) &:= h^{x_0, u_0}(\gamma_2(s), t - s), & p_2(s) &:= \frac{\partial L}{\partial \dot{x}}(\gamma_2(s), u_2(s), \dot{\gamma}_2(s)). \end{aligned}$$

Then $(x_1(s), u_1(s), p_1(s))$ and $(x_2(s), u_2(s), p_2(s))$ satisfy equations (1.2) with

$$\begin{aligned} x_1(0) &= x_0, & x_1(t) &= x, & \lim_{s \rightarrow 0^+} u_1(s) &= u_0, \\ x_2(0) &= x, & x_2(t) &= x_0, & \lim_{s \rightarrow t^-} u_2(s) &= u_0. \end{aligned}$$

We call $h_{x_0, u_0}(x, t)$ (resp. $h^{x_0, u_0}(x, t)$) a forward (resp. backward) implicit action function associated with L and the curves achieving the infimums in (2.1) (resp. (2.2)) minimizers of $h_{x_0, u_0}(x, t)$ (resp. $h^{x_0, u_0}(x, t)$). The relation between forward and backward implicit action functions is as follows: for any given $x_0, x \in M, u_0, u \in \mathbb{R}$ and $t > 0$,

$$h_{x_0, u_0}(x, t) = u \quad \text{if and only if} \quad h^{x, u}(x_0, t) = u_0. \quad (2.3)$$

See [8] for another formulation of variational principles from the optimal control point of view. This viewpoint is strongly reminiscent of Herglotz' variational principle [26]. The following result is a direct consequence of (2.3).

Proposition 2. For any $x, y \in M$, any $u \in \mathbb{R}$ and any $t > 0$,

$$(1) \quad h^{y, h_{x, u}(y, t)}(x, t) = u,$$

$$(2) \quad h_{y, h^{x, u}(y, t)}(x, t) = u.$$

• **Implicit action functions.** We now collect some basic properties of implicit action functions.

Proposition 3.

(1) (Monotonicity). Given $x_0 \in M, u_0, u_1, u_2 \in \mathbb{R}$, Lagrangians L_1 and L_2 satisfying (L1)-(L3),

(i) if $u_1 < u_2$, then $h_{x_0, u_1}(x, t) < h_{x_0, u_2}(x, t)$, for all $(x, t) \in M \times (0, +\infty)$;

(ii) if $L_1 < L_2$, then $h_{x_0, u_0}^{L_1}(x, t) < h_{x_0, u_0}^{L_2}(x, t)$, for all $(x, t) \in M \times (0, +\infty)$ where $h_{x_0, u_0}^{L_i}(x, t)$ denotes the forward implicit action function associated with $L_i, i = 1, 2$.

(2) (Lipschitz continuity). The function $(x_0, u_0, x, t) \mapsto h_{x_0, u_0}(x, t)$ is Lipschitz continuous on $M \times [a, b] \times M \times [c, d]$ for all real numbers a, b, c, d with $a < b$ and $0 < c < d$.

(3) (Minimality). Given $x_0, x \in M, u_0 \in \mathbb{R}$ and $t > 0$, let $S_{x_0, u_0}^{x, t}$ be the set of the solutions $(x(s), u(s), p(s))$ of (1.2) on $[0, t]$ with $x(0) = x_0, x(t) = x, u(0) = u_0$. Then

$$h_{x_0, u_0}(x, t) = \inf\{u(t) : (x(s), u(s), p(s)) \in S_{x_0, u_0}^{x, t}\}, \quad \forall (x, t) \in M \times (0, +\infty).$$

(4) (Markov property). Given $x_0 \in M, u_0 \in \mathbb{R}$,

$$h_{x_0, u_0}(x, t + s) = \inf_{y \in M} h_{y, h_{x_0, u_0}(y, t)}(x, s)$$

for all $s, t > 0$ and all $x \in M$. Moreover, the infimum is attained at y if and only if there exists a minimizer γ of $h_{x_0, u_0}(x, t + s)$ with $\gamma(t) = y$.

(5) (Reversibility). Given $x_0, x \in M$ and $t > 0$, for each $u \in \mathbb{R}$, there exists a unique $u_0 \in \mathbb{R}$ such that

$$h_{x_0, u_0}(x, t) = u.$$

Proposition 4.

(1) (Monotonicity). Given $x_0 \in M$ and $u_1, u_2 \in \mathbb{R}$, Lagrangians L_1, L_2 satisfying (L1)-(L3),

(i) if $u_1 < u_2$, then $h^{x_0, u_1}(x, t) < h^{x_0, u_2}(x, t)$, for all $(x, t) \in M \times (0, +\infty)$;

(ii) if $L_1 > L_2$, then $h_{L_1}^{x_0, u_0}(x, t) < h_{L_2}^{x_0, u_0}(x, t)$, for all $(x, t) \in M \times (0, +\infty)$, where $h_{L_i}^{x_0, u_0}(x, t)$ denotes the backward implicit action function associated with $L_i, i = 1, 2$.

(2) (Lipschitz continuity). The function $(x_0, u_0, x, t) \mapsto h^{x_0, u_0}(x, t)$ is Lipschitz continuous on $M \times [a, b] \times M \times [c, d]$ for all real numbers a, b, c, d with $a < b$ and $0 < c < d$.

(3) (Maximality). Given $x_0, x \in M, u_0 \in \mathbb{R}$ and $t > 0$, let $S_{x, t}^{x_0, u_0}$ be the set of the solutions $(x(s), u(s), p(s))$ of (1.2) on $[0, t]$ with $x(0) = x, x(t) = x_0, u(t) = u_0$. Then

$$h^{x_0, u_0}(x, t) = \sup\{u(0) : (x(s), u(s), p(s)) \in S_{x, t}^{x_0, u_0}\}, \quad \forall (x, t) \in M \times (0, +\infty).$$

(4) (Markov property). Given $x_0 \in M, u_0 \in \mathbb{R}$,

$$h^{x_0, u_0}(x, t + s) = \sup_{y \in M} h^{y, h^{x_0, u_0}(y, t)}(x, s)$$

for all $s, t > 0$ and all $x \in M$. Moreover, the supremum is attained at y if and only if there exists a minimizer γ of $h^{x_0, u_0}(x, t + s)$, such that $\gamma(t) = y$.

(5) (Reversibility). Given $x_0, x \in M$, and $t > 0$, for each $u \in \mathbb{R}$, there exists a unique $u_0 \in \mathbb{R}$ such that

$$h^{x_0, u_0}(x, t) = u.$$

We will use the following result to prove Proposition 8 below. See the Appendix for the proof.

Proposition 5. Let $(x(t), u(t)) : \mathbb{R} \rightarrow M \times \mathbb{R}$ be a locally Lipschitz curve satisfying

$$u(t_2) = h_{x(t_1), u(t_1)}(x(t_2), t_2 - t_1)$$

for all $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Then $x(t)$ is of class C^1 . Let $p(t) := \frac{\partial L}{\partial \dot{x}}(x(t), u(t), \dot{x}(t))$. Then $(x(t), u(t), p(t))$ is a solution of (1.2). Moreover, for each $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, $x(t)|_{[t_1, t_2]}$ is a minimizer of $h_{x(t_1), u(t_1)}(x(t_2), t_2 - t_1)$.

• **Solution semigroups.** We collect some basic properties of the solution semigroups.

Proposition 6. Let $\varphi, \psi \in C(M, \mathbb{R})$.

(1) (Monotonicity). If $\psi < \varphi$, then $T_t^\pm \psi < T_t^\pm \varphi, \forall t \geq 0$.

(2) (Local Lipschitz continuity). The function $(x, t) \mapsto T_t^\pm \varphi(x)$ is locally Lipschitz on $M \times (0, +\infty)$.

(3) ($e^{\lambda t}$ -expansiveness). $\|T_t^\pm \varphi - T_t^\pm \psi\|_\infty \leq e^{\lambda t} \cdot \|\varphi - \psi\|_\infty, \forall t \geq 0$.

(4) (Continuity at the origin). $\lim_{t \rightarrow 0^+} T_t^\pm \varphi = \varphi$.

(5) (Representation formula). For each $\varphi \in C(M, \mathbb{R})$,

$$(i) T_t^- \varphi(x) = \inf_{y \in M} h_{y, \varphi(y)}(x, t), \quad \forall (x, t) \in M \times (0, +\infty);$$

$$(ii) T_t^+ \varphi(x) = \sup_{y \in M} h^{y, \varphi(y)}(x, t), \quad \forall (x, t) \in M \times (0, +\infty).$$

(6) (Semigroup). $\{T_t^\pm\}_{t \geq 0}$ are one-parameter semigroup of operators. For all $x_0, x \in M$, all $u_0 \in \mathbb{R}$ and all $s, t > 0$,

$$(i) T_s^- h_{x_0, u_0}(x, t) = h_{x_0, u_0}(x, t + s), \quad T_{t+s}^- \varphi(x) = \inf_{y \in M} h_{y, T_s^- \varphi(y)}(x, t);$$

$$(ii) T_s^+ h^{x_0, u_0}(x, t) = h^{x_0, u_0}(x, t + s), \quad T_{t+s}^+ \varphi(x) = \sup_{y \in M} h^{y, T_s^+ \varphi(y)}(x, t).$$

• **Weak KAM solutions.** Following Fathi (see, for instance, [21]), one can define weak KAM solutions of equation

$$H(x, u(x), Du(x)) = 0 \tag{2.4}$$

as follows.

Definition 1. A function $u \in C(M, \mathbb{R})$ is called a backward weak KAM solution of (2.4) if

(1) for each continuous piecewise C^1 curve $\gamma : [t_1, t_2] \rightarrow M$, we have

$$u(\gamma(t_2)) - u(\gamma(t_1)) \leq \int_{t_1}^{t_2} L(\gamma(s), u(\gamma(s)), \dot{\gamma}(s)) ds; \quad (2.5)$$

(2) for each $x \in M$, there exists a C^1 curve $\gamma : (-\infty, 0] \rightarrow M$ with $\gamma(0) = x$ such that

$$u(x) - u(\gamma(t)) = \int_t^0 L(\gamma(s), u(\gamma(s)), \dot{\gamma}(s)) ds, \quad \forall t < 0. \quad (2.6)$$

Similarly, a function $v \in C(M, \mathbb{R})$ is called a forward weak KAM solution of (2.4) if it satisfies (1) and for each $x \in M$, there exists a C^1 curve $\gamma : [0, +\infty) \rightarrow M$ with $\gamma(0) = x$ such that

$$v(\gamma(t)) - v(x) = \int_0^t L(\gamma(s), v(\gamma(s)), \dot{\gamma}(s)) ds, \quad \forall t > 0. \quad (2.7)$$

We say that u in (2.5) is a dominated function by L , denoted by $u \prec L$. We call curves satisfying (2.6) (resp. (2.7)), $(u, L, 0)$ -calibrated curves (resp. $(v, L, 0)$ -calibrated curves).

Under assumptions (H1)-(H3), backward weak KAM solutions are viscosity solutions.

Proposition 7.

(1) $u \in \mathcal{S}_-$ if and only if $T_t^- u = u$ for all $t \geq 0$.

(2) $v \in \mathcal{S}_+$ if and only if $T_t^+ v = v$ for all $t \geq 0$.

The following result will be useful for the proof of Proposition 11. We give the proof in the Appendix 6.2, since it is quite lengthy.

Proposition 8. Let $u \in \mathcal{S}_-$. Given any $x \in M$, if $\gamma : (-\infty, 0] \rightarrow M$ is a $(u, L, 0)$ -calibrated curve with $\gamma(0) = x$, then $(\gamma(t), u(\gamma(t)), p(t))$ satisfies equations (1.2) on $(-\infty, 0)$, where $p(t) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), u(\gamma(t)), \dot{\gamma}(t))$. Moreover, we have

$$(\gamma(t+s), u(\gamma(t+s)), Du(\gamma(t+s))) = \Phi_s(\gamma(t), u(\gamma(t)), Du(\gamma(t))), \quad \forall t, s < 0,$$

and

$$H(\gamma(t), u(\gamma(t)), \frac{\partial L}{\partial \dot{x}}(\gamma(t), u(\gamma(t)), \dot{\gamma}(t))) = 0, \quad \forall t < 0.$$

Remark 2. A similar result holds true for $v \in \mathcal{S}_+$: let $v \in \mathcal{S}_+$. Given any $x \in M$, if $\gamma : [0, +\infty) \rightarrow M$ is a $(v, L, 0)$ -calibrated curve with $\gamma(0) = x$, then $(\gamma(t), v(\gamma(t)), p(t))$ satisfies equations (1.2) on $(0, +\infty)$, where $p(t) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), v(\gamma(t)), \dot{\gamma}(t))$. Moreover, we have

$$(\gamma(t+s), v(\gamma(t+s)), Dv(\gamma(t+s))) = \Phi_s(\gamma(t), v(\gamma(t)), Dv(\gamma(t))), \quad \forall t, s > 0.$$

Since the proof of the above result is quite similar to the one of Proposition 8, we omit it.

Let $u \in \mathcal{S}_-$ and $v \in \mathcal{S}_+$. In view of Lemma 3 in the Appendix 6.2, both u and v are Lipschitz continuous.

2.2 More on solution semigroups

Proposition 9. Let $\varphi \in C(M, \mathbb{R})$. If the function $(x, t) \mapsto T_t^- \varphi(x)$ is bounded on $M \times [0, +\infty)$, then $\varphi_\infty(x) := \liminf_{t \rightarrow +\infty} T_t^- \varphi(x)$ is a viscosity solution of (2.4).

Proof. Let K_1 be a positive constant such that

$$|T_t^- \varphi(x)| \leq K_1, \quad \forall (x, t) \in M \times [0, +\infty). \quad (2.8)$$

Recall Proposition 3(2), i.e., the function $(x_0, u_0, x, t) \mapsto h_{x_0, u_0}(x, t)$ is Lipschitz on $M \times [a, b] \times M \times [c, d]$ for all real numbers a, b, c, d with $a < b$ and $0 < c < d$.

First we show that $\{T_t^- \varphi(x)\}_{t>1}$ is equi-Lipschitz on M . Denote by $l_1 > 0$ a Lipschitz constant of the function $(x_0, u_0, x) \mapsto h_{x_0, u_0}(x, 1)$ on $M \times [-K_1, K_1] \times M$. From Proposition 6 (6)(i), we have

$$|T_t^- \varphi(x) - T_t^- \varphi(y)| \leq \sup_{z \in M} |h_{z, T_{t-1}^- \varphi(z)}(x, 1) - h_{z, T_{t-1}^- \varphi(z)}(y, 1)|$$

for all $x, y \in M$, and all $t > 1$. In view of (2.8), the above inequality implies that

$$|T_t^- \varphi(x) - T_t^- \varphi(y)| \leq l_1 \cdot d(x, y).$$

Then let $\varphi_\infty(x) := \liminf_{t \rightarrow +\infty} T_t^- \varphi(x)$. We show that φ_∞ is a fixed point of $\{T_t^-\}_{t \geq 0}$. Since $\{T_t^- \varphi(x)\}_{t>1}$ is equi-Lipschitz on M , it is easy to see that

$$\lim_{t \rightarrow +\infty} \inf_{s \geq t} T_s^- \varphi(x) = \varphi_\infty(x) \quad \text{uniformly on } x \in M. \quad (2.9)$$

Note that

$$\varphi_\infty(x) = \liminf_{t \rightarrow +\infty} T_s^- \circ T_t^- \varphi(x), \quad \forall s \geq 0.$$

By the definition of \liminf , we have

$$\begin{aligned}
\varphi_\infty(x) &= \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \min_{m \leq t \leq n} T_s^- \circ T_t^- \varphi(x) \\
&= \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} T_s^- \left(\min_{m \leq t \leq n} T_t^- \varphi \right)(x) \\
&= T_s^- \left(\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \min_{m \leq t \leq n} T_t^- \varphi \right)(x) \\
&= T_s^- \varphi_\infty(x).
\end{aligned}$$

□

Proposition 10. *Let $\varphi \in C(M, \mathbb{R})$. Then*

$$(1) \quad T_t^- \circ T_t^+ \varphi \geq \varphi, \quad \forall t > 0;$$

$$(2) \quad T_t^+ \circ T_t^- \varphi \leq \varphi, \quad \forall t > 0.$$

Proof. For any $x \in M$ and $t > 0$, we have that

$$T_t^- \circ T_t^+ \varphi(x) = \inf_{y \in M} h_{y, T_t^+ \varphi(y)}(x, t) \geq \inf_{y \in M} h_{y, h^{x, \varphi(x)}(y, t)}(x, t) = \varphi(x),$$

and

$$T_t^+ \circ T_t^- \varphi(x) = \sup_{y \in M} h^{y, T_t^- \varphi(y)}(x, t) \leq \sup_{y \in M} h^{y, h^{x, \varphi(x)}(y, t)}(x, t) = \varphi(x).$$

□

Proposition 11.

(1) *For each $u \in \mathcal{S}_-$, the uniform limit $\lim_{t \rightarrow +\infty} T_t^+ u =: v$ exists and $v \in \mathcal{S}_+$.*

(2) *For each $v \in \mathcal{S}_+$, the uniform limit $\lim_{t \rightarrow +\infty} T_t^- v =: u$ exists and $u \in \mathcal{S}_-$.*

Before showing Proposition 11, we need to prove some preliminary results.

Proposition 12. *Let $u, v \in C(M, \mathbb{R})$ and let $t \geq 0$. Then $v \leq T_t^- u$ if and only if $T_t^+ v \leq u$.*

Proof. If $v \leq T_t^- u$ for some $t \geq 0$, we will show $T_t^+ v \leq u$. It is clear that $T_0^+ v(x) = v(x)$. Fix $(x, t) \in M \times (0, +\infty)$. By Proposition 6(5)(ii), we have

$$T_t^+ v(x) = \sup_{y \in M} h^{y, v(y)}(x, t).$$

It suffices to prove that $h^{y,v(y)}(x, t) \leq u(x)$ for all $y \in M$. Let $\varphi(y) := h^{y,v(y)}(x, t)$ for all $y \in M$. Then by (2.3), we have that $v(y) = h_{x,\varphi(y)}(y, t)$ for all $y \in M$. In view of $v \leq T_t^- u$ and Proposition 6(5)(i), for each $y \in M$, we get that

$$v(y) \leq T_t^- u(y) = \inf_{z \in M} h_{z,u(z)}(y, t),$$

which implies $v(y) \leq h_{x,u(x)}(y, t)$, that is, $h_{x,\varphi(y)}(y, t) \leq h_{x,u(x)}(y, t)$. By Proposition 3(1)(i), we have $\varphi(y) \leq u(x)$ for each $y \in M$.

The converse implication can be proved in a similar manner. \square

The following result is a direct consequence of the above proposition.

Corollary 1. *Solution semigroups $\{T_t^-\}_{t \geq 0}$ and $\{T_t^+\}_{t \geq 0}$ preserve the set of viscosity subsolutions of equation (2.4).*

Proposition 13. *Let $u \in \mathcal{S}_-$. For each $x \in M$, let $\gamma : (-\infty, 0] \rightarrow M$ be a $(u, L, 0)$ -calibrated curve with $\gamma(0) = x$. Then, $T_t^+ u(\gamma(-t)) = u(\gamma(-t))$ for all $t \geq 0$.*

Proof. For any given $t > 0$, let $z = \gamma(-t)$ and $u_t = u(z)$. By Proposition 12, we only need to prove $T_t^+ u(z) \geq u_t$. By Proposition 6(5)(ii), we have

$$T_t^+ u(z) = \sup_{y \in M} h^{y,u(y)}(z, t) \geq h^{x,u(x)}(z, t). \quad (2.10)$$

So, it suffices to show $h^{x,u(x)}(z, t) \geq u_t$. By Proposition 8, $(\gamma(s), u(\gamma(s)), p(s))$ satisfies equations (1.2) on $(-\infty, 0)$, where $p(s) = \frac{\partial L}{\partial \dot{x}}(\gamma(s), u(\gamma(s)), \dot{\gamma}(s))$. Let $\mathbf{u}(s) := u(\gamma(s - t))$ for $s \in [0, t]$. Then $\mathbf{u}(0) = u(\gamma(-t)) = u_t$ and $\mathbf{u}(t) = u(\gamma(0)) = u(x)$. By Proposition 4(3), it is clear that

$$h^{x,\mathbf{u}(t)}(z, t) \geq u_t.$$

This completes the proof. \square

Corollary 2. *Let $u \in \mathcal{S}_-$. The family of functions $\{T_t^+ u\}_{t > 2}$ is uniformly bounded and equi-Lipschitz on M .*

Proof. In order to prove the corollary, we proceed in two steps.

Step 1: we first prove the uniform boundedness of $\{T_t^+ u\}_{t > 2}$. By Proposition 12 and the compactness of M , the function $(x, t) \mapsto T_t^+ u(x)$ is bounded from above on $M \times [0, +\infty)$.

On the other hand, since $u \in \mathcal{S}_-$, then for any given $y \in M$, there is a $(u, L, 0)$ -calibrated curve $\gamma : (-\infty, 0] \rightarrow M$ with $\gamma(0) = y$. By Proposition 13, $T_t^+ u(\gamma(-t)) = u(\gamma(-t))$ for all $t > 0$. For any $t > 1$ and any $x \in M$, from Proposition 6(6), (5)(ii), we deduce that

$$\begin{aligned} T_t^+ u(x) &= T_1^+ \circ T_{t-1}^+ u(x) = \sup_{z \in M} h^{z, T_{t-1}^+ u(z)}(x, 1) \\ &\geq h^{\gamma(-t), T_{t-1}^+ u(\gamma(-t))}(x, 1) \\ &= h^{\gamma(-t), u(\gamma(-t))}(x, 1). \end{aligned}$$

By Proposition 4(2), the function $h^{\cdot, \cdot}(\cdot, 1)$ is bounded on $M \times [-\|u\|_\infty, \|u\|_\infty] \times M$. Thus, the function $(x, t) \mapsto T_t^+ u(x)$ is bounded from below on $M \times (1, +\infty)$.

Step 2: we show the equi-Lipschitz property of $\{T_t^+ u\}_{t>2}$. Denote by $K_2 > 0$ a constant such that $\|T_t^+ u\|_\infty \leq K_2$ for all $t > 1$. In view of Proposition 6(6)(ii), for any $x, y \in M$, we get that

$$\begin{aligned} |T_t^+ u(x) - T_t^+ u(y)| &= \left| \sup_{z \in M} h^{z, T_{t-1}^+ u(z)}(x, 1) - \sup_{z \in M} h^{z, T_{t-1}^+ u(z)}(y, 1) \right| \\ &\leq \sup_{z \in M} |h^{z, T_{t-1}^+ u(z)}(x, 1) - h^{z, T_{t-1}^+ u(z)}(y, 1)|. \end{aligned}$$

By Proposition 4(2), the function $h^{\cdot, \cdot}(\cdot, 1)$ is Lipschitz on $M \times [-K_2, K_2] \times M$ with a Lipschitz constant $\kappa_1 > 0$, and thus we get that

$$|T_t^+ u(x) - T_t^+ u(y)| \leq \kappa_1 d(x, y), \quad \forall t > 2.$$

□

Proof of Proposition 11. (1) By Proposition 12 and Corollary 2, the uniform limit $\lim_{t \rightarrow +\infty} T_t^+ u$ exists. Define

$$v := \lim_{t \rightarrow +\infty} T_t^+ u.$$

It follows from Proposition 6(3) that, for any given $t \geq 0$, we get that

$$\|T_{t+s}^+ u - T_t^+ v\|_\infty \leq e^{\lambda t} \|T_s^+ u - v\|_\infty, \quad \forall s > 0.$$

Letting $s \rightarrow +\infty$, we have

$$T_t^+ v(x) = v(x), \quad \forall x \in M.$$

By Proposition 7(2), we deduce that $v \in \mathcal{S}_+$. The proof of Proposition 11(1) is complete.

(2) Now we turn to the proof of Proposition 11(2). Since the proof of (2) is quite similar to the one of (1), we only sketch the strategy of the proof.

Let $v \in \mathcal{S}_+$. By similar arguments used in the proofs of Propositions 12, 13 and Corollary 2, one can show that

- (a) $T_t^- v \geq v$ for all $t \geq 0$.
- (b) For each $x \in M$, let $\gamma : [0, +\infty) \rightarrow M$ be a $(v, L, 0)$ -calibrated curve with $\gamma(0) = x$. Then, $T_t^- v(\gamma(t)) = v(\gamma(t))$ for all $t \geq 0$.
- (c) The family of functions $\{T_t^- v\}_{t \geq 0}$ is uniformly bounded and equi-Lipschitz on M .

Using the above three facts, we deduce that the uniform limit $\lim_{t \rightarrow +\infty} T_t^- v =: u$ exists and that $u \in \mathcal{S}_-$.

The proof of Proposition 11 is now complete. \square

3 Existence of solutions of the generalized ergodic problem

We prove Theorem A in this section. Theorem A is a direct consequence of Proposition 14 and Proposition 16 below.

3.1 Necessary and sufficient conditions for the existence I

This part is devoted to the following result.

Proposition 14. *Let $c \in \mathbb{R}$. The following statements are equivalent.*

- (1) Equation (E_c) has viscosity solutions;
- (2) There exist $\varphi, \psi \in C(M, \mathbb{R})$ and $t_1, t_2 \in \mathbb{R}_+$ such that $T_{t_1}^{+,c} \varphi \leq \varphi$, $T_{t_2}^{+,c} \psi \geq \psi$.

In the rest of subsection 3.1, without any loss of generality we assume that $c = 0$. Consider the contact Hamilton-Jacobi equation

$$H(x, u(x), Du(x)) = 0. \quad (3.1)$$

Lemma 1. *Let $\varphi \in C(M, \mathbb{R})$. If $\varphi \leq T_t^- \varphi$ (resp. $\varphi \geq T_t^- \varphi$) for all $t \geq 0$, then either*

$$\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = +\infty \quad (\text{resp. } \lim_{t \rightarrow +\infty} T_t^- \varphi(x) = -\infty)$$

uniformly on $x \in M$, or

$$\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = \varphi_\infty(x)$$

uniformly on $x \in M$, where $\varphi_\infty(x)$ is a viscosity solution of (3.1).

Proof. We divide the proof in two steps.

Step 1: First, we consider the case $\varphi \leq T_t^- \varphi$ for all $t \geq 0$. In view of $\varphi \leq T_t^- \varphi$, if $\lim_{t \rightarrow +\infty} T_t^- \varphi(x_0) = B < +\infty$ for some $x_0 \in M$, then for each $x \in M$, we have

$$T_t^- \varphi(x) = (T_1^- \circ T_{t-1}^-) \varphi(x) = \inf_{y \in M} h_{y, T_{t-1}^- \varphi(y)}(x, 1) \leq h_{x_0, T_{t-1}^- \varphi(x_0)}(x, 1) \leq h_{x_0, B}(x, 1) < +\infty.$$

Thus, we deduce that for any $t > 1$,

$$-\|\varphi\|_\infty \leq T_t^- \varphi(x) \leq \max_{y, y' \in M} h_{y, B}(y', 1), \quad \forall x \in M. \quad (3.2)$$

So, there are two possibilities: (i) $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = \varphi_\infty(x)$ for all $x \in M$. In view of (3.2), by Proposition 9, $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, and φ_∞ is a viscosity solution of (3.1). (ii) $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = +\infty$ for all $x \in M$. Next, we prove $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = +\infty$ uniformly on $x \in M$. Suppose not. Then there are $K_0 > 0$, $\{t_n\} \nearrow +\infty$ and $x_n \in M$, such that

$$T_{t_n}^- \varphi(x_n) \leq K_0.$$

Then for any $n \in \mathbb{N}$, any $x \in M$

$$T_{t_n+1}^- \varphi(x) \leq h_{x_n, T_{t_n}^- \varphi(x_n)}(x, 1) \leq h_{x_n, K_0}(x, 1) \leq \max_{y', y'' \in M} h_{y', K_0}(y'', 1) < +\infty,$$

which contradicts $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = +\infty$ for all $x \in M$.

Step 2: Second, we deal with the case $\varphi \geq T_t^- \varphi$ for all $t \geq 0$. If $\lim_{t \rightarrow +\infty} T_t^- \varphi(x_0) = B' > -\infty$ for some $x_0 \in M$, then for any $y \in M$, we have that

$$v_t := h_{y, T_t^- \varphi(y)}(x_0, 1) \geq T_{t+1}^- \varphi(x_0) \geq B'.$$

So, we get that

$$h^{x_0, v_t}(y, 1) = T_t^- \varphi(y).$$

Thus, one can deduce that for any $t > 0$

$$\|\varphi\|_\infty \geq \varphi(y) \geq T_t^- \varphi(y) \geq h^{x_0, B'}(y, 1) \geq -\|h^{x_0, B'}(\cdot, 1)\|_\infty, \quad \forall y \in M. \quad (3.3)$$

So, there are two possibilities: (i) $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = \varphi'_\infty(x)$ for all $x \in M$. In view of (3.3), by Proposition 9, $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = \varphi'_\infty(x)$ uniformly on $x \in M$, and φ'_∞ is a viscosity solution of (3.1). (ii) $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = -\infty$ for all $x \in M$. Next, we prove $\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = -\infty$ uniformly on $x \in M$. Suppose not. Then there are $K'_0 < 0$, $\{t'_n\} \nearrow +\infty$ and $x'_n \in M$, such that

$$T_{t'_n}^- \varphi(x'_n) \geq K'_0.$$

Given any $y \in M$, let

$$v'_n := h_{y, T_{t'_n-1}^- \varphi(y)}(x'_n, 1) \geq T_{t'_n}^- \varphi(x'_n) \geq K'_0.$$

By (2.3),

$$h^{x'_n, v'_n}(y, 1) = T_{t'_n-1}^- \varphi(y).$$

Thus,

$$T_{t'_n-1}^- \varphi(y) \geq h^{x'_n, K'_0}(y, 1) \geq \min_{z, z' \in M} h^{z, K'_0}(z', 1) > -\infty,$$

which contradicts $\lim_{t \rightarrow +\infty} T_t^- \varphi(y) = -\infty$.

The proof is complete. \square

The following result is a direct consequence of Lemma 1. We omit the proof.

Corollary 3. *Let $\varphi \in C(M, \mathbb{R})$. If $\varphi \leq T_{t_0}^- \varphi$ (resp. $\varphi \geq T_{t_0}^- \varphi$) for some $t_0 > 0$, then either*

$$\lim_{n \rightarrow +\infty} T_{nt_0}^- \varphi(x) = +\infty \quad (\text{resp.} \quad \lim_{n \rightarrow +\infty} T_{nt_0}^- \varphi(x) = -\infty)$$

uniformly on $x \in M$, or $\lim_{n \rightarrow +\infty} T_{nt_0}^- \varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, where $\varphi_\infty(x) \in \text{Lip}(M)$.

Proof of Proposition 14. In view of Proposition 10, notice that $T_t^- \varphi \geq \varphi$ if and only if $T_t^+ \varphi \leq \varphi$.

If equation (3.1) has a viscosity solution φ , then by Proposition 11 one can deduce that $\psi := \lim_{t \rightarrow +\infty} T_t^+ \varphi$ is a forward weak KAM solution. Thus, $T_t^- \varphi = \varphi$ and $T_t^+ \psi = \psi$ for all $t \geq 0$, i.e., item (2) in Proposition 14 holds true.

Let $\varphi \in C(M, \mathbb{R})$ and $t_1 > 0$ be such that $T_{t_1}^- \varphi \geq \varphi$. Let $\psi \in C(M, \mathbb{R})$ and $t_2 > 0$ be such that $T_{t_2}^+ \psi \geq \psi$. We aim to prove that equation (3.1) has viscosity solutions.

Since $T_{t_1}^- \varphi \geq \varphi$, then by Corollary 3, either $\lim_{n \rightarrow +\infty} T_{nt_1}^- \varphi(x) = +\infty$ uniformly on $x \in M$, or $\lim_{n \rightarrow +\infty} T_{nt_1}^- \varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, where $\varphi_\infty(x) \in \text{Lip}(M)$.

Case 1: If $\lim_{n \rightarrow +\infty} T_{nt_1}^- \varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, then for any $s \in [0, t_1]$, $\lim_{n \rightarrow \infty} T_{nt_1+s}^- \varphi(x) = T_s^- \varphi_\infty(x)$. Hence, $T_t^- \varphi(x)$ is bounded on $M \times [0, +\infty)$. Thus, by Proposition 9,

$$\varphi'(x) := \liminf_{t \rightarrow +\infty} T_t^- \varphi(x)$$

is a viscosity solution of equation (3.1).

Case 2: If $\lim_{n \rightarrow +\infty} T_{nt_1}^- \varphi(x) = +\infty$ uniformly on $x \in M$, then there is $n_1 \in \mathbb{N}$ such that $T_{n_1 t_1}^- \varphi > \psi$, and $T_{n_1 t_1}^- \varphi > \varphi$. Choose $k_1, k_2 \in \mathbb{N}$ such that

$$s_0 := \frac{k_1}{k_2} t_2 - n_1 t_1 > 0$$

small enough with

$$T_{n_1 t_1 + s_0}^- \varphi > \psi, \quad T_{n_1 t_1 + s_0}^- \varphi > \varphi.$$

Let $t_0 := k_2(n_1 t_1 + s_0) = k_1 t_2$. Then

$$T_{t_0}^- \varphi > \varphi, \quad T_{t_0}^- \varphi > \psi, \quad T_{t_0}^+ \psi \geq \psi.$$

Then by Proposition 12,

$$T_{t_0}^+ \varphi \leq \varphi.$$

Let $\varphi' = T_{t_0}^- \varphi$. Then by Proposition 10, we have that

$$T_{t_0}^+ \varphi' = T_{t_0}^+ \circ T_{t_0}^- \varphi \leq \varphi,$$

and

$$T_{t_0}^+ \varphi' \geq T_{t_0}^+ \psi \geq \psi.$$

Therefore, we get that

$$\psi \leq T_{nt_0}^+ \psi \leq T_{nt_0}^+ \varphi' \leq \varphi.$$

So, $\{T_{nt_0}^+ \psi\}_{n \in \mathbb{N}}$ is bounded, and thus the uniform limit

$$\lim_{n \rightarrow \infty} T_{nt_0}^+ \psi =: \psi_\infty$$

exists. And for any $s \in [0, t_0]$,

$$\lim_{n \rightarrow \infty} T_{nt_0+s}^+ \psi(x) =: T_s^+ \psi_\infty(x), \quad x \in M.$$

It follows that the function $(x, t) \mapsto T_t^+ \psi(x)$ is bounded on $M \times [0, +\infty)$. Let

$$\psi'_\infty(x) := \limsup_{t \rightarrow +\infty} T_t^+ \psi(x).$$

We assert that ψ'_∞ is a forward weak KAM solution of equation (3.1). If the assertion is true, then by Proposition 11, one can deduce that $\mathcal{S}_- \neq \emptyset$.

So, it suffices to show the assertion. Let $K_3 > 0$ be such that $|T_t^+ \psi(x)| \leq K_3$ for all $(x, t) \in M \times [0, +\infty)$. First we show that $\{T_t^+ \psi(x)\}_{t > 1}$ is equi-Lipschitz on M . Denote by $\kappa_2 > 0$ a Lipschitz constant of the function $(x_0, u_0, x) \mapsto h^{x_0, u_0}(x, 1)$ on $M \times [-K_3, K_3] \times M$. From Proposition 6 (6)(ii), we have

$$|T_t^+ \psi(x) - T_t^+ \psi(y)| \leq \sup_{z \in M} |h^{z, T_{t-1}^+ \psi(z)}(x, 1) - h^{z, T_{t-1}^+ \psi(z)}(y, 1)|$$

for all $x, y \in M$, and all $t > 1$. The above inequality implies that

$$|T_t^+ \psi(x) - T_t^+ \psi(y)| \leq \kappa_2 \cdot d(x, y).$$

Next we show that ψ'_∞ is a fixed point of $\{T_t^+\}_{t \geq 0}$. Since $\{T_t^+\psi(x)\}_{t > 1}$ is equi-Lipschitz on M , it is easy to see that

$$\lim_{t \rightarrow +\infty} \sup_{s \geq t} T_s^+ \psi(x) = \psi'_\infty(x) \quad \text{uniformly on } x \in M.$$

For any $s \geq 0$, notice that

$$\begin{aligned} \psi'_\infty(x) &= \lim_{t \rightarrow +\infty} \sup T_t^+ \psi(x) = \lim_{t \rightarrow +\infty} \sup T_s^+ \circ T_t^+ \psi(x) \\ &= \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \max_{m \leq t \leq n} T_s^+ \circ T_t^+ \psi(x) \\ &= \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \max_{m \leq t \leq n} \sup_{y \in M} h^{y, T_t^+ \psi(y)}(x, s) \\ &= \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sup_{y \in M} h^{y, \max_{m \leq t \leq n} T_t^+ \psi(y)}(x, s) \\ &= \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} T_s^+ \left(\max_{m \leq t \leq n} T_t^+ \psi \right)(x) \\ &= T_s^+ \left(\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \max_{m \leq t \leq n} T_t^+ \psi \right)(x) \\ &= T_s^+ \psi'_\infty(x). \end{aligned}$$

The proof is complete. □

3.2 A key proposition

In order to complete the proof of Theorem A, we need to prove a technical proposition here which also provides certain information of the long time behavior of the viscosity solution of the Cauchy problem

$$w_t(x, t) + H(x, w(x, t), Dw(x, t)) = c \quad (3.4)$$

with $w(x, 0) = \varphi(x)$. Moreover, we will use this proposition again in the proof of Theorem B. Recall that the function $(x, t) \mapsto T_t^{-,c} \varphi(x)$ is the unique viscosity solution of the Cauchy problem.

Proposition 15. *Let $\varphi \in C(M, \mathbb{R})$ and let $c \in \mathbb{R}$.*

- (1) *if there is $t_0 > 0$ such that $T_{t_0}^{-,c} \varphi \geq \varphi$ (resp. $T_{t_0}^{-,c} \varphi \leq \varphi$), then for any $s \in [0, t_0]$, $\lim_{n \rightarrow +\infty} T_{nt_0+s}^{-,c} \varphi(x) = +\infty$ (resp. $\lim_{n \rightarrow +\infty} T_{nt_0+s}^{-,c} \varphi(x) = -\infty$) uniformly on $x \in M$, or $\lim_{n \rightarrow +\infty} T_{nt_0+s}^{-,c} \varphi(x) = u(x, s)$ uniformly in $(x, s) \in M \times [0, t_0]$, where $u(x, s)$ is a viscosity solution of equation (3.4) which is t_0 -periodic in time.*

(2) if there is $t_0 > 0$ such that $T_{t_0}^{-,c}\varphi > \varphi$ (resp. $T_{t_0}^{-,c}\varphi < \varphi$), then $\lim_{t \rightarrow +\infty} T_t^{-,c}\varphi(x) = +\infty$ (resp. $\lim_{t \rightarrow +\infty} T_t^{-,c}\varphi(x) = -\infty$) uniformly on $x \in M$, or $\lim_{t \rightarrow +\infty} T_t^{-,c}\varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, where φ_∞ is a viscosity solution of equation (E_c) .

(3) if for any $t > 0$,

$$\left\{x \in M : T_t^{-,c}\varphi(x) = \varphi(x)\right\} \neq \emptyset,$$

then $|T_t^{-,c}\varphi(x)| \leq K_\varphi$ for all $(x, t) \in M \times [0, +\infty)$ and for some constant $K_\varphi > 0$ depending on φ .

Proof of Proposition 15. We will split the proof into three steps.

Step 1: if there is $t_0 > 0$ such that $\varphi \leq T_{t_0}^{-,c}\varphi$, then by Corollary 3, either $\lim_{n \rightarrow +\infty} T_{nt_0}^{-,c}\varphi(x) = +\infty$ uniformly on $x \in M$, or, $\lim_{n \rightarrow +\infty} T_{nt_0}^{-,c}\varphi(x) = \varphi_\infty(x)$ uniformly on $x \in M$, where $\varphi_\infty(x)$ is a Lipschitz continuous function on M .

Case (i): If $\lim_{n \rightarrow +\infty} T_{nt_0}^{-,c}\varphi(x) = +\infty$ uniformly on $x \in M$, then we assert that for any $s \in [0, t_0]$,

$$\lim_{n \rightarrow +\infty} T_{nt_0+s}^{-,c}\varphi(x) = +\infty$$

for all $x \in M$. Note that if there are $s_0 \in [0, t_0]$ and $x_0 \in M$ such that

$$\lim_{n \rightarrow \infty} T_{nt_0+s_0}^{-,c}\varphi(x_0) = A'' < +\infty,$$

then for all $x \in M$,

$$T_{nt_0+s_0}^{-,c}\varphi(x) \leq h_{x_0, A''}^c(x, t_0) \leq \max_{y, y'' \in M} h_{y, A''}^c(y'', t_0) < +\infty.$$

So, in order to show the above assertion, we can assume by contradiction that there is $s_0 \in [0, t_0]$ such that

$$\lim_{n \rightarrow \infty} T_{nt_0+s_0}^{-,c}\varphi(x) = \varphi_\infty^{s_0}(x), \quad \forall x \in M,$$

where $\varphi_\infty^{s_0}(x)$ is a function defined on M . It is clear that $\{|T_{nt_0+s_0}^{-,c}\varphi(x)|\}_n$ is bounded by a constant $K > 0$. And thus, by similar arguments used in the proof of Proposition 9, $\{|T_{nt_0+s_0}^{-,c}\varphi(x)|\}_n$ is equi-Lipschitz. Therefore,

$$\lim_{n \rightarrow \infty} T_{nt_0+s_0}^{-,c}\varphi(x) = \varphi_\infty^{s_0}(x),$$

uniformly on $x \in M$, and $\varphi_\infty^{s_0} \in \text{Lip}(M)$. Note that

$$\|T_{t_0-s_0}^{-,c} \circ T_{nt_0+s_0}^{-,c}\varphi - T_{t_0-s_0}^{-,c}\varphi_\infty^{s_0}\|_\infty \leq e^{\lambda t_0} \|T_{nt_0+s_0}^{-,c}\varphi - \varphi_\infty^{s_0}\|_\infty.$$

Thus, we get that

$$+\infty = \lim_{n \rightarrow \infty} T_{t_0 - s_0}^{-,c} \circ T_{nt_0 + s_0}^{-,c} \varphi(x) = T_{t_0 - s_0}^{-,c} \varphi_\infty^{s_0}(x),$$

a contradiction. We have proved the assertion.

Next, we prove for any $s \in [0, t_0]$, $\lim_{t \rightarrow +\infty} T_{nt_0 + s}^{-,c} \varphi(x) = +\infty$ uniformly on $x \in M$. Suppose not. Then there are $s' \in [0, s_0]$, $K_0 > 0$, $\{n_k\} \nearrow +\infty$ and $x_k \in M$, such that

$$T_{n_k t_0 + s'}^{-,c} \varphi(x_k) \leq K_0.$$

Then for any $k \in \mathbb{N}$, any $x \in M$

$$T_{(n_k+1)t_0 + s'}^{-,c} \varphi(x) \leq h_{x_k, T_{n_k t_0 + s'}^{-,c} \varphi(x_{n_k})}^c(x, t_0) \leq h_{x_k, K_0}^c(x, t_0) \leq \max_{y', y'' \in M} h_{y', K_0}^c(y'', t_0) < +\infty,$$

which contradicts $\lim_{n \rightarrow \infty} T_{nt_0 + s'}^{-,c} \varphi(x) = +\infty$ for all $x \in M$.

Case (ii) : If $\lim_{n \rightarrow +\infty} T_{nt_0}^{-,c} \varphi(x) = \varphi_\infty(x)$ for all $x \in M$, where $\varphi_\infty(x)$ is a Lipschitz continuous function on M , then for any $s \in \mathbb{R}$,

$$\lim_{n \rightarrow +\infty} T_{nt_0 + s}^{-,c} \varphi(x) = T_s^{-,c} \varphi_\infty(x) =: u(x, s).$$

It is clear that $u(x, s + t_0) = u(x, s)$ for all $s \in \mathbb{R}$, and that $u(x, s)$ is a viscosity solution of (1.5a).

The proof for the case $\varphi \geq T_{t_0}^{-,c} \varphi$ for some $t_0 > 0$ is quite similar and thus we omit it.

Step 2: if there is $t_0 > 0$ such that $\varphi < T_{t_0}^{-,c} \varphi$, then there is $t_1 > 0$ close enough to t_0 such that t_1/t_0 is an irrational number and $\varphi < T_{t_1}^{-,c} \varphi$. Since t_1/t_0 is an irrational number, then for any $s \in [0, t_0]$, any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there are $m_0, m_1 \in \mathbb{N}$ with $m_0, m_1 > N$, such that

$$|m_1 t_1 - (m_0 t_0 + s)| < \varepsilon. \quad (3.5)$$

By the result obtained in Proposition 15 (1), then $\lim_{n \rightarrow +\infty} T_{nt_0 + s}^{-,c} \varphi(x) = +\infty$ for all $x \in M$ and all $s \in \mathbb{R}$, or $\lim_{n \rightarrow +\infty} T_{nt_0 + s}^{-,c} \varphi(x) = u(x, s)$ for all $x \in M$ and all $s \in \mathbb{R}$. If $\lim_{n \rightarrow +\infty} T_{nt_0 + s}^{-,c} \varphi(x) = u(x, s)$ for all $x \in M$ and all $s \in [0, t_0]$, then in view of (3.5), we get that $\lim_{n \rightarrow \infty} T_{nt_1}^{-,c} \varphi(x) =: \varphi'_\infty(x)$ for some $\varphi'_\infty \in \text{Lip}(M)$. By (3.5) again, $\varphi'_\infty(x) = u(x, s)$ for all $x \in M$ and $s \in [0, t_0]$. Thus, φ'_∞ is a viscosity solution of (E_c) .

The proof for the case $\varphi \geq T_{t_0}^{-,c} \varphi$ for some $t_0 > 0$ is quite similar and thus we omit it.

Step 3: if for any $t > 0$, there are $x_1, x_2 \in M$ such that $T_t^{-,c} \varphi(x_1) > \varphi(x_1)$ and $T_t^{-,c} \varphi(x_2) < \varphi(x_2)$, then for any $t > 0$, there is $x_t \in M$ such that $T_t^{-,c} \varphi(x_t) = \varphi(x_t)$. Note that

$$T_t^{-,c} \varphi(x) = T_1^{-,c} \circ T_{t-1}^{-,c} \varphi(x) \leq h_{x_{t-1}, T_{t-1}^{-,c} \varphi(x_{t-1})}^c(x, 1) = h_{x_{t-1}, \varphi(x_{t-1})}^c(x, 1).$$

Thus, $T_t^{-,c}\varphi(x)$ is bounded from above. Note that for any $y \in M$,

$$h_{y, T_t^{-,c}\varphi(y)}^c(x_{t+1}, 1) \geq T_{t+1}^{-,c}\varphi(x_{t+1}) = \varphi(x_{t+1}),$$

which implies that $T_t^{-,c}\varphi(y) \geq h_c^{x_{t+1}, \varphi(x_{t+1})}(y, 1)$ for all $y \in M$. Therefore, we get that for any $t > 1$, any $y \in M$,

$$\min_{(z, z') \in M \times M} h_c^{z, \varphi(z)}(z', 1) \leq h_c^{x_{t+1}, \varphi(x_{t+1})}(y, 1) \leq T_t^{-,c}\varphi(y) \leq \max_{(z, z') \in M \times M} h_{z, \varphi(z)}^c(z', 1).$$

Hence, $|T_t^{-,c}\varphi(y)|$ is bounded on $M \times [0, +\infty)$. \square

3.3 Necessary and sufficient conditions for the existence II

We prove the following result in this part.

Proposition 16. *Let $c \in \mathbb{R}$. The following statements are equivalent.*

- (1) *Equation (E_c) has viscosity solutions;*
- (2) *There exist $\varphi, \psi \in C(M, \mathbb{R})$ and $t_1, t_2 \in \mathbb{R}_+$ such that $T_{t_1}^{-,c}\varphi \geq \varphi$, $T_{t_2}^{-,c}\psi \leq \psi$;*
- (3) *There exist $\varphi, \psi \in C(M, \mathbb{R})$ such that $T_t^{-,c}\varphi$ is bounded from below and $T_t^{-,c}\psi$ is bounded from above on $M \times [0, +\infty)$.*

Proof. Without any loss of generality, we assume that $c = 0$. The strategy of our proof: we will prove the equivalence of items (2) and (3), and then the equivalence of items (3) and (1).

Step 1: we first show (2) \Leftrightarrow (3). If condition (2) holds true, then in view of Proposition 15 (1), one can deduce that $T_t^- \varphi$ is bounded from below and $T_t^- \psi$ is bounded from above on $M \times [0, +\infty)$.

In the rest of Step 1, we show that (3) \Rightarrow (2). Let $\varphi, \psi \in C(M, \mathbb{R})$ be such that $T_t^- \varphi$ is bounded from below and $T_t^- \psi$ is bounded from above on $M \times [0, +\infty)$.

If $T_t^- \varphi$ is also bounded from above, then by Proposition 9, $\varphi_\infty(x) = \liminf_{t \rightarrow +\infty} \varphi(x)$ is a viscosity solution of (3.1) and thus $\varphi_\infty = T_t^- \varphi_\infty$ for all $t \geq 0$. If $T_t^- \varphi$ is unbounded from above, then from Proposition 15 (3) we can get that there is $t' > 0$ such that

$$\{x \in M : T_{t'}^- \varphi(x) = \varphi(x)\} = \emptyset.$$

Thus, one can deduce that either $T_{t'}^- \varphi > \varphi$ or $T_{t'}^- \varphi < \varphi$. Since $T_t^- \varphi$ is bounded from below, by Proposition 15 (2) again we get that

$$\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = +\infty$$

uniformly in $x \in M$. Hence, there is $t_1 > 0$ such that $T_{t_1}^- \varphi \geq \varphi$.

If $T_t^- \psi$ is also bounded from below, then by Proposition 9, $\psi_\infty(x) = \liminf_{t \rightarrow +\infty} \psi(x)$ is a viscosity solution of (3.1) and thus $\psi_\infty = T_t^- \psi_\infty$ for all $t \geq 0$. If $T_t^- \psi$ is unbounded from below, then from Proposition 15 (3) we can get that there is $t'' > 0$ such that

$$\{x \in M : T_{t''}^- \psi(x) = \psi(x)\} = \emptyset.$$

Thus, one can deduce that either $T_{t''}^- \psi > \psi$ or $T_{t''}^- \psi < \psi$. Since $T_t^- \psi$ is bounded from above, by Proposition 15 (2) again we get that

$$\lim_{t \rightarrow +\infty} T_t^- \psi(x) = -\infty$$

uniformly in $x \in M$. Hence, there is $t_2 > 0$ such that $T_{t_2}^- \psi \leq \psi$.

Step 2: next we show (1) \Leftrightarrow (3). The fact that (1) \Rightarrow (3) is clear. It suffices to show that (3) \Rightarrow (1).

From the proof of (3) \Rightarrow (2), we only need to discuss the case:

$$\lim_{t \rightarrow +\infty} T_t^- \varphi(x) = +\infty, \quad \text{and} \quad \lim_{t \rightarrow +\infty} T_t^- \psi(x) = -\infty.$$

Let $u_\rho = \rho\varphi + (1 - \rho)\psi$, $\rho \in [0, 1]$ and let

$$\rho_0 = \inf\{\rho : \lim_{t \rightarrow +\infty} T_t^- u_\rho(x) = +\infty\}.$$

Consider $T_t^- u_{\rho_0}$. If $T_t^- u_{\rho_0}$ is bounded on $M \times [0, +\infty)$, then by Proposition 9 again, $\liminf_{t \rightarrow +\infty} T_t^- u_{\rho_0}$ is a viscosity solution of (3.1). If $T_t^- u_{\rho_0}$ is unbounded on $M \times [0, +\infty)$, then by similar arguments used in Step 1, one can deduce that either $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0} = +\infty$ uniformly in $x \in M$ or $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0} = -\infty$ uniformly in $x \in M$.

If $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0} = +\infty$ uniformly in $x \in M$, then there is $t_0 > 0$ such that $T_{t_0}^- u_{\rho_0} > u_{\rho_0}$. Thus, there is $\epsilon_0 > 0$ such that $T_{t_0}^- u_{\rho_0 - \epsilon_0} > u_{\rho_0 - \epsilon_0}$. Then by Proposition 15 (2) we get that either $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0 - \epsilon_0} = u_\infty$ uniformly in $x \in M$, or $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0 - \epsilon_0} = +\infty$ uniformly in $x \in M$. In view of the definition of ρ_0 , we deduce that $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0 - \epsilon_0} = u_\infty$ uniformly in $x \in M$ and u_∞ is a viscosity solution of (3.1).

If $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0} = -\infty$ uniformly in $x \in M$, then there is $t'_0 > 0$ such that $T_{t'_0}^- u_{\rho_0} < u_{\rho_0}$. Thus, there exists $\delta > 0$ such that

$$T_{t'_0}^- u_{\rho_0 + \epsilon} < u_{\rho_0 + \epsilon}$$

for all $\epsilon \in (0, \delta)$. Then for any $\epsilon \in (0, \delta)$, either $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0 + \epsilon} = -\infty$ uniformly in $x \in M$ or $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0 + \epsilon} = u'_\infty$ uniformly in $x \in M$, where u'_∞ is a viscosity solution of (3.1). Recall that $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0} = -\infty$ uniformly in $x \in M$. Hence, by the definition of ρ_0 there must be $\epsilon_0 \in (0, \delta)$ such that $\lim_{t \rightarrow +\infty} T_t^- u_{\rho_0 + \epsilon_0} = u''_\infty$ uniformly in $x \in M$, where u''_∞ is a viscosity solution of (3.1). \square

4 Analysis of the admissible set: min-max and max-min formulas

4.1 The admissible set is an interval

Before proving Theorem B, we recall a approximation and regularity result of Lipschitz functions by Czarnecki and Rifford [13].

Proposition 17. *Let $f \in \text{Lip}(M)$. Then there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C^\infty(M, \mathbb{R})$ such that*

$$\lim_{n \rightarrow +\infty} \|f_n - f\|_\infty = 0$$

and

$$\lim_{n \rightarrow +\infty} d_{\text{Haus}}(\text{graph}(Df_n), \text{graph}(\partial f)) = 0.$$

Here, $\partial f(x)$ denotes Clarke's generalized gradient of f at x :

$$\partial f(x) = \text{co} \left\{ \zeta \mid \exists (x_n)_{n \in \mathbb{N}} \subset \text{Dom}(Df), x_n \rightarrow x, Df(x_n) \rightarrow \zeta, n \rightarrow \infty \right\}$$

which is non-empty by Rademacher's theorem. And $\text{graph}(\partial f) := \{(x, p) \in T^*M : p \in \partial f(x)\}$.

Let S_1 and S_2 be two non-empty closed subsets of T^*M ,

$$d_{\text{Haus}}(S_1, S_2) := \sup \left\{ \sup_{(x,p) \in S_1} d_{S_2}(x, p), \sup_{(x,p) \in S_2} d_{S_1}(x, p) \right\}$$

denotes the Hausdorff distance, where $d_S(x, p) = \inf_{(x', p') \in S} d((x, p), (x', p'))$.

In view of Rademacher's theorem, $M \setminus \text{Dom}(Du)$ is negligible. Since $\|Du(x)\|_x$ is bounded by the Lipschitz constant of u and H is of class C^3 , then the following equality is a direct consequence of Proposition 17.

$$\inf_{u \in \text{Lip}(M)} \sup_{x \in \text{Dom}(Du)} H(x, u(x), Du(x)) = \inf_{u \in \text{SCL}^+(M)} \sup_{x \in \text{Dom}(Du)} H(x, u(x), Du(x)). \quad (4.1)$$

We omit the proof.

Proof of Theorem B. First we show that if $c < c_l$, then equation (E_c) has no viscosity subsolutions. Assume by contradiction that there is a viscosity subsolution of equation (E_c) . By classical results on viscosity solutions, we have that u is Lipschitz on M and satisfies $H(x, u(x), Du(x)) \leq c$ for a.e. $x \in M$. Thus, by (4.1)

$$c_l = \inf_{w \in \text{Lip}(M)} \sup_{x \in \text{Dom}(Dw)} H(x, w(x), Dw(x)) \leq c,$$

a contradiction.

Next we show that if $c > \mathfrak{c}_r$, then equation (E_c) has no viscosity solutions. If equation (E_c) admits a viscosity solution u^* , then by Proposition 11 one can deduce that $v^* = \lim_{t \rightarrow +\infty} T_t^{+,c} u^*$ is a forward weak KAM solution of (E_c) . Thus, $-v^*$ is a viscosity solution of $H(x, -w(x), -Dw(x)) = c$, which implies that $H(x, v^*(x), Dv^*(x)) = c$ a.e. $x \in M$. Hence, in view of the semiconvexity of v^* we get that

$$\mathfrak{c}_r = \sup_{u \in \text{SCL}^+(M)} \inf_{x \in \text{Dom}(Du)} H(x, u(x), Du(x)) \geq c.$$

Combining the above arguments and the non-emptiness of \mathfrak{C} , one have that

$$\mathfrak{c}_l \leq \mathfrak{c}_r, \quad \text{and} \quad \mathfrak{C} \subset [\mathfrak{c}_l, \mathfrak{c}_r].$$

To finish the proof, it suffices to show that for any $c \in (\mathfrak{c}_l, \mathfrak{c}_r)$, equation (E_c) admits at least a viscosity solution. Since $c > \mathfrak{c}_l$, then by definition there is $\varphi \in \text{Lip}(M)$ such that φ is a viscosity subsolution of (E_c) , and thus $\varphi \leq T_t^{-,c} \varphi$ for all $t \geq 0$.

In view of Theorem A (4), we only need to prove that there is $\psi \in C(M, \mathbb{R})$ such that $T_t^{-,c} \psi$ is bounded from above. Since $c < \mathfrak{c}_r$, then there are $\delta > 0$ and $\psi \in \text{SCL}^+(M)$ such that

$$H(x, \psi(x), D\psi(x)) - c \geq \delta, \quad \text{a.e. } x \in M.$$

If for any $t > 0$, $\{x \in M : T_t^{-,c} \psi(x) = \psi(x)\} \neq \emptyset$, then $T_t^{-,c} \psi$ is bounded. If there is $t_0 > 0$ such that either $T_{t_0}^{-,c} \psi < \psi$, or $T_{t_0}^{-,c} \psi > \psi$, we only need to take care of the case $T_{t_0}^{-,c} \psi > \psi$. By Proposition 15 (2), either $\lim_{t \rightarrow +\infty} T_t^{-,c} \psi = \psi_\infty$, where ψ_∞ is a viscosity solution of (E_c) , or $\lim_{t \rightarrow +\infty} T_t^{-,c} \psi = +\infty$. In the rest of the proof we show that the case $\lim_{t \rightarrow +\infty} T_t^{-,c} \psi = +\infty$ cannot happen.

Assume by contradiction that $\lim_{t \rightarrow +\infty} T_t^{-,c} \psi = +\infty$. Thus, there are $t_1 > 0$ and $x_0 \in M$ such that $T_{t_1}^{-,c} \psi \geq \psi$ for all $t \geq t_1$ and $T_{t_1}^{-,c} \psi(x_0) = \psi(x_0)$. Note that ψ is a semiconvex function and $T_{t_1}^{-,c} \psi$ is a semiconcave function [6, Theorem 3.2]. Then by Lemma 6 in the Appendix, both $\psi' := T_{t_1}^{-,c} \psi$ and ψ are differentiable at x_0 , and $\psi(x_0) = \psi'(x_0)$, $D\psi(x_0) = D\psi'(x_0)$. Let $u_0 = \psi(x_0) = \psi'(x_0)$, $p_0 = D\psi(x_0) = D\psi'(x_0)$. Let $(x(t), u(t), p(t))$ be the solution of (1.2) with $(x(0), u(0), p(0)) = (x_0, u_0, p_0)$ in a small neighbourhood of 0. Hence, we have that

$$u(t) = u_0 + (\langle p_0, \dot{x}(0) \rangle_{x_0} - H(x_0, u_0, p_0) + c)t + \mathfrak{o}(t),$$

and

$$\psi(x(t)) = u_0 + \langle p_0, \dot{x}(0) \rangle_{x_0} t + \mathfrak{o}(t).$$

Recall that

$$H(x_0, u_0, p_0) - c \geq \delta,$$

which implies that $\psi(x(t)) > u(t)$ in a small enough neighbourhood of 0. Thus we have

$$T_t^{-,c}\psi'(x(t)) \leq h_{x_0, u_0}^c(x(t), t) \leq u(t) < \psi(x(t))$$

for sufficiently small $t > 0$, which contradicts that

$$T_t^{-,c}\psi' \geq \psi, \quad \forall t \geq 0.$$

The proof of Theorem B is complete. \square

4.2 Examples

Let us discuss several illustrative examples of contact Hamiltonians satisfying (H1)-(H3) and describe the corresponding \mathfrak{c}_l , \mathfrak{c}_r . We discuss genuine contact Hamiltonians in the first two examples, while a classical Hamiltonian is studied in the last example. The classical case can be regarded as the critical case.

Let $h(x, p)$ denote a generic Tonelli Hamiltonian on T^*M in the following.

Example 1 ($\mathfrak{c}_l = -\infty$, $\mathfrak{c}_r = +\infty$). Let $H(x, u, p) = f(x)u + h(x, p)$ for all $(x, u, p) \in T^*M \times \mathbb{R}$, where f is a smooth function on M .

(i) If $f(x) > 0$ for all $x \in M$, then for any $a < 0$,

$$\sup_{x \in M} (f(x)a + h(x, 0)) \leq \sup_{x \in M} (f(x)a) + \sup_{x \in M} h(x, 0) = a \inf_{x \in M} f(x) + \sup_{x \in M} h(x, 0).$$

Letting $a \rightarrow -\infty$, we get that $\mathfrak{c}_l = -\infty$. Similarly, for any $a > 0$, we have that

$$\inf_{x \in M} (f(x)a + h(x, 0)) \geq \inf_{x \in M} (f(x)a) + \inf_{x \in M} h(x, 0) = a \inf_{x \in M} f(x) + \inf_{x \in M} h(x, 0).$$

Letting $a \rightarrow +\infty$, we get that $\mathfrak{c}_r = +\infty$.

(ii) For case $f(x) < 0$ for all $x \in M$, one can get the same results in a similar manner.

Example 2. Let $H(x, u, p) = V(u) + h(x, p)$ for all $(x, u, p) \in T^*M \times \mathbb{R}$, where $V(u)$ is a smooth function on \mathbb{R} and $\|V'\|_\infty \leq \lambda$.

(i) ($\mathfrak{c}_l \in \mathbb{R}$, $\mathfrak{c}_r = +\infty$). Assume, in addition, V is bounded from below and $\sup_{u \in \mathbb{R}} V(u) = +\infty$. Since $V(u) + h(x, p)$ is bounded from below, then

$$\mathfrak{c}_l = \inf_{u \in \text{Lip}(M)} \sup_{x \in \text{Dom}(Du)} V(u) + h(x, Du(x)) > -\infty,$$

and

$$\mathbf{c}_l \leq V(a) + \sup_{x \in M} h(x, 0), \quad \forall a \in \mathbb{R}.$$

Thus, $\mathbf{c}_l \in \mathbb{R}$. Note that,

$$\sup_{a \in \mathbb{R}} V(a) + \inf_{x \in M} h(x, 0) = +\infty.$$

Thus, we get that

$$\mathbf{c}_r = \sup_{u \in \text{SCL}^+(M)} \inf_{x \in \text{Dom}(Du)} \left(V(u) + h(x, Du(x)) \right) = +\infty.$$

(ii) ($\mathbf{c}_l = -\infty$, $\mathbf{c}_r \in \mathbb{R}$). Assume, in addition, V is bounded from above and $\inf_{u \in \mathbb{R}} V(u) = -\infty$.

$$\mathbf{c}_l \leq \inf_{a \in \mathbb{R}} (V(a) + \sup_{x \in M} h(x, 0)) = \inf_{a \in \mathbb{R}} V(a) + \sup_{x \in M} h(x, 0) = -\infty.$$

Notice that for any $u \in \text{SCL}^+(M)$,

$$\inf_{x \in M} (V(u(x)) + h(x, Du(x))) \leq \sup_{x \in M} V(u(x)) + \inf_{x \in M} h(x, Du(x)) \leq \sup_{x \in M} V(u(x)) + h(y, 0),$$

where y is an arbitrary point in M with $Du(y) = 0$. Hence, we deduce that $\mathbf{c}_r \in \mathbb{R}$.

Example 3 ($\mathbf{c}_l = \mathbf{c}_r = 0$). Let $H(x, u, p) = \|p\|_x^2$ for all $(x, u, p) \in T^*M \times \mathbb{R}$. For each $u \in \text{SCL}^+(M)$,

$$\inf_{x \in M} \|Du(x)\|_x^2 = 0,$$

which implies that

$$\mathbf{c}_r = \sup_{u \in \text{SCL}^+(M)} \inf_{x \in M} \|Du(x)\|_x^2 = 0.$$

By definition, it is direct to see that

$$\mathbf{c}_l = \inf_{u \in \text{SCL}^+(M)} \sup_{x \in M} \|Du(x)\|_x^2 = 0.$$

In this example, $\mathfrak{C} = \{0\}$.

5 Appendix

5.1 Proof of Proposition 5

In order to prove the proposition 5, we provide a preliminary lemma.

Lemma 2. *Given any $x_0, x \in M$, $u_0 \in \mathbb{R}$ and $t > 0$, let $\gamma : [0, t] \rightarrow M$ be a minimizer of $h_{x_0, u_0}(x, t)$. Then for each $t_0 \in (0, t)$, there is a unique minimizer of $h_{x_0, u_0}(\gamma(t_0), t_0)$.*

Proof. Since γ is a minimizer of $h_{x_0, u_0}(x, t)$, then $\gamma|_{[0, t_0]}$ is a minimizer $h_{x_0, u_0}(\gamma(t_0), t_0)$. If there is another minimizer $h_{x_0, u_0}(\gamma(t_0), t_0)$, denoted by α , then we will show that $\alpha = \gamma|_{[0, t_0]}$. Let

$$\beta(s) := \begin{cases} \alpha(s), & s \in [0, t_0], \\ \gamma(s), & s \in [t_0, t]. \end{cases}$$

Then we get

$$\begin{aligned} h_{x_0, u_0}(x, t) &= h_{x_0, u_0}(\gamma(t_0), t_0) + \int_{t_0}^t L(\gamma(s), h_{x_0, u_0}(\gamma(s), s), \dot{\gamma}(s)) ds \\ &= h_{x_0, u_0}(\alpha(t_0), t_0) + \int_{t_0}^t L(\gamma(s), h_{x_0, u_0}(\gamma(s), s), \dot{\gamma}(s)) ds \\ &= u_0 + \int_0^{t_0} L(\alpha(s), h_{x_0, u_0}(\alpha(s), s), \dot{\alpha}(s)) ds \\ &\quad + \int_{t_0}^t L(\gamma(s), h_{x_0, u_0}(\gamma(s), s), \dot{\gamma}(s)) ds \\ &= u_0 + \int_0^t L(\beta(s), h_{x_0, u_0}(\beta(s), s), \dot{\beta}(s)) ds, \end{aligned}$$

which implies that β is a minimizer of $h_{x_0, u_0}(x, t)$. From Proposition 1, γ and β are both of class C^1 . Therefore, we have $\dot{\gamma}(t_0) = \dot{\beta}(t_0)$. By Proposition 1 and the uniqueness of solutions of initial value problem of ordinary differential equations, we have $\alpha(s) = \gamma(s)$ for all $s \in [0, t_0]$, which completes the proof. \square

Proof of Proposition 5. We divide the proof in two steps.

Step 1: Given any $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $t_0 \in (t_1, t_2)$, since $(x(t), u(t))$ is globally minimizing, then we have

$$\begin{aligned} u(t_2) &= h_{x(t_1), u(t_1)}(x(t_2), t_2 - t_1), \\ u(t_2) &= h_{x(t_0), u(t_0)}(x(t_2), t_2 - t_0), \\ u(t_0) &= h_{x(t_1), u(t_1)}(x(t_0), t_0 - t_1). \end{aligned}$$

It follows that

$$h_{x(t_1), u(t_1)}(x(t_2), t_2 - t_1) = h_{x(t_0), u(t_0)}(x(t_2), t_2 - t_0) = h_{x(t_0), h_{x(t_1), u(t_1)}(x(t_0), t_0 - t_1)}(x(t_2), t_2 - t_0).$$

In view of Proposition 3(4), there is a minimizer of $h_{x(t_1), u(t_1)}(x(t_2), t_2 - t_1)$, denoted by γ , such that $\gamma(t_0) = x(t_0)$.

Step 2: From the above arguments, there exists a minimizer α of $h_{x(t_1), u(t_1)}(x(t_2+1), t_2-t_1+1)$ such that $x(t_2) = \alpha(t_2)$. By Lemma 2, $\alpha|_{[t_1, t_2]}$ is the unique minimizer of $h_{x(t_1), u(t_1)}(x(t_2), t_2-t_1)$. By the arguments used in Step 1 again, $x(s) = \alpha(s)$ for all $s \in [t_1, t_2]$. Thus, by Proposition 1 and the arbitrariness of t_1 and t_2 with $t_1 < t_2$, $x(t)$ is of class C^1 for $t \in \mathbb{R}$, and $(x(t), u(t), p(t))$ is a solution of (1.2), where $p(t) := \frac{\partial L}{\partial \dot{x}}(x(t), u(t), \dot{x}(t))$. Since

$$\dot{u}(t) = L(x(t), u(t), \dot{x}(t)),$$

it is easy to see that $x(t)|_{[t_1, t_2]}$ is a minimizer of $h_{x(t_1), u(t_1)}(x(t_2), t_2-t_1)$. □

5.2 Proof of Proposition 8

Lemma 3. *If $\varphi \prec L$, then φ is Lipschitz continuous on M .*

Proof. For each $x, y \in M$, let $\gamma : [0, d(x, y)] \rightarrow M$ be a geodesic of length $d(x, y)$, parameterized by arclength and connecting x to y . Since M is compact and φ is continuous, then

$$A_1 := \max_{x \in M} |\varphi(x)| \quad A_2 := \sup\{L(x, u, \dot{x}) \mid x \in M, |u| \leq A_1, \|\dot{x}\|_x = 1\}$$

are well-defined. Since $\|\dot{\gamma}(s)\|_{\gamma(s)} = 1$ for each $s \in [0, d(x, y)]$, we have $L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \leq A_2$. Then by $\varphi \prec L$,

$$\varphi(\gamma(d(x, y))) - \varphi(\gamma(0)) \leq \int_0^{d(x, y)} L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) ds \leq \int_0^{d(x, y)} A_2 ds = A_2 d(x, y).$$

We finish the proof by exchanging the roles of x and y . □

Lemma 4. *Let $\varphi \prec L$ and let $\gamma : [a, b] \rightarrow M$ be a $(\varphi, L, 0)$ -calibrated curve. If φ is differentiable at $\gamma(t)$ for some $t \in (a, b)$, then we have*

$$H(\gamma(t), \varphi(\gamma(t)), D\varphi(\gamma(t))) = 0, \quad D\varphi(\gamma(t)) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t)).$$

Proof. By Lemma 3, φ is Lipschitz continuous on M . We first show that at each point $x \in M$ where $D\varphi(x)$ exists, we have

$$H(x, \varphi(x), D\varphi(x)) \leq 0. \tag{5.1}$$

For any given $\varphi \in T_x M$, let $\alpha : [0, 1] \rightarrow M$ be a C^1 curve such that $\alpha(0) = x$, $\dot{\alpha}(0) = v$. By $\varphi \prec L$, for each $t \in [0, 1]$, we have

$$\varphi(\alpha(t)) - \varphi(\alpha(0)) \leq \int_0^t L(\alpha(s), \varphi(\alpha(s)), \dot{\alpha}(s)) ds.$$

Dividing by $t > 0$ and let $t \rightarrow 0^+$, we have $\langle D\varphi(x), v \rangle \leq L(x, \varphi(x), v)$, which implies $H(x, \varphi(x), D\varphi(x)) = \sup_{v \in T_x M} (\langle D\varphi(x), v \rangle_x - L(x, \varphi(x), v)) \leq 0$. Thus, (5.1) holds.

If φ is differentiable at $\gamma(t)$ for some $t \in (a, b)$, then for each $t' \in [a, b]$ with $t \leq t'$, we have $\varphi(\gamma(t')) - \varphi(\gamma(t)) = \int_t^{t'} L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) ds$, since $\gamma : [a, b] \rightarrow M$ is a $(\varphi, L, 0)$ -calibrated curve. Dividing by $t' - t$ and let $t' \rightarrow t^+$, we have $\langle D\varphi(\gamma(t)), \dot{\gamma}(t) \rangle_{\gamma(t)} = L(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t))$. Thus, we have

$$H(\gamma(t), \varphi(\gamma(t)), D\varphi(\gamma(t))) \geq \langle D\varphi(\gamma(t)), \dot{\gamma}(t) \rangle_{\gamma(t)} - L(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t)) = 0,$$

which together with (5.1) implies $H(\gamma(t), \varphi(\gamma(t)), D\varphi(\gamma(t))) = 0$ and

$$\langle D\varphi(\gamma(t)), \dot{\gamma}(t) \rangle_{\gamma(t)} = H(\gamma(t), \varphi(\gamma(t)), D\varphi(\gamma(t))) + L(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t)).$$

In view of Legendre transform, we get

$$D\varphi(\gamma(t)) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), \varphi(\gamma(t)), \dot{\gamma}(t)).$$

This completes the proof. \square

Lemma 5. *Given any $a > 0$, let $\varphi \prec L$ and let $\gamma : [-a, a] \rightarrow M$ be a $(\varphi, L, 0)$ -calibrated curve. Then φ is differentiable at $\gamma(0)$.*

Proof. It suffices to prove the lemma for the case when $M = U$ is an open subset of \mathbb{R}^n . Set $x = \gamma(0)$. In order to prove the differentiability of u at x , we only need to show for each $y \in U$, there holds

$$\limsup_{\lambda \rightarrow 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \leq \frac{\partial L}{\partial \dot{x}}(x, \varphi(x), \dot{\gamma}(0)) \cdot y \leq \liminf_{\lambda \rightarrow 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda}. \quad (5.2)$$

For $\lambda > 0$ and $0 < \varepsilon \leq a$, define $\gamma_\lambda : [-\varepsilon, 0] \rightarrow U$ by $\gamma_\lambda(s) = \gamma(s) + \frac{s+\varepsilon}{\varepsilon} \lambda y$. Then $\gamma_\lambda(0) = x + \lambda y$ and $\gamma_\lambda(-\varepsilon) = \gamma(-\varepsilon)$. Since $u \prec L$ and $\gamma : [-a, a] \rightarrow M$ is a $(\varphi, L, 0)$ -calibrated curve, we have

$$\varphi(x + \lambda y) - \varphi(\gamma(-\varepsilon)) \leq \int_{-\varepsilon}^0 L(\gamma_\lambda(s), \varphi(\gamma_\lambda(s)), \dot{\gamma}_\lambda(s)) ds,$$

and

$$\varphi(x) - \varphi(\gamma(-\varepsilon)) = \int_{-\varepsilon}^0 L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) ds.$$

It follows that

$$\frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \leq \frac{1}{\lambda} \int_{-\varepsilon}^0 \left(L(\gamma_\lambda(s), \varphi(\gamma_\lambda(s)), \dot{\gamma}_\lambda(s)) - L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \right) ds.$$

By Lemma 3, there exists $K > 0$ such that

$$|\varphi(\gamma_\lambda(s)) - \varphi(\gamma(s))| \leq K \|\gamma_\lambda(s) - \gamma(s)\| = K \cdot \frac{s + \varepsilon}{\varepsilon} \cdot \lambda \|y\|,$$

which implies

$$\begin{aligned} \limsup_{\lambda \rightarrow 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} &\leq \int_{-\varepsilon}^0 \left(\frac{s + \varepsilon}{\varepsilon} \cdot \frac{\partial L}{\partial x}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \cdot y \right. \\ &\quad \left. + K \frac{s + \varepsilon}{\varepsilon} \left| \frac{\partial L}{\partial \varphi}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \right| \|y\| \right. \\ &\quad \left. + \frac{1}{\varepsilon} \frac{\partial L}{\partial \dot{x}}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \cdot y \right) ds. \end{aligned}$$

If we let $\varepsilon \rightarrow 0^+$, we get the first inequality in (5.2).

Define $\gamma_\lambda : [0, \varepsilon] \rightarrow M$ by $\gamma_\lambda(s) = \gamma(s) + \frac{\varepsilon - s}{\varepsilon} \lambda y$. We have

$$\begin{aligned} \varphi(\gamma(\varepsilon)) - \varphi(x + \lambda y) &\leq \int_0^\varepsilon L(\gamma_\lambda(s), \varphi(\gamma_\lambda(s)), \dot{\gamma}_\lambda(s)) ds, \\ \varphi(\gamma(\varepsilon)) - \varphi(x) &= \int_0^\varepsilon L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) ds. \end{aligned}$$

It follows that

$$\frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \geq \frac{1}{\lambda} \int_0^\varepsilon \left(L(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) - L(\gamma_\lambda(s), \varphi(\gamma_\lambda(s)), \dot{\gamma}_\lambda(s)) \right) ds,$$

which implies

$$\begin{aligned} \liminf_{\lambda \rightarrow 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} &\geq \int_0^\varepsilon \left(\frac{s - \varepsilon}{\varepsilon} \frac{\partial L}{\partial x}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \cdot y \right. \\ &\quad \left. + K \frac{s - \varepsilon}{\varepsilon} \left| \frac{\partial L}{\partial \varphi}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \right| \|y\| \right. \\ &\quad \left. + \frac{1}{\varepsilon} \frac{\partial L}{\partial \dot{x}}(\gamma(s), \varphi(\gamma(s)), \dot{\gamma}(s)) \cdot y \right) ds. \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$, we obtain the second inequality in (5.2). This completes the proof. \square

Proof of Proposition 8. Let $\mathbf{u}(t) := u(\gamma(t))$ for $t \leq 0$. We assert that for each $s, t < 0$ with $s < t$, there holds

$$\mathbf{u}(t) = h_{\gamma(s), \mathbf{u}(s)}(\gamma(t), t - s). \quad (5.3)$$

If the assertion is true, then by Proposition 5, $(\gamma(t), \mathbf{u}(t), p(t))$ satisfies equations (1.2) on $(-\infty, 0)$, where $p(t) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), \mathbf{u}(t), \dot{\gamma}(t))$. Now we prove the assertion. Since u is a

backward weak KAM solution, then we have $T_\sigma^- u(x) = u(x), \forall x \in M, \forall \sigma \geq 0$. Recall that $T_\sigma^- u(x) = \inf_{y \in M} h_{y, u(y)}(x, \sigma)$ for all $\sigma > 0$. Given any $s < t \leq 0$, we get

$$\mathbf{u}(\tau) \leq h_{\gamma(s), \mathbf{u}(s)}(\gamma(\tau), \tau - s), \quad \forall \tau \in (s, t]. \quad (5.4)$$

Since $\gamma : (-\infty, 0] \rightarrow M$ is a $(u, L, 0)$ -calibrated curve, then we have

$$\mathbf{u}(t) - \mathbf{u}(s) = \int_s^t L(\gamma(\tau), \mathbf{u}(\tau), \dot{\gamma}(\tau)) d\tau,$$

which together with (5.4) implies

$$\mathbf{u}(t) \geq \mathbf{u}(s) + \int_s^t L(\gamma(\tau), h_{\gamma(s), \mathbf{u}(s)}(\gamma(\tau), \tau - s), \dot{\gamma}(\tau)) d\tau \geq h_{\gamma(s), \mathbf{u}(s)}(\gamma(t), t - s).$$

By (5.4) again, we have $\mathbf{u}(t) = h_{\gamma(s), \mathbf{u}(s)}(\gamma(t), t - s)$. Hence, (5.3) holds.

By Lemma 4 and Lemma 5, u is differentiable at $\gamma(t)$ for any $t < 0$ and

$$Du(\gamma(t)) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), u(\gamma(t)), \dot{\gamma}(t)).$$

Hence, $(\gamma(t+s), u(\gamma(t+s)), Du(\gamma(t+s))) = \Phi_s(\gamma(t), u(\gamma(t)), Du(\gamma(t))), \forall t, s < 0$. In view of Lemma 4, we have

$$H(\gamma(t), u(\gamma(t)), \frac{\partial L}{\partial \dot{x}}(\gamma(t), u(\gamma(t)), \dot{\gamma}(t))) = 0, \quad \forall t < 0,$$

which completes the proof. \square

5.3 Proof of Lemma 6

Lemma 6. *Let $\varphi \in \text{SCL}^-(M)$ and $\psi \in \text{SCL}^+(M)$. Let x_0 be a local minimum point of $\varphi - \psi$. Then both φ and ψ are differentiable at x_0 with $D\varphi(x_0) = D\psi(x_0)$.*

Proof. Since $\varphi, -\psi, \varphi - \psi \in \text{SCL}^-(M)$, then $D^+\varphi(x_0), D^+(-\psi)(x_0), D^+(\varphi - \psi)(x_0)$ are non-empty. Since x_0 is a local minimum point of $\varphi - \psi$, then

$$D^+\varphi(x_0) + D^+(-\psi)(x_0) \subset D^+(\varphi - \psi)(x_0) = D(\varphi - \psi)(x_0) = \{0\},$$

which implies that both $D^+\varphi(x_0)$ and $D^+(-\psi)(x_0)$ are singletons. \square

This paper has no associated data.

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References

- [1] M.-C. Arnaud, Pseudographs and the Lax-Oleinik semi-group: a geometric and dynamical interpretation, *Nonlinearity* **24** (2011), 71–78.
- [2] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control: Foundations & Applications. Birkhäuser, 1997.
- [3] G. Barles and P. E. Souganidis, On the large time behavior of solutions of Hamilton-Jacobi equations, *SIAM J. Math. Anal.* **31** (2000), 925–939.
- [4] P. Bernard, Existence of $C^{1,1}$ critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds, *Ann. Sci. École Norm. Sup.* **40** (2007), 445–452.
- [5] P. Bernard, The dynamics of pseudographs in convex Hamiltonian systems, *J. Amer. Math. Soc.* **21** (2008), 615–669.
- [6] P. Cannarsa and H. Soner, Generalized one-sided estimates for solutions of Hamilton-Jacobi equations and applications, *Nonlinear Anal.* **13** (1989), 305–323.
- [7] P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*. Progress in Nonlinear Differential Equations and their Applications, **58**, Birkhäuser Boston, Inc., Boston, MA, 2004.
- [8] P. Cannarsa, W. Cheng, L. Jin, K. Wang and J. Yan, Herglotz’ variational principle and Lax-Oleinik evolution, *J. Math. Pures Appl.* **141** (2020), 99–136.
- [9] G. Contreras, R. Iturriaga, G. Paternain and M. Paternain, Lagrangian graphs, minimizing measures and Mañé’s critical values, *Geom. Funct. Anal.* **8** (1998), 788–809.
- [10] G. Contreras, Action potential and weak KAM solutions, *Calc. Var. Partial Differential Equations* **13** (2001), 427–458.
- [11] M. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **277** (1983), 1–42.
- [12] M. Crandall, L. Evans and P.-L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **282** (1984), 487–502.
- [13] M.-O. Czarnecki and L. Rifford, Approximation and regularization of Lipschitz functions: convergence of the gradients, *Trans. Amer. Math. Soc.* **358** (2006), 4467–4520.
- [14] A. Davini and A. Siconolfi, A generalized dynamical approach to the large time behavior of solutions of Hamilton-Jacobi equations, *SIAM J. Math. Anal.* **38** (2006), 478–502.
- [15] A. Davini, A. Fathi, R. Iturriaga and M. Zavidovique, Convergence of the solutions of the discounted Hamilton-Jacobi equation: convergence of the discounted solutions, *Invent. Math.* **206** (2016), 29–55.

- [16] M. Entov, L. Polterovich, Contact topology and non-equilibrium thermodynamics, arXiv:2101.03770v2.
- [17] L. Evans, Periodic homogenisation of certain fully nonlinear partial differential equations, Proc. Roy. Soc. Edinburgh Sect. A **206** (1992), 245–265.
- [18] A. Fathi, Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), 1043–1046.
- [19] A. Fathi, Solutions KAM faibles conjuguées et barrières de Peierls, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), 649–652.
- [20] A. Fathi, Sur la convergence du semi-groupe de Lax-Oleinik, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), 267–270.
- [21] A. Fathi, Weak KAM Theorem in Lagrangian Dynamics, www.math.u-bordeaux.fr/~pthieull/Recherche/KamFaible/Publications/Fathi2008_01.
- [22] A. Fathi and A. Siconolfi, Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation, Invent. math. **155** (2004), 363–388.
- [23] A. Figalli and L. Rifford. *Aubry sets, Hamilton-Jacobi equations, and the Mañé conjecture*. Geometric analysis, mathematical relativity, and nonlinear partial differential equations 599 (2012): 83–104.
- [24] A. Figalli, D. Gomes and D. Marcon, Weak KAM theory for a weakly coupled system of Hamilton-Jacobi equations, Calc. Var. Partial Differential Equations **55** (2016), no. 4, Art. 79, 32 pp.
- [25] D. Gomes, Generalized Mather problem and selection principles for viscosity solutions and Mather measures, Adv. Calc. Var. **1** (2008), 291–307.
- [26] G. Herglotz, Berührungstransformationen, Lectures at the University of Göttingen, Göttingen, 1930.
- [27] H. Ishii, Asymptotic solutions for large time of Hamilton-Jacobi equations, International Congress of Mathematicians, Vol. III, 213–227, Eur. Math. Soc., Zürich, 2006.
- [28] W. Jing, H. Mitake and H. Tran, Generalized ergodic problems: existence and uniqueness structures of solutions, J. Differential Equations **268** (2020), 2886–2909.
- [29] V. Kaloshin, Mather theory, weak KAM theory, and viscosity solutions of Hamilton-Jacobi PDE's. (English summary) EQUADIFF 2003, 39-48, World Sci. Publ., Hackensack, NJ, 2005.
- [30] P.-L. Lions, Generalized solutions of Hamilton-Jacobi equations. Pitman, Boston, 1982.
- [31] P.-L. Lions, G. Papanicolaou and S. R. S. Varadhan, Homogenization of Hamilton-Jacobi Equations, unpublished.

- [32] E. Maderna and A. Venturelli, Viscosity solutions and hyperbolic motions: a new PDE method for the N-body problem, *Ann. of Math. (2)* **192** (2020), 499–550.
- [33] S. Marò and A. Sorrentino, Aubry-Mather theory for conformally symplectic systems, *Commun. Math. Phys.* **354** (2017), 775–808.
- [34] J. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, *Math. Z.* **207** (1991), 169–207.
- [35] H. Mitake and K. Soga, Weak KAM theory for discounted Hamilton-Jacobi equations and its application, *Calc. Var. Partial Differential Equations* **57** (2018), Paper No. 78, 32 pp.
- [36] G. Namah and J.-M. Roquejoffre, Remarks on the long time behaviour of the solutions of Hamilton-Jacobi equations, *Comm. Partial Differential Equations* **24** (1999), 883–893.
- [37] J.-M. Roquejoffre, Convergence to steady states or periodic solutions in a class of Hamilton-Jacobi equations, *J. Math. Pures Appl.* **80** (2001), 85–104.
- [38] A. Sorrentino, Action-minimizing methods in Hamiltonian dynamics. An introduction to Aubry-Mather theory. *Mathematical Notes*, 50. Princeton University Press, Princeton, NJ, 2015. xii+115 pp.
- [39] A. Sorrentino, On John Mather’s seminal contributions in Hamiltonian dynamics, *Methods Appl. Anal.* **26** (2019), 37–63.
- [40] H. Tran, Hamilton-Jacobi equations: viscosity and applications, <https://people.math.wisc.edu/~hung/HJ%20equations-viscosity%20solutions%20and%20>
- [41] K. Wang and J. Yan, A new kind of Lax-Oleinik type operator with parameters for time-periodic positive definite Lagrangian systems, *Comm. Math. Phys.* **309** (2012), 663–691.
- [42] K. Wang, L. Wang and J. Yan, Implicit variational principle for contact Hamiltonian systems, *Nonlinearity* **30** (2017), 492–515.
- [43] K. Wang, L. Wang and J. Yan, Variational principle for contact Hamiltonian systems and its applications, *J. Math. Pures Appl.* **123** (2019), 167–200.
- [44] K. Wang, L. Wang and J. Yan, Aubry-Mather theory for contact Hamiltonian systems, *Commun. Math. Phys.* **366** (2019), 981–1023.
- [45] K. Wang, L. Wang and J. Yan, Weak KAM solutions of Hamilton-Jacobi equations with decreasing dependence on unknown functions, *J. Differential Equations* **286** (2021), 411–432.