A class of solutions of the asymmetric May-Leonard model

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Abstract

The asymmetric May-Leonard model is a prototypical system of 3 nonlinearly coupled first-order Ordinary Differential Equations with second-degree polynomial right-hand sides. In this short paper we identify a class of *special* solutions of this system which do not seem to have been previously advertised in spite of their rather elementary character.

1 Introduction and results

The May-Leonard model [1] is a prototypical system, introduced almost half a century ago, of 3 nonlinearly coupled first-order Ordinary Differential Equations with second-degree polynomial right-hand sides. In self-evident notation it reads as follows:

$$\dot{\xi}_1(t) = \xi_1(t)\left[1 - \xi_1(t) - \alpha\xi_2(t) - \beta\xi_3(t)\right] \ , \tag{1a}$$

$$\dot{\xi}_2(t) = \xi_2(t)\left[1 - \beta x_1(t) - x_2(t) - \alpha x_3(t)\right] \ , \tag{1b}$$

$$\dot{\xi}_3(t) = \xi_3(t)\left[1 - \alpha x_1(t) - \beta x_2(t) - x_3(t)\right] \ . \tag{1c}$$

The *asymmetric* May-Leonard model (much investigated in the literature: see for instance [2] [3] [4], and references therein) is the following generalization—again, in self-evident notation—of the May-Leonard model (1):

$$\dot{x}_1(t) = x_1(t)\left[\eta - x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right] \ , \tag{2a}$$

$$\dot{x}_2(t) = x_2(t)\left[\eta - a_{21}x_1(t) - x_2(t) - a_{23}x_3(t)\right] \ , \tag{2b}$$

$$\dot{x}_3(t) = x_3(t)\left[\eta - a_{31}x_1(t) - a_{32}x_2(t) - x_3(t)\right] \ . \tag{2c}$$

Here and hereafter the 6 (*constant*) parameters a_{nm} ($n = 1, 2;\ m = 1, 2, 3$) are *a priori arbitrary* (except, of course, for the restrictions on their values identified below); and superimposed dots indicate differentiations with respect to the independent variable t ("time").

Remark 1. The original May-Leonard model (1) clearly corresponds—up to a trivial renaming of the dependent variables—to the asymmetric May-Leonard model (2) with

$$a_{12} = a_{23} = a_{31} = \alpha \ ; \quad a_{13} = a_{21} = a_{32} = \beta \ , \quad \eta = 1 \ . \quad \blacksquare \tag{3}$$

Remark 2. Often the asymmetric May-Leonard model is characterized by the system of 3 ODEs (2) with $\eta = 1$. It is of course trivial to reduce the system (2) to its more standard version with $\eta = 1$ via the following simultaneous rescaling of the independent variable t and the 3 dependent variables $x_n(t)$: $x_n(t) \Rightarrow \eta\tilde{x}_n(\tilde{t})$, $\tilde{t} = \eta t$. We prefer to keep the extra parameter η in view of its relevance to generate the *isochronous* variant of the asymmetric May-Leonard model, see below **Remark 4**. $\blacksquare$

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Via the following well-known (see, for instance, [5]) change of variables,

$$x_n(t) = \exp(\eta t)\, y_n(\tau) \ , \quad \tau = [\exp(\eta t) - 1]/\eta \ , \quad n = 1, 2, 3 \tag{4a}$$

implying

$$x_n(0) = y_n(0) \ , \tag{4b}$$

the new dependent variables $y_n(\tau)$ are easily seen to satisfy the following (still *autonomous*) system of 3 nonlinearly coupled ODEs:

$$y_1'(\tau) = -y_1(\tau)\,[y_1(\tau) + a_{12} y_2(\tau) + a_{13} y_3(\tau)] \ , \tag{5a}$$

$$y_2'(\tau) = -y_2(\tau)\,[a_{21} y_1(\tau) + y_2(\tau) + a_{23} y_3(\tau)] \ , \tag{5b}$$

$$y_3'(\tau) = -y_3(\tau)\,[a_{31} y_1(\tau) + a_{32} y_2(\tau) + y_3(\tau)] \ . \tag{5c}$$

Here and below appended primes indicated of course differentiations with respect to the independent (generally *complex*) variable τ.

The fact that the 3 polynomials in the right-hand sides of this *autonomous* system of 3 ODEs are *homogeneous* in the dependent variables $y_n(\tau)$ implies (see [6]) that this system features the following elementary solution of its initial-values problem:

$$y_n(\tau) = y_n(0)\,/\,(1 + z\tau) \ , \quad n = 1, 2, 3 \ , \tag{6}$$

provided the parameter z, the 3 initial-values $y_n(0)$ and the 6 parameters a_{nm} ($n = 1, 2;\ m = 1, 2, 3$) satisfy the following 3 simple algebraic relations:

$$z = [y_1(0) + a_{12} y_2(0) + a_{13} y_3(0)] \ , \tag{7a}$$

$$z = [a_{21} y_1(0) + y_2(0) + a_{23} y_3(0)] \ , \tag{7b}$$

$$z = [a_{31} y_1(0) + a_{32} y_2(0) + y_3(0)] \ . \tag{7c}$$

We can therefore formulate (via (4a)) our main finding.

Proposition: the asymmetric May-Leonard model (2) admits the following solution of its initial-values problem,

$$x_n(t) = x_n(0)\,/\,\{\exp(-\eta t) + z\,[1 - \exp(-\eta t)]/\eta\} \ , \quad n = 1, 2, 3 \ , \tag{8}$$

with

$$\begin{aligned} z &= x_1(0) + a_{12} x_2(0) + a_{13} x_3(0) \\ &= a_{21} x_1(0) + x_2(0) + a_{23} x_3(0) \\ &= a_{31} x_1(0) + a_{32} x_2(0) + x_3(0) \ , \end{aligned} \tag{9}$$

provided the 3 initial-values $x_n(0)$ and the 6 parameters a_{nm} ($n = 1, 2;\ m = 1, 2, 3$) satisfy the following 2 simple algebraic relations (implied by (9)):

$$\begin{aligned} & x_1(0) + a_{12} x_2(0) + a_{13} x_3(0) \\ = \ & a_{21} x_1(0) + x_2(0) + a_{23} x_3(0) \\ = \ & a_{31} x_1(0) . + a_{32} x_2(0) + x_3(0) \ . \quad \blacksquare \end{aligned} \tag{10}$$

Remark 3. For *any given* assignment of any subset of 7 out of the 9 parameters a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, a_{32}, $x_1(0)$, $x_2(0)$, $x_3(0)$, these relations (10) determine (easily and uniquely) the other 2 of these 9 parameters in terms of the 7 *arbitrarily* assigned; thereby identifying—via (8) with (9)—the corresponding *explicit* solution of the initial-values problem of the asymmetric May-Leonard model (2). $\blacksquare$

Remark 4. It is obvious that if η is an *imaginary* number, $\eta = \mathbf{i}\omega$ (with $\mathbf{i}$ the *imaginary* unit, $\mathbf{i}^2 = -1$, and ω an *arbitrary nonvanishing real* number), then *all* the solutions of the asymmetric May-Leonard model (2) given by (8) with (9) are *isochronous*, i. e. *completely periodic* with period $T = 2\pi/|\omega|$, $x_n(t+T) = x_n(t)$. However in this case one would be dealing with a model involving *complex* dependent variables, $x_n(t) = \mathrm{Re}[x_n(t)] + \mathbf{i}\,\mathrm{Im}[x_n(t)]$, i. e. 6 *real* variables rather than only 3; and it would be then reasonable to also *double* the number of *real* parameters by setting $a_{nm} = \mathrm{Re}\,[a_{nm}] + \mathbf{i}\,\mathrm{Im}\,[a_{nm}]$. And we leave to the interested reader to consider the behavior of the solution (8) in the more general case when the parameter η is itself a *complex* number, $\eta = \mathrm{Re}\,[\eta] + \mathbf{i}\,\mathrm{Im}\,[\eta]$. ■

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