Rigidity for circle diffeomorphisms with breaks satisfying a Zygmund smoothness condition [1]

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Abstract

Let f and $\tilde{f}$ be two circle diffeomorphisms with a break point, with the same irrational rotation number of bounded type, the same size of the break c and satisfying a certain Zygmund type smoothness condition depending on a parameter $\gamma > 2$. We prove that under a certain condition imposed on the break size c, the diffeomorphisms f and $\tilde{f}$ are $C^{1+\omega_\gamma}$-smoothly conjugate to each other, where $\omega_\gamma(\delta) = |\log \delta|^{-(\gamma/2-1)}$.

1 Introduction

The problem of smoothness of a conjugacy between two circle diffeomorphisms is a classical problem in one-dimensional dynamics. Arnol'd [2] proved that any analytic circle diffeomorphism with a Diophantine rotation number, sufficiently close to the rigid rotation $f_\rho \to x + \rho$ is analytically conjugate to f_ρ. First significant extension of Arnol'd's result was obtained by Herman [4]. He proved that C^∞-smooth circle diffeomorphism with a Diophantine rotation number is C^∞-conjugate to f_ρ. Last forty years Herman's result was developed by Yoccoz [21], Khanin and Sinai [11], Katznelson and Ornstein [5, 6], and Khanin and Teplinsky [14] in virtue of their great discoveries, new ideas, methods, and phenomena. Summarising thus far, if f is $C^{2+\nu}$ and the rotation number satisfies a certain Diophantine condition, then the conjugacy is $C^{1+\alpha}$ for some $0 < \alpha < \nu$. Moreover, in [6], the authors considered a class of circle diffeomorphismsm bigger than $C^{2+\nu}$. They proved that if Df absolutely continuous and $D \log Df \in L_p$, for some $p > 1$ then the conjugacy is is absolutely continuous provided its rotation number is bounded type. One of the last results on the progression of the regularity of conjugacy of circle diffeomorphisms have been contributed by Akhadkulov *et al* [1] by extending previous results for circle diffeomorphisms satisfying a certain Zygmund-type smoothness condition depending on a parameter $\gamma > 0$. It was shown that, if a circle diffeomorphism satisfies the Zygmund condition for $\gamma > 1/2$ then there exists a subset of irrational numbers of unbounded type such that the conjugacy is absolutely continuous provided its rotation number belongs to the above set. Moreover, if $\gamma > 1$ then the conjugacy is C^1-smooth for almost all irrational rotation numbers. It is important to remark that, in the case of diffeomorphisms, rigidity is guaranteed only when the rotation numbers satisfy a certain Diophantine condition. Recently, Khanin and Teplinsky [12] showed that in the presence of *critical points* or *break points* points the rigidity may be stronger, i.e., valid for a "large" set of rotation numbers. They have showed that for the diffeomorphisms of a circle with a single critical point, the *robust rigidity* holds, that is, the rigidity holds without any Diophantine conditions. The

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robust rigidity result depends on exponential convergence of renormalizations so called *renormalization* problem. The renormalization problem was proved by de Faria and de Melo for C^∞-smooth critical circle maps with irrational rotation numbers of bounded type [16, 17], and extended, in the analytic setting, by Yampolsky [18] to cover all irrational rotation numbers. Recently, a remarkable rigidity results also have been obtained by Guarino and de Melo [19] and Guarino *et al* [20] in the case of lower smoothness of critical circle maps. In [19], it was proven a $C^{1+\alpha}$ (for a universal $\alpha > 0$) rigidity result for any two C^3 critical circle maps with the same irrational rotation number of bounded type and the same odd criticality. In the case of the class is C^4, C^1-rigidity holds for any irrational rotation number and $C^{1+\alpha}$- rigidity holds for a full Lebesgue measure set of rotation numbers as shown in [20].

In the case of a break type singularity, the first rigidity results for $C^{2+\alpha}$ circle diffeomorphisms were obtained by Khanin and Khmelev [7], and Khanin and Teplinsky [13]. In [7], rigidity theorem was proved for irrational rotation numbers with periodic *partial quotients* and in [13], for *half bounded* (see the definition below) irrational rotation numbers. Note that the robust rigidity does not hold for circle diffeomorphisms with breaks. Indeed, as shown in [8], there are irrational rotation numbers, and pairs of analytic circle diffeomorphisms with breaks, with the same rotation number and the same size of the break, for which any conjugacy between them is not even Lipschitz continuous. The most remarkable results in this direction were obtained by Khanin and Kocìč [9] and Khanin *et al* [10]. In [9], it was shown that the renormalizations of any two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break point, with the same irrational rotation number and the same size of the break, approach each other exponentially fast in the C^2-topology. This result implies that for almost all irrational numbers, any two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break, with the same rotation number and the same size of the break, are C^1-smoothly conjugate to each other as shown in [10]. The interesting problems of circle maps are the rigidity and renormalizations problems on the less regularities, for instance these problems are open for $C^{2+\alpha}$-smooth critical circle maps and for circle diffeomorphisms with break points satisfying a Zygmund condition, even for bounded combinatorics. The renormalizations problem for circle diffeomorphisms with a break satisfying a certain Zygmund condition is partially solved in [3].

In this paper we study the rigidity problem of two circle diffeomorphisms f and $\tilde{f}$ with a break point, with the same irrational rotation number of bounded type, the same size of the break c and satisfying a certain Zygmund type smoothness condition depending on a parameter $\gamma > 2$. We prove that under a certain condition imposed on the break size c, the diffeomorphisms f and $\tilde{f}$ are $C^{1+\omega_\gamma}$-smoothly conjugate to each other, where $\omega_\gamma(\delta) = |\log \delta|^{-(\gamma/2-1)}$. The rest of this paper is organized as follows. In Section 2, the main notions and statement of main theorem are given. In Section 3, we show the existence of a solution of a cohomological equation for the break-equivalent diffeomorphisms. In Section 4, some universal estimates for the ratio of the lengths of the segments of dynamical partition are obtained. Sections 5 and 6 are devoted to study the renormalizations and closeness of rescaled points. Finally, in Section 7, the proof of main theorem is given.

2 General settings and statement of main Theorem

2.1 Dynamical partition

In this section, first we present some of the basic notations of circle maps and then we estimate the ratio of lengths of elements of the dynamical partition. Denote by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ unit circle. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a circle homeomorphism we denote its rotation number by $\rho(f)$. It can be expressed as a continued fraction

$$\rho(f) = 1/(k_1 + 1/(k_2 + ...)) := [k_1, k_2, ..., k_n, ...).$$

The sequence of positive integers (k_n) with $n \geq 1$ called *partial quotients* and it is infinite if and only if $\rho(f)$ is irrational. We call $\rho := \rho(f)$ is bounded type if $s(\rho) := \sup k_n < \infty$. Let $p_n/q_n = [k_1, k_2, ..., k_n]$ be the sequence of rational convergents of ρ. The coprime numbers p_n and q_n satisfy the recurrence relations

$$p_n = k_n p_{n-1} + p_{n-2}, \text{ and } \ q_n = k_n q_{n-1} + q_{n-2}$$

for $n \geq 1$, where$p_0 = 0$, $q_0 = 1$ and $p_{-1} = 1$, $q_{-1} = 0$. Let $\xi_0 \in \mathbb{S}^1$. Define nth *fundamental segment* $\Delta_0^{(n)} := \Delta_0^{(n)}(\xi_0)$ as the circle arc $[\xi_0, f^{q_n}(\xi_0)]$ if n is even and $[f^{q_n}(\xi_0), \xi_0]$ if n is odd. We shall also use the notations $\widehat{\Delta}_0^{(n-1)} = \Delta_0^{(n)} \cup \Delta_0^{(n-1)}$ and $\check{\Delta}_0^{(n-1)} = \Delta_0^{(n-1)} \setminus \Delta_0^{(n+1)}$. Certain number of images of fundamental segments $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, under the iterates of f, cover whole circle without overlapping beyond the endpoints and form nth *dynamical partition* of the circle $\mathbb{S}^1$

$$\mathcal{P}_n := \mathcal{P}_n(\xi_0, f) = \left\{\Delta_j^{(n)} := f^j(\Delta_0^{(n)}), 0 \leq j < q_{n-1}\right\} \bigcup \left\{\Delta_i^{(n-1)} := f^i(\Delta_0^{(n-1)}), 0 \leq i < q_n\right\}.$$

The partition $\mathcal{P}_{n+1}$ is a refinement of the partition $\mathcal{P}_n$. Indeed, the segments of order n belong to $\mathcal{P}_{n+1}$ and each segment $\Delta_i^{(n-1)}$, $0 \leq i < q_n$ is partitioned into $k_{n+1} + 1$ segments belonging to $\mathcal{P}_n$ such that

$$\Delta_i^{(n-1)} = \Delta_i^{(n+1)} \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}. \tag{1}$$

One can easily see that the endpoints of the segments from $\mathcal{P}_n$ form the set

$$\Xi_n = \{\xi_i := f^i(\xi_0),\ 0 \leq i < q_n + q_{n-1}\}.$$

We shall also use the extended set $\Xi_n^* = \Xi_n \cup \{\xi_{q_n+q_{n-1}}\}$. Now we formulate a lemma which will be used in the sequel.

Lemma 2.1. *For every $m > n$, we have the following decomposition*

$$\Xi_m \cap \check{\Delta}_0^{(n-1)} = \bigcup_{\xi_l \in \Xi_m \cap \Delta_0^{(n)} \setminus \{\xi_{q_n}\}} \ \bigcup_{s=0}^{k_{n+1}-1} \xi_{l+sq_n+q_{n-1}}. \tag{2}$$

Furthermore, for every $\xi_l \in \Xi_m \cap \Delta_0^{(n)} \setminus \{\xi_{q_n}\}$ we have $\xi_{l+k_{n+1}q_n+q_{n-1}} = \xi_{l+q_{n+1}} \in \Xi_m^* \cap \widehat{\Delta}_0^{(n)}$.

Proof. The proof of the lemma follows directly from the properties of dynamical partition. ∎

2.2 Circle diffeomorphisms with a break and Zygmund class

We recall the following definition.

Definition 2.2. $f : \mathbb{S}^1 \to \mathbb{S}^1$ is called a circle diffeomorphism with a single break point ξ_0 if the following conditions are satisfied:

(i) $f \in C^1([\xi_0, \xi_0 + 1])$;

(ii) $\inf_{\xi \neq \xi_0} Df(\xi) > 0$;

(iii) f has one-sided derivatives $Df(\xi_0 \pm 0) > 0$ and

$$c := c_f(\xi_0) = \sqrt{\frac{Df(\xi_0 - 0)}{Df(\xi_0 + 0)}} \neq 1.$$

The number c is called the *size of break* of f at ξ_0. Circle diffeomorphisms with a break were first studied by Khanin & Vul in [15]. It was proven that the renormalizations circle diffeomorphisms with a break approximate fractional linear transformations. Next we define a class of circle diffeomorphisms with breaks satisfying a Zygmund condition. Consider the function $\mathcal{Z}_\gamma : [0, 1) \to [0, +\infty)$ defined as

$$\mathcal{Z}_\gamma(x) = |\log x|^{-\gamma}, \quad \text{for } x \in (0, 1)$$

and $\mathcal{Z}_\gamma(0) = 0$, where $\gamma > 0$. Let f be a circle diffeomorphism with the break point ξ_0. Denote by $\nabla^2 f(\xi, \tau)$ the *second symmetric difference* of Df, that is

$$\nabla^2 f(\xi, \tau) = Df(\xi + \tau) + Df(\xi - \tau) - 2Df(\xi)$$

where $\xi \in \mathbb{S}^1 \setminus \{\xi_0\}$ and $\tau \in [0, \frac{1}{2}]$. Suppose that there exists a constant $C > 0$ such that

$$\|\nabla^2 f(\cdot, \tau)\|_{L^\infty(\mathbb{S}^1)} \leq C\tau \mathcal{Z}_\gamma(\tau). \tag{3}$$

In this work we study the class of circle diffeomerphisms f with break point ξ_0, whose derivatives Df have bounded variation and satisfy the inequality (3). We denote this class by $\mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$.

Remark 2.3. Note that the class $\mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$ is bigger than $C^{2+\epsilon}(\mathbb{S}^1 \setminus \{\xi_0\})$ for any positive γ and ϵ.

2.3 Statement of the main theorem

In this section we formulate our main theorem. For this, let us first define some necessary facts. Let $m \in \mathbb{N}$. Define

$$\mathfrak{D}_m^{(1)} = \{c \in \mathbb{R}_+ \setminus \{1\} : \ c^{4m} - c^2 < 1\}; \quad \mathfrak{D}_m^{(2)} = \{c \in \mathbb{R}_+ \setminus \{1\} : \ c^{4m+2} + c^{4m} > 1\}.$$

The following is our main theorem.

Theorem 2.4. *Let $\gamma > 2$ and $m \in \mathbb{N}$. Let f and $\tilde{f}$ be two circle diffeomorphisms with a break satisfying the following conditions:*

(a) $f, \tilde{f} \in \mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$;

(b) *f and $\tilde{f}$ have the same irrational rotation number ρ of bounded type such that $s(\rho) = m$;*

(c) *f and $\tilde{f}$ have the same size of the break $c \in \mathbb{R}_+ \setminus \{1\}$;*

(d) *$c \in \mathfrak{D}_m^{(1)}$ in case of $c > 1$ or $c \in \mathfrak{D}_m^{(2)}$ in case of $0 < c < 1$.*

Then there exists a C^1-smooth circle diffeomorphism h and a constant $A > 0$ such that $h \circ f = \tilde{f} \circ h$ and

$$|Dh(x) - Dh(y)| \leq A\omega_\gamma(|x - y|)$$

for any $x, y \in \mathbb{S}^1$ such that $x \neq y$.

Remark 2.5. The reason for the restriction c in condition (d) is purely technical. It enables us to get an algebraic estimate for the ratio of lengths of segments $\Delta^{n+\ell}$ and Δ^n satisfying $\Delta^{n+\ell} \subset \Delta^n$ of the dynamical partition $\mathcal{P}_n$ while ℓ has a form of the logarithm of n. We do not know if the statement of Theorem 2.4 holds when the restriction is removed.

3 Cohomological equation for the break-equivalent diffeomorphisms

In this section we show the existence of a solution of a cohomological equation for the break-equivalent diffeomorphisms. We begin from the following definition.

Definition 3.1. We say that two circle diffeomorphisms f and $\tilde{f}$ with a break ξ_0 are break-equivalents if there exists a topological conjugacy h such that $h(\xi_0) = \xi_0$ and $c_f(\xi_0) = c_{\tilde{f}}(h((\xi_0)))$.

Consider two break-equivalent circle diffeomorphisms f and $\tilde{f}$ with irrational rotation number. Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be the conjugacy between f and $\tilde{f}$, that is,

$$h \circ f = \tilde{f} \circ h. \tag{4}$$

The *cohomological equation* associated to (4) is

$$\zeta \circ f - \zeta = \log D\tilde{f} \circ h - \log Df \tag{5}$$

where $\zeta : \mathbb{S}^1 \to \mathbb{R}$ is called the solution of (5) if it exists. Note that here $D\tilde{f}(h(x))$ means the derivative of $\tilde{f}$ at $h(x)$. Define

$$\Lambda_n(x) = \log Df^{q_n}(x) - \log D\tilde{f}^{q_n}(h(x)), \quad x \in \widehat{\Delta}_0^{(n-1)}.$$

Since f and $\tilde{f}$ are break-equivalents one-side limits of Λ_n at the break point ξ_0 are equal that is, $\Lambda_n(\xi_0 - 0) = \Lambda_n(\xi_0 + 0)$. Therefore Λ_n is continuous on $\widehat{\Delta}_0^{(n-1)}$ and it can be decomposed as

$$\Lambda_n(x) = \sum_{s=0}^{q_n-1} \log Df(f^s(x)) - \log D\tilde{f}(h \circ f^s(x)), \qquad x \in \widehat{\Delta}_0^{(n-1)}.$$

Denote $\Lambda_n = \max_{x \in \widehat{\Delta}_0^{(n-1)}} |\Lambda_n(x)|$. The following theorem will be used in the proof of main theorem.

Theorem 3.2. *Let f and $\tilde{f}$ be two break-equivalent circle diffeomorphisms with a break and with identical irrational rotation number $\rho = [k_1, k_2, ..., k_n, ...]$. If*

$$\sum_{n=0}^{\infty} k_{n+1}\Lambda_n < \infty$$

then the cohomological equation (5) has a continuous solution.

Proof. Let $i_n : \mathbb{S}^1 \to \mathbb{N}_0$ be the first entrance time of x in $\widehat{\Delta}_0^{(n-1)}$; that is,

$$i_n(x) = \min\{i \geq 0 : \ f^i(x) \in \widehat{\Delta}_0^{(n-1)}\}.$$

Define $\zeta_n : \mathbb{S}^1 \to \mathbb{R}$ as follows

$$\zeta_n(x) = \sum_{s=0}^{i_n(x)-1} \log Df(f^s(x)) - \log D\tilde{f}(h \circ f^s(x)).$$

Next we show that ζ_n is a Cauchy. For this, first we estimate $\|\zeta_{n+1} - \zeta_n\|_\infty$. To estimate this we distinguish the following three cases:

Case I. Suppose $x \in \mathbb{S}^1 \setminus \Xi_{n+1}$. By the definition of i_n we have

$$i_n(x) = \begin{cases} 0, & \text{if } \ x \in \widehat{\Delta}_0^{(n-1)} \\ q_{n-1} - j, & \text{if } \ x \in \Delta_j^{(n)} \\ q_n - i, & \text{if } \ x \in \Delta_i^{(n-1)} \end{cases}$$

where $0 < j < q_{n-1}$ and $0 < i < q_n$. Using the properties of dynamical partition we can show that

$$i_{n+1}(x) - i_n(x) = \begin{cases} 0, & \text{if } \ x \in \widehat{\Delta}_0^{(n)} \cup \Delta_i^{(n+1)} \\ k_{n+1}q_n, & \text{if } \ x \in \Delta_j^{(n)} \\ (k_{n+1} - \ell - 1)q_n, & \text{if } \ x \in \Delta_{i+q_{n-1}+\ell q_n}^{(n)} \end{cases}$$

where $0 < j < q_{n-1}$, $0 < i < q_n$ and $0 \leq \ell < k_{n+1}$. Therefore $|\zeta_{n+1}(x) - \zeta_n(x)| = 0$ if $x \in \widehat{\Delta}_0^{(n)} \cup \Delta_i^{(n+1)}$, $0 < i < q_n$ and

$$\begin{aligned} |\zeta_{n+1}(x) - \zeta_n(x)| &= \Big| \sum_{s=i_n(x)}^{i_{n+1}(x)-1} \log Df(f^s(x)) - \log D\tilde{f}(h \circ f^s(x)) \Big| \\ &= \Big| \sum_{s=0}^{i_{n+1}(x)-i_n(x)-1} \log Df(f^s(x_{i_n}) - \log D\tilde{f}(h \circ f^s(x_{i_n})) \Big| \\ &\leq \Big| \sum_{s=0}^{q_n-1} \log Df(f^s(x_{i_n}) - \log D\tilde{f}(h \circ f^s(x_{i_n})) \Big| \\ &+ \Big| \sum_{s=q_n}^{2q_n-1} \log Df(f^s(x_{i_n}) - \log D\tilde{f}(h \circ f^s(x_{i_n})) \Big| \\ &\vdots \\ &+ \Big| \sum_{s=i_{n+1}(x)-i_n(x)-q_n}^{i_{n+1}(x)-i_n(x)-1} \log Df(f^s(x_{i_n}) - \log D\tilde{f}(h \circ f^s(x_{i_n})) \Big| \end{aligned} \tag{6}$$

if $x \in \Delta_j^{(n)}$, $0 < j < q_{n+1}$ where $x_{i_n} = f^{i_n(x)}(x)$. Clearly f^{i_n} maps $\mathbb{S}^1$ into $\widehat{\Delta}_0^{(n-1)}$ and the points $x_{i_n}, f^{q_n}(x_{i_n}), ..., f^{i_{n+1}(x)-i_n(x)-q_n}(x_{i_n})$ lie in the interval $\widehat{\Delta}_0^{(n-1)}$. Therefore the right hand side of (6) can be estimated as follows

$$|\zeta_{n+1}(x) - \zeta_n(x)| \leq \sum_{s=0}^{i_{n+1}(x)-i_n(x)-q_n} \left|\Lambda_n\Big(f^{sq_n}(x_{i_n})\Big)\right| \leq k_{n+1}\Lambda_n.$$

Hence

$$\|\zeta_{n+1} - \zeta_n\|_\infty \leq k_{n+1}\Lambda_n. \tag{7}$$

Case II. Suppose $x = \xi_i \in \Xi_n$. For $i = 0$, it is clear that $|\zeta_{n+1}(\xi_0) - \zeta_n(\xi_0)| = 0$. For $i \geq 1$, one can easily see

$$i_n(\xi_i) = \begin{cases} q_{n-1} - i, & \text{if } 1 \leq i \leq q_{n-1} \\ q_n - i, & \text{if } q_{n-1} < i \leq q_n \\ q_n + q_{n-1} - i, & \text{if } q_n < i < q_n + q_{n-1}. \end{cases}$$

Consequently, we get

$$i_{n+1}(\xi_i) = \begin{cases} q_n - i, & \text{if } 1 \leq i \leq q_n \\ q_{n+1} - i, & \text{if } q_n < i < q_n + q_{n-1}. \end{cases}$$

Therefore

$$i_{n+1}(\xi_i) - i_n(\xi_i) = \begin{cases} q_n - q_{n-1}, & \text{if } 1 \leq i \leq q_{n-1} \\ 0, & \text{if } q_{n-1} < i \leq q_n \\ (k_{n+1} - 1)q_n, & \text{if } q_n < i < q_n + q_{n-1}. \end{cases}$$

This and by the definition of ζ_n we have $|\zeta_{n+1}(\xi_i) - \zeta_n(\xi_i)| = 0$ if $q_{n-1} < i \leq q_n$, and

$$|\zeta_{n+1}(\xi_i) - \zeta_n(\xi_i)| = |\Lambda_n(\xi_0) - \Lambda_{n-1}(\xi_0)| \leq \Lambda_n + \Lambda_{n-1} \tag{8}$$

if $1 \leq i \leq q_{n-1}$ and

$$|\zeta_{n+1}(\xi_i) - \zeta_n(\xi_i)| = \Big|\sum_{s=1}^{k_{n+1}-1} \Lambda_n(\xi_{sq_n+q_{n-1}})\Big| \leq k_{n+1}\Lambda_n. \tag{9}$$

if $q_n < i < q_n + q_{n-1}$.

Case III. Suppose $x = \xi_i \in \Xi_{n+1} \setminus \Xi_n$. In this case we consider the following sub-cases:

a) $i \in L_n := \{\ell q_n + q_{n-1},\ 1 \leq \ell < k_{n+1}\}$, *b)* $i \in (q_n + q_{n-1}, q_{n+1}) \setminus L_n$,

c) $i = q_{n+1}$, *d)* $i \in (q_{n+1}, q_{n+1} + q_n)$.

It is easy to check that $i_n(\xi_i) = 0$ and $i_{n+1}(\xi_i) = (k_{n+1} - \ell)q_n$ in the sub-case of *a*). Thus one gets

$$|\zeta_{n+1}(\xi_i) - \zeta_n(\xi_i)| = \Big|\sum_{s=\ell}^{k_{n+1}-1} \Lambda_n(\xi_{sq_n+q_{n-1}})\Big| \leq k_{n+1}\Lambda_n. \tag{10}$$

Consider the sub-case b). It is clear that i can be written as $i = \ell_1 q_n + q_{n-1} + i_1$ for some $1 \leq \ell_1 < k_{n+1}$ and $1 \leq i_1 < q_n$. By the definition of i_n we have $i_n(\xi_i) = q_n - i_1$ and $i_{n+1}(\xi_i) = q_{n+1} - i = (k_{n+1} - \ell_1)q_n - i_1$. It implies

$$|\zeta_{n+1}(\xi_i) - \zeta_n(\xi_i)| = \Big| \sum_{s=\ell_1}^{k_{n+1}-1} \Lambda_n(\xi_{sq_n+q_{n-1}}) \Big| \leq k_{n+1}\Lambda_n. \tag{11}$$

The sub-case c) is clear because of both functions Λ_n and Λ_{n+1} are zero at ξ_i. Finally, consider the sub-case d). In this case i can be written as $i = q_{n+1} + i_1$ for some $1 \leq i_1 < q_n$. One can easily see $i_n(\xi_i) = q_n - i_1$ and $i_{n+1}(\xi_i) = q_{n+1} + q_n - i = q_{n+1} + q_n - (q_{n+1} + i_1) = q_n - i_1$ which implies

$$|\zeta_{n+1}(\xi_i) - \zeta_n(\xi_i)| = 0. \tag{12}$$

Combining the inequalities (7)-(12) we obtain, finally,

$$\|\zeta_{n+1} - \zeta_n\|_\infty \leq k_n\Lambda_{n-1} + k_{n+1}\Lambda_n. \tag{13}$$

From this it follows that

$$\|\zeta_{n+p} - \zeta_n\|_\infty \leq 2\sum_{m=n}^{n+p} k_m\Lambda_{m-1}. \tag{14}$$

Thus ζ_n is a Cauchy. Let $\zeta(x) = \lim_{n\to\infty} \zeta_n(x)$. Next we show that the function $\zeta : \mathbb{S}^1 \to \mathbb{R}$ is continuous and satisfies the cohomological equation (5). First we show that ζ satisfies (5). It is easy to see that for any $x \in \mathbb{S}^1 \setminus \{\xi_0\}$ there exists $n_0 := n_0(x)$ such that $i_n(f(x)) = i_n(x) - 1$ for all $n \geq n_0$. This and by the definition of ζ_n we get

$$\zeta_n \circ f - \zeta_n = \log D\tilde{f} \circ h - \log Df$$

for all $n \geq n_0$. Taking the limit as $n \to \infty$ we get (5). Let $x = \xi_0$. It is easy to see that $\zeta_n(\xi_0) = 0$ and

$$\begin{aligned}
\zeta_n(f(\xi_0)) &= \sum_{s=0}^{i_n(f(\xi_0))-1} \log Df(f^{s+1}(\xi_0)) - \log D\tilde{f}(h \circ f^{s+1}(\xi_0)) \\
&= \sum_{s=0}^{q_{n-1}-2} \log Df(f^{s+1}(\xi_0)) - \log D\tilde{f}(h \circ f^{s+1}(\xi_0)) \\
&= \Lambda_{n-1}(\xi_0) + \log D\tilde{f}(h(\xi_0)) - \log Df(\xi_0).
\end{aligned} \tag{15}$$

Taking the limit as $n \to \infty$ we again get (5). Next we show that ζ is continuous at $x = \xi_0$. Since $\zeta_n(\xi_0) = 0$ for all $n \geq 1$ we have $\zeta(\xi_0) = 0$. Take any $z \in \widehat{\Delta}_0^{(n-1)}$. It is obvious that $i_j(z) = 0$ for every $j \leq n$, so $\zeta_j(z) = 0$ for every $j \leq n$. In particular

$$\zeta_{n+p}(z) = \sum_{m=0}^{p-1} \zeta_{n+m+1}(z) - \zeta_{n+m}(z).$$

This and relation (13) imply

$$|\zeta_{n+p}(z)| \leq 2\sum_{m=n}^{n+p} k_m\Lambda_{m-1}.$$

Consequently

$$\lim_{n\to\infty} \sup_{z\in\widehat{\Delta}_0^{(n-1)}} |\zeta(z)| = 0.$$

Hence ζ is continuous at $x = \xi_0$. Denote by $\Xi = \{\xi_i := f^i(\xi_0),\, i \in \mathbb{N}\}$ the positive trajectory of ξ_0. Since ζ is continuous at $x = \xi_0$ and $\log D\tilde{f} \circ h - \log Df$ is continuous on $\mathbb{S}^1$, by

$$\zeta \circ f - \zeta = \log D\tilde{f} \circ h - \log Df$$

it implies that ζ is continuous on Ξ. Note that $i_n : \mathbb{S}^1 \to \mathbb{R}$ is continuous in the interior of each element of the partition $\mathcal{P}_n$ for every $n \geq 1$. As a consequence ζ_n is continuous in the interior of each element of the partition $\mathcal{P}_n$ for every $n \geq 1$. Thus the limit function ζ is continuous on $x \in \mathbb{S}^1 \setminus \Xi$. □

Remark 3.3. It is important to remark that Theorem 3.2 holds true for any two break-equivalent circle diffeomorphisms with any countable number of break points.

4 Renormalizations of circle diffeomorphisms with a break

In this section we will discuss on convergence of renormalizations of two circle diffeomorphisms with a break. Let us recall first the definition of renormalization of circle maps. The segment $\widehat{\Delta}_0^{(n-1)}$ is called the n^{th} *renormalization neighborhood* of ξ_0. On $\widehat{\Delta}_0^{(n-1)}$ we define the Poincaré map $\pi_n = (f^{q_n}, f^{q_{n-1}}) : \widehat{\Delta}_0^{(n-1)} \to \widehat{\Delta}_0^{(n-1)}$ as follows

$$\pi_n(\xi) = \begin{cases} f^{q_n}(\xi), & \text{if} \quad \xi \in \Delta_0^{(n-1)}, \\ f^{q_{n-1}}(\xi), & \text{if} \quad \xi \in \Delta_0^{(n)}. \end{cases}$$

Next we define the renormalization of f as follows. Let $\mathcal{A}_n : \mathbb{R} \to \mathbb{S}^1$ be an affine covering map such that $\mathcal{A}_n([-1, 0]) = \Delta_0^{(n-1)}$, with $\mathcal{A}_n(0) = \xi_0$ and $\mathcal{A}_n(-1) = f^{q_{n-1}}(\xi_0)$. We define $a_n \in \mathbb{R}$ to be a positive number such that $\mathcal{A}_n(a_n) = f^{q_n}(\xi_0)$. It is obvious that $\mathcal{A}_n : [0, a_n] \to \Delta_0^{(n)}$ and $\mathcal{A}_n : [-1, 0] \to \Delta_0^{(n-1)}$. A pair of functions $(f_n, g_n) : [-1, a_n] \to [-1, a_n]$ defined by $(f_n, g_n) = \mathcal{A}_n^{-1} \circ \pi_n \circ \mathcal{A}_n$, is called the n^{th} *renormalization* of f, where $\mathcal{A}_n^{-1}$ is the inverse branch that maps $\widehat{\Delta}_0^{(n-1)}$ onto $[-1, a_n]$. Define the following Möbius transformation

$$F_n := F_{a_n, v_n, c_n} : z \to \frac{a_n + c_n z}{1 - v_n z}$$

where $c_n = c$ if n is even, $c_n = c^{-1}$ if n is odd, and

$$a_n = \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}, \qquad v_n = \frac{c_n - a_n - b_n}{b_n}, \qquad b_n = \frac{|\Delta_0^{(n-1)}| - |\Delta_{q_{n-1}}^{(n)}|}{|\Delta_0^{(n-1)}|}.$$

The following theorem has been proved in [3].

Theorem 4.1. *Let $f \in \mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$ and $\gamma > 1$. Suppose the rotation number of f is irrational. There exists a constant $C = C(f) > 0$ and a natural number $N_0 = N_0(f)$ such that*

$$\|f_n - F_n\|_{C^1([-1,0])} \leq \frac{C}{n^\gamma}, \qquad \|D^2 f_n - D^2 F_n\|_{C^0([-1,0])} \leq \frac{C}{n^{\gamma-1}}$$

for all $n \geq N_0$.

The following lemma will be used in the subsequent sections.

Lemma 4.2. *Let $f \in \mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$ and $\gamma > 1$. Suppose the rotation number of f is irrational. There exists a constant $Q = Q(f) > 0$ such that*

$$\|f_n\|_{C^2([-1,0])} \leq Q.$$

Proof. The proof of the lemma implies from Theorem 4.1 and Proposition 7.1 stated in [3]. ◻

Half-bounded rotation numbers. The half-bounded rotation numbers were defined by Khanin and Teplinsky in [13] as follows. Denote by M_o and M_e the class of all irrational rotation numbers $\rho = [k_1, k_2, ...)$, such that

$$M_o = \{\rho : (\exists C > 0)\, (\forall m \in \mathbb{N})\ k_{2m-1} \leq C\}, \quad M_e = \{\rho : (\exists C > 0)\, (\forall m \in \mathbb{N})\ k_{2m} \leq C\}.$$

Let us formulate the following theorem borrowed from [13].

Theorem 4.3. *Let f and $\tilde{f}$ be two $C^{2+\nu}$-smooth circle diffeomorphisms with breaks of the same size c and the same rotation number $\rho \in M_e$ in case of $c > 1$, or $\rho \in M_o$ in case of $0 < c < 1$. There exist constants $C = C(f, \tilde{f}) > 0$ and $\mu \in (0, 1)$ such that*

$$\|f_n - \tilde{f}_n\|_{C^2([-1,0])} \leq C\mu^n.$$

This theorem was extended by Khanin and Kocić [10] for all irrational rotation numbers and for the class of $\mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$ by Akhadkulov *et al* [3]. More precisely, in [3], it was proven the following

Theorem 4.4. *Let $f, \tilde{f} \in \mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$ and $\gamma > 1$. Assume that f and $\tilde{f}$ have the same break size c and the same rotation number $\rho \in M_e$ in the case of $c > 1$, or $\rho \in M_o$ in the case of $0 < c < 1$. There exists a constant $C = C(f, \tilde{f}) > 0$ and a natural number $N_0 = N_0(f, \tilde{f})$ such that*

$$\|f_n - \tilde{f}_n\|_{C^1([-1,0])} \leq \frac{C}{n^\gamma}, \qquad \|D^2 f_n - D^2 \tilde{f}_n\|_{C^0([-1,0])} \leq \frac{C}{n^{\gamma-1}}$$

for all $n \geq N_0$.

An estimate of Df_n. The following set plays an important role in the investigations of renormalizations of comuting pairs of Möbius transformations (see [13]).

$$\Phi_c^\varepsilon = \{(a, v) : \varepsilon < a < c - \varepsilon,\, \varepsilon < \frac{v}{c-1} < 1 - \varepsilon,\, v + a - c + 1 > \varepsilon\},\ \varepsilon > 0.$$

Lemma 4.5. *Let $f \in \mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$, $\gamma > 1$ be a circle diffeomorphism with irrational rotation ρ and the break size c. Assume that $\rho \in M_e$ if $c > 1$ or $\rho \in M_o$ if $0 < c < 1$. There exists a constant $\varepsilon = \varepsilon(f) > 0$ and a natural number $N_0 = N_0(f)$ such that the projection (a_n, v_n) of the renormalization (f_n, g_n) belongs to $\Phi_{c_n}^\varepsilon$ for all $n \geq N_0$.*

Proof. The proof follows from Proposition 7.1 in [3]. ◻

Lemma 4.6. *Let $f \in \mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$, $\gamma > 1$ be a circle diffeomorphism with irrational rotation ρ and the break size c. Assume that $\rho \in M_e$ if $c > 1$ or $\rho \in M_o$ if $0 < c < 1$. There exists a constant $\varepsilon = \varepsilon(f) > 0$ and a natural number $N_0 = N_0(f)$ such that, for all $n \geq N_0$, we have*

$$\frac{c_n}{(c_n + \varepsilon(1-c_n))^2} - \frac{C}{n^\gamma} \leq Df_n(z) \leq c_n^2 - \varepsilon(c_n^2 - 1 - \varepsilon(c_n - 1)) + \frac{C}{n^\gamma}$$

if $c_n > 1$ and

$$c_n^2 - \varepsilon(c_n^2 - 1 - \varepsilon(c_n - 1)) - \frac{C}{n^\gamma} \leq Df_n(z) \leq \frac{c_n}{(c_n + (1-c_n)\varepsilon)^2} + \frac{C}{n^\gamma}$$

if $c_n < 1$.

Proof. It is easy to see that $DF_n(z) = (c_n + a_n v_n)(1 - v_n z)^{-2}$. Let $c_n > 1$. Lemma 4.5 implies $(c_n - 1)\varepsilon < v_n < (c_n - 1)(1-\varepsilon)$ and hence $1 + (c_n - 1)\varepsilon < 1 + v_n < c_n - \varepsilon(c_n - 1)$. Using these inequalities we get

$$DF_n(z) \leq c_n + a_n v_n < c_n + (c_n - \varepsilon)(c_n - 1)(1-\varepsilon) = c_n^2 - \varepsilon(c_n^2 - 1 - \varepsilon(c_n - 1)) \tag{16}$$

and

$$DF_n(z) \geq \frac{c_n + a_n v_n}{(1+v_n)^2} > \frac{c_n + \varepsilon^2(c_n - 1)}{(c_n + \varepsilon(1-c_n))^2} > \frac{c_n}{(c_n - \varepsilon(c_n - 1))^2}. \tag{17}$$

Assume $c_n < 1$. By Lemma 4.5 we have $(c_n - 1)(1-\varepsilon) < v_n < (c_n - 1)\varepsilon$, which implies that $c_n + (1-c_n)\varepsilon < 1 + v_n < 1 + (c_n - 1)\varepsilon$ and $(1 - v_n z)^2 > (1+v_n)^2$. Hence we have

$$DF_n(z) \leq \frac{c_n + a_n v_n}{(1+v_n)^2} < \frac{c_n - (1-c_n)\varepsilon^2}{(c_n + (1-c_n)\varepsilon)^2} < \frac{c_n}{(c_n + (1-c_n)\varepsilon)^2} \tag{18}$$

and

$$DF_n(z) \geq c_n + a_n v_n > c_n + (c_n - \varepsilon)(c_n - 1)(1-\varepsilon) = c_n^2 - \varepsilon(c_n^2 - 1 - \varepsilon(c_n - 1)). \tag{19}$$

The proof of the lemma now follows from (16)-(19) and Theorem 4.1. $\square$

Denote $\mathfrak{c} = \max\{c, c^{-1}\}$. It follows from Lemma 4.6 the following

Corollary 4.7. *Let $f \in \mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \setminus \{\xi_0\})$, $\gamma > 1$ be a circle diffeomorphism with irrational rotation ρ and the break size c. Assume that $\rho \in M_e$ if $c > 1$ or $\rho \in M_o$ if $0 < c < 1$. There exists a natural number $N_0 = N_0(f)$ such that*

$$\frac{1}{\mathfrak{c}^2} - \frac{C}{n^\gamma} \leq Df_n(z) \leq \mathfrak{c}^2 + \frac{C}{n^\gamma}$$

for all $n \geq N_0$.

5 Universal estimates for the segments of $\mathcal{P}_n$

In this section we estimate the ratio of lengths of segments of dynamical partition of circle diffeomorphisms satisfying in the setting of rotation number is bounded type.

Lemma 5.1. *Let $f \in \mathrm{D}^{1+\mathcal{Z}_\gamma}(\mathbb{S}^1 \backslash \{\xi_0\})$, $\gamma > 1$ be a circle diffeomorphism with the break size c and irrational rotation number ρ of bounded type such that $s(\rho) = m$. Let $\Delta^{(n+k)} \in \mathcal{P}_{n+k}$ such that $\Delta^{(n+k)} \subset \widehat{\Delta}_0^{(n-1)}$ where $k \geq 1$. There exists a constant $C = C(f) > 0$ and a natural number $N_0 = N_0(f)$ such that*

$$\frac{|\Delta^{(n+k)}|}{|\widehat{\Delta}_0^{(n-1)}|} \leq C\lambda^k\Big(1 + \frac{1}{n^{\gamma-1}}\Big)$$

for all $n \geq N_0$, where $\lambda = \sqrt{\frac{\mathfrak{c}^2}{\mathfrak{c}^2+1}}$.

Proof. First we show that

$$\frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n-1)}|} \leq \lambda^2 + \frac{C}{n^\gamma} \tag{20}$$

for large enough n. One can verify that $\Delta_0^{(n+1)} \subset \Delta^{(n)}_{k_{n+1}q_n+q_{n-1}}$. By (1) and Corollary 4.7 we have

$$\begin{aligned} \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n-1)}|} &\leq \frac{1}{1 + \frac{|\Delta^{(n)}_{(k_{n+1}-1)q_n+q_{n-1}}|}{|\Delta_0^{(n+1)}|}} \leq \frac{1}{1 + \frac{|\Delta^{(n)}_{(k_{n+1}-1)q_n+q_{n-1}}|}{|\Delta^{(n)}_{k_{n+1}q_n+q_{n-1}}|}} \\ &\leq \frac{1}{1 + (Df^{q_n}(\hat{\xi}))^{-1}} = \frac{1}{1 + (Df_n(\hat{z}))^{-1}} \leq \frac{\mathfrak{c}^2}{\mathfrak{c}^2+1} + \frac{C}{n^\gamma} \end{aligned} \tag{21}$$

where $\hat{\xi} \in \Delta^{(n)}_{(k_{n+1}-1)q_n+q_{n-1}}$ and $\hat{z} \in (-1, 0)$ such that $\mathcal{A}_n(\hat{z}) = \hat{\xi}$. Inequality (21) yields

$$\frac{|\Delta_0^{(n+2l+1)}|}{|\Delta_0^{(n-1)}|} \leq \exp\left(\sum_{s=0}^{l} \ln\Big(\lambda^2 + \frac{C}{(n+2s)^\gamma}\Big)\right) \leq \lambda^{2(l+1)}\Big(1 + \frac{C}{n^{\gamma-1}}\Big). \tag{22}$$

Since the rotation number is bounded type we have

$$\frac{|\Delta_0^{(n+k)}|}{|\widehat{\Delta}_0^{(n-1)}|} \leq C\lambda^k\Big(1 + \frac{1}{n^{\gamma-1}}\Big) \tag{23}$$

for any $k \geq 1$ and for n large. Let $\Delta^{(n+k)}$ be any interval satisfying $\Delta^{(n+k)} \in \mathcal{P}_{n+k}$ and $\Delta^{(n+k)} \subset \widehat{\Delta}_0^{(n-1)}$ where $k \geq 1$. There exists i_0 such that $f^{i_0}(\Delta_0^{(n+k)}) = \Delta^{(n+k)}$. We claim that the length of intervals $\Delta_0^{(n+k)}$ and $f^{i_0}(\Delta_0^{(n+k)})$ are comparable, that is, there exists a constant $C > 1$ such that $C^{-1} \leq |\Delta_0^{(n+k)}|/|f^{i_0}(\Delta_0^{(n+k)})| \leq C$. Indeed, due to Finzi's inequality we have

$$e^{-v} \leq \frac{|\Delta_0^{(n+k)}|}{|f^{i_0}(\Delta_0^{(n+k)})|}\frac{|f^{i_0}(\widehat{\Delta}_0^{(n-1)})|}{|\widehat{\Delta}_0^{(n-1)}|} \leq e^v. \tag{24}$$

where v is the total variation of $\log Df$. On the other hand the length of intervals $f^{i_0}(\widehat{\Delta}_0^{(n-1)})$ and $\widehat{\Delta}_0^{(n-1)}$ are $(2e^v+1)$-comparable since

$$f^{i_0}(\widehat{\Delta}_0^{(n-1)}) \subset f^{-q_{n-1}}(\widehat{\Delta}_0^{(n-1)}) \cup \widehat{\Delta}_0^{(n-1)} \cup f^{q_{n-1}}(\widehat{\Delta}_0^{(n-1)})$$

and

$$\widehat{\Delta}_0^{(n-1)} \subset f^{-q_{n-1}+i_0}(\widehat{\Delta}_0^{(n-1)}) \cup f^{i_0}(\widehat{\Delta}_0^{(n-1)}) \cup f^{q_{n-1}+i_0}(\widehat{\Delta}_0^{(n-1)}).$$

Therefore the length of intervals $\Delta_0^{(n+k)}$ and $f^{i_0}(\Delta_0^{(n+k)})$ are comparable. This and inequality (23) imply

$$\frac{|\Delta^{(n+k)}|}{|\widehat{\Delta}_0^{(n-1)}|} \leq C\lambda^k\Big(1+\frac{1}{n^{\gamma-1}}\Big)$$

for $k \geq 1$ and large enough n. □

6 Closeness of rescaled points

Our aim in this section is to show the closeness of rescaled points of ξ and $h(\xi)$. Let f be a circle diffeomorphism with a break. Let $\mathcal{A}_n$ be the affine covering map of f. Denote by $\mathfrak{r}_n : \widehat{\Delta}_0^{(n-1)} \to [-1, a_n]$ the inverse of $\mathcal{A}_n$. The point $\mathfrak{r}_n(\xi)$ is called rescaled point of ξ. Next consider two circle diffeomorphisms f and $\tilde{f}$ with a break and with the identical irrational rotation number. Define the distance between appropriately rescaled points of ξ and $h(\xi)$:

$$\mathfrak{d}_n(\xi) = |\mathfrak{r}_n(\xi) - \tilde{\mathfrak{r}}_n(h(\xi))|.$$

We have

Lemma 6.1. *Let f and $\tilde{f}$ satisfy the assumptions of Theorem 2.4. Then for any $\alpha \in (0,\gamma)$ there exist $\kappa = \kappa(f,\tilde{f}) > 1$, $C = C(f,\tilde{f}) > 0$ and $N_0 = N_0(f,\tilde{f}) \in \mathbb{N}$ such that*

$$\mathfrak{d}_n(\xi) \leq \frac{C}{n^{\gamma-\alpha}}$$

for all $\xi \in \Xi_\ell^* \cap \widehat{\Delta}_0^{(n-1)}$ provided $n \leq \ell \leq n + [\alpha \log_\kappa n]$ for $n \geq N_0$ where $[\cdot]$ is the integer part of a number.

Proof. It is easy to verify that $\Xi_\ell^* \cap \widehat{\Delta}_0^{(n-1)} = \{\xi_{q_{n-1}}, \xi_{q_n+q_{n-1}}, \xi_0, \xi_{q_n}\}$ for $\ell = n$. One can easily see that $\mathfrak{d}_n(\xi_{q_{n-1}}) = \mathfrak{d}_n(\xi_0) = 0$, $\mathfrak{d}_n(\xi_{q_n+q_{n-1}}) = |f_n(-1) - \tilde{f}_n(-1)|$ and $\mathfrak{d}_n(\xi_{q_n}) = |f_n(0) - \tilde{f}_n(0)|$. Hence by Theorem 4.4 we get

$$\max_{\xi\in\Xi_n^*\cap\widehat{\Delta}_0^{(n-1)}} \mathfrak{d}_n(\xi) \leq \frac{C}{n^\gamma} \tag{25}$$

for large enough n. For fixed $\ell > n$ let us denote $\mathfrak{q}_n = \max_{\xi\in\Xi_\ell^*\cap\widehat{\Delta}_0^{(n-1)}} \mathfrak{d}_n(\xi)$. The obvious equality $\mathfrak{d}_n(\xi) = |f_n(0)\mathfrak{r}_{n+1}(\xi) - \tilde{f}_n(0)\tilde{\mathfrak{r}}_{n+1}(h(\xi))|$ and Theorem 4.4 imply

$$\mathfrak{d}_n(\xi) \leq a_n \mathfrak{d}_{n+1}(\xi) + \frac{C}{n^\gamma} \tag{26}$$

if $\xi \in \Xi_\ell^* \cap \widehat{\Delta}_0^{(n)}$ and n is large, where $a_n = f_n(0) = |\Delta_0^{(n)}|/|\Delta_0^{(n-1)}|$. Let $\xi \in \Xi_\ell \cap \check{\Delta}_0^{(n-1)}$. Consider an arbitrary thread in the decomposition (2) and denote $\eta_s = \mathfrak{r}_n(\xi_{l+sq_n+q_{n-1}})$,

$\tilde{\eta}_s = \tilde{\mathfrak{r}}_n(\tilde{\xi}_{l+sq_n+q_{n-1}})$, for $0 \le s \le k_{n+1}$, so that $\mathfrak{d}_n(\xi_{l+sq_n+q_{n-1}}) = |\eta_s - \tilde{\eta}_s|$ where $\tilde{\xi}_{l+sq_n+q_{n-1}} = h(\xi_{l+sq_n+q_{n-1}})$. It is easy to see that $\eta_{s+1} = f_n(\eta_s)$ and $\tilde{\eta}_{s+1} = \tilde{f}_n(\tilde{\eta}_s)$. First we consider the case $s = 0$. In this case, it is a simple matter to verify that

$$
\begin{aligned}
\mathfrak{d}_n(\xi_{l+q_{n-1}}) &= |\eta_0 - \tilde{\eta}_0| = \left|\frac{\mathfrak{r}_{n-1}(\xi_{l+q_{n-1}})}{f_{n-1}(0)} - \frac{\tilde{\mathfrak{r}}_{n-1}(\tilde{\xi}_{l+q_{n-1}})}{\tilde{f}_{n-1}(0)}\right| \\
&\le \frac{\mathfrak{d}_{n-1}(\xi_{l+q_{n-1}})}{f_{n-1}(0)} + \left|\frac{1}{f_{n-1}(0)} - \frac{1}{\tilde{f}_{n-1}(0)}\right|\left|\tilde{\mathfrak{r}}_{n-1}(\tilde{\xi}_{l+q_{n-1}})\right|, \\
\mathfrak{d}_{n-1}(\xi_{l+q_{n-1}}) &= |f_{n-1}(\mathfrak{r}_{n-1}(\xi_l)) - \tilde{f}_{n-1}(\tilde{\mathfrak{r}}_{n-1}(\tilde{\xi}_l))| \\
&\le Df_{n-1}(\mathfrak{r}^0)\mathfrak{d}_{n-1}(\xi_l) + |f_{n-1}(\tilde{\mathfrak{r}}_{n-1}(\xi_l)) - \tilde{f}_{n-1}(\tilde{\mathfrak{r}}_{n-1}(\xi_l))|, \\
\mathfrak{d}_{n-1}(\xi_l) &= |f_{n-1}(0)f_n(0)\mathfrak{r}_{n+1}(\xi_l) - \tilde{f}_{n-1}(0)\tilde{f}_n(0)\tilde{\mathfrak{r}}_{n+1}(\tilde{\xi}_l)| \\
&\le f_{n-1}(0)f_n(0)\mathfrak{d}_{n+1}(\xi_l) + |f_{n-1}(0)f_n(0) - \tilde{f}_{n-1}(0)\tilde{f}_n(0)||\tilde{\mathfrak{r}}_{n+1}(\tilde{\xi}_l)|,
\end{aligned}
\tag{27}
$$

where $\mathfrak{r}^0$ is a point between $\mathfrak{r}_{n-1}(\xi_l)$ and $\tilde{\mathfrak{r}}_{n-1}(\tilde{\xi}_l)$ such that $|f_{n-1}(\mathfrak{r}_{n-1}(\xi_l)) - f_{n-1}(\tilde{\mathfrak{r}}_{n-1}(\tilde{\xi}_l))| = Df_{n-1}(\mathfrak{r}^0)|\mathfrak{r}_{n-1}(\xi_l) - \tilde{\mathfrak{r}}_{n-1}(\tilde{\xi}_l)|$. Since the rotation number is bounded type, Theorem 4.4 and inequalities (20) and (27) imply that

$$
\mathfrak{d}_n(\xi_{l+q_{n-1}}) = |\eta_0 - \tilde{\eta}_0| \le a_n Df_{n-1}(\mathfrak{r}^0)\mathfrak{d}_{n+1}(\xi_l) + \frac{C}{n^\gamma} \tag{28}
$$

for n large. Now consider the case $0 < s < k_{n+1}$. Let $\mathfrak{r}^s$ be a point between η_{s-1} and $\tilde{\eta}_{s-1}$ such that $|f_n(\eta_{s-1}) - f_n(\tilde{\eta}_{s-1})| = Df_n(\mathfrak{r}^s)|\eta_{s-1} - \tilde{\eta}_{s-1}|$. Then we have

$$
\mathfrak{d}_n(\xi_{l+sq_n+q_{n-1}}) = |\eta_s - \tilde{\eta}_s| \le Df_n(\mathfrak{r}^s)|\eta_{s-1} - \tilde{\eta}_{s-1}| + \frac{C}{n^\gamma}
$$

for n large. Iterating into it we get

$$
\mathfrak{d}_n(\xi_{l+sq_n+q_{n-1}}) = |\eta_s - \tilde{\eta}_s| \le \prod_{i=1}^{s} Df_n(\mathfrak{r}^i)|\eta_0 - \tilde{\eta}_0| + \Big(1 + \sum_{j=2}^{s}\prod_{i=j}^{s} Df_n(\mathfrak{r}^i)\Big)\frac{C}{n^\gamma} \tag{29}
$$

Since the rotation number is bounded type the expressions $\Big(1 + \sum_{j=2}^{s}\prod_{i=j}^{s} Df_n(\mathfrak{r}^i)\Big)$ and $\prod_{i=1}^{s} Df_n(\mathfrak{r}^i)$ are bounded above by a universal constant. This and relations (28) and (29) imply

$$
\mathfrak{d}_n(\xi_{l+sq_n+q_{n-1}}) \le \prod_{i=1}^{s} Df_n(\mathfrak{r}^i)\Big(a_n Df_{n-1}(\mathfrak{r}^0)\Big)\mathfrak{d}_{n+1}(\xi_l) + \frac{C}{n^\gamma} \tag{30}
$$

for n large. Finally, consider the case $s = k_{n+1}$. In this case, it is easy to see that

$$
\begin{aligned}
\mathfrak{d}_n(\xi_{l+q_{n+1}}) &= \left|\mathfrak{r}_n(\xi_{l+q_{n+1}}) - \tilde{\mathfrak{r}}_n(\tilde{\xi}_{l+q_{n+1}})\right| \\
&= |f_n(0)f_{n+1}(\mathfrak{r}_{n+1}(\xi_l)) - \tilde{f}_n(0)\tilde{f}_{n+1}(\tilde{\mathfrak{r}}_{n+1}(\xi_l))| \\
&\le a_n Df_{n+1}(\mathfrak{r}^{k_{n+1}})\mathfrak{d}_{n+1}(\xi_l) + \frac{C}{n^\gamma}
\end{aligned}
\tag{31}
$$

for large enough n, where $\mathfrak{r}^{k_{n+1}}$ is a point between $\mathfrak{r}_{n+1}(\xi_l)$ and $\tilde{\mathfrak{r}}_{n+1}(\tilde{\xi}_l)$ such that $|f_{n+1}(\mathfrak{r}_{n+1}(\xi_l)) - f_{n+1}(\tilde{\mathfrak{r}}_{n+1}(\tilde{\xi}_l))| = Df_{n+1}(\mathfrak{r}^{k_{n+1}})|\mathfrak{r}_{n+1}(\xi_l) - \tilde{\mathfrak{r}}_{n+1}(\tilde{\xi}_l)|$. Combining Lemmas 4.5 and 4.6 we can easily obtain that $a_n Df_{n-1}(\mathfrak{r}^0)$, $a_n Df_{n+1}(\mathfrak{r}^{k_{n+1}}) \leq c_n^2 + Cn^{-\gamma}$ if $c_n > 1$ and $a_n Df_{n-1}(\mathfrak{r}^0)$, $a_n Df_{n+1}(\mathfrak{r}^{k_{n+1}}) \leq c_n^{-1} + Cn^{-\gamma}$ if $c_n < 1$ and

$$\prod_{i=1}^{s} Df_n(\mathfrak{r}^i)\Big(a_n Df_{n-1}(\mathfrak{r}^0)\Big) \leq \begin{cases} c_n^{2(s+1)} + Cn^{-\gamma}, & \text{if } c_n > 1 \\ c_n^{-(s+1)} + Cn^{-\gamma}, & \text{if } c_n < 1 \end{cases} \tag{32}$$

for n large. Let us denote $\mathfrak{c} = \max\{c, c^{-1}\}$ and $\kappa := \kappa(c, m) = \mathfrak{c}^{2m}$. It follows from the relations (26), (28), (30),(31) and (32) that

$$\mathfrak{q}_n \leq \kappa \mathfrak{q}_{n+1} + \frac{C}{n^\gamma} \tag{33}$$

for n large. Iterating (33) we get

$$\mathfrak{q}_n \leq \kappa^{\ell-n}\mathfrak{q}_\ell + C\sum_{j=n}^{\ell-1}\frac{\kappa^{j-n}}{j^\gamma}$$

for n large. Inequality (25) implies $\mathfrak{q}_\ell \leq C\ell^{-\gamma}$. Hence

$$\mathfrak{q}_n \leq C\sum_{j=n}^{\ell}\frac{\kappa^{j-n}}{j^\gamma} \leq \frac{C\kappa^{\ell-n}}{n^\gamma}. \tag{34}$$

The condition $n \leq \ell \leq n + [\alpha \log_\kappa n]$ makes it obvious that

$$\mathfrak{q}_n \leq \frac{C}{n^{\gamma-\alpha}}$$

for large enough n. Lemma 6.1 is proved. □

7 Proof of main theorem

In this section we prove our main theorem. For this, first we prove a preparatory lemma and then we prove C^1-smoothness of the conjugacy. Finally, we prove $C^{1+\omega_\gamma}$-smoothness of the conjugacy.

7.1 Preparatory lemma

We begin by proving the following lemma.

Lemma 7.1. *Let f and $\tilde{f}$ satisfy the assumptions of Theorem 2.4. Then there exists a constant $C := C(f, \tilde{f}) > 0$ and a natural number $N_0 := N_0(f, \tilde{f})$ such that*

$$\Lambda_n \leq \frac{C}{n^{\frac{\gamma}{2}}}$$

for all $n \geq N_0$.

Proof. One can see that

$$\begin{aligned} |\Lambda_n(\xi)| &= |\log Df^{q_n}(\xi) - \log D\tilde{f}^{q_n}(h(\xi))| = |\log Df_n(\mathfrak{r}_n(\xi)) - \log D\tilde{f}_n(\tilde{\mathfrak{r}}_n(\tilde{\xi}))| \\ &\leq |\log Df_n(\mathfrak{r}_n(\xi)) - \log Df_n(\tilde{\mathfrak{r}}_n(\tilde{\xi}))| + |\log Df_n(\tilde{\mathfrak{r}}_n(\tilde{\xi})) - \log D\tilde{f}_n(\tilde{\mathfrak{r}}_n(\tilde{\xi}))| \\ &\leq \|D\log Df_n\|_{C^0([-1,0])}\mathfrak{d}_n(\xi) + \frac{1}{\inf D\tilde{f}_n}\|Df_n - D\tilde{f}_n\|_{C^0([-1,0])}. \end{aligned} \tag{35}$$

By Lemma 4.2 we have $\|D\log Df_n\|_{C^0([-1,0])}\le Q$. Denjoy's inequality implies $(\inf D\tilde{f}_n)^{-1}\le e^{v_f}$. From Theorem 4.4 it follows that $\|Df_n - D\tilde{f}_n\|_{C^0([-1,0])}\le Cn^{-\gamma}$ for n large. Next we estimate $\mathfrak{d}_n(\xi)$ on $\widehat{\Delta}_0^{(n-1)}$. First we assume that $\xi\in\Xi^*_{n+[\frac{\gamma}{2}\log_\kappa n]}\cap\widehat{\Delta}_0^{(n-1)}$. Then, if we choose $\alpha=\gamma/2$ in Lemma 6.1 then for large enough n, the function $\mathfrak{d}_n(\xi)$ can be estimated as follows

$$\mathfrak{d}_n(\xi)\le\frac{C}{n^{\frac{\gamma}{2}}}. \tag{36}$$

Let ξ be any point of $\widehat{\Delta}_0^{(n-1)}$. Denote by $\Delta^{(n+[\frac{\gamma}{2}\log_\kappa n])}(\xi)$ the segment of $\mathcal{P}_{n+[\frac{\gamma}{2}\log_\kappa n]}$ containing the point ξ and $r_n(\xi):=r_{n+[\frac{\gamma}{2}\log_\kappa n]}(\xi)$ the right endpoint of $\Delta^{(n+[\frac{\gamma}{2}\log_\kappa n])}(\xi)$. A trivial reasoning shows that

$$\begin{aligned}\mathfrak{d}_n(\xi)&=|\mathfrak{r}_n(\xi)-\tilde{\mathfrak{r}}_n(h(\xi))|\\&\le\Big|\frac{\xi-r_n(\xi)}{|\Delta_0^{(n-1)}|}-\frac{h(\xi)-h(r_n(\xi))}{|\tilde{\Delta}_0^{(n-1)}|}\Big|+\mathfrak{d}_n(r_n(\xi))\\&\le\frac{|\Delta^{(n+[\frac{\gamma}{2}\log_\kappa n])}(\xi)|}{|\Delta_0^{(n-1)}|}+\frac{|\tilde{\Delta}^{(n+[\frac{\gamma}{2}\log_\kappa n])}(h(\xi))|}{|\tilde{\Delta}_0^{(n-1)}|}+\mathfrak{d}_n(r_n(\xi))\end{aligned} \tag{37}$$

where $\tilde{\Delta}^{(n+[\frac{\gamma}{2}\log_\kappa n])}(h(\xi))$ the segment of $\tilde{\mathcal{P}}_{n+[\frac{\gamma}{2}\log_\kappa n]}:=\tilde{\mathcal{P}}_{n+[\frac{\gamma}{2}\log_\kappa n]}(h(\xi_0),\tilde{f})$ containing the point $h(\xi)$. By (36) we have

$$\mathfrak{d}_n(r_n(\xi))\le\frac{C}{n^{\frac{\gamma}{2}}}. \tag{38}$$

It follows easily from Lemma 5.1 that

$$\frac{|\Delta^{(n+[\frac{\gamma}{2}\log_\kappa n])}(\xi)|}{|\Delta_0^{(n-1)}|}\le C\lambda^{\frac{\gamma}{2}\log_\kappa n}\Big(1+\frac{1}{n^{\gamma-1}}\Big), \tag{39}$$

and

$$\frac{|\tilde{\Delta}^{(n+[\frac{\gamma}{2}\log_\kappa n])}(h(\xi))|}{|\tilde{\Delta}_0^{(n-1)}|}\le C\lambda^{\frac{\gamma}{2}\log_\kappa n}\Big(1+\frac{1}{n^{\gamma-1}}\Big). \tag{40}$$

One can see that

$$\lambda^{\frac{\gamma}{2}\log_\kappa n}=\Big(\frac{1}{n^{\frac{\gamma}{2}}}\Big)^{\log_\kappa\frac{1}{\lambda}} \tag{41}$$

Hypothesis (d) of Theorem 2.4 implies that $\lambda^{-1}>\kappa$. Hence

$$\log_\kappa\frac{1}{\lambda}>1.$$

This implies

$$\lambda^{\frac{\gamma}{2}\log_\kappa n}\le\frac{1}{n^{\frac{\gamma}{2}}} \tag{42}$$

Combining (35)-(43) we conclude that

$$\Lambda_n\le\frac{C}{n^{\frac{\gamma}{2}}}$$

for large enough n. Lemma 7.1 is proved. □

7.2 C^1-smoothness of conjugacy

By the hypotheses of Theorem 2.4 the rotation number of f and $\tilde{f}$ is bounded type and $\gamma > 2$. Lemma 7.1 implies that

$$\sum_{n=0}^{\infty} k_{n+1}\Lambda_n < \infty.$$

Therefore, it follows from Theorem 3.2 that the cohomological equation (5) has a continuous solution ζ. Next we prove the following lemma.

Lemma 7.2. *There exists $\beta > 0$ such that*

$$Dh(\xi) = \beta e^{\zeta(\xi)}, \qquad \textit{for all} \qquad \xi \in \mathbb{S}^1.$$

Proof. Denote by $\beta_n = |\tilde{\Delta}_0^{(n)}|/|\Delta_0^{(n)}|$. Since the rotation number of f and $\tilde{f}$ is bounded type, Theorem 4.4 and inequality (20) imply that

$$|\ln\beta_n - \ln\beta_{n-1}| = |\ln f_n(0) - \ln\tilde{f}_n(0)| \leq \frac{1}{L_m(\hat{v})}|f_n(0) - \tilde{f}_n(0)| \leq \frac{C}{n^\gamma}. \tag{43}$$

Since $\gamma > 2$ the sequence $(\ln\beta_n)_n$ and as well as $(\beta_n)_n$ is convergent. Let $\beta = \lim_{n\to\infty}\beta_n$. It follows from

$$\lim_{n\to\infty}\frac{|\tilde{\Delta}_0^{(n)}|}{|\Delta_0^{(n)}|} = \lim_{n\to\infty}\frac{|h(\Delta_0^{(n)})|}{|\Delta_0^{(n)}|} = Dh(\xi_0)$$

and $\zeta(\xi_0) = 0$ that $Dh(\xi_0) = \beta e^{\zeta(\xi_0)}$. From this and the equality $h \circ f = \tilde{f} \circ h$ we deduce

$$\log Dh(\xi_i) - \log Dh(\xi_{i-1}) = \log D\tilde{f}(h(\xi_{i-1})) - \log Df(\xi_{i-1}). \tag{44}$$

for any $\xi_i \in \Xi$ where $\xi_i = f^i(\xi_0)$, $i \geq 1$. The cohomological equation (5) implies that

$$\zeta(\xi_i) - \zeta(\xi_{i-1}) = \log D\tilde{f}(h(\xi_{i-1})) - \log Df(\xi_{i-1}). \tag{45}$$

Combining (44) and (45) we get

$$\log Dh(\xi_i) - \zeta(\xi_i) = \log Dh(\xi_{i-1}) - \zeta(\xi_{i-1})$$

which implies

$$\log Dh(\xi_i) - \zeta(\xi_i) = \log Dh(\xi_0) - \zeta(\xi_0).$$

Hence

$$Dh(\xi_i) = \beta e^{\zeta(\xi_i)} \tag{46}$$

for any $\xi_i \in \Xi$. Since ζ is continuous and Ξ is dense in $\mathbb{S}^1$ the function Dh can be continuously extended to the whole of $\mathbb{S}^1$ verifying the equality (46). This proves Lemma 7.2 and concludes the C^1-smoothness of the conjugacy. $\square$

7.3 $C^{1+\omega_\gamma}$-smoothness of conjugacy

It follow from C^1-smoothness of conjugacy and the equality $h \circ f = \tilde{f} \circ h$ that

$$\log Dh \circ f - \log Dh = \log D\tilde{f} \circ h - \log Df. \tag{47}$$

Consider the points ξ_i and $\xi_{i+q_{n-1}+sq_n}$ where $1 \leq s \leq k_{n+1}$. It is clear that $\xi_i, \xi_{i+q_{n-1}+sq_n} \in \Delta_i^{(n-1)}$. The relation (47) implies

$$|\log Dh(\xi_{i+q_{n-1}+sq_n}) - \log Dh(\xi_i)| \leq s\Lambda_n + \Lambda_{n-1}.$$

Consequently, for any $\xi_j \in \Xi \cap \check{\Delta}_i^{(n-1)}$ we have

$$|\log Dh(\xi_j) - \log Dh(\xi_i)| \leq C \sum_{\ell=n}^{\infty} k_{\ell+1}\Lambda_\ell.$$

Since $k_{\ell+1}$ is bounded from Lemma 7.1 it implies that

$$|\log Dh(\xi_j) - \log Dh(\xi_i)| \leq C \sum_{\ell=n}^{\infty} \frac{1}{\ell^{\frac{\gamma}{2}}} \leq \frac{C}{n^{\frac{\gamma}{2}-1}}. \tag{48}$$

It is obvious that

$$|\Delta_i^{(n+1)}| \leq |\xi_j - \xi_i| \leq |\Delta_i^{(n-1)}|. \tag{49}$$

Lemma 5.1 implies that there exist $\mu_1, \mu_2 \in (0,1)$ verifying $\mu_1 < \mu_2$ such that

$$\mu_1^n \leq |\Delta^{(n)}| \leq \mu_2^n \tag{50}$$

for any $\Delta^{(n)} \in \mathcal{P}_n$. Relations (49) and (50) imply

$$n = \mathcal{O}\Big(\frac{1}{|\log|\xi_j - \xi_i||}\Big). \tag{51}$$

Combining (48) with (51) we can assert that

$$|\log Dh(\xi_j) - \log Dh(\xi_i)| \leq \frac{C}{\big|\log|\xi_j - \xi_i|\big|^{\frac{\gamma}{2}-1}}. \tag{52}$$

Since Ξ is dense in $\mathbb{S}^1$, the function Dh can be continuously extended to the whole of $\mathbb{S}^1$ verifying the inequality (52). This proves $C^{1+\omega_\gamma}$-smoothness of the conjugacy. Theorem 2.4 is proved.

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