

Shift-like Operators on $L^p(X)$

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Abstract

In this article we develop a general technique which takes a known characterization of a property for weighted backward shifts and lifts it up to a characterization of that property for a large class of operators on $L^p(X)$. We call these operators “shift-like”. The properties of interest include chaotic properties such as Li-Yorke chaos, hypercyclicity, frequent hypercyclicity as well as properties related to hyperbolic dynamics such as shadowing, expansivity and generalized hyperbolicity. Shift-like operators appear naturally as composition operators on $L^p(X)$ when the underlying space is a dissipative measure system. In the process of proving the main theorem, we provide some results concerning when a property is shared by a linear dynamical system and its factors.

1 Introduction

Weighted shifts is an important class of operators in linear dynamics. It is a tool for constructing examples and counterexamples as well as inspiring new conjectures. Many dynamical properties are studied and characterized for weighted shifts before the general theory of the dynamical property in question reveals itself.

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A systematic study of composition operators on $L^p(X)$ in the setting of linear dynamics was initiated in [1, 7]. The motivation for the study of composition operators is to have a concrete and large class of operators which can be utilized as examples and counterexamples in linear dynamics. The class of composition operators includes weighted shifts as a special case. It also includes other large classes of operators, such as composition operators induced by measures on odometers [10].

Let us briefly recall the composition operators on $L^p(X)$, for $1 \leq p < \infty$. We start with a σ -finite measure space (X, \mathcal{B}, μ) and a bimeasurable invertible map $f : X \rightarrow X$ such that the Radon-Nikodym derivative of $\mu \circ f$ with respect to μ is bounded below. Then, $T_f : L^p(X) \rightarrow L^p(X)$ defined by

$$T_f(\varphi) = \varphi \circ f$$

is a well-defined bounded linear operator on $L^p(X)$ known as the composition operator.

For a general $(X, \mathcal{B}, \mu, f, T_f)$ as above, Bayart, Pires and the second author [1] gave necessary and sufficient conditions on f that guarantee hypercyclicity and mixing of T_f . Bernardes, Pires and the second author [7] gave necessary and sufficient conditions on f that guarantee that T_f is Li-Yorke chaotic. These are very general characterizations and, as a specific case, when $X = \mathbb{Z}$ and f is the +1-map, they yield well-known characterizations of hypercyclicity, mixing and Li-Yorke chaos for weighted backward shifts.

Unfortunately, the behaviors of frequent hypercyclicity and the shadowing property in the setting of composition operators are complicated. This is not unexpected as it took some time to complete the characterization of frequent hypercyclicity even for weighted backward shifts. Using ideas of Bayart and Ruzsa [3], the second author and Pires [13] gave a characterization of frequently hypercyclic operators among composition operators on $L^p(X)$ when the measure system is dissipative and of bounded distortion.

In a parallel development concerning the shadowing property, Bernardes and Messaoudi [9] gave a characterization of the shadowing property for the class of weighted backward shifts. The current authors of this article [12] lifted their result to the setting of composition operators where the measure system is dissipative and of bounded distortion.

Motivated by these results, we develop a general method which takes a characterization of a linear dynamical property for weighted backward shifts and translates it into the setting of composition operators on $L^p(X)$ where the underlying system (X, \mathcal{B}, μ, f) is a dissipative, measurable system of bounded distortion.

Before stating the main result of the article, let us fix some notation and terminology. Let (X, \mathcal{B}, μ) be a σ -finite measure space and $f : X \rightarrow X$ be

a bimeasurable invertible map such that the Radon-Nikodym derivatives of $\mu \circ f$ and $\mu \circ f^{-1}$ with respect to μ are bounded below. In particular, this implies that the composition operators T_f and $T_{f^{-1}}$ are well-defined, invertible, bounded linear operators with $T_f^{-1} = T_{f^{-1}}$. We say that (X, \mathcal{B}, μ, f) is a *dissipative system generated by W* if $W \in \mathcal{B}$, $0 < \mu(W) < \infty$ and $X = \dot{\cup}_{k=-\infty}^{+\infty} f^k(W)$, where the symbol $\dot{\cup}$ denotes pairwise disjoint union (we are slightly deviating from the usual definition of dissipative system, i.e., $\mu(W) < \infty$ is usually not required). Moreover, if there exists $K > 0$ such that

$$\frac{1}{K} \frac{\mu(f^k(W))}{\mu(W)} \leq \frac{\mu(f^k(B))}{\mu(B)} \leq K \frac{\mu(f^k(W))}{\mu(W)} \quad (\diamond)$$

for all $k \in \mathbb{Z}$ and all measurable $B \subseteq W$ with $\mu(B) > 0$, then we say that (X, \mathcal{B}, μ, f) is a *dissipative system of bounded distortion generated by W* . We will call $(X, \mathcal{B}, \mu, f, T_f)$ a *dissipative composition dynamical system of bounded distortion, generated by set W* [12, Definition 2.6.3]. It is easy to see a connection between such a system and weighted backward shifts. Namely, if we let B_w be the bilateral weighted backward shift on $\ell^p(\mathbb{Z})$, with the weight sequence

$$w_k = \left(\frac{\mu(f^{k-1}(W))}{\mu(f^k(W))} \right)^{\frac{1}{p}}, \quad (1)$$

then B_w is a linear factor of T_f in a natural way. More specifically, $\Gamma : L^p(X) \rightarrow \ell^p(\mathbb{Z})$, defined by

$$\Gamma(\varphi)(k) = \frac{\mu(f^k(W))^{\frac{1}{p}}}{\mu(W)} \int_W \varphi \circ f^k d\mu, \quad (2)$$

shows that B_w is a linear factor of T_f [12, Lemma 4.2.3], i.e., the following diagram commutes:

$$\begin{array}{ccc} L^p(X) & \xrightarrow{T_f} & L^p(X) \\ \downarrow \Gamma & & \downarrow \Gamma \\ \ell^p(\mathbb{Z}) & \xrightarrow{B_w} & \ell^p(\mathbb{Z}) \end{array}$$

We now state the main result of the paper.

Theorem M. *The operator T_f has Property P, if and only if B_w has Property P where Property P denotes one of: (1) Li-Yorke chaos; (2) hypercyclicity; (3) mixing; (4) chaos; (5) frequent hypercyclicity; (6) expansivity; (7) uniform expansivity; (8) the shadowing property.*

As the operator T_f of a dissipative composition dynamical system of bounded distortion $(X, \mathcal{B}, \mu, f, T_f)$ behaves similarly to shifts, we call these types of operators *shift-like*.

We refer the reader to [12, 13] for basic definitions and a fuller discussion of the relationship between weighted shifts, composition operators and the role played by dissipativity and bounded distortion.

2 Preliminary Definitions

In this section we briefly recall basic definitions and terminology. Next to the definitions, we provide references where one may find fuller and detailed discussions concerning the treated concepts.

Let $\{w_i\}_{i \in \mathbb{Z}}$ be a bounded sequence of positive numbers. A *weighted backward shift with weights* $\{w_i\}_{i \in \mathbb{Z}}$ is a bounded linear operator $B_w : \ell^p(\mathbb{Z}) \rightarrow \ell^p(\mathbb{Z})$ defined by $B_w(\mathbf{x})(i) = w_{i+1}x_{i+1}$. Moreover, if $\{w_i\}_{i \in \mathbb{Z}}$ is bounded away from zero, then B_w is invertible.

Throughout the paper, we frequently use the composition operator representation of the weighted backward shift B_w . More specifically, we have the following proposition.

Proposition 2.1. *Every weighted backward shift B_w is conjugate, by an isometry, to the composition operator $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \nu, g, T_g)$, where*

$$g(i) = i + 1,$$

$$\nu(0) = 1, \quad \nu(i) = \begin{cases} \frac{1}{(w_1 \cdots w_i)^p}, & i > 0 \\ (w_{i+1} \cdots w_0)^p, & i < 0. \end{cases}$$

Moreover, for every $i \in \mathbb{Z}$,

$$w_i = \left(\frac{\nu(i-1)}{\nu(i)} \right)^{\frac{1}{p}},$$

and when B_w is given by (1), we have

$$\nu(i) = \frac{\mu(f^i(W))}{\mu(W)}.$$

Proof. That T_g is conjugate to B_w is witnessed by the isometry

$$\mathbf{x} \in \ell^p(\mathbb{Z}) \mapsto \{x_i(\nu(i))^{\frac{1}{p}}\}_{i \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$$

□

Let $T : X \rightarrow X$ be a bounded linear operator acting on a separable Banach space X .

Definition 2.2. *The operator $T : X \rightarrow X$ is said to be*

- [4, Theorem 5] Li-Yorke chaotic *if T admits an irregular vector, that is, a vector $x \in X$ such that $\underline{\lim}_{n \rightarrow \infty} \|T^n(x)\| = 0$ and $\overline{\lim}_{n \rightarrow \infty} \|T^n(x)\| = \infty$;*
- hypercyclic *if T admits a hypercyclic vector, i.e., if there exists x in X such that $\{T^n(x) : n \in \mathbb{N}\}$ is dense in X . Moreover if the set of periodic points of T is dense in X , then T is said to be chaotic;*
- topologically mixing *if for any pair of non-empty open subsets U, V of X , there is $k_0 \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$, for all $k \geq k_0$;*
- frequently hypercyclic *if T admits a frequently hypercyclic vector, i.e., a vector $x \in X$ such that for each non-empty open subset U of X ,*

$$\underline{\lim}_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : T^n(x) \in U\} > 0.$$

The definition of Li-Yorke chaos stated above is equivalent to the standard definition of Li-Yorke chaos ([4, 5]). We refer the reader to [2, 14] for further information concerning the other definitions stated above.

Expansivity and the shadowing property play a fundamental role in hyperbolic dynamics. In the context of linear dynamics, they have alternate formulations equivalent to the ones we use [6].

Definition 2.3. *An invertible operator $T : X \rightarrow X$*

- *is expansive if for each x with $\|x\| = 1$, there exists $n \in \mathbb{Z}$ such that $\|T^n x\| \geq 2$;*
- *is uniformly expansive if there exists $n \in \mathbb{N}$ such that*

$$z : \|z\| = 1 \implies \|T^n z\| \geq 2 \text{ or } \|T^{-n} z\| \geq 2;$$

- *has the shadowing property if there is a constant $K > 0$ such that, for every bounded sequence $\{z_n\}_{n \in \mathbb{Z}}$ in X , there is a sequence $\{y_n\}_{n \in \mathbb{Z}}$ in X such that*

$$\sup_{n \in \mathbb{Z}} \|y_n\| \leq K \sup_{n \in \mathbb{Z}} \|z_n\| \quad \text{and} \quad y_{n+1} = Ty_n + z_n, \text{ for all } n \in \mathbb{Z}.$$

In the definitions of expansivity and uniform expansivity, the number 2 can be replaced by any number $c > 1$. For further information on this topic, we refer the reader to [6, 8, 9, 11].

3 Properties Preserved By Factors

In this section, we discuss and state some results when a linear dynamical property is shared by factor maps. Throughout this section, let X and Y be separable Banach spaces, S and T be bounded, invertible, linear operators on X and Y , respectively, and $\Pi : X \rightarrow Y$ be a factor map, i.e., a linear, bounded, surjection for which the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{S} & X \\ \downarrow \Pi & & \downarrow \Pi \\ Y & \xrightarrow{T} & Y \end{array}$$

Next, we state a result which shows when a property carries over to the factor map. It is well-known and easy to show that if S is hypercyclic, mixing, chaotic or frequently hypercyclic, then so is T [14]. We show that if some extra conditions hold on Π , then many of the dynamical properties pass from S to T .

In the sequel, given two sequences of positive reals, $\{a_k\}_{k \in \mathbb{Z}}$ and $\{b_k\}_{k \in \mathbb{Z}}$, and a constant $L \geq 1$, by $a_k \sim_L b_k$, we mean that, for every $k \in \mathbb{Z}$,

$$a_k \leq Lb_k \quad \& \quad b_k \leq La_k.$$

Definition 3.1. *We say that Π admits a strong bounded selector if*

$$\exists L \geq 1 \text{ s.t., } \forall y \in Y, \exists x \in \Pi^{-1}(y) \text{ with } \|S^n(x)\| \sim_L \|T^n(y)\|.$$

We have the following result.

Lemma 3.2. *For an arbitrary factor map Π ,*

(1) *if S has the shadowing property, also T has the shadowing property.*

If the factor map Π admits a strong bounded selector, the following statements hold:

(2) *if S is expansive, then T is expansive;*

(3) *if S is uniformly expansive, then T is uniformly expansive;*

Proof. (1). This was proved in [12, Lemma 4.2.2] with the additional hypothesis that Π admits a bounded selector, i.e., there exists $L > 0$ such that for all $y \in Y$, there exists $x \in \Pi^{-1}(y)$ with $\|x\| \leq L\|y\|$. We thank Nilson Bernardes for pointing out that the Open Mapping Theorem implies that this

is always the case. Indeed, let $\delta > 0$ be such that $\overline{B_Y(0, \delta)} \subseteq \Pi(B_X(0, 1))$. Then, for any $y \in Y$, we have that there is $x \in \Pi^{-1}(y)$ with $\|x\| \leq \frac{1}{\delta}\|y\|$.

(2). By hypothesis, there exists a constant $L \geq 1$ such that

$$\forall y \in Y, \exists x \in \Pi^{-1}(y) \text{ with } \|S^n(x)\| \sim_L \|T^n(y)\|.$$

Recall that, in general, an operator S is expansive if and only if $\sup_{n \in \mathbb{Z}} \|S^n(x)\| = \infty$, for every $x \in X \setminus \{0\}$ [6, Proposition 19]. As S is expansive and $\|S^n(x)\| \sim_L \|T^n(y)\|$, then $\sup_{n \in \mathbb{Z}} \|T^n(y)\| = \infty$, i.e., T is expansive.

(3). Let K be a constant such that $\min\{\frac{K}{L}, \frac{K}{L^2}\} \geq 2$. As S is uniformly expansive, let $m \in \mathbb{N}$ be such that, for each $x \in X$ with $\|x\| = 1$, we have $\|S^m(x)\| \geq K$ or $\|S^{-m}(x)\| \geq K$. Now, let $y \in Y$ with $\|y\| = 1$. Then, there exists $x \in \Pi^{-1}(y)$ such that

$$\|S^n(x)\| \sim_L \|T^n(y)\|.$$

We distinguish two cases.

Case 1. $\|x\| = 1$. Then, for some $j \in \{m, -m\}$,

$$\|T^j(y)\| \geq \frac{1}{L}\|S^j(x)\| \geq \frac{K}{L} \geq 2.$$

Case 2. $\|x\| \neq 1$. As $\left\|\frac{x}{\|x\|}\right\| = 1$, then, for some $j \in \{m, -m\}$, we have

$$\begin{aligned} \|T^j(y)\| &\geq \frac{1}{L}\|S^j(x)\| = \frac{1}{L}\left\|S^j\left(\frac{x}{\|x\|}\right)\right\|\|x\| \\ &\geq \frac{1}{L}K\|x\|. \end{aligned}$$

As $\Pi(x) = y$, by the strong bounded selector condition, we have $\|x\| \geq \frac{1}{L}\|y\| = \frac{1}{L}$, implying

$$\|T^j(y)\| \geq \frac{1}{L}K\frac{1}{L} = \frac{K}{L^2} \geq 2.$$

□

Now, we prove that, in some cases, if a factor enjoys a property, then so does its extension.

Lemma 3.3. *Suppose that Π admits a strong bounded selector. If T is Li-Yorke chaotic, then S is Li-Yorke chaotic.*

Proof. By hypothesis, T is Li-Yorke chaotic, that is, T admits an irregular vector $y \in Y$, meaning that there exists $y \in Y$ such that

$$\underline{\lim}_{n \rightarrow \infty} \|T^n y\| = 0 \quad \& \quad \overline{\lim}_{n \rightarrow \infty} \|T^n y\| = +\infty.$$

By hypothesis, there exists a constant $L \geq 1$ and there exists $x \in \Pi^{-1}(y)$ with $\|S^n(x)\| \sim_L \|T^n(y)\|$. Clearly, x is an irregular vector for S , that is, S is Li-Yorke chaotic. \square

Next, we show that Γ , as defined in the introduction, satisfies the strong bounded selector condition.

Lemma 3.4. *The factor map $\Gamma : L^p(X) \rightarrow \ell^p(\mathbb{Z})$ admits a strong bounded selector.*

Proof. By a small modification of Lemma 4.2.3 of [12], we show that Γ admits a strong bounded selector.

Let $\mathbf{x} = \{x_k\}_{k \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$. We let φ be as in the proof of Lemma 4.2.3 of [12], i.e.,

$$\varphi = \sum_{k \in \mathbb{Z}} \frac{x_k}{\mu(f^k(W))^{\frac{1}{p}}} \chi_{f^k(W)}.$$

It was shown there that $\Gamma(\varphi) = \mathbf{x}$. Hence, we only need to show that there exists $L \geq 1$ such that $\|T_f^n(\varphi)\|_p^p \sim_L \|B_w^n(\mathbf{x})\|_p^p$. In fact, we will show that $\|T_f^n(\varphi)\|_p^p = \|B_w^n(\mathbf{x})\|_p^p$.

Recalling that $w_k = \left(\frac{\mu(f^{k-1}(W))}{\mu(f^k(W))} \right)^{\frac{1}{p}}$, we have

$$\|B_w^n(\mathbf{x})\|_p^p = \sum_{k \in \mathbb{Z}} |w_{k+1} \cdots w_{k+n} x_{k+n}|^p = \sum_{k \in \mathbb{Z}} \frac{\mu(f^k(W))}{\mu(f^{k+n}(W))} |x_{k+n}|^p.$$

Moreover,

$$\begin{aligned}
\|T_f^n(\varphi)\|_p^p &= \int_X |\varphi|^p \circ f^n d\mu \\
&= \int_{\cup_{h \in \mathbb{Z}} f^h(W)} \left| \sum_{k \in \mathbb{Z}} \frac{x_k}{\mu(f^k(W))^{\frac{1}{p}}} \chi_{f^k(W)} \right|^p \circ f^n d\mu \\
&= \sum_{h \in \mathbb{Z}} \int_{f^h(W)} \left| \sum_{k \in \mathbb{Z}} \frac{x_k}{\mu(f^k(W))^{\frac{1}{p}}} \chi_{f^k(W)} \right|^p \circ f^n d\mu \\
&= \sum_{h \in \mathbb{Z}} \int_{f^h(W)} \left| \frac{x_{h+n}}{\mu(f^{h+n}(W))^{\frac{1}{p}}} \chi_{f^{h+n}(W)} \right|^p \circ f^n d\mu \\
&= \sum_{h \in \mathbb{Z}} \frac{|x_{h+n}|^p}{\mu(f^{h+n}(W))} \int_{f^h(W)} \chi_{f^{h+n}(W)} \circ f^n d\mu \\
&= \sum_{h \in \mathbb{Z}} \frac{|x_{h+n}|^p}{\mu(f^{h+n}(W))} \mu(f^h(W)) \\
&= \|B_w^n(\mathbf{x})\|_p^p.
\end{aligned}$$

□

4 Proof for Li-Yorke Chaos

By Lemma 3.3 and Lemma 3.4, we have that if B_w is Li-Yorke chaotic, then so is T_f . We will now show that the converse is also true. However, we prove a lemma first.

Lemma 4.1. *There exists $L \in \mathbb{R}$ such that for all $n \in \mathbb{Z}$ and $\varphi \in L^p(X)$, we have that*

$$\int_{f^k(W)} |\varphi|^p \circ f^n d\mu \sim_L \frac{\mu(f^k(W))}{\mu(f^{k+n}(W))} \left(\int_{f^{k+n}(W)} |\varphi|^p d\mu \right).$$

Proof. As the system is of bounded distortion and it is generated by W , there exists $L \in \mathbb{R}$, [12, Proposition 2.6.5], such that

$$\frac{1}{L} \frac{\mu(f^m(W))}{\mu(f^l(W))} \leq \frac{\mu(f^m(B))}{\mu(f^l(B))} \leq L \frac{\mu(f^m(W))}{\mu(f^l(W))}, \quad (\diamond\diamond)$$

for all $l, m \in \mathbb{Z}$, and all measurable $B \subseteq W$ with $\mu(B) > 0$.

Let $\varphi \in L^p(X)$ and $n, k \in \mathbb{Z}$. Let us first prove the result for the case $\varphi|_{f^{k+n}(W)} = \sum_{i=1}^t a_i \chi_{A_i}$ with $f^{k+n}(W) = \dot{\cup}_{i=1}^t A_i$, and letting $B_i = f^{-(k+n)}(A_i)$. Observe that

$$\begin{aligned}
\int_{f^k(W)} |\varphi|^p \circ f^n d\mu &= \int_{f^{k+n}(W)} |\varphi|^p d\mu(f^{-n}) \\
&= \sum_{i=1}^t |a_i|^p \mu(f^{-n}(A_i)) \\
&= \sum_{i=1}^t |a_i|^p \mu(f^{-n}(f^{k+n}(B_i))) \\
&= \sum_{i=1}^t |a_i|^p \mu(f^k(B_i)). \tag{*}
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{f^{k+n}(W)} |\varphi|^p d\mu &= \sum_{i=1}^t |a_i|^p \mu(A_i) \\
&= \sum_{i=1}^t |a_i|^p \mu(f^{k+n}(B_i)). \tag{**}
\end{aligned}$$

Now, applying $(\diamond\diamond)$ to B_i with $m = k$ and $l = k + n$, we have

$$\begin{aligned}
\frac{1}{L} \sum_{i=1}^t |a_i|^p \frac{\mu(f^k(W))}{\mu(f^{k+n}(W))} \mu(f^{k+n}(B_i)) &\leq \sum_{i=1}^t |a_i|^p \mu(f^k(B_i)) \\
&\leq L \sum_{i=1}^t |a_i|^p \frac{\mu(f^k(W))}{\mu(f^{k+n}(W))} \mu(f^{k+n}(B_i)).
\end{aligned}$$

Putting the last inequality together with $(*)$ and $(**)$, we obtain

$$\begin{aligned}
\frac{1}{L} \frac{\mu(f^k(W))}{\mu(f^{k+n}(W))} \left(\int_{f^{k+n}(W)} |\varphi|^p d\mu \right) &\leq \int_{f^k(W)} |\varphi|^p \circ f^n d\mu \\
&\leq L \frac{\mu(f^k(W))}{\mu(f^{k+n}(W))} \left(\int_{f^{k+n}(W)} |\varphi|^p d\mu \right),
\end{aligned}$$

i.e.,

$$\int_{f^k(W)} |\varphi|^p \circ f^n d\mu \sim_L \frac{\mu(f^k(W))}{\mu(f^{k+n}(W))} \left(\int_{f^{k+n}(W)} |\varphi|^p d\mu \right).$$

We obtain the proof for an arbitrary φ by passing through the limit and applying the Lebesgue dominated convergence theorem. \square

Proof: T_f Li-Yorke chaotic $\Rightarrow B_w$ Li-Yorke chaotic. By hypothesis T_f admits an irregular vector $\varphi \in L^p(X)$, meaning that there exists $\varphi \in L^p(X)$ such that

$$\underline{\lim}_{n \rightarrow \infty} \|T_f^n \varphi\|_p = 0 \quad \& \quad \overline{\lim}_{n \rightarrow \infty} \|T_f^n \varphi\|_p = +\infty.$$

We want to apply Corollary 1.5 in [7] to show that T_g , as defined in Proposition 2.1, is Li-Yorke chaotic, or equivalently that B_w is Li-Yorke chaotic. Note that, using Lemma 4.1, we have, for each $n, k \in \mathbb{Z}$,

$$\begin{aligned} \|T_f^n \varphi\|_p^p &= \int_X |\varphi|^p \circ f^n d\mu \geq \int_{f^k(W)} |\varphi|^p \circ f^n d\mu \\ &\geq \frac{1}{L} \frac{\mu(f^k(W))}{\mu(f^{k+n}(W))} \left(\int_{f^{k+n}(W)} |\varphi|^p d\mu \right), \end{aligned}$$

and, in particular, for $k = -n$,

$$\|T_f^n \varphi\|_p^p \geq \frac{1}{L} \frac{\mu(f^{-n}(W))}{\mu(W)} \left(\int_W |\varphi|^p d\mu \right).$$

As $\underline{\lim}_{n \rightarrow \infty} \|T_f^n \varphi\|_p = 0$, then $\underline{\lim}_{n \rightarrow \infty} \mu(f^{-n}(W)) = 0$, and, hence,

$$\underline{\lim}_{n \rightarrow \infty} \nu(g^{-n}(0)) = \underline{\lim}_{n \rightarrow \infty} \nu(-n) = \underline{\lim}_{n \rightarrow \infty} \frac{\mu(f^{-n}(W))}{\mu(W)} = 0,$$

that is, Condition (a) of Corollary 1.5 in [7] is satisfied, i.e. $\underline{\lim}_{n \rightarrow -\infty} \nu(n) = 0$.

Now, we show that Condition (b) of Corollary 1.5 in [7] is satisfied, i.e.,

$$\sup \left\{ \frac{\nu(h)}{\nu(h+n)}, h \in \mathbb{Z}, n \in \mathbb{N} \right\} = \infty,$$

or, equivalently, $\sup \left\{ \frac{\mu(f^h(W))}{\mu(f^{h+n}(W))}, h \in \mathbb{Z}, n \in \mathbb{N} \right\} = \infty$. To obtain a contradiction, assume that $\frac{\mu(f^h(W))}{\mu(f^{h+n}(W))}$ is bounded above by a constant H . Hence, from Lemma 4.1, it follows

$$\int_{f^h(W)} |\varphi|^p \circ f^n d\mu \leq LH \int_{f^{h+n}(W)} |\varphi|^p d\mu \quad \text{for every } h \in \mathbb{Z}, n \in \mathbb{N},$$

implying

$$\int_X |\varphi|^p \circ f^n d\mu \leq LH \int_X |\varphi|^p d\mu, \quad \text{for every } n \in \mathbb{N},$$

contradicting the fact that φ is an irregular vector.

5 Proof for Hypercyclicity and Mixing

If T_f is hypercyclic or mixing, then, using Propositions 1.13 and 1.40 of [14], it follows that B_w is hypercyclic or mixing, respectively. To see the converse, we need the following two conditions.

Proposition 5.1. *The following statements hold.*

- (1) *If, for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists $k \geq 1$ such that, for $h \in \{k, -k\}$,*

$$\mu(f^h(\cup_{|j| \leq N} f^j(W))) < \epsilon,$$

then T_f is hypercyclic.

- (2) *If, for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$, such that, for each $k \geq k_0$ and $h \in \{k, -k\}$,*

$$\mu(f^h(\cup_{|j| \leq N} f^j(W))) < \epsilon,$$

then T_f is mixing.

Proof. We will use characterizations of hypercyclicity and mixing given in [1, Theorems 1.1 and 1.2].

(1). Let $\epsilon > 0$ and $B \in \mathcal{B}$, with $0 < \mu(B) < \infty$. Let $N \in \mathbb{N}$ be so large that

$$\mu\left(B \setminus \left(\bigcup_{j=-N}^N (B \cap f^j(W))\right)\right) < \epsilon.$$

Define $B' = \cup_{j=-N}^N (B \cap f^j(W))$. Then, $\mu(B \setminus B') < \epsilon$. By hypothesis, there exists $k \geq 1$ such that, for $h \in \{k, -k\}$,

$$\mu(f^h(\cup_{|j| \leq N} f^j(W))) < \epsilon.$$

Hence,

$$\mu(f^h(B')) \leq \mu(f^h(\cup_{j=-N}^N f^j(W))) < \epsilon.$$

We have just shown that, for all $\epsilon > 0$ and for all $B \in \mathcal{B}$ with $0 < \mu(B) < \infty$, there exist $B' \subset B$ and $k \geq 1$ such that

$$\mu(B \setminus B') < \epsilon, \quad \mu(f^{-k}(B')) < \epsilon \quad \text{and} \quad \mu(f^k(B')) < \epsilon.$$

Now, it follows from Theorem 1.1 in [1] that T_f is hypercyclic.

(2). Let ϵ , B , N , and B' be as in part (1). Let k_0 be the integer guaranteed by the hypothesis. Letting, for every $k > k_0$, $B_k = B'$, we have, for all $h \in \{k, -k\}$,

$$\mu(B \setminus B_k) < \epsilon \text{ and } \mu(f^h(B_k)) < \epsilon.$$

Now, by Theorem 1.2 of [1], we have that T_f is mixing. □

Proof: B_w hypercyclic $\Rightarrow T_f$ is hypercyclic. The operator T_g is hypercyclic by Proposition 2.1. We will show that hypothesis (1) of Proposition 5.1 is satisfied by T_f , so that it is hypercyclic. To this end, let $\epsilon > 0$ and $N \in \mathbb{N}$. Applying Theorem 1.1 of [1] to $B = \cup_{j=-N}^N g^j(0)$ and $\tilde{\epsilon} = \min_{-N \leq j \leq N} \frac{1}{2} \left\{ \frac{\epsilon}{\mu(W)}, \nu(g^j(0)) \right\}$, we have that there exist $B' \subseteq B$ and $k \geq 1$ such that, for $h \in \{k, -k\}$,

$$\nu(B \setminus B') < \tilde{\epsilon}, \nu(g^h(B')) < \tilde{\epsilon}.$$

By our choice of $\tilde{\epsilon}$, we have that $B' = B = \cup_{j=-N}^N g^j(0)$. In particular, we have, for $h \in \{k, -k\}$,

$$\nu(g^h(\cup_{j=-N}^N g^j(0))) = \sum_{j=-N}^N \nu(g^{h+j}(0)) < \tilde{\epsilon} < \frac{\epsilon}{\mu(W)}. \quad (\heartsuit)$$

Substituting

$$\nu(g^n(0)) = \nu(n) = \frac{\mu(f^n(W))}{\mu(W)}$$

in (\heartsuit) , we obtain

$$\sum_{j=-N}^N \nu(g^{h+j}(0)) = \sum_{j=-N}^N \frac{\mu(f^{h+j}(W))}{\mu(W)} = \frac{\mu(f^h(\cup_{j=-N}^N f^j(W)))}{\mu(W)} < \frac{\epsilon}{\mu(W)},$$

i.e.,

$$\mu(f^h(\cup_{j=-N}^N f^j(W))) < \epsilon.$$

By Proposition 5.1-(1), T_f is hypercyclic.

Proof: B_w mixing $\Rightarrow T_f$ is mixing. We will show that hypothesis of (2) of Proposition 5.1 is satisfied, so that T_f is mixing. We proceed as in the proof of hypercyclicity. Let $\epsilon > 0$ and $N \in \mathbb{N}$. Let B and $\tilde{\epsilon}$ as in the proof of

hypercyclicity. As B_w is mixing, applying Theorem 1.2 of [1] to B and $\tilde{\epsilon}$, there exist $k_0 \geq 1$ and a sequence $\{B_k\}$ such that, for each $k \geq k_0$, $h \in \{-k, k\}$,

$$\nu(B \setminus B_k) < \tilde{\epsilon}, \quad \nu(g^h(B_k)) < \tilde{\epsilon}.$$

Then, by the choice of $\tilde{\epsilon}$, it must be $B_k = B$ for each $k \geq k_0$. Hence, for each $k \geq k_0$ and $h \in \{k, -k\}$, we have

$$\nu(g^h(\cup_{j=-N}^N g^j(0))) = \sum_{j=-N}^N \nu(g^{h+j}(0)) < \tilde{\epsilon} < \frac{\epsilon}{\mu(W)}.$$

Hence, (\heartsuit) is satisfied for each $k \geq k_0$. Now, proceeding as in the proof of hypercyclicity, we have that the hypothesis of (2) of Proposition 5.1 is satisfied.

6 Proof for Chaos and Frequent Hypercyclicity

It follows from Theorem 3.7 [13] that T_f is chaotic if and only if T_f is frequently hypercyclic. Hence, it suffices to show the result for frequently hypercyclic operators.

If T_f is frequently hypercyclic, then that B_w is frequently hypercyclic follows from the fact that frequent hypercyclicity is preserved by factor maps [14, Proposition 9.4]. To see the converse, assume that B_w , equivalently T_g , is frequently hypercyclic. Note that ν is atomic and g is ergodic on \mathbb{Z} . Applying Corollary 3.9 of [13], we have $\nu(\mathbb{Z}) < \infty$. However, $\nu(\mathbb{Z}) = \frac{\mu(X)}{\mu(W)}$, implying $\mu(X) < \infty$. Now, applying Theorem 3.3 of [13], we have that T_f is frequently hypercyclic, completing the proof.

7 Proof for Expansivity and Uniform Expansivity

By Lemma 3.2 and Lemma 3.4, it follows that if T_f is expansive or uniform expansive, then so is B_w . Now we will prove the converse.

Proof: B_w expansive $\Rightarrow T_f$ expansive. Assume that B_w , equivalently T_g , is expansive, i.e. $\sup_{n \in \mathbb{Z}} \|T_g^n \varphi\|_p = \infty$, [6, Proposition 19] for each $\varphi \in$

$L^p(\mathbb{Z}) \setminus \{0\}$. Hence, taking $\varphi = \chi_{\{0\}}$, it follows that

$$\begin{aligned} \infty &= \sup_{n \in \mathbb{Z}} \|T_g^n \varphi\|_p^p = \sup_{n \in \mathbb{Z}} \int_{\mathbb{Z}} \chi_{\{0\}} \circ g^n d\nu = \sup_{n \in \mathbb{Z}} \nu(g^{-n}(0)) \\ &= \sup_{n \in \mathbb{Z}} \nu(-n) = \sup_{n \in \mathbb{Z}} \frac{\mu(f^{-n}(W))}{\mu(W)}, \end{aligned}$$

i.e., $\sup_{n \in \mathbb{Z}} \mu(f^n(W)) = \infty$. This will imply that T_f is expansive. In fact, using Lemma 4.1 with $k = -n$, for each $\varphi \in L^p(X) \setminus \{0\}$, we have

$$\begin{aligned} \|T_f^n \varphi\|_p^p &= \int_X |\varphi|^p \circ f^n d\mu \geq \int_{f^{-n}(W)} |\varphi|^p \circ f^n d\mu \\ &\geq \frac{1}{L} \frac{\mu(f^{-n}(W))}{\mu(W)} \left(\int_W |\varphi|^p d\mu \right). \end{aligned}$$

Now,

$$\sup_{n \in \mathbb{Z}} \|T_f^n \varphi\|_p^p \geq \frac{1}{L} \frac{\int_W |\varphi|^p d\mu}{\mu(W)} \sup_{n \in \mathbb{Z}} (\mu(f^{-n}(W))) = \infty,$$

i.e., T_f is expansive.

Proof: B_w uniform expansive $\Rightarrow T_f$ uniform expansive. We use the definition of uniform expansivity. Let L be as in the statement of Lemma 4.1. Note that $L > 1$. By the definition of uniform expansivity, there exists $n \in \mathbb{N}$ such that for all $\varphi \in \ell^p(\mathbb{Z})$ with $\|\varphi\|_p = 1$, one of the following holds:

$$\|T_g^n \varphi\|_p^p \geq 4L \text{ or } \|T_g^{-n} \varphi\|_p^p \geq 4L.$$

Taking $\varphi = \frac{\chi_{\{i\}}}{\nu(i)^{\frac{1}{p}}}$, we obtain

$$\|T_g^n \varphi\|_p^p = \frac{\nu(g^{-n}(i))}{\nu(i)} \quad \& \quad \|T_g^{-n} \varphi\|_p^p = \frac{\nu(g^n(i))}{\nu(i)}.$$

Recalling that $\frac{\nu(g^k(i))}{\nu(i)} = \frac{\mu(f^{k+i}(W))}{\mu(f^i(W))}$, we have that, for each $i \in \mathbb{Z}$, either

$$\frac{\mu(f^{n+i}(W))}{\mu(f^i(W))} \geq 4L \quad \text{or} \quad \frac{\mu(f^{-n+i}(W))}{\mu(f^i(W))} \geq 4L.$$

Let N_+ be the set of $i \in \mathbb{Z}$ such that the first holds and N_- be the set of $i \in \mathbb{Z}$ where the second holds. Clearly, $\mathbb{Z} = N_+ \cup N_-$.

Let $\varphi \in L^p(X)$ with $\|\varphi\|_p = 1$. It will suffice to show that either

$$\|T_f^n \varphi\|_p^p > 2 \quad \text{or} \quad \|T_f^{-n} \varphi\|_p^p > 2.$$

As $\|\varphi\|_p = 1$, then either

$$\sum_{i \in \mathbb{N}_+} \int_{f^i(W)} |\varphi|^p d\mu \geq 1/2 \quad \text{or} \quad \sum_{i \in \mathbb{N}_-} \int_{f^i(W)} |\varphi|^p d\mu \geq 1/2.$$

Without loss of generality, assume that the first holds. Now, using Lemma 4.1, we have

$$\begin{aligned} \|T_f^n \varphi\|_p^p &= \int_X |\varphi|^p \circ f^n d\mu \geq \sum_{i \in \mathbb{N}_+} \int_{f^{-n+i}(W)} |\varphi|^p \circ f^n d\mu \\ &\geq \frac{1}{L} \sum_{i \in \mathbb{N}_+} \frac{\mu(f^{-n+i}(W))}{\mu(f^i(W))} \left(\int_{f^i(W)} |\varphi|^p d\mu \right) \\ &\geq \frac{1}{L} 4L \sum_{i \in \mathbb{N}_+} \left(\int_{f^i(W)} |\varphi|^p d\mu \right) \\ &\geq 2. \end{aligned}$$

8 Proof for the Shadowing Property

By Corollary SC of [12], we have that T_f has the shadowing property if and only if one of Conditions \mathcal{HC} , \mathcal{HD} or \mathcal{GH} in it is satisfied.

By Theorem 18 of [9], we have that B_w has the shadowing property if and only if one of Conditions (A), (B) or (C) in it is satisfied.

As

$$w_k = \left(\frac{\mu(f^{k-1}(W))}{\mu(f^k(W))} \right)^{\frac{1}{p}},$$

we have that Condition \mathcal{HC} is equivalent to Condition (A), \mathcal{HD} is equivalent to Condition (B) and \mathcal{GH} is equivalent to Condition (C), implying that T_f has the shadowing property if and only if B_w does.

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