HIGHER SYSTOLIC INEQUALITIES FOR 3-DIMENSIONAL CONTACT MANIFOLDS

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ABSTRACT. A contact form is called Besse when the associated Reeb flow is periodic. We prove that Besse contact forms on closed connected 3-manifolds are the local maximizers of suitable higher systolic ratios. Our result extends earlier ones for Zoll contact forms, that is, contact forms whose Reeb flow defines a free circle action.

1. INTRODUCTION

1.1. **Background and main result.** The aim of this paper is to prove some sharp inequalities involving the periods of closed orbits of Reeb flows on 3-manifolds and the contact volume. Let Y be a closed, connected, orientable 3-manifold. We recall that a one-form λ on Y is called a contact form when $\lambda \wedge d\lambda$ is nowhere vanishing. The contact form λ induces a vector field R_λ, which is called Reeb vector field of λ, by the identities $R_\lambda \lrcorner\, d\lambda = 0$ and $R_\lambda \lrcorner\, \lambda = 1$. The flow of R_λ is called the Reeb flow, and we will denote it by ϕ_λ^t. It preserves the contact form λ, and in particular the volume form $\lambda \wedge d\lambda$. Reeb flows are also called contact flows in the literature. Reeb flows on 3-manifolds constitute a special class of volume preserving flows with the remarkable feature of always having closed orbits: the Weinstein conjectures postulates that Reeb flows on arbitrary closed contact manifolds admit closed orbits, and this conjecture has been confirmed in dimension 3 by Taubes, see [Tau07].

We denote by $\tau_1(\lambda)$ the minimum of all periods of closed Reeb orbits and define the systolic ratio of λ as the quotient

$$\rho_1(\lambda) := \frac{\tau_1(\lambda)^2}{\mathrm{vol}(Y,\lambda)}, \tag{1.1}$$

where the contact volume $\mathrm{vol}(Y,\lambda)$ is defined as the integral of the volume form $\lambda \wedge d\lambda$ over Y. The choice of the power 2 in the numerator of (1.1) makes ρ_1 invariant under rescaling: $\rho_1(c\lambda) = \rho_1(\lambda)$ for every non-zero constant c. As observed in [CK94, Lemma 2.1], different contact forms on Y inducing the same Reeb vector field give the same contact volume. Therefore, the systolic ratio ρ_1 is a dynamical invariant of Reeb flows. It is actually invariant by smooth conjugacies and linear time rescalings.

The term "systolic ratio" is borrowed from metric geometry: the systolic ratio of a Riemannian metric on a closed surface is the ratio between the square of the length of the shortest closed geodesic and the Riemannian area. Geodesic flows are particular Reeb flows, and the metric systolic ratio coincides with 2π-times the

Date: July 25, 2021.
2010 *Mathematics Subject Classification.* 53D10.
Key words and phrases. systolic inequalities, Besse contact forms, Seifert fibrations, Calabi homomorphism.

contact systolic ratio defined above. Indeed, the length of any closed geodesic agrees with its period as closed Reeb orbit, and the contact volume of the unit tangent bundle of a Riemannian surface is 2π-times the Riemannian area.

Still borrowing the terminology from Riemannian geometry, a contact form λ on Y is called Zoll if all its Reeb orbits are closed and have the same minimal period. In this case, the Reeb flow of λ induces a free S^1-action on Y, and the systolic ratio of λ has the value $-1/e$, where the negative integer e is the Euler number of the S^1-bundle which is induced by this S^1-action.

Zoll contact forms are precisely the local maximizers of the systolic ratio ρ_1 in the C^3-topology of contact forms: this was proven for arbitrarily closed 3-manifolds by Benedetti and Kang in [BK21], generalizing a result of the first author together with Bramham, Hryniewicz and Salomão in [ABHS18] for the 3-sphere. Recently, this result has been extended to manifolds of arbitrary dimension by the first author and Benedetti, see [AB19]. We refer the reader to the latter paper and to [APB14] for a discussion on some consequences of the local systolic maximality of Zoll contact forms in metric and systolic geometry.

We denote by $\sigma(\lambda)$ the action spectrum (or period spectrum) of the Reeb flow of λ, i.e. the set

$$\sigma(\lambda) = \big\{ t > 0 \ \big| \ \mathrm{fix}(\phi_\lambda^t) \neq \varnothing \big\}.$$

Note that every closed Reeb orbit contributes to $\sigma(\lambda)$ with all the multiples of its minimal period. In general, $\sigma(\lambda)$ is a non-empty closed set of Lebesgue measure zero, and for generic contact forms it is discrete.

The number $\tau_1(\lambda)$ is the minimum of $\sigma(\lambda)$, and we would like to define $\tau_k(\lambda)$ as the k-th element of $\sigma(\lambda)$, where the elements of $\sigma(\lambda)$ are ordered increasingly and are counted with multiplicity given by the number of closed orbits having a given period. Since in general $\sigma(\lambda)$ is not discrete, a correct definition of $\tau_k(\lambda)$ is the following: $\tau_k(\lambda)$ is the infimum of all positive real numbers τ such that there exist at least k closed Reeb orbits with period less than or equal to τ; here, each iterate of a closed Reeb orbit contributes to the count. In formulas,

$$\tau_k(\lambda) := \inf \left\{ \tau > 0 \ \middle| \ \sum_{0<t\leq\tau} \#\big(\mathrm{fix}(\phi_\lambda^t)/\sim\big) \geq k \right\}, \tag{1.2}$$

where $\sim$ is the equivalence relation on Y identifying points on the same Reeb orbit, i.e. $z_0 \sim z_1$ if and only if $z_1 = \phi_\lambda^t(z_0)$ for some $t \in \mathbb{R}$. Note that the sequence of values $\tau_k(\lambda)$, $k \geq 1$, is (not necessarily strictly) increasing and consists of elements of $\sigma(\lambda)$.

If $\sigma(\lambda)$ is discrete and for any $\tau \in \sigma(\lambda)$ there are finitely many Reeb orbits of period τ, then $k \mapsto \tau_k(\lambda)$ is a surjective map from $\mathbb{N}$ to $\sigma(\lambda)$. If instead there are infinitely many periodic orbits of (not necessarily minimal) period $\tau_k(\lambda)$ for some k, or a strictly decreasing sequence in $\sigma(\lambda)$ converging to $\tau_k(\lambda)$, then $\tau_h(\lambda) = \tau_k(\lambda)$ for every $h \geq k$.

We now define the k-th systolic ratio of the contact form λ as the positive number

$$\rho_k(\lambda) := \frac{\tau_k(\lambda)^2}{\mathrm{vol}(Y,\lambda)}.$$

The aim of this paper is to give a complete characterization of local maximizers of the k-th systolic ratio ρ_k.

Borrowing once more the terminology from Riemannian geometry, a contact form λ on Y is called Besse if all its Reeb orbits are closed. Here, different Reeb orbits are not required to have the same minimal period, and therefore Besse contact forms constitute a larger class than Zoll forms. Thanks to a theorem of Wadsley [Wad75] or, in the special case of dimension 3, an earlier theorem of Epstein [Eps72], Besse Reeb flows are periodic (see also [Sul78]). In our case, since Y has dimension 3, Epstein's theorem implies that all Reeb orbits of the Besse contact form λ have the same minimal period T except for finitely many ones, whose minimal period divides T. The orbits of the first kind are called regular, whereas the finitely many exceptional orbits with smaller minimal period are called singular.

In Riemannian geometry, suitable lens spaces have a geodesic flow that is Besse but not Zoll. Nevertheless, on simply connected manifolds, Besse geodesic flows are conjectured to be Zoll: this was confirmed for the 2-sphere, thanks to a classical result of Gromoll and Grove [GG81], and for n-spheres of dimension $n \geq 4$, by a recent result of Radeschi and Wilking [RW17]. In the more general class of Finsler geodesic flows, and in the even larger class of Reeb flows, there are plenty of examples of flows that are Besse but not Zoll: the simplest ones are the geodesic flows of rational Katok's Finsler metrics on the 2-sphere, see [Kat73, Zil83], and the Reeb flows on rational ellipsoids in $\mathbb{C}^2$; other examples are the geodesic flows on certain Riemannian orbifolds, see [Bes78, Lan20, LS21].

The theory of Seifert fibrations leads to the construction of many more examples and to a full classification of Besse Reeb flows in dimension 3, see [KL21] and Section 3.2 below. Indeed, the Reeb flow of a Besse contact form λ on Y induces a locally free S^1-action, whose quotient projection $\pi : Y \to B$ is a Seifert fibration over a 2-dimensional orbifold B. The Euler number e of such a Seifert fibration is rational and negative, see [LM04], and conversely any Seifert fibration with negative Euler number can be realized in this way. Moreover

$$\mathrm{vol}(Y, \lambda) = -T^2 e, \tag{1.3}$$

where T is the minimal common period of the Reeb orbits of λ, see [Gei20, Cor. 6.3] or Lemma 3.2 below.

If λ is a Besse contact form on the closed 3-manifold Y, the sequence $\tau_k(\lambda)$ which we introduced above stabilizes: denoting by T the minimal common period of the Reeb orbits, by $\gamma_1, \ldots, \gamma_h$ the singular Reeb orbits, and by $\alpha_1, \ldots, \alpha_h$ the integers greater than 1 such that γ_i has minimal period T/α_i, we find that $\tau_k(\lambda) = T$ for every $k \geq k_0(\lambda)$, where

$$k_0(\lambda) := \alpha_1 + \cdots + \alpha_h - h + 1,$$

and $k_0(\lambda)$ is the minimal integer with this property. Indeed, the Reeb flow of λ has a continuum of orbits of minimal period T and precisely $\alpha_1 + \cdots + \alpha_h - h$ orbits of period strictly less than T, given by the iterates γ_i^j for $1 \leq j \leq \alpha_i - 1$ of the singular orbits.

Together with (1.3), the above considerations yield the following formula for the $k_0(\lambda)$-th systolic ratio of the Besse contact form λ:

$$\rho_{k_0(\lambda)}(\lambda) = -\frac{1}{e},$$

where e is the Euler number of the Seifert fibration $\pi : Y \to B$ induced by λ.

We now state the main result of this paper, which characterizes Besse contact forms as local maximizers of the higher systolic ratios.

Theorem A. *Let Y be a closed, connected, orientable 3-manifold and k a positive integer.*

(*i*) *If a contact form λ_0 on Y is a local maximizer of the k-th systolic ratio ρ_k in the C^∞-topology, then λ_0 is Besse with $k_0(\lambda_0) = k$.*

(*ii*) *Every Besse contact form λ_0 on Y such that $k_0(\lambda_0) = k$ has a C^3-neighborhood $\mathcal{U}$ in the space of contact forms on Y such that*

$$\rho_k(\lambda) \leq \rho_k(\lambda_0), \qquad \forall \lambda \in \mathcal{U},$$

with equality if and only if there exists a diffeomorphism $\theta : Y \to Y$ such that $\theta^*\lambda = c\lambda_0$ for some $c > 0$.

We remark that Besse contact forms are never global maximizers of ρ_k on the space of contact forms inducing a given contact structure ξ on the closed 3-manifold Y: indeed, $\rho_k \geq \rho_1$ and ρ_1 is unbounded from above on the space of all contact forms on (Y, ξ). See [ABHS19] for the case of 3-dimensional contact manifolds and [Sağ21] for the general case.

Example 1.1. It is instructive to consider Theorem A in the case $Y = S^3$. Any Besse contact form on S^3 coincides, up to a diffeomorphism and multiplication by a positive number, with the restriction of the standard Liouville 1-form

$$\lambda_0 := \frac{1}{2} \sum_{j=1}^{2} \big(x_j \, dy_j - y_j \, dx_j\big)$$

of $\mathbb{R}^4$ to the boundary of the solid ellipsoid

$$E(p,q) := \left\{ z \in \mathbb{C}^2 \;\middle|\; \frac{|z_1|^2}{p} + \frac{|z_2|^2}{q} \leq \frac{1}{\pi} \right\} \subset \mathbb{C}^2 = \mathbb{R}^4,$$

where $p \leq q$ are coprime positive integers, see for instance [GL18, Prop. 5.2] and [MR20, Th. 1.1]. The Reeb flow of the contact form

$$\lambda_{p,q} := \lambda_0|_{\partial E(p,q)}$$

has a closed orbit of minimal period p, a closed orbit of minimal period q and all other orbits have minimal period pq. Therefore,

$$k_0(\lambda_{p,q}) = p + q - 1,$$

and, for $k_0 := k_0(\lambda_{p,q})$,

$$\rho_{k_0}(\lambda_{p,q}) = pq.$$

In particular, $k_0(\lambda_{1,k}) = k$ and, according to Theorem A, for every $k \geq 1$ the contact form $\lambda_{1,k}$ is a local maximizer of ρ_k. For $k = 1, 2, 3, 5$, this is the only local maximizer of ρ_k on S^3, but for all the other values of the positive integer k the linear Diophantine equation $p + q - 1 = k$ is easily seen to have more positive solutions $p \leq q$ that are coprime. For instance, ρ_4 is locally maximized by both $\lambda_{1,4}$ and $\lambda_{2,3}$, with $\rho_4(\lambda_{1,4}) = 4$ and $\rho_4(\lambda_{2,3}) = 6$. The number of local mazimizers of ρ_k on contact forms on S^3 diverges for $k \to \infty$. □

Example 1.2. Other natural applications of Theorem A concern geodesic flows on Riemannian 2-orbifolds. Consider for instance the spindle orbifold $S^2(m,n)$ whose underlying space is S^2 and which has two conic singularities of order m and n, respectively. Here, m and n are positive integers and a conic singularity of order m corresponds to the local model $\mathbb{R}^2/\mathbb{Z}_m$, where the cyclic group $\mathbb{Z}_m$ acts by rotations. The case $m = n = 1$ gives us the standard smooth 2-sphere. Let us assume $m+n > 2$, so that we have at least one singular point. The geodesic flow of any Riemannian metric on $S^2(m,n)$ can be seen as a smooth Reeb flow on the lens space $L(m+n,1)$, i.e. the quotient of $S^3 \subset \mathbb{C}^2$ by the free action of $\mathbb{Z}_{m+n}$ which is generated by the diffeomorphism

$$(z_1, z_2) \mapsto \left(e^{\frac{2\pi i}{m+n}} z_1, e^{\frac{2\pi i}{m+n}} z_2\right),$$

see [Lan20]. The spindle orbifold $S^2(m,n)$ admits a Besse Riemannian metric turning it into a Tannery surface: the spindle orbifold is realized as a sphere of revolution having the two cone singularities at the poles, see [Bes78, Chapter 4]. The equator is a closed geodesic of length 2π and all other geodesics are closed with length $2\pi a$, where $a := m+n$ if $m+n$ is odd and $a := \frac{m+n}{2}$ if $m+n$ is even. Here, meridians are seen as geodesic segments belonging to closed geodesics of length $2\pi a$.

The geodesic flow of this Tannery surface has two periodic orbits of minimal period 2π, corresponding to the two orientations of the equator, and all other orbits are closed with minimal period $2\pi a$. Therefore, the integer k_0 associated with the corresponding Besse contact form on $L(m+n,1)$ is

$$k_0 := 2a - 1.$$

The Tannery surface is a local maximizer in the C^3-topology of Riemannian metrics on $S^2(m,n)$ of the k_0-th systolic ratio given by the square of the length of the k_0-th shortest closed geodesic, where closed geodesics are counted with multiplicity as in (1.2), and the Riemannian area of the orbifold. In other words, if the Riemannian metric of the Tannery surface is modified by a C^3-small perturbation not affecting the Riemannian area, then the new geodesic flow is either still Besse, and in this case is smoothly conjugate to the Tannery geodesic flow, or the following holds: if the closed geodesic which is obtained by continuation from the equator (which is non-degenerate in the case $m+n > 2$ we are considering here) is not shorter than 2π, then there exists a closed geodesic of minimal length close to $2\pi a$ and smaller than this number.

An analogous result holds for Finsler perturbations of the Tannery surface, where now the two closed geodesics which are obtained by continuation from the equator might be geometrically distinct and have different lengths, if the Finsler perturbation is not reversible.

Actually, the second author and Soethe [LS21] proved that, within the class of Riemannian rotationally symmetric spindle 2-orbifolds, the Besse ones are even the global maximizers of the suitable higher systolic ratio. □

1.2. **Sketch of the proof of Theorem A.** We conclude this introduction by giving an informal sketch of the proof of Theorem A.

The proof of statement (i) is elementary. First we show that all the Reeb orbits of a contact form λ_0 which locally maximize ρ_k are closed and have minimal period not exceeding $\tau_k(\lambda_0)$: if there is a point $x \in Y$ whose orbit violates this assertion, we can deform λ_0 in a neighborhood of x and make the volume smaller without introducing closed orbits of period smaller than $\tau_k(\lambda_0)$. This shows that λ_0 is Besse with $k_0(\lambda_0) \leq k$. It remains to show that a Besse contact form λ does not locally maximize ρ_k if $k > k_0(\lambda_0)$. This can be done by considering explicit perturbations of λ_0 of the form $(1+\epsilon\, h \circ \pi)\lambda_0$, where $\pi : Y \to B$ is the quotient projection induced by the locally free S^1-action given by the Reeb flow of λ_0 and h is a suitable smooth real function on B.

The proof of statement (ii) is based on global surfaces of section and on a quantitative fixed point theorem for Hamiltonian diffeomorphisms of compact surfaces that are close to the identity. This kind of arguments has already been used in [ABHS18, BK21] in order to prove that Zoll contact forms are local maximizers of ρ_1 on closed 3-manifolds, but here we need two new ingredients which may be of independent interest.

We sketch the argument in the case of a Besse contact form λ_0 that is not Zoll, and hence $k_0 := k_0(\lambda_0) > 1$, but in the detailed proof we give in Section 4 we shall recover also the case in which λ_0 is Zoll. In this paper, by a global surface of section for the flow of the Reeb vector field R_λ we mean a smooth map $\iota : \Sigma \to Y$ from an oriented compact surface Σ whose restriction to each component of the boundary $\partial\Sigma$ is a positive covering of some periodic orbit of R_λ, whose restriction to the interior of Σ is an embedding into $Y \setminus \iota(\partial\Sigma)$ transversal to R_λ, and such that every orbit of R_λ intersects $\iota(\Sigma)$ in positive and negative time. The first new ingredient is the following result.

Theorem B. *If λ_0 is a Besse contact form on the closed 3-manifold Y and γ is any orbit of R_{λ_0}, then the Reeb flow of λ_0 admits a global surface of section* (*as in the previous paragraph*) *with $\iota(\partial\Sigma) = \gamma$.*

See Theorem 3.1 below for a more detailed statement. We remark that the boundary of Σ may have several components, but they are all mapped onto γ by ι. See also [AG21] for related results about global surfaces of section for general flows on 3-manifolds defining a Seifert fibration.

We normalize λ_0 so that all its regular orbits have minimal period 1, that is, $\tau_{k_0}(\lambda_0) = 1$. We apply Theorem B to some singular orbit γ_1 of period $1/\alpha_1$ of the Reeb flow of λ_0, which we fix once and for all. The embedded surface $\iota(\mathrm{int}(\Sigma))$ intersects each regular orbit of R_{λ_0} exactly α times, for some $\alpha \in \mathbb{N}$ which can be derived from the invariants of the Seifert fibration induced by λ_0.

Now consider a contact form λ which is suitably close to λ_0. Since the singular orbits of Besse Reeb flows are non-degenerate, the Reeb flow of λ has a closed orbit which is close to γ_1. Up to multiplying λ by a constant and applying a diffeomorphism to it, we can assume that R_λ coincides with R_{λ_0} on γ_1, which is therefore a closed orbit of both flows, with the same period $1/\alpha_1$. In this case, we can show that $\iota : \Sigma \to Y$ is a global surface of section also for the Reeb flow of λ, provided that λ is close enough to λ_0.

We now consider the diffeomorphism

$$\phi : \Sigma \to \Sigma$$

which is given by the α-th iterate of the first return map of the flow of R_λ to Σ. This map is actually defined only in the interior of Σ, but we will show that it extends to a diffeomorphism on Σ. The exact smooth 2-form $\omega := \iota^*(d\lambda)$ is symplectic in the interior of Σ and vanishes with order 1 on the boundary. The map ϕ is an exact symplectomorphism on (Σ, ω) and actually

$$\phi^*\lambda - \lambda = d\tau,$$

where $\tau : \Sigma \to (0, +\infty)$ is the α-th return time of the flow of R_λ (or, more precisely, the smooth extension to Σ of this function, which is defined in the interior of Σ).

The volume of (Y,λ) can be recovered by τ thanks to the identity

$$\mathrm{vol}(Y,\lambda) = \frac{1}{\alpha}\int_\Sigma \tau\,\omega.$$

The exact symplectomorphism ϕ lifts to a unique element $\tilde\phi$ of $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ which is C^1-close to the identity. Here, $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ denotes the subgroup of the universal cover of the group of Hamiltonian diffeomorphisms of (Σ,ω) consisting of isotopy classes $[\{\phi_t\}]$ starting at the identity which have vanishing flux on any curve connecting pairs of points on $\partial\Sigma$. The zero flux condition is important here and holds because we are considering a global surface of section with boundary on just one closed orbit.

Elements $\tilde\psi$ of $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ have a well-defined action

$$a_{\tilde\psi,\nu} : \Sigma \to \mathbb{R}, \qquad \psi^*\nu - \nu = da_{\tilde\psi,\nu},$$

with respect to any primitive ν of ω, where ψ denotes the projection of $\tilde\psi$ to the Hamiltonian group. The action at contractible fixed points is independent of ν, and so is the integral of the action on (Σ,ω), which defines the normalized Calabi invariant of $\tilde\psi$, i.e. the number

$$\widehat{\mathrm{Cal}}(\tilde\psi) := \frac{1}{\mathrm{area}(\Sigma,\omega)}\int_\Sigma a_{\tilde\psi,\nu}\,\omega.$$

In the case of the lift $\tilde\phi$ of the α-th return map ϕ and of the primitive $\nu := \iota^*\lambda$ of ω, we obtain the identities

$$a_{\tilde\phi,\nu} = \tau - 1, \qquad \widehat{\mathrm{Cal}}(\tilde\phi) = \frac{\mathrm{vol}(Y,\lambda)}{\mathrm{vol}(Y,\lambda_0)} - 1. \tag{1.4}$$

The second new ingredient of this paper is the following fixed point theorem.

Theorem C. *Let ω be a smooth exact 2-form on the compact surface Σ which is symplectic in the interior and vanishes with order 1 on the boundary. For every $c > 0$ there exists a C^1-neighborhood $\mathcal{U} \subset \widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ of the identity such that every $\tilde\psi \in \mathcal{U}$ with $\widehat{\mathrm{Cal}}(\tilde\psi) \leq 0$ has a contractible interior fixed point z such that*

$$a_{\tilde\psi}(z) + c\,a_{\tilde\psi}(z)^2 \leq \frac{1}{2}\widehat{\mathrm{Cal}}(\tilde\psi),$$

with the equality holding if and only if $\tilde\psi$ is the identity.

See Theorem 2.5 below and the discussion preceding it for the precise definition of all the notions involved in this theorem. The novelty here is the presence of the term which is quadratic in the action. Indeed, the weaker inequality without that term is proven in [ABHS18] when Σ is the disk and in [BK21] when Σ has just one boundary component, but the case of more boundary components can be taken care of similarly thanks to the zero-flux assumption. The constant $\frac{1}{2}$ is sharp in the above inequality, and the presence of the quadratic term is crucial in the conclusion of the argument that we sketch below.

Since we are assuming that γ_1 is a closed orbit of R_λ with minimal period $1/\alpha_1$, and since all the other singular orbits of R_{λ_0} correspond to closed orbits of R_λ of nearby period, the strict inequality $\rho_{k_0}(\lambda) < \rho_{k_0}(\lambda_0)$ holds trivially when $\mathrm{vol}(Y,\lambda) > \mathrm{vol}(Y,\lambda_0)$. Therefore, we can assume that $\mathrm{vol}(Y,\lambda) \leq \mathrm{vol}(Y,\lambda_0)$, which by (1.4) implies $\widehat{\mathrm{Cal}}(\tilde\phi) \leq 0$. If λ is C^3-close to λ_0, then $\tilde\phi$ is C^1-close to the identity and from Theorem C with $c = \frac{1}{2}$ we obtain the existence of an interior contractible fixed point z of $\tilde{\phi}$ with

$$a_{\tilde{\phi}}(z) + \frac{1}{2}\, a_{\tilde{\phi}}(z)^2 \leq \frac{1}{2}\, \widehat{\mathrm{Cal}}(\tilde{\phi}). \tag{1.5}$$

By (1.4), the fixed point z corresponds to a closed orbit $\gamma \neq \gamma_1^{\alpha_1}$ of R_λ with (not necessarily minimal) period

$$\tau(z) = 1 + a_{\tilde{\phi}}(z).$$

Since $\tau(z)$ is close to 1, this orbit is either the β-th iterate of the orbit of R_λ corresponding to some singular orbit of R_{λ_0} of minimal period $1/\beta$ other than γ_1, or is an orbit of minimal period $\tau(z)$ bifurcating from the set of regular orbits of R_{λ_0}. In both cases, its presence implies that $\tau_{k_0}(\lambda) \leq \tau(z)$ and by (1.5) we find

$$\begin{aligned} \rho_{k_0}(\lambda) &= \frac{\tau_{k_0}(\lambda)^2}{\mathrm{vol}(Y,\lambda)} \leq \frac{\tau(z)^2}{\mathrm{vol}(Y,\lambda)} = \frac{(1+a_{\tilde{\phi}}(z))^2}{\mathrm{vol}(Y,\lambda)} = \frac{1 + 2a_{\tilde{\phi}}(z) + a_{\tilde{\phi}}(z)^2}{\mathrm{vol}(Y,\lambda)} \\ &\leq \frac{1 + \frac{\mathrm{vol}(Y,\lambda)}{\mathrm{vol}(Y,\lambda_0)} - 1}{\mathrm{vol}(Y,\lambda)} = \frac{1}{\mathrm{vol}(Y,\lambda_0)} = \frac{\tau_{k_0}(\lambda_0)^2}{\mathrm{vol}(Y,\lambda_0)} = \rho_{k_0}(\lambda_0). \end{aligned}$$

This shows that λ_0 is a local maximizer of ρ_{k_0} in the C^3-topology. Finally, if this inequality is an equality, then the equality holds in (1.5) and hence $\tilde{\phi}$ is the identity. This implies that λ is Besse with regular orbits having minimal period 1, and from the local rigidity of Seifert fibrations and Moser's trick we obtain a diffeomorphism $\theta : Y \to Y$ such that $\theta^*\lambda = \lambda_0$. This concludes the sketch of the proof of Theorem A.

1.3. **Organization of the paper.** In Section 2, we review the notions of flux, action and Calabi invariant for symplectomorphisms of surfaces and prove Theorem C. In Section 3, we prove Theorem B and show how the resulting global surface of section survives to small perturbations of the contact form. In Section 4, we prove Theorem A.

1.4. **Acknowledgments.** We thank Hansjörg Geiges and Umberto Hryniewicz for discussions concerning surfaces of section, and Gabriele Benedetti for discussing with us the fixed point theorem in [BK21].

A. Abbondandolo and M. Mazzucchelli are partially supported by the IEA-International Emerging Actions project IEA00549 from CNRS. A. Abbondandolo is also partially supported by the SFB/TRR 191 'Symplectic Structures in Geometry, Algebra and Dynamics', funded by the DFG (Projektnummer 281071066 – TRR 191).

2. A fixed point theorem

In this section, we prove a refinement of a fixed point theorem due to Benedetti-Kang [BK21, Section 4.4]. Our version allows us to deal with compact surfaces with possibly disconnected boundary and gives a more precise upper bound on the action of the fixed point, which will play a crucial role in the proof of Theorem A.

2.1. **Preliminaries: action, flux, and Calabi homomorphism.** Before stating the theorem, we review some facts about the action of exact symplectomorphisms, the flux and the Calabi homomorphism in a setting which is slightly different than the one considered in classical references such as [Cal70, Ban78, Ban97, MS98].

Throughout this section, we consider a compact connected surface Σ with non-empty boundary and an exact two-form ω on Σ which is symplectic (i.e. nowhere vanishing) in the interior of Σ. In the fixed points theorem below, we will assume that ω vanishes on the boundary of Σ in a certain precise way, but in order to introduce the objects this theorem is about we do not need this assumption. As we shall see in Section 3.3, allowing symplectic forms to vanish on the boundary is important when dealing with global surfaces of section of Reeb flows, see also [ABHS18, BK21] and, for a more general approach in any dimension, the theory of ideal Liouville domains in [Gir20].

By a symplectomorphism of (Σ, ω) we mean a diffeomorphism $\phi : \Sigma \to \Sigma$ such that $\phi^*\omega = \omega$. In other words, ϕ is a diffeomorphism of Σ which restricts to a symplectomorphism of the open symplectic manifold $\mathrm{int}(\Sigma)$.

Let $\{\phi_t\}_{t\in[0,1]}$ be an isotopy on Σ starting at the identity; we will always tacitly require that every $\phi_t : \Sigma \to \Sigma$ is surjective (i.e. a diffeomorphism, and not simply an embedding). We denote by X_t the generating vector field, which is uniquely determined by the equation

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

The isotopy $\{\phi_t\}$ consists of symplectomorphisms if and only if the one-form $X_t \lrcorner\, \omega$ is closed for every $t \in [0, 1]$. When these one-forms are exact, i.e.

$$X_t \lrcorner\, \omega = dH_t, \qquad \forall t \in [0, 1],$$

for some $H \in C^\infty([0, 1] \times \Sigma)$, then X_t is called a Hamiltonian vector field, $\{\phi_t\}$ a Hamiltonian isotopy, and H a generating Hamiltonian. Generating Hamiltonians are uniquely defined up to the addition of a function of t. The fact that X_t is tangent to the boundary of Σ forces each H_t to be constant on each boundary component. By adding a suitable function of time, we could assume that H_t vanishes on a chosen component of the boundary of Σ, but in general H_t will not necessarily vanish on the other components.

Note that a smooth function $H : \Sigma \to \mathbb{R}$ defines a vector field on the interior of Σ through the identity $X \lrcorner\, \omega = dH$, but in general one needs further assumptions on H in order to guarantee that X extends smoothly to the boundary of Σ. This will not be a reason of concern for us here, as we will construct Hamiltonians from vector fields and not the other way around.

A symplectomorphism $\phi : \Sigma \to \Sigma$ is said to be Hamiltonian if $\phi = \phi_1$ for some Hamiltonian isotopy $\{\phi_t\}$. If $\{\phi_t\}$ and $\{\psi_t\}$ are Hamiltonian isotopies generated by the vector fields X_t and Y_t with Hamiltonians H_t and K_t, then the composition $\{\psi_t \circ \phi_t\}$ is generated by the vector field $Y_t + (\psi_t)_* X_t$, which is Hamiltonian with generating Hamiltonian

$$K_t + H_t \circ \psi_t^{-1}. \tag{2.1}$$

Therefore, Hamiltonian diffeomorphisms form a group, which we denote by $\mathrm{Ham}(\Sigma, \omega)$. Note that we are not requiring the diffeomorphisms in $\mathrm{Ham}(\Sigma, \omega)$ to be supported in the interior of Σ.

Every Hamiltonian diffeomorphism ϕ is exact, meaning that the one-form $\phi^*\nu - \nu$ is exact for one (and hence any) primitive ν of ω. Indeed, every isotopy $\phi_t : \Sigma \to \Sigma$

with $\phi_0 = \mathrm{id}$ is Hamiltonian if and only if it is exact for every t, see [MS98, Proposition 9.3.1]. A function $a : \Sigma \to \mathbb{R}$ satisfying

$$\phi^*\nu - \nu = da$$

is called action of the Hamiltonian diffeomorphism ϕ with respect to the primitive ν of ω. Once a primitive of ω has been fixed, the action is uniquely determined up to an additive constant. If $\phi = \phi_1$ where $\{\phi_t\}$ is a Hamiltonian isotopy with generating Hamiltonian H_t, then the formula

$$a_{H,\nu}(z) := \int_{\{t\mapsto\phi_t(z)\}} \nu + \int_0^1 H_t\big(\phi_t(z)\big)\, dt, \qquad \forall z \in \Sigma, \tag{2.2}$$

defines an action of ϕ with respect to ν.

If $\{\phi_t\}$ is a symplectic isotopy starting at the identity and generated by the vector field X_t and $\gamma : [0,1] \to \Sigma$ a smooth curve, the flux of $\{\phi_t\}$ through γ is defined as the symplectic area swept out by the path γ under the isotopy $\{\phi_t\}$, i.e. the quantity

$$\begin{aligned}\mathrm{Flux}(\{\phi_t\})(\gamma) &:= \int_{[0,1]\times[0,1]} h^*\omega = \int_0^1 \int_0^1 \omega\big(X_t(\phi_t(\gamma(s))), d\phi_t(\gamma(s))[\dot\gamma(s)]\big)\, ds\, dt\\ &= \int_0^1 \int_{[0,1]} \gamma^*\big((\phi_t^* X_t) \lrcorner\, \omega\big)\, dt,\end{aligned}$$

where $h(t,s) := \phi_t(\gamma(s))$ and in the last identity we have used the fact that the diffeomorphisms ϕ_t are symplectic. The fact that the one-forms $(\phi_t^* X_t)\lrcorner\,\omega$ are closed implies that $\mathrm{Flux}(\{\phi_t\})(\gamma)$ only depends on the homotopy class of γ relative to the endpoints, or on the free homotopy class of the closed curve γ.

Moreover, if ν is a primitive of ω we find by Stokes theorem

$$\mathrm{Flux}(\{\phi_t\})(\gamma) = \int_\gamma (\phi_1^*\nu - \nu) + \int_{\{t\mapsto\phi_t(\gamma(0))\}} \nu - \int_{\{t\mapsto\phi_t(\gamma(1))\}} \nu.$$

The above identity shows that if γ is a curve joining two points on the boundary of Σ, then $\mathrm{Flux}(\{\phi_t\})(\gamma)$ does not vary under homotopies of $\{\phi_t\}$ fixing the endpoints. If γ is a closed curve, then $\mathrm{Flux}(\{\phi_t\})(\gamma)$ depends only on the homology class of the closed one-form $\phi_1^*\nu - \nu$. In particular, $\mathrm{Flux}(\{\phi_t\})(\gamma)$ vanishes on closed curves when ϕ_1 is Hamiltonian. Actually, any symplectic isotopy with vanishing flux through every closed curve is homotopic to a Hamiltonian isotopy, see [MS98, Theorem 10.2.5].

When the isotopy $\{\phi_t\}$ is Hamiltonian with generating Hamiltonian H_t, we find the identity

$$\mathrm{Flux}(\{\phi_t\})(\gamma) = \int_0^1 H_t(\phi_t(\gamma(1)))\, dt - \int_0^1 H_t(\phi_t(\gamma(0)))\, dt.$$

If γ is a curve connecting two boundary points, we have

$$\mathrm{Flux}(\{\phi_t\})(\gamma) = \int_0^1 H_t(C_1)\, dt - \int_0^1 H_t(C_0)\, dt, \tag{2.3}$$

where C_0 and C_1 are the connected components of $\partial\Sigma$ containing the points $\gamma(0)$ and $\gamma(1)$, respectively, and $H_t(C)$ denotes the common value of H_t on the component $C \subset \partial\Sigma$ (recall that each H_t is constant on every boundary component).

We denote by
$$\pi : \widetilde{\mathrm{Ham}}(\Sigma,\omega) \to \mathrm{Ham}(\Sigma,\omega)$$
the universal cover of $\mathrm{Ham}(\Sigma,\omega)$. The group $\mathrm{Ham}(\Sigma,\omega)$ is endowed with the C^1 topology which is induced by the inclusion in the space of C^1 maps from Σ to itself. The C^1 topology on $\mathrm{Ham}(\Sigma,\omega)$ induces a C^1 topology on $\widetilde{\mathrm{Ham}}(\Sigma,\omega)$ so that, with respect to these topologies, the covering map π is a local homeomorphism. As usual, we identify the elements of $\widetilde{\mathrm{Ham}}(\Sigma,\omega)$ with homotopy classes with fixed endpoints of Hamiltonian isotopies $\{\phi_t\}$ starting at the identity, so that $\pi([\{\phi_t\}]) = \phi_1$. By the invariance of the flux under homotopies with fixed endpoints of the isotopy and (2.3), we deduce that the flux induces a map
$$\widetilde{\mathrm{Flux}} : \widetilde{\mathrm{Ham}}(\Sigma,\omega) \times H_0(\partial\Sigma)^2 \to \mathbb{R},$$
$$\widetilde{\mathrm{Flux}}([\{\phi_t\}], C_0, C_1) = \int_0^1 H_t(C_1)\,dt - \int_0^1 H_t(C_0)\,dt,$$
which for any pair (C_0, C_1) restricts to a homomorphism from $\mathrm{Ham}(\Sigma,\omega)$ to $\mathbb{R}$, thanks to the form (2.1) of the Hamiltonian generating the product of two Hamiltonian isotopies.

Remark 2.1. The above considerations can be restated slightly more abstractly by seeing the flux as a homomorphism from the universal cover of the identity component of the symplectomorphism group of Σ to $H^1(\Sigma,\partial\Sigma)$. See [MS98, Section 10.2] for the case of a closed symplectic manifold. □

We shall be particularly interested in the subgroup $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ of $\widetilde{\mathrm{Ham}}(\Sigma,\omega)$ consisting of Hamiltonian isotopies whose flux through any curve with endpoints on the boundary of Σ vanishes, i.e.
$$\widetilde{\mathrm{Ham}}_0(\Sigma,\omega) := \Big\{ \tilde\phi \in \widetilde{\mathrm{Ham}}(\Sigma,\omega) \;\Big|\; \widetilde{\mathrm{Flux}}(\tilde\phi, C_0, C_1) = 0 \quad \forall C_0, C_1 \in H_0(\partial\Sigma) \Big\}.$$
This is a normal subgroup of $\widetilde{\mathrm{Ham}}(\Sigma,\omega)$ and a proper subgroup whenever $\partial\Sigma$ has more than one connected component.

Remark 2.2. An element $\tilde\phi$ of $\widetilde{\mathrm{Ham}}(\Sigma,\omega)$ belongs to $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ if and only if we can normalize the Hamiltonian H_t generating any isotopy $\{\phi_t\}$ representing $\tilde\phi$ by requiring
$$\int_0^1 H_t(z)\,dt = 0, \qquad \forall z \in \partial\Sigma. \tag{2.4}$$
Similarly, $\tilde\phi = [\{\phi_t\}]$ belongs to $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ if and only if we can normalize the action a of ϕ_1 with respect to any primitive ν of ω by requiring
$$a(z) = \int_{\{t\mapsto\phi_t(z)\}} \nu, \qquad \forall z \in \partial\Sigma. \tag{2.5}$$
Indeed, (2.5) corresponds to the choice $a = a_{H,\nu}$ of (2.2), where H is normalized as in (2.4). □

When $\tilde\phi$ belongs to $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ and ν is a primitive of ω, we shall denote by
$$a_{\tilde\phi,\nu} : \Sigma \to \mathbb{R}$$
the action of $\pi(\tilde\phi)$ normalized as in (2.5). As the notation suggests, this action does not depend on the choice of the Hamiltonian isotopy representing $\tilde\phi$.

If $\tilde{\phi} = [\{\phi_t\}]$ and $\tilde{\psi} = [\{\psi_t\}]$ are in $\widetilde{\mathrm{Ham}}_0(\Sigma, \omega)$ and ν is any primitive of ω, we have the identity

$$a_{\tilde{\psi}\circ\tilde{\phi},\nu} = a_{\tilde{\phi},\psi_1^*\nu} + a_{\tilde{\psi},\nu}. \tag{2.6}$$

Indeed, one readily checks that the function $a := a_{\tilde{\phi},\psi_1^*\nu} + a_{\tilde{\psi},\nu}$ satisfies

$$\begin{aligned} da &= (\psi_1 \circ \phi_1)^*\nu - \nu, \\ a(z) &= \int_{\{t\mapsto\psi_t(\phi_t(z))\}} \nu, \qquad \forall z \in \partial\Sigma. \end{aligned}$$

A fixed point z of $\tilde{\phi} = [\{\phi_t\}] \in \widetilde{\mathrm{Ham}}(\Sigma, \omega)$ is by definition a fixed point of the map ϕ_1. Such a fixed point is said to be contractible if the loop $t \mapsto \phi_t(z)$ is contractible in Σ. The latter condition is clearly independent on the choice of the Hamiltonian isotopy representing $\tilde{\phi}$.

The normalized action $a_{\tilde{\phi},\nu}(z)$ of any contractible fixed point z of $\tilde{\phi} \in \widetilde{\mathrm{Ham}}_0(\Sigma, \omega)$ is independent on the choice of the primitive ν. Indeed, if $\tilde{\phi} = [\{\phi_t\}]$ and H_t is the Hamiltonian normalized by (2.4) generating ϕ_t, then the identity $a_{\tilde{\phi},\nu} = a_{H,\nu}$ and Stokes' theorem imply

$$a_{\tilde{\phi},\nu}(z) = \int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(\phi_t(z))\, dt, \tag{2.7}$$

where $u : \mathbb{D} \to \Sigma$ is a capping of the contractible closed curve $t \mapsto \phi_t(z)$. In (2.7), the dependence on ν disappears. Therefore, we shall denote the normalized action of the contractible fixed point z of $\tilde{\phi}$ simply as $a_{\tilde{\phi}}(z)$.

Finally, we define the Calabi homomorphism

$$\mathrm{Cal} : \widetilde{\mathrm{Ham}}_0(\Sigma, \omega) \to \mathbb{R}, \qquad \mathrm{Cal}(\tilde{\phi}) := \int_\Sigma a_{\tilde{\phi},\nu}\, \omega = 2\int_0^1 \left(\int_\Sigma H_t\, \omega \right) dt.$$

The equality of the above two expressions is proven in the lemma below. Notice that the above double representation implies that $\mathrm{Cal}(\tilde{\phi})$ is independent of the choice of the primitive of ν defining the normalized action $a_{\tilde{\phi},\nu}$ and of the choice of the Hamiltonian isotopy representing $\tilde{\phi}$ and defining H_t. The fact that Cal is a homomorphism can be proven by either using the representation in terms of action together with (2.6), or the Hamiltonian representation together with (2.1).

Lemma 2.3. *For each $\tilde{\phi} = [(\phi_t)] \in \widetilde{\mathrm{Ham}}_0(\Sigma, \omega)$, if H_t is the Hamiltonian normalized by (2.4) generating the isotopy ϕ_t, we have*

$$\int_\Sigma a_{\tilde{\phi},\nu}\, \omega = 2\int_0^1 \left(\int_\Sigma H_t\, \omega \right) dt.$$

Proof. From the identity $a_{\tilde{\phi},\nu} = a_{H,\nu}$ we find

$$\begin{aligned} \int_\Sigma a_{\tilde{\phi},\nu}\, \omega &= \int_\Sigma \left(\int_0^1 (X_t \lrcorner\, \nu + H_t) \circ \phi_t\, dt \right) \omega = \int_0^1 \left(\int_\Sigma (X_t \lrcorner\, \nu + H_t) \circ \phi_t\, \omega \right) dt \\ &= \int_0^1 \left(\int_\Sigma (X_t \lrcorner\, \nu + H_t)\, \omega \right) dt = \int_0^1 \left(\int_\Sigma \nu \wedge dH_t + H_t\, \omega \right) dt, \end{aligned}$$

where we have used the fact that ϕ_t preserves ω, and the identity $(X_t \lrcorner \nu)\omega = \nu \wedge dH_t$. By Stokes theorem, we find

$$\begin{aligned}\int_0^1 \left(\int_\Sigma \nu \wedge dH_t \right) dt &= \int_0^1 \left(\int_\Sigma \big(H_t\, d\nu - d(H_t \nu) \big) \right) dt \\ &= \int_0^1 \left(\int_\Sigma H_t\, \omega \right) dt - \int_0^1 \left(\int_{\partial\Sigma} H_t \nu \right) dt,\end{aligned}$$

and the latter integral vanishes thanks to the normalization condition (2.4):

$$\int_0^1 \left(\int_{\partial\Sigma} H_t \nu \right) dt = \sum_{C \in \pi_0(\partial\Sigma)} \left(\int_0^1 H_t(C)\, dt \right) \left(\int_C \nu \right) = 0. \qquad \square$$

2.2. **The fixed point theorem.** We now prescribe the way in which the two-form ω, which is assumed to be symplectic in the interior of Σ, vanishes on the boundary:

Assumption 2.4. Every connected component C of the boundary $\partial\Sigma$ has a collar neighborhood $A_C \subset \Sigma$ and an identification $A_C \equiv [0, \rho) \times S^1$, for some $\rho > 0$ such that

$$\omega|_{A_C} = -r\, dr \wedge ds.$$

Here, we are identifying S^1 with $\mathbb{R}/\mathbb{Z}$, and (r, s) denotes a point in $[0, \rho) \times S^1$. Note that the orientation of $\partial\Sigma$ as boundary of the oriented surface (Σ, ω) coincides, under the above identification of each component $C \subset \partial\Sigma$ with $\{0\} \times S^1$, with the orientation given by ds. $\square$

The main result of this section is the following fixed point theorem, which is stated as Theorem C in the Introduction and in which we are denoting by

$$\widehat{\mathrm{Cal}}(\tilde{\phi}) := \frac{\mathrm{Cal}(\tilde{\phi})}{\mathrm{area}(\Sigma, \omega)}$$

the normalized Calabi invariant of $\tilde{\phi} \in \widetilde{\mathrm{Ham}}_0(\Sigma, \omega)$.

Theorem 2.5. *Assume that the exact two-form ω on the compact surface Σ is symplectic on $\mathrm{int}(\Sigma)$ and satisfies Assumption 2.4. For every $c > 0$ there exists a C^1-neighborhood $\mathcal{U}$ of the identity in $\widetilde{\mathrm{Ham}}_0(\Sigma, \omega)$ such that every $\tilde{\phi}$ in $\mathcal{U}$ with $\mathrm{Cal}(\tilde{\phi}) \leq 0$ has a contractible fixed point $z \in \mathrm{int}(\Sigma)$ whose normalized action satisfies*

$$a_{\tilde{\phi}}(z) + c\, a_{\tilde{\phi}}(z)^2 \leq \frac{1}{2} \widehat{\mathrm{Cal}}(\tilde{\phi}), \tag{2.8}$$

with equality if and only if $\tilde{\phi}$ is the identity.

In particular, Theorem 2.5 implies that any $\tilde{\phi} \in \widetilde{\mathrm{Ham}}_0(\Sigma, \omega) \setminus \{\mathrm{id}\}$ which is sufficiently C^1-close to the identity and satisfies $\mathrm{Cal}(\tilde{\phi}) \leq 0$ has a contractible interior fixed point z with negative action satisfying

$$a_{\tilde{\phi}}(z) < \frac{1}{2} \widehat{\mathrm{Cal}}(\tilde{\phi}). \tag{2.9}$$

For the special case when Σ is the disk, the weaker conclusion (2.9) is deduced in [ABHS18, Corollary 5] from a non-perturbative statement. For arbitrary compact surfaces Σ having one boundary component, (2.9) is proven in [BK21, Corollary 4.16]. The more precise bound which we prove here involving the square of the action turns out to be important in order to prove systolic inequalities for Reeb flows using quite general global surfaces of section (see Remark 4.1 below for more about this).

Remark 2.6. The upper bound (2.8) can be restated as

$$a_{\tilde{\phi}}(z) \leq f_c\big(\widehat{\mathrm{Cal}}(\tilde{\phi})\big),$$

where

$$f_c(s) := \frac{1}{2c}\left(\sqrt{1+2cs}-1\right) = \frac{1}{2}s - \frac{c}{4}s^2 + O(s^3) \qquad \text{for } s \to 0.$$

As already observed in [ABHS18, Remark 2.21], the constant $\frac{1}{2}$ in front of the linear term in s is optimal, meaning that it cannot be replaced by a larger constant (recall that the argument of f_c is non-positive): the example that is contained there can be easily modified to produce, for every $\eta > \frac{1}{2}$, an element $\tilde{\phi}$ of $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ which is arbitrarily close to the identity in any C^k norm, has negative Calabi invariant but no contractible fixed point z satisfying

$$a_{\tilde{\phi}}(z) \leq \eta \cdot \widehat{\mathrm{Cal}}(\tilde{\phi}).$$

Therefore, (2.9) can be improved only by considering higher order terms in s; the bound (2.8) is such an improvement. □

Remark 2.7. By applying Theorem 2.5 to $\tilde{\phi}^{-1}$, we obtain the following statement: *For every $c > 0$ there exists a C^1-neighborhood $\mathcal{U}$ of the identity in $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ such that every $\tilde{\phi}$ in $\mathcal{U}$ with $\mathrm{Cal}(\tilde{\phi}) \geq 0$ has a contractible fixed point $z \in \mathrm{int}(\Sigma)$ whose normalized action satisfies*

$$a_{\tilde{\phi}}(z) - c\, a_{\tilde{\phi}}(z)^2 \geq \frac{1}{2}\widehat{\mathrm{Cal}}(\tilde{\phi}),$$

with equality if and only if $\tilde{\phi}$ is the identity. □

The proof of Theorem 2.5 uses quasi-autonomous Hamiltonians: We recall that the time-dependent Hamiltonian $H_t : \Sigma \to \mathbb{R}$ is called quasi-autonomous if there exist $z_{\min}, z_{\max} \in \Sigma$ such that

$$H_t(z_{\min}) = \min_{\Sigma} H_t, \quad H_t(z_{\max}) = \max_{\Sigma} H_t, \qquad \forall t \in [0,1].$$

Note that, if the Hamiltonian isotopy $\{\phi_t\}$ is generated by a quasi-autonomous Hamiltonian H_t as above and $z_{\min}$ and $z_{\max}$ belong to the interior $\mathrm{int}(\Sigma)$, then these points are contractible fixed points of $\tilde{\phi} = [\{\phi_t\}]$.

Exact symplectomorphisms of Σ that are C^1-close to the identity are generated by a quasi-autonomous Hamiltonian. More precisely, we have the following result.

Theorem 2.8. *Assume that the exact two-form ω on the compact surface Σ is symplectic on $\mathrm{int}(\Sigma)$ and satisfies Assumption 2.4. Let $\phi : \Sigma \to \Sigma$ be an exact symplectomorphism that is sufficiently C^1-close to the identity. Then there exists a Hamiltonian isotopy $\{\phi_t\}$ from id to ϕ whose generating Hamiltonian H_t is quasi-autonomous. Moreover, for every $\epsilon > 0$ there exists $\delta > 0$ such that, if $\mathrm{dist}_{C^1}(\phi, \mathrm{id}) < \delta$, then:*

(*i*) *$\|H_t\|_{C^1} < \epsilon$ and $\mathrm{dist}_{C^1}(\phi_t, \mathrm{id}) < \epsilon$ for every $t \in [0,1]$;*
(*ii*) *in the collar neighborhood $A_C \equiv [0,\rho) \times S^1$ of each boundary component C of Σ as in Assumption 2.4, the Hamiltonian H_t has the form*

$$H_t(r,s) = b_C + r^2 h_C(t,r,s)$$

for some real number b_C and some smooth function $h_C : [0,1] \times A \to \mathbb{R}$ such that $|b_C| < \epsilon$ and $\|h_C\|_{C^0} < \epsilon$.

In statements (i) and (ii), the C^1 distances and norm are measured with respect to an arbitrary Riemannian metric on Σ.

Remark 2.9. Theorem 2.8 has other interesting applications. For instance, it implies that the identity in $\mathrm{Ham}(\Sigma,\omega)$ has a C^1-neighborhood on which the Hofer metric is flat. See [BP94] for more about this in the setting of compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ and [LM95] for the case of compactly supported Hamiltonian diffeomorphisms of more general symplectic manifolds. □

This theorem is proven in the next section. Here we will show how the fixed point theorem can be deduced from it.

Proof of Theorem 2.5. If $\tilde{\phi}$ is the identity, then any point $z \in \mathrm{int}(\Sigma)$ is a contractible fixed point of $\tilde{\phi}$ and $a_{\tilde{\phi}}(z) = \mathrm{Cal}(\tilde{\phi}) = 0$. Therefore, we must prove that if $\mathcal{U}$ is a sufficiently small C^1-neighborhood of the identity in $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ then any $\tilde{\phi} \in \mathcal{U} \setminus \{\mathrm{id}\}$ with $\mathrm{Cal}(\tilde{\phi}) \leq 0$ has a contractible fixed point $z \in \mathrm{int}(\Sigma)$ satisfying the strict inequality in (2.8).

We fix

$$\epsilon := \frac{N}{4\,\mathrm{area}(\Sigma,\omega)\,c} > 0, \tag{2.10}$$

where $N \geq 1$ is the number of connected components of $\partial\Sigma$ and c is the arbitrary positive number which appears in the statement we are proving. By Theorem 2.8, if $\mathcal{U}$ is sufficiently small then any $\tilde{\phi} \in \mathcal{U} \setminus \{\mathrm{id}\}$ is represented by a Hamiltonian isotopy $\{\phi_t\}$ which is generated by a quasi-autonomous Hamiltonian H_t satisfying the bounds (i) and (ii) for the ϵ given by (2.10).

Since $\tilde{\phi}$ belongs to $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega)$ and H is constant on $[0,1]\times C$ for every connected component C of $\partial\Sigma$, up to adding a suitable constant we may assume that H_t vanishes on $\partial\Sigma$ for every $t \in [0,1]$. By Theorem 2.8(ii), on the collar neighborhood $A_C \equiv [0,\rho) \times S^1$ of every connected component C of $\partial\Sigma$ the Hamiltonian H_t has the form

$$H_t(r,s) = r^2 h_C(t,r,s), \qquad \text{where} \quad \|h_C\|_{C^0} < \epsilon. \tag{2.11}$$

Since H_t is quasi-autonomous, there exists $z_{\min} \in \Sigma$ which minimizes H_t for every $t \in [0,1]$. Since H_t vanishes on $\partial\Sigma$, we have $H_t(z_{\min}) \leq 0$ for every $t \in [0,1]$. Since

$$\int_0^1 \left(\int_\Sigma H_t\,\omega \right) dt = \frac{1}{2}\,\mathrm{Cal}(\tilde{\phi}) \leq 0,$$

and since H does not vanish identically, because $\tilde{\phi} \neq \mathrm{id}$, $H_t(z_{\min})$ is strictly negative for some $t \in [0,1]$. In particular, $z_{\min}$ belongs to the interior of Σ and hence is a contractible fixed point of $\tilde{\phi}$ of action

$$a_{\tilde{\phi}}(z_{\min}) = \int_0^1 H_t(z_{\min})\,dt < 0.$$

In order to estimate this action, we introduce the function

$$K : \Sigma \to \mathbb{R}, \qquad K(z) := \int_0^1 H_t(z)\,dt.$$

The point $z_{\min}$ minimizes K, and

$$-m := K(z_{\min}) = a_{\tilde{\phi}}(z_{\min}) < 0.$$

Consider the collar neighborhood $A_C \equiv [0,\rho) \times S^1$ of some connected component C of $\partial\Sigma$. By (2.11), we have

$$|K(r,s)| \leq \epsilon r^2, \qquad \forall (r,s) \in A_C.$$

Together with the fact that $K \geq -m$ on Σ, we deduce that

$$K(r,s) \geq \max\{-\epsilon r^2, -m\} \qquad \forall (r,s) \in A_C. \tag{2.12}$$

Up to reducing if necessary the neighborhood $\mathcal{U}$, we can make the C^0-norm of H as small as we wish and hence we may assume that $m < \epsilon\rho^2$. Therefore,

$$\max\{-\epsilon r^2, -m\} = \begin{cases} -\epsilon r^2, & \forall r \in \left[0, \sqrt{m/\epsilon}\right], \\ -m, & \forall r \in \left[\sqrt{m/\epsilon}, \rho\right). \end{cases}$$

By integrating (2.12) over A_C, we infer

$$\begin{aligned} \int_{A_C} K\,\omega &> \int_{[0,\rho)\times S^1} \max\{-\epsilon r^2, -m\} r\,dr \wedge ds = \int_0^\rho \max\{-\epsilon r^2, -m\} r\,dr \\ &= -\epsilon \int_0^{\sqrt{m/\epsilon}} r^3\,dr - m \int_{\sqrt{m/\epsilon}}^\rho r\,dr = -\frac{m}{2}\rho^2 + \frac{m^2}{4\epsilon} \\ &= -m\,\mathrm{area}(A_C,\omega) + \frac{m^2}{4\epsilon}, \end{aligned}$$

Note that we have written a strict inequality here because the inequality in (2.12) cannot be everywhere an equality, as the right-hand side is not a differentiable function of r at $r = \sqrt{m/\epsilon} \in (0,\rho)$. On the other hand, on $\Sigma \setminus U$, where U denotes the union of the collar neighborhoods A_C of the N components of $\partial\Sigma$ we have

$$\int_{\Sigma\setminus U} K\,\omega \geq -m\,\mathrm{area}(\Sigma \setminus U, \omega).$$

Putting these inequalities together, we find

$$\int_\Sigma K\,\omega > -m\,\mathrm{area}(\Sigma,\omega) + N\frac{m^2}{4\epsilon}. \tag{2.13}$$

Since $-m = a_{\tilde{\phi}}(z_{\min})$ and the integral of K on Σ is $\frac{1}{2}\,\mathrm{Cal}(\tilde{\phi})$, the inequality (2.13) and our choice of ϵ in (2.10) give us the desired conclusion:

$$a_{\tilde{\phi}}(z_{\min}) + c\,a_{\tilde{\phi}}(z_{\min})^2 < \frac{\mathrm{Cal}(\tilde{\phi})}{2\,\mathrm{area}(\Sigma,\omega)} = \frac{1}{2}\widehat{\mathrm{Cal}}(\tilde{\phi}). \qquad \square$$

2.3. **Construction of a generating quasi-autonomous Hamiltonian.** The aim of this section is to prove Theorem 2.8. The proof closely follows the argument of [BK21, Section 4], but for sake of completeness we work out the details.

By Assumption 2.4 and up to reducing the positive number ρ appearing there, we can find a primitive ν_0 of ω which on the collar neighborhood $A_C \equiv [0,\rho) \times S^1$ of each component C of the boundary of Σ has the form

$$\nu_0|_{A_C} = \left(a_C - \tfrac{1}{2}r^2\right) ds, \tag{2.14}$$

for some $a_C \in \mathbb{R}$. Indeed, if ν is any primitive of ω and a_C is its integral on the boundary component $C = \{0\} \times S^1$, then the one-form above differs from $\nu|_{A_C}$ by the differential of a function g_C. By adding to ν the differential of a function on Σ that agrees with g_C on a slighltly reduced collar neighborhood of every component C of $\partial\Sigma$, we obtain the desired primitive ν_0.

For $i = 1, 2$, we consider the one-forms $\nu_i := \mathrm{pr}_i^* \nu_0$ on $\Sigma \times \Sigma$, where

$$\mathrm{pr}_i : \Sigma \times \Sigma \to \Sigma, \qquad \mathrm{pr}_i(z_1, z_2) = z_i,$$

and the standard Liouville form λ_{std} on the cotangent bundle $\mathrm{T}^*\Sigma$, which is uniquely defined by the equation $\alpha^*\lambda_{\mathrm{std}} = \alpha$ for all one-forms α on Σ.

If z_1 and z_2 are points on the same connected component C of $\partial\Sigma$, then they are identified with pairs $(0, s_1)$, $(0, s_2)$ in $A_C = [0, \rho) \times S^1$. When s_2 and s_1 are not antipodal in $S^1 = \mathbb{R}/\mathbb{Z}$, meaning that $|s_2 - s_1| < \frac{1}{2}$ for suitable lifts to $\mathbb{R}$, we denote by $[z_1, z_2]$ the unique shortest oriented arc in C from z_1 to z_2, so that

$$\int_{[z_1,z_2]} ds < \tfrac{1}{2}.$$

The next result is a version of Weinstein tubular neighborhood theorem in our setting.

Lemma 2.10 (Weinstein tubular neighborhood)**.** *There exists an open neighborhood $U \subset \Sigma \times \Sigma$ of the diagonal $\Delta_\Sigma = \{(z, z) \mid z \in \Sigma\}$, an open neighborhood $V \subset \mathrm{T}^*\Sigma$ of the 0-section $0_\Sigma \subset \mathrm{T}^*\Sigma$, and a smooth map $\psi : U \to V$ that restricts to a diffeomorphism $\psi : U \cap \mathrm{int}(\Sigma \times \Sigma) \to V \cap \mathrm{T}^*\mathrm{int}(\Sigma)$, and satisfies*

$$\psi(\Delta_\Sigma) = 0_\Sigma, \qquad \psi^*\lambda_{\mathrm{std}} = \nu_2 - \nu_1 + df,$$

where $f : \Sigma \times \Sigma \to \mathbb{R}$ is a smooth function such that $f|_{\Delta_\Sigma} \equiv 0$ and

$$df(z_1, z_2) = (\nu_1 - \nu_2)_{(z_1,z_2)}, \qquad f(z_1, z_2) = -\int_{[z_1,z_2]} \nu_0,$$

for every pair $(z_1, z_2) \in U \cap (\partial\Sigma \times \partial\Sigma)$. If $A_C \equiv [0, \rho) \times S^1$ is the collar neighborhood of a connected component C of $\partial\Sigma$ on which ν_0 has the form (2.14), the restriction $\psi|_{U\cap(A_C\times A_C)}$ has the form

$$\begin{aligned} \psi(r, s, R, S) = \big(R, s,\ R\,(S - s), \tfrac{1}{2}(r^2 - R^2)\big),& \\ \forall (r, s, R, S) \in U \cap (A_C \times A_C).& \end{aligned} \tag{2.15}$$

Proof. We first provide the construction within the collar neighborhood $A = A_C$ of each connected component C of $\partial\Sigma$. We consider a small enough neighborhood $W \subset A \times A$ of the diagonal $\Delta_A = \{(z, z) \mid z \in A\}$ so that, for each $(r, s, R, S) \in W$, the points $s, S \in S^1$ are not antipodal. We define the map

$$\kappa_0 : W \to \underbrace{\mathrm{T}^*A}_{A\times\mathbb{R}^2}, \qquad \kappa_0(r, s, R, S) = \big(R, s,\ R\,(S - s), \tfrac{1}{2}(r^2 - R^2)\big).$$

This map restricts to a diffeomorphism onto its image

$$\kappa_0 : W \cap \mathrm{int}(A \times A) \to \mathrm{int}(\mathrm{T}^*A),$$

and satisfies $\kappa_0(\Delta_A) = 0_A$ and

$$\begin{aligned}\kappa_0^*\lambda_{\mathrm{std}} &= R\,(S-s)\,dR + \tfrac{1}{2}(r^2 - R^2)\,ds\\ &= -\tfrac{1}{2}\,R^2\,dS + \tfrac{1}{2}\,r^2\,ds + d\big(\tfrac{1}{2}(S-s)\,R^2\big)\\ &= -\nu_1 + \nu_2 + d\big((a_C - \tfrac{1}{2}R^2)(s-S)\big)\\ &= -\nu_1 + \nu_2 + df_0,\end{aligned}$$

where

$$f_0 : W \to \mathbb{R}, \qquad f_0(r,s,R,S) = (a_C - \tfrac{1}{2}R^2)(s-S).$$

Notice that $f_0|_{\Delta_A} \equiv 0$ and

$$\begin{aligned}f_0(0,s,0,S) &= a_C(s-S) = -\int_{[z_1,z_2]} \nu_0,\\ df_0(0,s,0,S) &= a_C\,ds - a_C\,dS = \nu_1 - \nu_2,\end{aligned}$$

where $z_1 = (0,s)$ and $z_2 = (0,S)$.

For some sufficiently small neighborhood $U \subset \Sigma \times \Sigma$ of the diagonal Δ_Σ, we choose an arbitrary smooth function $f_1 : U \to \mathbb{R}$ that coincides with f_0 on a neighborhood of $U \cap (\partial\Sigma \times \partial\Sigma)$ and satisfies $f_1|_{\Delta_\Sigma} \equiv 0$ and $df_1 = \nu_1 - \nu_2$ at all points of the diagonal Δ_Σ. Up to further shrinking the neighborhood U, we also choose a smooth map $\kappa_1 : U \to \mathrm{T}^*\Sigma$ that coincides with κ_0 on a neighborhood of $W \cap (\partial\Sigma \times \partial\Sigma)$, restricts to a diffeomorphism onto its image $\kappa_1 : U \cap \mathrm{int}(\Sigma\times\Sigma) \to \mathrm{T}^*(\mathrm{int}(\Sigma))$, and such that $\kappa_1(\Delta_\Sigma) = 0_\Sigma$.

We now conclude the proof by means of a typical application of Moser's trick. We set

$$\mu_t := t\kappa_1^*(\lambda_{\mathrm{std}}) + (1-t)(\nu_2 - \nu_1),$$

and we look for an isotopy $\phi_t : U \to \Sigma\times\Sigma$, defined after possibly further shrinking the neighborhood U, such that $\phi_0 = \mathrm{id}$ and $\phi_t^*\mu_t - \mu_0$ is exact. We denote by X_t the time-dependent vector field generating ϕ_t and compute

$$\begin{aligned}\tfrac{d}{dt}\phi_t^*\mu_t &= \phi_t^*(L_{X_t}\mu_t + \tfrac{d}{dt}\mu_t) = \phi_t^*\big(X_t \lrcorner\, d\mu_t + d(\mu_t(X_t)) + \mu_1 - \mu_0\big)\\ &= \phi_t^*\big(X_t \lrcorner\, d\mu_t + \kappa_1^*\lambda_{\mathrm{std}} - \nu_2 + \nu_1 - df_1 + d(\mu_t(X_t) + f_1)\big).\end{aligned}$$

Notice that the symplectic forms $\kappa_1^* d\lambda_{\mathrm{std}}$ and $-d\nu_1 + d\nu_2$ define the same orientation, since they coincide on a neighborhood of $U \cap (\partial\Sigma\times\partial\Sigma)$. Therefore $d\mu_t$ is symplectic away from $\partial(\Sigma\times\Sigma)$ for every $t \in [0,1]$. We choose the vector field X_t so that

$$X_t \lrcorner\, d\mu_t + \kappa_1^*\lambda_{\mathrm{std}} - \nu_2 + \nu_1 - df_1 = 0.$$

Notice that X_t vanishes on the diagonal Δ_Σ and on a neighborhood of $U \cap (\partial\Sigma\times\partial\Sigma)$. Moreover

$$\tfrac{d}{dt}\phi_t^*\mu_t = \phi_t^* d(\mu_t(X_t) + f_1).$$

Up to shrinking the neighborhood U, we obtain a well defined isotopy $\phi_t : U \to \mathrm{T}^*\Sigma$ that coincides with κ_0 on a neighborhood of $U \cap (\partial\Sigma \times \partial\Sigma)$ and satisfies

$$\phi_1^*\kappa_1^*(\lambda_{\mathrm{std}}) = -\nu_1 + \nu_2 + df,$$

where

$$f(w) = \int_0^1 \big(\mu_t(X_t) + f_1\big) \circ \phi_t(w)\,dt.$$

The desired map is $\psi := \kappa_1 \circ \phi_1$. □

Proof of Theorem 2.8. We still work with the special primitive ν_0 of ω satisfying (2.14). By assumption, the diffeomorphism $\phi : \Sigma \to \Sigma$ satisfies

$$\phi^*\nu_0 - \nu_0 = da$$

for some smooth function a on Σ. Note that a is C^1-small when ϕ is C^1-close to the identity. We consider the associated map

$$\Phi : \Sigma \to \Sigma \times \Sigma, \qquad \Phi(z) = (z, \phi(z)).$$

We require ϕ to be sufficiently C^1-close to the identity so that the image of Φ is contained in the domain of the map $\psi : U \to \mathrm{T}^*\Sigma$ provided by Lemma 2.10, and the image of $\psi \circ \Phi$ is a section of the cotangent bundle $\mathrm{T}^*\Sigma$. Namely, if we denote by $\pi : \mathrm{T}^*\Sigma \to \Sigma$ the projection onto the base of the cotangent bundle, the map

$$\widetilde{\phi} : \Sigma \to \Sigma, \qquad \widetilde{\phi}(z) = \pi \circ \psi \circ \Phi(z)$$

is a diffeomorphism. We consider the smooth function $f : \Sigma \times \Sigma \to \mathbb{R}$ provided by Lemma 2.10. Since

$$(\psi \circ \Phi)^*\lambda_{\mathrm{std}} = \Phi^*(\nu_2 - \nu_1 + df) = \phi^*\nu - \nu + d(f \circ \Phi) = d(a + f \circ \Phi),$$

we have that

$$\psi \circ \Phi(z) = (\widetilde{\phi}(z), dF(\widetilde{\phi}(z))), \tag{2.16}$$

where $F : \Sigma \to \mathbb{R}$ is the smooth generating function

$$F(w) = (a + f \circ \Phi) \circ \widetilde{\phi}^{-1}(w).$$

Identity (2.16) implies that F is C^2-small when ϕ is C^1-close to the identity.

Consider now the collar neighborhood $A = A_C \equiv [0, \rho) \times S^1$ of a connected component C of $\partial\Sigma$ as in (2.14). For all $(r, s) \in A \cap \phi^{-1}(A)$, if we set $(R, S) = \phi(r, s)$, we have $\widetilde{\phi}(r, s) = (R, s)$ and

$$R\,(S - s) = \partial_R F(R, s), \qquad \tfrac{1}{2}(r^2 - R^2) = \partial_s F(R, s). \tag{2.17}$$

This implies that $dF = 0$ at all points of $\partial\Sigma$. In particular, F is constant on each component C of $\partial\Sigma$ and in a neighborhood of this component we can write F as

$$F(R, s) = b + R^2 G(R, s), \tag{2.18}$$

where $b = b_C$ is a real number and $G = G_C$ is a smooth function. More precisely,

$$b = \lim_{R \to 0} \frac{\partial_R F(R, s)}{R} = S(0, s) - s$$

and the Taylor theorem with integral remainder gives us the formula

$$F(R, s) = b + R^2 \int_0^1 \partial_R^2 F(tR, s)(1 - t)\, dt.$$

By differentiating the first identity in (2.17) with respect to R, we find

$$G(R, s) = \int_0^1 \partial_R^2 F(tR, s)(1 - t)\, dt = \int_0^1 \big(-s + S(tR, s) + tR\,\partial_R S(tR, s)\big)(1 - t)\, dt.$$

The above formulas for b and G imply that $|b|$ and $\|G\|_{C^0}$ are both small if ϕ is C^1-close to the identity.

We now consider the isotopy

$$\phi_t : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma), \ \ t \in [0, 1],$$

with the associated map $\Phi_t : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma \times \Sigma)$, $\Phi_t(z) = (z, \phi_t(z))$, whose image $\psi \circ \Phi_t(\mathrm{int}(\Sigma))$ is the graph of $t\,dF$. Notice that ϕ_t defines an associated diffeomorphism

$$\widetilde{\phi}_t := \pi \circ \psi \circ \Phi_t : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma),$$

and

$$\psi \circ \Phi_t(z) = (\widetilde{\phi}_t(z), t\,dF(\widetilde{\phi}_t(z))). \tag{2.19}$$

The endpoints of the isotopy are $\phi_0 = \mathrm{id}$ and $\phi_1 = \phi$. We claim that ϕ_t extends as a smooth isotopy $\phi_t : \Sigma \to \Sigma$ that is C^1-close to the identity. In order to prove this, let us focus on the collar neighborhood $A_C \equiv [0, \rho) \times S^1$ of a connected component C of $\partial\Sigma$. If we write $(R_t, S_t) := \phi_t(r, s)$, then Equation (2.19) in the annulus $\mathrm{int}(A_C)$ becomes

$$R_t\,(S_t - s) = t\,\partial_R F(R_t, s), \qquad \tfrac{1}{2}(r^2 - R_t^2) = t\,\partial_s F(R_t, s),$$

that is, using (2.18),

$$S_t = s + \underbrace{\big(2G(R_t, s) + R_t\,\partial_R G(R_t, s)\big)t}_{(*)}, \qquad r = R_t\,\underbrace{\sqrt{1 + 2t\,\partial_s G(R_t, s)}}_{(**)}.$$

The term $(*)$ is C^1-small and the term $(**)$ is C^1-close to 1 as functions of (R_t, s). This shows that the isotopy $(R_t, s) \mapsto (r, S_t)$ is C^1-close to the identity and extends smoothly to the boundary C by $(0, s) \mapsto (0, s + 2t\,G(0, s))$. Therefore, we obtain a C^1-close to the identity smooth extension $\phi_t : \Sigma \to \Sigma$ as well.

Let X_t bt the time dependent vector field generating the isotopy ϕ_t. We claim that X_t is Hamiltonian with Hamiltonian function

$$H_t : \Sigma \to \mathbb{R}, \qquad H_t(z) := F \circ \pi \circ \psi(\phi_t^{-1}(z), z). \tag{2.20}$$

Indeed, consider an arbitrary $v \in \mathrm{T}_z\Sigma$ and set

$$w := d\widetilde{\phi}_t(z)v, \qquad q_t := \widetilde{\phi}_t(z), \qquad y_t := \tfrac{d}{dt}\widetilde{\phi}_t(z).$$

In Darboux coordinates, we locally see $\mathrm{T}^*\Sigma$ as $\Sigma \times \mathbb{R}^2$, and compute

$$\begin{aligned}
d\nu\big(X_t(\phi_t(z)), d\phi_t(z)v\big) &= \psi^* d\lambda_{\mathrm{std}}\big((0, X_t(\phi_t(z))), (v, d\phi_t(z)v)\big)\\
&= d\lambda_{\mathrm{std}}\big((y_t, dF(q_t) + t\,d^2F(q_t)y_t), (w, t\,d^2F(q_t)w)\big)\\
&= dF(q_t)w + t\,d^2F(q_t)[y_t, w] - t\,d^2F(q_t)[w, y_t]\\
&= dF(q_t)w = d(F \circ \widetilde{\phi}_t)(z)v\\
&= d(F \circ \widetilde{\phi}_t \circ \phi_t^{-1})(\phi_t(z))d\phi_t(z)v\\
&= dH_t(\phi_t(z))d\phi_t(z)v,
\end{aligned}$$

proving our claim.

Note that for every $t \in [0, 1]$, the maximum and minimum of H_t on Σ coincide with those of F. Note also that by (2.18) we have

$$H_t(z) = F(z) = b_C \qquad \forall z \in C, \tag{2.21}$$

for every connected component C of $\partial\Sigma$. Moreover, the previous considerations on F, b_C and G_C imply that if ϕ is C^1-close to the identity, then H_t is C^1-small and on A_C has the form

$$H_t(r, s) = b_C + r^2 h_C(t, r, s),$$

where both $|b_C|$ and $\|h_C\|_{C^0}$ are small. Indeed, the above identity and the C^0-smallness of h_C follow from (2.15) and (2.20), which give us the identity

$$H_t(r,s) = F(r, \overline{S}_t(r,s)) = b_C + r^2 G_C(r, \overline{S}_t(r,s))$$

where $\phi_t^{-1}(r,s)) = (\overline{R}_t(r,s), \overline{S}_t(r,s))$. Together with the already mentioned C^1-closeness of ϕ_t to the identity, this proves statements (i) and (ii) in Theorem 2.8.

Let us check that for every $t \in [0,1]$ the function H_t achieves its minimum at some point $z_{\min} \in \Sigma$ which is independent of t. If F achieves its minimum on $\partial\Sigma$, this follows from the identity $\min H_t = \min F$ and (2.21). So let us assume that F achieves its minimum at an interior point $q_{\min}$. Then $dF(q_{\min}) = 0$ and, since the inverse image of the zero-section in $T^*\mathrm{int}(\Sigma)$ is the diagonal in $\mathrm{int}(\Sigma) \times \mathrm{int}(\Sigma)$, we have

$$\psi(z_{\min}, z_{\min}) = (q_{\min}, 0)$$

for some $z_{\min} \in \mathrm{int}(\Sigma)$. Since ψ maps the graph of ϕ_t to the graph of $t\,dF$, we have

$$\phi_t(z_{\min}) = z_{\min} \qquad \forall t \in [0,1],$$

and hence

$$H_t(z_{\min}) = F \circ \pi \circ \psi(z_{\min}, z_{\min}) = F \circ \pi(q_{\min}, 0) = F(q_{\min}).$$

This shows that

$$H_t(z_{\min}) = \min_{\Sigma} H_t \qquad \forall t \in [0,1].$$

The argument for the maximum is analogous, and we conclude that H_t is quasi-autonomous. □

3. Global surfaces of section for nearly Besse contact forms on 3-manifolds

3.1. **Global surfaces of section.** In this paper, a global surface of section for a contact form λ on a 3-manifold Y is a smooth map $\iota : \Sigma \to Y$, where Σ is an oriented connected compact surface with non-empty and possibly disconnected boundary, with the following properties:

- (*Boundary*) The restriction $\iota|_{\partial\Sigma}$ is an immersion positively tangent to the Reeb vector field R_λ. Namely, with the orientation on the boundary $\partial\Sigma$ induced by the one of Σ, the restriction of ι to any connected component of $\partial\Sigma$ is an orientation preserving covering map of a closed Reeb orbit of (Y, λ).
- (*Transversality*) The restriction $\iota|_{\mathrm{int}(\Sigma)}$ is an embedding into $Y \setminus \iota(\partial\Sigma)$ transverse to the Reeb vector field R_λ. In particular, the 2-form $\iota^* d\lambda$ is nowhere vanishing on $\mathrm{int}(\Sigma)$, and we assume that it is a positive area form on the oriented surface $\mathrm{int}(\Sigma)$.
- (*Globality*) For each point $z \in Y$, the Reeb orbit $t \mapsto \phi_\lambda^t(z)$ intersects Σ in both positive and negative time.

We stress that, in the literature, the notion of global surface of section may be slightly different than the one given here: for instance, the map $\iota : \Sigma \to Y$ may be required to be an embedding, or the restriction of ι to some connected component of Σ may be allowed to be an orientation reversing covering map of a closed Reeb orbit.

3.2. **The surgery description of a Besse contact form on a 3-manifold.** The proof of Theorem A will require suitable surfaces of section for the Reeb flows of contact forms sufficiently C^3-close to a Besse one. As a preliminary step, in the next subsection we shall construct global surfaces of sections for the Reeb flow of Besse contact 3-manifolds. It will be useful to employ the surgery description of Besse contact 3-manifolds as Seifert fibered spaces, which we now recall. We refer the reader to, e.g., [Orl72, JN83] for more details.

A Seifert fibration $\pi : Y \to B$, in the generality that we need for the study of Besse contact 3-manifolds, is defined up to a suitable notion of isomorphism by a genus and $k \geq 1$ pairs of coprime integers $(\alpha_1, \beta_1), ..., (\alpha_k, \beta_k) \in \mathbb{N} \times \mathbb{Z}$. Here, $\mathbb{N}$ denotes the set of positive integers and the coprimeness assumption implies that $\alpha_j = 1$ if $\beta_j = 0$. We denote by B_0 an oriented compact connected surface of the given genus with k boundary components. We write its oriented boundary as $\partial B_0 = \partial_1 B_0 \cup ... \cup \partial_k B_0$, where each $\partial_j B_0$ is a connected component oriented as the boundary of B_0. Over B_0, we consider the trivial S^1-bundle

$$\pi : Y_0 := B_0 \times S^1 \to B_0, \qquad \pi(z,t) = z,$$

with its associated free S^1-action

$$t \cdot (z,s) = (z, s+t), \qquad \forall t \in S^1,\ (z,s) \in Y_0.$$

Here and elsewhere in the paper, $S^1 = \mathbb{R}/\mathbb{Z}$. Next, we consider k disjoint copies $B_j \subset \mathbb{C}$, $j = 1, ..., k$ of the disk of some radius $\rho > 0$ centered at the origin, and the solid tori together with their base projections

$$\pi : Y_j := B_i \times S^1 \to B_j, \qquad \pi(z,t) = z.$$

By the Bézout identity, we can find pairs of coprime integers $(\alpha_j', \beta_j') \in \mathbb{Z} \times \mathbb{Z}$ such that

$$\det \begin{pmatrix} \alpha_j & \alpha_j' \\ \beta_j & \beta_j' \end{pmatrix} = 1, \qquad \forall j = 1, ..., k. \tag{3.1}$$

The pair (α_j', β_j') is not uniquely determined by (α_j, β_j) (except if $(\alpha_j, \beta_j) = (1, 0)$, in which we necessarily have $(\alpha_j', \beta_j') = (0, 1)$). We introduce the oriented curves

$$m_j = \partial B_j \times \{*\} \subset \partial Y_j, \qquad\qquad l_j = \{*\} \times S^1 \subset \partial Y_j,$$
$$h_j = \{*\} \times S^1 \subset \partial_j B_0 \times S^1 \subset \partial Y_0, \qquad f_j = -\partial_j B_0 \times \{*\} \subset \partial Y_0.$$

Here, we used the symbol $*$ to denote an arbitrary point of a space. We glue $Y_0, Y_1, ..., Y_k$ along their boundaries by identifying

$$m_j \equiv \alpha_j f_j + \beta_j h_j, \qquad l_j \equiv \alpha_j' f_j + \beta_j' h_j, \qquad \forall j = 1, ..., k$$

and denote by Y the resulting closed 3-manifold. Here, we mean that $m_j \subset \partial Y_j$ is identified with an oriented embedded circle in $\partial_j Y_0$ that is homologous to $\alpha_j [f_j] + \beta_j [h_j]$, and analogously for l_j. The free S^1-action on Y_0 extends to an S^1-action on the whole Y, which on the solid tori Y_j has the form

$$t \cdot (z,s) = \big(z e^{-2\pi \alpha_j' t i}, s + \alpha_j t\big), \qquad \forall t \in S^1,\ (z,s) \in Y_j.$$

If $\alpha_j > 1$, the S^1-action is not free on the orbit

$$\gamma_j := \{0\} \times S^1 \subset Y_j,$$

and in this case we call such an orbit singular. All those S^1-orbits that are not singular are called regular. The surfaces B_j are glued accordingly to form a closed surface B. The maps π are glued as well, and the obtained $\pi : Y \to B$ is the quotient projection of the S^1-action on Y. We can always assume without loss of generality that the number of Seifert pairs (α_j, β_j) is $k \geq 2$; indeed, adding the trivial Seifert pair $(1, 0)$ does not affect the Seifert fibration.

The locally free S^1-action that is induced by a Besse Reeb flow on Y can be seen as the S^1-action on the total space of a Seifert fibration $\pi : Y \to B$ as above. In this case, the Euler number

$$e(Y) := -\frac{\beta_1}{\alpha_1} - ... - \frac{\beta_k}{\alpha_k}$$

is negative, as shown in [LM04] (see also [KL21, Theorem 1.4]).

3.3. **A Global surface of section for a Besse contact form on a 3-manifold.** The existence of global surfaces of section in Seifert spaces was thoroughly investigated by Albach-Geiges [AG21]. In this subsection, we prove a statement which may be of independent interest (Theorem 3.1) asserting that on a Besse contact 3-manifold any given closed Reeb orbit is the (multiply covered) boundary of a surface of section as defined in Subsection 3.1

Let (Y, λ_0) be a Besse contact 3-manifold whose Reeb orbits have minimal common period 1, and γ_1 be an arbitrary closed Reeb orbit. The Reeb flow $\phi^t_{\lambda_0}$ defines a locally free S^1-action on Y. We can see this action as the S^1-action of a Seifert fibered structure which we describe with the same notation of the previous subsection: in particular, for each $j = 1, ..., k$, we denote by γ_j the closed Reeb orbit corresponding to the Seifert pair (α_j, β_j). Notice in particular that we are assuming without loss of generality that our given γ_1 is the closed Reeb orbit corresponding to the Seifert pair (α_1, β_1). We recall that γ_1 has minimal period $1/\alpha_1$, and therefore it is a singular orbit if and only if $\alpha_1 > 1$.

We consider the tubular neighborhood $Y_1 \subset Y$ of γ_1, which was realized as $Y_1 \equiv B_1 \times S^1$, and under this identification we have $\gamma_1 = \{0\} \times S^1 \subset Y_1$. We equip Y_1 with the contact form

$$\lambda_0' = \tfrac{1}{\alpha_1} r^2 d\theta + \tfrac{1}{\alpha_1}\big(1 + 2\pi \tfrac{\alpha_1'}{\alpha_1} r^2\big) ds,$$

where (r, θ) are polar coordinates on the disk B_1, and $s \in S^1 = \mathbb{R}/\mathbb{Z}$. The associated Reeb vector field is given by

$$R_{\lambda_0'} = -2\pi\alpha_1' \partial_\theta + \alpha_1 \partial_s = R_{\lambda_0}|_{Y_1},$$

and therefore the associated Reeb flow agrees with the Seifert S^1-action

$$\phi^t_{\lambda_0'}(z, s) = \phi^t_{\lambda_0}(z, s) = t \cdot (z, s) = \big(z e^{-2\pi\alpha_1' t i}, s + \alpha_1 t\big),$$

Up to pulling back λ_0 by an S^1-equivariant diffeomorphism, we can assume that

$$\lambda_0|_{Y_1} = \lambda_0' = \tfrac{1}{\alpha_1} r^2 d\theta + \tfrac{1}{\alpha_1}\big(1 + 2\pi \tfrac{\alpha_1'}{\alpha_1} r^2\big) ds$$

This can be obtained by means of a Moser's trick, see [CGM20, Lemma 4.5]. Notice that every orbit of the Reeb flow $\phi^t_{\lambda_0}$ on Y_1 has minimal period 1, except possibly $\gamma_1 = \{0\} \times S^1$ that has minimal period $1/\alpha_1$ (the case in which γ_1 is regular is allowed, and corresponds to the Seifert invariants $(\alpha_1, \beta_1) = (1, 0)$).

For any pair of coprime integers $p_0 \neq 0$ and q_0 such that

$$\frac{q_0}{p_0} < -\frac{\alpha_1'}{\alpha_1}, \tag{3.2}$$

and for any $s_0 \in S^1$, we introduce the map

$$\iota : [0,\rho) \times S^1 \to B_1 \times S^1, \qquad \iota(r,s) = \big(re^{2\pi(s_0+q_0 s)i}, p_0 s\big),$$

which satisfies the following properties:

- (*Transversality*) The restriction $\iota|_{(0,\rho)\times S^1}$ is an embedding transverse to the Reeb vector field R_{λ_0}, and the image $\iota((0,\rho)\times S^1)$ intersects $-\alpha_1 q_0 - p_0\alpha_1' > 0$ times every Reeb orbit in Y_1 other than γ_1. Therefore, the 2-form $\iota^* d\lambda_0$ is nowhere vanishing on the interior $(0,\rho)\times S^1$, and we employ it to orient the annulus $[0,\rho)\times S^1$.
- (*Boundary*) The restriction $\iota|_{\{0\}\times S^1}$ is an orientation preserving p_0-th fold covering map of the closed Reeb orbit γ_1. Here, the boundary circle $\{0\}\times S^1$ is oriented by means of the 1-form $\iota^*\lambda_0$.

We call $\iota : [0,\rho)\times S^1 \to B^2 \times S^1$ a (p_0,q_0)-local surface of section with boundary on the orbit γ_1.

We now provide the construction of a suitable global surface of section for the Besse contact 3-manifold (Y,λ) with boundary on the orbit γ_1. An alternative construction in the special case of a regular orbit in S^3 was provided by Albach-Geiges [AG21, Example 5.6]. The next result is a more precise version of Theorem B from the Introduction.

Theorem 3.1. *Let (Y,λ_0) be a Besse contact 3-manifold with minimal common Reeb period 1, and γ_1 be any of its closed Reeb orbits. We denote by $\alpha_1 > 0$ the integer whose reciprocal $1/\alpha_1$ is the minimal period of γ_1. Then, there exist integers $b > 0$, $p_0 > 0$, and q_0 with $\gcd(p_0,q_0) = 1$, a compact connected oriented surface Σ with b boundary components and a global surface of section $\iota : \Sigma \to Y$ for (Y,λ_0) satisfying the following properties:*

(*i*) *Any connected component C of the boundary $\partial\Sigma$ has a collar neighborhood $A_C \cong [0,\rho)\times S^1$ such that the restriction $\iota|_{A_C}$ is a (p_0,q_0)-local surface of section with boundary on the orbit γ_1. In particular, all boundary components are positively oriented.*

(*ii*) *Any regular closed Reeb orbit in $Y \setminus \gamma_1$ intersects the image $\iota(\mathrm{int}(\Sigma))$ in α points, where*

$$\alpha := -\frac{b\,p_0}{e(Y)\,\alpha_1} > 0.$$

Here, $e(Y)$ is the Euler number of (Y,λ_0).

(*iii*) *The restriction $\iota|_{\partial\Sigma}$ is a covering map of γ_1 of degree $b\,p_0$.*

Proof. Let $(\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k)$ be the Seifert invariants of (Y,λ_0). Here, we assume without loss of generality that γ_1 is the Reeb orbit corresponding to the Seifert pair (α_1,β_1) (as we already pointed out, γ_1 is allowed to be a regular orbit, and in that case we have $(\alpha_1,\beta_1) = (1,0)$). For every Seifert pair (α_j,β_j), we denote by (α_j',β_j')

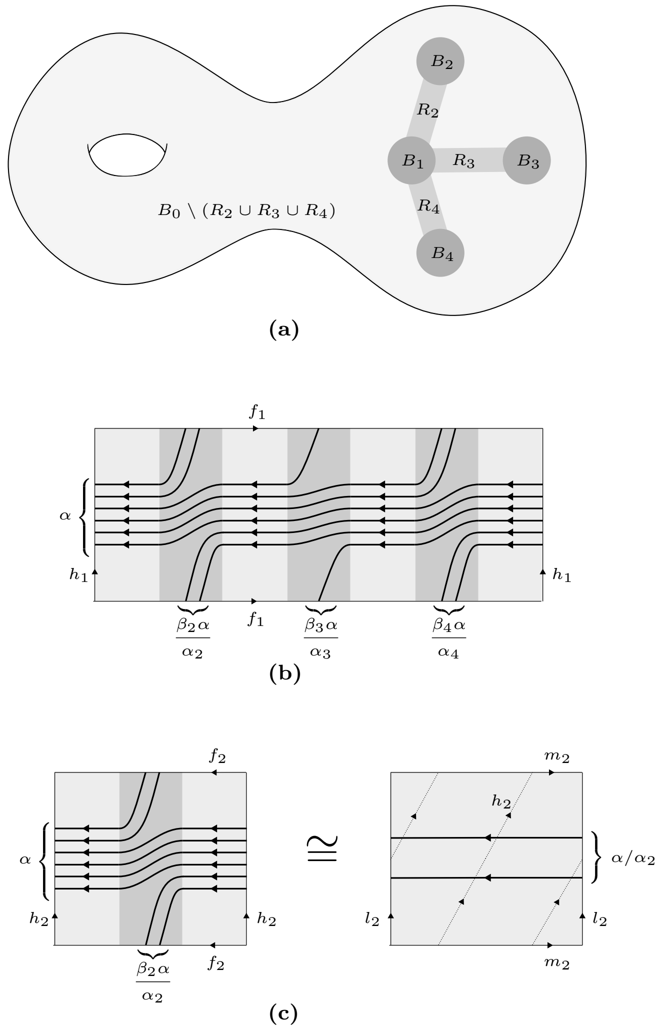

FIGURE 1. Construction of a global surface of section.

the dual pair satisfying (3.1). We set

$$
\begin{aligned}
\alpha &:= \operatorname{lcm}(\alpha_2, ..., \alpha_k) > 0, \\
\beta &:= \Big(\tfrac{\beta_2}{\alpha_2} + ... + \tfrac{\beta_k}{\alpha_k}\Big)\alpha, \\
p &:= \beta_1 \alpha + \alpha_1 \beta = -e(Y)\, \alpha_1 \alpha > 0, && (3.3)\\
q &:= -\beta_1' \alpha - \alpha_1' \beta, && (3.4)\\
b &:= \gcd(p, q), \\
p_0 &:= p/b, \\
q_0 &:= q/b.
\end{aligned}
$$

By inverting the linear Equations (3.3) and (3.4), we have

$$-\alpha = \alpha_1 q + \alpha_1' p,$$
$$\beta = \beta_1 q + \beta_1' p.$$

Notice that (3.2) is satisfied, for

$$\frac{q_0}{p_0} + \frac{\alpha_1'}{\alpha_1} = \frac{q}{p} + \frac{\alpha_1'}{\alpha_1} = \frac{\alpha_1 q + \alpha_1' p}{\alpha_1 p} = -\frac{\alpha}{\alpha_1 p} < 0$$

Moreover,

$$\alpha = -\frac{b\, p_0}{e(Y)\, \alpha_1}.$$

With the notation of Section 3.2, we consider the small disks $B_i \subset B$ containing $\pi \circ \gamma_i$ in their interior. We connect B_1 with every other B_i by means of a rectangle R_i as shown in Figure 1(a). We first define the intersection of our desired surface of section with the solid torus $Y_1 = \pi^{-1}(B_1)$ as b-many (p_0, q_0)-local surfaces of section with boundary on γ_1; of course, away from their boundary on γ_1, such local surfaces of section are disjoint. We shall extend these b-many components outside Y_1 in such a way to create a (connected) global surface of section.

On the torus $\pi^{-1}(\partial_1 B_0) = \pi^{-1}(\partial B_1)$, our defined surface of section winds $-\alpha$-times around f_1 and β-times around h_1. We distribute the windings around h_1 into $k-1$ groups as in the example of Figure 1(b), where the shaded regions are the annuli $\pi^{-1}(\partial_1 B_0 \cap R_i)$, $i = 2, ..., k$. We set $B_0' := B_0 \setminus (R_2 \cup ... \cup R_k)$, and extend the surface of section to $\pi^{-1}(B_0') \cong B_0' \times S^1$ as α distinct constant sections

$$B_0' \times \{t_1\}, ..., B_0' \times \{t_\alpha\};$$

On $\pi^{-1}(R_i)$, we extend the section in $\pi^{-1}(\partial_1 B_0 \cap R_i)$ by taking the product with an interval. As a result, in each torus $\pi^{-1}(\partial B_i)$, $i = 2, ..., k$, our surface of section winds α times around f_i, and $\alpha\beta_i/\alpha_i$ times around h_i (Figure 1(c)); this means that the surface of section winds α/α_i times around m_i, and zero times around l_i. Therefore, we can extend it to $\pi^{-1}(B_i) \cong B_i \times S^1$ as α/α_i many distinct constant sections

$$B_i \times \{t_{i,1}\}, ..., B_i \times \{t_{i,\alpha/\alpha_i}\}.$$

It remains to show that the resulting surface of section is connected. For each $i = 2, ..., k$, in $\pi^{-1}(\partial B_i)$ as in the left of Figure 1(c) outside the shaded region, the j-th horizontal line is connected with the $(j + \beta_i \alpha/\alpha_i)$-th one modulo α. Hence, in order the prove the claim, it suffices to remark that

$$\gcd\left(\alpha, \tfrac{\beta_2 \alpha}{\alpha_2}, ..., \tfrac{\beta_k \alpha}{\alpha_k}\right) = 1.$$

This latter equality follows from the fact that α_i and β_i are coprime and $\alpha = \mathrm{lcm}(\alpha_2, ..., \alpha_k)$. □

Now, we consider the global surface of section $\iota : \Sigma \to Y$ given by Theorem 3.1, and the differential forms

$$\nu_0 := \iota^* \lambda_0, \qquad \omega_0 := d\nu_0.$$

Since the Reeb vector field R_{λ_0} is transverse to $\iota(\mathrm{int}(\Sigma))$, we readily infer that ω_0 is symplectic on $\mathrm{int}(\Sigma)$. A simple computation shows that ω_0 vanishes at all points of $\partial\Sigma$, and indeed satisfies Assumption 2.4 from Section 2.2. This is a consequence of the fact that our surface of section restricts to a (p_0, q_0)-local surface of section near any connected component C of $\partial\Sigma$ (Theorem 3.1(i)). Indeed, consider a collar neighborhood $A_C \subset \Sigma$ of C, and the solid cylinder neighborhood $Y_1 \subset Y$ of γ_1. Under suitable identifications

$$A_C \equiv [0,\rho) \times S^1, \qquad Y_1 \equiv B_1 \times S^1,$$

the restriction $\iota|_{A_C} : A_C \to Y$ has the form

$$\iota|_{A_C}(r,s) = \big(re^{2\pi q_0 s i}, p_0 s\big),$$

and $\omega_0|_{A_C}$ can be written as

$$\omega_0|_{A_C} = \underbrace{\tfrac{4\pi}{\alpha_1}\big(q_0 + \tfrac{\alpha_1'}{\alpha_1}p_0\big)}_{(*)}\, r\, dr \wedge ds. \tag{3.5}$$

The constant $(*)$ is strictly negative, and up to rescaling in the r variable, it can be replaced by -1, as in Assumption 2.4.

The global surface of section $\iota : \Sigma \to Y$ induces a surjective map

$$\tilde\iota : \Sigma \times S^1 \to Y, \qquad \tilde\iota(z,t) = \phi^t_{\lambda_0}(\iota(z)),$$

which restricts to an α-th fold branched covering map

$$\tilde\iota|_{\mathrm{int}(\Sigma)\times S^1} : \mathrm{int}(\Sigma) \times S^1 \to Y \setminus \gamma_1. \tag{3.6}$$

Here,

$$\alpha = -\frac{b\, p_0}{e(Y)\, \alpha_1} > 0$$

is the number of intersections of any regular Reeb orbit in $Y \setminus \gamma_1$ with $\iota(\mathrm{int}(\Sigma))$, according to Theorem 3.1(ii). The branch set has the form $\Sigma_{\mathrm{sing}} \times S^1$, where Σ_{sing} is the finite set of points in $\mathrm{int}(\Sigma)$ which are mapped by ι to points on singular orbits. The map $\tilde\iota$ is a local diffeomorphism also at the branch points.

We denote by $\mathrm{pr}_1 : \Sigma \times S^1 \to \Sigma$ the projection onto the first factor. The one-form

$$\tilde\lambda_0 := \tilde\iota^* \lambda_0 = dt + \mathrm{pr}_1^* \nu_0 \tag{3.7}$$

is a contact form on the interior $\mathrm{int}(\Sigma) \times S^1$, with associated Reeb vector field

$$R_{\tilde\lambda_0} = \partial_t.$$

Notice in particular that $R_{\tilde\lambda_0}$ is well-defined and smooth up to the boundary $\partial\Sigma \times S^1$ as well.

The global surface of section allows to compute the volume of our Besse contact manifold. This was pointed out by Geiges [Gei20], and we provide the details here for the reader's convenience.

Lemma 3.2. *If (Y,λ_0) is a Besse contact 3-manifold whose closed Reeb orbits have minimal common period 1, its volume is equal to the negative of its Euler number. More precisely, if $\iota : \Sigma \to Y$ is a global surface of section as above, we have:*

$$\mathrm{vol}(Y,\lambda_0) = \frac{1}{\alpha}\mathrm{area}(\Sigma,\omega_0) = -e(Y).$$

Proof. The contact volume form $\lambda_0 \wedge d\lambda_0$ is pulled back to

$$\tilde\iota^*(\lambda_0 \wedge \lambda_0) = \tilde\lambda_0 \wedge d\tilde\lambda_0 = dt \wedge \mathrm{pr}_1^* d\nu_0.$$

We recall that $\tilde\iota$ restricts to an α-th fold branched covering map (3.6). Therefore

$$\alpha\,\mathrm{vol}(Y,\lambda_0)=\alpha\int_Y\lambda_0\wedge d\lambda_0=\int_{\Sigma\times S^1}\tilde\iota^*(\lambda_0\wedge d\lambda_0)=\int_{\Sigma\times S^1}dt\wedge\mathrm{pr}_1^*\omega_0=\int_\Sigma\omega_0,$$

and the latter term is precisely $\mathrm{area}(\Sigma,\omega_0)$. Moreover, since $\iota|_{\partial\Sigma}$ is an orientation preserving $b\,p_0$-th fold covering map of the $1/\alpha_1$-periodic Reeb orbit γ_1, Stokes' theorem implies

$$\int_\Sigma\omega_0=\int_{\partial\Sigma}\nu_0=b\,p_0\int_{\gamma_1}\lambda_0=\frac{b\,p_0}{\alpha_1}=-\alpha\,e(Y).\qquad\square$$

3.4. **From Besse to nearly Besse contact forms.** Once the existence of a global surface of section for Besse contact 3-manifolds established, the construction of global surfaces of section for nearly-Besse contact 3-manifolds will be a generalization of the one of nearly-Zoll contact 3-manifolds provided in [ABHS18, BK21]. We work out the details in this section.

We consider the global surface of section $\iota:\Sigma\to Y$ of the Besse contact manifold (Y,λ_0) with boundary on γ_1, and its related objects from the previous subsection. Let λ be a contact form on Y such that

$$R_\lambda|_{\gamma_1}=R_{\lambda_0}|_{\gamma_1} \tag{3.8}$$

We set

$$\tilde\lambda:=\tilde\iota^*\lambda,\quad \nu:=\iota^*\lambda,\quad \omega:=d\nu=\iota^*d\lambda.$$

We recall that the analogous differential forms for the Besse contact form λ_0 are denoted by $\tilde\lambda_0$, ν_0, and ω_0. Since $\tilde\iota$ is a local diffeomorphism on $\mathrm{int}(\Sigma)\times S^1$, $\tilde\lambda$ is a contact form on this open manifold.

The next lemma is analogous to [ABHS18, Prop. 3.6] and [BK21, Prop. 3.10].

Lemma 3.3.

(i) *The pull-back of ν via the inclusion $\partial\Sigma\hookrightarrow\Sigma$ satisfies*

$$\nu|_{\partial\Sigma}=\nu_0|_{\partial\Sigma}.$$

(ii) *The 2-form ω vanishes at all points of $\partial\Sigma$, i.e.*

$$\omega_z=0,\qquad\forall z\in\partial\Sigma.$$

(iii) *The Reeb vector field $R_{\tilde\lambda}$, a priori only defined on the interior of $\Sigma\times S^1$, admits a smooth extension to $\Sigma\times S^1$ that is tangent to the boundary $\partial\Sigma\times S^1$.*

(iv) *For all $\epsilon>0$ there exists $\delta>0$ such that, if the above contact form λ further satisfies $\|R_\lambda-R_{\lambda_0}\|_{C^2}<\delta$, then $\|R_{\tilde\lambda}-\partial_t\|_{C^1}<\epsilon$. Here, the norms are associated with arbitrary fixed Riemannian metrics.*

Proof. Let us consider a collar neighborhood $A_C\times S^1$ of a connected component $C\times S^1$ of $\partial\Sigma\times S^1$, and the solid cylinder neighborhood $Y_1\subset Y$ of γ_1. With the identifications $A_C\equiv[0,\rho)\times S^1$, $Y_1\equiv B_1\times S^1$, and $\gamma_1\equiv\{0\}\times S^1$, as in the previous subsection, the restriction of the map $\tilde\iota$ can be written as

$$\tilde\iota|_{A_C\times S^1}:A_C\times S^1\to Y_1\equiv B_1\times S^1,\qquad \tilde\iota(r,s,t)=\big(re^{2\pi(q_0s-\alpha_1't)i},p_0s+\alpha_1t\big).$$

and $\tilde\lambda_0|_{A_C\times S^1}$ can be written as

$$\tilde\lambda_0|_{A_C\times S^1}:=dt+\Big(\tfrac{p_0}{\alpha_1}+\tfrac{2\pi}{\alpha_1}\big(q_0+\tfrac{\alpha_1'}{\alpha_1}p_0\big)r^2\Big)ds. \tag{3.9}$$

The computations

$$\tilde\iota_*\partial_t|_{C\times S^1} = R_{\lambda_0} = R_\lambda, \qquad \tilde\iota_*\partial_s|_{C\times S^1} = p_0\partial_s = \tfrac{p_0}{\alpha_1} R_{\lambda_0} = \tfrac{p_0}{\alpha_1} R_\lambda,$$

imply

$$\tilde\lambda(\partial_t)|_{C\times S^1} = \tilde\lambda_0(\partial_t)|_{C\times S^1} \equiv 1, \qquad \tilde\lambda(\partial_s)|_{C\times S^1} = \tilde\lambda_0(\partial_s)|_{C\times S^1} \equiv \tfrac{p_0}{\alpha_1},$$

and

$$\begin{aligned}(\partial_t \lrcorner\, d\tilde\lambda)_z &= \tilde\iota^*(R_{\lambda_0} \lrcorner\, d\lambda)_z = \tilde\iota^*(R_\lambda \lrcorner\, d\lambda)_z \equiv 0,\\ (\partial_s \lrcorner\, d\tilde\lambda)_z &= \tfrac{p_0}{\alpha_1}\tilde\iota^*(R_{\lambda_0} \lrcorner\, d\lambda)_z = \tfrac{p_0}{\alpha_1}\tilde\iota^*(R_\lambda \lrcorner\, d\lambda)_z \equiv 0,\\ &\qquad\qquad\qquad\qquad \forall z \in C\times S^1.\end{aligned}$$

These identities, together with $\nu_0 = \tilde\lambda_0|_{\Sigma\times\{0\}}$ and $\nu = \tilde\lambda|_{\Sigma\times\{0\}}$, readily imply points (i) and (ii).

As for point (iii), let us write $R_\lambda|_{Y_1}$ in coordinates $(x+iy,s) = (re^{i\theta}, s)$ as

$$\begin{aligned}R_\lambda|_{Y_1} &= F_1\partial_x + F_2\partial_y + \partial_s\\ &= (F_1\cos(\theta) + F_2\sin(\theta))\,\partial_r + \tfrac{1}{r}(F_2\cos(\theta) - F_1\sin(\theta))\partial_\theta + F_3\,\partial_s,\end{aligned}$$

for some smooth functions $F_j : Y_1 \to \mathbb{R}$. Since R_λ and R_{λ_0} coincide along the closed Reeb orbit γ_1, we have

$$F_1(0) = F_2(0) = 0, \qquad F_3(0) = \alpha_1.$$

Since

$$\tilde\iota_*\partial_r = \partial_r, \qquad \tilde\iota_*\partial_s = 2\pi q_0\,\partial_\theta + p_0\,\partial_s, \qquad \iota_*\partial_t = -2\pi\alpha_1'\,\partial_\theta + \alpha_1\,\partial_s,$$

the Reeb vector fiels $R_{\tilde\lambda}$ on the interior $\mathrm{int}(A_C)\times S^1$ is given by

$$R_{\tilde\lambda} = f_1\,\partial_r + f_2\,\partial_s + f_3\,\partial_t,$$

where, if we set

$$\theta(s,t) := q_0 s - \alpha_1' t, \qquad G_j(r,s,t) := \frac{F_j\circ\iota(r,s,t)}{r}, \quad j=1,2,$$

the functions $f_j : \mathrm{int}(A_C)\times S^1 \to \mathbb{R}$ are given by

$$f_1 = \cos(\theta)F_1\circ\tilde\iota + \sin(\theta)F_2\circ\tilde\iota, \tag{3.10}$$

$$f_2 = \big(2\pi\big(q_0 + \tfrac{\alpha_1'}{\alpha_1}p_0\big)\big)^{-1}\Big(-\sin(\theta)G_1 + \cos(\theta)G_2 + 2\pi\tfrac{\alpha_1'}{\alpha_1}F_3\circ\tilde\iota\Big), \tag{3.11}$$

$$f_3 = \tfrac{1}{\alpha_1}\big(F_3\circ\tilde\iota - p_0\,f_2\big). \tag{3.12}$$

Since $F_1\circ\tilde\iota|_{C\times S^1} = F_2\circ\tilde\iota|_{C\times S^1} \equiv 0$, the functions G_1 and G_2 extend smoothly to the whole $A_C\times S^1$, and so do the functions f_1, f_2, f_3. Moreover, $f_1|_{C\times S^1}\equiv 0$. This proves that $R_{\tilde\lambda}$ extends smoothly to a vector field on $\Sigma\times S^1$ that is tangent to the boundary $\partial\Sigma\times S^1$.

Finally, assume that R_λ is C^2-close to R_{λ_0}. Away from any fixed neighborhood of $\partial\Sigma\times S^1$, $R_{\tilde\lambda}$ is C^2-close to $R_{\tilde\lambda_0} = \partial_t$. On Y_1, since $R_\lambda = -2\pi\alpha_1'\partial_\theta + \alpha_1\partial_s$, the functions

$$F_1 - 2\pi\alpha_1' r\sin(\theta), \qquad F_2 + 2\pi\alpha_1' r\cos(\theta), \qquad F_3 - \alpha_1$$

are C^2-small and vanish at $\partial\Sigma\times S^1$. Therefore, the function

$$-\sin(\theta)G_1 + \cos(\theta)G_2 + 2\pi\tfrac{\alpha_1'}{\alpha_1}F_3\circ\tilde\iota$$

is C^1-small. Equation (3.10) implies that f_1 is C^2-small, and Equations (3.11) and (3.12) imply that f_2 and $f_3 - 1$ are C^1-small. Overall, we conclude that $R_{\tilde\lambda}$ is C^1-close to $R_{\tilde\lambda_0} = \partial_t$. □

Besides (3.8), we now assume:

Assumption 3.4. The vector field $R_{\tilde\lambda}$ on $\Sigma \times S^1$ is transverse to $\Sigma \times \{s\}$ and oriented as ∂_s, for every $s \in S^1$. □

Thanks to Lemma 3.3(iv), Assumption 3.4 is implied by the C^2-closeness of R_λ to R_{λ_0}. By Assumption 3.4, the first-return time

$$\tau : \Sigma \to (0,\infty), \qquad \tau(z) := \min \big\{ t > 0 \bigm| \phi^t_{\tilde\lambda}(z,0) \in \Sigma \times \{0\} \big\}, \tag{3.13}$$

is a well-defined smooth map, and the first-return map

$$\phi : \Sigma \to \Sigma, \qquad (\phi(z), 0) = \phi^{\tau(z)}_{\tilde\lambda}(z,0), \tag{3.14}$$

is a diffeomorphism. The diffeomorphism ϕ is isotopic to the identity through the isotopy $\{\phi_s\}$ defined by $\phi_0 := \mathrm{id}$ and, for $s \in (0,1]$,

$$(\phi_s(z), s) = \phi^{\tau_s(z)}_{\tilde\lambda}(z,0), \tag{3.15}$$

where

$$\tau_s(z) := \min \big\{ t > 0 \bigm| \phi^t_{\tilde\lambda}(z,0) \in \Sigma \times \{s\} \big\}.$$

In the particular case $\lambda = \lambda_0$, we would get that τ is identically equal to 1 and ϕ_s equals the identity on Σ for every $s \in [0,1]$.

The next lemma relates the volume of (Y,λ) to the integral of the first return time.

Lemma 3.5. *The contact volume of (Y,λ) is given by*

$$\mathrm{vol}(Y,\lambda) = \frac{1}{\alpha} \int_\Sigma \tau\, \omega.$$

Proof. The bijective map

$$j : \Sigma \times [0,1) \to \Sigma \times S^1, \qquad j(z,s) := \phi^{s\tau(z)}_{\tilde\lambda}(z,0),$$

satisfies

$$j^*(\tilde\lambda \wedge d\tilde\lambda) = \tau\, ds \wedge \mathrm{pr}_1^* \omega,$$

where $\mathrm{pr}_1 : \Sigma \times [0,1) \to \Sigma$ is the projection onto the first factor. Together with the fact that the restriction of $\tilde\imath$ to the interior of $\Sigma \times S^1$ is an α-th fold branched covering map of a full measure subset of Y pulling the volume form $\lambda \wedge d\lambda$ back to $\tilde\lambda \wedge d\tilde\lambda$, we obtain

$$\begin{aligned} \alpha\, \mathrm{vol}(Y,\lambda) &= \alpha \int_Y \lambda \wedge d\lambda = \int_{\Sigma \times S^1} \tilde\lambda \wedge d\tilde\lambda = \int_{\Sigma \times [0,1)} j^*(\tilde\lambda \wedge d\tilde\lambda) \\ &= \int_{\Sigma \times [0,1)} \tau\, ds \wedge \mathrm{pr}_1^* \omega = \int_\Sigma \tau\, \omega. \end{aligned}$$

□

The next lemma relates the first return map ϕ to the first return time τ via the 1-form ν.

Lemma 3.6. *The first-return map ϕ is an exact symplectomorphism of (Σ, ω), and more precisely*

$$\phi^*\nu = \nu + d\tau.$$

The boundary restriction of the first return time τ is given by

$$\tau(z) = 1 + \int_{\{s \mapsto \phi_s(z)\}} \nu, \qquad \forall z \in \partial\Sigma.$$

Proof. The first statement follows by the well-known computation

$$\phi^*\nu = (\phi_{\tilde{\lambda}}^{\tau(z)})^* \tilde{\lambda}|_{\Sigma\times\{0\}} + \phi^*(\tilde{\lambda}(R_{\tilde{\lambda}}))d\tau = \nu + d\tau.$$

For each $z \in \partial\Sigma$, the curve

$$\zeta_z : [0,1] \to \partial\Sigma \times S^1, \qquad \zeta_z(s) := (\phi_s(z), s) = \phi_{\tilde{\lambda}}^{\tau_s(z)}(z, 0),$$

is a reparametrization of the restriction of the orbit of $(z, 0)$ by the Reeb flow of $\tilde{\lambda}$ to the interval $[0, \tau(z)]$, which makes one full turn around the second factor of $\partial\Sigma \times S^1$. Therefore,

$$\tau(z) = \int_{\zeta_z} \tilde{\lambda} = \int_{\zeta_z} \tilde{\lambda}_0 = \int_{\zeta_z} \big(ds + \mathrm{pr}_1^*\nu\big) = \int_{\zeta_z} ds + \int_{\mathrm{pr}_1\circ\zeta_z} \nu = 1 + \int_{\{s\mapsto\phi_s(z)\}} \nu,$$

where the second equality follows by Lemma 3.3(i), and the third one from (3.7). □

In order to prove Theorem A, we will need to apply the fixed point Theorem 2.5, which concerns symplectomorphisms that are C^1-close to the identity on a surface Σ equipped with a fixed 2-form symplectic in the interior and vanishing in a suitable way at the boundary. By Lemma 3.6, the diffeomorphism $\phi : \Sigma \to \Sigma$ is symplectic with respect to the 2-form $\omega = \iota^* d\lambda$, which varies with λ. However, assumption (3.8) and its consequence $\nu|_{\partial\Sigma} = \nu_0|_{\partial\Sigma}$ from Lemma 3.3(i) imply that

$$\mathrm{area}(\Sigma, \omega) = \int_{\partial\Sigma} \nu = \int_{\partial\Sigma} \nu_0 = \mathrm{area}(\Sigma, \omega_0).$$

Therefore, we can conjugate ϕ by a diffeomorphism $\kappa : \Sigma \to \Sigma$ pulling ω back to ω_0 and obtain a symplectomorphism with respect to the fixed 2-form ω_0 on Σ. The construction of this diffeomorphism and the proof of its further properties are based as usual on Moser's trick but require a bit of care, since we are working on a surface with boundary. We work out the details in the following lemma, which is a variation of [BK21, Prop. 3.9].

Lemma 3.7. *If λ is C^2-close enough to λ_0, then there exists a diffeomorphism $\kappa : \Sigma \to \Sigma$ such that $\kappa|_{\partial\Sigma} = \mathrm{id}$ and $\kappa^*\omega = \omega_0$. Moreover, κ C^1-converges to the identity as λ C^2-converges to λ_0.*

Proof. Note that the smallness of $\|\lambda - \lambda_0\|_{C^2}$ implies the smallness of $\|\nu - \nu_0\|_{C^2}$ and $\|\omega - \omega_0\|_{C^1}$. Assumption 3.4 guarantees that ω is a symplectic form in the interior of Σ inducing the same orientation as ω_0. Therefore, the 2-forms $\omega_t := t\omega + (1-t)\omega_0$ are symplectic on $\mathrm{int}(\Sigma)$ for every $t \in [0,1]$. They are actually uniformly C^1-close to ω_0 when $\|\lambda - \lambda_0\|_{C^2}$ is small.

We look for an isotopy $\kappa_t : \Sigma \to \Sigma$ such that $\kappa_t|_{\partial\Sigma} \equiv \mathrm{id}$ and $\kappa_t^*\omega_t = \omega_0$. We build the time-dependent vector field X_t realizing such isotopy, i.e. $\frac{d}{dt}\kappa_t = X_t \circ \kappa_t$. By differentiating $\kappa_t^*\omega_t$ with respect to t, we obtain

$$0 = \tfrac{d}{dt}\kappa_t^*\omega_t = \kappa_t^*\big(d(X \lrcorner\, \omega_t) + \omega - \omega_0\big) = \kappa_t^* d\big(X \lrcorner\, \omega_t + \nu - \nu_0\big). \tag{3.16}$$

We define X_t on $\mathrm{int}(\Sigma)$ by the equation

$$X_t \lrcorner\, \omega_t = \nu_0 - \nu + df_t \tag{3.17}$$

for a suitable C^2-small smooth function $f : \Sigma \times [0,1] \to \mathbb{R}$ to be determined. A suitable choice of f will guarantee that X_t has a smooth extension to the whole Σ with $X_t|_{\partial\Sigma} \equiv 0$.

For every connected component C of $\partial\Sigma$, we fix a collar neighborhood $A_C \subset \Sigma$ so that, with the usual suitable identification $[0,\rho) \times S^1$, the differential forms ω_0 can be written as in (3.5). Actually, up to rescaling the interval $[0,\rho)$, we can even write $\omega_0|_{A_C}$ as

$$\omega_0|_{A_C} = -r\, dr \wedge ds,$$

where $r \in [0,\rho)$ and $s \in S^1$. Lemma 3.3(i-ii) implies $\nu|_C = \nu_0|_C$ and $\omega_z = (\omega_0)_z$ for all $z \in C$. Therefore we can write

$$(\nu_0 - \nu)|_{A_C} = h_1\, dr + r\, h_2\, ds, \qquad \omega|_{A_C} = -r\, h_3\, dr \wedge ds,$$

for some smooth functions $h_i : A_C \to \mathbb{R}$. If ν_0 and ν are C^2-close, the function h_1 is C^2-small, while h_2 is C^1-small. Moreover, since ω_0 and ω are C^1-close, the function $1 - h_3(0,\cdot)$ is C^0-small. In particular, up to choosing the annulus A_C small enough, h_3 is strictly positive on the whole A_C. Since $d(\nu_0 - \nu) = \omega_0 - \omega$, we have

$$\partial_r(r\, h_2) - \partial_s h_1 = r(h_3 - 1). \tag{3.18}$$

The two-form $\omega_t|_{A_C}$ is given by

$$\omega_t|_{A_C} = -r\,(t(h_3 - 1) + 1)\, dr \wedge ds,$$

and if we write the vector field $X_t|_{A_C}$ in (r,s) coordinates as $X_t = R_t\, \partial_r + S_t\, \partial_s$, Equation (3.17) becomes

$$R_t = -\frac{r\, h_2 + \partial_s f}{r\,(t(h_3 - 1) + 1)}, \qquad S_t = \frac{h_1 + \partial_r f}{r\,(t(h_3 - 1) + 1)}.$$

We now choose $f : \Sigma \to \mathbb{R}$ to be a smooth function such that $f|_{\partial\Sigma} \equiv 0$ and, on any collar neighborhood $A_C = [0,\rho) \times S^1$ as above, satisfies

$$f(r,s) = \begin{cases} -\displaystyle\int_0^r h_1(x,s)\, dx, & \text{if } r \leq \tfrac{1}{3}\rho, \\ 0, & \text{if } r \geq \tfrac{2}{3}\rho. \end{cases}$$

We shall choose such an f so that $\|f\|_{C^2} \leq \mathrm{const}\, \|h_1\|_{C^2}$, and in particular f is C^2-small since ν and ν_0 are C^2-close. With this choice of f, we have $S_t(r,s) = 0$ if $r \leq \frac{1}{3}\rho$. As for the function R_t, for all $r \leq \frac{1}{3}\rho$ Equation (3.18) implies

$$\begin{aligned} R_t(r,s) &= -\frac{r\, h_2(r,s) + \partial_s f(r,s)}{r\,(t(h_3 - 1) + 1)} = -\frac{r\, h_2(r,s) - \int_0^r \partial_s h_1(x,s)\, dx}{r\,(t(h_3 - 1) + 1)} \\ &= -\frac{r\, h_2(r,s) + \int_0^r \big(x(h_3(x,s) - 1) - \partial_x(x\, h_2(x,s))\big)\, dx}{r\,(t(h_3 - 1) + 1)} \\ &= -\frac{\frac{1}{r}\int_0^r x(h_3(x,s) - 1)\, dx}{t(h_3 - 1) + 1}. \end{aligned}$$

We already know that the function $t(h_3 - 1) + 1$ appearing in the denominator of the above equation is nowhere vanishing. As for the numerator, we can rewrite the integral as

$$\int_0^r x(h_3(x,s) - 1)\, dx = r^2\, h_4(r,s)$$

for some C^0-small smooth function $h_4 : A_C \to \mathbb{R}$ such that $h_4(0,s) = h_3(0,s) - 1$. Therefore

$$R_t(r,s) = -\frac{r\, h_4(r,s)}{t\,(h_3 - 1) + 1}.$$

From this expression we readily infer that R_t is C^1-small, extends smoothly to the whole Σ, and $R_t|_{\partial\Sigma} \equiv 0$. Summing up, we obtained a C^1-small smooth vector field X_t on Σ satisfying (3.17) and $X|_{\partial\Sigma} \equiv 0$. Its flow κ_t is C^1-small for all $t \in [0,1]$, and satisfies $\kappa_t|_{\partial\Sigma} = \mathrm{id}$ and, by (3.16), $\kappa_t^* \omega_t = \omega_0$. □

The following proposition sums up the arguments of this section and will play a crucial role in the proof of Theorem A(ii).

Proposition 3.8. *Let λ_0 be a Besse contact form on the closed manifold Y whose closed Reeb orbits have minimal common period 1, and let γ_1 be any orbit of R_{λ_0}. Then there exists a closed surface with boundary Σ endowed with an exact 2-form ω_0 which is symplectic on the interior of Σ and satisfies Assumption 2.4 such that the following holds. For every $\epsilon > 0$ small enough and for every C^1-neighborhood $\mathcal{U}$ of the identity in $\widetilde{\mathrm{Ham}}_0(\Sigma, \omega_0)$ there exist $\delta > 0$ and, for each contact form λ on Y such that*

$$R_\lambda|_{\gamma_1} = R_{\lambda_0}|_{\gamma_1}, \qquad \|\lambda - \lambda_0\|_{C^2} < \delta, \qquad \|R_\lambda - R_{\lambda_0}\|_{C^2} < \delta,$$

a global surface of section

$$j : \Sigma \to Y$$

for R_λ mapping each component of $\partial\Sigma$ onto some positive iterate of γ_1 and an element

$$\tilde{\psi} \in \mathcal{U}$$

with the following properties:

(i) *The normalized Calabi invariant of $\tilde{\psi}$ is related to the volumes of (Y, λ) and (Y, λ_0) by*

$$\widehat{\mathrm{Cal}}(\widetilde{\psi}) = \frac{\mathrm{vol}(Y, \lambda)}{\mathrm{vol}(Y, \lambda_0)} - 1.$$

(ii) *A point $z \in \mathrm{int}(\Sigma)$ is a contractible fixed point of $\tilde{\psi}$ if and only if*

$$\gamma_z(t) := \phi_\lambda^t(j(z))$$

is a closed Reeb orbit of R_λ in $Y \setminus \gamma_1$ with (not necessarily minimal) period

$$1 + a_{\tilde{\psi}}(z) \in (1 - \epsilon, 1 + \epsilon).$$

Here, $a_{\tilde{\psi}}(z)$ is the normalized action of the contractible fixed point z.

(iii) *The element $\tilde{\psi}$ is the identity in $\widetilde{\mathrm{Ham}}_0(\Sigma, \omega_0)$ if and only if (Y, λ) is Besse and its Reeb orbits have common period 1.*

Proof. We consider a global surface of section

$$\iota : \Sigma \to Y$$

for the Reeb flow of λ_0 as in Theorem 3.1 and the corresponding map

$$\tilde\iota : \Sigma \times S^1 \to Y, \qquad \tilde\iota(z,t) := \phi^t_{\lambda_0}(\iota(z)).$$

The 2-form

$$\omega_0 := \iota^* d\lambda_0,$$

is symplectic in the interior of Σ, satisfies Assumption 2.4 (see the discussion after the proof of Theorem 3.1) and, thanks to Lemma 3.2, has total area

$$\mathrm{area}(\Sigma,\omega_0) = \alpha\,\mathrm{vol}(Y,\lambda_0), \tag{3.19}$$

where α is the positive integer appearing in Theorem 3.1. Given another contact form λ on Y, we set as before

$$\tilde\lambda := \tilde\iota^*\lambda, \qquad \nu := \iota^*\lambda, \qquad \omega := d\nu = \iota^* d\lambda.$$

Here, we are assuming that $R_\lambda|_{\gamma_1} = R_{\lambda_0}|_{\gamma_1}$, which is exactly condition (3.8), and that $\|R_\lambda - R_{\lambda_0}\|_{C^2}$ is small enough, so that also Assumption 3.4 holds thanks to Lemma 3.3(iv). In particular, ι is also a global surface of section for the Reeb flow of λ. Moreover, by Lemma 3.3 the 1-form $\tilde\lambda$ defines a flow on $\Sigma\times S^1$ having $\Sigma\times\{0\}$ as global surface of section and we denote by τ and ϕ the corresponding first return time and first return map, see (3.13) and (3.14). By Lemma 3.3(iv), the map ϕ is C^1-close to the identity when $\|R_\lambda - R_{\lambda_0}\|_{C^2}$ is small.

By further assuming that $\|\lambda-\lambda_0\|_{C^2}$ is small enough, we can use Lemma 3.7 to find a diffeomorphism $\kappa : \Sigma\to\Sigma$ that is C^1-close to the identity and satisfies $\kappa^*\omega = \omega_0$. Up to conjugating ϕ by κ and replacing ι by $j := \iota\circ\kappa$, which is still a global surface of section for the Reeb flow of λ, we may assume that ω equals ω_0.

In this case, ν is a primitive of ω_0 and the equality

$$\phi^*\nu - \nu = d\tau \tag{3.20}$$

proved in Lemma 3.6 shows that ϕ is an exact symplectomorphism on (Σ,ω_0). Being C^1-close to the identity, ϕ is the image under the universal cover

$$\pi : \widetilde{\mathrm{Ham}}(\Sigma,\omega_0) \to \mathrm{Ham}(\Sigma,\omega_0)$$

of a unique $\tilde\psi = [\{\psi_t\}]$ which is also C^1-close to the identity (see Theorem 2.8). Moreover, the C^1-closeness to the identity implies that the Hamiltonian isotopy $\{\psi_t\}$ is homotopic with fixed ends to the (non necessarily symplectic) isotopy $\{\phi_t\}$ which is defined in (3.15), and hence Lemma 3.6 gives us the identity

$$\tau(z) = 1 + \int_{\{t\mapsto\psi_t(z)\}} \nu, \qquad \forall z\in\partial\Sigma. \tag{3.21}$$

Identities (3.20) and (3.21) imply that $\tilde\psi$ has vanishing flux (see Remark 2.2), so we may assume that it belongs to the C^1-neighborhood $\mathcal{U}$ of the identity in $\widetilde{\mathrm{Ham}}_0(\Sigma,\omega_0)$, and give us the following relationship between the function τ and the normalized action of $\tilde\psi$ with respect to the primitive ν of ω_0:

$$\tau = 1 + a_{\tilde\psi,\nu}. \tag{3.22}$$

Therefore, (3.19) and Lemma 3.5 imply the identity

$$\widehat{\mathrm{Cal}}(\tilde\psi) = \frac{1}{\mathrm{area}(\Sigma,\omega_0)}\int_\Sigma (\tau-1)\,\omega_0 = \frac{\mathrm{vol}(Y,\lambda)}{\mathrm{vol}(Y,\lambda_0)} - 1,$$

which proves (i). Moreover, if $z \in \mathrm{int}(\Sigma)$ is a contractible fixed point of $\tilde\psi$, then the Reeb orbit

$$\gamma_z(t) := \phi_\lambda^t(j(z))$$

is different from γ_1 and closes up at time $\tau(z) = 1 + a_{\tilde\psi,\nu}(z)$. This number belongs to the interval $(1-\epsilon, 1+\epsilon)$ when $\|R_\lambda - R_{\lambda_0}\|_{C^2}$ is small enough, again by Lemma 3.3(iv). Conversely, if ϵ is small enough then any closed orbit of R_λ other than γ_1 and with (non necessarily minimal) period in the interval $(1-\epsilon, 1+\epsilon)$ correesponds to an interior fixed point of $\phi = \pi(\tilde\psi)$. All fixed points of ϕ are contractible as fixed points of the lift $\tilde\psi$, as this is C^1-close to the identity. This proves (ii).

If $\tilde\psi$ is the identity, then every orbit of the Reeb flow of λ is closed and, since the action $a_{\tilde\psi,\nu}$ vanishes identically, has (non necessarily minimal) period 1 by (3.22). Therefore, (Y,λ) is Besse with orbits having common period 1. Conversely, if (Y,λ) has this property then the fact that τ is close to 1 and the closeness of $\tilde\psi$ to the identity imply that $\tilde\psi$ is the identity. This proves (iii). □

Remark 3.9. The above result can be generalized to a more general situation in which the Reeb flows of λ and λ_0 have more closed orbits $\gamma_1, \dots, \gamma_h$ in common and the boundary of Σ is mapped onto their union, but with a caveat: If $R_\lambda|_{\gamma_i} = c_i R_{\lambda_0}|_{\gamma_i}$ then the flux of the Hamiltonian isotopy defining $\tilde\psi$ is in general non zero, unless all numbers c_i coincide.

4. Proof of Theorem A

Proof of Theorem A(i). Let λ be a contact form on Y such that there exists a point $z \in Y$ whose Reeb orbit is open or has minimal period strictly larger than $\tau_k(\lambda)$. The same must be true for all points in a sufficiently small compact neighborhood $U \subset Y$ of z. Let $f : Y \to (-\infty, 0]$ be a non-positive smooth function supported in U and such that $f(z) < 0$. For each $\epsilon > 0$ small enough, the contact form $\lambda_\epsilon := e^{\epsilon f}\lambda$ satisfies $\mathrm{fix}(\phi^t_{\lambda_\epsilon}) = \mathrm{fix}(\phi^t_\lambda)$ for all $t \in [0, \tau_k(\lambda)]$. In particular, $\tau_k(\lambda_\epsilon) = \tau_k(\lambda)$. Since

$$\mathrm{vol}(Y, \lambda_\epsilon) = \int_Y e^{2\epsilon f}\lambda \wedge d\lambda < \int_Y \lambda \wedge d\lambda = \mathrm{vol}(Y,\lambda),$$

we have that $\rho_k(\lambda_\epsilon) > \rho_k(\lambda)$, and therefore λ is not a local maximizer of ρ_k. This proves that each local maximizer of ρ_k is a Besse contact form λ_0 such that $k_0(\lambda_0) \le k$.

Now, let λ_0 be a Besse contact form on Y with $k_0 := k_0(\lambda_0)$. It remains to show that λ_0 is not a local maximizer of ρ_k for any $k > k_0$. Without loss of generality, we can assume that $\tau_{k_0}(\lambda_0) = 1$, so that the Reeb flow of λ_0 defines a locally free $S^1 = \mathbb{R}/\mathbb{Z}$-action on Y and

$$\tau_k(\lambda_0) = \tau_{k_0}(\lambda_0) = 1 \qquad \forall k \ge k_0.$$

We denote by $\gamma_1, \dots, \gamma_h$ the singular orbits of R_{λ_0} and by $\alpha_1, \dots, \alpha_h$ the integers greater than 1 such that γ_i has minimal period $1/\alpha_i$ (if λ_0 is Zoll, we have $h = 0$). Then

$$k_0 = \alpha_1 + \dots + \alpha_h - h + 1.$$

We denote by $B := Y/S^1$ the quotient orbifold and by $\pi : Y \to B$ the quotient projection. We choose a small open disk $D \subset B$ with smooth boundary such that, for all $b \in D$, the preimage $\pi^{-1}(b)$ is a closed Reeb orbit of minimal period 1. We can identify D with the open disk of radius ρ in $\mathbb{C}$ and assume that the restriction of λ_0 to $\pi^{-1}(D)$ has the form

$$\lambda_0 = ds + \frac{r^2}{2}\, d\theta, \qquad \forall (re^{i\theta}, s) \in D \times S^1, \tag{4.1}$$

where r, θ are polar coordinates on $\mathbb{C}$. We now choose a smooth function $h : B \to \mathbb{R}$ such that

(i) $\int_Y h \circ \pi\, \lambda_0 \wedge d\lambda_0 = 0$;
(ii) h equals a positive constant c_+ on $Y \setminus D$;
(iii) on D, h has the form $h = \chi(r)$ where $\chi : [0, \rho] \to \mathbb{R}$ is a smooth function such that $-c_- := \chi(0) < 0$, $\chi(\rho) = c_+$ and $\chi'(r) > 0$ for every $r \in (0, \rho)$.

For every $\epsilon > 0$, we consider the 1-form

$$\lambda_\epsilon := (1 + \epsilon\, h \circ \pi)\lambda_0,$$

which is a contact form for ϵ small enough. By (i), we have

$$\mathrm{vol}(Y, \lambda_\epsilon) = \int_Y (1 + \epsilon\, h \circ \pi)^2\, \lambda_0 \wedge d\lambda_0 = \mathrm{vol}(Y, \lambda_0) + c\,\epsilon^2, \tag{4.2}$$

where

$$c := \int_Y (h \circ \pi)^2\, \lambda_0 \wedge d\lambda_0.$$

Let $\epsilon > 0$ be so small that λ_ϵ is a contact form. Condition (ii) implies that the set $\pi^{-1}(B \setminus D)$ is invariant under the Reeb flow of λ_ϵ, and hence the same is true for its complement $\pi^{-1}(D)$. The Reeb orbits of λ_ϵ on $\pi^{-1}(B \setminus D)$ are exactly the Reeb orbits of λ_0 reparametrized in such a way that their period gets multiplied by $1 + \epsilon c_+$. In particular, on $\pi^{-1}(B \setminus D)$ the Reeb flow of λ_ϵ has exactly

$$k_0 - 1 = \alpha_1 + \cdots + \alpha_h - h$$

closed orbits with period strictly less that $1 + \epsilon c_+$: the iterates γ_i^m with $1 \le m \le \alpha_i - 1$.

On $\pi^{-1}(D)$, the Reeb flow of λ_ϵ has an orbit of minimal period $1 - \epsilon c_-$, which is given by the inverse image by π of the center of D, and all other orbits are either non-periodic or have a very large minimal period when ϵ is small. The latter assertion follows from (4.1) and (iii), which imply that the Reeb orbits of λ_ϵ in $\pi^{-1}(D)$ are lifts of Hamiltonian orbits on D defined by the standard symplectic form $r\, dr \wedge d\theta$ and the radial Hamiltonian χ. These orbits wind around the circle of radius $r \in (0, \rho)$ with frequency $\frac{\epsilon \chi'(r)}{2\pi r} > 0$, which by the mean value theorem has the upper bound

$$\frac{\epsilon \chi'(r)}{2\pi r} \le \frac{\epsilon}{2\pi} \max_{r \in [0,\rho]} |\chi''(r)|, \qquad \forall r \in (0, \rho).$$

If ϵ is so small that the above upper bound is smaller than $(1 + \epsilon c_+)^{-1}$ and

$$2(1 - \epsilon c_-) \ge 1 + \epsilon c_+,$$

we conclude that on $\pi^{-1}(D)$ the Reeb flow of λ_ϵ has precisely one closed orbit whose period is strictly less that $1 + \epsilon c_+$.

Summing up, λ_ϵ has k_0 many orbits whose period is strictly less that $1 + \epsilon c_+$. Together with the fact that this Reeb flow has infinitely many closed orbits of minimal period $1 + \epsilon c_+$, we deduce that

$$\tau_k(\lambda_\epsilon) = 1 + \epsilon c_+ \qquad \forall k > k_0,$$

when $\epsilon > 0$ is small enough. By (4.2), we conclude that for every $k > k_0$ the k-th systolic ratio of λ_ϵ has the lower bound

$$\rho_k(\lambda_\epsilon) = \frac{\tau_k(\lambda_\epsilon)^2}{\mathrm{vol}(Y,\lambda_\epsilon)} = \frac{(1+\epsilon c_+)^2}{\mathrm{vol}(Y,\lambda_0)+c\,\epsilon^2} \geq \frac{1+2\epsilon c_+}{\mathrm{vol}(Y,\lambda_0)+c\,\epsilon^2},$$

which is strictly larger than

$$\frac{1}{\mathrm{vol}(Y,\lambda_0)} = \rho_k(\lambda_0)$$

if ϵ is small enough. This shows that λ_0 is not a local maximizer of ρ_k in the C^∞-topology if $k > k_0$. □

Proof of Theorem A(ii). Let (Y,λ_0) be a Besse contact 3-manifold. We recall that the positive integer

$$k_0 := k_0(\lambda_0)$$

is the minimal k so that the Reeb orbits of (Y,λ_0) have minimal common period $\tau_k(\lambda_0)$. Without loss of generality, up to multiplying λ_0 with a positive constant, we can assume that

$$\tau_{k_0}(\lambda_0) = 1.$$

We first carry out the proof under the assumption that (Y,λ_0) is not Zoll, so that $k_0 > 1$. We denote by $\gamma_1,...,\gamma_h$ the singular Reeb orbits of (Y,λ_0), that is, the closed Reeb orbits with minimal period strictly less than 1. We denote by $\alpha_i > 1$ the positive integer whose reciprocal $1/\alpha_i$ is the minimal period of γ_i, and by γ_i^m the closed Reeb orbit γ_i seen as a m/α_i-periodic orbit. Therefore,

$$k_0 = \alpha_1 + ... + \alpha_h - h + 1. \tag{4.3}$$

It is well known that all the periodic orbits γ_i^m with $1 \leq m \leq \alpha_i - 1$ are non-degenerate, i.e.

$$\ker(d\phi_{\lambda_0}^{m/\alpha_i}(\gamma_i(0)) - I) = \mathrm{span}\{R_{\lambda_0}(\gamma_i(0))\}, \qquad \forall m = 1,...,\alpha_i - 1.$$

We refer the reader to [CGM20, Section 4.1] for a proof of this fact. Standard results about perturbation of vector fields imply that, for every

$$\epsilon \in \big(0, \tfrac{1}{2\max\{\alpha_1,...,\alpha_h\}}\big),$$

there is a C^3-neighborhood $\mathcal{V}$ of λ_0 such that every $\lambda \in \mathcal{V}$ satisfies the following properties.

(i) R_λ has pairwise distinct closed orbits $\tilde\gamma_i$, $i = 1,\dots,h$, such that $\tilde\gamma_i$ has minimal period in $(\frac{1}{\alpha_i} - \epsilon, \frac{1}{\alpha_i} + \epsilon)$ and is C^2-close to γ_i.

(ii) The family of possibly iterated closed orbits of R_λ of period less than or equal to $1-\epsilon$ is

$$\big\{\tilde\gamma_i^m \;\big|\; i \in \{1,...,h\},\ m \in \{1,...,\alpha_i - 1\}\big\}.$$

Here, we say that two closed curves $\gamma : \mathbb{R}/p\mathbb{Z} \to Y$ and $\tilde\gamma : \mathbb{R}/\tilde p\mathbb{Z} \to Y$ are C^2-close if γ and $\tilde\gamma$ are C^2-close on $[0, \max\{p,\tilde p\}]$.

In particular, for any $\lambda \in \mathcal{V}$, the closed Reeb orbit γ_1 of (Y,λ_0) with minimal period $1/\alpha_1$ is C^2-close to some closed Reeb orbit $\tilde\gamma_1$ of (Y,λ) with minimal period $T \in (\frac{1}{\alpha_1} - \epsilon, \frac{1}{\alpha_1} + \epsilon)$. Up to multiplying the contact form λ with a constant close to 1, we can assume that $T = 1/\alpha_1$, that is, γ_1 and $\tilde\gamma_1$ have the same minimal period $1/\alpha_1$. By an argument analogous to [ABHS18, Prop. 3.10], there exists a diffeomorphism $\upsilon : Y \to Y$ such that $\upsilon \circ \gamma_1 = \tilde\gamma_1$ and the quantities $\|\upsilon^*\lambda - \lambda_0\|_{C^2}$ and $\|R_{\upsilon^*\lambda} - R_{\lambda_0}\|_{C^2}$ are small. Therefore, up to pulling back λ by υ, we can assume that

$$R_\lambda|_{\gamma_1} = R_{\lambda_0}|_{\gamma_1},$$

that is, γ_1 is a closed orbit of minimal period $1/\alpha_1$ for both R_λ and R_{λ_0}. After this modification, we can assume that λ and R_λ are arbitrarily C^2-close to λ_0 and R_{λ_0} respectively, so that the assumptions of Proposition 3.8 are fulfilled.

The contact form λ satisfies

$$\tau_{k_0}(\lambda) \le \tau_{k_0}(\lambda_0) = 1;$$

as a consequence of (4.3), of point (ii) above and of the fact that $\gamma_1^{\alpha_1}$ is an orbit of R_λ of period 1. If $\mathrm{vol}(Y,\lambda) > \mathrm{vol}(Y,\lambda_0)$, we have $\rho_{k_0}(\lambda) < \rho_{k_0}(\lambda_0)$, and we are done. Therefore, it remains to consider the case in which

$$\mathrm{vol}(Y,\lambda) \le \mathrm{vol}(Y,\lambda_0). \tag{4.4}$$

We now apply Proposition 3.8 (using the objects and terminology introduced therein), choosing a C^1-neighborhood $\mathcal{U} \subset \widetilde{\mathrm{Ham}}_0(\Sigma,\omega_0)$ of the identity such that the conclusion of the fixed point Theorem 2.5 with $c := \frac{1}{2}$ holds for all elements of $\mathcal{U}$. We require λ and R_λ to be sufficiently C^2-close to λ_0 and R_{λ_0} respectively, so that the element $\tilde\psi \in \widetilde{\mathrm{Ham}}_0(\Sigma,\omega_0)$ provided by Proposition 3.8 is contained in $\mathcal{U}$. By Proposition 3.8(i) and (4.4), the normalized Calabi invariant of $\tilde\psi$ has the value

$$\widehat{\mathrm{Cal}}(\tilde\psi) = \frac{\mathrm{vol}(Y,\lambda)}{\mathrm{vol}(Y,\lambda_0)} - 1 \le 0.$$

By Theorem 2.5, $\tilde\psi$ has a contractible fixed point $z \in \mathrm{int}(\Sigma)$ whose normalized action satisfies

$$a_{\tilde\psi}(z) + \frac{1}{2} a_{\tilde\psi}(z)^2 \le \frac{1}{2}\widehat{\mathrm{Cal}}(\tilde\psi) = \frac{1}{2}\left(\frac{\mathrm{vol}(Y,\lambda)}{\mathrm{vol}(Y,\lambda_0)} - 1\right). \tag{4.5}$$

Moreover, if $\tilde\psi$ is not the identity, then the above inequality is strict.

Assume first that $\tilde\psi$ is not the identity. By Proposition 3.8(ii), the contact manifold (Y,λ) has a closed Reeb orbit of period $1 + a_{\tilde\psi}(z) \in (1-\epsilon, 1+\epsilon)$. By the strict inequality in (4.5), we obtain the desired strict upper bound

$$\begin{aligned}\rho_{k_0}(\lambda) = \frac{\tau_{k_0}(\lambda)^2}{\mathrm{vol}(Y,\lambda)} &\le \frac{(1+a_{\tilde\psi}(z))^2}{\mathrm{vol}(Y,\lambda)} = \frac{1 + 2a_{\tilde\psi}(z) + a_{\tilde\psi}(z)^2}{\mathrm{vol}(Y,\lambda)} \\ &< \frac{1 + \frac{\mathrm{vol}(Y,\lambda)}{\mathrm{vol}(Y,\lambda_0)} - 1}{\mathrm{vol}(Y,\lambda)} = \frac{1}{\mathrm{vol}(Y,\lambda_0)} = \frac{\tau_{k_0}(\lambda_0)^2}{\mathrm{vol}(Y,\lambda_0)} = \rho_{k_0}(\lambda_0).\end{aligned}$$

Assume now that $\tilde\psi$ is the identity. By Proposition 3.8(iii), (Y,λ) is Besse and its Reeb orbits have common period 1. Since the only closed Reeb orbits of (Y,λ) with minimal period less than 1 are $\tilde\gamma_1, ..., \tilde\gamma_h$, we infer that 1 is the minimal common period of the closed Reeb orbits of (Y,λ). Every $\tilde\gamma_i$ is C^2-close to the corresponding γ_i and has minimal period close to the minimal period $1/\alpha_i$ of γ_i. This, together with the Besse property, implies that $\tilde\gamma_i$ and γ_i have the same minimal period (once again, provided λ is sufficiently C^3-close to λ_0). We conclude that the C^2-close Besse flows of λ and λ_0 have the same common period 1 and there is a period preserving bijection between their singular orbits. Thanks to the local rigidity of

Seifert fibrations, we can find a diffeomorphism $\theta : Y \to Y$ such that $\theta^* R_\lambda = R_{\lambda_0}$. Deforming θ by means of a Moser's trick, we can actually ensure that $\theta^*\lambda = \lambda_0$.

Actually, the existence of a diffeomorphism $\theta : Y \to Y$ with the latter property follows also from a theorem of Cristofaro-Gardiner and the third author, stating that the prime action spectrum determines Besse contact forms on closed 3-manifolds. Here, the prime action spectrum $\sigma_{\mathrm{p}}(\lambda)$ is the set of minimal periods of the Reeb orbits of λ, and the above discussion implies in particular that $\sigma_{\mathrm{p}}(\lambda) = \sigma_{\mathrm{p}}(\lambda_0)$. According [CGM20, Theorem 1.5], the equality $\sigma_{\mathrm{p}}(\lambda) = \sigma_{\mathrm{p}}(\lambda_0)$ implies the existence of a diffeomorphism $\theta : Y \to Y$ such that $\theta^*\lambda = \lambda_0$, also without assuming λ to be close to λ_0.

It remains to consider the case in which (Y, λ_0) is Zoll, for which $k_0 = 1$. This case was already treated by Benedetti-Kang [BK21], generalizing the result of the first author together with Bramham-Hryniewicz-Salomão [ABHS18] for the special case $Y = S^3$, but we add some details here for the reader's convenience. The argument provided above in the non-Zoll case goes through in the Zoll case as well, except for the existence of the closed orbit $\tilde\gamma_1$, which now cannot be obtained perturbatively starting from a non-degenerate orbit of R_{λ_0} as in (i) above. Since all orbits of R_{λ_0} have the same minimal period 1, we choose γ_1 to be any one of them. This is the orbit we will apply Proposition 3.8 to. Note that if γ is any other orbit of R_{λ_0}, then we can find a diffeomorphism $\eta_\gamma : Y \to Y$ such that $\eta_\gamma^*\lambda_0 = \lambda_0$ and $\eta_\gamma \circ \gamma = \gamma_1$. Moreover, the set of these diffeomorphisms can be chosen to be pre-compact in the C^k-topology for every $k \in \mathbb{N}$.

We now consider a perturbation λ of λ_0. If $\lambda - \lambda_0$ is C^3-small, then R_λ admits a closed orbit $\tilde\gamma_1$ of period close to 1 which is C^2-close to some orbit γ of R_{λ_0}. This is a consequence of the fact that the space of 1-periodic closed Reeb orbits of the Zoll contact form λ_0 is Morse-Bott non-degenerate (see, e.g., [Wei73], [Bot80] or [Gin87]). Up to replacing λ by $\eta_\gamma^*\lambda$, which is still C^3-close to $\lambda_0 = \eta_\gamma^*\lambda_0$, we may assume that $\tilde\gamma_1$ is C^2-close to γ_1. The rest of the proof continues as in the non Zoll case. □

Remark 4.1. There is a key point in which the proof of Theorem A(ii) above differs from the proofs of the local systolic maximality of Zoll contact forms in [ABHS18] and [BK21]. The proofs from these two papers use the weaker version (2.9) of the fixed point Theorem 2.5, and in this case it is crucial that the boundary of the global surface of section is given by a closed orbit of λ having minimal period. In the Besse case, the same argument would require us to have the boundary of the global surface of section on an orbit γ which realizes $\tau_{k_0}(\lambda)$, where $k_0 = k_0(\lambda_0)$. This orbit might be close to a singular orbit of λ_0, and hence be one of the orbits that are considered in assertion (i) of the above proof, but could also be an orbit of minimal period close to 1 bifurcating from the set of regular orbits of λ_0. In the latter case, finding a global surface of section with boundary on γ and first return map C^1-close to the identity seems problematic: we could apply a diffeomorphism bringing this orbit to a fixed regular orbit of R_{λ_0}, but we cannot hope to have a uniform bound on the C^k norms of this diffeomorphism, because the set of regular orbits of λ_0 is not compact and γ could be very close to some iterate of a singular orbit of λ_0. This issue is overcome by the more precise fixed point Theorem 2.5 which we proved here, whose use does not require the boundary periodic orbit to have any minimality property. □

References

[AB19] A. Abbondandolo and G. Benedetti, *On the local systolic optimality of Zoll contact forms*, arXiv:1912.04187, 2019.

[ABHS18] A. Abbondandolo, B. Bramham, U. L. Hryniewicz, and P. A. S. Salomão, *Sharp systolic inequalities for Reeb flows on the three-sphere*, Invent. Math. **211** (2018), no. 2, 687–778.

[ABHS19] A. Abbondandolo, B. Bramham, U. L. Hryniewicz, and P. A. S. Salomão, *Contact forms with large systolic ratio in dimension three*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19** (2019), 1561–1582.

[AG21] B. Albach and H. Geiges, *Surfaces of section for Seifert fibrations*, Arnold Math. J. **7** (2021), no. 4, 573–597.

[APB14] J. C. Álvarez Paiva and F. Balacheff, *Contact geometry and isosystolic inequalities*, Geom. Funct. Anal. **24** (2014), 648–669.

[Ban78] A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. **53** (1978), 174–227.

[Ban97] A. Banyaga, *The structure of classical diffeomorphism groups*, Kluwer, 1997.

[BK21] G. Benedetti and J. Kang, *A local contact systolic inequality in dimension three*, J. Eur. Math. Soc. (JEMS) **23** (2021), 721–764.

[Bes78] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 93, Springer-Verlag, Berlin-New York, 1978.

[BP94] M. Bialy and L. Polterovich, *Geodesics of Hofer's metric on the group of Hamiltonian diffeomorphisms*, Duke Math. J. **76** (1994), 273–292.

[Bot80] M. Bottkol, *Bifurcation of periodic orbits on manifolds and Hamiltonian systems*, J. Differential Equations **37** (1980), 12–22.

[Cal70] E. Calabi, *On the group of automorphisms of a symplectic manifold*, Problems in analysis (R. Gunning, ed.), Princeton University Press, 1970, pp. 1–26.

[CGM20] D. Cristofaro-Gardiner and M. Mazzucchelli, *The action spectrum characterizes closed contact 3-manifolds all of whose Reeb orbits are closed*, Comment. Math. Helv. **95** (2020), 461–481.

[CK94] C. B. Croke and B. Kleiner, *Conjugacy and rigidity for manifolds with a parallel vector field*, J. Differential Geom. **39** (1994), 659–680.

[Eps72] D. B. A. Epstein, *Periodic flows on 3-manifolds*, Ann. of Math. (2) **95** (1972), 66–82.

[Gei20] H. Geiges, *What does a vector field know about volume?*, J. Fixed Point Theory Appl. **24** (2022), no. 2, Paper No. 23, 26 pp.

[GL18] H. Geiges and C. Lange, *Seifert fibrations of lens spaces*, Abh. Math. Semin. Univ. Hambg. **88** (2018), no. 1, 1–22. Correction: Ibid. **91** (2021), no. 1, 145–150.

[GG81] D. Gromoll and K. Grove, *On metrics on S^2 all of whose geodesics are closed*, Invent. Math. **65** (1981), 175–177.

[Gin87] V. L. Ginzburg, *New generalizations of Poincaré's geometric theorem*, Funktsional. Anal. i Prilozhen. **21** (1987), 16–22, 96.

[Gir20] E. Giroux, *Ideal Liouville domains, a cool gadget*, J. Symplectic Geom. **18** (2020), 769–790.

[JN83] M. Jankins and W. D. Neumann, *Lectures on Seifert manifolds*, Brandeis Lecture Notes, vol. 2, Brandeis University, Waltham, MA, 1983.

[Kat73] A. B. Katok, *Ergodic perturbations of degenerate integrable Hamiltonian systems*, Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 539–576.

[KL21] M. Kegel and C. Lange, *A Boothby-Wang theorem for Besse contact manifolds*, Arnold Math. J. **7** (2021), no. 2, 225–241.

[LM95] F. Lalonde and D. McDuff, *Hofer's L^∞-geometry: energy and stability of Hamiltonian flows. i, ii.*, Invent. Math. **122** (1995), 1–33, 35–69.

[Lan20] C. Lange, *On metrics on 2-orbifolds all of whose geodesics are closed*, J. Reine Angew. Math. **758** (2020), 67–94.

[LS21] C. Lange and T. Soethe, *Sharp systolic inequalities for rotationally symmetric* 2-*orbifolds*, arXiv:2108.13183, 2021.

[LM04] P. Lisca and G. Matić, *Transverse contact structures on Seifert 3-manifolds*, Algebr. Geom. Topol. **4** (2004), 1125–1144.

[MR20] M. Mazzucchelli and M. Radeschi, *On the structure of Besse convex contact spheres*, arXiv:2012.05389, 2020.

[MS98] D. McDuff and D. Salamon, *Introduction to symplectic topology*, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998.

[Orl72] P. Orlik, *Seifert manifolds*, Lecture Notes in Mathematics, Vol. 291, Springer-Verlag, Berlin-New York, 1972.

[RW17] M. Radeschi and B. Wilking, *On the Berger conjecture for manifolds all of whose geodesics are closed*, Invent. Math. **210** (2017), 911–962.

[Sağ21] M. Sağlam, *Contact forms with large systolic ratio in arbitrary dimensions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **22** (2021), 1265–1308.

[Sul78] D. Sullivan, *A foliation of geodesics is characterized by having no "tangent homologies"*, J. Pure Appl. Algebra **13** (1978), 101–104.

[Tau07] C. H. Taubes, *The Seiberg-Witten equations and the Weinstein conjecture*, Geom. Topol. **11** (2007), 2117–2202.

[Wad75] A. W. Wadsley, *Geodesic foliations by circles*, J. Differ. Geom. **10** (1975), 541–549.

[Wei73] A. Weinstein, *Normal modes for Hamiltonian systems*, Invent. Math. **20** (1973), 47–57.

[Zil83] W. Ziller, *Geometry of the Katok examples*, Ergodic Theory Dynam. Systems **3** (1983), 135–157.

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