TREES AND HOMOGENEOUS LOTS

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Abstract. We describe those complete linearly ordered topological spaces X which are homogeneous (=CHLOTS). That is, X is order isomorphic with any nonempty open interval in X. Using countable tail-like ordinals as indices, we build towers of distinct CHLOTS. Using tree constructions we are able to extend the towers and to describe an inductive procedure which yields every CHLOTS.

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1. Introduction

At first glance the Cantor Set $C \subset [0,1]$ does not appear to be homogeneous. Aside from the maximum and minimum there is the countable family of endpoint pairs, each of which forms a gap $x^- < x^+$ such that the intersection $C \cap [x^+, \infty)$ is clopen in C. Then there is the uncountable residuum of points whose very existence is not obvious until the bijection from the set of zero/one sequences to C is revealed. This bijection is in fact a homeomorphism of C with a topological group from which topological homogeneity is clear as the automorphism group $H(C)$ contains all the translations of the group.

The original impression of non-homogeneity comes from the order structure on C inherited from $\mathbb{R}$. Indeed, if you restrict to $H_+(C)$, the subgroup of order preserving automorphisms, then there are five equivalence classes with respect to the action: $\{max\}$, $\{min\}$, the set of left endpoints $\{x^-\}$, the set of right endpoints $\{x^+\}$, and the remaining residual subset. If you allow order reversing automorphisms, using $H_\pm(C)$ which contains $H_+(C)$ as a subgroup of index two, then the *max* and *min* pair up and the left and right endpoints are equivalent, leading to three classes. Ignoring the extrema we focus on the distinction between the gap pairs and the rest.

There is an interesting construction called the *Alexandrov-Sorgenfrey Double Arrow*

$$\mathbb{R}' =_{def} \mathbb{R} \times \{-1, +1\} \tag{1.1}$$

in which we denote by t^- the point $(t,-1)$ and similarly $t^+ = (t,+1)$. On $\mathbb{R}'$ we introduce the lexicographic ordering and use the associated order topology. For every $t \in \mathbb{R}$ $t^- < t^+$ is a gap pair in $\mathbb{R}'$ and so every point is either a left or a right endpoint. Every closed, bounded subset of $\mathbb{R}'$ is compact and so the space is locally compact and σ-compact. The family of clopen intervals $\mathcal{B} = \{[t^+, s^-] : t < s \text{ in } \mathbb{R}\}$ is uncountable and so $\mathbb{R}'$ is not metrizable although it is clearly separable. Since $\mathcal{B}$ is a basis for the topology the space is zero-dimensional. The space $\mathbb{R}'$ is a famous example in part because the subset $\{t^- : t \in \mathbb{R}\}$ is order isomorphic with $\mathbb{R}$ and so the order topology is the usual one on $\mathbb{R}$, but the subspace topology induced from $\mathbb{R}'$ is the nonmetrizable, not locally compact topology on $\mathbb{R}$ with basis the right-closed, left-open intervals.

We denote by $\bullet\mathbb{R}'\bullet$ the two point compactification obtained by attaching a minimum m and a maximum M to $\mathbb{R}'$. We call this space the *Fat Cantor Set*. For every $t < s$ in $\mathbb{R}$ there is an order preserving homeomorphism

$$f : \bullet\mathbb{R}'\bullet \quad \to \quad [t^+, s^-]. \tag{1.2}$$

Clearly, the group $H_\pm(\mathbb{R}')$ acts transitively on $\mathbb{R}'$.

Our work began with an analogy question: The Cantor Set is to the real line as the Fat Cantor Set is to what?

There should be a linearly ordered space X with the order topology, that is, a *LOTS*,which is connected, which contains the Fat Cantor Set and which is homogeneous in the sense that for all $a < b$ in X there exists an order preserving homeomorphism

$$f : X \to (a, b). \tag{1.3}$$

Our first thought was to use $\mathbb{R} \times J$ with $J = [-1,+1] \subset \mathbb{R}$ with the lexicographic ordering. This LOTS is connected but it is not homogeneous. However, equipped with the lexicographic ordering the countably infinite product

$$\mathbb{R}_\omega \quad = \quad \mathbb{R} \times J \times J \times ... \tag{1.4}$$

with ω the first infinite ordinal. This is a connected and homogeneous LOTS, which we call a CHLOTS. Define for $t \in J$

$$j(t^-) \ = \ (t,-1,-1,...) \qquad \text{and} \qquad j(t^+) \ = \ (t,+1,+1,...). \tag{1.5}$$

Then $j : [(-1)^+, (+1)^-] \to \mathbb{R}_\omega$ is an order preserving, topological embedding onto a closed subset. Thus, $\mathbb{R}_\omega$ naturally contains the Fat Cantor Set.

Looking for other examples of CHLOTS led us to look at products like (1.4) but indexed by more general countable ordinals than ω. Over each CHLOTS we constructed a tower of distinct CHLOTS, increasing in size in a suitable sense. The tower is indexed by the countable ordinals, i.e. the ordinals less than Ω, the first uncountable ordinal.

In this way we re-discovered a construction begun by Arens [4] and [5] and extended by Babcock [6], see also [20]. Their procedure was also re-discovered by others, see e.g. [9].

We had nearly completed the initial phase of our work in 2001, [3], when we were directed to the paper of Hart and van Mill [10] whose work is complementary to ours, and of course to that of Arens and Babcock. Hart and van Mill construct an uncountable family of distinct CHLOTS no two of which have comparable size but all of which are bigger than $\mathbb{R}$ but smaller than $\mathbb{R}_\omega$. So their class of examples extends horizontally where ours proceeds vertically.

We now apply trees to the study of CHLOTS.

A *tree* is a partially ordered set such that the set of the predecessors of each point is well-ordered by the induced order and so is isomorphic to an ordinal. The relation between trees and general LOTS is well-known, e.g. [19] and [7]. By using trees we develop a number of constructions for building CHLOTS and extend the tower over a CHLOTS to one indexed by $\Omega \times \Omega$. In addition, we describe an inductive tree construction from which every CHLOTS can be obtained.

We would like to thank Richard Wilson for some helpful discussions as we began this work.

We now provide a brief sketch of what follows.

In Section 2 we introduce the elementary properties of LOTS and ordinals.

A subset J of a LOTS is *convex* when $a < c < b$ in X and $a, b \in J$ implies $c \in J$. An open or closed interval is convex. A LOTS X is *order dense* when $a < b$ in X implies that the interval (a, b) is infinite and X is *complete* when every bounded set has a supremum and infimum. A LOTS is connected iff it is a complete and order dense LOTS. An order dense LOTS X is contained as a dense subset in an essentially unique connected LOTS $\hat{X}$ called its *completion*.

An order dense LOTS which is not complete contains holes. There is a *hole* between a and b if there is a clopen convex set which contains a and not b. A LOTS has *dense holes* when between any pair of distinct points there are holes.

We call a LOTS *unbounded* when it has neither a maximum nor a minimum and *bounded* when it has both.

There is a rough notion of *size* for LOTS. In comparing two LOTS X and X_1 we say that X *injects into* X_1 if there exists a, not necessarily continuous, injective order map from X into X_1. We say that X_1 is *bigger* than X if X order injects into X_1 but not vice-versa. For example, any CHLOTS which is not $\mathbb{R}$ itself is bigger than $\mathbb{R}$. If neither injects into the other we say that they are not comparable.

In outlining the use of ordinals we pay special attention to those which are tail-like.

An ordinal α is *tail-like* if $\beta < \alpha$ implies $\beta + \alpha = \alpha$, or, equivalently, $\alpha = \omega^\gamma$ for some ordinal γ. For ordinal exponentiation, γ countable implies ω^γ is countable and so there are uncountably many countable tail-like ordinals, indexed by the countable ordinals γ. We describe the Cantor Normal Form which writes any ordinal uniquely as a sum of a finite non-increasing sequence of tail-like ordinals.

In Section 3 we define various transitivity and homogeneity properties for LOTS.

A LOTS X is *transitive* when $a, b \in X$ implies there exists an order isomorphism f on X with $f(a) = b$, i.e. the group of order automorphisms acts transitively on X. A LOTS X is *homogeneous* when X is order isomorphic with any nonempty, open, convex subset of X in which case we call it a *HLOTS*. In particular, we call X a *CHLOTS* when it is a connected homogenous LOTS. If a HLOTS is not complete, then it has dense holes and we call it an incomplete homogeneous LOTS, an *IHLOTS*. An IHLOTS is order-dense and its completion is a CHLOTS.

Any HLOTS is first countable and σ-bounded. A CHLOTS is locally compact and σ-compact.

In Section 4 we build over a given HLOTS X, a tower indexed by the countable ordinals, i.e. by Ω , the first uncountable ordinal.

For a LOTS X and a positive ordinal α define X^α to be the set of maps from α to X, thought of as the lexicographically ordered product. In X we select a nontrivial closed interval J and then define $X_\alpha = \{s \in X^\alpha : s(i) \in J \text{ for all } i > 0\}$. If X is order dense then X^α and X_α are order dense. If, in addition, X is complete, then X_α is complete and so is connected.

Call a LOTS X $\mathbb{R}$-*bounded* if there exists an order injection of X into $\mathbb{R}_\delta$ for some countable ordinal δ.

Theorem 1.1. *Assume that X is a HLOTS.*

(a) *If α is a countable, tail-like ordinal, then X_α is a HLOTS and so its completion $\widehat{X_\alpha}$ is a CHLOTS. In particular, if X is a CHLOTS, then X_α is a CHLOTS for each countable, tail-like ordinal.*

(b) *Assume X is a CHLOTS, e.g. $X = \mathbb{R}$. If α and β are positive ordinals with $\alpha > \beta$, then X_α is bigger than X_β. That is, there is an order injection from X_β into X_a but no order injection in the other direction. In addition, X_α is not homeomorphic to X_β.*

(c) *Assume X is an $\mathbb{R}$-bounded IHLOTS. If α and β are sufficiently large countable, tail-like ordinals with $\alpha > \beta$, then $\widehat{X_\alpha}$ is bigger than $\widehat{X_\beta}$.*

Part (a) is essentially the result of Arens [4], [5] and Babcock [6]. In addition, they showed that for $X = \mathbb{R}$ the elements of the tower at different heights are not isomorphic.

In Section 5 we describe the definitions and elementary results for trees.

Let T be a tree. For a vertex $p \in T$ we let A_p denote the set of predecessors of p. This is isomorphic to a unique ordinal $o(p)$, the *order* of p. We let T_p consist of p together with all its successors. The immediate successors of p are those $q \in T_p$ with $o(q) = o(p)+1$; the set of all immediate successors of p is denoted S_p. For any subset A of T the *height* $h(A)$ is the smallest ordinal greater than $o(p)$ for all $p \in A$.

A subset $T_1 \subset T$ is a *subtree* if $p \in T_1$ implies that all the T predecessors lie in T_1. So, for example, T_p is not a subtree if $o(p) > 0$. On the other hand, the *truncation* $T^\alpha = \{p \in T : o(p) < \alpha\}$ is a subtree.

Our trees are all assumed to be at least *semi-normal* meaning:

- There is a unique root, $0 \in T$, with $o(0) = 0$.
- For all $p \in T$, the set S_p is either empty or contains at least two points.
- If for $p, q \in T$, $A_p = A_q$ and $o(p) = o(q)$ is a limit ordinal, then $p = q$.

Notice that $p, q \in T$, $A_p = A_q$ and $o(p) = o(q) = \beta + 1$ iff $p, q \in S_r$ for some $r \in T$ with $o(r) = \beta$.

A tree is *normal* if, in addition:

- If $p \in T$ and α is an ordinal with $o(p) < \alpha < h(T)$, then there exists a successor q of p with $o(q) = \alpha$.

A *branch* is a maximal linearly ordered subset $x \subset T$. We denote by $X(T)$ the *branch space*, i.e. the set of branches of T. We let $x_i \in T$ denote the element of x with $o(x_i) = i$.

We call the tree T Ω*-bounded* when $h(x) < \Omega$ for every $x \in X(T)$. This implies $h(T) \leq \Omega$, but a tree of height Ω can be Ω-bounded.

The tree T is *bi-ordered* when the successor set S_p has the structure of a LOTS for all $p \in T$. In that case there is an induced order on $X(T)$. For distinct branches x, y we write $x < y$ when for some ϵ, $x_\epsilon = y_\epsilon$ and $x_{\epsilon+1} < y_{\epsilon+1}$. The latter ordering is the LOTS ordering in S_p with $p = x_\epsilon = y_\epsilon$. With the induced order $X(T)$ is a LOTS.

- $X(T)$ is order dense if either every S_p is order dense, or else, every S_p is unbounded and $h(T)$ is a limit ordinal.
- $X(T)$ is complete if every S_p is complete and $o(p) > 0$ implies S_p is bounded.

A bijection $h : T_1 \to T_2$ is a *tree isomorphism* when it preserves both orders. A tree isomorphism induces an order isomorphism between the branch spaces.

A tree T is *reproductive* if for all $p \in T$ there is an isomorphism from T to T_p. A reproductive tree has height a tail-like ordinal. For a reproductive tree, every S_p is order isomorphic to the LOTS S_0.

Theorem 1.2. *If T is an Ω-bounded, reproductive tree with S_0 a HLOTS, then $X(T)$ is a HLOTS.*

In Section 6 a number of tree constructions are presented.

If X is a LOTS and α is a positive ordinal, then the *simple tree* on X, α has vertices X^i at level $i < \alpha$, and for $s \in X^i$ the predecessor at level $j < i$ is the restriction $s|j$. The simple tree has height α and we can identify the branch space with X^α.

For $p \in X^i, q \in X^j$ we define $p + q \in X^{i+j}$ by

$$(p+q)(k) \;=\; \begin{cases} p(k) & \text{for } k < i, \\ q(k \setminus i) & \text{for } i \leq k < i+j. \end{cases} \tag{1.6}$$

Here $k \setminus i = \{\ell : i \leq \ell < k\}$ is identified with the ordinal with which it is isomorphic so that, e.g. $(i+j) \setminus i = j$.

A subtree T of the simple tree on X is called an *additive tree* if

- $p, q \in T \quad \Longleftrightarrow \quad p + q \in T.$

The map $a_p : T \to T_p$ given by $a_p(q) = p + q$ is then an isomorphism from T to T_p and so an additive tree is reproductive. In particular, the height of an additive tree is tail-like.

For a CHLOTS X let $\bullet X \bullet$ be the two point compactification of X obtained by adding a minimum m and a maximum M. If $p \in X^\alpha$ we define $\hat{p} \in (\bullet X \bullet)^{\alpha+1}$ by

$$\hat{p}(0) = m, \quad \hat{p}(i) = sup\{p(j) : j < i\} \text{ for } 0 < i \leq \alpha. \tag{1.7}$$

We call p *sharply increasing* when for all $i < \alpha$, $\hat{p}(i) < p(i)$. We define the *order tree* $T(X)$ to be the subtree of the simple tree on X, Ω whose elements of order α are the bounded, sharply increasing elements of X^α.

Theorem 1.3. *If X is a CHLOTS the order tree $T(X)$ is an Ω-bounded reproductive tree of height Ω with $S_0 \cong X$. The branch space of $T(X)$ is an IHLOTS with completion a CHLOTS.*

We will denote by $a(X)$ the completion of the branch space of $T(X)$.

In Section 7 we build the Double Tower over a CHLOTS X.

Inductively, we let $a_0(X) = X$ and $a_{\alpha+1}(X) = a(a_\alpha(X))$. If α is a countable limit ordinal we define the CHLOTS $a_\alpha(X)$ as an inverse limit of $\{a_\beta(X) : \beta < \alpha\}$. For $(\alpha, \beta) \in \Omega \times \Omega$ we obtain the CHLOTS $(a_\alpha(X))_\beta$.

Theorem 1.4. *If $(\alpha', \beta') > (\alpha, \beta)$ in the lexicographical ordering on $\Omega \times \Omega$, then $(a_{\alpha'}(X))_{\beta'}$ is bigger than $(a_\alpha(X))_\beta$.*

In particular, $a(X)$ is bigger than X_β for any countable ordinal β. It follows that $a(X)$ is not $\mathbb{R}$-bounded.

In Section 8 for any given CHLOTS X we construct an Ω-bounded, additive subtree of the simple tree on $\mathbb{Z}, \Omega$ the completion of whose branch space is isomorphic to X. From it we obtain the following.

Theorem 1.5. *If X is a LOTS, then the following are equivalent.*

- *X is a CHLOTS.*
- *There exists an Ω-bounded, additive subtree of the simple tree on $\mathbb{Z}, \Omega$ the completion of whose branch space is isomorphic to X.*
- *There exists an IHLOTS D, dense in $\mathbb{R}$ and an Ω-bounded, additive subtree of the simple tree on D, Ω the completion of whose branch space is isomorphic to X.*

- *There exists an Ω-bounded, reproductive tree with S_0 a HLOTS the completion of whose branch space is isomorphic to X.*

In addition, we show that a CHLOTS X is $\mathbb{R}$-bounded if and only if there exists a tree of countable height the completion of whose branch space is isomorphic to X.

In Section 9 we describe the Hart-van Mill results.

A subset $Y \subset \mathbb{R}$ is a *Bernstein subset* (a *B-set*) if it meets every Cantor subset of $\mathbb{R}$. It is a *Bi-Bernstein subset* (a BB-set) when, in addition, its complement is a Bernstein subset.

Let $\mathcal{G}$ denote the group of positive, rational, affine transformations on $\mathbb{R}$, i.e. maps of the form $t \mapsto at + b$ with $a, b \in \mathbb{Q}$ and $a > 0$. We observe that nonempty subset of $\mathbb{R}$ which is $\mathcal{G}$ invariant is a HLOTS.

Let $\mathbf{c} = 2^{\aleph_0}$ denote the cardinality of $\mathbb{R}$.

Definition 1.6. *A Hart-van Mill collection $\mathcal{H}$ with base V is a set of cardinality $\mathbf{c}$ which satisfies the following conditions.*

- *$\mathcal{H} \cup \{V\}$ is a collection of pairwise disjoint BB-subsets of $\mathbb{R}$ each of which is $\mathcal{G}$ invariant.*
- *$\mathbb{Q} \subset V$.*
- *$Y \in \mathcal{H}$ implies $-Y \in \mathcal{H}$ with $-Y = \{-x : x \in Y\}$ distinct from Y.*
- *Assume $Y \in \mathcal{H}$ and f is an order automorphism of $\mathbb{R}$. If $f(Y) \setminus Y$ has cardinality $\mathbf{c}$, then $f(Y) \cap V$ has cardinality $\mathbf{c}$.*

For any $\mathcal{J} \subset \mathcal{H}$, let $X(\mathcal{J})$ denote the complement of the union of the elements of $\mathcal{J}$. Because it contains V and so $\mathbb{Q}$, and because it is $\mathcal{G}$ invariant, $X(\mathcal{J})$ is a HLOTS.

Hart and van Mill show that such collections exist and prove the following.

Theorem 1.7. *For a Hart-van Mill collection, let $\mathcal{J}, \mathcal{J}_1$ be distinct subsets of $\mathcal{H}$ with associated HLOTS X, X_1. For countable tail-like ordinals α, α_1 the CHLOTS $\widehat{X_\alpha}$ and $\widehat{(X_1)_{\alpha_1}}$ are not order isomorphic. If $\alpha = \alpha_1 = \omega$, then the two do not even have the same size.*

It follows from Theorem 1.1 that for any subset $\mathcal{J}$ of $\mathcal{H}$, we obtain a tower $\widehat{X(\mathcal{J})_\alpha}$, indexed by the countable tail-like ordinals α. Each tower consists of CHLOTS and is nondecreasing in size. For the $2^{\mathbf{c}}$ distinct subsets, Theorem 1.7 implies that the towers are distinct, i.e. no two contain any pairwise isomorphic elements.

In Section 10 we conclude with some results on complete, perfect, zero-dimensional LOTS, that is, complete LOTS with no isolated points for which the clopen intervals form a basis. Thus, we return to generalizations of the Cantor Set which motivated our original inquiry.

If T is a normal tree of type $2 = \{0, 1\}$ and of height a limit ordinal, then the branch space is a compact, perfect, zero-dimensional LOTS which is first countable if the tree is Ω-bounded. We let $\bar{0}$ and $\bar{1}$ denote the minimum and maximum branches in $X(T)$, so that $\bar{0}_i = 0$, or $\bar{1}_i = 1$, for every successor ordinal $i < h(\bar{0})$, resp. $i < h(\bar{1})$.

We focus on those zero-dimensional LOTS which satisfy the *clopen interval condition* which holds when any two clopen intervals are isomorphic. In particular, we prove the following.

Theorem 1.8. *If T is an additive tree of type $2 = \{0, 1\}$ with height α a tail-like ordinal, such that $h(\bar{0}) = h(\bar{1}) = \alpha$, then the branch space $X(T)$ is a compact, perfect, zero-dimensional LOTS which satisfies the clopen interval condition. In addition, if α is countable, then $X(T)$ is topologically homogeneous.*

In particular, from this we recover results of Maurice [15], [16] on the product space 2^α for α a countable tail-like ordinal.

2. **LOTS and Ordinals**

2.1. **The Category of LOTS.** For a totally ordered set X we will use the usual notation (a, b) for the *open interval* and $[a, b]$ for the *closed interval* with *endpoints* $a \leq b$ in X and we will write $(a, +\infty)$ and $(-\infty, b)$ for unbounded open intervals. An interval is called *proper* when it contains more than one point. A subset A of X is *bounded* when $A \subset [a, b]$ for some $a, b \in X$. So X itself is bounded iff it has a maximum and a minimum (hereafter *max* and *min*). Somewhat abusively, we will call X *unbounded* when it has neither *max* nor *min*.

As usual, we will let $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ stand for the set of reals, rationals, integers and non-negative integers, respectively. In particular, $0 \in \mathbb{N}$. They are all equipped with the usual order.

A linearly ordered topological space (hereafter a LOTS) is a totally ordered set equipped with the order topology. That is, the set of open intervals is a base for the topology. The topology is Hausdorff and the order and topology properties are closely related.

A LOTS X is called *order complete* (hereafter *complete*)when every bounded subset A has a supremum and an infimum (denoted *sup* A

and $inf\ A$), or, equivalently, when every closed bounded interval $[a, b]$ is compact. Thus, a complete LOTS is locally compact. In particular, X is compact iff it is complete and bounded. On the other hand, local compactness is not sufficient for completeness. For example, $X = \mathbb{R} \setminus \mathbb{Z}$ is locally compact, but not complete.

A LOTS X is called *order dense* when between any two points of X there lie other points of X, or, equivalently, when every nonempty open interval in X is infinite. X is connected iff it is complete and order-dense, in which case, every subinterval is connected. If X is not order-dense, then there exists a *gap pair*, $a < b$ in X with $(a, b) = \emptyset$. The point a is then called the *left endpoint* and b is called the *right endpoint* of the pair. By convention the *max* of X, if it exists, is a left endpoint and the *min* is a right endpoint. Thus, $a \in X$ is a left (or right) endpoint iff the closed interval $(-\infty, a]$ (resp. $[a, +\infty)$) is open. A point a is *isolated*, i.e. $\{a\}$ is clopen, iff it is both a left and a right endpoint.

Lemma 2.1. *A subset A of a LOTS X is closed iff for all $B \subset X$, $a = sup(A \cap B)$ or $a = inf(A \cap B)$ implies $a \in A$.*

Proof. Assume the conditions hold and that a is a point of the closure of A. Let $a_1 = sup(A \cap (-\infty, a))$ and $a_2 = inf(A \cap (a, \infty))$. If $a = a_1$ or $a = a_2$, then $a \in A$ by hypothesis. If a is neither of these, then $a \in (a_1, a_2)$ and $A \cap (a_1, a_2) \setminus \{a\} = \emptyset$. So $a \in \overline{A}$ implies $a \in A$.

The converse is clear.

□

A subset A of a LOTS X is *convex* if $a < c < b$ and $a, b \in A$ imply $c \in A$. Intervals are convex subsets and if X is complete then every convex subset is an interval. If a convex set J contains at least three points $a < c < b$, then the nonempty open interval (a, b) is a subset of J and so J has a nonempty interior.

A *Dedekind cut* in X is a partition of X by a pair of nonempty disjoint sets (A_1, A_2) such that for all $a < b$ in X, $b \in A_1$ implies $a \in A_1$ and so $a \in A_2$ implies $b \in A_2$. A *hole* between a and b is a Dedekind cut (A_1, A_2) with $a \in A_1$, $b \in A_2$ such that A_1 and A_2 are clopen. For example, if $a < b$ is a gap pair then $((-\infty, a], [b, +\infty))$ is a hole between a and b. On the other hand, while the LOTS $\mathbb{Q}$ of rational numbers is order dense, every irrational number creates a hole in $\mathbb{Q}$. We say that a LOTS X has *dense holes* if there is a hole between every pair $a < b$ in X. Thus $\mathbb{Q}$ and the Cantor Set have dense holes.

Lemma 2.2. *Let X_1 be a subset of a LOTS X. If X is order dense and the subset X_1 is dense in a convex subset of X, then, regarded as a LOTS in its own right, X_1 is order dense.*

Proof. If X_1 is dense in J, a convex subset of X, and $a < b$ in X_1, then because X is order dense the interval (a, b) in X is infinite. Since J is convex, (a, b) is a subset of J. Between any two points of J there are points of X_1 because X_1 is dense in J. Hence, $(a, b) \cap X_1$ is infinite.

□

The *reverse* of a LOTS X, denoted X^*, is the set equipped with the reverse order. Clearly, the intervals, and so the topology, for X and X^* are the same.

A function $f : X_1 \to X_2$ between LOTS is an *order map* if it is order preserving, i.e. $a \leq b$ implies $f(a) \leq f(b)$, while f is called an *order* map* if it is order reversing, i.e. $f : X_1^* \to X_2$ is an order map or, equivalently, if $f : X_1 \to X_2^*$ is an order map. An injective (or surjective) order map is called an *order injection* (resp. an *order surjection*). A bijective order map is called an *order isomorphism* or just an *isomorphism*. This is the isomorphism concept for the category of LOTS with order maps. We say that order isomorphic LOTS X_1, X_2 have the same *order type* and we write $X_1 \cong X_2$.

While an order isomorphism is a homeomorphism, an order map need not be continuous. In particular, if X_1 is a subset of a LOTS X, then with the induced order X_1 is itself a LOTS. However, the LOTS topology on X_1 need not be the topology induced from X and the inclusion map might not be continuous. For example, consider $X_1 = [-\infty, 0] \cup (1, \infty)$ in $X = \mathbb{R}$. Clearly, the disconnected subset X_1 is order isomorphic to $\mathbb{R}$ itself.

We will call a map $f : X_1 \to X_2$ an *order embedding* if it is an order map which is a topological embedding, i.e. $f : X_1 \to f(X_1)$ is a homeomorphism with the topology on $f(X_1)$ induced from X_2.

Proposition 2.3. *Let $f : X_1 \to X_2$ be an order map.*

- (a) *Assume f is surjective.*
  - (i) *If each point-inverse is closed, then f is continuous. If each point-inverse is compact, then the map f is closed as well as continuous. If each point-inverse is compact and X_1 is complete, then the map f is topologically proper, i.e. the preimage of every compact subset of X_2 is a compact subset of X_1.*
  - (ii) *If X_2 is order dense, then f is continuous.*

(iii) *If X_2 is unbounded, then X_1 is unbounded and the preimage of every bounded subset of X_2 is a bounded subset of X_1.*
(iv) *If X_1 is complete, X_2 is unbounded and f is continuous, then f is topologically proper.*

(b) *Assume f is injective. If f is continuous, then it is an order embedding. This occurs if one of the following holds.*
(i) *The image $f(X_1)$ is convex in X_2.*
(ii) *X_2 is complete and $f(X_1)$ is closed in X_2.*
(iii) *X_2 is order dense and $f(X_1)$ is dense in X_2.*

(c) *Let A be a subset of X_1. If A is bounded in X_1 then the image $f(A)$ is bounded in X_2. If f is continuous and $x = infA$ (or $= supA$) then $f(x) = inff(A)$ (resp. $= supf(A)$) in X_2. Conversely, if for every bounded subset A of X_1 $x = infA$ (or $= supA$) implies $f(x) = inff(A)$ (resp. $= supf(A)$) in X_2, then f is continuous.*

(d) *If f is surjective, then there exists a map $g : X_2 \to X_1$ such that $f \circ g = 1_{X_2}$. Any such map g is a (not usually continuous) order injection.*

(e) *If X_1 is order dense and f is injective on a dense subset D of X_1, then f is injective.*

Proof. (a): If $f(a_1) = a_2$ and $f(b_1) = b_2$ then

$$f^{-1}((a_2, b_2)) = (a_1, b_1) \setminus (f^{-1}(a_2) \cup f^{-1}(b_2)) \tag{2.1}$$

which is open if f has closed point inverses.

Now with f continuous assume that A is a subset of X_1 with y a limit point of $f(A)$ not in $f(A)$. By replacing A by $A \cap f^{-1}((-\infty, y])$ or by $A \cap f^{-1}([y, \infty))$ (which are closed when A is) we may assume that $y = supf(A)$ or $= inff(A)$. Assume the first. Since $f^{-1}(y)$ is compact it has infimum which we denote x. So $a \in A$ implies $f(a) < y = f(x)$ and so $a < x$. For any $z < x$, $f(z) < f(x) = y$ because $x = inff^{-1}(y)$. Since $y = supf(A)$ there exists $a_z \in A$ such that $f(z) < f(a_z) < f(x)$ and so $z < a_z < x$. This means that $x = supA$. Since $x \notin A$, it follows that A is not closed. Contrapositively, when all point inverses are compact, A closed implies that $f(A)$ contains all its limit points and so is closed.

Now assume that all point inverses are compact and X_1 is complete. If B is a compact subset of X_2, then it has a supremum y. Since $f^{-1}(y)$ is compact, it has a supremum x. It is clear that $x = supf^{-1}(B)$. Similarly, $f^{-1}(B)$ has an infimum. Since f is continuous, $f^{-1}(B)$ is closed as well as bounded and so is compact by completeness of X_1.

(ii): If $f(x) \in (a_2, b_2)$ and X_2 is order dense then there exist a_3, b_3 in X_1 such that

$$a_2 < f(a_3) < f(x) < f(b_3) < b_2 \\ \text{and so} \\ x \in (a_3, b_3) \subset f^{-1}((a_2, b_2)). \tag{2.2}$$

Thus, the latter is a neighborhood of x in X.

(iii), (iv): If M is an upper bound for $A \subset X_1$ then $f(M)$ is an upper bound for $f(A)$. In particular, if $M = maxX_1$ then $f(M) = maxX_2$ since f is surjective. So if X_2 is unbounded then X_1 is. Furthermore, if B is bounded above in X_2, then because X_2 has no max and f is surjective, there exists $a \in X_1$ such that $y < f(a)$ for all $y \in B$. Hence, $x < a$ for all $x \in f^{-1}(B)$. If B is compact, then it is closed and bounded in X_2. If f is continuous, then $f^{-1}(B)$ is closed as well as bounded and so is compact if X_1 is complete.

(b): Let $A = f(X_1) \subset X_2$. The order injection f is an order isomorphism of X_1 with A regarded as LOTS and so is a homeomorphism. The problem concerns the comparison between the order topology on A and the topology induced from X_2. If $a, b \in A$ then the interval (a, b) in A is the intersection of A with the corresponding interval in X_2. Hence the topology on A is included in the topology induced from X_2 and the two topologies agree exactly when the inclusion is continuous.

If $a \in A, x \in X_2 \setminus A$ and $a < x$ then $(-\infty, x) \cap A$ is a neighborhood in the induced topology. If x is an upper bound for A then the intersection is A. Otherwise, we require a point $\tilde{a} \in A$ with $a < \tilde{a}$ such that $(-\infty, x) \cap A \supset (-\infty, \tilde{a}) \cap A$. Such a point exists iff the following condition holds:

$$a \in A,\ x \in X_2,\ a < x,\ (a, x] \cap A = \emptyset,\ [x, \infty) \cap A \neq \emptyset \\ \Longrightarrow \\ \exists b \in A \text{ such that } x \leq b \text{ and } (x, b) \cap A = \emptyset. \tag{2.3}$$

This condition and its analogue for the reverse orders are those required for the two topologies to agree.

If A is convex then $(a, x] \cap A = \emptyset$ implies $[x, \infty) \cap A = \emptyset$. If X_2 is order dense and A is dense in X_2 then $a < x$ implies $(a, x) \cap A \neq \emptyset$. So the conditions hold vacuously in these cases.

If X_2 is complete and A is closed in X_2, then $b = inf[x, \infty) \cap A$ is a point of A and $(x, b) \cap A = \emptyset$. So the conditions hold in this case as well.

(c): If x is a lower bound for A, then $f(x)$ is a lower bound for $f(A)$. If $f(x) < a_2$ and a_2 is a lower bound for $f(A)$, then $f(x)$ is not in the closure of $f(A)$ and so by continuity x is not $inf\ A$.

For the converse, if A is a closed subset of X_2 and for $B \subset X_1$ $x = inf(B \cap f^{-1}(A))$ or $x = sup(B \cap f^{-1}(A))$, then, by assumption, $f(x) = inf f(B \cap f^{-1}(A))$ or $f(x) = sup f(B \cap f^{-1}(A))$ and so $f(x)$ is in the closed set A. Hence, $x \in f^{-1}(A)$ and so $f^{-1}(A)$ is closed by Lemma 2.1.

(d): We can define $g(x)$ by choosing any element of $f^{-1}(x)$. Such choices exactly define the functions g such that $f \circ g = 1_X$. In that case, if $g(x_1) \leq g(x)$ then $x_1 = f(g(x_1)) \leq f(g(x_2)) = x_2$. Contrapositively, $x_1 > x_2$ implies $g(x_1) > g(x_2)$.

(e): If $y_1 < y_2$ in X_1, then there exist x_1, x_2 in the dense subset with $y_1 < x_1 < x_2 < y_2$. Then $f(y_1) \leq f(x_1) < f(x_2) \leq f(y_2)$. Thus, f is injective on X_1.

□

Corollary 2.4. *If $f : X_1 \to X$ is an order map with X order dense and $f(X_1)$ is dense in a convex subset of X, then f is continuous.*

Proof. Assume J is a convex subset of X and $f(X_1)$ is dense in J. By Lemma 2.2, regarded as LOTS in their own right, both J and $f(X)$ are order dense. By Proposition 2.3(a)(ii) the order surjection $f : X_1 \to f(X_1)$ is continuous. The inclusions $f(X_1) \to J$ and $J \to X$ are continuous by (b)(iii) and (b)(i), respectively. Hence, the composition $f : X_1 \to X$ is continuous.

□

If I is a LOTS and $\{X_i : i \in I\}$ is a family of nonempty LOTS indexed by I (a *LOTS indexed family*), then we define the *order space sum*:

$$\Sigma_{i \in I} X_i \quad = \bigcup_{i \in I} \{i\} \times X_i \tag{2.4}$$

with $(i, x) < (j, y)$ if $i < j$ or $i = j$ and $x < y$ in X_i. If $I = \{0, 1\}$ then we write $X_0 + X_1$ for the sum.

If $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ are LOTS indexed families of nonempty LOTS and $f : I \to J$ and $\{g_i : X_i \to Y_{f(i)}\}$ are order maps with f injective, then the *sum order map* is the map obtained by *putting together the family* $\{g_i\}$:

$$g = \Sigma_f g_i : \Sigma_{i\in I} X_i \to \Sigma_{j\in J} Y_j \qquad (2.5)$$
$$g(i,x) = (f(i), g_i(x)).$$

This is clearly an order map which is injective/surjective/bijective if f and each g_i satisfies the corresponding property.

The projection map

$$\pi : \Sigma_{i\in I} X_i \to I \qquad (2.6)$$
$$\pi(i,x) = i$$

can be thought of as the special case of putting together the surjections from X_i to the singleton LOTS $\{i\}$. By Proposition 2.3(a), π is continuous if I is order dense, or, more generally, when each X_i is a closed subset of the sum.

Proposition 2.5. *Let* $\{X_i : i \in I\}$ *be a LOTS indexed family of nonempty LOTS and let* $X = \Sigma_{i\in I} X_i$.

(a) *A pair* $(i,x) < (j,y)$ *is a gap pair in* X *iff either* $i = j$ *and* $x < y$ *is a gap pair in* X_i *or* $i < j$ *is a gap pair in* I *and* $x = maxX_i$ *and* $y = minX_j$. *So if* X *is order dense, then each* X_i *is order dense. Conversely, assume that each* X_i *is order dense. If* I *is also order dense or if each* X_i *is unbounded, then* X *is order dense.*

(b) *If* I *is complete and each* X_i *is compact then* X *is complete.*

Proof. (a): Obvious.

(b): If $A \subset X$ is bounded then $\pi(A)$ in I is by Proposition 2.3(c). Let $i = inf\ \pi(A)$. Because X_i is compact, $\pi^{-1}(i) \cap A$ is bounded in X_i. Let x be its inf in X_i. If $\pi^{-1}(i) \cap A = \emptyset$ let $x = max\ X_i$. Clearly, $(i,x) = inf\ A$.

$\square$

When it is identified with $\pi^{-1}(i) = \{i\} \times X_i, X_i$ is a convex subset of $\Sigma_{i\in I} X_i$. By Proposition 2.3(b) the inclusion of X_i into the sum is an embedding.

On the other hand, let $\{X_i : i \in I\}$ be a family of nonempty convex subsets which partition a LOTS X, i.e. an I indexed *convex partition* of X. Observe that if A and B are disjoint convex subsets of X, then $x_1 < y_1$ for some pair $x_1 \in A, y_1 \in B$ implies $x < y$ for all $x \in A, y \in B$. Hence, I is a LOTS with ordering uniquely defined by $i < j$ when $x < y$ for all $x \in X_i, y \in X_j$. A convex partition of X indexed by a LOTS I is equivalent to an order surjection $\pi : X \to I$.

Clearly,we have an order isomorphism:

$$\begin{aligned} \Sigma_{i \in I} X_i \ &\cong \ X \\ (i, x) \ &\mapsto \ x, \end{aligned} \tag{2.7}$$

which we can regard as an identification.

This will allow us to put together a family of order isomorphisms between elements of convex partitions, to obtain an order isomorphism between the partitioned spaces.

2.2. **Ordinal Constructions.** Of special interest are the ordinals. As usual we let $0 = \emptyset$ and define the ordinal α to be the set of ordinals smaller than α, with the ordering by set inclusion. The *successor* $\alpha + 1$ of α is $\alpha \cup \{\alpha\}$. If $A \subset \alpha$ then $inf\ A = \bigcap\ A$ is the first element of A and $sup\ A = \bigcup\ A$. Thus, any ordinal is a complete LOTS.

Any well-ordered set has the order type of an ordinal. If $A \subset \alpha$ then there is a unique order isomorphism of A onto an ordinal $\beta \leq \alpha$. We will usually identify the subset A with the ordinal β whose order type is that of A.

We let ω denote the first infinite ordinal and Ω denote the first uncountable ordinal. We identify the ordinal ω with the set $\mathbb{N}$ by letting n label the n^{th} (finite) ordinal. Thus $n = \{0, 1, \dots, n-1\}$.

For ordinal results we follow Rosenstein [17] and Jech [13]. The arithmetic of ordinals is defined inductively so that $\alpha + \beta, \alpha \cdot \beta$ and α^β are continuous in the β variable.

$$\begin{aligned} \alpha + 0 = \alpha \quad &\text{and} \quad \alpha + (\beta + 1) = (\alpha + \beta) + 1. \\ \alpha \cdot 0 = 0 \quad &\text{and} \quad \alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha. \\ \alpha^0 = 1 \quad &\text{and} \quad \alpha^{(\beta+1)} = (\alpha^\beta) \cdot \alpha. \end{aligned} \tag{2.8}$$

In particular, if α and β are countable ordinals then the results of all of these operations are countable ordinals. We will use the usual order type sloppiness, writing $A + B$ for well-ordered sets A and B to mean the ordinal which is the sum of the ordinals having order types A and B.

Notice that the ordinal sum is a special case of the two term order space sum defined above. Also the product $\alpha \cdot \beta$ is isomorphic to the order space sum of the β indexed family of copies of α.

Ordinal addition and multiplication are associative and by induction on β_2 the following arithmetic identities hold

$$\begin{aligned} \alpha \cdot (\beta_1 + \beta_2) &= \alpha \cdot \beta_1 + \alpha \cdot \beta_2, \\ \alpha^{\beta_1 + \beta_2} &= \alpha^{\beta_1} \cdot \alpha^{\beta_2}, \\ (\alpha^{\beta_1})^{\beta_2} &= \alpha^{\beta_1 \cdot \beta_2}. \end{aligned} \tag{2.9}$$

In addition, we have

$$\begin{aligned} 0 < \alpha, \beta \quad &\Longrightarrow \quad \alpha < \alpha + 1 \leq \alpha + \beta, \\ 0 < \alpha \text{ and } 1 < \beta \quad &\Longrightarrow \quad \alpha < \alpha + \alpha = \alpha \cdot 2 \leq \alpha \cdot \beta, \\ 1 < \alpha, \beta \quad &\Longrightarrow \quad \alpha < \alpha \cdot 2 \leq \alpha \cdot \alpha = \alpha^2 \leq \alpha^\beta, \\ 1 < \alpha \text{ and } \beta_1 < \beta_2 \quad &\Longrightarrow \quad \alpha^{\beta_1} < a^{\beta_1} \cdot \alpha = a^{\beta_1 + 1} \leq a^{\beta_2}. \end{aligned} \tag{2.10}$$

If α is an ordinal and $\beta < \alpha$ then the *tail*

$$\begin{aligned} \alpha \setminus \beta = \{i : \beta \leq i < \alpha\} \subset \alpha, \\ \text{so that} \quad \beta + (\alpha \setminus \beta) = \alpha. \end{aligned} \tag{2.11}$$

As usual, we identify the subset $\alpha \setminus \beta$ with the ordinal having the same order type.

An ordinal α is called *tail-like* if it is positive and all of the tails of α have order type α, i.e. $\alpha \setminus \beta = \alpha$ for all $\beta < \alpha$. Observe that if $\beta < \alpha$, then $\alpha \setminus \beta = \alpha$ is equivalent to $\beta + \alpha = \alpha$. Thus, α is tail-like iff $\beta_1 + \beta_2 < \alpha$ for all $\beta_1, \beta_2 < \alpha$.

We recall from [17] Theorem 3.46 the *Cantor Normal Form Theorem.*

Proposition 2.6. *An ordinal α is tail-like iff $\alpha = \omega^\beta$ for some ordinal β.*

Any positive ordinal α can be written uniquely as the sum

$$\begin{aligned} \alpha = \omega^{\beta_1} + ... + \omega^{\beta_N} \\ \textit{with} \quad \beta_1 \geq ... \geq \beta_N \end{aligned} \tag{2.12}$$

Proof. First observe that if $\gamma < \beta < \alpha$ and $\alpha \setminus \beta = \alpha$, then

$$\alpha = \alpha \setminus \beta \leq \alpha \setminus \gamma \leq \alpha. \tag{2.13}$$

Clearly, $1 = \omega^0$ is tail-like. It is the only tail-like ordinal which is not a limit ordinal.

Inductively, we have, for $\epsilon < \beta$ and $N < \omega$:

$$\begin{aligned} \omega^\epsilon + \omega^\beta = \omega^\epsilon \cdot (1 + \omega^{\beta \setminus \epsilon}) = \omega^\epsilon \cdot \omega^{\beta \setminus \epsilon} = \omega^\beta, \\ \omega^\beta \cdot N + \omega^{\beta+1} = \omega^\beta \cdot (N + \omega) = \omega^\beta \cdot \omega = \omega^{\beta+1} \end{aligned} \tag{2.14}$$

Thus, $\omega^\beta \setminus \omega^\epsilon = \omega^\beta$ and $\omega^{\beta+1} \setminus \omega^\beta \cdot N = \omega^{\beta+1}$. It then follows from (2.13) that ω^β is tail-like for any ordinal β.

Continuity in β implies that the set $\{\beta : \omega^\beta \leq \alpha\}$ is closed and so we can choose ω^{β_1} to be the largest ordinal of this form less than or equal to α. We see that β_1 is the unique ordinal such that $\omega^{\beta_1} \leq \alpha < \omega^{\beta_1+1}$.

We proceed by induction on β_1. Observe that if α satisfies (2.12), then $\omega^{\beta_1} \leq \alpha \leq \omega^{\beta_1} \cdot N < \omega^{\beta_1+1}$. Thus, β_1 is uniquely determined by this inequality.

If $\alpha = \omega^{\beta_1}$, then we have Cantor Normal Form Theorem for α with $N = 1$.

If $\alpha > \omega^{\beta_1}$, then since $\alpha < \omega^{\beta_1+1}$, there exists $k \in \omega$ such that $\alpha < \omega^{\beta_1} \cdot (k+2)$. The minimum such value $k = k(\alpha)$ is uniquely determined by α. We have $\alpha \geq \omega^{\beta_1} \cdot (k+1)$ and $\gamma = \alpha \backslash \omega^{\beta_1} \cdot (k+1) < \omega^{\beta_1}$. Let $\beta_i = \beta_1$ for $1 \leq i \leq k+1$.

Applying the induction hypothesis, let $\gamma = \omega^{\beta_{k+2}} + ... + \omega^{\beta_N}$ be the unique Cantor Normal Form for γ. We have $\omega^{\beta_{k+2}} \leq \gamma < \omega^{\beta_1} = \omega^{\beta_{k+1}}$ and so by (2.10) $\beta_{k+2} < \beta_{k+1}$. Summing the two decompositions we obtain the unique normal form for α.

Finally, observe that if $N > 1$, then by (2.10) $\omega^{\beta_1} < \alpha$. Clearly $\alpha \setminus \omega^{\beta_1}$ is equal to $\omega^{\beta_2} + ... + \omega^{\beta_N}$ and by uniqueness of the Cantor Normal Form this does not equal α. Hence, α is not tail-like.

It follows that the only tail-like ordinals are of the form ω^γ.

□

Corollary 2.7. *An ordinal α is a limit ordinal iff $\alpha = \omega \cdot \beta$ for some positive ordinal β.*

Any infinite ordinal α can be written uniquely as $\alpha = \beta + k$ with β a limit ordinal and $k < \omega$.

Proof. If $\omega^{\beta_1} + ... + \omega^{\beta_N}$ is Cantor Normal Form for α, then $\omega^{\beta_1} + ... + \omega^{\beta_N} + \omega^0$ is Cantor Normal Form for $\alpha + 1$. So we can uniquely write α as

$$\omega \cdot (\omega^{\beta_1 - 1} + ... + \omega^{\beta_{N-k} - 1}) + k \tag{2.15}$$

with $\beta_{N-k} > 0$ and $\beta_i = 0$ for $N - k < i \leq N$. For α an infinite ordinal, e.g. a limit ordinal, $\beta_1 > 0$. It is a limit ordinal iff $k = 0$.

□

A *cardinal* $\aleph$ is the ordinal which is minimum among the ordinals of that cardinality. That is, if $\alpha < \aleph$, then the cardinality of α is strictly less than that of $\aleph$. By mapping $\beta + k$ to $\beta + 2k$ or to $\beta + 2k + 1$ for β any limit ordinal less than $\aleph$, we see that an infinite cardinal can be written as the disjoint union of two sets of the same cardinality. It follows that any infinite cardinal $\aleph$ is tail-like.

The elements of ω are the finite cardinals, and ω, Ω are the infinite cardinals $\aleph_0$ and $\aleph_1$, respectively.

If α is a positive ordinal and $\{X_i : i \in \alpha\}$ is an α indexed family of nonempty LOTS then we define the *order space product* to be the set $\Pi_{i\in\alpha} X_i$ with the lexicographic ordering . That is, for $x \neq y$ in the product

$$x < y \quad \Longleftrightarrow \quad x_\beta < y_\beta \text{ with } \beta = min\{j : x_j \neq y_j\}. \tag{2.16}$$

If $\alpha = 2$, we write $X_0 \times X_1$ for the product.

When $X_i = X$ for all i then we obtain X^α, the space of functions from α to X, as a LOTS. Observe that the LOTS topology is *not* the product topology.

If $X_i = X$ for all $i \in I$ in an I indexed family $\{X_i : i \in I\}$ then the order sum $\Sigma_{i\in I} X_i$ is the product $I \times X$.

If Y is a LOTS then $(\bigcup_{i\in I}\{i\} \times X_i) \times Y = \bigcup_{i\in I}(\{i\} \times X_i) \times Y)$ implies

$$(\Sigma_{i\in I} X_i) \times Y \quad \cong \quad \Sigma_{i\in I}(X_i \times Y). \tag{2.17}$$

Furthermore, if $\{X_i : i \in I\}$ is an arbitrary family of subsets of a LOTS X and Y is a LOTS, then $\{X_i \times Y : i \in I\}$ is a family of subsets of $X \times Y$ and regarded as LOTS in their own right, we have

$$(\bigcup_{i\in I} X_i) \times Y \quad \cong \quad \bigcup_{i\in I}(X_i \times Y). \tag{2.18}$$

In contrast with the sum, the ordinal conventions for products and powers given by (2.8) disagree with these new definitions.

It follows by induction on β using (2.17) and (2.18) that the order space product $\beta \times \alpha$ is the ordinal product $\alpha \cdot \beta$.

The order space power 2^ω is the uncountable space of all zero/one valued sequences. The ordinal 2^ω is the limit of the finite ordinals 2^N and so is just ω.

If $\{X_i : i \in \alpha\}$ is an ordinal indexed family of nonempty LOTS and $0 < \beta \leq \alpha$ then we can write Π_β for the subproduct $\Pi_{i\in\beta} X_i$ and if $0 < \epsilon \leq \beta \leq \alpha$ then we denote by

$$\pi^\beta_\epsilon : \Pi_\beta \longrightarrow \Pi_\epsilon \tag{2.19}$$

the projection map obtained by forgetting the coordinates in $\beta \setminus \epsilon$. Since ϵ is an initial segment of β it is clear that π^β_ϵ is an order surjection. However, it need not be continuous. The first coordinate projection $\omega \times \omega \to \omega$ does not have closed point inverses.

Proposition 2.8. *Let $X = \Pi_{i \in \alpha} X_i$ be an order space product of LOTS.*

(a) *If each X_i is order dense, then X is order dense. In that case, each projection π^β_ϵ for $0 < \epsilon \leq \beta \leq \alpha$ is continuous.*

(b) *If each X_i for $i > 0$ is bounded, then each projection π^β_ϵ is continuous. If, in addition, each X_i is complete, then X is complete.*

Proof. (a) If $x < y$ in X, then with $\beta = min\{j : x_j \neq y_j\}$ we can choose $z_\beta \in X_\beta$ such that $x_\beta < z_\beta < y_\beta$. Define $z_j = x_j$ for all $j \neq \beta$. Then $x < z < y$ in X. The projections are continuous by Proposition 2.3(a)(ii).

(b) Given $a < b$ in Π_ϵ with $\epsilon > 0$ define $a+$ and $b-$ in Π_β by:

$$(a+)_i = \begin{cases} a_i & i \in \epsilon \\ max\ X_i & i \in \beta \setminus \epsilon \end{cases} \tag{2.20}$$

$$(b-)_i = \begin{cases} b_i & i \in \epsilon \\ min\ X_i & i \in \beta \setminus \epsilon \end{cases}$$

Note that $i \in \beta \setminus \epsilon$ is positive and so X_i is bounded. Clearly,

$$\begin{aligned} (\pi^\beta_\epsilon)^{-1}((a,b)) &= (a+, b-), \\ (\pi^\beta_\epsilon)^{-1}([a,b]) &= [a-, b+]. \end{aligned} \tag{2.21}$$

Hence, π^β_ϵ is continuous.

If $A \subset X$ is contained in $[a,b]$ then by replacing X_0 by $[a_0, b_0]$ we can assume that every X_i is bounded. We prove by induction on β that if every X_i is compact, then $A \subset \Pi_\beta$ has an inf. With similar results for sup, completeness follows.

If $\beta = 1$ then $\Pi_\beta = X_0$ which is compact.

Now assume that the result holds for all $\epsilon < \beta$ and let $x^\epsilon = inf\ \pi^\beta_\epsilon(A)$ in Π_ϵ. By Proposition 2.3(c) it is clear that $\delta \leq \epsilon$ implies

$$\pi^\epsilon_\delta(x^\epsilon) = x^\delta. \tag{2.22}$$

Case 1: If β is a limit ordinal then define x^β so that

$$x^\beta_i = x^\epsilon_i \qquad \text{for } i < \epsilon < \beta, \tag{2.23}$$

which is well-defined by (2.22). Clearly, $x^\beta = inf\ A$ in this case.

Case 2: If $\beta = \epsilon + 1$ then Π_β is the order space sum of copies of the compact LOTS X_ϵ indexed by the points of Π_ϵ. By induction hypothesis and Proposition 2.5(b) Π_β is complete. Hence, $inf\ A$ exists in this case as well.

□

We will also use the inverse limit construction under special assumptions which we will refer to as a *special inverse system*. We begin with an ordinal indexed family $\{X_i : i \in \alpha\}$ of connected LOTS. For $i < j < \alpha$ we are given order surjections $p_i^j : X_j \to X_i$. such that

$$p_k^i \circ p_i^j = p_k^j \qquad \text{for } k < i < j < \alpha \tag{2.24}$$

We assume that each p_i^j has compact point inverses and so each p_i^j is continuous and topologically proper by Proposition 2.3(a). If each X_i is unbounded, i.e. has no *max* or *min*, then we call the system *unbounded* and Proposition 2.3(a) implies directly that each p_i^j is continuous and topologically proper, i.e. the compact point inverses condition is automatically satisfied.

We then define the *inverse limit*

$$\overleftarrow{Lim}\{X_i\} \ = \ \{x \in \Pi_{i \in \alpha} X_i : x_i = p_i^j(x_j) \text{ for all } i < j < \alpha\}. \tag{2.25}$$

We denote by $p_j : \overleftarrow{Lim}\{X_i\} \to X_j$ the projection to the j coordinate.

If $x < y$ in $\overleftarrow{Lim}$ then with $\epsilon = min\{j : x_j \neq y_j\}$, $x_\epsilon < y_\epsilon$ and $x_i = y_i$ for all $i < \epsilon$. On the other hand, if $j > \epsilon$ then $x_j < y_j$ because p_ϵ^j is an order map. It follows that each p_j is an order map and we have

$$x < y \iff \exists \epsilon < \alpha \text{ such that} \begin{cases} x_i = y_i & \text{for all } \ i < \epsilon, \\ x_i < y_i & \text{for all } \ i \geq \epsilon. \end{cases} \tag{2.26}$$

Proposition 2.9. *If* $(\{X_i : i \in \alpha\}, \{p_i^j : i < j < \alpha\})$ *is a special inverse system, then the inverse limit* $\overleftarrow{Lim}\{X_i\}$ *is a connected LOTS and each projection* p_j *is a continuous, topologically proper order surjection. If the system is unbounded, then the inverse limit is unbounded. If each* X_i *is compact, then the inverse limit is compact.*

Proof. Given $a \in X_j$ we construct by the usual compactness argument $x \in \overleftarrow{Lim}\{X_i\}$ such that $p_j(x) = a$. We use the topological product topology on $Z = \prod_i X_i$. For $j \leq i < \alpha$ let

$$\begin{aligned} Q_i = \{z \in Z : z_k \in (p_i^k)^{-1}(z_i) \text{ for } k > i, \\ z_k = p_k^i(x_i) \text{ for } k \leq i, \ \ \text{and with } z_j = a\} \end{aligned} \tag{2.27}$$

Because each p_k^i is a continuous, topologically proper map $\{Q_i\}$ is a filterbase of nonempty compact sets in the topological product. Hence, the intersection, which is $(p_j)^{-1}(a)$ is nonempty.

By Proposition 2.3(a) each p_j is continuous and $\overleftarrow{Lim}$ is unbounded if some X_j is unbounded.

If $x < y$ in $\overleftarrow{Lim}$, then let ϵ satisfy (2.26). Because X_ϵ is order dense, we can choose $z_\epsilon \in X_\epsilon$ such that $x_\epsilon < z_\epsilon < y_\epsilon$. Choose $z \in \overleftarrow{Lim}$ such that $p_\epsilon(z) = z_\epsilon$. Because p^ϵ_i is order preserving, $x_i = z_i = y_i$ for all $i < \epsilon$. Hence, $x < z < y$ in $\overleftarrow{Lim}$ and so the inverse limit is order dense.

If $A \subset \overleftarrow{Lim}$, let $A_j = p_j(A)$. Clearly, $i < j$ implies $A_i = p^j_i(A_j)$. If A is bounded in $\overleftarrow{Lim}$ then by Proposition 2.3(c) each A_j is bounded in X_j. If each A_j is bounded and if $x_j = inf A_j$ then $x_i = p^j_i(x_j)$. This defines a point x of $\overleftarrow{Lim}$ which is $inf A$. With a similar argument for the sup we see that $\overleftarrow{Lim}$ is complete and so is connected.

If $C \subset A_j$ is bounded, then let $A = (p_j)^{-1}(C)$ and $A_i = p_i(A)$ for all i. In particular, since p_j is surjective, $A_j = C$, $A_i = p^j_i(C)$ for $i < j$ and $A_i = (p^i_j)^{-1}(C)$ for $i > j$. Thus, each A_j is bounded because every p^i_j is a topologically proper, continuous map. It follows that Hence, $A = (p_j)^{-1}(C)$ is bounded as well as closed and so is compact by completeness. Thus, each p_j is topologically proper.

It follows that if X_j is compact, then the inverse limit $(= (p_j)^{-1}(X_j))$ is compact.

$\square$

Remarks. While it is easy to show that $\overleftarrow{Lim}\{X_i\}$ is a closed subset of the order space product $\Pi_{i \in \alpha} X_i$, the topology on the inverse limit is not the relative topology induced from that product. For example, if $X_0 = X_1 = \mathbb{R}$ and p^1_0 is the identity map then the inverse limit is the diagonal $\{(t,t) : t \in \mathbb{R}\}$ which is a discrete subset of the order space product $\mathbb{R} \times \mathbb{R}$. The topology on the inverse limit is instead the relative topology induced from the topological product space.

Absent the assumption of connectedness, the inverse limit projections need not be continuous. Define the ω indexed inverse limit system by

$$X_n = \{1, \dots, n\} \cup \{\omega\} \quad \text{and } p^{n+1}_n(k) = \begin{cases} k \text{ for } k \leq n, k = \omega, \\ n \text{ for } k = n+1. \end{cases} \tag{2.28}$$

The inverse limit is isomorphic to $\omega + 1$ with $(p_1)^{-1}(1) = \omega$.

Any LOTS X can be regarded as a subset of a smallest complete LOTS $\hat{X}$ called its *completion*, see [17] Theorem 2.32. We will only need $\hat{X}$ in the case when X is order dense.

First, assume that X is unbounded. In that case, define $\hat{X}$ to be the set of open subsets A of X such that $(A, X \setminus A)$ is a Dedekind cut in X, i.e. A is a proper open subset of X and $x \in A$ implies $(-\infty, x) \subset A$. Order $\hat{X}$ by inclusion and identify $x \in X$ with $(-\infty, x) \in \hat{X}$. Since X is unbounded and $A, X \setminus A$ are nonempty, $\hat{X}$ is unbounded. The *sup* of a bounded subset of $\hat{X}$ is its union and the *inf* is the interior of its intersection. If $A_2 < A_1$ in $\hat{X}$, then because A_1 is open, there exist $x < y$ in $A_1 \setminus A_2$. So $A_2 < (-\infty, y) < A_1$ because $x \in (-\infty, y) \setminus A_2$ and $y \in A_1 \setminus (-\infty, y)$. It follows that $\hat{X}$ is order dense, and so is connected, and that X is dense in $\hat{X}$.

If X has a *max* or *min* then we obtain $\hat{X}$ by removing such endpoints, completing, and then reattaching them. If M is $maxX$ then we regard M as $max\hat{X}$ and if $m = minX$ then we regard m as $min\hat{X}$.

Let Y be a complete LOTS and $f : X \to Y$ be an order map with X order dense. Define $\hat{f} : \hat{X} \to Y$ so that for $\hat{x} \in \hat{X}$

$$\hat{f}(\hat{x}) \quad = \quad sup\{f(x) : x \in X \cap (-\infty, \hat{x}]\}. \tag{2.29}$$

Clearly, $\hat{f}$ is an order map which extends f.

Proposition 2.10. *Assume Y is a complete LOTS and $f : X \to Y$ is an order map with X order dense.*

(a) *If f is injective, then $\hat{f}$ is injective.*
(b) *If $f(X)$ is dense in a connected subset of Y, then the map f and its extension $\hat{f}$ are continuous order maps with $\hat{f}(\hat{X})$ connected.*
(c) *Assume Y is connected and $f(X)$ is dense in Y. If Y is unbounded, then $\hat{f}$ is surjective and X is unbounded. More generally, if*

$$\begin{aligned} max\ Y\ exists &\implies max\ X\ exists\ and \\ min\ Y\ exists &\implies min\ X\ exists, \end{aligned} \tag{2.30}$$

then $\hat{f}$ is surjective.

In particular, if X is a dense subset of a connected LOTS Y, then X is order dense. If Y is unbounded or (2.30) holds, then $\hat{X} \cong Y$.

Proof. (a): This follows from Proposition 2.3(e).

(b): Let Y_0 be a connected subset of Y in which $f(X)$ is dense. Since the closure of a connected set is connected, we may assume that Y_0 is closed and so $Y_0 = \overline{f(X)}$. Clearly, $f(X) \subset \hat{f}(\hat{X}) \subset Y_0$. It then follows from Corollary 2.4 that $f : X \to Y_0$ and $\hat{f} : \hat{X} \to Y_0$ are continuous maps. By Proposition 2.3(b)(ii) the inclusion of Y_0 into Y is continuous.

Composing we see that f and $\hat{f}$ are continuous. Since $\hat{X}$ is connected, its image is connected.

(c): By (b) $\hat{f}$ is continuous and the subset $\hat{f}(\hat{X})$ is connected and dense in Y. If Y is unbounded, $\hat{f}(\hat{X}) = Y$ and so $\hat{f}$ is surjective. Also X is unbounded by Proposition 2.3(c). If $maxX$ exists, then by Proposition 2.3(c) again $\hat{f}(maxX) = maxY$ and similarly for the minimum. Thus, (2.30) implies that $\hat{f}(\hat{X}) \subset Y$ is dense and connected and contains $maxY$ or $minY$ when they exist. Hence, $\hat{f}(\hat{X}) = Y$.

In particular, if f is injective and (2.30) holds, then from (a) it follows that $\hat{f}$ is bijective and so is an order isomorphism.

□

A bounded, convex subset J in a complete space is an interval with endpoints $infJ$ and $supJ$. With $X \subset \hat{X}$ as above, a bounded, convex subset $J \subset X$ is equal to $\hat{J} \cap X$ where $z \in \hat{J}$ iff there exist $x_1, x_2 \in J$ such that $x_1 \leq z \leq x_2$. It follows that a convex set in X is the intersection of an interval of $\hat{X}$ with X.

2.3. **Countability Conditions.** A topological space X is *separable* if it has a countable dense subset. X satisfies the *countable chain condition*, hereafter denoted *c.c.c*, if any collection of pairwise disjoint, nonempty open subsets is countable. A LOTS X satisfies c.c.c. if any collection of pairwise disjoint, nonempty open subintervals is countable because the subintervals form a base for the topology.

A subset A of a LOTS X is *cofinal* if for any $x \in X$ there exists $a \in A$ such that $x \leq a$. If $M = maxX$ exists then A is cofinal iff $M \in A$. If X has no *max* then any dense subset of X is cofinal. A is *coinitial* if it is cofinal for the reverse X^*. A is $\pm$*cofinal* if it is both cofinal and coinitial. X is called σ-*bounded* if it admits countable $\pm$cofinal subsets. If X is complete then it is σ-bounded iff it is σ-compact.

A point x in a LOTS X has a countable neighborhood base iff the interval $(-\infty, x)$ has a countable cofinal subset and the interval $(x, +\infty)$ has a countable coinitial subset. Thus, X is first countable iff for every point $x \in X, x$ is either a right endpoint or the limit of an increasing sequence, and x is either a left endpoint or the limit of a decreasing sequence. Equivalently, X is first countable iff every bounded interval in X is a σ-bounded LOTS in its own right.

We say that a LOTS X has *countable type* if every convex subset of X is a σ-bounded LOTS. It clearly suffices that every open, convex subset of X be σ-bounded.

Proposition 2.11. *Let X be a LOTS.*

(a) *X satisfies c.c.c. iff there does not exist an order injection into X of a LOTS with uncountably many isolated points.*

(b) *X is of countable type iff there does not exist an injective order map or order* map of the first uncountable ordinal Ω into X.*

(c) *If X satisfies c.c.c., then it has only countably many isolated points.*

(d) *If X is of countable type, then it is first countable and σ-bounded. If X is complete, first countable and σ-bounded then it is of countable type.*

(e) *If X is separable, then it satisfies c.c.c. If X satisfies c.c.c., then it is of countable type.*

(f) *Let $f : X_1 \to X$ be an order injection. If X is separable, satisfies c.c.c. or is of countable type, then X_1 satisfies the corresponding property.*

(g) *Let $f : X \to X_1$ be an order surjection. If X is separable, satisfies c.c.c. or is of countable type, then X_1 satisfies the corresponding property.*

(h) *Let $f : X \to X_1$ be an order map with image $f(X)$ dense in X_1. If X is of countable type, then X_1 is. If X_1 has only countably many isolated points and X is separable or satisfies c.c.c., then X_1 satisfies the corresponding property. If f is continuous and X is separable or satisfies c.c.c., then X_1 satisfies the corresponding property.*

Proof. (a): Suppose $f : A \to X$ is an order injection and with $I \subset A$ defined to be the set of isolated points in A, excluding the max and min if any, suppose that I is uncountable. For each $x \in I$ there are unique $x-, x+ \in A$ such that $x- < x < x+$ and x is the only point in the interval $(x-, x+)$. By Zorn's Lemma we can choose $\tilde{I} \subset I$ maximal with respect to the property that $x \in \tilde{I} \Rightarrow x-, x+ \not\in \tilde{I}$. By maximality $x \in I$ implies that either $x, x-$ or $x+ \in \tilde{I}$. Hence, $\tilde{I}$ is uncountable. If $x_1 < x_2$ in $\tilde{I}$, then $x_1+ \leq x_2$ and the inequality is strict because $x_1+ \not\in \tilde{I}$. Hence, $x_1+ \leq x_2-$. It follows that $\{(f(x-), f(x+)) : i \in \tilde{I}\}$ is an uncountable family of pairwise disjoint, nonempty open intervals in X. Thus, X does not satisfy c.c.c.

Now assume that X does not satisfy c.c.c. If X has uncountably many isolated points, then the identity on X is the required order injection. So we can assume that X itself has only countably many isolated points and that $\{J_i\}$ is an uncountable family of pairwise disjoint open subintervals in X. If J_i is finite, then it consists of isolated points and there are only countably many such. Thus, we can assume that each J_i is infinite. For each i choose points $x_i- < x_i < x_i+$ in J_i. The collection of all these points is a subset A of X and the inclusion $f : A \to X$ is a, not necessarily continuous, order injection of LOTS. In A each x_i is an isolated point and so A has uncountably many isolated points.

(b): If $f : \Omega \to X$ is an order injection, then $J = \{x : x < f(i)$ for some $i \in \Omega\}$ is a nonempty, open convex subset of X and if A is a countable subset of J, then we can choose for each $a \in A, i(a) \in \Omega$ such that $a < f(i(a))$. Let $j \in \Omega$ with $j > sup\ \{i(a) : a \in A\}$. Then $f(j) > a$ for all $a \in A$ and $f(j) < f(j+1)$ so that $f(j) \in J$. Thus, A is not cofinal in J.

Conversely, if J is a nonempty, open convex subset of X with no countable cofinal subset, then we can construct an order injection $f : \Omega \to J$ inductively by choosing $f(j) \in J$ larger than all the $f(i)$ previously chosen for $i < j$.

Similarly, every nonempty, open convex subset of X has countable coinitial subsets iff there does not exist an order injection of the reverse Ω^* into X.

(c): This follows from (a) or directly because the isolated points form a pairwise disjoint collection of open singletons.

(d): If X is of countable type, then X is a convex subset and so is σ-bounded. Every interval is σ-bounded and so X is first countable. Conversely, if X is first countable and complete then every bounded convex set is an interval and so is σ-bounded by first countability. If, in addition, X is σ-bounded, then every interval is σ-bounded and so X is of countable type.

(e): Disjoint nonempty open sets contain distinct elements of any dense subset. Hence separability implies c.c.c. Ω and Ω^* contain uncountably many isolated points and so c.c.c. implies countable type by (a) and (b).

(f): If $g : A \to X_1$ is an order injection, then $f \circ g : A \to X$ is an order injection. By (a) and (b) X is not c.c.c. or of countable type if X_1 is not.

Now assume that D is a countable dense subset of X. For every pair $d_1 < d_2$ in D with $f(X_1) \cap (d_1, d_2) \neq \emptyset$ choose a point $x \in X$ such that $d_1 < f(x) < d_2$. The collection of such points is a countable subset D_1 of X_1. Since X is separable it satisfies c.c.c. and so X_1 satisfies c.c.c. by what we have already shown. Thus, the set D_2 of isolated points of X_1 is countable by (c). We show that the countable set $D_1 \cup D_2$ is dense in X_1. Let $a < b$ in X_1. If the interval $(a, b) \subset X_1$ is finite but nonempty, then it consists of isolated points and so meets D_2. If it is infinite then we can choose $c_1 < c_2 < c_3$ in (a, b). The open subintervals $(f(a), f(c_2))$ and $(f(c_2), f(b))$ contain $f(c_1)$ and $f(c_3)$ respectively and so they meet D. That is, there exist $d_1, d_2 \in D$ such that $f(a) < d_1 < f(c_2) < d_2 < f(b)$. By definition there exists $x \in D_1$ such that $f(x) \in (d_1, d_2)$. Hence, $a < x < b$ and so (a, b) meets D_1.

(g): Choose for each $x \in X_1$ a point $g(x) \in f^{-1}(x) \subset X$, which is nonempty since f is surjective. We obtain an order injection $g : X_1 \to X$ and so the results follow from (f) applied to g.

(h): Suppose that $\tilde{g} : \Omega \to X$ is an order injection. We will define, by induction, an order injection $g : \Omega \to X_1$ such that

$$\tilde{g}(i) < f \circ g(i) < \tilde{g}(i+2) \quad \text{for all } i \in \Omega. \tag{2.31}$$

For $i = 0$ or i a limit ordinal, $\tilde{g}(i+1)$ in the interval $(\tilde{g}(i), \tilde{g}(i+2))$ implies that the dense set $f(X_1)$ meets the interval and so we can choose $g(i)$ so that $f(g(i))$ is in the interval. If $i = \beta + 1$ and g has been defined through β, then

$$f \circ g(\beta) < \tilde{g}(\beta + 2) = \tilde{g}(i+1) \tag{2.32}$$

implies that $\tilde{g}(i+1)$ lies in the interval between $max\ (\tilde{g}((i), f \circ g(\beta))$ and $\tilde{g}(i+2)$. Choose $g(i)$ so that $f(g(i))$ lies in this interval. With a similar argument for Ω^* we use (b) to see that if X is not of countable type then X_1 is not of countable type.

Now assume that X_1 has only countably many isolated points and let D_2 denote the set of isolated points of X_1 together with its max and min if any. So D_2 is countable by assumption.

If X_1 does not satisfy c.c.c., then there exists an uncountable family $\{J_i\}$ of pairwise disjoint, nonempty open intervals in X_1. Since D_2 is countable we can assume that $J_i \cap D_2 = \emptyset$ for all i and so each J_i is infinite. In J_i choose $y_1 < ... < y_4$. Because $f(X)$ is dense there exist x_1, x_2, x_3 in X such that

$$y_1 < f(x_1) < y_2 < f(x_2) < y_3 < f(x_3) < y_4. \tag{2.33}$$

Hence, (x_1, x_3) is a nonempty open interval contained in $f^{-1}(J_i)$. Thus, we have constructed a pairwise disjoint,uncountable family of nonempty open intervals in X. Thus, X does not satisfy c.c.c.

Now assume that X contains a countable dense subset D. I claim that $D_2 \cup f(D)$ is dense in X_1, which will prove that X_1 is separable.

If J is a nonempty open subinterval of X_1 and J is finite, then J meets D_2. Choose $y_1 < ... < y_4$ in J as before and get x_1, x_2, x_3 in X which satisfies (2.33). The interval (x_1, x_3) in X is nonempty and so meets D. Thus, J meets $f(D)$.

If f is continuous and D is dense in X, then $f(D)$ is dense in $f(X)$ and hence in X_1. If $\{O_1\}$ is a pairwise disjoint family of nonempty open subsets of X_1 then $\{f^{-1}(O_i\}$ is such a family in X. So if f is continuous with a dense image then X_1 is separable or c.c.c. if X is. This does not even require that f be order preserving.

□

Remark. Here is an example which illustrates why the extra condition is required in part (h). The order space product $X = \mathbb{R} \times \{-1, +1\}$ is separable but the points $(t, 0)$ in $X_1 = \mathbb{R} \times \{-1, 0, +1\}$ are all isolated. Mapping $(t, -1)$ to $(t, 0)$ and $(t, +1)$ to $(t, +1)$ for all $t \in \mathbb{R}$ we obtain a discontinuous order injection $f : X \to X_1$ with a dense image.

Corollary 2.12. *If $f : \Omega \to X$ is an order map or an order* map and X is of countable type, then f is eventually constant. That is, there exists $\alpha \in \Omega$ such that $f(i) = f(\alpha)$ for all $i \in \Omega$ with $i \geq \alpha$.*

Proof. If f is not eventually constant, then we can define an order map $q : \Omega \to \Omega$ so that $f \circ g$ is injective. We consider the case when f is order preserving. Define $q(0) = 0$. If q has been defined for all $j < i$, then let $i^* = sup\{q(i) : i < j\}$ and define $q(i) = min\{k \in \Omega : f(k) > f(i^*)\}$. This set is nonempty because f is not eventually constant. Clearly, $f \circ g$ is strictly increasing and so is an order injection. Since X is of countable type, this contradicts Proposition 2.11(b).

□

Proposition 2.13. *Let $\{X_i : i \in I\}$ be a LOTS indexed family of nonempty LOTS.*

(a) *The order space sum $\Sigma_{i \in I} X_i$ is of countable type iff I and each X_i is of countable type.*
(b) *Assume $I = \alpha$ is a positive ordinal. If α is countable, then the order space product $\Pi_{i \in I} X_i$ is of countable type iff each X_i is of countable type. If each X_i is nontrivial and $\Pi_{i \in I} X_i$ is of countable type then α is countable.*
(c) *Assume $I = \alpha$ is a positive ordinal and that $(\{X_i : i \in \alpha\}, \{p_i^j : i < j < \alpha\})$ is a special inverse family. If α is countable and each X_i is of countable type, then the inverse limit $\overleftarrow{Lim}\{X_i\}$ is of countable type. If $\overleftarrow{Lim}\{X_i\}$ is of countable type, then each X_i is of countable type.*

Proof. (a): Each X_i is a convex subset of Σ and the sum admits an order surjection $\pi : \Sigma \to I$. So if the sum is of countable type, I and each X_i is of countable type by Proposition 2.11(f),(g).

Conversely, assume that I and each X_i is of countable type. Suppose that $f : \Omega \to \Sigma$ is an order map. By Corollary 2.12 the order map $\pi \circ f$ is eventually constant, i.e. for some $\beta \in \Omega$, $\pi \circ f$ takes the constant value $i \in I$ on the tail $\Omega \setminus \beta$. Then on the tail $\Omega \setminus \beta$, f itself maps into X_i, regarded as a subset of the sum. Because $\Omega \setminus \beta \cong \Omega$ Corollary 2.12 applied to X_i implies that f is eventually constant. Applying a similar argument to the reverse Ω^* it follows from Proposition 2.11(b) that Σ is of countable type.

(b): Assume that α is a countable ordinal. We prove the result by induction on α. If $\alpha = 1$, then the product Π is X_0 and so the product is of countable type iff X_0 is.

Now assume, inductively, that the equivalence holds for all $\beta < \alpha$.

Case 1: If $\alpha = \beta + 1$, then $\Pi_{i\in\alpha} X_i$ is order isomorphic to the order sum indexed by $\Pi_{i\in\beta} X_i$ with each term a copy of X_β. So by part (a) $\Pi_{i\in\alpha} X_i$ is of countable type iff X_β and $\Pi_{i\in\beta} X_i$ are. By the inductive hypothesis this holds iff X_i is of countable type for all $i \in \alpha = \beta \cup \{\beta\}$.

Case 2: Assume now that α is a limit ordinal.

If $\Pi_{i\in\alpha} X_i$ is of countable type, then by Proposition 2.11(g) applied to the order surjections $\pi^\beta_\alpha : \Pi_{i\in\alpha} X_i \to \Pi_{i\in\beta} X_i$ the latter are all of countable type. By inductive hypothesis this implies that X_i is of countable type for all $i < \beta < \alpha$ and so for all $i \in \alpha$ since α is a limit ordinal.

On the other hand, if each X_i is of countable type and $f : \Omega \to \Pi_{i\in\alpha} X_i$ is an order map, then for each $\beta < \alpha$ we can apply the inductive hypothesis and Corollary 2.12 to see that each $\pi^\beta_\alpha \circ f : \Omega \to \Pi_{i\in\beta} X_i$ is eventually constant, i.e. there exists $\epsilon(\beta)$ such that $\pi^\beta_\alpha \circ f$ is constant on the tail $\Omega \setminus \epsilon(\beta)$. Because α is a countable ordinal $\epsilon(\alpha) = sup\{\epsilon(\beta) : \beta < \alpha\}$ is a countable ordinal. Clearly, f is constant on the tail $\Omega \setminus \epsilon(\alpha)$. Together with a similar argument for Ω^*, this shows that $\Pi_{i\in\alpha} X_i$ is of countable type.

This completes the induction for the case when α is assumed to be countable.

Finally, suppose that each X_i is nontrivial. There exist $a, b \in \Pi_{i\in\alpha} X_i$ such that $a_i < b_i$ for all $i \in \alpha$. Define $f : \alpha + 1 \to \Pi_{i\in\alpha} X_i$ by

$$f(\beta)_i = \begin{cases} b_i & i < \beta \\ a_i & \beta \le i < \alpha. \end{cases} \tag{2.34}$$

f is an order injection and if α is uncountable then it restricts to an order injection of Ω into Π.

(c): If $\overleftarrow{Lim}$ is of countable type, then by Proposition 2.11(g) applied to the surjection $p_j : \overleftarrow{Lim} \to X_j$, each X_j is of countable type.

Now assume that α is countable and each X_j is of countable type. If $f : \Omega \to \overleftarrow{Lim}$ is an order map or an order* map, then by Corollary 2.12 each $p_j \circ f$ is eventually constant. That is, there exist $\epsilon(j) \in \Omega$ and $x_j \in X_j$ such that $p_j f(\beta) = x_j$ for all $\beta \in \Omega \setminus \epsilon(j)$. Since $i < j$ implies $p_i^j \circ p_j = p_i$, it follows that $p_i^j(x_j) = x_i$. Hence, there is a unique $x \in \overleftarrow{Lim}$ such that $p_j(x) = x_j$ for all $j \in \Omega$. Since α is countable there exists $\epsilon \in \Omega$ such that $\epsilon > \epsilon(j)$ for all $j \in \alpha$. Clearly, $f(\beta) = x$ for all $\beta \in \Omega \setminus \epsilon$. By Proposition 2.11(b), $\overleftarrow{Lim}$ is of countable type.

□

Proposition 2.14. *Let $\{X_i : i \in \alpha\}$ be an ordinal indexed family of nonempty LOTS with α countable. If each X_i is first countable, then $X = \Pi_{i\in\alpha} X_i$ is first countable and order dense.*

Proof. For $x \in X$ with $x \neq maxX$, we show that x is the limit of a decreasing sequence in X.

Let $K = \{i < \alpha : x_i \neq maxX_i\}$. Since x is not a maximum point, $K \subset \alpha$ is nonempty and so is isomorphic to an ordinal γ.

Case 1: $\gamma = \beta + 1$. If x_β a left endpoint of a gap with associated right end-point a, then with z $z_i = x_i$ for $i \neq \beta$ and $z_\beta = a$, the pair $x < z$ is a gap pair.

Case 2: $\gamma = \beta + 1$. If x_β not the left endpoint of a gap, then we can choose $\{a^n\}$ be a sequence in X_β decreasing and with limit x_β. Define $y_i^n = x_i$ for $i \neq \beta$ and $y_\beta^n = a^n$.

Case 3: γ is a limit ordinal. Choose β_n an increasing sequence in K converging to γ. For each β_n choose $a^n \in X_{\beta_n}$ with $x_{\beta_n} < a^n$. Define $y_i^n = x_i$ for $i \neq \beta_n$ and $y_{\beta_n}^n = a^n$.

In cases 2 and 3, $\{y^n\}$ is a decreasing sequence in X converging to x.

Similarly, for $x \neq minX$, we can construct an increasing sequence in X converging to x or find a left endpoint for a gap pair with x on the right.

□

Proposition 2.15. *Let X be an unbounded, order dense LOTS. Let $\hat{X}$ be the completion of X and let $a < b$ be points in X.*

(a) *If A is a countable LOTS, then there exists an order injection $f : A \to (a,b)$. If X is countable and A is order dense, then f can be chosen to be an order isomorphism.*
(b) *There exists a continuous order map $f : [a,b] \to I$ where I is the unit interval in $\mathbb{R}$ and with $f((a,b))$ dense in I. If X is complete, then f is surjective. If X is complete and separable, then f can be chosen to be an isomorphism.*
(c) *If α is a positive ordinal and $f : \alpha + 1 \to X$ is an order embedding with $f(0) = a$ and $f(\alpha) = b$, then $\{[f(i), f(i+1)) : i \in \alpha\}$ is a convex partition of $[a,b)$ and so we can identify*

$$[a,b) \cong \Sigma_{i\in\alpha}\ [f(i), f(i+1)) \tag{2.35}$$

expressing $[a,b)$ as the order sum of an α indexed family of intervals.
(d) *If X is first countable and α is a positive, countable ordinal, then there exists an order embedding $f : \alpha+1 \to X$ with $f(0) = a$ and $f(\alpha) = b$.*
(e) *The following conditions are equivalent.*
  (i) *X is of countable type.*
  (ii) *$\hat{X}$ is of countable type.*
  (iii) *$\hat{X}$ is first countable and σ-bounded.*
  (iv) *There does not exist $f : \Omega \to \hat{X}$ a continuous injective map which is either order preserving or order reversing.*

Proof. (a): This is a standard inductive argument using a counting of the points of A. If A is order dense and X, too, is countable, then one counts X as well and proceeds back and forth between A and X to build the isomorphism.

(b): With A the set of rationals in I, use (a) to get an order injection $g : A \to [a,b]$ such that $g(0) = a$ and $g(1) = b$. Define for $x \in [a,b]$

$$f(x) \ = \ sup\ g^{-1}([a,x]) \ = \ inf\ g^{-1}([x,b]). \tag{2.36}$$

By Proposition 2.3(a), f is continuous with $f(g(c)) = c$ for $c \in A$.

If X is complete, then it is connected and so the connected dense image is I.

If X is separable, choose g with a dense image in $[a,b]$, then f is injective by Proposition 2.3 (e).

(c): If $x \in [a,b)$, then $b = f(\alpha) > x$. Let $\beta = min\{j \in \alpha+1 : f(j) > x\}$. Since $a = f(0) \leq x$, β is positive and by continuity of f, β is not a limit ordinal. Hence, $\beta = i+1$ for some $i \in \alpha$ and $x \in [f(i), f(i+1))$.

(d): We construct the embedding f by induction on α. If $\alpha = 1$, let $f(0) = a$ and $f(1) = b$. Now assume the result is true for all $\beta < \alpha$.

Case 1: If $\alpha = \beta + 1$, then choose $\tilde{b}$ so that $a < \tilde{b} < b$. By inductive hypothesis there exists an order embedding $\tilde{f} : \beta + 1 \to [a, \tilde{b}]$ with $\tilde{f}(0) = a$ and $\tilde{f}(\beta) = \tilde{b}$. Extend the definition by $f(\alpha) = b$ to get f.

Case 2: If α is a limit ordinal, then because it is countable, there exists an increasing cofinal sequence $\{\beta_n\}$ in α. Because X is first countable and order dense there exists an increasing sequence $\{x_n\}$ in (a, b) with limit b. By inductive hypothesis there exists an order embedding of the interval $(\beta_n, \beta_{n+1}]$ in α to $(x_n, x_{n+1}]$ with β_{n+1} mapped to x_{n+1}. Put these together and map 0 to a and α to b to get f.

(e): (1)$\Leftrightarrow$(ii) by Proposition 2.11(f)(h) since the inclusion of X into $\hat{X}$ is continuous by Proposition 2.3(a).

(ii)$\Leftrightarrow$(iii) by Proposition 2.11(d).

(ii)$\Rightarrow$(iv) by Proposition 2.11(b).

(iv)$\Rightarrow$(ii) If $\hat{X}$ is not of countable type, then by Proposition 2.11(b) there exists an injective map $\tilde{f} : \Omega \to \hat{X}$ which is either order preserving or order reversing. Without loss of generality assume that $\tilde{f}$ is an order map. Define $f : \Omega \to \hat{X}$ by

$$f(\beta) = \begin{cases} sup\ \{\tilde{f}(i) : i < \beta\} & \beta \text{ is a limit ordinal} \\ \tilde{f}(\beta) & \text{otherwise.} \end{cases} \tag{2.37}$$

It is easy to check that f is a continuous, injective order map.

□

3. **Complete Homogeneous LOTS**

3.1. **Doubly Transitive and Homogeneous LOTS.** If a group G acts on a set S, then we say that s_1 and s_2 in S are G *equivalent* if $g(s_1) = s_2$ for some $g \in G$. We say that G *acts transitively* on S when all points are G equivalent.

For a topological space X we let $H(X)$ denote the automorphism group of X, i.e. the group of homeomorphisms from X to itself. We call X *topologically homogeneous* if $H(X)$ acts transitively on X. If X is a LOTS, then any order automorphism of X, i.e. order preserving bijection of X to itself, is a homeomorphism. We denote by $H_+(X)$ the subgroup of order automorphisms and by $H_\pm(X)$ the subgroup of bijections which either preserve or reverse order. If X admits an order reversing homeomorphism then we call X a *symmetric* LOTS in which case $H_+(X)$ is a subgroup of $H_\pm(X)$ of index 2. Otherwise, $H_+(X) = H_\pm(X)$.

We call a LOTS X *±transitive* if $H_\pm(X)$ acts transitively on X and *transitive* if $H_+(X)$ acts transitively on X. X is *doubly transitive* if $H_+(X)$ acts transitively, via the diagonal action, on the set $\{(x_1, x_2) : x_1 < x_2\} \subset X \times X$. Since any order automorphism of X is also an order automorphism of the reverse X^*, i.e.$H_+(X) = H_+(X^*)$, the reverse LOTS X^* is ±transitive, transitive or doubly transitive when the corresponding property holds for X.

Lemma 3.1. *If X_1 and X_2 are LOTS with X_1 connected and $f : X_1 \to X_2$ is a continuous map then the image $f(X_1)$ is a convex subset of X_2. If, in addition f is injective, then it is either order preserving or order reversing.*

Proof. The image of f is connected by continuity. Any connected subset of a LOTS is convex.

Assume now that f is injective. If $f(a) < f(c) < f(b)$, then the image of the open interval between a and b is connected and so contains $f(c)$. Since f is injective, c must therefore lie in the interval. That is, either $a < c < b$ or $a > c > b$. Thus, on each triple of points in X_1 f either preserves or reverses order. If f preserves the order of some pair a, b in X_1 then it preserves the order of every triple which includes a, b and so of every pair which includes either a or b. So it preserves every triple which includes a and so preserves every pair. The remaining possibility is that f reverses every pair.

□

Proposition 3.2. *Let X be a LOTS with at least three points.*

(a) *If X is connected, then $H(X) = H_\pm(X)$ and any order reversing homeomorphism has a unique fixed point. In addition, X is transitive if it is topologically homogeneous.*

(b) *If X is ±transitive, then it is unbounded. If X is transitive, then it is ±transitive.*

(c) *X is transitive iff for all $a, b \in X$*

$$(a, \infty) \cong (b, \infty) \qquad \text{and} \qquad (-\infty, a) \cong (-\infty, b). \tag{3.1}$$

If X is transitive, then it is either discrete, i.e. every point is isolated, or it is order dense. If X is transitive and complete, then X is first countable and it is either order isomorphic to $\mathbb{Z}$, the LOTS of integers, or it is connected.

(d) *The following conditions are equivalent.*

(i) *X is doubly transitive.*
(ii) *X and every nonempty open subinterval of X are transitive.*
(iii) *X has no min and (a, ∞) is transitive for every $a \in X$.*
(iv) *X has no max and $(-\infty, a)$ is transitive for every $a \in X$.*
(v) *X is transitive and (a, ∞) is transitive for some $a \in X$.*
(vi) *X is transitive and $(-\infty, a)$ is transitive for some $a \in X$.*
(vii) *X is unbounded and any two nontrivial, closed, bounded subintervals of X are order isomorphic.*
(viii) *For every positive integer n, $H_+(X)$ acts transitively on*

$$\{(x_1, ..., x_n) : x_1 < ... < x_n\} \subset X^n.$$

(e) *If X is doubly transitive, then it is order dense and unbounded and every nonempty convex open subset is doubly transitive.*

Proof. (a): $H(X) = H_\pm(X)$ by Lemma 3.1. If f is order reversing on X and $y = f(x) > x$, then $f(y) < f(x) = y$. Since f is not the identity $\{x : f(x) > x\}$ and $\{x : f(x) < x\}$ are disjoint nonempty open subsets of X. Because X is connected their union is a proper subset and so some fixed point e exists. If $x > e$, then $f(x) < f(e) = e$ and so x is not a fixed point. Similarly, if $x < e$. Thus, e is the unique fixed point.

If $H(X)$ acts transitively, then $H(X) = H_\pm(X)$ implies X is $\pm$transitive. If $H_+ = H_\pm$, then X is transitive. On the other hand, if X is symmetric, let f_0 be an order reversing homeomorphism with fixed point e. If $x \in X$, then there exists $f \in H_\pm$ such that $f(x) = e$. Also, $f_0 \circ f(x) = e$ and either f or $f_0 \circ f$ is in $H_+(X)$. Thus, every $x \in X$ is $H_+(X)$ equivalent to e and so X is transitive.

(b): If X has a *max*, then the *max* is fixed by any element of $H_+(X)$ and is mapped to *min* by any order reversing isomorphism. Hence, if X is $\pm$transitive and has at least three points, then it is unbounded.

Since $H_+ \subset H_\pm$, transitivity implies $\pm$transitivity.

(c): If $f \in H_+(X)$ maps a to b, then it restricts to an order isomorphism of (a, ∞) with (b, ∞) and of $(-\infty, a)$ with $(-\infty, b)$. Conversely, we can put together isomorphisms $(a, \infty) \cong (b, \infty)$ and $(-\infty, a) \cong (-\infty, b)$ to get an automorphism which maps a to b.

Now assume that X is transitive but not order dense. There exists a gap pair with $a < b$ left and right endpoints, respectively. As all points of X are H_+ equivalent all points are both left and right endpoints. That is, every point of X is isolated. If, in addition, X is complete, choose $f(0) \in X$ and inductively define $f(n+1) = inf(f(n), \infty)$ and $f(-n-1) = sup(-\infty, f(-n))$ for every nonnegative integer n. This defines an order injection $f : \mathbb{Z} \to X$. Since X is discrete, $f(\mathbb{Z})$ has no *sup* or *inf* and so is $\pm$cofinal in X. For $x \in X$, $f(n) < x$ for some $n \in \mathbb{Z}$ but not for all. If n is the largest such, then $f(n+1) \leq x$ since the interval $(f(n), f(n+1))$ is empty. By maximality of n, $x = f(n+1)$. Hence, f is surjective and so is an order isomorphism of $\mathbb{Z}$ with X. On the other hand, if X is order dense and complete then it is connected.

$\mathbb{Z}$ is first countable and in the connected case there exists a bounded increasing sequence which converges to some point a by completeness. Similarly, some bounded decreasing sequence converges to a point b. As every $x \in X$ is H_+ equivalent to both a and b, every x is the limit of increasing and decreasing sequences. Thus, X is first countable.

(d) (i)$\Rightarrow$(iii)&(iv): If $a < b < c$ in X then the pair a, b is H_+ equivalent to b, c by double transitivity. Hence, a is not the *min* and c is not the *max*. Thus, X is unbounded. If $x_1, x_2 \in (a, \infty)$ and $f \in H_+$ maps the pair a, x_1 to the pair a, x_2, then f restricts to an automorphism of (a, ∞) which maps x_1 to x_2, proving (iii). Similarly, for (iv).

(ii)$\Rightarrow$(iii)&(iv): By (b) X is unbounded. Hence, (a, ∞) and $(-\infty, a)$ are open, nonempty subintervals of X and they are transitive by assumption (ii).

(iii)$\Leftrightarrow$ (v) and (iv)$\Leftrightarrow$ (vi): If X is transitive, then it is unbounded and all (a, ∞) intervals are isomorphic. Hence (v) $\Rightarrow$ (iii). On the other hand, if $x < y$ in X and X has no *min*, then there exists $a < x$ and so (iii) implies there exists an automorphism of (a, ∞) which maps x to y. Extend by the identity on $(-\infty, a]$. Hence, X is transitive. The proof of (iv)$\Leftrightarrow$ (vi) is similar.

(iii)$\Rightarrow$(viii): Given $x_1 < ... < x_n$ and $y_1 < ... < y_n$ choose a smaller than all of them. We construct, by induction on n, $f \in H_+((a, \infty))$ such that $f(x_i) = y_i$ for $i = 1, ..., n$. Assume that $f_1 \in H_+((a, \infty))$ satisfies $f(x_i) = y_i$ for $i = 1, ...n-1$ and let $\tilde{x}_n = f_1(x_n) > f_1(x_{n-1}) = y_{n-1}$. Choose $f_2 \in H_+((y_{n-1}, \infty))$ such that $f_2(\tilde{x}_n) = y_n$ and extend f_2 by the identity on $(a, y_{n-1}]$ to get $f_2 \in H_+((a, \infty))$. Let $f = f_2 \circ f_1$. Having obtained f, extend by the identity on $(-\infty, a]$ to get $f \in H_+(X)$ mapping $x_1 < ... < x_n$ to $y_1 < ... < y_n$.

(iv)$\Rightarrow$(viii): Use a similar proof or apply (iii)$\Rightarrow$ (viii) to X^*.

(viii)$\Rightarrow$(vii): Since condition (viii) clearly implies double transitivity, X is unbounded by (i)$\Rightarrow$(iii)&(iv). If $x_1 < x_2$ and $y_1 < y_2$, then $f \in H_+(X)$ which maps the pair x_1, x_2 to y_1, y_2 restricts to an isomorphism from $[x_1, x_2]$ to $[y_1, y_2]$, proving (vii).

(vii)$\Rightarrow$ X is order dense and (i)&(ii): Given $x_1, x_2 \in X$ we can choose $a, b \in X$ with $a < x_1, x_2 < b$ because X is unbounded. Put together order isomorphisms

$$[a, x_1] \cong [a, x_2] \qquad [x_1, b] \cong [x_2, b] \tag{3.2}$$

and extend by the identity outside (a, b) to get $f \in H_+(X)$ which maps x_1 to x_2. This proves that X is transitive.

If X were discrete, then we could choose points $a < b < c$ with $(a, b) = (b, c) = \emptyset$. But then $[a, b] = \{a, b\}$ is not be isomorphic to $[a, c] = \{a, b, c\}$. So by (c) X is order dense.

Now we show that if J is a nonempty, open convex subset of X, then J is doubly transitive. and $x_1, x_2 \in J$, then because X is unbounded and order dense, we can choose $a, b \in J$ such that $a < x_1, x_2 < b$ and just as before get $f \in H_+((a, b))$ mapping x_1 to x_2. Extend by the identity outside of (a, b) to get $f \in H_+(J)$. Hence, J is transitive. Assume $x_1 < x_2$ and $y_1 < y_2$ in J, choose a smaller and b larger than all four of them. Put together isomorphisms

$$[a, x_1] \cong [a, y_1] \qquad [x_1, x_2] \cong [y_1, y_2] \qquad [x_2, b] \cong [y_2, b] \tag{3.3}$$

and extend by the identity outside (a, b) to get $f \in H_+(J)$ which maps the pair x_1, x_2 to y_1, y_2. Observe that all these intervals are nonempty because X is order dense. Thus, J is doubly transitive.

In particulary, all this proves (e).

□

We call X *weakly homogeneous* if it has at least three points and is order isomorphic with every nonempty, bounded, open subinterval of itself. Clearly, the reverse LOTS X^* is weakly homogeneous when X is.

Proposition 3.3. *Let X be a LOTS with at least three points.*

(a) *X is weakly homogeneous iff for every $a \in X$*

$$(a, \infty) \ \cong \ X \ \cong \ (-\infty, a). \tag{3.4}$$

If X is weakly homogeneous, then it is doubly transitive and it is order isomorphic with every nonempty, open subinterval of itself (whether bounded or not).

(b) *If X is doubly transitive, then every nonempty, bounded, open subinterval of X is weakly homogeneous. If X is doubly transitive and first countable, then it is weakly homogeneous iff it is σ-bounded.*

(c) *Assume X is transitive. If for $a \in X$, $(-\infty, a)$ is symmetric, then X is doubly transitive. If for $a \in X$, $(-\infty, a)$ and (a, ∞) are symmetric, then X is weakly homogeneous.*

Proof. (a): Assume that for all $a \in X$, $(-\infty, a) \cong X \cong (a, \infty)$. If $a < b$ in X, let $f : (a, \infty) \to X$ be an order isomorphism and let $\tilde{b} = f(b)$. Let $\tilde{f} : X \to (-\infty, \tilde{b})$ be an order isomorphism. Then $f^{-1} \circ \tilde{f} : X \to (a, b)$ is an order isomorphism. It follows that X is weakly homogeneous.

Conversely, assume that X is weakly homogeneous. We prove first that if $a < x < b$ in X then x is not an isolated point. If it were then we could choose a and b so that $(a, x) = (x, b) = \emptyset$ and so $(a, b) = \{x\}$ could not be isomorphic to X. It follows that X is infinite. If b were the max of X then we could choose a and x so that (a, x) and (x, b) are infinite. Since $(a, x) \cong X$ the interval (a, x) has a max which we call y. Hence, $(y, x) = \emptyset$ and (a, y) is infinite. Since $(a, y) \cong X$ the interval (a, y) has a max and so the point y is isolated. This contradiction implies that X has no max. Similarly, there is no min. Now Proposition 3.2(d) (vii)$\Rightarrow$(i) shows that X is doubly transitive and so it is order dense by Proposition 3.2(e).

Now given $a \in X$, (a, ∞) and $(-\infty, a)$ are nonempty since a is neither max nor min. Choose $f : X \to (b, c)$ an isomorphism with $b < c$ in X and let $\tilde{a} = f(a)$. f induces isomorphisms $(-\infty, a) \cong (b, \tilde{a})$ and $(a, \infty) \cong (\tilde{a}, c)$. Since neither is empty, each is isomorphic with X. Thus, X is isomorphic with every nonempty, open subinterval of itself.

(b): If X is doubly transitive and $a < x_1 < x_2 < b$ then there exists $f \in H_+(X)$ which maps the pair a, b to x_1, x_2. This restricts to an order isomorphism $(a, b) \cong (x_1, x_2)$. Hence, (a, b) is weakly homogeneous.

If X is first countable then we can use Proposition 2.15(d) to construct an order injection $f : \mathbb{Z} \to (a, b)$ whose image is $\pm$cofinal. Similarly, if X is σ-bounded we can construct an order injection $\tilde{f} : \mathbb{Z} \to X$ whose image is $\pm$cofinal. If, in addition, X is doubly transitive, then we can choose for each $i \in \mathbb{Z}$ an order isomorphism $[f(i), f(i + 1)] \cong [\tilde{f}(i), \tilde{f}(i + 1)]$ and put them together to get an order isomorphism $(a, b) \cong X$. Hence, X is weakly homogeneous.

If X is weakly homogeneous then $X \cong (a, b)$ for $a < b$ and the latter is σ-bounded if X is first countable.

(c): Since X is transitive it is unbounded. In addition, given $a_1 < b_1$ and $a_2 < b_2$ in X we can apply an element of $H_+(X)$ to move the pair a_1, b_1 so that we can assume $b_1 = b_2$ (hereafter denoted b). Let q be an orientation reversing homeomorphism of $(-\infty, b)$. Define $\tilde{a}_i = q(a_i)$ for $i = 1, 2$. Since X is transitive there exists an isomorphism $f : (-\infty, \tilde{a}_1) \to (-\infty, \tilde{a}_2)$. Then $q^{-1} \circ f \circ q$ restricts to an isomorphism $(a_1, b) \cong (a_2, b)$. So X is doubly transitive by Proposition 3.2(d) (vii)$\Rightarrow$(i). If, instead, we use an isomorphism $f_1 : (-\infty, \tilde{a}_1) \to (-\infty, b)$ then $q^{-1} \circ f_1 \circ q$ restricts to an isomorphism $(a_1, b) \cong (-\infty, b)$.

Now if (b, ∞) is symmetric as well then we can similarly construct an isomorphism $(b, a_3) \cong (b, \infty)$ with $b < a_3$. Putting this together with the isomorphism $(a_1, b) \cong (-\infty, b)$ above we get an isomorphism $(a_1, a_3) \cong X$. Thus, X is isomorphic to nonempty, bounded, open subintervals of itself and so is weakly homogeneous. Since $X \cong (a, b) \cong (-\infty, b)$ it is symmetric as well.

$\square$

Since a nontrivial transitive LOTS X is unbounded, it cannot be compact. There is a condition due to G. D. Birkhoff, [8] page 47, which we will call the *closed interval condition*. A LOTS X satisfies the closed interval condition when any two nontrivial, closed bounded subintervals are isomorphic. Clearly, any nontrivial convex subset of such a LOTS satisfies the closed interval condition as well.

By Proposition 3.2(d) an unbounded LOTS satisfies the closed interval condition iff it is doubly transitive. If X is bounded with $min = m$ and $max = M$ and X satisfies the closed interval condition, then $X \setminus \{m.M\}$ is weakly homogeneous. Conversely, if X_0 is weakly homogeneous and $X = \{m\} + X_0 + \{M\}$, then X satisfies the closed interval condition.

Example 3.4. *Linearly ordered groups, fields and products*

An ordered group is a group with a linear order such that the translation maps are order isomorphisms. Since a group acts transitively on itself by translation, it follows that an ordered group, like $\mathbb{Z}$, is a transitive LOTS. An ordered field is a field with a linear order such that the additive translation maps, and multiplication by positive elements are order isomorphisms. An ordered field, like $\mathbb{Q}$ and $\mathbb{R}$, is doubly transitive. Observe that if $x_1 < x_2$ in the field, then $x \mapsto (x - x_1) \cdot (x_2 - x_1)^{-1}$ maps (x_1, x_2) isomorphically onto $(0, 1)$. In fact, an ordered field is weakly homogeneous, because the map:

$$f(x) \ = \ \begin{cases} x \cdot (1+x)^{-1} & \text{for} \quad -1 < x \leq 0, \\ x \cdot (1-x)^{-1} & \text{for} \quad 0 \leq x < 1, \end{cases} \tag{3.5}$$

is an isomorphism from $(-1, 1)$ to the entire field.

If $\{X_i : i \in I\}$ is a family of LOTS indexed by an ordinal I, and h_i is an automorphism of X_i for each i, then $\prod_i \ h_i$ is an automorphism of the order space product $\prod_i X_i$. It follows if each X_i is transitive, then the product is transitive as well. If each X_i is an ordered group, then the product is an ordered group with coordinate-wise addition.

On the other hand, since a discrete LOTS with at least three points is never doubly transitive, it follows that the discrete, linearly ordered groups $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{R} \times \mathbb{Z}$ are not doubly transitive.

The groups $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{R} \times \mathbb{R}$ are order dense. Because $\mathbb{Q} \times \mathbb{Q}$ is a countable order dense LOTS, it is order isomorphic to the ordered field $\mathbb{Q}$ by Proposition 2.15 (a) and so it is doubly transitive. On the other hand, $\mathbb{R} \times \mathbb{R}$ is not doubly transitive. Let $x_1 = (0, 0), x_2 = (0, 1), x_3 = (1, 0)$. The map $t \mapsto (0, t)$ is an order isomorphism from the unit interval $(0, 1)$ in $\mathbb{R}$ onto the interval (x_1, x_2) and, in particular, the latter interval is separable. On the other hand, if D is a countable subset of (x_1, x_3), then we can choose t such that $0 < t < 1$ and t is not the first coordinate of a point of D. Hence, the open interval $((t, 0), (t, 1))$ is disjoint from D. It follows that (x_1, x_3) is not separable and so is not isomorphic to (x_1, x_2).

With $-1, +1 \in \mathbb{R}$ the order space product $\mathbb{R} \times (-1, +1)$ is isomorphic to $\mathbb{R} \times \mathbb{R}$ and so is transitive, locally connected and of countable type. Its completion is the order product $\mathbb{R} \times [-1, +1]$ which is connected and of countable type, but not transitive. Observe that the union of the separable open subsets of $\mathbb{R} \times [-1, +1]$ is $\mathbb{R} \times (-1, +1)$. That is, a point of the form $(t, \pm 1)$ does not have any separable neighborhood.

The product $\mathbb{Z} \times \mathbb{R}$ is isomorphic to the complement of $\mathbb{Z}$ in $\mathbb{R}$ and $\mathbb{Q} \times \mathbb{R}$ is isomorphic to the complement of the Cantor Set C in $[0, 1]$. Both of these are transitive, but not doubly transitive. Any homeomorphism maps components to components and so cannot map a pair of points contained in a component to a pair in different components. Both of these provide examples of dense, transitive subsets of $\mathbb{R}$ which are not doubly transitive. However, neither of these has dense holes.

Finally, we will see below that because its complement $\mathbb{Q}$ is doubly transitive, the set $\mathbb{I}$ of irrationals is doubly transitive. Hence, the product $\mathbb{Q} \times \mathbb{I}$ is transitive. It is isomorphic to $X = [0, 1] \setminus (\mathbb{Q} \cup C)$. Between any two points of X in the same component of $[0, 1] \setminus C$ there are only countably many holes, whereas between two points of X in different components of $[0, 1] \setminus C$ there are uncountably many holes. Hence, X provides an example in $(0, 1) \cong \mathbb{R}$ of a dense, transitive subset with dense holes, but which is not doubly transitive.

We do not know whether or not a dense additive subgroup of $\mathbb{R}$ is necessarily doubly transitive.

In contrast with the above examples Treybig proved the following beautiful result which we state using our language.

Theorem 3.5. *If X is a connected, unbounded transitive LOTS, then X is doubly transitive.*

Proof. See [20]. As we will not need the result below, we include the proof in an Appendix.

□

The following uses the easy part of an argument due to Babcock [6].

Theorem 3.6. *If α is a tail-like ordinal and X is doubly transitive, then X^α is doubly transitive.*

Proof. Choose two points of X which we label $-1 < +1$. Let $x^\pm$ be defined by $x^+_i = +1, x^-_i = -1$ for all $i < \alpha$. Using transitivity of X we choose for any $a \in X$ automorphisms $g^\pm_a$ of X with $g^+_a(+1) = a, g^-_a(-1) = a$. Given $y < z \in X^\alpha$ we construct an isomorphism $f : [x^-, x^+] \to [y, z]$.

Let $\beta = min\{i : y_i \neq z_i\} < \alpha$ so that $y_i = z_i$ for all $i < \beta$ and $y_\beta < z_\beta$. Let $g^0 : [-1, +1] \to [y_\beta, z_\beta]$ be an isomorphism. Because α is tail-like there is an isomorphism $\gamma : \alpha \to \alpha \setminus \beta$. For $x \in [x-, x+]$ define $f(x)$ by:

- $f(x)_i = y_i = z_i$ for $i < \beta$.
- $f(x)_\beta = g^0(x_0) = g^0(x_{\gamma(\beta)})$.
- $f(x)_i = g^-_{y_i}(x_{\gamma(i)})$ if $\beta < i < \alpha$ and $x_{\gamma(j)} = -1$ for all $\beta \leq j < i$.
- $f(x)_i = g^+_{z_i}(x_{\gamma(i)})$ if $\beta < i < \alpha$ and $x_{\gamma(j)} = +1$ for all $\beta \leq j < i$.
- $f(x)_i = x_{\gamma(i)}$ otherwise.

For $\beta \leq i < \alpha$, $f(x)_i$ depends only on the values of x_k with $k \leq \gamma(i)$ and so it easily follows that f is an order injection of $[x-, x+]$ into $[y, z]$. On the other hand, it is easy to reverse the procedure to see that f is surjective and so is an isomorphism.

□

Remark. Since α need not be countable, beginning with $X = \mathbb{R}$ this constructs doubly transitive LOTS of arbitrary cardinality.

Using the completion $\hat{X}$ for an order dense LOTS X, we now describe the strong homogeneity condition that we want to focus on.

Proposition 3.7. *Let $\hat{X}$ be the completion of X, a nontrivial, order dense LOTS.*

(a) *The following conditions are equivalent.*

(i) *X and $\hat{X}$ are doubly transitive.*

(ii) *X is doubly transitive and $\hat{X}$ is first countable.*
(iii) *Any two nonempty, open, bounded, convex subsets of X are order isomorphic.*

(b) *The following conditions are equivalent.*
(i) *X and $\hat{X}$ are weakly homogeneous.*
(ii) *X is doubly transitive and of countable type.*
(iii) *X is order isomorphic with every nonempty, open, convex subset of X.*

Proof. Since X is order dense and nontrivial, it is infinite.

(a)(i)$\Rightarrow$(ii): Since $\hat{X}$ is doubly transitive, it is transitive. Completeness implies first countability by Proposition 3.2(c).

(ii)$\Rightarrow$(iii): Bounded open convex sets are of the form $(z_1, z_2) \cap X$ and $(w_1, w_2) \cap X$ with $z_1 < z_2$ and $w_1 < w_2$ in $\hat{X}$. Because $\hat{X}$ is first countable and connected there exist order injections $\tilde{g}, \tilde{f} : \mathbb{Z} \to \hat{X}$ whose images are $\pm$cofinal in (z_1, z_2) and (w_1, w_2), respectively. We choose for each $i \in \mathbb{Z}$, $f(i) \in (\tilde{f}(i), \tilde{f}(i+1)) \cap X$ and $g(i)$ similarly so as to get $g, f : \mathbb{Z} \to X$. In the now familiar way we put together isomorphisms $(f(i), f(i+1)) \cong (g(i), g(i+1))$ to get the required isomorphism between the open convex sets.

(iii)$\Rightarrow$(i): If X had a $max = b$ and $a < b$ then (a, b) and $(a, b]$ are both bounded, open convex subsets of X. Since X is order dense (a, b) has no max and so cannot be isomorphic to $(a, b]$. Thus, condition (iii) implies that X has no max and similarly no min. If $z_1 < z_2$ and $w_1 < w_2$ in $\hat{X}$, then any isomorphism $(z_1, z_2) \cap X \cong (w_1, w_2) \cap X$ extends as in (2.29) to an isomorphism $(z_1, z_2) \cong (w_1, w_2)$ in $\hat{X}$. Hence, both X and $\hat{X}$ satisfy condition (vii) of Proposition 3.2(d). Since (vii)$\Rightarrow$(i) there, X and $\hat{X}$ are doubly transitive.

(b): By Proposition 3.3(a) (i) here implies condition (i) of (a). By Proposition 2.15(d) (ii) here implies condition (ii) of (a). Clearly, (iii) here implies condition (iii) of (a). Hence, in proving the equivalences we can use all of the conditions of (a).

(i)$\Rightarrow$(ii): By weak homogeneity of $\hat{X}$, $\hat{X}$ is isomorphic to any bounded, nonempty, open interval and such an interval is σ-bounded because $\hat{X}$ is first countable. Hence, $\hat{X}$ is first countable and σ-bounded. Thus, X is of countable type by Proposition 2.15(d).

(ii)$\Rightarrow$(iii): Because $\hat{X}$ is first countable, X is and X is σ-bounded as well as doubly transitive by assumption. Therefore, it is weakly homogeneous by Proposition 3.3(b). Let J be any nonempty, open, convex subset of X and $a < b$ in X. By weak homogeneity there is an isomorphism $f : X \to (a, b)$ and $f(J)$ is a bounded, open, convex subset of X. So by condition (iii) of (a) $f(J) \cong (a, b)$. So $J \cong f(J) \cong (a, b) \cong X$.

(iii)⇒(i): Clearly, X is weakly homogeneous. If $w_1 < w_2$ in $\hat{X}$ then $(w_1, w_2) \cap X$ is a nonempty, open, convex subset which is isomorphic to X. The isomorphism extends to the completions so that the interval (w_1, w_2) in $\hat{X}$ is isomorphic to $\hat{X}$. Thus, $\hat{X}$ is weakly homogeneous as well.

□

We call X a *homogeneous LOTS*, written HLOTS, if X contains at least three points and it is order isomorphic with every nonempty, open, convex subset of itself. If a HLOTS is complete then we call it a CHLOTS, a *complete homogeneous LOTS*. Otherwise, we call it an IHLOTS, an *incomplete homogeneous LOTS*.

Proposition 3.8. *Let X be a nontrivial LOTS.*

(a) *X is a HLOTS if and only if X is doubly transitive, of countable type and contains at least three points.*
(b) *If X is a HLOTS, then it is order dense, unbounded, σ-bounded and first countable.*
(c) *The completion $\hat{X}$ of a HLOTS X is a CHLOTS and a HLOTS X is a CHLOTS iff $X = \hat{X}$. If X is a CHLOTS, then it is connected, locally compact and σ-compact.*
(d) *A HLOTS X is an IHLOTS iff it is a proper subset of $\hat{X}$. If X is an IHLOTS then it has dense holes. Furthermore, $\hat{X} \setminus X$ is an IHLOTS which is dense in $\hat{X}$, and so the completion of $\hat{X} \setminus X$ is $\hat{X}$.*

Proof. (a), (b): A HLOTS is clearly weakly homogeneous. A finite LOTS is not isomorphic to its singleton subsets and so a HLOTS is infinite. It follows from Proposition 3.3(a) that a HLOTS is doubly transitive and so by Proposition 3.2(e) it is order dense and unbounded. Now Proposition 3.7(b) (ii)⇔(iii) implies the equivalence in (a). Since a HLOTS is of countable type, it is first countable and σ-bounded by Proposition 2.11(d).

(c), (d): X is complete iff $X = \hat{X}$. Completeness implies local compactness. If, in addition, it is order dense, then it is connected and then σ-boundedness implies σ-compactness. In general, if X is a HLOTS, then by (a) and (b) it is order dense, doubly transitive and of countable type. By Proposition 2.15(e) $\hat{X}$ is of countable type and by

Proposition 3.7(a) it is doubly transitive. By (a) above, $\hat{X}$ is a HLOTS and so is a CHLOTS.

Now assume that X is a proper subset of $\hat{X}$ with $z \in \hat{X} \setminus X$. Choose $a, b \in X$ so that $a < z < b$. For any $c < d$ in X there exists $f \in H_+(X)$ mapping the pair a, b to c, d. The completion $\hat{f} \in H_+(\hat{X})$ maps z to a point of $\hat{X} \setminus X$ between c and d. Thus, X has dense holes. That is, $\hat{X} \setminus X$ is dense in $\hat{X}$ and so it is order dense with completion $\hat{X}$. Now if $\tilde{J}$ is a nonempty, open, convex subset of $\hat{X} \setminus X$, then there exists an open, convex subset J of $\hat{X}$ such that $\tilde{J} = J \cap (\hat{X} \setminus X)$. There exists an isomorphism f of the IHLOTS X with the nonempty, open, convex subset $J \cap X$ of itself. The completion $\hat{f} : \hat{X} \cong J$ restricts to an isomorphism $\hat{X} \setminus X \cong \tilde{J}$. Thus, $\hat{X} \setminus X$ is a HLOTS, indeed an IHLOTS since X is nonempty.

$\square$

Remark: From Treybig's Theorem 3.5 it follows that a nontrivial, transitive, connected, σ-compact LOTS is a CHLOTS.

The motivating example of an IHLOTS is the set of rationals $\mathbb{Q}$ with completion the CHLOTS $\mathbb{R}$ and with complementary IHLOTS the irrationals $\mathbb{I}$ in $\mathbb{R}$.

Example 3.9. *HLOTS ultraproduct*

On ω, the first infinite ordinal, we choose an ultrafilter $\mathcal{U}$ which is nonprincipal, i.e. $\mathcal{U}$ contains all cofinite sets. For a CHLOTS F we define the ultraproduct construction on F associated with $\mathcal{U}$.

On the, unordered, set of maps F^ω we define the equivalence relation $\equiv_\mathcal{U}$

$$a \equiv_\mathcal{U} b \qquad \Longleftrightarrow \qquad \{i : a_i = b_i\} \in \mathcal{U}, \tag{3.6}$$

which is an equivalence relation because $\mathcal{U}$ is a filter. We let $F^\mathcal{U}$ denote the LOTS of equivalence classes with the order

$$a < b \qquad \Longleftrightarrow \qquad \{i : a_i < b_i\} \in \mathcal{U}. \tag{3.7}$$

This definition is independent of the choice of a and b in the equivalence classes, but we can then choose representatives such that $a_i < b_i$ for all $i \in \omega$. The relation on $F^\mathcal{U}$ is a total order because $\mathcal{U}$ is an ultrafilter, i.e. if $D_1 \cup ... \cup D_n \in \mathcal{U}$, then $D_k \in \mathcal{U}$ for some $k = 1, ..., n$. In particular, if $a_i < b_i$ for all i, then we can choose for each i an order isomorphism $f_i : F \cong (a_i, b_i)$. The product $\prod_i \ f_i$ is an order isomorphism $F^\mathcal{U} \cong$

$(a,b) \subset F^{\mathcal{U}}$. Thus, $F^{\mathcal{U}}$ is weakly homogeneous. In the case $F = \mathbb{R}$ this is the ordered field of hyperreal numbers.

$F^{\mathcal{U}}$ is order dense but not complete. Its completion is not even transitive.

We show that if $A = \{a^1, a^2, ..\}, B = \{b^1, b^2, ..\}$ are nonempty countable subsets of $F^{\mathcal{U}}$ with $a < b$ for all $a \in A, b \in B$, then there exists $w \in F^{\mathcal{U}}$ with $a < w < b$ for all $a \in A, b \in B$.

By replacing a^k by $max_{\ell \leq k} a^\ell$ and eliminating any repeats we may assume that $\{a^k\}$ is a finite or infinite increasing sequence. Similarly, we may assume that $\{b^k\}$ is a finite or infinite decreasing sequence. Assume that representatives of a^k and b^k have been chosen so that $a^k_i < b^k_i$ for all $i \in \omega$. Now choose a representative a^{k+1} so that $a^k_i < a^{k+1}_i < b^k_i$ for all i. Then choose a representative b^{k+1} so that $a^{k+1}_i < b^{k+1}_i < b^k_i$ for all i. We do the obvious adjustments if either sequence is finite, ending at the k level.

We have thus obtained representatives of the elements of A and B so that $a_i < b_i$ for all $a \in A, b \in B, i \in \omega$. Choose w_i so that

$$a^k_i \ < \ w_i \ < \ b^k_i \qquad \text{for } k = 1, ..., i. \tag{3.8}$$

For each k, $\{i : a^k_i < w_i\}$ includes all $i \geq k$ and so is in $\mathcal{U}$. Hence, each $a^k < w$ for each k. Similarly, $b^k > w$ for each k.

An obvious adjustment covers the case when A or B is empty.

Using the case when A or B is a singleton, we see that no increasing or decreasing sequence converges in $F^{\mathcal{U}}$.

If x in the completion $\widehat{F^{\mathcal{U}}}$ is the limit of an increasing sequence $\{a^k\}$, then we can choose the sequence to be in $F^{\mathcal{U}}$ because the latter meets (a^k, a^{k+1}). So if $\{b^k\}$ is any decreasing sequence with $b^k > x$ for all k, then there exists $w \in F^{\mathcal{U}}$ with $a^k < w < b^k$ for all k. Since $x \not\in F^{\mathcal{U}}$, $w \neq x$. Since $\{a^k\}$ converges to x, it cannot be that $w < x$. Hence, $w > x$ and so $\{b^k\}$ does not converge to x.

Thus, $\widehat{F^{\mathcal{U}}}$ is partitioned by three dense sets, namely $\widehat{F^{\mathcal{U}}}_-$ consisting of the limits of increasing sequences, $\widehat{F^{\mathcal{U}}}_+$ the limits of decreasing sequences and the remaining set $\widehat{F^{\mathcal{U}}}_0$ which contains $F^{\mathcal{U}}$. Each automorphism of $\widehat{F^{\mathcal{U}}}$ preserves each of these sets.

In general, Hausdorff defines an *α-set* X to be a linearly ordered set of cardinality $\aleph_\alpha$ such that $A, B \subset X$ with $A < B$ and the cardinality of $A \cup B$ less than $\aleph_\alpha$ implies there exists $x \in X$ such that $A < x < B$. See [11] and also section 4 or [12]. The isomorphism between any two α-sets is proved using the transfinite analogue of the back and forth argument which gives the uniqueness up to isomorphism of countable, unbounded, order dense sets, see, e.g. Proposition 2.15(a).

The cardinality of $F^{\mathcal{U}}$ is $\mathbf{c}$ which equals $\aleph_1$ if one assumes the Continuum Hypothesis. It then follows that $F^{\mathcal{U}}$ is a 1-set (assuming CH). Any open convex subset of $F^{\mathcal{U}}$ which has neither a countable cofinal nor a countable coinitial subset is also a 1-set and so is isomorphic to $F^{\mathcal{U}}$ by uniqueness. Of course, we already know that if $a < b \in F^{\mathcal{U}}$, then the interval is a isomorphic to $F^{\mathcal{U}}$ and so is a 1-set, but for $a < b \in \widehat{F^{\mathcal{U}}}_0$ the interval (a, b) is a 1-set as well.

Now suppose that $a < b \in \widehat{F^{\mathcal{U}}}_-$. Let $\{a_n\}$ be an increasing sequence in $F^{\mathcal{U}}$ which converges to a and $\{b_n\}$ be an increasing sequence in $(a, b) \cap F^{\mathcal{U}}$ which converges to b. Each of the open intervals

$$(-\infty, a_1), \ldots, (a_k, a_{k+1}), \ldots, (a, b_1), \ldots, (b_k, b_{k+1}), \ldots, (b, \infty) \tag{3.9}$$

is a 1-set.

It easily follows that for $a < b, c < d \in \widehat{F^{\mathcal{U}}}_-$ we can complete an automorphism of $F^{\mathcal{U}}$ to obtain an automorphism of $\widehat{F^{\mathcal{U}}}$ which maps $[a, b]$ onto $[c, d]$.

With a similar argument for $\widehat{F^{\mathcal{U}}}_+$, it follows that three sets $\widehat{F^{\mathcal{U}}}_\pm, \widehat{F^{\mathcal{U}}}_0$ are the orbits of the action of $H_+(\widehat{F^{\mathcal{U}}})$ on $\widehat{F^{\mathcal{U}}}$. Separately, each is a doubly transitive LOTS.

Thus, assuming CH, we see that the LOTS $\widehat{F^{\mathcal{U}}}_-$ is a doubly transitive LOTS with every increasing sequence convergent, but with no convergent decreasing sequences.

3.2. **The Double Arrow of a LOTS.** For any LOTS X we define the *Alexandrov-Sorgenfrey Double Arrow* of X, hereafter the *AS double* of X, to be the order space product

$$X' \quad = \quad X \times \{-1, +1\}, \tag{3.10}$$

regarding $\{-1, +1\}$ as a two point LOTS. For x in X we denote by $x^{\pm}$ the pair $(x, \pm 1)$ and we define the first coordinate projection map

$$\begin{aligned} \pi' &: X' \to X \\ \pi'(x^{\pm}) &= x. \end{aligned} \tag{3.11}$$

Clearly, we have for $a < b$ in X:

$$\begin{aligned} (\pi')^{-1}((a, b)) &= (a^+, b^-) \\ (\pi')^{-1}([a, b]) &= [a^-, b^+]. \end{aligned} \tag{3.12}$$

Since $(\pi')^{-1}(a)$ is the compact set $\{a^-, a^+\}$, Proposition 2.3(a) implies that the surjective order map π' is closed and continuous.

For each $x \in X$, $x^- < x^+$ is a gap pair in X' so that x^- is a left endpoint and x^+ is a right endpoint. x^- is also a right endpoint, and so is an isolated point of X', iff x is a right endpoint in X, i.e. $[x, \infty)$ is open. For if $[x.\infty)$ is open then by (3.12), $\{x^-\} = (-\infty, x^+) \cap [x^-, \infty)$ is open, while if $A \subset X \setminus \{x\}$ with $x = supA$ then $x^- = sup(\pi')^{-1}(A)$. Similarly, x^+ is isolated iff x is a left endpoint in X. In particular, if X has a $max = M$ (or a $min = m$) then M^+ (resp. m^-) is an isolated point in X'.

If $f : X_1 \to X_2$ is an order injection then we define

$$\begin{aligned} f' &: X_1' \to X_2' \\ f'(x^{\pm}) &= f(x)^{\pm}. \end{aligned} \tag{3.13}$$

Clearly, f' is the unique order injection such that the diagram

$$\begin{tikzcd} X_1' \arrow[r, "f'"] \arrow[d, "\pi'"'] & X_2' \arrow[d, "\pi'"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

commutes. If f is not injective then the map defined by (3.13) does not preserve order. If f is a continuous, noninjective, order map we can obtain many continuous order maps f' which make the above diagram commute. If X_1 is complete, then for each $y \in f(X_1)$ the set $f^{-1}(y)$ is a closed interval $[x_1, x_2]$. Choose a point $\tilde{x} \in [x_1, x_2] = f^{-1}(y)$ and then map $[x_1^-, \tilde{x}^-]$ to y^- and $[\tilde{x}^+, x_2^+]$ to y^+. In general,$f^{-1}(y)$ is a closed, convex set and with the choice of $\tilde{x} \in f^{-1}(y)$ we map $(-\infty, \tilde{x}^-] \cap (f \circ \pi')^{-1}(y)$ to y^- and $[\tilde{x}^+, \infty) \cap (f \circ \pi')^{-1}(y)$ to y^+.

To check that such maps f' are continuous, observe that

$$\begin{aligned} &(f')^{-1}((-\infty, y^-)) = (f \circ \pi')^{-1}((-\infty, y)) \\ &(f')^{-1}((-\infty, y^+)) = (f')^{-1}((-\infty, y^-)) \quad \text{if } y \notin f(X), \\ &(f')^{-1}((-\infty, y^+)) = (-\infty, \tilde{x}^+) \quad \text{if } y \in f(X). \end{aligned} \tag{3.14}$$

Use a similar argument for the open intervals unbounded above.

If $r : X_1 \to X_2$ is an injective order* map then we define the continuous injective order* map

$$\begin{aligned} r^* &: X_1' \to X_2' \\ r^*(x^{\pm}) &= r(x)^{\mp}. \end{aligned} \tag{3.15}$$

If f (or r) is bijective, then f' (resp. r^*) is.

Lemma 3.10. *Let X be an unbounded, order dense LOTS.*

(a) *Let X_1 be a LOTS and $g : X' \to X_1'$ be a homeomorphism. The LOTS X_1 is order dense and unbounded. If g is an order map, then there exists $f : X \to X_1$ an order isomorphism such that $g = f'$. If g is an order* map, then there exists $r : X \to X_1$ an order* homeomorphism such that $g = r^*$.*

In particular, we have

$$X' \;\cong\; X_1' \quad \Longleftrightarrow \quad X \;\cong\; X_1. \tag{3.16}$$

(b) *Assume that X is complete and so is connected. The bounded clopen subintervals of X' are compact sets which form a base for X', and so X' is zero-dimensional. If C is any nonempty, bounded, clopen subset of X', then there is a unique finite sequence $a_1 < b_1 < a_2 ... < b_n$ in X such that*

$$C = \bigcup_{i=1}^{n} [a_i^+, b_i^-]. \tag{3.17}$$

Proof. (a): X' has no isolated points and so X_1' has none. Hence, X_1 has no left or right endpoints. If g is either order preserving or order reversing then gap pairs are mapped to gap pairs. Hence g induces a bijection from X to X_1 which preserves or reverses order according to which g does.

(b): Since X is connected with no *max* or *min*, any bounded clopen interval of X' is of the form $[a^+, b^-]$ with $a < b$ in X. These form a base.

Let C be a nonempty, bounded, clopen subset of X'. Since C is open it is a union of such subintervals and since C compact it is a finite union of them. If two such intervals intersect, or if the *max* of one and the *min* of another form a gap pair, then the union of the two is a clopen interval. Combining in this way we obtain C as the finite, disjoint union of clopen intervals with points of $X' \setminus C$ between any two successive intervals, i.e. (3.17) holds. Furthermore, the intervals $[a_i, b_i]$ are the components of the image $\pi'(C)$ in X. Uniqueness follows. □

We now describe the gap between ±transitivity and transitivity in the complete case.

Proposition 3.11. *A complete LOTS X is ±transitive but not transitive iff $X \cong X_0'$ with X_0 a connected, symmetric, transitive LOTS.*

Proof. If X were trivial it would be transitive. If X is the two point LOTS, then $X \cong X_0'$ with X_0 trivial. So we can assume that X has at least three points so that Proposition 3.2 applies.

Assume that X_0 is a nontrivial, order dense, symmetric, transitive LOTS. If $f \in H_+(X_0)$ maps x to y, then f' in $H_+(X_0')$ maps x^+ to y^+ and x^- to y^-. Thus, the right endpoints x^+ and the left endpoints x^- in X_0' form $H_+(X_0')$ classes which are distinct because X_0' has no isolated points. Thus, X_0' is not transitive. If r is an order reversing homeomorphism on X_0, then r^* maps x^+ to $r(x)^-$ and so X_0' consists of a single $H_{\pm}(X_0')$ equivalence class. If, in addition, X_0 is connected, then X_0' is complete.

Conversely, if X is $\pm$transitive, complete, not transitive and contains at least three points, then by Proposition 3.2(b) it is unbounded. If there were any isolated points, then by transitivity X would be discrete and we could apply the argument in the proof of Proposition 3.2(c) to show that $X \cong \mathbb{Z}$ which is transitive. If X were connected, then by Proposition 3.2(a) it would be transitive. Hence, X contains gap pairs but no isolated points. It follows from $\pm$transitivity that every point is either a left or right endpoint and that no point is both. Since X is unbounded, all of the endpoints occur in gap pairs. Let X_0 be the LOTS of gap pairs. Every element of $H_{\pm}(X)$ induces an element of $H_{\pm}(X_0)$ and so X_0 is $\pm$transitive. Completeness of X_0 follows from completeness of X. For each pair z in X_0 let z^+ be the right and z^- be the left endpoint of the pair. This yields an isomorphism $X \cong X_0'$. Since X has no isolated points, X_0 is order dense and so is connected. Hence, X_0 is transitive by Proposition 3.2(a). Finally, since X is $\pm$transitive but not transitive, it admits some order reversing homeomorphism g. By Lemma 3.10(a) $g = r^*$ for r an order reversing homeomorphism of X_0. Hence, X_0 is symmetric.

$\square$

Remark. Notice that for the case of $\mathbb{Z}$ we can define the order isomorphism:

$$\begin{gathered} f : \mathbb{Z}' \to \mathbb{Z} \\ f(n^-) = 2n \qquad \text{and} \qquad f(n^+) = 2n+1. \end{gathered} \tag{3.18}$$

For X a complete LOTS with no *max* or *min* the *two-point compactification*, denoted $\bullet X \bullet$, is the LOTS obtained by attaching a *max* and a *min*, i.e. the order sum

$$\bullet X \bullet \quad = \quad \{m\} + X + \{M\}. \tag{3.19}$$

Proposition 3.12. *If X is a CHLOTS and C is a nonempty, bounded, clopen subset of X' then*

$$C \quad \cong \quad \bullet X' \bullet \quad . \tag{3.20}$$

Proof. Let $a_1 < b_1 < a_2 ... < b_n$ be the sequence in X such that (3.17) holds. Let $f_i : [a_i, b_i] \cong [a_i, a_{i+1}]$ for $i = 1, ..., n-1$. Each f_i' restricts to an isomorphism $[a_i^+, b_i^-] \cong [a_i^+, a_{i+1}^-]$. Put them together to get an isomorphism:

$$C \quad \cong \quad (\bigcup_i^{n-1} [a_i^+, a_{i+1}^-]) \cup [a_n^+, b_n^-] \quad = \quad [a_1^+, b_n^-]. \tag{3.21}$$

If $g : X \to (a_1, b_n)$ is an order isomorphism, then so is

$$g' : X' \to (a_1, b_n)' = (a_1^+, b_n^-). \tag{3.22}$$

Attaching the endpoints we obtain an isomorphism $\bullet X' \bullet \cong [a_1^+, b_n^-]$. □

If X has no isolated points, then the isolated points of X' are of the form x^+ where x is a left endpoint of X and x^- where x is a right endpoint. We define for a LOTS X'' with no isolated points:
(3.23)

$$X'' = X' \setminus (\{x^+ : x \ \text{ a left endpoint}\} \cup \{x^- : x \ \text{ a right endpoint}\}).$$

That is, X'' is the subset of X' obtained by removing the isolated points. Note that if X is unbounded and order dense then $X'' = X'$.

In general, if $x_1 < x_2$ is a gap pair in X, then since x_1 is not a right endpoint and x_2 is not a left endpoint (neither is isolated) we see that $x_1^- < x_2^+$ is a gap pair in X''. If $M = maxX$, then $M^- = maxX''$ and similarly, $m^+ = minX''$ if $m = minX$. If x is not an endpoint in X, then $x^- < x^+$ is a gap pair in X''. Thus, every point of X'' is a right or a left endpoint. Let

$$\pi'' : X'' \to X \tag{3.24}$$

denote the restriction of the projection $\pi' : X' \to X$. For every $x \in X$ either x^+ or x^- or both lies in X'' and so π'' is an order surjection. Since $(\pi'')^{-1}(x)$ is either a singleton or a pair for every $x \in X$ it follows from Proposition 2.3(a) that π'' is continuous and closed.

If $x = supA$ with $A \subset (-\infty, x)$ in X, then x is not a right endpoint and so $x^- \in X''$. Furthermore,

$$x^- \quad = \quad sup\pi''^{-1}(A). \tag{3.25}$$

Similarly, $x = inf B$ with $B \subset (x, \infty) \subset X$ implies

$$x^+ \quad = \quad inf \pi''^{-1}(B). \tag{3.26}$$

In particular, X'' has no isolated points.

Now for a useful construction:

Assume X is a complete LOTS with no isolated points, X_1 is an unbounded LOTS with no isolated points, A is a dense subset of X_1 and $g : A \to X$ is an order injection.

First, define

$$\begin{aligned} G : X_1' \to X, &\qquad \text{by} \\ G(x^-) = sup g((-\infty, x) \cap A) & \\ G(x^+) = inf g((x, \infty) \cap A). & \end{aligned} \tag{3.27}$$

So that $G(x^-) \leq G(x^+)$.

Assume $x_1 < x_2$ in X_1.

If x_1, x_2 is not a gap pair, then the interval (x_1, x_2) in X_1 is nonempty and so is infinite because X_1 has no finite nonempty open set. Because A is dense, $(x_1, x_2) \cap A$ is infinite. Thus, there exist $a_1, a_2 \in A$ such that $x_1 < a_1 < a_2 < x_2$. It follows that

$$G(x_1^-) \leq G(x_1^+) \leq g(a_1) < g(a_2) \leq G(x_2^-) \leq G(x_2^+). \tag{3.28}$$

If x_1, x_2 is a gap pair, then at least

$$G(x_1^-) \quad \leq \quad G(x_2^+). \tag{3.29}$$

It can happen that in the gap pair case, $G(x_2^-) < G(x_1^+)$. On the other hand, if x_1, x_2 is a gap pair, then $x_1^+, x_2^- \not\in X_1''$.

It follows that the restriction of G to X_1'' is an order map.

Now define the lift g''.

$$\begin{aligned} g'' : X_1'' \to X'', &\qquad \text{by} \\ g''(x^-) = sup\ (\pi'')^{-1}(g((-\infty, x) \cap A)) & \\ g''(x^+) = inf\ (\pi'')^{-1}(g((x, \infty) \cap A)). & \end{aligned} \tag{3.30}$$

Notice that if $x^- \in X_1''$, then x is not a right endpoint and so $x = sup\ ((-\infty, x) \cap A)$ and similarly for x^+.

It follows from (3.25) and (3.26) that on $X_1'' \subset X_1'$ we have

$$\begin{aligned} g''(x^-) \quad &= \quad G(x^-)^- \\ g''(x^+) \quad &= \quad G(x^+)^+. \\ \text{and so} \qquad \pi'' \circ g'' \quad &= \quad G. \end{aligned} \tag{3.31}$$

Assume that $x_1 < x_2$ in X_1.

If $x_1^-, x_2^+ \in X_1''$, then by (3.31) together with (3.28) or (3.29)

$$g''(x_1^-) \quad < \quad g''(x_2^+). \tag{3.32}$$

If x_1^+ or $x_2^- \in X_1''$, then x_1, x_2 is not a gap pair So from (3.28) and (3.31) again, we have

$$g''(x_1^-) \ < \ g''(x_1^+) \ < \ g''(x_2^-) \ < \ g''(x_2^+) \tag{3.33}$$

omitting whichever terms are undefined.

It follows that g'' is an order injection.

Now assume that $x \in X_1$ is not a right endpoint so that $x^- \in X_1''$. By definition, $G(x^-)$ is not a right endpoint and so $g''(x^-) = G(x^-)^- = sup\{y^- : y < G(x^-)\}$. For $y < G(x^-)$ the interval $(y, G(x^-))$ is infinite and we need only consider such points y which are also not right endpoints. There exists $a \in (-\infty, x) \cap A$ such that $y < g(a)$. There exists $x_1 \in (a, x) \subset X_1$ not a right endpoint so that $x_1^- \in X_1''$ as well. Since x_1, x is not a gap pair, (3.28) implies that

$$\begin{gathered} y < g(a) \leq G(x_1^-) < G(x^-), \quad \text{and so} \\ y^- < G(x_1^-)^- < G(x^-)^-, \ \text{i.e. } y^- < g''(x_1^-) < g''(x^-), \end{gathered} \tag{3.34}$$

and so

$$g''(x^-) \quad = \quad sup\ \{g''(x_1^-) : x_1^- \in X_1'', \text{ and } x_1 < x\}. \tag{3.35}$$

From this together with a similar argument for $g''(x^+)$ when x is not a left endpoint, it follows that g'' is continuous and so is an order embedding. Since π'' is continuous, it follows from (3.31) that $G : X_1'' \to X$ is a continuous order map.

In particular, we obtain:

Lemma 3.13. *If X and X_1 are connected LOTS, $A \subset X$ is dense and $g : A \to X_1$ is an order injection, then $g'' : X' \to X_1'$ is an order embedding.*

Now we apply this construction.

Theorem 3.14. *(a) If X is a connected, nontrivial LOTS, then X contains a closed subset A which is a compact, separable LOTS with no isolated points.*

(b) *If C is a separable, compact LOTS with no isolated points, then C'' has the order type of the* Fat Cantor Set, *i.e.*

$$C'' \quad \cong \quad \bullet\mathbb{R}'\bullet\,. \tag{3.36}$$

Proof. (a): Let $a < b$ in X. By Proposition 2.15(a) there exists an order injection $g : \mathbb{Q} \to (a, b)$ where $\mathbb{Q}$ is the LOTS of rationals in $\mathbb{R}$. By replacing a, b if necessary we can assume that $g(\mathbb{Q})$ is $\pm$cofinal in (a, b). Since $\mathbb{R}$ and (a, b) are order dense and unbounded, $\mathbb{R}' = \mathbb{R}''$ and $(a, b)' = (a, b)''$. The above construction yields an order embedding

$$g'' : \mathbb{R}' \to (a, b)'. \tag{3.37}$$

Extend to $\bullet\mathbb{R}'\bullet$ by mapping m to a^+ and M to b^-. Because the image of g is $\pm$cofinal we obtain an order embedding of $\bullet\mathbb{R}'\bullet$ into $[a^+, b^-] \subset X'$. $A = (\pi' \circ g'')(\bullet\mathbb{R}'\bullet)$ is compact and separable by continuity. Since the Fat Cantor Set, $\bullet\mathbb{R}'\bullet$, has no isolated points and π' has finite point inverses, A has no isolated points.

(b): If C is compact with no isolated points, let

$$L \quad = \quad C \setminus (\{\text{right endpoints of } C\} \cup \{max\ C\}). \tag{3.38}$$

Regarded as a LOTS L is unbounded and it is order dense because if $x_1 < x_2$ in L, then x_2 is not a right endpoint and so (x_1, x_2) is infinite in C. If $y_1 < y_2$ are both right endpoints in this interval, then between them there is a left endpoint. Since C has no isolated points it follows that (x_1, x_2) meets L. Similarly, if y is a right endpoint of C and $y < x$ in C, then the infinite interval (y, x) meets L. Finally, if $x \in C$ is not the maximum, then the infinite interval (x, max) meets L. It follows that L is dense in C.

If C is separable, then by Proposition 2.11(f) L is separable and so contains a countable dense set D. So D is countable, order dense and unbounded. By Proposition 2.15(a) there exists an order isomorphism $g : \mathbb{Q} \to D \subset C$. Regarded as a not necessarily continuous order injection into C, g induces, as above, $g'' : \mathbb{R}' \to C''$ an order embedding. Extend by mapping m to $(min\ C)^+$ and M to $(max\ C)^-$. We thus have an order embedding $g'' : \bullet\mathbb{R}'\bullet \to C''$. By continuity the image is compact and hence closed in C''. If $x^- \in C''$, then x is the limit of an increasing sequence in D. If $x \neq max\ C$, then the sequence is bounded in D. Let $t \in \mathbb{R}$ be the limit of the corresponding increasing sequence in $\mathbb{Q}$. Clearly, $x^- = g''(t^-)$. With a similar argument for x^+ in C'' with $x \neq min\ A$, we see that g'' is onto. It is thus an order isomorphism.

$\square$

Example 3.15. *Embedding the Fat Cantor Set.*

Suppose that $g : \mathbb{Q} \to X$ is an order injection with image a bounded, countable discrete subset. For example, if $X = \mathbb{R} \times [-1, 1]$ we can let $g(a) = (a, 0)$ for $a \in \mathbb{Q} \cap (0, 1) \subset \mathbb{R}$. Then for each $t \in (0, 1)$,

$G(t^-) < G(t^+)$ and so $G : (0,1)' = (0,1)'' \to X$ is injective by (3.28). Since $g'' : \mathbb{R}' \to X'' = X'$ is an order embedding, $G = \pi'' \circ g''$ is continuous and so is an order embedding of $(0,1)'$. Extending to the two-point compactification we obtain an embedding of the Fat Cantor Set into X. In the case of $X = \mathbb{R} \times [-1,1]$ with $g(a) = (a,0)$ the embedding is given by $G(t^{\pm}) = (t, \pm 1)$ for $t \in [0,1]$ (omitting 0^- and 1^+).

3.3. **The CHLOTS Long Line.** In a CHLOTS F we label two points $-1 < +1$. With Ω the first uncountable ordinal we use the order product to define

$$F^{\Omega\infty} \quad = \quad (\Omega \times [-1,+1)) \setminus \{(0,-1)\} \tag{3.39}$$

In the case when $F = \mathbb{R}$ this is the *Long Line*.

If $\alpha < \Omega$, then α is a countable ordinal and so by Proposition 2.15(d) there exists an order embedding $f_\alpha : \alpha + 1 \to [-1,+1]$ with $f_\alpha(0) = -1$ and $f_\alpha(\alpha) = +1$. Choose for each $i \in \alpha$ an order isomorphism $\{i\} \times [-1,+1) \cong [f_\alpha(i), f_\alpha(i+1))$ and put them together to get an order isomorphism between the intervals $(-\infty, (\alpha,-1)) \subset F^{\Omega\infty}$ and $(-1.+1) \subset F$ which is isomorphic to F itself. It follows that every nonempty, open interval which is bounded above in $F^{\Omega\infty}$ is isomorphic to F. By Proposition 3.2(d)(vii)$\Rightarrow$(i) $F^{\Omega\infty}$ is doubly transitive. It is connected and first countable, but not σ-compact. It has countable coinitial subsets but every countable subset is bounded above.

If we apply the above construction to F^* and then reverse it, then we obtain ${}^{-\infty\Omega}F$ which has countable cofinal sets but no countable coinitial sets. Using the order sum we define

$${}^{-\infty\Omega}F^{\Omega\infty} \quad = \quad {}^{-\infty\Omega}F + \{0\} + F^{\Omega\infty} \tag{3.40}$$

with every countable subset bounded. In each case all of the nonempty, bounded, open subintervals are isomorphic to F and the entire space is doubly transitive but not weakly homogeneous.

We can extend Proposition 3.3 to completely describe the gap between double transitivity and homogeneity in the complete case.

Theorem 3.16. *Let X be a complete, doubly transitive LOTS and let F be a nonempty, bounded, open subinterval of X. F is a CHLOTS and X is order isomorphic to exactly one of the four spaces: $F, F^{\Omega\infty}$, ${}^{-\infty\Omega}F$, or ${}^{-\infty\Omega}F^{\Omega\infty}$.*

Proof. For a complete LOTS homogeneity is equivalent to weak homogeneity. Since X is complete, F is complete and so by Proposition

3.3(b) F is a CHLOTS. Also if X has countable coinitial and cofinal subsets, then it is σ-bounded and so is homogeneous by Proposition 3.3(b) again. In that case, $X \cong F$. Similarly, if X has countable coinitial subsets, then $X \cong (-\infty, a)$ for every $a \in X$.

Now assume that X has countable coinitial subsets but that every countable subset is bounded above. Inductively, we can construct an order embedding $f : \Omega \to X$. Since X is first countable the image of Ω cannot be bounded in X and so it is cofinal in X. By Proposition 2.15(c) continuity of f implies that

$$[f(1), \infty) = \bigcup \{[f(i), f(i+1)) : 0 < i < \Omega\}. \tag{3.41}$$

With points $-1 < +1$ fixed in F, choose for each i an isomorphism $\{i\} \times [-1, +1) \cong [f(i), f(i+1))$ and put them together to get an isomorphism from $[(1, -1), \infty) \subset F^{\Omega\infty}$ onto $[f(1), \infty) \subset X$. Put this together with an isomorphism $\{0\} \times (-1, +1) \cong (-\infty, f(1))$ to get an isomorphism $F^{\Omega\infty} \cong X$.

Applying this result to X^* we see that if X has countable cofinal subsets but no countable coinitial subsets, then ${}^{-\infty\Omega}F \cong X$. Finally, if every countable subset is bounded in X, then we can pick $a \in X$ and apply the previous results to show $(a, \infty) \cong F^{\Omega\infty}$ and $(-\infty, a) \cong {}^{-\infty\Omega}F$. So directly from (3.40) we see that $X \cong {}^{-\infty\Omega}F^{\Omega\infty}$ in this case.

$\square$

Example 3.17. *A first countable, doubly transitive LOTS whose completion is not first countable and so is not transitive.*

With X one of the long line examples above and α a countable tail-like ordinal with $\alpha > 1$, $Z = X^\alpha$ is first countable by Proposition 2.14 and is doubly transitive by Theorem 3.6.

Consider $X = F^{\Omega\infty}$. There is an embedding $g : \Omega \to X$. Now choose $x \in X$ and let $\tilde{g} : \Omega \to Z$ be defined for $\beta < \Omega$ by $\tilde{g}(\beta)_1 = g(\beta)$ and $\tilde{g}(\beta)_i = x$ for $i \neq 1$. In particular, $\tilde{g}(\beta)_0 = x$ for all $\beta < \Omega$ and so the image of $\tilde{g}$ is bounded in Z. Let $\hat{x}$ be the supremum of the image in the completion $\hat{Z}$. Clearly, $\hat{x}$ is not the limit of any countable increasing sequence.

4. **Towers of CHLOTS**

4.1. **The LOTS X_α.** In an unbounded LOTS X we pick out a distinguished closed, bounded subinterval J containing at least three points. We label the endpoints ± 1. Thus, $J = [-1, +1]$ and its interior $J^\circ = (-1, +1)$ is nonempty.

For every positive ordinal α we define the subset of the order space product X^α

$$X_\alpha = \{x \in X^\alpha : x_i \in J \text{ for all } 0 < i < \alpha\}. \tag{4.1}$$

Thus, X_α is the order space product indexed by α with the first factor X and the remaining factors copies of J. In particular,

$$\begin{aligned} X_1 &= X \\ X_2 &= X \times J \\ X_\omega &= X \times J \times J \times \ldots \end{aligned} \tag{4.2}$$

When X is weakly homogeneous, e.g. if X is a HLOTS, then it is order isomorphic to J° and so the space X_α is independent of the choice of interval J.

For $0 < \beta \leq \alpha$ we have projections $\pi^\alpha_\beta : X_\alpha \to X_\beta$. Identifying X_1 with X we have the special case $\pi^\alpha : X_\alpha \to X$, the projection to the first coordinate.

Following (2.20) we define for $z \in X_\beta$ with $0 < \beta < \alpha$ the points $z+$ and $z-$ in X_α by

$$(z\pm)_i = \begin{cases} z_i & i < \beta \\ \pm 1 & \beta \leq i < \alpha. \end{cases} \tag{4.3}$$

As in (2.21) we have for $a < b$ in X_β

$$\begin{aligned} (\pi^\alpha_\beta)^{-1}((a, b)) &= (a+, b-) \\ (\pi^\alpha_\beta)^{-1}([a, b]) &= [a-, b+]. \end{aligned} \tag{4.4}$$

Using (4.3) we define for $0 < \beta < \alpha$

$$\begin{aligned} j^\beta_\alpha &: (X_\beta)' \to X_\alpha \\ j^\beta_\alpha(z^\pm) &= z\pm. \end{aligned} \tag{4.5}$$

Proposition 4.1. *Let X be an unbounded, σ-bounded, order dense LOTS with distinguished subinterval $J = [-1, +1]$ and let α be a positive ordinal.*

(a) *X_α is unbounded, order dense and σ-bounded.*
(b) *If X is an IHLOTS, then X_α has dense holes.*

(c) *If X is connected, then X_α is connected and σ-compact.*
(d) *If X has countable type and α is countable, then X_α is of countable type. If α is uncountable then X_α is not even first countable.*
(e) *If $0 < \beta < \alpha$, then π^α_β is a continuous order surjection, closed when X is connected, and j^β_α is an order embedding onto a closed subset of X_α.*
(f) *If X is a HLOTS and $a < b$ in X, then $(a+, b-) \subset X_\alpha$ is order isomorphic with X_α itself.*

Proof. (a): X_α is order dense by Proposition 2.8(a) which also shows that each π^α_β is a continuous order surjection. Since X is unbounded and σ-bounded, X_α is unbounded and σ-bounded by Proposition 2.3(a) applied to the order surjection $\pi^\alpha : X_\alpha \to X$.

(b): If $z < w$ in X_α, let $\beta = min\{j : z_j \neq w_j\}$ so that $z_\beta < w_\beta$ and $z_i = w_i$ for $i < \beta$. By Proposition 3.8(d) X has dense holes and so there exists a clopen subset A of X such that $z_\beta \in A, w_\beta \notin A$, and $x \in A \ \Rightarrow \ (-\infty, x) \subset A$. Define

$$\tilde{A} = \{x \in X_\alpha : x < z\} \cup \{x \in X_\alpha : x_i = z_i \text{ for } i < \beta \text{ and } x_\beta \in A\}. \tag{4.6}$$

It is clear that $\tilde{A}$ defines a hole in X_α between z and w.

(c): If X is complete, then J is compact and so X_α is complete by Proposition 2.8(b). As X_α is order dense and σ-bounded, it is connected and σ-compact.

(d): Since X and J are of countable type, the first result follows from Proposition 2.13(b). If α is uncountable, then define $f : \alpha + 1 \to X_\alpha$ as in (2.34) with $a = (-1)-$ and $b = (+1)+$. Then f is an order embedding and since $\Omega \leq \alpha$ the point $f(\Omega)$ is defined in X_α. It is not the limit of any increasing sequence.

(e): For $z \in X_\beta$ the pre-image $(\pi^\alpha_\beta)^{-1}(z) = [z-, z+]$ which is closed and is compact when X, and therefore X_α, are connected. So π^α_β is continuous, and is closed in the connected case, by Proposition 2.3(a).

The complement of the image of j^β_α is the union of the collection of open intervals $\{(z-, z+) : z \in X_\beta\}$. In the case when X is complete, continuity follows from Proposition 2.3(b). In the general case, one checks directly that condition (2.2) holds for the image, i.e. if $a \in j^\beta_\alpha(X_\beta)$ and $x \in X_\alpha$ with $a < x$ and $(a, x] \cap j^\beta_\alpha(X_\beta) = \emptyset$ then $a = z-$ and $z_i = x_i$ for all $i \in \beta$. The required b is $z+$.

(f): If $\tilde{f} : X \to (a, b)$ is an order isomorphism, then

$$f(x)_i = \begin{cases} \tilde{f}(x_0) & i = 0 \\ x_i & 0 < i < \alpha \end{cases} \tag{4.7}$$

defines the required isomorphism $f : X_\alpha \to (a+, b-)$.

□

Theorem 4.2. *Let α be a countable, tail-like ordinal and X be a LOTS.*

(i) *If X is doubly transitive and first countable, then X_α is doubly transitive and first countable.*
(ii) *If X is an IHLOTS, then X_α is an IHLOTS.*
(iii) *If X is a CHLOTS, then X_α is a CHLOTS.*

Proof. In any case X_α is first countable by Proposition 2.14. By Proposition 4.1 X_α has dense holes if X is an IHLOTS and it is complete if X is a CHLOTS. Furthermore, in the HLOTS cases X_α has countable type. We will show that if α is countable and tail-like and X is doubly transitive, then X_α is doubly transitive. The HLOTS results then follow from Proposition 3.8.

If $\alpha = 1$ then $X_\alpha = X$ which is doubly transitive. Thus, we can assume that $\alpha > 1$ and so that α is a limit ordinal.

Choose $0 \in J^\circ$ so that $-1 < 0 < +1$. Define $\tilde{a} < \tilde{c} < \tilde{b}$ in X_α by

$$(4.8) \qquad \begin{array}{cccc} \tilde{a}_0 = -1 & \tilde{c}_0 = 0 & \tilde{b}_0 = +1 & \\ \tilde{a}_i = +1 & \tilde{c}_i = 0 & \tilde{b}_i = -1 & 0 < i < \alpha. \end{array}$$

Given an arbitrary pair $a < b$ in X_α, it suffices to prove $(a, b) \cong (\tilde{a}, \tilde{b})$.

Let $\beta = min\ \{j : a_j \neq b_j\}$, so that $a_\beta < b_\beta$ and $a_j = b_j$ for $j < \beta$. Because X is order dense, we can choose $c \in X_\alpha$ such that

$$(4.9) \qquad \begin{array}{ll} a_i = c_i = b_i & \text{for } i < \beta \\ a_\beta < c_\beta < b_\beta & \text{for } i = \beta \\ c_i = 0 & \text{for } \beta < i < \alpha. \end{array}$$

We will construct an order isomorphism $[c, b) \cong [\tilde{c}, \tilde{b})$. Applying a similar argument (or the same argument to X^*) we obtain an isomorphism $(a, c] \cong (\tilde{a}, \tilde{c}]$. Putting them together we get $(a, b) \cong (\tilde{a}, \tilde{b})$ as required.

Now define for the purposes of this proof (i.e. ignore (3.10) for the duration)

$$(4.10) \qquad \begin{aligned} K &= \{\beta\} \cup \{k : \beta < k < \alpha \text{ and } b_k > -1 \text{ in } J\} \\ K' &= K \cup \{\alpha\}. \end{aligned}$$

Thus, K' is a subset of $\alpha + 1$ and so is a countable well-ordered set. We define $f : K' \to X_\alpha$ by

$$\begin{aligned} f(\beta) &= c \\ f(k)_i &= \begin{cases} b_i & i < k \\ -1 & k \leq i < \alpha \end{cases} \qquad \text{for} \quad \beta < k \leq \alpha. \end{aligned} \tag{4.11}$$

In particular, $f(\alpha) = b$.

For $k \in K'$ let k' denote its successor in the well-ordered set K'. For $k \in K' \setminus \{\beta\}$ we have

$$\begin{aligned} f(k)_i \;&=\; f(k')_i \;=\; b_i \qquad \text{for} \quad i < k \\ -1 \;=\; f(k)_k \;&<\; f(k')_k \;=\; b_k \\ f(k)_i \;=\; -1 \;&=\; b_i \;=\; f(k')_i \qquad \text{for} \quad k < i < k' \\ f(k)_i \;&=\; -1 \;=\; f(k')_i \qquad \text{for} \quad k' \leq i < \alpha. \end{aligned} \tag{4.12}$$

Thus, f is an order injection.

Notice that if $x \in X_\alpha$ and $k \in K$ with $k > \beta$, then $x_j \geq f(k)_j = -1$ for all $j \geq k$ and $x_j \geq -1$ for all $j > \beta$ with $j \notin K$.

Thus, if k is a limit element of K and $x \in X_\alpha$ with $x < f(k)$ and j the minimum at which $x_j \neq f(k)_j$, then $x_j < f(k)_j$. So either $j \leq \beta < k$ or else $j \in K$ with $j < k$. In either case, there exists some $\tilde{k} \in K$ with $max(j, \beta) < \tilde{k} < k$ so that $x_j < f(\tilde{k})_j = f(k)_j$ and so $x < f(\tilde{k})$. It follows that f is continuous and so Proposition 2.15(c) implies that

$$[c, b) \;=\; \bigcup_{k \in K} [f(k), f(k')). \tag{4.13}$$

By Proposition 2.15(d) there exists an order embedding $\tilde{f}_0 : K' \to [0, +1]$ with $\tilde{f}_0(\beta) = 0$ and $\tilde{f}_0(\alpha) = +1$. Now define $\tilde{f} : K' \to X_\alpha$ by

$$\begin{aligned} \tilde{f}(k)_0 \;&=\; \tilde{f}_0(k) \qquad \text{for} \quad k \in K' \\ \tilde{f}(\beta)_i \;&=\; 0 \qquad \text{for} \quad 0 < i < \alpha \\ \tilde{f}(k)_i \;&=\; -1 \qquad \text{for} \quad 0 < i < \alpha \quad \text{and} \quad k \in K' \setminus \{\beta\}. \end{aligned} \tag{4.14}$$

Clearly, $\tilde{f}$ is an order embedding and again continuity implies

$$[\tilde{c}, \tilde{b}) \;=\; \bigcup_{k \in K} [\tilde{f}(k), \tilde{f}(k')). \tag{4.15}$$

Notice that $\tilde{f}(\beta) = \tilde{c}$ and $\tilde{f}(\alpha) = \tilde{b}$.

Because X is doubly transitive, we can choose for each $k \in K$ an order isomorphism between intervals in J :
(4.16)
$$q_k : [f(k)_k, f(k')_k) = [-1, b_k) \to [\tilde{f}(k)_0, \tilde{f}(k')_0) = [\tilde{f}_0(k), \tilde{f}_0(k')).$$

Because α is tail-like, there exists for each $k \in K$ a unique order isomorphism $\tau^k : \alpha \to \alpha \setminus k = \{\epsilon : k \leq \epsilon < \alpha\}$. Define for each $k \in K$ the map between intervals of X_α

$$Q_k : [f(k), f(k')) \to [\tilde{f}(k), \tilde{f}(k'))$$

(4.17)
$$Q_k(x)_i = \begin{cases} q_k(x_k) & \text{for } i = 0 \\ x_{\tau^k(i)} & \text{for } 0 < i < \alpha. \end{cases}$$

Notice that by (4.12) $x_i = b_i$ for all $i < k$ when $x \in [f(k), f(k'))$. It follows that each Q_k is an order isomorphism.

Putting together these isomorphisms we obtain the required isomorphism $[c, b) \cong [\tilde{c}, \tilde{b})$.

□

4.2. **Size Comparisons.** In distinguishing between CHLOTS we define a rough order of size.

Definition 4.3. *For LOTS X and X_1 we say that X* injects into X_1 *if there exists an order injection $g : X \to X_1$. We say that X_1 is* bigger than X *if X injects into X_1 but not the reverse. When neither injects into the other we say their sizes are* not comparable. *On the other hand, we say that X* has the same size as X_1 *when each injects into the other. Finally, we say that the size of X* lies between X_1 *and X_2 when X_1 injects into X and X injects into X_2.*

This is the usual crude partial ordering used to compare order types. We will now see that for CHLOTS X and X_1, X injects into X_1 iff there exists an order surjection $f : X_1 \to X$. By Proposition 2.3(a) such a surjection is always continuous. On the other hand, there exists a continuous order injection, i.e. an order embedding, $g : X \to X_1$ iff $X \cong X_1$.

Proposition 4.4. (a) *If $f : X_1 \to X$ is an order surjection of LOTS then there exists a map $g : X \to X_1$ such that $f \circ g = 1_X$. Any such map g is an order injection.*

(b) *Assume X is connected. If $g : X \to X_1$ is an order injection with image $g(X)$ $\pm$cofinal in X_1, then there exists a continuous order surjection $f : X_1 \to X$ such that $f \circ g = 1_X$.*
(c) *Assume that X_1 is a HLOTS and that X is unbounded. If there exists an order injection of X into X_1, then there exists an order injection with image $\pm$cofinal in X_1.*
(d) *Assume that X_1 and X are CHLOTS. If there exists a non-constant, continuous order map from X_1 to X, then X injects into X_1. If there exists an order embedding of X into X_1, then $X_1 \cong X$.*

Proof. (a): This is a repeat of Proposition 2.3(d).

(b): Define for $x \in X$ the closed convex set $J_x \subset X_1$ by

$$X_1 \setminus J_x \;=\; (\bigcup\{(-\infty, g(a)) : a < x\}) \cup (\bigcup\{(g(b), \infty) : b > x\}). \tag{4.18}$$

If $x_1 < x_2$ in X then because X is order dense we can choose a, b such that $x_1 < b < a < x_2$. Since $g(b) < g(a)$

$$(-\infty, g(a)) \cup (g(b), \infty) \quad = \quad X_1 \tag{4.19}$$

and so $J_{x_1} \cap J_{x_2} = \emptyset$.

If $y \in X_1$ equals $g(x)$ then $y \in J_x$. In particular, J_x is nonempty. If $y \not\in g(X)$ then because the image is cofinal and coinitial, the pair $g^{-1}((-\infty, y)), g^{-1}((y, \infty))$ is a partition of X by nonempty convex sets. By completeness of X we can define $x = inf \ g^{-1}((y, \infty))$. If $g(b) < y$, then b is a lower bound for $g^{-1}((y, \infty))$ and so $b \leq x$. Contrapositively, $b > x$ implies $g(b) > y$. If $g(a) > y$, then $a \in g^{-1}((y, \infty))$ and so $x \leq a$. Contrapositively, $a < x$ implies $g(a) \leq y$. Thus, $y \in J_x$.

It follows that $\{J_x : x \in X\}$ is an X indexed family of nonempty, closed convex sets with union X_1. So mapping J_x to x defines an order surjection $f : X_1 \to X$ which is continuous by Proposition 2.3(a) because each $f^{-1}(x) = J_x$ is closed. If $x \in X$, then $g(x) \in J_x$ implies $f(g(x)) = x$.

(c): Define $J = \{y \in X_1 : \text{for some } x_1, x_2 \in X \quad g(x_1) < y < g(x_2)\}$. J is an open, convex subset of X_1. Because X is unbounded, $g(X)$ is a subset of J which is $\pm$cofinal in J. Because X_1 is a HLOTS there exists an order isomorphism $q : J \to X_1$. Replace g by $q \circ g$.

(d): If $f : X_1 \to X$ is a continuous order map and $c < d$ in the image $f(X_1)$, then let $a = sup f^{-1}(c)$ and $b = inf f^{-1}(d)$. The interval $[a, b]$ is compact and connected and so by continuity of f the image is as well. Since f is an order map $f([a, b])$ is a compact, connected subset of $[c, d]$ which contains c and d. Hence, $f([a, b]) = [c, d]$ and so, by definition of a and b, $f((a, b)) = (c, d)$. Let $h_1 : X_1 \to (a, b)$ and $h : (c, d) \to X$

be order isomorphisms. Then $f_1 = h \circ f \circ h_1$ is an order surjection of X_1 onto X and so X injects into X_1 by (a).

If, in addition, f is injective then the restriction to (a, b) is an order isomorphism to (c, d) and so f_1 is an isomorphism.

□

As an immediate consequence we obtain the following.

Corollary 4.5. *If X and X_1 are CHLOTS then X injects into X_1 iff there exists an order surjection $g : X_1 \to X$. Such an order surjection is necessarily continuous.*

We can strengthen this result.

Theorem 4.6. *If X and X_1 are connected LOTS and $f : X_1 \to X$ is a non-constant continuous map, then either f is an order* map onto a non-trivial interval in X, or there exists an order surjection from X_1 onto a non-trivial interval in X. If X is a CHLOTS, then there exists an injection from X to X_1 which is either order-preserving or order-reversing.*

Proof. Because the LOTS are connected and f is continuous and non-constant the image of f is an interval in X in any case. If f is order-reversing then by applying Proposition 4.4 to the reverse order, we obtain an order* injection to X_1 from an interval in X.

Assume f is not order-reversing so that there exist $a < b$ in X_1 such that with $c = f(a), d = f(b)$ we have $c < d$. We will construct an order preserving injection $g : [c, d] \to [a, b]$. Let $g(c) = a, g(d) = b$. For $x \in (c, d)$, let $g(x) = sup f^{-1}(x) \cap [a, b]$ which is not empty because $f([a, b])$ is connected and contains $[c, d]$. Because f is continuous, $f^{-1}(x)$ is closed and so $f(g(x)) = x$. If $x < x_1 < d$ then $f([g(x), b])$ contains $[x, d]$ and so there exist points of $f^{-1}(x_1)$ between $g(x)$ and b. Hence, $g(x) < g(x_1) < b$. That is, g is an order injection.

It now follows from Proposition 4.4(b) that with $a_1 = inf g((c, d))$ and $b_1 = sup g((c, d))$ there exists a - necessarily continuous - order surjection from $[a_1, b_1]$ to $[c, d]$. Extending by constants below a_1 and above b_1 we obtain an order surjection from X_1 onto $[c, d]$.

Thus, we obtain an injection to X_1 from a non-trivial open interval in X. If X is a CHLOTS we can precede by an isomorphism from X to the open interval and get an injection from X to X_1.

□

Proposition 4.7. *Let X and X_1 be LOTS.*

(a) *Let $f : X \to X_1$ be an order map. Assume that X is order dense and that D is a dense subset of X. If the restriction $f|D$ is injective, then f is an order injection, i.e. it is injective on all of X.*

(b) *Assume that X is order dense and unbounded and that X_1 is complete. If X injects into X_1, then the completion $\hat{X}$ injects into X_1.*

(c) *Let $\alpha \leq \beta$ be positive ordinals. If X injects into X_1 then X^α injects into X_1^β. If, in addition, X and X_1 are HLOTS then X_α injects into $(X_1)_\beta$.*

(d) *If α and β are positive ordinals then*

$$(X^\alpha)^\beta \quad \cong X^{\alpha\cdot\beta}. \tag{4.20}$$

If, in addition, X is a HLOTS then

$$(X_\alpha)_\beta \quad \cong X_{\alpha\cdot\beta}. \tag{4.21}$$

Proof. (a): This is a repeat of Proposition 2.3 (e).

(b): If $f : X \to X_1$ is an order injection, then $\hat{f}$ is an order injection by Proposition 2.9.

(c): If $f : X \to X_1$ is an order injection and $z \in X_1$ then we can define the injection $\tilde{f} : X^\beta \to X_1^\alpha$ by

$$\tilde{f}(x)_i = \begin{cases} f(x_i) & \text{for } i < \beta \\ z & \text{for } \beta \leq i < \alpha. \end{cases} \tag{4.22}$$

In the HLOTS case, we can assume that the distinguished intervals have been chosen so that $f(J) \subset J_1$ and that $z \in J_1$. Then $\tilde{f}$ restricts to an injection of X_β into $(X_1)_\alpha$.

(d): We see first that the natural bijection $q : (X^\alpha)^\beta \to X^{\beta\times\alpha}$ given by $q(x)(i,j) = x(i)(j)$ preserves the orders.

For $q(x_1) < q(x_2)$ means that for some $(i,j) \in \beta \times \alpha$ $q(x_1)(i,j) < q(x_2)(i,j)$ and $q(x_1)(k,\ell) = q(x_2)(k,\ell)$ for all $(k,\ell) < (i,j)$. So for $k < i, x_1(k) = x_2(k)$. Furthermore, $x_1(i)(\ell) = x_2(i)(\ell)$ for $\ell < j$, while $x_1(i)(j) < x_2(i)(j)$. That is, $x_1(i) < x_2(i)$. Consequently, $x_1 < x_2$. Since q is a bijection it is an order isomorphism.

The result follows because $\beta \times \alpha = \alpha \cdot \beta$.

In the HLOTS case with J the distinguished interval in X we choose $J^\alpha = (\pi^\alpha)^{-1}(J)$ as the distinguished interval in X^α. It is an interval by (4.4). Then q maps $(X_\alpha)_\beta$ onto $X_{\alpha\cdot\beta}$.

□

Remark. Notice that for an unbounded, connected LOTS X the projection $\pi' : X' \to X$ is injective on the dense subset $D = X \times \{-1\}$ but not itself injective. Thus, the order dense hypothesis in (a) is required.

We will now show that for a CHLOTS X and positive ordinals $\alpha > \beta$ it is always true that X_α is bigger than X_β.

Definition 4.8. *A LOTS X is called* order simple *if X' is bigger than X where X' is the AS double of X.*

It is always true that X injects into X', e.g. use $x \mapsto x^-$. So X is order simple when X' does not inject into X.

Proposition 4.9. (a) *If X is an uncountable, order dense LOTS which satisfies the countable chain condition, then X is order simple.*
(b) *If X_1 and X_2 are LOTS of the same size, then X_1 is order simple iff X_2 is.*
(c) *If X is order simple, then the reverse X^* is order simple.*
(d) *If X is order simple, then there does not exist an injective order* map from X' into X.*

Proof. (a) If $f : X' \to X$ is an order injection for any LOTS X, then $\{(f(z^-), f(z^+)) : z \in X\}$ is a family of open intervals in X and each is nonempty if X is order dense. If $z_1 < z_2$ in X, then $f(z_1^+) < f(z_2^-)$ so that the intervals are pairwise disjoint. If X satisfies c.c.c., then X must be countable.

(b) If $f : X_2 \to X_1$ is an order injection, then from (3.13) we obtain the order injection $f' : X_2' \to X_1'$. So if X_1 is at least as big as X_2 then X_1' is at least as big as X_2'. Thus, if X_1 and X_2 have the same size and X_1' has the same size as X_1, then X_2' has the same size as well.

(c) It is clear that

$$(X^*)' \quad = \quad (X')^*. \tag{4.23}$$

So any order injection of X' into X is an order injection of $(X^*)'$ into X^*.

(d) Define the map

$$\begin{aligned} q : X' &\to (X')' \\ q(x^\pm) \quad &= \quad (x^\pm)^\pm. \end{aligned} \tag{4.24}$$

This is an order embedding of X' onto the closed set whose complement is the open set of isolated points $\{(x^+)^-, (x^-)^+ : x \in X\}$. Now if

$g : X' \to X$ is an order reversing injection, then we use (3.15) to define an order injection from X' to X as the composition:

$$X' \xrightarrow{q} (X')' \xrightarrow{g^*} X' \xrightarrow{g} X.$$

$\square$

Remark. If X is unbounded and order dense so that X' has no isolated points, then the map q of (4.24) is actually an order isomorphism of X' onto $(X')''$ defined via (3.23).

In the Remark after Proposition 3.11 we observed that $\mathbb{Z}' \cong \mathbb{Z}$ and so $\mathbb{Z}$ is not order simple. By Proposition 2.15(a) $\mathbb{Q}$ is not order simple. There exist connected LOTS which are not order simple as well. With $J = [-1, +1] \subset \mathbb{R}$ define

$$\begin{aligned} X_n &= [n, n+1) \times J^n \\ X &= \Sigma_{n \in \omega} X_n. \end{aligned} \tag{4.25}$$

It is easy to see that X is connected with $min = 0$ in $X_0 = [0, 1)$. The order isomorphisms

$$\begin{gathered} f_n : X_n \times J \to X_{n+1} \qquad \text{for} \quad n \in \omega \\ f_n(x, t)_i = \begin{cases} x_0 + 1 & i = 0 \\ x_i & 0 < i \leq n \\ t & i = n+1 \end{cases} \end{gathered} \tag{4.26}$$

can be put together to get an order embedding of $X \times J$ into X. Since $X' \subset X \times J$, it follows that X is not order simple. Notice that X has no max.

Lemma 4.10. ***(The Shift Lemma)*** *Let X be a complete LOTS with $min = m$. If $f : X' \to X$ is an order injection, then for all $x \in X$*

$$f(x^+) \quad > \quad x. \tag{4.27}$$

In particular, X has no max.

Proof. Define $S = \{a \in X : f(x^+) > x \text{ for all } x \leq a\}$. Hence, $a \in S$ implies $(-\infty, a] \subset S$. Since $m = min\ X$

$$m \quad \leq \quad f(m^-) \quad < \quad f(m^+) \tag{4.28}$$

and so $m \in S$. It suffices to show that S is unbounded as this implies $S = X$.

Assume S is bounded and let $z = sup\ S$.

First we show that $z \in S$. If not, then $a < z$ for all $a \in S$ and so $a^+ < z^-$. Hence

$$a \quad < \quad f(a^+) \quad < \quad f(z^-) \quad < \quad f(z^+). \tag{4.29}$$

Thus, $f(z^-)$ is an upper bound for S. Since $z = sup\ S$

$$z \quad \leq \quad f(z^-) \quad < \quad f(z^+). \tag{4.30}$$

As all $x < z$ are not upper bounds for S they are elements of S. It follows that $z \in S$ after all.

Now let $y = f(z^+) > z$. If $z < x \leq y$ then $z^+ < x^+$ and so

$$x \quad \leq \quad y \quad = \quad f(z^+) \quad < \quad f(x^+). \tag{4.31}$$

Thus, $y \in S$. Since $y > z$, this contradicts the assumption that $z = sup\ S$.

X has no *max* because S is unbounded. Also, $x = max$ could not satisfy (4.27).

$\square$

Corollary 4.11. *A LOTS X is order simple if it satisfies one of the following conditions:*

(i) *X is compact.*
(ii) *X is doubly transitive and complete.*
(iii) *$X \cong (X_1)_\alpha$ where X_1 is any CHLOTS and α is any positive ordinal.*

Proof. (i) The Shift Lemma implies that if a complete LOTS X is not order simple and has a *min*, then it has no *max*. In particular, a compact LOTS is order simple.

(ii) Assume that $f : X' \to X$ is an order map with X doubly transitive and complete. For any pair $a < b$ in X define $\tilde{a} = f(a^-)$ and $\tilde{b} = f(b^+)$. If $\tilde{a} = \tilde{b}$, then f is not injective. If $\tilde{a} \neq \tilde{b}$, then f maps $[a, b]'$ into $[\tilde{a}, \tilde{b}]$. Because X is doubly transitive we can choose an order isomorphism $g : [\tilde{a}, \tilde{b}] \to [a, b]$. The composition $g \circ f : [a, b]' \to [a, b]$ is an order map. Since $[a, b]$ is compact, (i) implies that this map is not injective and so f is not injective. It follows that X is order simple.

(iii) For the closed interval J in X_1,the subset J^α of $(X_1)_\alpha$ is compact and so is order simple. If $a < b$ in J then Proposition 4.1(f) implies that $(X_1)_\alpha$ is order isomorphic to the subset $(a+, b-)$ of J^α. Hence, $(X_1)_\alpha$ has the same size as J^α and so is order simple by Proposition 4.9(b).

$\square$

Theorem 4.12. *Assume that X is a CHLOTS. If $\alpha > \beta$ are positive ordinals then X_α is bigger than X_β. Furthermore, X_α is not homeomorphic to X_β.*

Proof. Recall that j^β_α injects X'_β into X_α, see (4.5). If $f : X_\alpha \to X_\beta$ is an injective map then the composite $f \circ j^\beta_\alpha : X'_\beta \to X_\beta$ is an injection which is order preserving or order reversing if f is. Because X_β is order simple by Corollary 4.11, f cannot be order preserving and by Proposition 4.9(d) it cannot be order reversing either.

If $f : X_\alpha \to X_\beta$ were a homeomorphism, then by Lemma 3.1 it would be either order preserving or reversing.

□

Corollary 4.13. *If X is a CHLOTS and α is a positive ordinal such that X_α is transitive, then α is countable and tail-like.*

Proof. Since X_α is connected, transitivity implies first countability by Proposition 3.2(c). By Proposition 4.1(d), α is countable.

Now assume that α is not tail-like. This means that there exists $\beta < \alpha$ and $\epsilon < \alpha$ such that $\epsilon \cong \alpha \setminus \beta = \{i : \beta \leq i < \alpha\}$. Choose $b \in X_\alpha$ such that $b_\beta \in J^\circ$, i.e. $-1 < b_\beta < +1$ in X. Define $a = (-1)-$ in X_α, i.e. $a_i = -1$ for all $i < \alpha$. We will obtain a contradiction from the assumption that there exists $f \in H_+(X_\alpha)$ such that $f(a) = b$.

Define $c, d \in X_\alpha$ by

$$\begin{aligned} c_i \ = \ d_i \ &= \ b_i \qquad \text{for} \quad i < \beta \\ c_\beta \ = \ -1, \qquad & d_\beta \ = \ +1 \\ c_i \ = \ +1, \qquad d_i \ &= \ -1 \qquad \text{for} \quad \beta < i < \alpha. \end{aligned} \tag{4.32}$$

Clearly, we have $c < b < d$ and since $\epsilon \cong \alpha \setminus \beta$ Proposition 4.1 (f) implies

$$(c, d) \quad \cong \quad X_\epsilon. \tag{4.33}$$

By continuity of f, $f^{-1}((c, d))$ contains a neighborhood of a and so contains some interval (x, a) with $x < a$ in X_α. This implies $x_0 < a_0 = -1$ in X and so we can choose $\tilde{a}, \tilde{b} \in X$ such that $x_0 < \tilde{a} < \tilde{b} < a_0$ in X. Because $x < \tilde{a}+ < \tilde{b}- < a$ in X_α we have

$$(\tilde{a}+, \tilde{b}-) \quad \subset \quad f^{-1}((c, d)). \tag{4.34}$$

By Proposition 4.1(f) again

$$X_\alpha \quad \cong \quad (\tilde{a}+, \tilde{b}-). \tag{4.35}$$

Composing the isomorphism of (4.35) with the restriction of f and the isomorphism of (4.33) we get an order injection of X_α into X_ϵ contradicting Theorem 4.12.

$\square$

For any connected unbounded LOTS X, e.g. any CHLOTS, the associated *Cantor Space for* X, denoted $C(X)$, is the two point compactification

$$C(X) \quad = \quad \bullet X' \bullet \tag{4.36}$$

where X' is the AS double. In particular, the Cantor Space for $\mathbb{R}$ is the Fat Cantor Set.

Because an isomorphism maps *max* to *max* and *min* to *min*,(3.16) implies for connected unbounded LOTS X, X_1

$$C(X) \;\cong\; C(X_1) \quad \Longleftrightarrow \quad X \;\cong\; X_1. \tag{4.37}$$

For any positive ordinal α we have

$$C(X_\alpha) \quad = \quad \bullet (X_\alpha)' \bullet . \tag{4.38}$$

If X is a CHLOTS, then for $a < b$ in X the isomorphism $f : X_\alpha \to (a+, b-)$ of Proposition 4.1(f) induces the isomorphism $f' : (X_\alpha)' \to ((a+)^+, (b-)^-)$ which extends to the two-point compactification to show

$$C(X_\alpha) \quad \cong \quad [(a+)^+, (b-)^-]. \tag{4.39}$$

Theorem 4.14. *Assume that X is a CHLOTS. If $\alpha > \beta$ are positive ordinals then $C(X_\alpha)$ is bigger than X_α which is bigger than $C(X_\beta)$.*

Proof. Because X_α is order simple, there is no order injection from $(X_\alpha)'$ into it. $C(X_\alpha)$ projects onto X_α and contains $(X_\alpha)'$. Hence, $C(X_\alpha)$ is bigger than X_α.

By (4.39) $C(X_\beta)$ is the same size as $(X_\beta)'$. Now choose $a < b$ in $J^\circ \subset X$. Clearly, we have

$$\begin{aligned} \{-1,+1\} \times \{-1,+1\} \quad &\cong \quad \{-1,a,b,+1\} \qquad \text{and so} \\ ((X_\beta)')' \quad &\cong \quad X_\beta \times \{-1,a,b,+1\} \quad \subset \quad X_{\beta+1}. \end{aligned} \tag{4.40}$$

Hence, there is an order injection from $C(X_\beta)'$ into $X_{\beta+1}$ which injects into X_α. If X_α were to inject into $C(X_\beta)$, then $C(X_\beta)$ would not be order simple, contradicting Corollary 4.11.

$\square$

Remark. Because $C(X_\alpha)$ and $C(X_\beta)$ are not connected, we cannot use Lemma 3.1 to show that they are topologically distinct. As far as we know it may happen that for some $\alpha \neq \beta$ $C(X_\alpha)$ and $C(X_\beta)$ are homeomorphic. On the other hand, it is clear that $C(X)$ is separable or satisfies c.c.c. iff X satisfies the corresponding property. So the original Fat Cantor Set $C(\mathbb{R})$ is the only Cantor Space of a CHLOTS which is separable.

In summary, we have the following.

Theorem 4.15. *If F is a CHLOTS, then F_{ω^γ} is a tower of CHLOTS strictly increasing in size and no two distinct members of which are homeomorphic. The tower $C(F_{\omega^\gamma})$ of CHLOTS Cantor Spaces is also strictly increasing in size.*

5. **Trees**

5.1. **Trees and Bi-Ordered Trees.** For the theory of trees we follow [13] Section 22.

If $(T, \succ)$ is a partially ordered set then for $p \in T$ we define the *tail set*, the *predecessor set* and the *successor set* of p:

$$\begin{aligned} T_p &= \{q \in T : q \succeq p\} \\ A_p &= \{q \in T : q \prec p\} \\ S_p &= \{q \in T : q \succ p \text{ and } \nexists r \in T \text{ such that } q \succ r \succ p\}. \end{aligned} \tag{5.1}$$

A non-empty, partially ordered set $(T, \succ)$ is called a *tree* when A_p is well-ordered by $\prec$ for each $p \in T$. The elements of T are then called the *vertices* of T. If $p \in T$ then the *order* of p, denoted $o(p)$ is the ordinal whose order type is that of A_p, i.e. there is a unique order isomorphism from $o(p)$ onto A_p. The bijection can be extended to $o(p) + 1$ by mapping $o(p)$ to p. For any ordinal α the *level* α set is

$$L_\alpha = \{p \in T : o(p) = \alpha\}. \tag{5.2}$$

The successors of p are the points of T_p at the next level, i.e.

$$o(p) = \alpha \quad \Longrightarrow \quad S_p = T_p \cap L_{\alpha+1}. \tag{5.3}$$

If A is a nonempty subset of T then we define its *height* by

$$\begin{aligned} h(A) &= sup\{o(p) + 1 : p \in A\} \\ &= min\{\alpha : o(p) < \alpha \text{ for all } p \in A\}. \end{aligned} \tag{5.4}$$

Any subset of a tree is a tree in its own right, leading to different notions of order and height for elements of the subset. The two concepts agree when $R \subset T$ is a *subtree* defined by the condition

$$p \in R \quad \Longrightarrow \quad A_p \subset R. \tag{5.5}$$

For example, for each positive ordinal α we define the α *truncation* subtree

$$T^\alpha \ = \ \{p \in T : o(p) < \alpha\} \ = \ \bigcup_{\beta < \alpha} L_\beta. \tag{5.6}$$

A *branch* x of a tree T is a maximal, linearly ordered subset of T. Because the set of predecessors of any vertex p of T is totally ordered, any branch of T is a subtree by maximality. A branch x is a well-ordered set whose order type is that of the ordinal $h(x)$. By Zorn's Lemma every subset of T linearly ordered by $\succ$ is contained in some branch. In particular, each vertex lies in some branch.

We denote by $X(T)$ the *branch space* of the tree T, i.e. the set of branches of T.

Since there is a branch through every vertex

$$h(T) \quad = \quad sup\{h(x) : x \in X(T)\}. \tag{5.7}$$

We define for a branch x and any ordinal $\alpha < h(x)$ the vertex x_α to be the - unique - level α element of x, i.e.

$$\{x_\alpha\} \quad = \quad x \cap L_\alpha. \tag{5.8}$$

Let p be a vertex of T. The tail set T_p is a tree but not a subtree of T. For $q \in T_p$ let $o_p(q)$ denote its order in the tree T_p. Clearly,

$$o(q) \quad = \quad o(p) \ + \ o_p(q), \tag{5.9}$$

using ordinal addition.

If x is a branch of T then

$$\begin{gathered} x \cap T_p \neq \emptyset \quad \Longleftrightarrow \quad p \in x \\ \text{in which case} \quad p = x_{o(p)}. \end{gathered} \tag{5.10}$$

In that case, $x \cap T_p$ is a branch of T_p. On the other hand, if y is a branch of T_p then

$$j_p(y) \quad = \quad y \cup A_p \tag{5.11}$$

is a branch of T. Hence, (5.11) defines a bijection

$$j_p : X(T_p) \to \{x \in X(T) : p \in x\}, \tag{5.12}$$

and we have for all $y \in X(T_p)$

$$h(j_p(y)) \quad = \quad o(p) + h_p(y), \tag{5.13}$$

where h_p denotes the height with respect to T_p.

We call a subset A of a tree T an *antichain* if no two vertices in A are comparable with respect to $\succ$. For example, L_α is an antichain for any ordinal α.

Let $\#S$ denote the cardinality of a set S. Following [13] Chapter 4:

Definition 5.1. *A tree T is called a* semi-normal tree *when it satisfies the following conditions*

(i) $\#L_0 = 1$*, i.e. T has a* root *which we denote $0 \in T$.*
(ii) *For all $p \in T$, $\#S_p \neq 1$.*
(iii) *If $p, q \in T$ with $o(p) = o(q)$ a limit ordinal and with $A_p = A_q$, then $p = q$.*

T is called a normal *tree when it is semi-normal and, in addition, satisfies the condition*

(iv) *If $p \in T$ and α is an ordinal with $o(p) \leq \alpha < h(T)$, then there exists $q \in T_p$ with $o(q) = \alpha$.*

This implies

(v) *If $p \in T$ and $o(p) + 1 < h(T)$, then $S_p \neq \emptyset$ and so $\#S_p > 1$.*

We call a tree Ω-bounded *when it satisfies*

(vi) *$h(x) < \Omega$ for all $x \in X(T)$ and so $h(T) \leq \Omega$.*

An Aronszajn tree *is a normal tree of height Ω, the first uncountable ordinal, which satisfies (vi) and*

(vii) *If $p \in T$ and $o(p) + 1 < h(T)$, then S_p is an infinite set.*
(viii) *L_α is a countable set for each $\alpha < \Omega$.*

A Suslin tree *is a normal tree of height Ω which satisfies (vii) and*

(ix) *Every antichain in T is a countable set.*

N. B. From now on we will assume that all trees are at least semi-normal, unless otherwise mentioned.

Notice that a branch x is a subtree which is not semi-normal as each vertex $p \in x$ with $o(p) + 1 < h(x)$ has a single successor in the branch.

Condition (ii) says that if $p \in T$ has any successors then it has at least two. Thus, a vertex is either terminal, i.e. $S_p = \emptyset$ or the tree branches in at least two directions after p. For a normal tree, the latter always happens unless $o(p) + 1 = h(T)$.

If T is a normal tree, $0 < \alpha < h(T)$ and $p \in L_\alpha$ then T^α and T_p are normal trees with

$$\begin{aligned} h(T^\alpha) &= \alpha \\ h(T_p) &\cong h(T) \setminus \alpha. \end{aligned} \tag{5.14}$$

Since L_α is an antichain, (ix) implies (viii) in Definition 5.1. Furthermore, if $x \in X(T)$ then we can choose for each α with $\alpha + 1 < h(x)$ a successor y_α of x_α different from $x_{\alpha+1}$. The set $\{y_\alpha : \alpha + 1 < h(x)\}$ is an antichain. It follows that (ix) implies (vi). Thus, every Suslin tree is Aronszajn.

If T is normal and $x \in X(T)$, then by (v) either $h(x) = h(T)$ or $h(x)$ is a limit ordinal less than $h(T)$.

If x and y are two distinct branches of a tree T, then by (i) the set

$$Eq(x,y) = \{i : x_i = y_i\} \neq \emptyset. \tag{5.15}$$

Furthermore, $i \in Eq(x,y)$ and $j < i$ imply $j \in Eq(x,y)$ and so $Eq(x,y)$ is an ordinal and by condition (iii) it is not a limit ordinal. So we can define

$$\begin{aligned} \epsilon(x,y) &= max Eq(x,y) \qquad \text{so that} \\ Eq(x,y) &= \epsilon(x,y) + 1 \cong x \cap y. \end{aligned} \tag{5.16}$$

We call $\epsilon(x,y)$ the *equality level* of the pair x, y. Clearly, the equality level ϵ is the unique ordinal ϵ such that

$$x_\epsilon = y_\epsilon \qquad \text{and} \qquad x_{\epsilon+1} \neq y_{\epsilon+1}. \tag{5.17}$$

Since both x and y extend to the $\epsilon + 1$ level we have

$$\epsilon(x,y) + 1 < h(x), h(y). \tag{5.18}$$

Definition 5.2. *A* bi-ordered tree T *is a tree of height greater than* 1 *with a linear order on each nonempty set of successors S_p.*

We will say that a bi-ordered tree T is of Y type *for a LOTS Y if for $p \in T$*

$$S_p \neq \emptyset \quad \Rightarrow \quad S_p \cong Y. \tag{5.19}$$

We will say that a bi-ordered tree T is of unbounded type*,* of dense type*,* of separable type *or* of countable type *if each nonempty successor set is a LOTS which is unbounded, order dense, separable or of countable type, respectively.*

For example, a normal bi-ordered tree is of $\mathbb{Q}$ type if $p \in T$ and $o(p) + 1 < h(T)$ implies that S_p is an unbounded, countable, order dense LOTS.

For a bi-ordered tree T the *induced order* on the branch space $X(T)$ is defined by

$$x < y \quad \Longleftrightarrow \quad x_\epsilon = y_\epsilon \text{ and } x_{\epsilon+1} < y_{\epsilon+1} \text{ for some ordinal } \epsilon. \tag{5.20}$$

Note that at the $\epsilon + 1$ level we are using the LOTS ordering. By (5.17) the ordinal ϵ is the equality level of x, y. Conversely, if $x \neq y$ and x, y have equality level ϵ, then both $x_{\epsilon+1}$ and $y_{\epsilon+1}$ are successors of x_ϵ and so either $x_{\epsilon+1} < y_{\epsilon+1}$ or the reverse. Furthermore, it is easy to check that

$$x < z < y \qquad \Rightarrow \qquad \epsilon(x,y) = min(\epsilon(x,z), \epsilon(z,y)) \tag{5.21}$$

and from this that $x < y$. Consequently, with the induced order $X(T)$ is a LOTS.

If T is a bi-ordered tree, then by retaining the LOTS structure on each S_p we give each tail tree T_p and any subtree, e.g. any truncated tree T^α, the structure of a bi-ordered tree of the same type (i.e. normal, of Y type, or of unbounded, dense, separable or countable type).

Proposition 5.3. *Let T be a bi-ordered tree with $p \in T$ and R a subtree of T.*

(a) *The injection $j_p : X(T_p) \to X(T)$ of (5.12) is an order embedding with a convex image.*
(b) *The map $\pi : X(T) \to X(R)$ defined by $x \mapsto x \cap R$ is an order surjection.*

Proof. (a): If $x < z < y$ in T with $x, y \in T_p$, then $\epsilon(x,y) \geq o(p)$ and so by (5.21), $z \in T_p$ and so $x < z < y$ in T_p. Thus, j_p is an order injection with a convex image and so an embedding by Proposition 2.3(b).

(b): Because R is a subtree it is clear that $\pi(x) < \pi(y)$ in R implies $x < y$ in T. It follows that π is an order map. By Zorn's Lemma any branch of R extends to a branch of T. Hence, π is surjective.

$\square$

Remark. It follows from Proposition 2.3(d) that there exists an order injection from $X(R)$ into $X(T)$ for any subtree R of T.

In particular, for $0 < \beta \leq \alpha \leq h(T)$ the projection map

$$\begin{aligned} \pi^\alpha_\beta : X(T^\alpha) &\to X(T^\beta) \\ \pi^\alpha_\beta(z) &= z \cap T^\beta \end{aligned} \tag{5.22}$$

is an order surjection. When $\alpha = h(T)$ we will omit the superscript, writing $\pi_\beta : X(T) \to X(T^\beta)$.

If T is a bi-ordered tree, then we denote by T^* the same tree with the reverse LOTS order on each nonempty S_p. It is clear the T^* is a bi-ordered tree of Y^* type if T is of Y type and is otherwise of the same type (normal, unbounded, dense, etc.). The branch space $X(T^*)$ is the branch space $X(T)$ with the reverse ordering, ie. $X(T^*) = X(T)^*$.

Proposition 5.4. *If T is a bi-ordered tree, then $X(T)$ is an order dense LOTS when either of the following two conditions hold.*

(i) *The tree T is of dense type, i.e. each nonempty S_p is order dense.*

(ii) *T is normal and there exists a limit ordinal α such that $h(T) = \alpha$ or $h(T) = \alpha + 1$ and S_p has no max for any $p \in T$ (e.g. if T is of unbounded type).*

Proof. For $x < y$ in $X(T)$ let ϵ be the x, y equality level. Because $x \neq y$, $\epsilon < h(x)$.

(i): If S_p is order dense, then with $p = x_\epsilon = y_\epsilon$ we can choose $q \in S_p$ so that

$$x_{\epsilon+1} \quad < \quad q \quad < \quad y_{\epsilon+1}. \tag{5.23}$$

(ii): If $h(x) = \alpha + 1$, then condition (iii) of Definition 5.1 implies that $\epsilon < \alpha$. If $h(x) \leq \alpha$, then $h(x)$ is a limit ordinal because T is normal. So in either case, $\epsilon + 2 < h(x)$. We have $x_{\epsilon+1} < y_{\epsilon+1}$ and with $p = x_{\epsilon+1}$, $o(p) + 1 < h(x)$ implies that S_p is nonempty. If S_p has no max, then we can choose $q \in S_p$ such that $x_{\epsilon+2} < q$.

In either case, if z is a branch through q, then $x < z < y$.

□

When $X(T)$ is an order dense LOTS, we will denote by $\widehat{X(T)}$ its completion. In particular when T is of dense type, we will write $\hat{\pi}^\alpha_\beta : \widehat{X(T^\alpha)} \to \widehat{X(T^\beta)}$ and $\hat{j}_p : \widehat{X(T_p)} \to \widehat{X(T)}$ for the extensions to the completions of the maps defined above.

Proposition 5.5. *Let T be a bi-ordered tree.*

(a) *For each $p \in T$ the inclusion j_p is an order embedding onto a convex subset of $X(T)$.*

If $o(p) + 1 = h(T)$, then $T_p = \{p\}$ and the image of j_p is the unique branch through p.

Assume that for every $p \in T$ with $o(p) + 1 < h(T)$, $\#S_p > 1$ (i.e. condition (v) of Definition 5.1 holds), e.g. T is normal. If $o(p) + 2 < h(T)$ or if $o(p) + 2 = h(T)$ and $\#S_p > 2$, then $j_p(X(T_p))$ has a nonempty interior in $X(T)$.

(b) *The following equivalence holds:*

$$\begin{aligned} &\text{For all } p \in T, \alpha \text{ with } o(p)+1 \leq \alpha \leq h(T) \\ &\qquad j_p(X(T_p^\alpha)) \text{ is a closed subset of } X(T^\alpha). \\ &\qquad\qquad \Longleftrightarrow \\ &\text{For all } \beta \leq \alpha \leq h(T), \quad \pi_\beta^\alpha : X(T^\alpha) \to X(T^\beta) \quad \text{is continuous .} \end{aligned} \tag{5.24}$$

These are both true when any of the following three conditions hold.

(i) *Every successor set S_p with $o(p) > 1$ is either empty or bounded.*
(ii) *T is normal and of dense type.*
(iii) *T is normal and of unbounded type.*

(c) *Assume T is normal and of unbounded type, and $p \in T$ with $o(p)+1 < h(T)$. The image $j_p(X(T_p))$ is a nonempty, infinite, clopen, convex set in $X(T)$. If, in addition, $X(T)$ is order dense, then $X(T_p)$ is order dense and the image $\hat{j}_p(\widehat{X(T_p)})$ is a nonempty, open interval in $\widehat{X(T)}$.*

(d) *If S_0, the successor set to the root, is unbounded, then $X(T)$ is unbounded. If S_0 is unbounded and is σ-bounded, then $X(T)$ is σ-bounded. If, in addition, $X(T)$ is order dense, then $\widehat{X(T)}$ is unbounded and σ-compact.*

(e) *Assume there exists a limit ordinal α such that $h(T) = \alpha$ or $h(T) = \alpha + 1$. If T is normal and of unbounded type, then $X(T)$ is order dense and has dense holes.*

(f) *Assume T is normal and of dense type. For each $0 < \beta \leq \alpha \leq h(T)$ the projection π_β^α and its extension $\hat{\pi}_\beta^\alpha$ to the completions are continuous order surjections. The extension $\hat{j}_p$ is an order embedding onto an interval in $\widehat{X(T)}$.*

(g) *A subset W of $X(T)$ is dense in $X(T)$ if for every $p \in T$ with $o(p)+1 < h(T)$ the set $\{q \in S_p : \exists x \in W \text{ with } q \in x\}$ is dense in S_p. In particular, if for every $p \in T$ there exists $x \in W$ such that $p \in x$, then W is dense in $X(T)$.*

Conversely, assume for every $p \in T$ with $o(p)+1 < h(T)$, $\#S_p > 1$ and if $o(p)+2 = h(T)$, then $\#S_p > 2$. If $W \subset X(T)$ is dense in $X(T)$, then for every $p \in T$ there exists $x \in W$ such that $p \in x$.

Proof. (a): By Proposition 5.3(a) j_p is an order embedding with a convex image.

Now assume that (v) of Definition 5.1 holds. If $o(p) + 2 < h(T)$, then we can choose $q_1 < q_4 \in S_p$ and $q_2 < q_3 \in S_{q_4}$. If S_p contains at least three points, then there exist $q_1 < q_2 < q_3$ in S_p. In either case, let $z_i \in X(T)$ be a branch containing q_i for $i = 1, 2, 3$. So $z_1 < z_2 < z_3$ and thus the interval $(z_1, z_3) \subset j_p(X(T_p))$ is nonempty.

(b): We are writing T^α_p for $(T^\alpha)_p$.

Assume that every j_p has a closed image.

For $x \in T^\beta$, $(\pi^\alpha_\beta)^{-1}(x) = \bigcap_{p \in x} j_p(X(T^\alpha_p))$. Hence, the order surjection π^α_β has closed point-inverses and so is continuous by Proposition 2.3(a).

Assume that every π^α_β is continuous. If $p = 0$, then $X(T_p) = X(T)$ is closed. If $o(p) + 1 = h(T)$, then $j_p(X(T_p))$ is a singleton and so is closed.

Assume that $1 \leq o(p), o(p) + 1 < h(T)$. Let $\beta = o(p) + 1$ and $x = x(p)$. $j_p(X(T^\alpha_p)) = (\pi^\alpha_\beta)^{-1}(x)$ and so is closed.

Assume (i): Replacing T by T^α we prove that $j_p(X(T_p))$ is closed. We may assume $1 \leq o(p), o(p) + 1 < h(T)$.

We show that the convex set $j_p(X(T_p))$ has a max and min and so is closed.

Inductively, we define a collection of points p_α totally ordered by $\succ$ and use it to define the max M of $j_p(X(T_p))$.

Begin with $\alpha = o(p)$ and $p_\alpha = p$.

If $\alpha = \beta+1$ and $S_{p_\beta} = \emptyset$ then the process stops and we let $M = x(p_\beta)$. Otherwise, let p_α be the maximum element of S_{p_β}.

If α is a limit ordinal, then $\{p_\beta : o(p) \leq \beta < \alpha\}$ is contained in a branch $x \in X(T)$. Clearly, $x \in j_p(X(T_p))$. If $h(x) = \alpha$, then the process stops and we let $M = x$. Otherwise, let $p_\alpha = x_\alpha$.

The process stops at or before $\alpha = h(T)$ and defines M. If $y \in X(T_p)$, then $\epsilon = \epsilon(y, M)$ has $o(p) \leq \epsilon$. Hence, $y_\epsilon = p_\epsilon$ and so $y_{\epsilon+1} < M_{\epsilon+1}$ since the latter is the maximum element of S_{p_ϵ}. Thus, M is the maximum element of $j_p(X(T_p))$.

Assume (ii): By Proposition 5.4, $X(T^\beta)$ is order dense and so the surjection π^α_β is continuous by Proposition 2.3(a).

Assume (iii): Replacing T by T^α we prove that $j_p(X(T_p))$ is closed. We may assume $1 \leq o(p), o(p) + 1 < h(T)$.

If $x < y, p \in x$ and $p \notin y$, then with $\epsilon = \epsilon(x, y)$, the equality level of x, y, $\epsilon + 1 \leq o(p)$. By assumption $o(p) + 1 < h(T)$, Either $h(y) = h(T)$ or $h(y)$ is a limit ordinal. In either case, with $p_1 = y_{\epsilon+1}$, $y_{\epsilon+2} \in S_{p_1}$ and, by assumption, S_{p_1} is unbounded. So we can choose $q \in S_{p_1}$ with

$q < y_{\epsilon+2}$. A branch z through a point q satisfies $x < z < y$ and $p \notin z$, because $p_1 = y_{\epsilon+1} \in z$. It follows that the open interval (z, ∞) in $X(T)$ contains y and is disjoint from the image of j_p. Arguing similarly if $y < x$ we see that j_p has a closed image in X(T).

(c): Since $o(p)+1 < h(T), p \in x$ and S_p is unbounded, for $x \in X(T_p)$ we can choose $q_1, q_2 \in S_p$ so that $q_1 < x_{\alpha+1} < q_2$ where $\alpha = o(p)$. Choosing branches z_i through q_i for $i = 1, 2$ we see that the image of j_p contains the open interval (z_1, z_2). Hence, the image of j_p is open. It is closed by (b).

If $X(T)$ is order dense, then the convex subset $j_p(X(T_p))$ is order dense and it is isomorphic to $X(T_p)$ via j_p. Hence, $X(T_p)$ is order dense.

Furthermore, the image of $\hat{j}_p$ is the completion of the image of j_p and so it is an interval in $\widehat{X(T)}$. The completion of an open convex set in $X(T)$ is an open interval in $\widehat{X(T)}$. Hence, the image of $\hat{j}_p$ is open in $\widehat{X(T)}$.

Since $o(p) + 1 < h(T)$ S_p is unbounded and so T_p is infinite. Hence, the image of j_p is infinite.

(d): Since the root 0 is the unique element of level 0 in T, $X(T^2) \cong S_0$.

Since $\pi_2 : X(T) \to X(T^2)$ is an order surjection the results for $X(T)$ follow from Proposition 2.3(a). If $X(T)$ is unbounded, then its completion is. If $X(T)$ is σ-bounded then its completion is,too, and so is σ-compact.

(e): If T is of unbounded type, $X(T)$ is order dense by Proposition 5.4. It is unbounded by (d).

Let x, y be elements of $X(T)$ with $x < y$, $\epsilon = \epsilon(x, y)$, and $p = x_\epsilon = y_\epsilon$. Since $X(T)$ is order dense we choose $z \in X(T)$ such that $x < z < y$. By (5.21) $\epsilon_1 = \epsilon(z, y) \geq \epsilon$. With $q = z_{\epsilon_1+1}$ we have $q < y_{\epsilon_1+1}$.

Define the set

$$G \; = \; (-\infty, z) \cup j_q(T_q) \; = \; (-\infty, z] \cup j_q(T_q) \; \subset \; X(T). \tag{5.25}$$

G is a convex set in $X(T)$ which contains x but not the upper bound y.

If $h(y) = \alpha + 1$, then $\epsilon_1 + 1 \leq h(y)$ implies $\epsilon_1 \leq \alpha$. But (iii) of Definition 5.1 then implies $\epsilon_1 < \alpha$. If $h(y) \leq \alpha$, then $\epsilon_1 + 1 \leq h(y)$ implies $\epsilon_1 < \alpha$. Consequently, $\epsilon_1 + 2 < \alpha \leq h(T)$ because α is a limit ordinal. Because $o(q) + 1 = \epsilon_1 + 2 < h(T)$, $j_q(T_q)$ is clopen in $X(T)$ by (c). Since $(-\infty, z)$ is open and $(-\infty, z]$ is closed, it follows that G is clopen in $X(T)$.

We show that G has no supremum and so reveals a hole between x and y.

Since G is closed, if w were the supremum of G, then w would be an element of G and so would be the max of G.

On the other hand, $X(T)$ is unbounded. So $w \in G$ and G open implies there exist $z_1, z_2 \in X(T)$ such that $w \in (z_1, z_2) \subset G$. Since $X(T)$ is order dense, we can take z_1, z_2 to lie in G. Hence, G has no max.

(f): Each π^{α}_{β} is continuous by (b). The extensions to the completions are therefore well-defined, surjective and continuous by Proposition 2.10.

(g): Assume that $\{q \in S_p : \exists z \in W \text{ with } q \in z\}$ is dense in S_p for all p.

For $x < y$ in $X(T)$ with (x, y) nonempty there exists $z \in X(T)$ such that $x < z < y$. Let $\epsilon_1 = \epsilon(z, y) \geq \epsilon(x, y)$ and $p = z_{\epsilon_1} = y_{\epsilon_1}$.

If $\epsilon_1 > \epsilon$, then $z_{\epsilon_1+1} < y_{\epsilon_1+1}$ implies that there exists $w \in W$ with $w_{\epsilon_1+1} < y_{\epsilon_1+1}$ in S_p. Because $\epsilon_1 > \epsilon$, $x_{\epsilon+1} < y_{\epsilon+1} = w_{\epsilon+1}$ and so $x < w < y$.

If $\epsilon_1 = \epsilon$, then $x_{\epsilon+1} < z_{\epsilon+1} < y_{\epsilon+1}$ implies that there exists $w \in W$ with $x_{\epsilon+1} < w_{\epsilon+1} < y_{\epsilon_1+1}$ in S_p and so again $x < w < y$. So the condition is sufficient for density.

In particular, if $p \in T$ implies there exists $x \in W$ with $p \in x$, then W is dense.

Now assume that $o(p) + 1 < h(T)$ implies $\#S_p > 1$ and $o(p) + 2 = h(T)$ implies $\#S_p > 2$. Further, assume that W is dense.

By (a) $j_p(X(T_p))$ has a nonempty interior and so meets W. It follows that there exists $w \in W$ with $p \in w$.

$\square$

If $\alpha + 1 \leq h(T)$ and $p \in L_\alpha$, then

$$x(p) \quad = \quad \{p\} \cup A_p \tag{5.26}$$

is a branch in $X(T^{\alpha+1})$ of height $\alpha + 1$. So $p \mapsto x(p)$ defines an injective map from L_α into $X(T^{\alpha+1})$ and so into $\widehat{X(T^{\alpha+1})}$ when T is of dense type. Regarding this map as an inclusion, we will regard L_α as a subset of these LOTS and so induce an order upon it. Since $(\pi_{\alpha+1})^{-1}(x(p))$ in $X(T)$ consists of the branches which contain p we have

$$j_p(X(T_p)) \ = \ (\pi_{\alpha+1})^{-1}(x(p)) \ \subset \ X(T) \qquad \text{with } \alpha = o(p). \tag{5.27}$$

If α is a limit ordinal with $\alpha + 1 \leq h(T)$ then condition (iii) of Definition 5.1 implies that

$$\pi_{\alpha}^{\alpha+1} : X(T^{\alpha+1}) \ \cong \ X(T^{\alpha}). \tag{5.28}$$

$\pi_{\alpha}^{\alpha+1}$ maps $x(p) = \{p\} \cup A_p$ to A_p which is a branch in $X(T^{\alpha})$ of height α.

When T is of dense type, this order isomorphism extends to $\hat{\pi}_{\alpha}^{\alpha+1}$, an order isomorphism between the completions.

If $\alpha \leq h(T)$ is a limit ordinal then we define

$$\tilde{L}_{\alpha} \quad = \quad \{x \in X(T^{\alpha}) : h(x) = \alpha\}. \tag{5.29}$$

If $\alpha < h(T)$, then

$$\pi_{\alpha}^{\alpha+1}(L_{\alpha}) \ \subset \ \tilde{L}_{\alpha}. \tag{5.30}$$

Lemma 5.6. *Let T be a bi-ordered normal tree and let α be an ordinal.*

(a) *If $\alpha < h(T)$, then for all $p \in T$ there exists $x \in X(T)$ such that $p \in x$ and $h(x) > \alpha$. Furthermore, the subset $(\pi_{\alpha+1})^{-1}(L_{\alpha})$ is dense in $X(T)$ (and hence in $\widehat{X(T)}$ when $X(T)$ is order dense).*

(b) *If $\alpha = h(T)$ and α is a countable limit ordinal, then for all $p \in T$ there exists $x \in X(T)$ such that $p \in x$ and $h(x) = \alpha$. Furthermore, the subset $\tilde{L}_{\alpha}$ is dense in $X(T)$ (and hence in $\widehat{X(T)}$ when $X(T)$ is order dense).*

Proof. (a) If $o(p) < \alpha$ then by condition (iv) of Definition 5.1 there exists $q \in L_{\alpha}$ such that $p \prec q$. If $o(p) \geq \alpha$ then there exists a unique $q \in L_{\alpha}$ such that $p \succeq q$. Any branch x containing p and q satisfies $h(x) > \alpha$. Density follows from Proposition 5.5(g).

(b) Let $\{\alpha_n\}$ be an increasing sequence of ordinals with $o(p) < \alpha_1$ and $sup\{\alpha_n\} = \alpha$. Apply condition (iv) inductively to choose a sequence of vertices $\{p_n\}$ such that

$$\begin{aligned} p \ = \ p_0 \ \prec \ p_1 \ \prec ... \\ o(p_n) \quad = \quad \alpha_n. \end{aligned} \tag{5.31}$$

The unique branch x which contains $\{p_0, p_1, ...\}$ has height α. Density again follows from Proposition 5.5(g)

□

Proposition 5.7. *Let T be a bi-ordered tree.*

If for each $p \in T$ the successor set S_p is a, possibly empty, complete LOTS and is bounded and so compact if $o(p) > 0$, then $X(T)$ is complete. If, in addition, T is of dense type, then $X(T)$ is connected.

Proof. By Proposition 5.4 $X(T)$ is order dense when T is of dense type. So $X(T)$ is connected when it is complete.

We prove by induction on α that $X(T^\alpha)$ is complete and connected when the successor sets are connected.

Case 1: $\alpha = 2$: $X(T^\alpha) \cong S_0$ and so it is complete.

Case 2: $\alpha = \beta + 1$ with β a limit ordinal: then $\pi^\alpha_\beta : X(T^\alpha) \to X(T^\beta)$ is an order isomorphism and so $X(T^\alpha)$ is complete by induction hypothesis.

Case 3: $\alpha = \beta + 1$ with $\beta = \epsilon + 1$: we define a family of compact LOTS indexed by the LOTS $X(T^\beta)$. For $x \in X(T^\beta)$ with $h(x) < \beta$ let $X_x = \{x\}$ the trivial LOTS. If $x \in X(T^\beta)$ with $h(x) = \beta$, then $x = x(p)$ with $p \in T$ and $o(p) = \epsilon$. Let $X_x = \{x\}$ if $S_p = \emptyset$ and $X_x = S_p$ otherwise. It is easy to see that

$$X(T^\alpha) \quad \cong \quad \Sigma\{X_x : x \in X(T^\beta)\}. \tag{5.32}$$

$X(T^\alpha)$ is complete by Proposition 2.5.

Case 4: α is a limit ordinal: Let $A \subset X(T^\alpha)$ which is bounded. For each $\beta < \alpha$ $A_\beta = \pi^\alpha_\beta(A)$ is bounded and so has a supremum $s_\beta \in X(T^\beta)$ by inductive hypothesis. If $\beta_1 < \beta < \alpha$, then $A_{\beta_1} = \pi^\beta_{\beta_1}(A_\beta)$. By Proposition 5.5(b) each $\pi^\beta_{\beta_1}$ is continuous and so by Proposition 2.3 (c) $s_{\beta_1} = \pi^\beta_{\beta_1}(s_\beta)$. Hence, $s = \bigcup_{\beta<\alpha} s_\beta$ is a branch of $X(T^\alpha)$. If $y < s \in X(T^\alpha)$, then with $\epsilon = \epsilon(y,s)$, $y_{\epsilon+1} < s_{\epsilon+1}$ and so $\pi^\alpha_{\epsilon+1}(y) < \pi^\alpha_{\epsilon+1}(s) = s_{\epsilon+1}$. So there exists $a \in A$ such that $\pi^\alpha_{\epsilon+1}(y) < \pi^\alpha_{\epsilon+1}(a)$ and so $y < a$. Similarly, one shows that $y \in A$ with $y \neq s$ implies $y > s$. Hence, $s = supA$. With a similar argument for the infimum we see that $X(T^\alpha)$ is complete.

$\square$

It will be helpful to describe the completion of the branch space of a bi-ordered normal tree T of dense type as a branch space itself. Recall that the completion $\hat{X}$ of an order dense LOTS X is a connected LOTS which has a *max* or *min* iff X does. We call a LOTS Y the *completion with endpoints of* X if Y is the completion of X with *max* (and *min*) attached if X did not already have a *max* (resp. a *min*). So Y is compact as well as connected. If X had neither *max* nor *min* to begin with, then the completion with endpoints is the two-point compactification $\bullet\hat{X}\bullet$. In general, if X is a dense subset of a compact LOTS Y, then Y is isomorphic to the completion with endpoints of X.

Definition 5.8. *Let T be a bi-ordered normal tree of dense type. We define its* completion, *denoted $\hat{T}$, to be the tree which contains T as follows:*

(a) *For the root 0 in T and $\hat{T}$, the successor set in $\hat{T}$, denoted $\hat{S}_0$, is the completion of the order dense LOTS S_0.*
(b) *For $p \in T$ with $o(p) > 0$ the successor set in $\hat{T}$, denoted $\hat{S}_p$, is the completion with endpoints of the order dense LOTS S_p.*
(c) *If $q \in \hat{S}_p \setminus S_p$ for any $p \in T$, then q is called a* new vertex *of $\hat{T}$. If q is a new vertex of $\hat{T}$, then its successor set in $\hat{T}$, denoted $\hat{S}_q$, is empty.*

Proposition 5.9. *Let T be a bi-ordered normal tree of dense type and let $\hat{T}$ be its completion.*

(a) *$\hat{T}$ is a bi-ordered tree with $h(\hat{T}) = h(T)$.*
(b) *For each $p \in T$ the successor set $\hat{S}_p$ is a connected LOTS which is compact if $o(p) > 0$.*
(c) *If $q \in \hat{S}_p$ is a new vertex in $\hat{T}$, then q is the end-point of a unique branch of $\hat{T}$ namely $x(q) = \{q\} \cup A_q$ with*

$$h(x(q)) \ = \ o(q) + 1 \ = \ o(p) + 2, \tag{5.33}$$

and so $h(x(q))$ is a successor ordinal. No two new vertices lie on the same branch, i.e. the set of new vertices is an anti-chain in $\hat{T}$.
(d) *Each branch of T is a branch of $\hat{T}$ of the same height, i.e. $X(T) \subset X(\hat{T})$.*
(e) *If S_0 is unbounded, then so are $\hat{S}_0, X(T)$ and $X(\hat{T})$.*
(f) *If S_0 is unbounded, then $X(\hat{T})$ is the completion $\widehat{X(T)}$. If S_0 has both max and min, then $X(\hat{T})$ is compact and is the completion with endpoints of $X(T)$. In either case, $\hat{X}(T)$ is connected.*
(g) *If for each $p \in T$ the successor set S_p is a connected LOTS which is compact if $o(p) > 0$, then $\hat{T} = T$ and so $X(T)$ is connected.*

Proof. It is clear that $\hat{T}$ is semi-normal and so is a bi-ordered tree, although it is usually not normal. If q is a new vertex, then $x(q)$ is a branch since the successor set for q is empty. The results of (a),(b),(c) and (d) follow easily.

(e): If S_0 is unbounded, then its completion $\hat{S}_0$ is and so $X(T)$ and $X(\hat{T})$ are by Proposition 5.5(d).

(f): From Proposition 5.7 (b) it follows that $X(\hat{T})$ is connected.

If $q \in \hat{S}_p$ is a new vertex and $x(q) < x \in X(\hat{T})$, then let $\epsilon = \epsilon(x(q), x)$ and $r = x_\epsilon \in \{p\} \cup A_p$. If $r = p$, then because S_p is order dense and dense in $\hat{S}_p$ we can choose $q_1 \in S_p$ and $x_1 \in X(T)$ with $q_1 \in x_1$ so that $q < q_1 < x_{\epsilon+1}$ and so $x(q) < x_1 < x$. On the other hand, if $\epsilon = o(r) < o(p)$, then because S_r is order dense we can choose $q_1 \in S_r$ so that $x(q)_{\epsilon+1} = x(p)_{\epsilon+1} < q_1 < x_{\epsilon+1}$. Again if $x_1 \in X(T)$ with $q_1 \in x_1$, then $x(q) < x_1 < x$. Similarly, if $x(q) > x \in X(\hat{T})$, then there exists $x_1 \in X(T)$ with $x(q) > x_1 > x$. Hence, $X(T)$ is dense in $X(\hat{T})$.

Thus, $X(\hat{T})$ is connected and contains the order dense LOTS $X(T)$ as a dense subset. If S_0 is unbounded, it follows that $X(\hat{T})$ is the completion of $X(T)$.

Now assume that S_0 has a maximum M. Define $x_M \in X(\hat{T})$ inductively with $(x_M)_1 = M$. If $(x_M)_i$ is defined and is a new vertex or $i+1 = h(T)$, then x_M terminates with height $i+1$. If $p = (x_M)_i \in T$ with $i+1 < h(T)$, then $(x_M)_{i+1}$ is chosen to be the maximum of $\hat{S}_p$. If for a limit ordinal $\alpha \leq h(T)$ $(x_M)_i$ is defined for all $i < \alpha$, then all such $(x_M)_i \in T$. By (iii) of Definition 5.1 there exists at most one vertex $p \in T$ with $o(p) = \alpha$ and $p \succ (x_M)_i$ for all $i < \alpha$. If no such p exists, then $\{(x_M)_i\}$ defines the branch x_M of height α. If such a p exists, let $(x_M)_\alpha = p$.

It is easy to see that $x_M = maxX(\hat{T})$ and so is $maxX(T)$ if it does not terminate at a new vertex. It it does terminate at a new vertex $(x_M)_{i+1}$, then with $p = (x_M)_i \in T$, S_p has no maximum and so $X(T)$ has no maximum.

With a similar construction for the minimum, we see that if S_0 has both a maximum and a minimum, then $X(\hat{T})$ does so as well and so is compact as well as connected. It follows that $X(\hat{T})$ is the completion with endpoints of $X(T)$.

(g): Obvious.

□

Remark. If T is a bi-ordered normal tree of dense type and $x \in X(\hat{T})$ with $h(x) < h(T)$, then $h(x)$ is a limit ordinal iff $x \in X(T)$.

Clearly, for any ordinal α the completion of T^α is $(\hat{T})^\alpha$ with the new vertices attached which have order less than α. The situation for the tail trees requires a bit of quibbling.

Lemma 5.10. *Let T be a bi-ordered normal tree of dense, unbounded type. If $p \in T$ with $o(p) > 0$ then $X((\hat{T})_p)$ is a compact, connected LOTS. It is the two-point compactification of the completion of $X(T_p)$. That is,*

$$X((\hat{T})_p) \quad \cong \quad \bullet X(\widehat{T_p}) \bullet . \tag{5.34}$$

Proof. Since $o(p) > 0$ the successor space $\hat{S}_p$ is the compact LOTS with the $max = M_p$ and $min = m_p$ attached as two new vertices. In T_p the vertex p is the root and so the max and min are not included in the completion. That is,

$$(\hat{T})_p \quad = \quad \widehat{T_p} \cup \{m_p, M_p\} \tag{5.35}$$

from which the result clearly follows.

□

If $x < y$ in $X(T)$ and $q \in L_{\epsilon+1}$ is between $x_{\epsilon+1}$ and $y_{\epsilon+1}$ with $\epsilon = \epsilon(x,y)$, then every branch through q lies between x and y. Furthermore, since the completion $\hat{\pi}_{\epsilon+2}$ is order preserving we have

$$x_{\epsilon+1} \ < \ q \ < \ y_{\epsilon+1} \quad \Longrightarrow \quad (\hat{\pi}_{\epsilon+2})^{-1}(x(q)) \ \subset \ (x,y) \ \subset \ \widehat{X(T)}. \tag{5.36}$$

Lemma 5.11. *Assume that T is a bi-ordered normal tree of dense type and that S_0 is unbounded. Let α be a positive ordinal with $\alpha+1 < h(T)$. Let $\hat{\pi}_{\alpha+1} : \widehat{X(T)} \to \widehat{X(T^{\alpha+1})}$ be the canonical projection.*

If $x \in \widehat{X(T^{\alpha+1})}$ but $x \neq x(p)$ for any $p \in L_\alpha$ then x is the image under $\hat{\pi}_{\alpha+1}$ of a unique point in $\hat{X}(T)$, i.e.

$$\#(\hat{\pi}_{\alpha+1})^{-1}(x) \quad = \quad 1. \tag{5.37}$$

The collection $\{(\hat{\pi}_{\alpha+1})^{-1}(x(p)) : p \in L_\alpha\}$ is a pairwise disjoint family of closed, nontrivial subintervals of $\widehat{X(T)}$, and the open set

$$O_{\alpha+1} \quad = \quad \bigcup\{[(\hat{\pi}_{\alpha+1})^{-1}(x(p))]^\circ : p \in L_\alpha\} \tag{5.38}$$

is dense in $\widehat{X(T)}$.

Furthermore, the open set

$$O^{\alpha+2}_{\alpha+1} \quad = \quad \bigcup\{[(\hat{\pi}^{\alpha+2}_{\alpha+1})^{-1}(x(p))]^\circ : p \in L_\alpha\} \tag{5.39}$$

is dense in $\widehat{X(T^{\alpha+2})}$ and the restriction

$$\hat{\pi}_{\alpha+2} : \widehat{X(T)} \setminus O_{\alpha+1} \to \widehat{X(T^{\alpha+2})} \setminus O^{\alpha+2}_{\alpha+1} \tag{5.40}$$

is a homeomorphism.

Proof. By Proposition 5.9 $\widehat{X(T)} = X(\hat{T})$. Let x be a branch of $\hat{T}$. If $h(x) \leq \alpha$ then x regarded as a branch of $\hat{T}$ is the unique branch which contains x regarded as a branch of $\hat{T}^{\alpha+1}$. If $h(x) > \alpha$ then $p = x_\alpha \in \hat{T}$ is defined and $\hat{\pi}^{\alpha+1}$ maps x to $x(p)$. If p is a new vertex of $\hat{T}$ then $x = x(p)$ is the unique branch which contains p. Otherwise, $p \in L_\alpha$.

By (5.27) and Proposition 5.5 (a) the interval $(\hat{\pi}_{\alpha+1})^{-1}(x(p))$ is nontrivial for $p \in L_\alpha$ because $\alpha + 1 < h(T)$ and by Lemma 5.6(a) the union of these intervals is dense in $\widehat{X(T)}$. If the union of a family of nontrivial intervals is dense then the union of the interiors is dense, since each interval is contained in the closure of its interior, provided that the LOTS is order dense.

We can apply the result to $T^{\alpha+2}$ whose height is $\alpha + 2 > \alpha + 1$. By (5.37) the map in (5.40) is a bijection. The closed subsets of the completions are locally compact spaces and the map is topologically proper. Hence, it is a homeomorphism.

$\square$

5.2. **Countability Conditions.**

Proposition 5.12. *Let T be a bi-ordered normal tree of dense type with S_0 unbounded. Assume that the height of T is not the successor of a limit ordinal. Define $\epsilon = h(T)$ if $h(T)$ is a limit ordinal and by $\epsilon + 2 = h(T)$ if the height is a successor.*

(a) *The following conditions are equivalent.*
  - (i) *$X(T)$ is separable.*
  - (ii) *$\widehat{X(T)}$ is separable.*
  - (iii) $\widehat{X(T)} \quad \cong \quad \mathbb{R}$.
  - (iv) *$h(T)$ is a countable ordinal, for all $\alpha \leq \epsilon$ the level set L_α is countable, and for each $p \in L_\epsilon$ the LOTS S_p is separable.*
  - (v) *$T^{\epsilon+1}$ is a countable tree and for each $p \in L_\epsilon$ the LOTS S_p is separable.*

(b) *The following conditions are equivalent.*
  - (i) *$X(T)$ satisfies the countable chain condition.*
  - (ii) *$\widehat{X(T)}$ satisfies the countable chain condition.*
  - (iii) *The tree $T^{\epsilon+1}$ satisfies condition (ix) of Definition 5.1, i.e. every anti-chain is countable, and for each $p \in L_\epsilon$ the LOTS S_p satisfies the countable chain condition.*

These conditions imply that $T^{\epsilon+1}$ satisfies conditions (vi) and (viii) of Definition 5.1 and, in particular, that $h(T) \leq \Omega$. If $h(T) = \Omega$, then T is a Suslin tree.

(c) *The following conditions are equivalent.*
  (i) *$X(T)$ is of countable type.*
  (ii) *$\widehat{X(T)}$ is first countable and σ-compact.*
  (iii) *The tree T is of countable type and is Ω bounded, i.e. $h(x) < \Omega$ for all $x \in X(T)$ (condition (vi) of Definition 5.1).*
  (iv) *The tree T is of countable type and $h(x) < \Omega$ for all $x \in X(\hat{T})$.*

Proof. When $h(T) = \epsilon + 2$, let $S_x = S_p$ for $x = x(p)$ with $o(p) = \epsilon$ and let $S_x = \{x\}$ for $x \in X(T^{\epsilon+1}) \setminus L_\epsilon$. Each branch in $X(T)$ extends its projection $x \in X(T^{\epsilon+1})$ by a point in S_x. Thus, in this case we have the order sum isomorphism

$$X(T) \quad \cong \quad \Sigma\{S_x : x \in X(T^{\epsilon+1})\}. \tag{5.41}$$

(b) (i)$\Leftrightarrow$(ii): By Proposition 2.3(b) the inclusion of $X(T)$ into $\widehat{X(T)}$ is continuous and so the equivalence follows from Proposition 2.11(f),(h).

(iii)$\Rightarrow$(i): Let $\{J_i : i \in I\}$ be a pairwise disjoint family of nonempty open intervals in $X(T)$. By (5.36) there exists a vertex $q_i \in T$ such that every branch through q_i is contained in J_i. If $o(q_i) < \epsilon+1$ then let $p(q_i) = q_i$. If $o(q_i) = \epsilon + 1$ then let $p(q_i)$ be the immediate predecessor of q_i. In that case, J_i meets $S_{p(q_i)}$ which satisfies c.c.c. Consequently, for each $p \in L_\epsilon$ the set $\{i \in I : p(q_i) = p\}$ is countable. The set $\{p(q_i) : i \in I\}$ is an anti-chain in $T^{\epsilon+1}$ since the intervals J_i are disjoint. So by assumption this set is countable and consequently I itself is countable.

(i)$\Rightarrow$(iii): By (5.41) each S_p for $p \in L_\epsilon$ is isomorphic to a subinterval of $X(T)$. So if $X(T)$ satisfies c.c.c. then each such S_p does. If A is an anti-chain in $T^{\epsilon+1}$ then $o(p)+1 < h(T)$ implies that $\{[(\pi^{o(p)+1})^{-1}(x(p))]^\circ : p \in A\}$ is a family of nonempty open intervals in $X(T)$ by Lemma 5.11. The family is pairwise disjoint since A is an anti-chain. Because $X(T)$ satisfies c.c.c. A is countable.

We proved after Definition 5.1 that condition (ix) implies (vi) and (viii). Thus $h(X(T^{\epsilon+1})) \leq \Omega$. So if $h(T)$ is not a limit ordinal, then $\epsilon + 1 < \Omega$ and so $h(T) = \epsilon + 2 < \Omega$.

(a) (i)$\Rightarrow$(ii): $X(T)$ is dense in $\widehat{X(T)}$.

(ii)$\Rightarrow$(iii): This follows from Proposition 2.15(b) since $\widehat{X(T)}$ is separable, connected and is unbounded.

(iii)⇒(i): By Proposition 2.3(b) the topology on $X(T)$ is inherited from $\widehat{X(T)}$. Any subset of $\mathbb{R}$ is second countable and so is separable.

(i)⇒(iv): Since separability implies c.c.c. the results of (b) can be applied and so $T^{\epsilon+1}$ satisfies (vi) and (viii) of Definition 5.1. In particular, for each $\alpha \leq \epsilon$ the set L_α is countable. Let D be a countable dense subset of $X(T)$. If J is any nontrivial interval in $X(T)$, then $J^\circ \cap D$ is dense in J° and so in J. By (5.41) S_p is separable for each $p \in L_\epsilon$. Now let $\alpha = sup\{h(x) : x \in D\}$. By condition (vi) each $h(x)$ is countable and so $\alpha < \Omega$. I claim that $h(T) \leq \alpha + 1$, for if $\alpha + 1 < h(T)$ and $o(p) = \alpha$, then by Lemma 5.11 $[\pi^{\alpha+1})^{-1}(x(p))]^\circ$ is a nonempty open interval and so contains some point $y \in D$. But then $p \in y$ implies $h(y) \geq \alpha + 1$ which contradicts the definition of α.

(iv)⇒(v): $T^{\epsilon+1}$ has countably many levels and each level is countable.

(v)⇒(i): If $h(T) = \epsilon + 2$, then we choose for each p in the countable set L_ϵ a countable dense subset of S_p. The union is a countable set which is dense in $O_{\epsilon+1}$ (see Equation (5.38)) which is dense in $X(T)$ by Lemma 5.11. If $h(T) = \epsilon$ is a limit ordinal, then $T = T^{\epsilon+1}$ is countable. Choose a branch through each vertex to get a countable set which is dense in $X(T)$ by Proposition 5.5(g).

(c) (i)⇔(ii): By Proposition 2.15(d).

(i)⇒(iii): Assume that $f : \Omega \to S_p$ is an order preserving or reversing injection for some $p \in T$. For each $i \in \Omega$ choose $\tilde{f}(i)$ a branch through $f(i)$. Then $\tilde{f} : \Omega \to X(T)$ is an injection which similarly preserves or reverses order. Hence, $X(T)$ is not of countable type.

On the other hand, suppose that $x \in X(T)$ with $h(x) \geq \Omega$. Define

$$\begin{aligned} K_+ &= \{j \in \Omega : x_{j+1} \neq minS_{x_j}\} \\ K_- &= \{j \in \Omega : x_{j+1} \neq maxS_{x_j}\}. \end{aligned} \tag{5.42}$$

Either K_+ or K_- is uncountable since $max \neq min$ for any S_p. Suppose K_+ is uncountable. For each $i \in K_+$ choose $f(i) \in X(T)$ such that

$$f(i)_{i+1} \in S_{x_i} \qquad \text{with} \qquad f(i)_{i+1} < x_{i+1}. \tag{5.43}$$

If $i < j$ in K_+, then the equality level for $f(i), f(j)$ is i and

$$f(i)_{i+1} \quad < \quad x_{i+1} \quad = \quad f(j)_{i+1}. \tag{5.44}$$

Thus, $f : K_+ \to X(T)$ is an order injection. Since $K_+ \cong \Omega$, $X(T)$ is not of countable type.

(iii)⇔(iv): It is clear from Proposition 5.9 that $h(x) < \Omega$ for all $x \in X(T)$ implies $h(x) < \Omega$ for all $x \in X(\hat{T})$ and the converse is obvious.

(iii)⇒(i): If $X(T)$ is not of countable type, then there exists an injective map $f : \Omega \to X(T)$ which we can assume without loss of generality to be order preserving.

We now prove by induction on $\alpha \in \Omega$ that there exist $\epsilon(\alpha) \in \Omega$ and $p(\alpha) \in L_\alpha$ such that

$$\pi_{\alpha+1} \circ f(\beta) \quad = \quad x(p(\alpha)) \qquad \text{for all } \beta \in \Omega \setminus \epsilon(\alpha). \tag{5.45}$$

If $\alpha = 0$, then $\pi_1 \circ f$ is constantly the root 0. Let $\epsilon(0) = 0$ and $p(0) = 0$.

Now assume that for all $i < \alpha$, $\epsilon(i) \in \Omega$ and $p(i) \in L_i$ have been defined so that (5.45) holds with α replaced by i.

Case 1: If $\alpha = \tilde{\alpha} + 1$, then $\pi_{\tilde{\alpha}+1} \circ f$ is constant on the tail $\Omega \setminus \epsilon(\tilde{\alpha})$ with value $x(p(\tilde{\alpha}))$. Hence, if $j < k$ in the tail, then the equality level of $f(j)$ and $f(k)$ is at least $\tilde{\alpha}$. Hence, $f(j) < f(k)$ in $X(T)$ implies $f(j)_\alpha \leq f(k)_\alpha$. Thus, $j \mapsto f(j)_\alpha$ defines an order map from $\Omega \setminus \epsilon(\tilde{\alpha}) \cong \Omega$ to the LOTS of countable type S_p with $p = p(\tilde{\alpha})$. By Corollary 2.12 this map is eventually constant. Hence there exists $\epsilon(\alpha) \geq \epsilon(\tilde{\alpha})$ such that for all $j \in \Omega \setminus \epsilon(\alpha)$ $f(j)_\alpha$ is a common vertex $p(\alpha) \in S_{p(\tilde{\alpha})}$ and so (5.45) holds.

Case 2: If α is a limit ordinal, then define $\epsilon(\alpha) = sup\{\epsilon(i) : i < \alpha\}$. For $\beta > \epsilon(\alpha)$, $f(\beta)$ and $f(\epsilon(\alpha))$ are distinct points with $f(\beta)_i = f(\epsilon(\alpha))_i$ for all $i < \alpha$. Hence, the equality level is at least α. In particular, $f(\beta)_\alpha = f(\epsilon(\alpha))_\alpha$. Let $p(\alpha)$ be this common vertex. Again (5.45) holds.

Having proved (5.45) for all α we can restate it as

$$f(\beta)_\alpha \quad = \quad p(\alpha) \qquad \text{for all } \beta \in \Omega \setminus \epsilon(\alpha). \tag{5.46}$$

It follows that if $\tilde{\alpha} < \alpha$ and $\beta \geq \epsilon(\alpha)$, then $p(\tilde{\alpha})$ and $p(\alpha)$ both lie on the branch $f(\beta)$. Hence, $p(\tilde{\alpha}) \prec p(\alpha)$. Thus, $\{p(\alpha) : \alpha \in \Omega\}$ is a linearly ordered collection of vertices . Hence, it is contained in some branch x and any such branch satisfies $h(x) \geq \Omega$. This completes the proof that (iii) ⇒ (i).

□

Remark. In the case excluded by the hypothesis, $h(T) = \epsilon + 1$ with ϵ a limit ordinal, $\pi_\epsilon : X(T) \to X(T^\epsilon)$ is an order isomorphism and $h(T^\epsilon) = \epsilon$. We obtain results for T in this case by applying the proposition to T^ϵ.

Corollary 5.13. *If T is a bi-ordered Aronszajn tree of $\mathbb{Q}$ type, then $X(T)$ is an order dense LOTS of countable type with dense holes and $\widehat{X(T)}$ is a σ-compact, connected, first countable LOTS. Neither is separable. For each $1 < \alpha < \Omega$, $\widehat{X(T^\alpha)} \cong \mathbb{R}$. If T is a Suslin tree, then $\widehat{X(T)}$ satisfies the countable chain condition.*

Proof. $X(T)$ is order dense, unbounded and with dense holes by Proposition 5.5(d) and (e). It is of countable type by Proposition 5.12(c). Hence, its completion $\widehat{X(T)}$ is σ-compact, connected and first countable. Because $h(T) = \Omega$, it is not separable by Proposition 5.12(a) and the same result implies that for countable α, $\widehat{X(T^\alpha)} \cong \mathbb{R}$. By Proposition 5.12(b), $\widehat{X(T)}$ for a Suslin tree satisfies c.c.c.

$\square$

5.3. **Homogeneous and Reproductive Trees.** If T_1 and T_2 are bi-ordered trees, then an *isomorphism* $f : T_1 \to T_2$ is a bijection which preserves both orders, i.e.

$$\begin{aligned} p \prec_1 q \qquad &\Longleftrightarrow \qquad f(p) \prec_2 f(q) \\ q <_1 r \quad \text{in } S_p \qquad &\Longleftrightarrow \qquad f(q) <_2 f(r) \quad \text{in } S_{f(p)}. \end{aligned} \tag{5.47}$$

The first condition says that f relates the tree structures. Hence, it maps the vertices of level α in T_1 to those of level α in T_2 and also f induces a bijection of branch spaces, denoted $f_* : X(T_1) \to X(T_2)$. The second condition implies that f_* is an order isomorphism with respect to the induced orders and so, in the order dense case, extends to an order isomorphism on the completions, which we will also denote by f_*. When $T_1 = T_2$ such an isomorphism is called an *automorphism*.

Definition 5.14. *A bi-ordered tree T is called* homogeneous *if for all $p, q \in T$ such that $o(p) = o(q)$ there exists an automorphism f of T such that $f(p) = q$.*

A bi-ordered tree T is called reproductive *if for all $p \in T$ the tail tree T_p is isomorphic to T.*

Theorem 5.15. *If T is a homogeneous, bi-ordered Aronszajn tree of dense type, then $\widehat{X(T)}$ is a nonseparable CHLOTS.*

Proof. Homogeneity clearly implies that no S_p has a *max* or *min* and so T is of $\mathbb{Q}$ type. By Corollary 5.13 $\widehat{X(T)}$ is first countable, σ-compact and nonseparable. By Proposition 3.8(a) it suffices to prove that $\widehat{X(T)}$ is doubly transitive. Given $x < y$ and $z < w$ in $\widehat{X(T)}$ we construct an order isomorphism on $\widehat{X(T)}$ which maps the pair x, y to z, w.

Because T is an Aronszajn tree, it satisfies condition (vi) of Definition 5.1 and so every height $h < \Omega$ on $\widehat{X(T)} = X(\hat{T})$. Choose a countable ordinal $\alpha > h(x), h(y), h(z), h(w)$ so that

$$\hat{\pi}_{\alpha+1}(r) \ \in \ \widehat{X(T^{\alpha+1})} \setminus L_\alpha \qquad \text{for } r = x, y, z, w. \tag{5.48}$$

By Lemma 5.11 $\hat{\pi}_{\alpha+1}$ is injective on the complement of $(\hat{\pi}_{\alpha+1})^{-1}(L_\alpha)$ we have

$$\hat{\pi}_{\alpha+1}(x) \ < \ \hat{\pi}_{\alpha+1}(y) \quad \text{and} \quad \hat{\pi}_{\alpha+1}(z) \ < \ \hat{\pi}_{\alpha+1}(w). \tag{5.49}$$

By Corollary 5.13 $\widehat{X(T^\alpha)} \cong \mathbb{R}$ and by condition (vii) or Definition 5.1 and Lemma 5.6(a), L_α is a countable dense subset of $\widehat{X(T^\alpha)}$. We can choose an order isomorphism of L_α which maps the convex set $L_\alpha \cap (\hat{\pi}_{\alpha+1}(x), \hat{\pi}_{\alpha+1}(y))$ to $L_\alpha \cap (\hat{\pi}_{\alpha+1}(z), \hat{\pi}_{\alpha+1}(w))$ because L_α is order isomorphic to the IHLOTS $\mathbb{Q}$. Extending to the completion we obtain an order isomorphism $\tilde{f} : \widehat{X(T^\alpha)} \to \widehat{X(T^\alpha)}$ such that

$$\tilde{f}(L_\alpha) = L_\alpha, \quad \tilde{f}(\hat{\pi}_{\alpha+1}(x)) = \hat{\pi}_{\alpha+1}(y), \quad \tilde{f}(\hat{\pi}_{\alpha+1}(z)) = \hat{\pi}_{\alpha+1}(w). \tag{5.50}$$

We can regard the LOTS $\widehat{X(T)}$ as the order space sum

$$\widehat{X(T)} \ = \ \Sigma\{(\hat{\pi}_{\alpha+1})^{-1}(p) : p \in \widehat{X(T^{\alpha+1})}\}, \tag{5.51}$$

where for $p \notin L_\alpha$, $(\hat{\pi}_{\alpha+1})^{-1}(p)$ is a single point and for $p \in L_\alpha$ it is a closed interval with nonempty interior, consisting of the branches of $\hat{T}$ which contain p.

For each $p \in L_\alpha$ choose an automorphism $\tilde{g}_p$ of T which maps the vertex p to the vertex $\tilde{f}(p)$. We can restrict the order isomorphism $(\tilde{g}_p)_*$ on $\hat{X}(T)$ to define for $p \in L_\alpha$ an order isomorphism

$$g_p : (\hat{\pi}_{\alpha+1})^{-1}(p) \to (\hat{\pi}_{\alpha+1})^{-1}(\tilde{f}(p)) \tag{5.52}$$

and for $p \notin L_\alpha$ let g_p denote the unique map between the singletons.

The required isomorphism on $\widehat{X(T)}$ is

$$g \ = \ \Sigma\{g_p : p \in \widehat{X(T^\alpha)}\}. \tag{5.53}$$

□

Proposition 5.16. *Let T be a bi-ordered Aronszajn tree of $\mathbb{Q}$ type.*

(a) *Let* $g : Y \to \widehat{X(T)}$ *be a continuous, injective map with* Y *an arbitrary separable topological space. There exists a countable ordinal* β *such that* $\hat{\pi}_{\beta+1} \circ g : Y \to \widehat{X(T^{\beta+1})}$ *is injective. In particular, any separable space which can be continuously injected into* $\widehat{X(T)}$ *can be continuously injected as well into* $\mathbb{R}$*.*

(b) *Let the LOTS* X_1 *be an uncountable, dense subset of* $\mathbb{R}$ *. The AS double* X_1' *does not inject into* $\widehat{X(T)}$*.*

Proof. (a): Let D be a countable dense subset of Y. Since $h < \Omega$ on $\widehat{X(T)} = X(\hat{T})$,

$$\alpha \quad = \quad sup\{h(g(y)) : y \in D\} \tag{5.54}$$

is a countable ordinal and we have

$$\hat{\pi}_{\alpha+1}(g(y)) \ \in \ \widehat{X(T^{\alpha+1})} \setminus L_\alpha \qquad \text{for } y \in D. \tag{5.55}$$

In the notation of Lemma 5.10, this implies that $g(D)$ is disjoint from the open set $O_{\alpha+1}$. Because D is dense in Y and g is continuous it follows that $g(Y)$ is disjoint from $O_{\alpha+1}$. By Lemma 5.10 the restriction of $\hat{\pi}_{\alpha+2}$ to $\widehat{X(T)} \setminus O_{\alpha+1}$ is injective and so the result follows with $\beta = \alpha + 2$.

(b): We will assume that $G_1 : X_1' \to \widehat{X(T)}$ is an order injection and use it to construct a separable, compact nonmetrizable subset C of $\widehat{X(T)}$. Since a compact space which continuously injects into $\mathbb{R}$ is second countable this will contradict part (a).

We will use the fact that a second countable, compact space has only countably many clopen sets. If $\mathcal{B}$ is a countable basis, then any clopen set U is a union of members of $\mathcal{B}$. Since U is closed, and hence compact, it is a union of finitely many members of $\mathcal{B}$. So the cardinality of the set of clopens is bounded by that of the collection of finite subsets of $\mathcal{B}$ which is countable.

The LOTS X_1 is order dense. Let D be a countable, dense subset of X_1 which we can identify with the countable dense subset $D \times \{-1\}$ of X_1' so define the order injection $g : D \to \widehat{X(T)}$ by $g(t) = G_1(t^-)$. Use g to define $G : X_1' \to \widehat{X(T)}$ by using (3.27). Because G_1 is injective, we have for any $x \in X_1$

$$G(x^-) \quad \leq \quad G_1(x^-) \quad < \quad G_1(x^+) \quad \leq \quad G(x^+). \tag{5.56}$$

Combined with (3.28) we see that the continuous order map G is injective. Choose $a < b \in X_1$ and let C be the closure in $\widehat{X(T)}$ of the image $G([a^+, b^-])$. For any $x \in (a, b)$ (5.56) implies that $G(x^-) < G(x^+)$ is a gap pair in the image and so in C. Since X_1' is separable and G is continuous, C is separable. Since $\widehat{X(T)}$ is complete and C is bounded, it is compact. Since it has uncountably many gap pairs, it has uncountably many clopen subsets and so is not metrizable.

$\square$

For the following recall from (4.1) the definition of Y_α for a nontrivial connected LOTS Y.

Corollary 5.17. *Let T be a bi-ordered Aronszajn tree of $\mathbb{Q}$ type, and $\alpha > 1$ be an ordinal.*

(a) *If Y is a nontrivial, connected LOTS, then Y_α does not inject into $\widehat{X(T)}$.*
(b) *If a LOTS Y is an uncountable, dense subset of $\mathbb{R}$, then $\widehat{Y_\alpha}$ does not inject into $\widehat{X(T)}$.*

Proof. (a): By Theorem 3.14, Y contains a compact subset A such that $A'' \subset A'$ is order isomorphic to the Fat Cantor Set $\bullet\mathbb{R}'\bullet$. Since $\alpha > 1$, $\hat{j}^1_\alpha : Y' \to Y_\alpha$ restricts to an order embedding of A'' into Y_α. So an order injection of Y_α into $\widehat{X(T)}$ would restrict to an order injection of A'' into $\widehat{X(T)}$. But by Proposition 5.16(b) the Fat Cantor Set cannot be order injected into $\widehat{X(T)}$.

(b): Similarly, an order injection of $\widehat{Y_\alpha}$ into $\widehat{X(T)}$ would restrict to an order injection of Y' into $\widehat{X(T)}$ which again contradicts Proposition 5.16(b).

$\square$

Lemma 5.18. *Let T be a bi-ordered normal tree, $p \in T$ and α a positive ordinal with $\alpha < h(T)$.*

(a) *Assume T is homogeneous. If $o(p) + 1 < h(T)$, then S_p is a transitive LOTS and so is unbounded, i.e. T is of unbounded type, and the tail tree T_p is homogeneous. If $q \in T$ with $o(q) = o(p)$, then as LOTS $S_p \cong S_q$. The subtree T^α is homogeneous. Regarded as a subset of $X(T^{\alpha+1})$, L_α is a transitive LOTS.*
(b) *Assume T is reproductive. The height $h(T)$ is a tail-like ordinal. As LOTS $S_p \cong S_0$ where 0 is the root of T. Hence, with $Y \cong S_0$ T is of Y type. The tail tree T_p is reproductive and if α is tail-like, then T^α is reproductive.*

(c) *Assume T is homogeneous and reproductive. The tail tree T_p is homogeneous and reproductive and if α is tail-like, then T^α is homogeneous and reproductive.*

Proof. (a): If $q_1, q_2 \in S_p$, then $o(q_1) = o(q_2) = o(p) + 1$ and so there exists an automorphism f of T such that $f(q_1) = q_2$. Since f preserves $\prec$, $f(S_p) = S_p$ and so f restricts to an order automorphism on S_p. Similarly, if $f(p) = q$ then f restricts to an order isomorphism $S_p \cong S_q$.

In general, if $q_1, q_2 \in T_p$ with $o_p(q_1) = o_p(q_2)$, then $o(q_1) = o(q_2)$ and so there exists an automorphism f of T such that $f(q_1) = q_2$. Since f preserves $\prec$, f fixes p and so maps T_p to itself. Hence, T_p is homogeneous.

Since any automorphism of T preserves T^α, the latter is homogeneous. Using the induced isomorphisms on $X(T^{\alpha+1})$, which preserve L_α, we see that L_α is transitive as well.

(b): An isomorphism of T_p with T restricts to an isomorphism of S_p with S_0. If $h(T) = 1$, then $T = \{0\}$ and $S_0 = \emptyset$. Otherwise, all successor sets are of S_0 type and $h(T)$ is an infinite limit ordinal. Furthermore, by (5.11)

$$h(T) = h(T_p) \cong h(T) \setminus o(p). \tag{5.57}$$

Thus, $h(T)$ is tail-like.

Now assume that α is an infinite tail-like ordinal (the case $\alpha = 1$ is trivial). If $o(p) < \alpha \leq \beta$, then (5.9) implies that $o(q) < \beta$ iff $o_p(q) < \beta$. This shows that for α tail-like:

$$o(p) \ < \ \alpha \ \leq \ \beta \quad \Rightarrow \quad (T^\beta)_p \ = \ (T_p)^\beta. \tag{5.58}$$

In particular, if $p \in T^\alpha$ then an isomorphism from T to T_p restricts to an isomorphism from T^α to $(T^\alpha)_p$ which shows that T^α is reproductive.

Since T_p is isomorphic to T it is reproductive.

(c): If α is tail-like, then T^α is homogeneous and reproductive by (a) and (b). Since T_p is isomorphic to T it is homogeneous and reproductive.

$\square$

Proposition 5.19. *If T is a bi-ordered tree which is reproductive and with S_0 a transitive LOTS, then T is a normal, homogeneous tree of unbounded type with height an infinite, tail-like ordinal. Furthermore, if $x, y \in X(T)$ with $h(x) = h(y)$, then there exists an automorphism f of T such that $f_*(x) = y$.*

Proof. Since S_0 is a transitive LOTS it is nonempty. By condition (ii) of Definition 5.1 it contains at least two points. As the two point

LOTS is not transitive, Proposition 3.2(b) implies that S_0 is infinite with no *max* or *min*. For every $p \in T$, there exists an isomorphism $f : T \to T_p$ and so $S_p = f(S_0)$ is infinite. Furthermore, for every ordinal $\alpha < h(T)$, there exists $q \in T$ with $o(q) = \alpha$ and so $f(q) \in T_p$ with $o(f(q)) = o(p) + o(q)$. Thus, condition (iv) of Definition 5.1 holds and so T is a normal tree. Since T is reproductive, $h(T)$ is tail-like and since $S_0 \neq \emptyset$, $h(T) > 1$. Hence, $h(T)$ is infinite.

We now have to show that if $p, q \in T$ with $o(p) = o(q)$ then there exists an automorphism f of T such that $f(p) = q$. We will call this the *vertex case* and use x, y to stand for the linearly ordered sets $x = \{p\} \cup A_p$ and $y = \{q\} \cup A_q$. Let $\alpha = h(x) = h(y) = o(p) + 1$ so that in the vertex case α is a successor. When x and y are branches, that is the *branch case*, then $\alpha = h(x) = h(y)$ is a limit ordinal. In the vertex case, let $T_x = T_p$ and $T_y = T_q$. In the branch case, let $T_x = T_y = \emptyset$. In either case,

$$
\begin{aligned}
T_x &= \{r \in T : x \subset \{r\} \cup A_r\} \\
T_y &= \{r \in T : y \subset \{r\} \cup A_r\}.
\end{aligned}
\tag{5.59}
$$

For any $r \in T$ we can define by analogy with (5.16) the r, y equality level to be $\epsilon = \epsilon(r, y)$ where

$$
y \cap (\{r\} \cup A_r) \quad \cong \quad \epsilon + 1. \tag{5.60}
$$

So $\epsilon = \alpha$ if $r \in T_y$ and $\epsilon < \alpha$ otherwise. Similarly, define $\epsilon(r, x)$.

Since the results are obvious for the trivial tree we will assume that $h(T)$ is an infinite, tail-like ordinal and $S_r \cong S_0$ is infinite for each $r \in T$.

By induction on $\beta \leq \alpha$ we will construct automorphisms f_β of T which satisfy:

$$
\delta \leq \beta \quad \Rightarrow \quad f_\beta(y_\delta) = x_\delta, \tag{5.61}
$$

and for all $r \in T$

$$
\epsilon(r, y) \leq \delta < \beta \quad \Rightarrow \quad f_\delta(r) = f_\beta(r) \tag{5.62}
$$

In the final step, when $\beta = \alpha$, the inequality in (5.61) is replaced by the strict inequality $\delta < \alpha$ since y_α and x_α are not defined. The final automorphism f_α is the one which maps x to y (and p to q in the vertex case).

Begin with $f_0 = 1_T$, the identity. Condition (5.61) holds since $y_0 = x_0 = 0$ and condition (5.62) holds vacuously. Now assume that f_δ has been constructed for all $\delta < \beta$.

Case 1: If $\beta = \delta + 1$, then the automorphism f_δ maps y_δ to x_δ and so x_β and $f_\delta(y_\beta)$ both lie in the transitive LOTS S_p with $p = x_\delta$. Choose g a LOTS automorphism of S_p which maps $f_\delta(y_\beta)$ to x_β.

Because T is reproductive we can choose for each $r \in S_p$ a tree isomorphism $g_r : T_r \to T_{g(r)}$. We define the tree automorphism $\tilde{g}$ to be g_r on each such T_r and to be the identity on the rest of T. Then define $f_\beta = \tilde{g} \circ f_\delta$. By construction (5.61) holds and if $\epsilon(r, y) < \beta$ then $f_\delta(r)$ does not lie in $T_p \setminus \{p\}$ and so $\tilde{g}$ is the identity on $f_\delta(r)$ from which (5.62) follows.

Case 2: If β is a limit ordinal, then define $f_\beta(r) = f_\delta(r)$ whenever $\epsilon(r, y) \leq \delta < \beta$. By (5.62) this definition is independent of the choice of δ and defines f_β whenever $\epsilon(r, y) < \beta$. Furthermore, by (5.62) $f_\beta(y_\delta) = x_\delta$ for all $\delta < \beta$.

If $\beta = \alpha$, then, since β is a limit ordinal, we are in the branch case with $T_y = T_x = \emptyset$. So f_β is defined on all of T and satisfies (5.62) and the adjusted version of (5.61).

If $\beta < \alpha$, then f_β maps the predecessors of y_β to the corresponding predecessors of x_β. Because T is reproductive we can finish our definition of f_β by choosing an isomorphism $f_\beta : T_{y_\beta} \to T_{x_\beta}$. This tree isomorphism maps the root y_β to the root x_β and so (5.61) holds.

This finishes the inductive construction and so completes the proof. $\square$

Lemma 5.20. *Let T be a reproductive, bi-ordered tree with S_0 a doubly transitive LOTS containing at least three points.*

(a) *Let $\alpha = h(T)$. If $x, y, z, w \in \tilde{L}_\alpha$ with $x < y$ and $z < w$, then there exists an order isomorphism $k : (x, y) \to (z, w)$ (intervals in $X(T)$) such that*

$$k((x, y) \cap \tilde{L}_\alpha) \quad = \quad (z, w) \cap \tilde{L}_\alpha. \tag{5.63}$$

(b) *Let α be an infinite, tail-like ordinal with $\alpha < h(T)$. If $x, y, z, w \in L_\alpha$ with $x < y$ and $z < w$, then there exists an order isomorphism $k : (x, y) \to (z, w)$ (intervals in $X(T^{\alpha+1})$) such that*

$$k((x, y) \cap L_\alpha) \quad = \quad (z, w) \cap L_\alpha. \tag{5.64}$$

Proof. (a): With β and ϵ the x, y and z, w equality levels, respectively, we let

$$p \;=\; x_\beta \;=\; y_\beta \qquad \text{and} \qquad q \;=\; z_\epsilon \;=\; w_\epsilon. \tag{5.65}$$

Note that $x < y$ and $z < w$ imply

$$\begin{aligned} \beta, \epsilon \ &< \ \alpha \\ x_{\beta+1} \ &< \ y_{\beta+1} \qquad \text{in } S_p \\ z_{\epsilon+1} \ &< \ w_{\epsilon+1} \qquad \text{in } S_q. \end{aligned} \tag{5.66}$$

Because T is reproductive there exist isomorphisms $g_1 : T_p \to T$ and $g_2 : T_q \to T$. Define

$$\begin{aligned} \tilde{x} \ &= \ g_1(x \cap T_p) \qquad \tilde{y} \ = \ g_1(y \cap T_p) \\ \tilde{z} \ &= \ g_2(z \cap T_q) \qquad \tilde{w} \ = \ g_2(w \cap T_q). \end{aligned} \tag{5.67}$$

Because α is tail-like, $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ are branches in $X(T)$ of height α. The pairs $\tilde{x}, \tilde{y}$ and $\tilde{z}, \tilde{w}$ have equality level 0 and by (5.66)

$$\tilde{x}_1 \ < \ \tilde{y}_1 \quad \text{and} \quad \tilde{z}_1 \ < \ \tilde{w}_1 \qquad \text{in } S_0. \tag{5.68}$$

Because S_0 is doubly transitive there exists a LOTS automorphism g of S_0 such that

$$g(\tilde{x}_1) \ = \ \tilde{z}_1 \quad \text{and} \quad g(\tilde{y}_1) \ = \ \tilde{w}_1. \tag{5.69}$$

Now choose for each $r \in S_0$ a tree isomorphism

$$g_r : T_r \to T_{g(r)}. \tag{5.70}$$

Before putting these together in the now familiar way, we make one last pair of adjustments.

With $r_1 = \tilde{x}_1$ and $r_2 = \tilde{z}_1$ we have that $(g_{r_1})_*(\tilde{x} \cap T_{r_1})$ and $\tilde{z} \cap T_{r_2}$ are both branches of height α in the tree $T_{r_2} \cong T$. By Proposition 5.19 there is an automorphism of T_{r_2} which maps one branch to the other. By composing with such an automorphism we can adjust g_{r_1} so that

$$(g_{\tilde{x}_1})_*(\tilde{x} \cap T_{\tilde{x}_1}) \ = \ \tilde{z} \cap T_{\tilde{z}_1}. \tag{5.71}$$

Similarly, since $\tilde{w}_1 \neq \tilde{z}_1$ we can make an independent adjustment on $T_{\tilde{w}_1}$ to get

$$(g_{\tilde{y}_1})_*(\tilde{y} \cap T_{\tilde{y}_1}) \ = \ \tilde{w} \cap T_{\tilde{w}_1}. \tag{5.72}$$

Now put together the tree isomorphisms g_r to get a tree automorphism g of T such that

$$g_*(\tilde{x}) \ = \ \tilde{z} \qquad \text{and} \qquad g_*(\tilde{y}) \ = \ \tilde{w}. \tag{5.73}$$

Define $f = (g_2)^{-1} \circ g \circ g_1 : T_p \to T_q$ a tree isomorphism such that

$$f_*(x \cap T_p) \ = \ z \cap T_q \qquad \text{and} \qquad f_*(y \cap T_p) \ = \ w \cap T_q. \tag{5.74}$$

Let $k = j_q \circ f_* \circ (j_p)^{-1}$ so that k is an order isomorphism from $j_p(X(T_p)) \subset X(T)$ to $j_q(X(T_q)) \subset X(T)$. From (5.74) and the definition (5.8) and (5.9) of j_p and j_q we have

$$k(x) \ = \ z \qquad \text{and} \qquad k(y) \ = \ w. \tag{5.75}$$

By (5.27) the open interval (x, y) is contained in $j_p(X(T_p))$ and k restricts to an isomorphism of (x, y) to (z, w). Since j_p, f_* and j_q map branches of height α to branches of height α we have that

$$k(j_p(X(T_p) \cap \tilde{L}_\alpha)) \ = \ X(T_q) \cap \tilde{L}_\alpha \tag{5.76}$$

which implies (5.63).

(b): In this case the branches x, y, z, w in $X(T^{\alpha+1})$ correspond to vertices in T of level α. We mimic the proof of part (a) defining β, ϵ and vertices p, q as before. $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ are branches of height α in $X(T^{\alpha+1})$. However, when we define the tree isomorphisms g_r as before they are isomorphisms of the entire tail tree not just of the level α truncation. In making the adjustment to g_{r_1} with $r_1 = \tilde{x}_1$, we think of $\tilde{x}$ and $\tilde{z}$ not as branches but as vertices of the tree T at level α. Then $g_{r_1}(\tilde{x})$ and $\tilde{z}$ are both level α vertices of $T_{r_2} \cong T$. By Proposition 5.19 we may use homogeneity of T to adjust g_{r_1} and similarly adjust g_{r_2} so that

$$g_{\tilde{x}_1}(\tilde{x}) \ = \ \tilde{z} \qquad \text{and} \qquad g_{\tilde{y}_1}(\tilde{y}) \ = \ \tilde{w}. \tag{5.77}$$

Assemble the maps g_r to form the automorphism g of T and define $f = (g_2)^{-1} \circ g \circ g_1 : T_p \to T_q$ as before. Regarding x, y, z, w as vertices, f satisfies

$$f(x) \ = \ z \qquad \text{and} \qquad f(y) \ = \ w. \tag{5.78}$$

Because α is tail-like and $o(p) < \alpha$ (5.58) implies that $(T^{\alpha+1})_p = (T_p)^{\alpha+1}$ and similarly for q since $o(q) < \alpha$ as well.

Thus, we can define $f_* : X((T^{\alpha+1})_p) \to X((T^{\alpha+1})_q)$ and let $k = j_q \circ f_* \circ (j_p)^{-1}$ an order isomorphism between the intervals $j_p(X((T^{\alpha+1})_p))$ and $j_q(X((T^{\alpha+1})_q))$ in $X(T^{\alpha+1})$ so that

$$k(j_p(X((T^{\alpha+1})_p)) \cap L_\alpha) \ = \ j_q(X((T^{\alpha+1})_q)) \cap L_\alpha. \tag{5.79}$$

As before, (5.78) implies that k maps the branches x and y to z and w, respectively. Finally, (5.27) again implies that the interval (x, y) is contained in $j_p(X((T^{\alpha+1})_p))$.

$\square$

We illustrate the use of this result with the following.

Proposition 5.21. *Let T be a reproductive, bi-ordered tree with S_0 a doubly transitive LOTS with at least three elements. Let $\alpha = h(T)$. If $\tilde{L}_\alpha = \{x \in X(T) : h(x) = \alpha\}$ is nonempty, then it is a doubly transitive LOTS which is dense in $X(T)$.*

Proof. If $x \in \tilde{L}_\alpha$ and $p \in T$, then there exists $q \in x$ such that $o(q) = o(p)$. By Proposition 5.19 there exists an automorphism f of T such that $f(q) = p$. Then $f_*(x) \in \tilde{L}_\alpha$ with $p \in f_*(x)$. It follows from Proposition 5.5(g) that $\tilde{L}_\alpha$ is dense in $X(T)$. Since S_0 is doubly transitive and infinite it is unbounded. Hence, by Proposition 5.5(d), $X(T)$ is unbounded. Since $\tilde{L}_\alpha$ is dense in $X(T)$ it is unbounded. That $\tilde{L}_\alpha$ is doubly transitive then follows from Lemma 5.20(a) and Proposition 3.2(d) (vii)$\Rightarrow$(i).

□

Our main application of Lemma 5.20 requires the following technical result.

Lemma 5.22. *Let X be an unbounded, order dense LOTS of countable type and let W be a dense subset of X. If for every $x < y$, $z < w$ in W there exists an order isomorphism $k : (x, y) \to (z, w)$ (intervals in X) such that*

$$k((x, y) \cap W) \ = \ (z, w) \cap W, \tag{5.80}$$

then the same is true for any $x < y$, $z < w$ in X. In particular, W and X are HLOTS.

Proof. W is unbounded and so is doubly transitive by Proposition 3.2(d)(v)$\Rightarrow$(i). By Proposition 2.11(f), W has countable type and so is a HLOTS by Proposition 3.8(a).

If $x < y$, $z < w$ in X, then since X is order dense, W is dense in X and X has countable type, there exist $f, g : \mathbb{Z} \to W$ with $f(\mathbb{Z})$ ±cofinal in (x, y) and with $g(\mathbb{Z})$ ±cofinal in (z, w). Put together isomorphisms $(f(i), f(i + 1)) \cong (g(i), g(i + 1))$ to get the required k. X is doubly transitive and so is a HLOTS by Proposition 3.2(d) and Proposition 3.8(a) again.

□

Theorem 5.23. *If T is a reproductive, Ω bounded, bi-ordered tree with S_0 a HLOTS, then $X(T)$ is an IHLOTS and $\widehat{X(T)}$ is a CHLOTS.*

Proof. By Lemma 5.18(b) T is of S_0 type. Proposition 5.12(c) and Proposition 2.15(c) then imply that $X(T)$ and $\widehat{X(T)}$ are of countable type. By Proposition 5.5 $X(T)$ is unbounded, order dense and has dense holes. So the completion $\widehat{X(T)}$ is unbounded and connected. If W is any dense subset of $X(T)$, then it is of countable type by Proposition 2.11(f). Since such a W is a proper subset of its completion $\widehat{X(T)}$ it is an IHLOTS if it is doubly transitive by Proposition 3.8. Then the completion $\widehat{X(T)}$ is a CHLOTS by the same proposition.

Since T is Ω bounded, we have $h(T) \leq \Omega$ with $h(T)$ an infinite tail-like ordinal. There are two cases.

Case 1: If $\alpha = h(T)$ is countable, then by Lemma 5.6(b) $\tilde{L}_\alpha$ is dense in $X(T)$. As in Proposition 5.21, Lemma 5.20(a) allows us to conclude that $\tilde{L}_\alpha$ is doubly transitive, using Proposition 3.2(d). Lemma 5.20(a) together with Lemma 5.22 implies that $\tilde{L}_\alpha$ and $X(T)$ are HLOTS in this case.

Case 2: If $h(T) = \Omega$, then given $x < y$ and $z < w$ in $X(T)$ we can choose α a countable tail-like ordinal such that

$$\alpha \quad > \quad h(x), h(y), h(z), h(w) \tag{5.81}$$

because h takes only countable values on $X(T)$ and $h(T) = \Omega$ is the limit of countable, tail-like ordinals.

Now we apply Lemma 5.20(b) to see that $W = L_\alpha \subset X(T^{\alpha+1})$ satisfies the hypotheses of Lemma 5.22. From (5.81) we clearly have

$$\pi^{\alpha+1}(x) \ < \ \pi^{\alpha+1}(y) \quad \text{and} \quad \pi^{\alpha+1}(z) \ < \ \pi^{\alpha+1}(w) \quad \text{in } X(T^{\alpha+1}). \tag{5.82}$$

It follows that there exists an order isomorphism $k : (\pi^{\alpha+1}(x), \pi^{\alpha+1}(y)) \to (\pi^{\alpha+1}(z), \pi^{\alpha+1}(w))$, intervals in $X(T^{\alpha+1})$, which preserves L_α.

We finish up as in the proof of Theorem 5.15. As we did there, write

$$X(T) \ = \ \Sigma\{(\pi^{\alpha+1})^{-1}(r) : r \in X(T^{\alpha+1})\}. \tag{5.83}$$

Recall that for $r = x(p)$ with $o(p) = \alpha$, $(\pi^{\alpha+1})^{-1}(r) = j_p(T_p)$ while for $r \not\in L_\alpha$ $(\pi^{\alpha+1})^{-1}(r)$ is a singleton.

For $x(p) \in (\pi^{\alpha+1}(x), \pi^{\alpha+1}(y)) \cap L_\alpha$ let $k(p)$ denote the vertex of level α such that $k(x(p)) = x(k(p))$. For each such p choose a tree isomorphism $g_p : T_p \to T_{k(p)}$.

For $r = x(p)$ let

$$k_r = j_{k(p)} \circ (g_p)_* \circ (j_p)^{-1} : (\pi^{\alpha+1})^{-1}(r) \to (\pi^{\alpha+1})^{-1}(k(r)). \tag{5.84}$$

and for $r \notin L_\alpha$ let k_r be the map of singletons given by $(\pi^{\alpha+1})^{-1}(r) \mapsto (\pi^{\alpha+1})^{-1}(k(\pi^{\alpha+1}(r)))$. Putting these together using (5.83) we obtain an isomorphism $\tilde{k} : (x, y) \to (z, w)$, intervals in $X(T)$.

Thus, $X(T)$ is doubly transitive in this case as well by Proposition 3.2(d).

□

Adapting the proof, we obtain the first part of the following.

Theorem 5.24. *Assume that T is a reproductive, bi-ordered tree of height Ω with S_0 a HLOTS.*

(a) *If $\tilde{L}_\Omega$ is nonempty, i.e. there exist branches of height Ω, then $\tilde{L}_\Omega$ is dense in $X(T)$ and is doubly transitive. If, in addition, $X(T) \setminus \tilde{L}_\Omega$ is nonempty, then $X(T) \setminus \tilde{L}_\Omega$ is dense in $X(T)$ and is first countable and weakly homogeneous.*

(b) *Assume that $\tilde{L}_\Omega$ and $X(T) \setminus \tilde{L}_\Omega$ are nonempty and that $X = \widehat{X(T)} \setminus \tilde{L}_\Omega$ with $\widehat{X(T)}$ the completion of $X(T)$. The LOTS X is first countable and weakly homogeneous. In addition, X is countably complete. That is, if $A \subset X$ is a bounded, countable set, then $sup\ A,\ inf\ A\ \in X$. On the other hand, X is not complete.*

Proof. (a): If $\tilde{L}_\Omega$ is nonempty, then it is dense in $X(T)$ and is doubly transitive by Proposition 5.21.

Since S_0 is a HLOTS, $X(T)$ is unbounded and is σ-bounded. If $x \in X(T)$ with $h(x) < \Omega$, then because $X(T)$ is unbounded and $\tilde{L}_\Omega$ is dense there exist $z_1, z_2 \in \tilde{L}_\Omega$ such that $z_1 < x < z_2$. Now let $w_1 < w_2 \in \tilde{L}_\Omega$ be arbitrary. By Lemma 5.20 there exists an order isomorphism $k : (z_1, z_2) \to (w_1, w_2)$ which maps $(z_1, z_2) \cap \tilde{L}_\Omega$ to $(w_1, w_2) \cap \tilde{L}_\Omega$. Hence, $k(x) \notin \tilde{L}_\Omega$, i.e. $w_1 < k(x) < w_2$ with $h(k(x)) < \Omega$. It follows that $X(T) \setminus \tilde{L}_\Omega$ is dense in $X(T)$.

The proof of Case 2 in the proof of Theorem 5.23 directly shows that $X(T) \setminus \tilde{L}_\Omega$ is doubly transitive.

Let α be a countable tail-like ordinal with $\alpha > h(x)$. By Lemma 5.18 (b), T^α is reproductive and so by Theorem 5.23 $X(T^\alpha)$ is a HLOTS and so is first countable. Hence, x is the limit of increasing and decreasing sequences in $X(T^\alpha)$. So there exist sequences in $X(T)$ whose projections via π^α are strictly monotone sequences $X(T^\alpha)$ converging to x. Furthermore, $(\hat{\pi}^\alpha)^{-1}(x)$ is a singleton in the complete LOTS $\widehat{X(T)}$. It follows that the sequences in $X(T)$ converge to x. Thus,

$X(T) \setminus \tilde{L}_\Omega$ is doubly transitive, first countable and σ-bounded. It is weakly homogeneous by Proposition 3.3(b).

(b): Applying Proposition 5.9 we identify the completion $\widehat{X(T)}$ with the branch space $X(\hat{T})$ where $\hat{T}$ is the tree completion of Definition 5.8. Every branch of $X(\hat{T}) \setminus X(T)$ is of the form $x(q)$ with q a new vertex. These branches have countable height and each is clearly the limit of increasing and decreasing sequences because S_0 is of countable type. From (a) it follows that X is first countable.

On the other hand, let $x \in \tilde{L}_\Omega$ and A be a countable subset of $X(\hat{T})$ with $a < x$ for all $a \in A$. The equality level of a and x is a countable ordinal for each a. We can choose $\alpha < \Omega$ which is greater than all of these equality levels. The successor set in T for x_α is isomorphic to S_0 and so is unbounded. Choose y in the successor set with $y < x_{\alpha+1}$. Thus, a branch b of $X(T)$ which contains y satisfies $b < x$ and $a < b$ for all $a \in A$. Thus, $sup\ A$ in the complete space $X(\hat{T})$ is not equal to x. Hence, $sup\ A \in X$. Similarly, if A is bounded below then $inf\ A$ lies in X. That is, X is countably compact. In particular, x is not a limit of an increasing or decreasing sequence in $X(\hat{T})$. It follows that any automorphism of $X(\hat{T})$ maps $\tilde{L}_\Omega$ to itself and so restricts to an automorphism of X.

If $a < b, c < d$ in $X(T) \setminus \tilde{L}_\Omega$, then from (a) there exists an automorphism of $X(T) \setminus \tilde{L}_\Omega$ which maps $[a, b]$ to $[c, d]$. Because $X(T) \setminus \tilde{L}_\Omega$ is dense in $X(T)$ which is dense in $X(\hat{T})$ the automorphism extends to an automorphism of $X(\hat{T})$ and then restricts to an automorphism of X.

Now let $a < b$ in X. Because $X(T)$ is σ-bounded and $X(T) \setminus \tilde{L}_\Omega$ is dense in $X(T)$ we can choose an embedding $q : \mathbb{Z} \to X(T) \setminus \tilde{L}_\Omega$ which is $\pm$ cofinal in X. Similarly, because X is first countable, we can choose an embedding $r : \mathbb{Z} \to (a, b) \cap (X(T) \setminus \tilde{L}_\Omega)$ which is $\pm$ cofinal in (a, b). Choose an isomorphism from $[q(i), q(i + 1)]$ to $[r(i), r(i + 1)]$ for each $i \in \mathbb{Z}$ and concatenate to obtain an isomorphism from X to $(a, b) \cap X$. Thus, X is weakly homogeneous.

□

Remark. Since X is invariant for every automorphism of $\widehat{X(T)}$ it follows that $X(T)$ and its completion are not transitive. If $x < z \in \tilde{L}_\Omega$, then the convex set $X_1 = (x, z) \cap X$ is not σ-bounded. So while X_1 is unbounded, every countable subset is bounded and so has a sup and inf in X_1. X_1 is doubly transitive, but not weakly homogeneous.

6. **Tree Constructions**

6.1. **The Simple Trees on a LOTS.** Throughout this section X will be a nontrivial LOTS, i.e. with at least two points. For any ordinal α, the set of maps X^α is a LOTS with the order space product structure, see (2.16). If $p \in X^\alpha$ and $\beta \leq \alpha$, then the restriction $p|\beta \in X^\beta$. For $p \in X^\alpha, q \in X^\beta$ we write

$$q \prec p \quad \Longleftrightarrow \quad \beta < \alpha \text{ and } p|\beta = q. \tag{6.1}$$

Accordingly, we define the *simple tree on* X, α by letting X^i be the set of vertices of level i for $i < \alpha$ and using the restriction partial order of (6.1). If $p \in X^i$ and $i+1 < \alpha$ then we can identify the successor set $S_p \subset X^{i+1}$ with X by

$$\begin{aligned} S_p \ &\cong \ X \\ q \ &\mapsto \ q(i). \end{aligned} \tag{6.2}$$

We use the LOTS ordering on X and the identification of (6.2) to give the simple tree on X, α the structure of a normal tree, bi-ordered and of X type. Conditions (i)-(iv) of Definition 5.1 are easy to check. Every branch has height α. Such a branch is a coherent collection $\{p_\epsilon : \epsilon < \alpha\}$ of functions whose union is a unique element $x \in X^\alpha$. Thus, we can identify X^α with the branch space of the simple tree on X, α.

For $p \in X^\alpha$ and $q \in X^\beta$ we define the *sum* $p + q \in X^{\alpha+\beta}$ by

$$(p+q)(i) \ = \ \begin{cases} p(i) & i < \alpha \\ q(i \setminus \alpha) & \alpha \leq i < \alpha + \beta \end{cases} \tag{6.3}$$

where, as usual, we identify the tail set $i \setminus \alpha$ with the ordinal which has its order type.

For ordinals $\alpha < \epsilon$ we define the *translation map*

$$\begin{aligned} \tau_\alpha : X^\epsilon &\to X^{\epsilon \setminus \alpha} \\ \tau_\alpha(r)(i) \ &= \ r(\alpha + i) \qquad \text{for } i < \epsilon \setminus \alpha. \end{aligned} \tag{6.4}$$

If $p \in X^\alpha$, $q \in X^\beta$ and $p + q \in X^\epsilon$ so that $\epsilon = \alpha + \beta$, then $\beta = \epsilon \setminus \alpha$ and

$$(p+q)|\alpha \ = \ p \qquad \text{and} \qquad \tau_\alpha(p+q) \ = \ q. \tag{6.5}$$

. Conversely, if $r \in X^\epsilon$ and $\epsilon = \alpha + \beta$, then $\beta = \epsilon \setminus \alpha$ and

$$r \ = \ (r|\alpha) + \tau_\alpha(r). \tag{6.6}$$

Clearly, if $o(p) = \alpha$ and $o(r) = \epsilon$, then

$$p \prec r \quad \Longleftrightarrow \quad r = p + q \tag{6.7}$$

for some $q \in X^{\epsilon \setminus \alpha}$ in which case, by (6.5) $\tau_\alpha(r) = q$.

The map τ_α is clearly surjective, but is not order preserving and need not be continuous.

Proposition 6.1. *Let α be a positive ordinal.*

(a) *If X is transitive, then the simple tree on X, α is homogeneous.*
(b) *If α is tail-like, then the simple tree on X, α is reproductive.*

Proof. (a) If $p, q \in X^i$, then there exists for each $j < i$, $g_j \in H_+(X)$ such that $g_j(p(j)) = q(j)$. For $i \leq j < \alpha$ let g_j be the identity. On X^k for $k < \alpha$ the product map $\Pi_{j<k} g_j$ defines a tree automorphism which maps p to q.

(b) For p in the simple tree $o(p) < \alpha$ and α tail-like implies that $o(p) + \beta < \alpha$ whenever $\beta < \alpha$. So we can define the *canonical tree isomorphism at* p by

$$\begin{aligned} a_p &: T \to T_p \\ a_p(q) &= p + q. \end{aligned} \tag{6.8}$$

□

As a corollary we obtain a tree proof of Theorem 3.6.

Corollary 6.2. *Let α be an infinite, tail-like ordinal.*

(a) *If X is doubly transitive, then X^α is doubly transitive.*
(b) *If α is countable and X is a HLOTS, then X^α is a HLOTS.*

Proof. X^α is the branch space of the simple tree on X, α. By Proposition 6.1(b) the simple tree is reproductive.

In this case every branch has height α, i.e. $X^\alpha = \tilde{L}_\alpha$. If X is doubly transitive, then X^α is doubly transitive by Proposition 5.21. If, in addition, α is countable and X has countable type, then by Proposition 2.13(b), X^α has countable type. So it is a HLOTS by Proposition 3.8(a).

□

Recall from Section 4 that if X is a HLOTS, then we choose a non-trivial, bounded subinterval $J = [-1, +1]$ in X to define X_α for every positive ordinal α. If α is an infinite limit ordinal, then we define

$$X_{<\alpha} = \bigcup_{\beta<\alpha} j^\beta_\alpha((X_\beta)') \tag{6.9}$$

using the inclusion maps of (4.5). So $x \in X_{<\alpha}$ iff there exists $\beta < \alpha$ such that

$$x_i = -1 \text{ for all } \beta \leq i < \alpha \qquad \text{or} \qquad x_i = +1 \text{ for all } \beta \leq i < \alpha. \tag{6.10}$$

Clearly, $X_{<\alpha}$ and its complement are dense in X_α.

Proposition 6.3. *If X is a HLOTS and α is an infinite, tail-like ordinal, then*

$$X^\alpha \quad \cong \quad X_\alpha \setminus X_{<\alpha}. \tag{6.11}$$

In particular, $X^\alpha, X_{<\alpha}, X_\alpha \setminus X_{<\alpha}$ and X_α all have order isomorphic completions.

Proof. For $i \in \alpha \setminus 1 = \{j : 0 < j < \alpha\}$ we denote by i^* the corresponding element of the reverse $(\alpha \setminus 1)^*$ and we write

$$|i| \ = \ i \ = \ |i^*| \qquad \text{for } i \in \alpha. \tag{6.12}$$

Apply Proposition 2.15(d) to $\bullet X \bullet$ to get an order embedding

$$\begin{gathered} \tilde{g} : (\alpha \setminus 1)^* + (\alpha \setminus 1) \to X \quad \text{with} \\ \tilde{g}(1) \ = \ +1, \qquad \tilde{g}(\alpha \setminus 1) \text{ cofinal in } X, \\ \tilde{g}(1^*) \ = \ -1, \quad \text{and} \quad \tilde{g}((\alpha \setminus 1)^*) \text{ coinitial in } X. \end{gathered} \tag{6.13}$$

We define the intervals $\{J_{i^*} : i^* \in (\alpha \setminus 1)^*\} \cup \{J_0\} \cup \{J_i : i \in (\alpha \setminus 1)\}$ and choose isomorphisms:

$$\begin{gathered} g_0 \ = \ \text{identity} \quad \text{ on } J_0 \ = \ (-1, +1). \\ g_i : [-1, +1) \to J_i \ = \ [\tilde{g}(i), \tilde{g}(i+1)) \\ g_{i^*} : (-1, +1] \to J_{i^*} \ = \ (\tilde{g}((i+1)^*), \tilde{g}(i^*)] \end{gathered} \tag{6.14}$$

This defines an $(\alpha \setminus 1)^* + \{0\} + (\alpha \setminus 1)$ indexed family of pairwise disjoint, nontrivial subintervals of X with union X.

Now for each $x \in X_\alpha \setminus X_{<\alpha}$ we define an order embedding from α into itself:

$$\begin{gathered} \beta(x, 0) = 0 \\ \beta(x, i) = sup\{\beta(x, j) : j < i\} \quad \text{when } i \text{ is a limit ordinal.} \end{gathered} \tag{6.15}$$

The remaining values are defined inductively. Assuming $\beta(x, i)$ is defined we construct $\beta(x, i + 1)$.

First, we set $m(x, 0) = 0$ and for $i \in \alpha \setminus 1$:

(6.16)

$$\begin{gathered} m(x, i) = 0 \quad \text{if} \quad -1 < x_{\beta(x,i)} < +1, \\ m(x, i) = j \quad \text{if} \quad x_{\beta(x,i)} = +1 \text{ and } j = min\ \{k : x_{\beta(x,i)+k} < +1\}, \\ m(x, i) = j^* \quad \text{if} \quad x_{\beta(x,i)} = -1 \text{ and } j = min\ \{k : x_{\beta(x,i)+k} > -1\}. \end{gathered}$$

Observe that $x \notin X_{<\alpha}$ implies that the sets used to define j in the latter cases are nonempty. Complete the inductive definition of β by

$$\beta(x, i+1) \;=\; \beta(x, i) + |m(x, i)| + 1, \tag{6.17}$$

so that, for example, $\beta(x, 1) = 1$.

Since $x \in X_\alpha$, $x_0 \in X$ and $x_i \in J = [-1, +1]$ for all $i \in \alpha \setminus 1$. We define $f(x) \in X^\alpha$ by

$$f(x)_0 = x_0 \quad \text{and} \quad f(x)_i = g_{m(x,i)}(x_{\beta(x,i)+|m(x,i)|}) \quad \text{for } i \in \alpha \setminus 1. \tag{6.18}$$

If $x_{\beta(x,i)} \in (-1, +1)$, then $m(x, i) = 0$ and $f(x)_i = x_{\beta(x,i)}$.

If $x_{\beta(x,i)} = +1$, then $j = m(x, i) \in \alpha \setminus 1$ and $x_{\beta(x,i)+k} = +1$ for all $k < j$ while $x_{\beta(x,i)+j} \in [-1, +1)$. The map g_j moves $x_{\beta(x,i)+j}$ to the interval J_j in X.

Similarly, if $x_{\beta(x,i)} = -1$ then $j^* = m(x, i) \in (\alpha \setminus 1)^*$ and g_{j^*} moves $x_{\beta(x,i)+j}$ to J_{j^*}.

Thus, we have defined a map $f : X_\alpha \setminus X_{<\alpha} \to X^\alpha$.

Suppose that $x < y$. If $x_0 < y_0$, then $f(x) < f(y)$. Otherwise, let $\epsilon = min\ \{j : x_j \neq y_j\}$ so that $\epsilon \in \alpha \setminus 1$ and $x_\epsilon < y_\epsilon$. By Proposition 2.15(c) applied to the order embedding $i \mapsto \beta(x, i)$ there exists $i \in \alpha$ such that

$$\beta(x, i) \;\leq\; \epsilon \;<\; \beta(x, i+1). \tag{6.19}$$

Since $\epsilon \in \alpha \setminus 1$ and $\beta(x, 1) = 1$, we have $i \in \alpha \setminus 1$. Notice that the inductive definitions imply that

$$m(x, j) = m(y, j) \quad j < i \qquad \text{and} \qquad \beta(x, j) = \beta(y, j) \quad j \leq i \tag{6.20}$$

and so we have $f(x)_j = f(y)_j$ for all $j < i$ and by considering the various cases we check that $f(x)_i < f(y)_i$.

Case 1: If $\beta(x, i) = \beta(y, i) = \epsilon$, then $x_{\beta(x,i)} < y_{\beta(y,i)}$ and so $m(y, i) \in \alpha$ and $m(x, i) \in \alpha^*$. Hence, $m(x, i) \leq m(y, i)$.

Case 2: If $\beta(x, i) = \beta(y, i) < \epsilon$, then $x_{\beta(x,i)} = y_{\beta(y,i)}$ and the common value cannot lie in $(-1, +1)$ since $\epsilon < \beta(x, i+1)$. If $x_{\beta(x,i)} = y_{\beta(y,i)} = +1$, then $m(x, i) < m(y, i) \in \alpha \setminus 1$. If $x_{\beta(x,i)} = y_{\beta(y,i)} = -1$, then $m(x, i) < m(y, i) \in (\alpha \setminus 1)^*$.

In either case, $f(x)_i < f(y)_i$ and so $f(x) < f(y)$. Thus, f is an order injection.

To reverse the procedure, start with $z \in X^\alpha$ and let $x_0 = z_0$. Then for any $i \in \alpha \setminus 1$, $\{z_j : 0 \leq j < i\}$ determines, inductively, the sets $\{m(x, j) : j < i\}, \{\beta(x, j) : j \leq i\}$ and $\{x_k : k < \beta(x, i)\}$. Now if $z_i \in J_0$, then $m(x, i) = 0$, $x_{\beta(x,i)} = z_i$ and $\beta(x, i+1) = \beta(x, i) + 1$. If $z_i \in J_j$ for some $j \in \alpha \setminus 1$ then $m(x, i) = j$, $x_{\beta(x,i)+k} = +1$ for

$0 \leq k < j$, $x_{\beta(x,i)+j} = (g_j)^{-1}(z_i)$ and $\beta(x, i+1) = \beta(x,i) + j + 1$. Similarly, if $z_i \in J_{j^*}$.

Thus, f is surjective and so is an order isomorphism.

□

We can use this result to get an alternative proof of the CHLOTS portion of Theorem 4.2.

Corollary 6.4. *Let X be a CHLOTS and α be a countably infinite, tail-like ordinal. X^α and $X_{<\alpha}$ are IHLOTS with completion isomorphic to the CHLOTS X_α.*

Proof. X^α is a HLOTS by Corollary 6.2(b) and it is isomorphic to a dense proper subset of X_α by Proposition 6.3. Since X is complete, X_α is complete by Proposition 2.8(b) and so X_α is the completion of the image of X^α. By Proposition 3.8 X_α is a CHLOTS and X^α and its complement in X_α, which is $X_{<\alpha}$, are IHLOTS.

□

Proposition 6.5. *Assume X is a LOTS. Let T be a bi-ordered tree of Y type. If Y injects into X and the height of the tree is at most α, then the branch space $X(T)$ injects into X^α. If, in addition, X is a CHLOTS, Y is unbounded and $h(T)$ is a limit ordinal, then $X(T)$ and its completion inject into X_α.*

Proof. We identify each nonempty successor set S_p with Y. Let $j : Y \to X$ be an order injection and let $z \in X$.

Define $j^\alpha : X(T) \to X^\alpha$ by

$$j^\alpha(x)(i) = \begin{cases} j(x(i)) & \text{for } i < h(x), \\ z & \text{for } h(x) \leq i < \alpha. \end{cases} \tag{6.21}$$

This is clearly an order injection.

If X is a CHLOTS, then $X \cong J^\circ$ implies that X^α injects into X_α. Hence, $X(T)$ injects into X_α.

If Y is unbounded and $h(T)$ is a limit ordinal, then $X(T)$ is order dense by Proposition 5.4. Since X_α is complete, the order injection $\widehat{j^\alpha} : \widehat{X(T)} \to X_\alpha$ is defined by Proposition 2.10.

□

Contrast these results with those for X_Ω. We can split $X_{<\Omega}$ into two disjoint pieces $X^{\pm}_{<\Omega}$ with

$$x \in X^{+}_{<\Omega} \quad \Longleftrightarrow \quad x_i = +1 \quad \text{for } i \in \Omega \text{ sufficiently large,} \tag{6.22}$$

and similarly, for $X^{-}_{<\Omega}$. Each of these is a dense subset of X_Ω and by Proposition 6.3 we can identify X^Ω with the complement of their union. It is easy to check that $z \in X_\Omega$ is the limit of some increasing (or of some decreasing) sequence iff $z \in X^{-}_{<\Omega}$ (resp. $z \in X^{+}_{<\Omega}$). It follows that if $f \in H_+(X_\Omega)$, then

$$f(X^{-}_{<\Omega}) \;=\; X^{-}_{<\Omega}, \quad f(X^\Omega) = X^\Omega, \quad f(X^{+}_{<\Omega}) \;=\; X^{+}_{<\Omega}. \tag{6.23}$$

That is, the decomposition of X_Ω into three pairwise disjoint, dense subsets is invariant with respect to the action of the automorphism group $H_+(X_\Omega)$.

By Corollary 6.2(a), X^Ω is doubly transitive. Using the proof of Proposition 3.8(d) and H_+ invariance it is easy to show that $X^{+}_{<\Omega}$ and $X^{-}_{<\Omega}$ are doubly transitive as well.

Moreover, if X is symmetric, e.g. $X = \mathbb{R}$, then X_Ω is symmetric and any orientation reversing isomorphism interchanges $X^{+}_{<\Omega}$ and $X^{-}_{<\Omega}$ while preserving X^Ω. In that case, $X_{<\Omega}$ is $\pm$transitive but not transitive. Any dense subset of X_Ω is order dense and so has no gap pairs.

Contrast this with the complete case in Proposition 3.11.

6.2. **Additive Trees.** Now we introduce an important class of subtrees of the simple tree.

Definition 6.6. *For a positive ordinal α let T be a subset of the simple tree on X, α. T is called an* additive X tree *when T contains the root* 0*, the level* 1 *vertices X^1 of the simple tree and for all vertices p, q of the simple tree*

$$p + q \in T \quad \Longleftrightarrow \quad p, q \in T. \tag{6.24}$$

T is called a partially additive X tree *when T contains $\{0\} \cup X^1$ and for all vertices p, q of the simple tree*

$$p + q \in T \quad \Longleftrightarrow \quad p, q \in T \quad \text{and} \quad o(p) + o(q) < h(T). \tag{6.25}$$

Clearly, $p + q \in T$ always implies

$$o(p) + o(q) \;=\; o(p + q) \;<\; h(T), \tag{6.26}$$

and so, as the name suggests, condition (6.25) is a weakening of condition (6.24).

Proposition 6.7. *Let $\alpha, \beta > 1$ be ordinals.*

(a) *The simple tree on X, α is a partially additive X tree. It is additive iff α is tail-like.*
(b) *If T is contained in the simple tree on X, α, then $h(T) \leq \alpha$. Conversely, if T is a partially additive X tree with $h(T) \leq \alpha$, then T is a normal subtree of the simple tree on X, α.*
(c) *If T is a partially additive X tree, then T^β is a partially additive X tree.*
(d) *Assume T is a partially additive X tree and $p \in T$. If $\beta < o(p)$, then*

$$p|\beta, \tau_\beta(p) \ \in \ T. \tag{6.27}$$

If $o(p) + 1 < h(T)$, then the successor set S_p in T consists of all successors of p in the simple tree.
(e) *A partially additive tree T is additive iff $h(T)$ is a tail-like ordinal.*
(f) *If T is an additive X tree, then T is a reproductive tree and for every finite n the set X^n of level n vertices in the simple tree is the set of level n vertices in T.*

Proof. (a),(b),(d): That the simple tree is partially additive is obvious. Condition (6.25) and (6.6) imply (6.27) which implies that a partially additive T is a subtree of the simple tree on X, α when $\alpha \geq h(T)$. Clearly, for p in the simple tree the successor set is given by

$$S_p \ = \ \{p + q : q \in X^1\}. \tag{6.28}$$

So $p \in T$ and $o(p) + 1 < h(T)$ implies this is a subset of T by (6.25). Furthermore, if $p \in T$ and $o(p) < \alpha < h(T)$, then there exists $q \in T$ with $o(q) = \alpha \setminus o(p)$ and so $p + q \in T$ by (6.25). Hence, T is a normal tree.

(c): Since $\beta > 1$, $X^1 \subset T^\beta$. Condition (6.25) for T^β follows from the condition on T.

(e),(f): If T is additive, then the canonical isomorphism (6.8) restricts to an isomorphism of T with T_p for any $p \in T$. Hence, T is reproductive and so $h(T)$ is tail-like by Lemma 5.18(b). Since $h(T) > 1$ it is infinite and so by induction on n using (d), $X^n \subset T$ for every finite n.

On the other hand, if $h(T)$ is tail-like, then $p, q \in T$ implies $o(p) + o(q) < h(T)$ and so $p + q \in T$ by (6.25) and so (6.24) holds.

In particular, the simple tree T on X, α is additive iff $\alpha = h(T)$ is tail-like.

□

Corollary 6.8. *If X is a HLOTS and T is an additive Ω bounded X tree, then $X(T)$ is an IHLOTS with completion $\widehat{X(T)}$ a CHLOTS.*

Proof. T is reproductive by Proposition 6.7(f) and so the result follows from Theorem 5.23.

□

Lemma 6.9. *Let T be an additive X tree. We can identify the branch space $X(T)$ with the set*

$$\begin{aligned}(6.29)\qquad \{x \in X^{\beta} : \beta \textit{ is an infinite limit ordinal}, x \notin T,\\ \textit{and } x|\epsilon \in T \textit{ for all } \epsilon < \beta\}.\end{aligned}$$

With this identification we have, for $p \in T$ and $x \in X^{\beta}$

$$(6.30)\qquad x \in X(T) \qquad \Longleftrightarrow \qquad p + x \in X(T).$$

For $x \in X^{\beta}$ and $\epsilon < \beta$

$$(6.31)\qquad x \in X(T) \qquad \Longleftrightarrow \qquad x|\epsilon \in T \quad \textit{and} \quad \tau_{\epsilon}(x) \in X(T).$$

Proof. Any branch of height less than $h(T)$ has height an infinite limit ordinal. Since T is reproductive, $h(T)$ is also an infinite limit ordinal. A branch of height β is a coherent collection $\{x_{\epsilon} \in X^{\epsilon} : \epsilon < \beta\}$ which fits together to define an element $x \in X^{\beta}$ such that $x|\epsilon = x_{\epsilon}$ for all $\epsilon < \beta$. Hence the restrictions are all in T. If x itself were in T, then we could extend the branch by adjoining x which violates the maximality condition of the branch. Hence, $x \notin T$. Conversely, if x lies in the set described by (6.29), then $\{x|\epsilon : \epsilon < \beta\}$ is a maximal totally ordered set of vertices of T and so is the corresponding branch.

The characterization of (6.29) together with (6.24) easily implies (6.30). Then (6.31) follows from (6.6).

□

We now present the inductive construction which shows how all additive trees are built. We use the translation maps defined by (6.4). Recall that if ϵ is tail-like and $\alpha < \epsilon$, then $\epsilon = \epsilon \setminus \alpha$ and so τ_{α} maps X^{ϵ} to itself. If ϵ is tail-like and $W \subset X^{\epsilon}$, then we call W *translation invariant* if

$$(6.32)\qquad \tau_{\alpha}(W) \ \subset \ W \qquad \text{for all } \alpha < \epsilon.$$

A collection of trees $\{T_\delta : \beta \leq \delta < \epsilon\}$ is called a *coherent collection of trees indexed by* $[\beta, \epsilon)$ if

$$\begin{aligned} h(T_\delta) \ &= \ \delta \qquad \text{for} \quad \beta \leq \delta < \epsilon, \qquad \text{and} \\ T_\delta = (T_\rho)^\delta & \qquad \text{for} \quad \beta \leq \delta \leq \rho < \epsilon. \end{aligned} \tag{6.33}$$

Theorem 6.10. *Let α be a tail-like ordinal and T be a partially additive X tree with $h(T) = \beta < \alpha$.*

(a) *Let ϵ be a limit ordinal with $\beta < \epsilon \leq \alpha$. If $\{T_\delta\}$ is a coherent collection of partially additive X trees indexed by $[\beta, \epsilon)$ then*

$$T_\epsilon \ = \ \bigcup \{T_\delta : \beta \leq \delta < \epsilon\} \tag{6.34}$$

is a partially additive X tree. It defines the unique tree such that $\{T_\delta : \beta \leq \delta \leq \epsilon\}$ is a coherent collection of trees indexed by $[\beta, \epsilon] = [\beta, \epsilon + 1)$.

(b) *Let $\epsilon(\beta) = min\ \{i : \beta \leq i \leq \alpha$ and i is tail-like $\}$. For each δ such that $\beta \leq \delta \leq \epsilon(\beta)$ there is a unique partially additive X tree T_δ of height δ and such that*

$$(T_\delta)^\beta \ = \ T \qquad for \quad \beta \leq \delta \leq \epsilon(\beta). \tag{6.35}$$

The collection $\{T_\delta : \beta \leq \delta \leq \epsilon(\beta)\}$ is a coherent collection of trees indexed by $[\beta, \epsilon(\beta)]$. The tree $T_{\epsilon(\beta)}$ is an additive X tree with

$$T_{\epsilon(\beta)} \ = \ \{p_1 + \cdots + p_n : p_i \in T_\beta \ \ for \ \ i = 1, \ldots, n < \omega\}. \tag{6.36}$$

(c) *Assume that T is an additive X tree so that the height β is tail-like.*

The set $\tilde{L}_\beta = \{x \in X(T) : h(x) = \beta\}$ is a translation invariant subset of X^β.

If $\tilde{L}_\beta = \emptyset$, then the only subtree $\tilde{T}$ of the simple tree on X, α which satisfies $(\tilde{T})^\beta = T$ is $\tilde{T} = T$ itself.

If $\tilde{L}_\beta \neq \emptyset$ and W is any nonempty, translation invariant subset of $\tilde{L}_\beta$, then

$$T_{\beta+1} \ = \ T \cup \{p + x : p \in T \quad and \quad x \in W\} \tag{6.37}$$

is a partially additive tree of height $\beta + 1$ such that

$$(T_{\beta+1})^\beta \ = \ T. \tag{6.38}$$

Conversely, if $T_{\beta+1}$ is a partially additive tree of height $\beta + 1$ which satisfies (6.38), then

$$W \ = \ \{q|\beta : q \in T_{\beta+1} \ with \ o(q) = \beta\} \tag{6.39}$$

is a nonempty, translation invariant subset of $\tilde{L}_\beta$ *and* $p + x \in T_{\beta+1}$ *for all* $p \in T$ *and* $x \in W$*.*

Proof. (a): Since each T_δ has height δ, T_ϵ has height $\epsilon = sup\ \{\delta : \delta < \epsilon\}$ and $(T_\epsilon)^\delta = T_\delta$ is clear for $\beta \leq \delta \leq \epsilon$.

If $p + q \in T_\epsilon$, then for some $\delta < \epsilon$ $p + q \in T_\delta$ and so $p, q \in T_\delta \subset T_\epsilon$ by partial additivity of T_δ. On the other hand, if $p, q \in T_\epsilon$ and $o(p) + o(q) = o(p+q) < \epsilon$, then for some $\delta < \epsilon$ $o(p) + o(q) < \delta$. Hence, $p + q \in T_\delta \subset T_\epsilon$. Thus, T_ϵ is partially additive.

(b): By induction on ρ with $\beta \leq \rho \leq \epsilon(\beta)$ we construct the trees T_ρ verifying (6.35) and uniqueness and coherence at each stage.

Begin with $T_\beta = T$ and assume that T_δ has been constructed for $\beta \leq \delta < \rho$.

Case 1: If ρ is a limit ordinal, define T_ρ using (6.34) with $\rho = \epsilon$. By (a) it is a partially additive X tree and the collection is coherent, indexed by $[\beta, \rho]$.

If $\tilde{T}$ is a partially additive X tree of height ϵ with $\tilde{T}^\beta = T$, then for $\delta \in [\beta, \epsilon)$ $\tilde{T}^\delta$ is partially additive and satisfies $(\tilde{T}^\delta)^\beta = \tilde{T}^\beta = T$. So by uniqueness at the δ level $\tilde{T}^\delta = T_\delta$. Since ρ is a limit ordinal, $h(\tilde{T}) = \rho$ implies

$$\tilde{T} \ = \ \bigcup\{\tilde{T}^\delta\} \ = \ \bigcup\{T_\delta\} \ = \ T_\epsilon. \tag{6.40}$$

Uniqueness follows.

Case 2: If $\rho = \delta + 1$, define

$$T_\rho \ = \ T_\delta \cup \{p + q : p, q \in T_\delta \text{ and } o(p) + o(q) = \delta\}. \tag{6.41}$$

Since $\beta \leq \delta < \epsilon(\beta)$, δ is not a tail-like ordinal and so there exist ordinals $i, j < \delta$ with $i + j = \delta$. Because $h(T_\delta) = \delta$ there exist $p, q \in T_\delta$ with $o(p) = i$ and $o(q) = j$. Hence, $p + q \in T_\rho$ with $o(p+q) = \delta$. Hence, T_ρ has height ρ and coherence is clear. In particular, $(T_\rho)^\beta = T$.

If $o(p) + o(q) < \delta$, then $p, q \in T_\rho$ iff $p, q \in T_\delta$ iff $p + q \in T_\delta$ (by partial additivity of T_δ) iff $p + q \in T_\rho$. Thus in checking (6.25) for T_ρ we can restrict attention to the case $o(p+q) = o(p) + o(q) = \delta$ and $o(q) > 0$ so that $o(p) < \delta$.

If $p, q \in T_\rho$, then $o(p) < \delta$ implies that $p \in T_\delta$. If $o(q) < \delta$, then $q \in T_\delta$ and so $p + q \in T_\rho$ by (6.41). If $o(q) = \delta$, then by (6.41) $q = \tilde{p} + \tilde{q}$ with $\tilde{p}, \tilde{q} \in T_\delta$. Since $o(q) = \delta$, $o(\tilde{q}) > 0$.

$$p + q \ = \ (p + \tilde{p}) + \tilde{q}. \tag{6.42}$$

Since $o(\tilde{q}) > 0$, $o(p + \tilde{p}) = o(p) + o(\tilde{p}) < \delta$. By partial additivity of T_δ, $p + \tilde{p} \in T_\delta$. By (6.42) and (6.41) $p + q \in T_\rho$.

On the other hand, if $p+q \in T_\rho$ with $o(p)+o(q) = \delta$, then by (6.41) there exist $\tilde{p}, \tilde{q} \in T_\delta$ such that

$$p + q \ = \ \tilde{p} + \tilde{q}. \tag{6.43}$$

Now if $o(p) = i \geq o(\tilde{p})$ and $\tilde{i} = i \setminus o(\tilde{p})$, then

$$p \ = \ \tilde{p} + (\tilde{q}|\tilde{i}) \quad \text{and} \quad q \ = \ \tau_{\tilde{i}}(\tilde{q}). \tag{6.44}$$

Since $\tilde{p}, \tilde{q}|\tilde{i} \in T_\delta$ and $o(p) < \delta$, $p \in T_\delta$ because T_δ is partially additive. By (6.27) $q \in T_\delta$ as well.

If, instead, $o(p) = i < o(\tilde{p})$, then

$$p \ = \ \tilde{p}|i \qquad \text{and} \qquad q \ = \ \tau_i(\tilde{p}) + \tilde{q}. \tag{6.45}$$

By (6.27) $p, \tau_i(\tilde{p}) \in T_\delta$. Since $\tilde{q} \in T_\delta$, $o(q) < \delta$ implies $q \in T_\delta$ because T_δ is partially additive, while $o(q) = \delta$ implies $q \in T_\rho$ by (6.45).

Thus, T_ρ is partially additive.

On the other hand, if $\tilde{T}$ has height ρ and $\tilde{T}^\beta = T$, then as in the limit case $\tilde{T}^\delta = T_\delta$ by uniqueness at the δ height. If $p, q \in T_\delta$ with $o(p) + o(q) = \delta < \rho$, then $p + q \in \tilde{T}$ because $\tilde{T}$ is partially additive. Hence, $T_\rho \subset \tilde{T}$. On the other hand, if $r \in \tilde{T}$ with $o(r) = \delta$, then we can choose $i < \delta$ such that $\delta \setminus i < \delta$, because δ is not tail-like. By (6.6) $r = r|i + \tau_i(r)$. Because $\tilde{T}$ is partially additive, (6.27) implies that $r|i, \tau_i(r) \in \tilde{T}^\delta = T_\delta$. Hence, $r \in T_\epsilon$ by (6.41). Hence, $\tilde{T} \subset T_\rho$ which proves uniqueness.

This completes the induction. At the final stage, $T_{\epsilon(\beta)}$ is partially additive with height $\epsilon(\beta)$ tail-like so that it is additive by Proposition 6.7(e).

Since T_ϵ is additive and contains T_β it clearly contains any finite sum $p_1 + \cdots + p_n$ with $p_i \in T_\beta$.

Conversely, if $p \in T_\epsilon$, then $o(p) < \epsilon$ and we can use Cantor Normal Form, Proposition 2.6, to write $o(p) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ with $\beta \geq \alpha_1 \geq \alpha_2 \cdots \geq \alpha_n$. Let $\sigma_0 = 0$ and $\sigma_i = \alpha_1 + \ldots \alpha_i$ for $i = 1, \ldots, n-1$. $p = p_1 + p_2 + \ldots p_n$ where $p_i = (\tau_{\sigma_{i-1}}(p))|\alpha_i$ for $i = 1, \ldots, n$. By additivity, each $p_i \in T_\epsilon$. Since $o(p_i) = \alpha_i \leq \beta$ and $(T_\epsilon)^\beta = T_\beta$ it follows that each $p_i \in T_\beta$.

(c): If $x \in \tilde{L}_\beta$ and $i < \beta$, then $\beta = \beta \setminus i$ and $\tau_i(x) \in X(T)$ by (6.31). Hence, $\tau_i(x) \in \tilde{L}_\beta$. Thus, $\tilde{L}_\beta$ is translation invariant.

If $\tilde{T}$ is a subtree of the simple tree with $\tilde{T}^\beta = T$ and $p \in \tilde{T} \setminus T$, then $o(p) \geq \beta$ and so $A_p \cap \tilde{T}^\beta = A_p \cap T$ is a branch of T with height β. So if such a $\tilde{T}$ exists, then $\tilde{L}_\beta \neq \emptyset$.

Assume W is a nonempty, translation invariant subset of $\tilde{L}_\beta$ and that $T_{\beta+1}$ is defined by (6.37). Clearly, (6.38) holds. It remains to verify (6.25).

Consider vertices p, q of the simple tree. As before it suffices to consider the case $o(p) + o(q) = \beta$ and $o(q) > 0$ so that $o(p) < \beta$.

If $p, q \in T_{\beta+1}$, then $o(p) < \beta$ implies $p \in T$. Since $o(p) + o(q) = \beta$ and β is tail-like, $o(q) = \beta$ and so $q = \tilde{p} + x$ with $\tilde{p} \in T$ and $x \in W$. Since T is additive, $p + \tilde{p} \in T$ and so

$$p + q \ = \ (p + \tilde{p}) + x \ \in T_{\beta+1}, \tag{6.46}$$

by definition (6.37).

On the other hand, if $p + q \in T_{\beta+1}$, then $o(p+q) = o(p) + o(q) = \beta$ implies $p + q = \tilde{p} + x$ for some $\tilde{p} \in T$ and $x \in W$.

As with (6.44), $o(p) \geq o(\tilde{p})$ and $\tilde{i} = i \setminus o(p)$ implies

$$p \ = \ \tilde{p} + (x|\tilde{i}) \quad \text{and} \quad q \ = \ \tau_{\tilde{i}}(x). \tag{6.47}$$

Since $\tilde{i} < \beta$, $x|\tilde{i} \in T$ by Lemma 6.9 and so $p \in T$ by additivity. Because W is translation invariant $q \in W \subset T_{\beta+1}$.

As in (6.45), $o(p) = i < o(\tilde{p})$ implies

$$p \ = \ \tilde{p}|i \qquad \text{and} \qquad q \ = \ \tau_i(\tilde{p}) + x. \tag{6.48}$$

Hence, $p, \tau_i(\tilde{p}) \in T$ and so $q \in T_{\beta+1}$.

This completes the proof of partial additivity for $T_{\beta+1}$

For the converse, it is clear that W of (6.39) is a subset of $\tilde{L}_\beta$. It is nonempty since $T_{\beta+1}$ has height β. For $q \in T_{\beta+1}$ with $o(q) = \beta$ and $i < \beta$, $\tau_i(q) \in T_{\beta+1}$ by partial additivity and $o(\tau_i(q)) = \beta \setminus i = \beta$. So $\tau_i(q)|\beta = \tau_i(q|\beta) \in W$. Thus, W is translation invariant. If $p \in T$ and $x \in T_{\beta+1}$ with $o(x) = \beta$, then by partial additivity $o(p + x) = o(p) + \beta = \beta$ implies $p + x \in T_{\beta+1}$.

$\square$

Remark. Recall that if T is a tree of height α a countable limit ordinal, then $\tilde{L}_\alpha \neq \emptyset$ by Lemma 5.6(b). Thus, the extension process of part (c) can go beyond any countable level. On the other hand, an additive tree of height Ω is Ω-bounded precisely when it cannot be continued to level $\Omega + 1$.

Corollary 6.11. *There exists a LOTS X which is weakly homogeneous, first countable and countably complete, but which is not complete.*

Proof. Because an additive tree is reproductive, it suffices by Theorem 5.24 to construct an additive tree of $\mathbb{Q}$ type which has height Ω and which contains branches of countable height and branches of height Ω.

Let $\bar{0} \in \mathbb{Q}^\Omega$ with $\bar{0}_i = 0 \in \mathbb{Q}$ for every $i < \Omega$. For $p \in \mathbb{Q}^\alpha$ with α a limit ordinal, we say that p *eventually equals* 0 if there exists $\beta < \alpha$ such that $p_i = 0$ for all i with $\beta \leq i < \alpha$.

We use that inductive construction of Theorem 6.10 to build a coherent collection T_α of partially additive trees such that $\bar{0}|\beta \in T_\alpha$ for all $\beta < \alpha$. This condition is clearly preserved in steps (a) and (b). For the choice step (c) with β tail-like, we have that $\bar{0}|\beta \in \tilde{L}_\beta$. The set $W = \{\bar{0}|\beta\}$ is translation invariant and we let $T_{\beta+1} = T \cup \{p + (\bar{0}|\beta) : p \in T\}$. Thus, L_β consists of the elements of $\tilde{L}_\beta$ which eventually equal 0.

With $T = \bigcup T_\alpha$ we identify $X(T)$ as in Lemma 6.9. So $\bar{0} \in X(T)$ with $h(\bar{0}) = \Omega$. On the other hand suppose $x \in \mathbb{Q}^\alpha$ with $\alpha < \Omega$ a tail-like ordinal and x is not eventually 0 and $x|\beta \in T$ for all $\beta < \alpha$, then x is a branch with height α, i.e. it is an element of $\tilde{L}_\alpha$ which is not in L_α. In particular, if $x \in \mathbb{Q}^\omega$ is not eventually 0, then $x \in X(T)$ with $h(x) = \omega$.

$\square$

This result answers a question raised by Babcock [6] Section 2. A closed, bounded interval in a LOTS X given by Corollary 6.11 satisfies his Linear Homogeneity Condition 2, but not his Linear Homogeneity Condition 3.

There is a different way of looking at additive trees.

Let $\tilde{\Omega}$ be the set of infinite tail-like ordinals in Ω so that

$$\tilde{\Omega} \; = \; \{\omega^\gamma : 0 < \gamma < \Omega\}. \tag{6.49}$$

We write $\epsilon(s,t)$ for the *equality level* for a pair $s, t \in X^\Omega$ so that with $\epsilon = \epsilon(s,t)$

$$s_i = t_i \;\text{ for }\; i < \epsilon, \quad \text{and} \quad s_\epsilon \neq t_\epsilon. \tag{6.50}$$

Thus, $s|\epsilon = t|\epsilon$ and $s|(\epsilon+1) \neq t|(\epsilon+1)$.

Definition 6.12. *Let H be a function from X^Ω to $\Omega \setminus 1$. We call H a* height function *when it satisfies:*

(i) *For all $s, t \in X^\Omega, \epsilon(s,t) \geq H(s)$ implies $H(t) = H(s)$.*

We call it an $\tilde{\Omega}$ height function when it takes values in $\tilde{\Omega}$.

The function H is called an additive height function *when it is a $\tilde{\Omega}$ height function which also satisfies:*

(ii) *For all $s \in X^{\Omega}, \alpha < H(s)$ implies $H(\tau_{\alpha}(s)) = H(s)$.*

Lemma 6.13. *Assume that H is an additive height function. Let $s, t \in X^{\Omega}, p \in X^{\alpha}$.*

(a) *If $p = t|\alpha$ with $\alpha < H(t)$, then $\alpha < H(p+s)$.*

(b) *If $\alpha < H(s)$, then $\alpha < H(p+s)$ implies $H(p+s) = H(s)$.*

Proof. (a): $\epsilon(p+s, t) \geq \alpha$, so $H(p+s) \leq \alpha$ would imply, by (i), $H(p+s) = H(t)$ which is larger than α.

(b): By (ii), $\alpha < H(p+s)$ implies that $H(s) = H(\tau_{\alpha}(p+s)) = H(p+s)$.

□

Theorem 6.14. *If T is an Ω-bounded subtree of the simple tree on X, Ω, then the associated height function is given by*

$$H(s) \ = \ min\{\alpha : s|\alpha \notin T\}. \tag{6.51}$$

Conversely, if H is a height function, then $T = \{s|\alpha : \alpha < H(s)\}$ is an Ω-bounded (not necessarily semi-normal) subtree of the simple tree on X, Ω. The branch space is given by

$$X(T) = \{s|H(s) : s \in X^{\Omega}\}. \tag{6.52}$$

The tree T is an additive, Ω-bounded subtree of the simple tree on X, Ω iff the associated height function H is an additive height function.

Proof. If T is an Ω-bounded subtree, and $\alpha < \beta < \Omega$, then $s|\alpha \notin T$ implies $s|\beta \notin T$. If T is Ω-bounded then for every $s \in X^{\Omega}$ $s|\alpha \notin T$ for some $\alpha < \Omega$. It follows that H defined by (6.51) is a height function.

Conversely, if H is a height function, then $s|\alpha = t|\alpha$ implies $\epsilon(s,t) \geq \alpha$. Hence, if $\alpha \geq H(s)$, then $H(s) = H(t)$ and so $\alpha \geq H(t)$. It follows that T is a subtree with associated height function H. The description of the branch space follows as in Lemma 6.9.

If T is additive, then the height of every branch is an infinite, tail-like ordinal. From the description (6.52) it follows that H is an $\tilde{\Omega}$ height function. Furthermore, if $x = s|H(s)$ is a branch and $\alpha < H(s)$, then by Lemma 6.9 $\tau_{\alpha}(x) = (\tau_{\alpha}(s))|H(s)$ is a branch and so $H(\tau_{\alpha}(s)) = H(s)$.

Now let H be an additive height function and that $p = t|o(p), q = s|o(q)$ with $s, t \in X^{\Omega}$ and $o(p), o(q) < \Omega$.

First, assume $p, q \in T$, so that $o(p) < H(t), o(q) < H(s)$.

By Lemma 6.13 (a) $H(p+s) > o(p)$ and so by Lemma 6.13 (b) $H(p+s) = H(s)$. Similarly, $H(q+t) > o(q)$ and $H(q+t) = H(t)$. Finally, $H(p+q+t) = H(q+t) = H(t)$.

If $o(p) < H(s)$, then since $H(s)$ is tail-like, $o(p+q) = o(p) + o(q) < H(s) = H(p+s)$. Hence, $p+q = (p+s)|(o(p)+o(q)) \in T$.

If, instead, $o(p) \geq H(s)$, then $H(t) > o(p) \geq H(s) > o(q)$ implies that $o(p)+o(q)$ is less than the tail-like ordinal $H(t) = H(p+q+t)$. Hence, $p+q = (p+q+t)|(o(p)+o(q)) \in T$.

Conversely, assume that $p+q \in T$ so that $p+q = r|(o(p)+o(q))$ with $o(p)+o(q) < H(r)$. Since $o(p) \leq o(p)+o(q) < H(r)$, $p = r|o(p) \in T$. By condition (ii) $H(\tau_{o(p)}r) = H(r)$. So $o(q) \leq o(p)+o(q) < H(r)$ implies $q = (\tau_{o(p)}(r))|o(q) \in T$.

It follows that the tree associated to the additive height function H is additive.

□

Proposition 6.15. *Let $H_0 : X^\Omega \to \Omega \setminus 1$ be an arbitrary function. There exists H a maximum height function with $H \leq H_0$. Furthermore, if H_0 takes values in $\tilde{\Omega}$, then H is an $\tilde{\Omega}$ height function.*

Proof. Let $\{H^i\}$ be the set of all height functions with $H^i < H_0$. This set is nonempty since the constant function with value 1 is in it. Define $H(s) = \sup_i H^i(s)$.

If $\epsilon(s,t) \geq H(s)$ then $\epsilon(s,t) \geq H^i(s)$ for all i and so $H^i(s) = H^i(t)$ since each H^i is a height function. Hence, $H(t) = H(s)$. That is, H is a height function.

We can describe H by a finitely inductive construction. For $n \geq 0$, define

$$H_{n+1}(s) = min\{H_n(t) : \epsilon(s,t) \geq min(H_n(s), H_n(t))\}. \tag{6.53}$$

In particular, $H_{n+1}(s) \leq H_n(s)$.

Define $H_\infty(s) = min_n H_n(s)$. By well-ordering, there exists for each s a positive integer N_s such that $H_\infty(s) = H_k(s)$ for all $k \geq N_s$.

If $H_n \geq H^i$ for a height function H^i, then $\epsilon(s,t) \geq min(H_n(s), H_n(t)) \geq min(H^i(s), H^i(t))$ implies $H_n(t) \geq H^i(t) = H^i(s)$. Hence, $H_{n+1}(s) \geq H^i(s)$. It follows that $H_\infty \geq H$.

Now suppose $\epsilon(s,t) \geq H_\infty(s)$ which equals $H_k(s) \geq min(H_k(s), H_k(t))$ for $k \geq max(N_s, N_t)$. It follows that $H_{k+1}(s) \leq H_k(t)$ and $H_{k+1}(t) \leq H_k(s)$. Since $k \geq max(N_s, N_t)$, $H_k(s) = H_{k+1}(s) = H_\infty(s)$ and $H_k(t) = H_{k+1}(t) = H_\infty(t)$. Thus, $H_\infty(s) = H_\infty(t)$. This means that H_∞ is a height function and so equals H.

Finally, if H_0 takes values in $\tilde{\Omega}$, then so does each H_n and so $H_\infty = H$ is an $\tilde{\Omega}$ height function.

$\square$

The difficulty with the inductive construction of Theorem 6.10 is that we don't have a convenient method for making the choices along the way so that the resulting tree is Ω-bounded.

There is an inconvenient method. If we begin with any function from X^Ω to $\tilde{\Omega}$, the construction of Proposition 6.15 yields an $\tilde{\Omega}$ height function with associated Ω-bounded tree R having all branch heights tail-like. Any Ω-bounded additive tree is such a tree. On the other hand, beginning with such a tree R we can use the construction of Theorem 6.10 to build an additive tree contained in R. The process will terminate at a countable tail-like ordinal β if either we cannot choose W as a nonempty, translation invariant subset of $\tilde{L}_\beta$, so that $\{p+x : p \in T \text{ and } x \in W\} \subset R$, or else, if such a choice is possible but the extension to the successor tail-like ordinal given by (6.36) is not contained in R. If the process does not terminate at a countable level, then we obtain an additive tree of height Ω, but which is Ω-bounded since it is contained in R.

6.3. **Special Trees for HLOTS.** For a HLOTS X there is a special class of trees whose construction parallels that of the additive trees.

For a HLOTS X the completion $\hat{X}$ is a CHLOTS and $\bullet\hat{X}\bullet = \{m\} + \hat{X} + \{M\}$ is its two point compactification. If $p \in X^\alpha$ for any positive ordinal α, define $\hat{p} \in (\bullet\hat{X}\bullet)^{\alpha+1}$ by

$$(6.54) \qquad \hat{p}(0) \ = \ m, \qquad \hat{p}(i) \ = \ sup \ \{p(j) : j < i\} \quad \text{for } 0 < i \leq \alpha.$$

We say that $p \in X^\alpha$ is *sharply increasing* when

$$(6.55) \qquad \hat{p}(i) \ < \ p(i) \quad \text{for } i < \alpha.$$

Notice that if $p : \alpha \to X$ is an order map, then

$$(6.56) \qquad \hat{p}(i+1) \ = \ p(i) \quad \text{for } i < \alpha,$$

and if p is injective, then $p(i) < p(i+1)$. That is, if p is any order injection then (6.55) holds for any successor ordinal $i < \alpha$. On the other hand, if p is an order embedding, then $\hat{p}(i) = p(i)$ for any limit ordinal $i < \alpha$. Thus, p is sharply increasing exactly when it is an order injection which is discontinuous at each limit ordinal. In general, $\hat{p}$ is a continuous order map and it is an order embedding into $\bullet\hat{X}\bullet$ if p is an injective order map.

We define the *order tree on* a HLOTS X, denoted $T(X)$, whose vertices at level α are the bounded, sharply increasing maps from α to X. That is, we define

$$L_\alpha(X) \ = \ \{p \in X^\alpha : \hat{p}(i) < p(i) \text{ for } i < \alpha \text{ and } \hat{p}(\alpha) < M\}. \tag{6.57}$$

Theorem 6.16. *For a HLOTS X the order tree $T(X)$ is a subtree of the simple tree on X, Ω. $T(X)$ is a reproductive tree of X type. The height of $T(X)$ is Ω but every branch has countable height, i.e. $T(X)$ is Ω-bounded. We can identify the branch space with the set*

$$\begin{aligned} &\{x \in X^\beta : \beta \textit{ is a countable, limit ordinal,} \\ &\quad \hat{x}(i) < x(i) \quad \textit{for } i < \beta \quad \textit{and} \quad \hat{x}(\beta) = M\}. \end{aligned} \tag{6.58}$$

The branch space of $T(X)$ is an IHLOTS and its completion is a CHLOTS.

Proof. Since X is of countable type, only countable ordinals admit order injections into X. Hence, $L_\alpha = \emptyset$ if α is uncountable. If $p \in L_\alpha$ and $\beta < \alpha$, then $p|\beta \in L_\beta(X)$ because for any $p \in X^\alpha$

$$\hat{p}|(\beta + 1) \ = \ (\widehat{p|\beta}). \tag{6.59}$$

Thus, $T(X)$ is a subtree of the simple tree on X, Ω.

If $p \in L_\alpha$ and $s = \hat{p}(\alpha)$, then because X is a HLOTS there exists an order isomorphism $f_s : X \to (s, M) \cap X$ and it extends to the isomorphism $\hat{f}_s : \bullet\hat{X}\bullet \to [s, M]$. For any $q \in X^\beta$

$$\hat{f}_s \circ \hat{q} \ = \ (\widehat{f_s \circ q}) \quad \text{on } \beta \setminus 1 \tag{6.60}$$

and so $q \in L_\beta$ iff $f_s \circ q \in L_\beta$.

Now define the analogue of the canonical inclusion of (6.8)

$$\begin{aligned} f_p : T(X) &\to T(X)_p \\ f_p(q) \ &= \ p + (f_s \circ q). \end{aligned} \tag{6.61}$$

Notice that

$$\hat{p}(\alpha) \ = \ s \ = \hat{f}_s(m) \ < \ f_s(q(0)) \tag{6.62}$$

and so $p + (f_s \circ q) \in L_{\alpha+\beta}$.

Conversely, $r \in T(X)_p$ implies

$$\hat{r}(\alpha) \ = \ \hat{p}(\alpha) \ = \ s \tag{6.63}$$

and so $f_s^{-1} \circ (\tau_\alpha(r)) \in T(X)$. Thus, f_p is a tree isomorphism and so $T(X)$ is reproductive.

Since $L_1 = X^1 \cong X$ we see that the tree is of X type because it is reproductive.

The identification of the branch space with the set described in (6.58) is now routine using an argument similar to that of Lemma 6.9. Because X is of countable type it follows that if $x \in X^\beta$ is a branch, then β is countable. Hence, every branch has countable height.

It follows from Theorem 5.23 that the branch space is an IHLOTS and its completion is a CHLOTS.

It remains to show that the height of $T(X)$ is Ω, i.e. $L_\alpha \neq \emptyset$ for every countable ordinal α. We use a construction which we will apply again later.

By Proposition 2.15(d) there exists an order embedding $\tilde{p} : \alpha + 1 \to X$. Choose for each $i < \alpha$, an isomorphism $g_i : X \to (\tilde{p}(i), \tilde{p}(i+1))$. For any $z \in X^\alpha$ define

$$p(z)(i) \;=\; g_i(z(i)). \tag{6.64}$$

It is easy to see that $p(z)$ is an order injection and for each limit ordinal $i \leq \alpha$

$$\widehat{p(z)}(i) \;=\; \tilde{p}(i) \tag{6.65}$$

and so if $i < \alpha$, $\widehat{p(z)}(i) < p(z)(i)$. Thus, $z \mapsto p(z)$ is an injective map from X^α into L_α.

$\square$

Remark. We can apply this last construction as well when $\tilde{p} : \alpha + 1 \to \bullet\hat{X}\bullet$ is an order embedding with

$$\tilde{p}(0) \;=\; m \qquad \text{and} \qquad \tilde{p}(\alpha) \;=\; M. \tag{6.66}$$

If α is a limit ordinal, then $z \mapsto p(z)$ defines an order injection from X^α into the branch space of $T(X)$ as identified in (6.58).

In order to define the analogue for $T(X)$ of additive subtrees, we need a piece of auxiliary equipment.

Definition 6.17. *For a HLOTS X a set $\mathcal{S}$ of maps from X to X is called a* special semigroup *when it satisfies the following conditions*

(i) *Each $f \in \mathcal{S}$ is either an order isomorphism $f : X \to X$ or an order isomorphism $f : X \to (x, \infty)$ with $x \in X$. The former are called the* invertible *elements of $\mathcal{S}$ and included among them is the identity 1_X.*

(ii) *If $f, g \in \mathcal{S}$, then $f \circ g \in \mathcal{S}$. If f is an invertible element of $\mathcal{S}$, then $f^{-1} \in \mathcal{S}$. Thus, the invertible elements of $\mathcal{S}$ form a group under composition.*
(iii) *The group of invertible elements acts transitively on X.*
(iv) *The set of noninvertible elements of $\mathcal{S}$ is nonempty.*

Notice that each $f \in \mathcal{S}$ is an order embedding by Proposition 2.3(b).

Lemma 6.18. *If $\mathcal{S}$ is a special semigroup for a HLOTS X, $x < y$ in X and $z \in X$, then there exists $f \in \mathcal{S}$ such that $f(X) = (x, \infty)$ and $f(z) = y$.*

Proof. There exists $f_1 : X \to (\tilde{x}, \infty)$ in $\mathcal{S}$ for some $\tilde{x} \in X$ by condition (iv). By (iii) there exists an invertible element f_2 such that $f_2(\tilde{x}) = x$. Let $\tilde{y} = (f_1^{-1} \circ f_2^{-1})(y)$. By (iii) again there exists an invertible element f_3 such that $f_3(z) = \tilde{y}$. The composite $f = f_2 \circ f_1 \circ f_3$ is the required element of $\mathcal{S}$.

□

A subset W of X^β is called $\mathcal{S}$ *invariant* if for all $f \in \mathcal{S}$

$$x \in W \qquad \Longleftrightarrow \qquad f \circ x \in W. \tag{6.67}$$

Definition 6.19. *Let $\mathcal{S}$ be a special semigroup for a HLOTS X and let T be a subset of the order tree $T(X)$. T is called an $\mathcal{S}$* tree *when it satisfies the following conditions.*

(i) *The root 0 and the level 1 vertices X^1 of $T(X)$ are contained in T.*
(ii) *For every $p \in T$ with $o(p) = \alpha$*

$$\hat{p}(i) \in X \quad \text{for } 1 \leq i \leq \alpha. \tag{6.68}$$

(iii) *For all $f \in \mathcal{S}$ and $p \in T(X)$*

$$p \in T \qquad \Longleftrightarrow \qquad f \circ p \in T. \tag{6.69}$$

(iv) *For all $p, q \in T(X)$ with $o(p) = \alpha$, $o(q) = \beta$*

$$p + q \in T \qquad \Longleftrightarrow \qquad p, q \in T, \ \hat{p}(\alpha) < q(0), \ \alpha + \beta < h(T). \tag{6.70}$$

Proposition 6.20. *Let T be an $\mathcal{S}$ tree with $\mathcal{S}$ a special semigroup for a HLOTS X.*

(a) *T is a normal subtree of $T(X)$.*

(b) *If $p \in T$ and $\beta \leq o(p)$, then*

$$p|\beta \in T \qquad \text{and} \qquad \tau_\beta(p) \in T. \tag{6.71}$$

If $o(p) + 1 < h(T)$, then the successor set S_p in T consists of all of the successors of p in $T(X)$.

(c) *T is a tree of X type and T is reproductive iff $h(T)$ is tail-like.*

(d) *If $1 < \beta \leq h(T)$, then T^β is an $\mathcal{S}$ tree.*

(e) *We can identify the branch space of T with the set*

$$\begin{aligned} \{x \in X^\beta : \beta = h(T) \ &\text{or an infinite limit ordinal,} \\ &x \notin T \quad \text{and} \quad x|\gamma \in T \text{ for all } \gamma < \beta\}. \end{aligned} \tag{6.72}$$

(f) *Assume that $h(T)$ is tail-like. For $p, q \in T(X)$ with $o(p) = \alpha$,*

$$p + q \in T \qquad \Longleftrightarrow \qquad p, q \in T \quad \text{and} \quad \hat{p}(\alpha) < q(0). \tag{6.73}$$

Furthermore, if $x \in X^\beta$ with $\beta > 0$, then

$$p + x \in X(T) \qquad \Longleftrightarrow \qquad p \in T, x \in X(T) \quad \text{and} \quad \hat{p}(\alpha) < x(0), \tag{6.74}$$

and so if $\gamma < \beta$, then

$$x \in X(T) \qquad \Longrightarrow \qquad x|\gamma \ \in T, \ \tau_\gamma(x) \in X(T). \tag{6.75}$$

(g) *If $h(T)$ is tail-like, then the branch space $X(T)$ is an IHLOTS and its completion $\hat{X}(T)$ is a CHLOTS.*

Proof. (a), (b), (d) and (6.71) follow from condition (iv) of Definition 6.19 and (6.6). Furthermore, (6.71) implies that T is a subtree of $T(X)$. For $p \in L_\alpha$ the successor set in $T(X)$ is given by

$$S_p \ = \ \{p + q : q \in X^1 \quad \text{and} \quad \hat{p}(\alpha) < q(0)\}. \tag{6.76}$$

If $p \in T$ and $o(p) + 1 < h(T)$, then conditions (i) and (iv) imply that this set is contained in T. Normality of the tree is clear from this and condition (iv).

With $\beta > 1$ to retain condition (i), (d) is obvious.

(e) and (f) follow as in Lemma 6.9. When $h(T)$ is tail-like $p, q \in T$ implies $o(p) + o(q) < h(T)$ and $x \in X(T)$ implies $o(p) + h(x) \leq h(T)$.

(c),(g): If T is reproductive, then $h(T)$ is tail-like. For the converse let $p \in T$ with $o(p) = \alpha$. By condition (ii) $s = \hat{p}(\alpha) \in X$. By Lemma 6.18 there exists $f_s : X \to (s, \infty)$ in $\mathcal{S}$. By (6.73) and condition (iii) the map f_p of (6.61) restricts to a tree isomorphism of T onto T_p. Thus, T is reproductive. Since $T \subset T(X)$, $h(x) < \Omega$ for all $x \in X(T)$. Hence, (g) follows from Theorem 5.23.

$\square$

When $\hat{X} = X$, i.e. X is a CHLOTS, then the order tree $T(X)$ is an $\mathcal{S}$ tree for any special semigroup $\mathcal{S}$. However, if X is an IHLOTS, then condition (ii) fails for $T(X)$. In that case, the *reduced order tree* is defined to be

$$\{p \in T(X) : \hat{p}(i) \in X \quad \text{for all } 1 \leq i \leq o(p)\} \tag{6.77}$$

is an $\mathcal{S}$ tree for any special semigroup $\mathcal{S}$ and it contains all other $\mathcal{S}$ trees.

In general, $T(X)^2 = \{0\} \cup X^1$ is the unique $\mathcal{S}$ tree of height 2 for any special semigroup and from $T(X)^2$ we can build all $\mathcal{S}$ trees by using an inductive construction completely analogous to that of Theorem 6.10.

Theorem 6.21. *Let $\mathcal{S}$ be a special semigroup for a HLOTS X, $\alpha \leq \Omega$ be a tail-like ordinal and T be an $\mathcal{S}$ tree with $h(T) = \beta < \alpha$.*

(a) *Let ϵ be a limit ordinal with $\beta < \epsilon \leq \alpha$. If $\{T_\delta\}$ is a coherent collection of $\mathcal{S}$ trees indexed by $[\beta, \epsilon)$, then*

$$T_\epsilon \;=\; \bigcup \{T_\delta : \beta \leq \delta < \epsilon\} \tag{6.78}$$

is an $\mathcal{S}$ tree. It defines the unique tree such that $\{T_\delta : \beta \leq \delta \leq \epsilon\}$ is a coherent collection of trees indexed by $[\beta, \epsilon] = [\beta, \epsilon + 1)$.

(b) *Let $\epsilon(\beta) = min\ \{i : \beta \leq i \leq \alpha$ and i is tail-like $\}$. For each δ such that $\beta \leq \delta \leq \epsilon(\beta)$ there is a unique $\mathcal{S}$ tree such that*

$$(T_\delta)^\beta \;=\; T \qquad \textit{for} \quad \beta \leq \delta \leq \epsilon(\beta). \tag{6.79}$$

The collection $\{T_\delta : \beta \leq \delta \leq \epsilon(\beta)\}$ is a coherent collection of trees indexed by $[\beta, \epsilon(\beta)]$.

(c) *Assume that $h(T) = \beta$ is tail-like. Using the description (6.54) we define $\ddot{L}_\beta = \{x \in X(T) : h(x) = \beta$ and $\hat{x}(\beta) \in X\}$, a subset of $\tilde{L}_\beta = \{x \in X(T) : h(x) = \beta\}$. Both $\ddot{L}_\beta$ and $\tilde{L}_\beta$ are nonempty, $\mathcal{S}$ invariant, translation invariant subsets of X^β.*

If W is any nonempty, $\mathcal{S}$ invariant, translation invariant subset of $\ddot{L}_\beta$, then

$$T_{\beta+1} \;=\; T \cup \{p + x : p \in T, x \in W, o(p) = \alpha \textit{ and } \hat{p}(\alpha) < x(0)\} \tag{6.80}$$

is an $\mathcal{S}$ tree of height $\beta + 1$ such that

$$(T_{\beta+1})^\beta \;=\; T. \tag{6.81}$$

Proof. The proof parallels that of Theorem 6.10 with a few adjustments which we will note.

(a): Conditions (ii) and (iii) are obviously preserved in the union. Condition (iv) and uniqueness are proved as in Theorem 6.10.

(b): In the inductive construction the limit stage follows from (a) as before and the uniqueness arguments in each case are completely analogous.

If $\epsilon = \delta + 1$, then we define

$$T_\epsilon = T_\delta \cup \{p + q : p, q \in T_\delta, o(p) = \alpha, o(q) = \beta, \hat{p}(\alpha) < q(0) \text{ and } \alpha + \beta = \delta\}. \tag{6.82}$$

Condition (ii) is clear and condition (iv) is checked as was partial additivity before.

If $f \in \mathcal{S}$ and $p, q \in T(X)$, then

$$f \circ (p + q) = (f \circ p) + (f \circ q), \tag{6.83}$$

and with $\hat{f} : \bullet\hat{X}\bullet \to \bullet\hat{X}\bullet$ the extension of f

$$(\hat{f} \circ \hat{p}) = \widehat{f \circ p}, \tag{6.84}$$

except at 0.

These let us verify condition (iii) for T_ϵ and so complete the inductive construction.

(c): From (6.72) we have for $f \in \mathcal{S}$ that

$$x \in X(T) \iff f \circ x \in X(T). \tag{6.85}$$

Notice that here we use the fact that

$$\hat{f}(M) = M \qquad \text{for all } f \in \mathcal{S}. \tag{6.86}$$

It follows that $\tilde{L}_\beta$ and $\ddot{L}_\beta$ are $\mathcal{S}$ invariant. Translation invariance follows from (6.75).

From (6.80), (6.81) is clear. We require $W \subset \ddot{L}_\beta$ so that condition (ii) holds for $T_{\beta+1}$. Condition (iii) uses (6.83), (6.84) and (6.85). Condition (iv) is checked just as partial additivity was in Theorem 6.10(c).

It remains to prove that $\ddot{L}_\beta \neq \emptyset$.

Choose a sequence $a_0 < a_1 < ...$ in X with limit $a \in X$. Because β is a countable limit ordinal we can choose a sequence $1 = \beta_0 < \beta_1 < ...$ in Ω with limit β. We inductively construct $p_0, p_1, ...$ in T such that for $i = 0, 1, ...$

$$o(p_i) = \beta_i, \quad p_{i+1}|\beta_i = p_i, \quad \hat{p}_i(\beta_i) = a_i. \tag{6.87}$$

There is, then, a unique branch x of height β with $x|\beta_i = p_i$. Since $\hat{x}(\beta) = a$, we have $x \in \ddot{L}_\beta$.

For the induction, begin with $p_0 \in X^1$ with $p_0(0) = a_0$. Assume that p_i is defined and choose $q \in T$ such that $o(q) = \beta_{i+1} \setminus \beta_i$ which exists because T has height β. By Lemma 5.12 and condition (ii) applied to q there exists $f \in \mathcal{S}$ such that $f(X) = (a_i, \infty)$ and

$$f(\hat{q}(\beta_{i+1} \setminus \beta_i)) = a_{i+1}. \tag{6.88}$$

Define

$$p_{i+1} = p_i + f \circ q \tag{6.89}$$

which lies in T by conditions (iii) and (iv). By (6.84) p_{i+1} satisfies (6.87) as required.

□

Corollary 6.22. *There exist countable special semigroups for the HLOTS of rationals $\mathbb{Q}$. For any countable special semigroup $\mathcal{S}$ for $\mathbb{Q}$ there exist $\mathcal{S}$ trees which are Aronszajn. In particular, reproductive Aronszajn trees exist.*

Proof. Choose $f_0 : \mathbb{Q} \to (0, \infty) \cap \mathbb{Q}$ an order isomorphism. Let $\mathcal{H}$ be the countable group of translations on $\mathbb{Q}$. Let $\mathcal{S}$ be the smallest semigroup containing $\{f_0\} \cup \mathcal{H}$. $\mathcal{S}$ is countable and is clearly a special semigroup for $\mathbb{Q}$.

Assume $\mathcal{S}$ is a countable special semigroup for $\mathbb{Q}$. We will use Theorem 6.21 to construct a coherent family $\{T_\delta : \delta < \Omega\}$ of countable $\mathcal{S}$ trees.

At the infinite, tail-like ordinal β stage, it suffices to construct $W \subset \ddot{L}_\beta$ which is countably infinite, $\mathcal{S}$ invariant and translation invariant.

To do so, choose $x \in \ddot{L}_\beta$ and let $W_0 = \{\tau_i(x) : i < \beta\}$ so that W_0 is a countably infinite, translation invariant subset of $\ddot{L}_\beta$. Then define $W_0 \subset W_1 \subset ...$ by

$$\begin{aligned} W_{n+1} \ &= \ \{f \circ x : x \in W_n \text{ and } f \in \mathcal{S}\} \cup \\ &\{x : \text{there exists } f \in \mathcal{S} \text{ such that } f \circ x \in W_n\}. \end{aligned} \tag{6.90}$$

Each W_{n+1} is translation invariant. Since $1_{\mathbb{Q}} \in \mathcal{S}$, $W_{n+1} \supset W_n$ and since $\ddot{L}_\beta$ is $\mathcal{S}$ invariant, $W_{n+1} \subset \ddot{L}_\beta$. $W = \cup_n W_n$ which is $\mathcal{S}$ invariant as well as translation invariant. Because $\mathcal{S}$ and W_0 are countable, each W_n is. Thus, W is the required countable subset of $\ddot{L}_\beta$.

By Theorem 6.21(a) $T = \cup\{T_\delta : \delta < \Omega\}$ is an $\mathcal{S}$ tree of height Ω. Since every branch of an $\mathcal{S}$ tree has countable height it follows that T is Aronszajn. □

6.4. **The Omega Thinning Construction.** Recall from Proposition 2.6 the Cantor Normal Form which implies that for an ordinal $\alpha \geq 1$:

$$\alpha = \omega^{\gamma_1} + \dots + \omega^{\gamma_{k+\ell}} \text{ with } k, \ell \in \mathbb{N}, k + \ell \geq 1,$$
$$\text{(6.91)} \qquad \text{and } \gamma_1 \geq \dots \gamma_k \geq \omega > \gamma_{k+1} \geq \dots \gamma_{k+\ell} \geq 0,$$
$$\Longrightarrow \quad \omega \cdot \alpha = \omega^{\gamma_1} + \dots + \omega^{\gamma_k} + \omega^{\gamma_{k+1}+1} + \dots + \omega^{\gamma_{k+\ell}+1}.$$

In particular, α is a limit ordinal iff $\alpha = \omega \cdot \beta$ for some $\beta \geq 1$, see Corollary 2.7.

Definition 6.23. *Assume that T is a normal bi-ordered tree with height $h(T)$ a tail-like ordinal satisfying $\omega < h(T) \leq \Omega$ and so $\omega^2 \leq h(T)$.*

Let the Omega Thinning *be the tree ωT with*

$$\text{(6.92)} \qquad \omega T \;=\; \{p \in T : o(p) \;\; \textit{is a limit ordinal}\}.$$

We will write $(\omega S)_p$ for the successor set of p in the tree ωT.

So $(\omega S)_0 = L_\omega$ and for $p \in \omega T$, $(\omega S)_p = T_p \cap L_{o(p)+\omega}$.

If $p \in \omega T$, then $o(p) = \omega \cdot \beta$ for some $\beta \geq 1$ and it follows that its order in ωT, $o_\omega(p) = \beta$.

So if $h(T) = \omega^\gamma$, then $h(\omega T) = h(T)$ if $\gamma \geq \omega$ and $h(\omega T) = \omega^{\gamma - 1}$ if $2 \leq \gamma < \omega$.

For $p \in T$, $o(p) < h(T)$ and $\omega < h(T)$. Since $h(T)$ is tail-like, $o(p) + \omega < h(T)$ and so normality implies there exists $q \in T_p$ with $o(q) = o(p) + \omega$. Thus condition (iv) of Definition 5.1 follows for ωT from the same condition for T. Thus, ωT is a normal tree.

Proposition 6.24. *If S_p has no max for each $p \in T$, e.g. if T is of unbounded type, then ωT is of dense type, i.e. $(\omega S)_p$ is order dense for each $p \in \omega T$.*

In particular, if T is of $\mathbb{Z}$ type or $\mathbb{N}$ type, then ωT is of dense type.

Proof. We proceed as in Proposition 5.4. Let $q < r \in (\omega S)_p$ for $p \in \omega T$. So $q, r \in T_p$ and with $\epsilon = \epsilon(q, r)$, we have $o(p) \leq \epsilon < o(p) + \omega$ and so $\epsilon + 2 < o(p) + \omega$.

$q_i = r_i$ for all $i \leq \epsilon$ and $q_{\epsilon+1} < r_{\epsilon+1}$. Because $S_{q_{\epsilon+1}}$ has no max, there exists $a \in S_{q_{\epsilon+1}}$ with $q_{\epsilon+2} < a$. By normality we can choose $s \in T$ with $o(s) = o(p) + \omega$ and $s_{\epsilon+2} = a$. It follows that $s \in (\omega S)_p$ with $q < s < r$. $\square$

Since $h(T)$ is assumed to be a limit ordinal, and T is normal, it follows that for any branch $x \in X(T)$ $h(x)$ is a limit ordinal and so from (6.91) $h(x) = \omega \cdot \beta$ with $\beta \geq 1$. A branch of ωT is of the form $x \cap \omega T$ for a unique branch x of T which has $h(x) = \omega^2 \cdot \beta$ with $\beta \geq 1$ and so that $h(x)$ is a limit of limit ordinals. The order of $x \cap \omega T$ in ωT is then $\omega \cdot \beta$.

The map defined by $x \cap \omega T \mapsto x$ is an order injection from $X(\omega T)$ into $X(T)$. We identify $X(\omega T)$ with its image under this map and so regard $X(\omega T) \subset X(T)$.

Corollary 6.25. *If S_p has no max for each $p \in T$, e.g. if T is of unbounded type, then $X(T)$ and $X(\omega T)$ are order dense LOTS and $X(\omega T)$ is a dense subset of $X(T)$. In particular, they have a common connected completion.*

Proof. Both $X(T)$ and $X(\omega T)$ are order dense by Proposition 5.4 and Proposition 6.24.

For any $p \in T$ we show there exists $x \in X(T)$ with $p \in x$ and with $h(x)$ a tail-like ordinal $\omega^\gamma, \gamma \geq 2$ and so $x \cap \omega T \in X(\omega T)$. It then follows from Proposition 5.5 (g) that $X(\omega T)$ is dense in $X(T)$.

Case 1 $h(T)$ is a countable ordinal: By Lemma 5.6, there exists $x \in X(T)$ such that $h(x) = h(T)$ and $p \in x$. Since $h(T)$ is a tail-like ordinal with $h(T) > \omega$, $x \in X(\omega T)$.

Case 2 $h(T) = \Omega = \omega^\Omega$: Choose $\gamma_1 > 2$ so that $o(p) < \omega^{\gamma_1}$. Choose $p(\gamma_1)$ such that $p \prec p(\gamma_1)$ and $o(p(\gamma_1)) = \omega^{\gamma_1}$. Assume that for all δ with $\gamma_1 \leq \delta < \gamma$, $p(\delta)$ has been chosen with $o(p(\delta)) = \omega^\delta$ and if $\delta_1 < \delta_2$, then $p(\delta_1) \prec p(\delta_2)$.

If $\gamma = \delta+1$, then choose $p(\gamma)$ such that $p(\delta) \prec p(\gamma)$ and $o(p(\gamma)) = \omega^\gamma$.

If γ is a limit ordinal, then there exists a branch $z \in X(T)$ with $p(\delta) \in z$ for all $\delta < \gamma$. If $h(z) = \omega^\gamma$, then the process stops and $z = x$ is the required element of $X(T)$. Otherwise, $h(z) > \omega^\gamma$ and we let $p(\gamma) = z_{\omega^\gamma}$.

If the process never stops, then we obtain $p(\gamma)$ for all γ with $\gamma_1 \leq \gamma < \Omega$ and these define a branch z of $X(T)$ with height Ω and $z = x$ is the required element of $X(T)$.

□

Proposition 6.26. *If T is homogeneous or reproductive, then ωT is homogeneous or reproductive, respectively.*

Proof. Any automorphism f of T restricts to an automorphism of ωT because $o(f(p)) = o(p)$. So if T is homogeneous, then ωT is. In particular, it then follows that each successor LOTS $(\omega S)_p$ is transitive.

Now assume that T is reproductive and that $p \in T$. The isomorphism $j_p : T \to T_p$ induces an isomorphism of ωT onto $\{p\} \cup [(\omega T) \cap T_p)]$. Notice that if $o(q)$ is a limit ordinal if and only if $o(j_p(q)) = o(p) + o(q)$ is a limit ordinal. If $p \in \omega T$, then $(\omega T)_p = \{p\} \cup [(\omega T) \cap T_p)]$. It follows that ωT is reproductive.

□

We now assume for a LOTS X that T is an additive subtree of the simple tree on X, Ω. We would like to say that ωT is then an additive tree and indeed it is, but to make sense of this requires a bit of preliminary work.

The set L_ω of vertices of T at level ω is a subset of $T^{\omega+1}$ and $\pi^{\omega+1}_\omega : T^{\omega+1} \to T^\omega$ is an isomorphism. Moreover, additivity implies that T^ω is the entire simple tree on X, ω by Proposition 6.7 (f). So $p \mapsto A_p$ is an order injection of L_ω into X^ω.

Because it is additive, T is reproductive by Proposition 6.7 (f) and so ωT is reproductive by Proposition 6.26. So it follows that ωT is a tree of $(\omega S)_0 = L_\omega$ type. We want to identify ωT with a subtree of the simple tree on L_ω, Ω.

Using (4.20) we can identify $(L_\omega)^\alpha \subset (X^\omega)^\alpha = X^{\omega \cdot \alpha}$.

Lemma 6.27. *For all α with $1 \leq \alpha < \Omega$, the set of level $\omega \cdot \alpha$ vertices of T, i.e. $L_{\omega\cdot\alpha} \subset X^{\omega\cdot\alpha}$, is a subset of $(L_\omega)^\alpha$.*

Proof. This is trivial for $\alpha = 1$.

If α is a limit ordinal, then $L_{\omega\cdot\alpha} \subset \{s \in X^{\omega\cdot\alpha} : s|(\omega\cdot\beta) \in L_{\omega\cdot\beta}$ for all $\beta < \alpha\}$. On the other hand, $(L_\omega)^\alpha = \{s \in X^{\omega\cdot\alpha} : s|(\omega\cdot\beta) \in (L_\omega)^\beta$ for all $\beta < \alpha\}$.

If $\alpha = \beta + 1$, then $L_{\omega\cdot\alpha} = \bigcup (T_p \cap L_{o(p)+\omega})$ with p varying over $L_{\omega\cdot\beta}$. By additivity, $T_p \cap L_{o(p)+\omega} = \{p + q : q \in L_\omega\}$. So $L_{\omega\cdot\alpha}$ consists of the successors of $L_{\omega\cdot\beta}$ in the L_ω, Ω simple tree.

Thus, the result follows by induction.

□

With these identifications we have:

Proposition 6.28. *If T is an additive subtree of the simple tree on X, Ω, then ωT is an additive subtree of the simple tree on L_ω, Ω. If, in addition, X is transitive, then ωT is homogeneous with L_ω transitive and order dense.*

Proof. Additivity is clear and only requires identifying addition in $X^{\omega\cdot\alpha}$ with addition in $(X^\omega)^\alpha$.

If X is transitive, then T is homogeneous by Proposition 5.19 and so ωT is homogeneous by Proposition 6.26. This implies that $(\omega S)_0 = L_\omega$ is transitive. Since X is transitive it has no max and so L_ω is order dense by Proposition 6.24. □

7. **The Double Tower for a CHLOTS**

Given a CHLOTS F we constructed in Section 4 a tower of CHLOTS, defining for every positive ordinal α

$$F_\alpha \quad = \quad \{x \in F^\alpha : x(i) \in J \quad \text{for } 1 \leq i < \alpha\}, \tag{7.1}$$

where $J = [-1, +1]$ is a distinguished, nontrivial, closed interval in F. By Theorem 4.2 each F_α is a CHLOTS when α is a countable, tail-like ordinal. In particular, with $\alpha = 1$, $F_\alpha = F$. By Proposition 6.3 F_α is order isomorphic to the completion of F^α.

In Section 6.3 we defined the order tree $T(F)$ associated with F. We define the *arborization* of a CHLOTS F to be the completion of the branch space of its order tree. We use the notation

$$a(F) \;=\; X(T(F)) \qquad \text{and} \qquad \widehat{a(F)} \;=\; \widehat{X(T(F)})). \tag{7.2}$$

The order tree is of type F with $X(T(F)^2) \cong S_0 \cong F$ in a natural way. Thus, from the projection map π_2 of (5.22) we can define the continuous order surjections

$$\pi_F : a(F) \to F \qquad \text{and} \qquad \hat{\pi}_F : \widehat{a(F)} \to F. \tag{7.3}$$

In Definition 4.3 we called a CHLOTS F at least as big as a CHLOTS F_1 if there exists an order injection of F_1 into F. By Corollary 4.5 this is equivalent to the existence of a, necessarily continuous, order surjection of F onto F_1. We summarize and extend some earlier results using this comparison concept.

Proposition 7.1. *Let F and F_1 be CHLOTS and $\mathbb{R}$ be the CHLOTS of real numbers.*

(a) *If F is not order isomorphic to $\mathbb{R}$, then F is bigger than $\mathbb{R}$.*
(b) *If F_1 is at least as big as F and F_1 satisfies the countable chain condition, then so does F.*
(c) *If F_1 is the completion of the branch space of an Aronszajn tree and α is an infinite, tail-like ordinal, then F_1 is not as big as F_α.*
(d) *If $\beta < \alpha$ are countable, tail-like ordinals, then F_α is bigger than F_β and $\widehat{a(F)}$ is bigger than F_α.*

Proof. (a), (b): If $f : F_2 \to F_1$ is an order surjection and F_2 is separable or satisfies c.c.c., then F_1 satisfies the corresponding condition by Proposition 2.11(g). If F is any CHLOTS, then by Proposition 2.15 there exists an order surjection of F onto $\mathbb{R}$ and so F is at least as big as $\mathbb{R}$. If $\mathbb{R}$ is as big as F, then F is separable and so is order isomorphic to $\mathbb{R}$ by Proposition 2.15.

(c): This is a restatement of Corollary 5.17.

(d): Since $\pi^\alpha_\beta : F_\alpha \to F_\beta$ is an order surjection, F_α is at least as big as F_β. By Theorem 4.12 F_β is not as big as F_α.

Now choose an order embedding $\tilde{p} : \alpha + 1 \to \bullet F \bullet$ with $\tilde{p}(0) = m$ and $\tilde{p}(\alpha) = M$ as in (6.66). The map $z \mapsto p(z)$ of (6.64) associates to each $z \in F^\alpha$ a branch of height α for the tree $T(F)$. We thus have an order injection from $F^\alpha \supset F_\alpha$ into $\tilde{L}_\alpha \subset a(F) \subset \widehat{a(F)}$. Thus, $\widehat{a(F)}$ is at least as big as F_α. Furthermore, if $\gamma > \alpha$ is a countable, tail-like ordinal, then $\widehat{a(F)}$ is at least as big as F_γ which is bigger than F_α. Hence, $\widehat{a(F)}$ is bigger than F_α.

$\square$

Theorem 7.2. *Let F be a CHLOTS. For every countable ordinal α there exists a HLOTS $a_\alpha(F)$ with completion $\widehat{a_\alpha(F)}$ and for each pair of countable ordinals $\beta < \alpha$ there exists an order surjection $p^\alpha_\beta : a_\alpha(F) \to \widehat{a_\beta(F)}$ with completion $\hat{p}^\alpha_\beta : \widehat{a_\alpha(F)} \to \widehat{a_\beta(F)}$ so that the following conditions hold.*

(i) *For $\gamma < \beta < \alpha < \Omega$ we have*

$$p^\alpha_\gamma = p^\beta_\gamma \circ p^\alpha_\beta. \tag{7.4}$$

(ii) *For $\beta < \alpha < \Omega$ define $a^\beta_\alpha(F) \subset a_\alpha(F)$ by*

$$a^\beta_\alpha(F) = \begin{cases} a_\alpha(F) & \text{for } \beta + 1 = \alpha \\ \cap\{(p^\alpha_i)^{-1}(a_i(F)) : \beta < i < \alpha\} & \text{for } \beta + 1 < \alpha. \end{cases} \tag{7.5}$$

The restriction of p^α_β to $a^\beta_\alpha(F)$ is surjective. That is,

$$p^\alpha_\beta(a^\beta_\alpha(F)) = \widehat{a_\beta(F)}. \tag{7.6}$$

(iii) $a_0(F) = \widehat{a_0(F)} = F$.

(iv) *If $\alpha = \gamma + 1$, then*

$$\begin{aligned} a_\alpha(F) &= a(\widehat{a_\gamma(F)}). \\ \widehat{a_\alpha(F)} &= \widehat{a}(\widehat{a_\gamma(F)}). \\ p^\alpha_\gamma &= \pi_{\widehat{a_\gamma(F)}}. \end{aligned} \tag{7.7}$$

(v) *If α is a countable limit ordinal, then* $(\{\widehat{a_i(F)} : i < \alpha\}, \{\hat{p}^i_j : j < i < \alpha\})$ *is an unbounded, special inverse system and*

$$\begin{aligned} \widehat{a_\alpha(F)} &= \overleftarrow{Lim}(\{\widehat{a_i(F)} : i < \alpha\}, \{\hat{p}^i_j : j < i < \alpha\}). \\ \hat{p}^\alpha_\beta &= \textit{coordinate projection to } \widehat{a_\beta(F)}. \\ a_\alpha(F) &= \bigcup_{i<\alpha} a^i_\alpha(F). \\ p^\alpha_\beta &= \hat{p}^\alpha_\beta | a_\alpha(F). \end{aligned} \tag{7.8}$$

Proof. Conditions (iii),(iv) and (v) define an inductive construction and we show, inductively that at each stage a_α the properties described above hold, i.e. each a_α is a HLOTS and each p^α_β is an order surjection which satisfies (7.4) and (7.6).

The construction begins with condition (iii).

Case 1: $\alpha = \gamma + 1$. a_α is an IHLOTS by Theorem 6.16, and p^α_γ is the surjective order map to the level 1 vertex set of the order tree. By Proposition 2.3(a) such a surjective order map is continuous and so its completion is defined. For $\beta < \gamma$ define

$$p^\alpha_\beta = p^\gamma_\beta \circ p^\alpha_\gamma. \tag{7.9}$$

As the composition of order surjections, each p^α_β is an order surjection and (7.4) for α follows from the corresponding condition for γ.

If $\beta = \gamma$, then $a^\beta_\alpha = a_\alpha$ and (7.6) is clear from what we have already shown. If $\beta < \gamma$ then by (7.9) and (7.5)

$$a^\beta_\alpha(F) = (p^\alpha_\gamma)^{-1}(a^\beta_\gamma(F)). \tag{7.10}$$

Hence, (7.6) for α follows from (7.6) for γ together with (7.9).

Case 2: α is a limit ordinal. Our inductive hypothesis implies that $(\{\widehat{a_i} : i < \alpha\}, \{\hat{p}^i_j : j < i < \alpha\})$ is an unbounded, special inverse limit system. By Proposition 2.9 the inverse limit $\widehat{a_\alpha}$ is an unbounded, connected LOTS and each $\hat{p}^\alpha_\beta$ is a continuous order surjection. By Proposition 2.13(c) $\alpha < \Omega$ implies that $\widehat{a_\alpha}$ is of countable type and hence, by Proposition 2.11(f), any subset has countable type.

For any $\beta < \alpha$ we can choose an increasing sequence of ordinals $\beta = \beta_0 < \beta_1 < ...$ with supremum α.

If $z_0 \in \widehat{a_\beta}$, then because $p^{\beta_n}_{\beta_{n-1}}$ satisfies (7.6) we can choose a sequence $z_1, z_2, ...$ such that

$$z_n \in a^{\beta_{n-1}}_{\beta_n} \quad \text{with} \quad p^{\beta_n}_{\beta_{n-1}}(z_n) = z_{n-1} \quad \text{for} \quad i = 1, 2, ... \tag{7.11}$$

Because the sequence $\{\beta_n\}$ is cofinal in α the sequence $\{z_n\}$ defines a unique element z of the inverse limit. By (7.11)

$$p_i^\alpha(z) \in a_i \qquad \text{for} \quad \beta_{n-1} < i \leq \beta_n,\ n = 1, 2, ... \tag{7.12}$$

So by definition (7.5) $z \in a_\alpha^\beta$ with $\hat{p}_\beta^\alpha(z) = z_0$. Strictly speaking the maps $\hat{p}_i^\alpha$ are used in (7.5) to define a_β^α because a_α and p_β^α are defined subsequently in (7.8). It follows that a_α defined in (7.8) is dense in $\widehat{a_\alpha}$ and so has completion $\widehat{a_\alpha}$. Furthermore, (7.6) holds at the α level. By definition of the inverse limit projections (7.4) holds for α.

It remains to prove that a_α is doubly transitive which will imply it is a HLOTS since it is of countable type.

Assume $x < y$ and $z < w$ in a_α. Because α is a limit ordinal there exists $\beta < \alpha$ such that $x, y, z, w \in a_\alpha^\beta$ with

$$p_\beta^\alpha(x) < p_\beta^\alpha(y) \qquad \text{and} \qquad p_\beta^\alpha(z) < p_\beta^\alpha(w). \tag{7.13}$$

By definition of a_α^β we have

$$p_i^\alpha(x), p_i^\alpha(y), p_i^\alpha(z), p_i^\alpha(w) \in a_\alpha^i \qquad \text{for} \quad \beta < i < \alpha. \tag{7.14}$$

Since $\widehat{a_\beta}$ is a CHLOTS there exists $\hat{f}_\beta \in H_+(\widehat{a_\beta})$ such that

$$\hat{f}_\beta(p_\beta^\alpha(x)) \ = \ p_\beta^\alpha(z) \qquad \text{and} \qquad \hat{f}_\beta(p_\beta^\alpha(y)) \ = \ p_\beta^\alpha(w). \tag{7.15}$$

By induction we will construct for $\beta < i \leq \alpha$, $f_i \in H_+(a_i)$ so that

$$p_\beta^i \circ f_i = \hat{f}_\beta \circ p_\beta^i \quad \text{and} \quad p_j^i \circ f_i = f_j \circ p_j^i \quad \text{for } \beta < j < i. \tag{7.16}$$

and

$$f_i(p_i^\alpha(x)) \ = \ p_i^\alpha(z) \qquad \text{and} \qquad f_i(p_i^\alpha(y)) \ = \ p_i^\alpha(w). \tag{7.17}$$

If i is a limit ordinal, e.g. $i = \alpha$, then we define $\hat{f}_i \in H_+(\widehat{a_i})$ to be the inverse limit of $\{\hat{f}_j : \beta < j < i|]$. That is, $\hat{f}_i(z)$ is the unique element of $\widehat{a_i}$ which projects via $\hat{p}_j^i$ to $\hat{f}_j(\hat{p}_j^\alpha(z))$. Since each $\hat{f}_j$ is the completion of an isomorphism f_j, it follows that $\hat{f}_i$ maps a_i to a_i. The restriction of $\hat{f}_i$ to a_i defines f_i so that (7.16) holds and the original $\hat{f}_i$ is the completion of f_i. Because $\widehat{a_i}$ is the inverse limit of its predecessors, $p_i^\alpha(r)$ is determined by the maps $p_j^\alpha(r)$ for $\beta < j < i$. Hence, (7.17) for i follows from the inductively assumed equations for $j < i$.

Finally, assume that $i = k+1$ and that f_j is defined for all $\beta < j \leq k$ so that (7.16) and (7.17) hold.

Now we use the tree structure: a_i is the branch space of the order tree on $\hat{a}_k$. We can regard $\hat{f}_k$ as an order isomorphism on the level 1 vertices of the tree. By (7.14) $p_i^\alpha(x), ..., p_i^\alpha(w)$ are branches in a_i with vertices at the 1 level $p_k^\alpha(x), ..., p_k^\alpha(w)$.

By Theorem 6.16 the order tree $T(\widehat{a_k})$ is reproductive and so is homogeneous by Proposition 5.19. We use a variation of the proof of Lemma 5.20(a).

For each $r \in S_0 = \widehat{a_k}$ we choose a tree isomorphism $g_r : T_r \to T_{\hat{f}_k(r)}$. With $r_1 = p_k^\alpha(x)$, $\hat{f}_k(r_1) = r_2 = p_k^\alpha(z)$. So the branches $(g_{r_1})_*(p_i^\alpha(x) \cap T_{r_1})$ and $p_i^\alpha(z) \cap T_{r_2}$ both lie in the branch space of the homogeneous tree T_{r_2}. We can adjust g_{r_1} so that these two are in fact equal. Similarly, for $p_i^\alpha(y)$ and $p_i^\alpha(w)$. Assemble the maps g_r to get a tree isomorphism g. Then $g_* \in H_+(a_i)$ so that

$$p_k^i \circ g_* \quad = \quad \hat{f}_k \circ p_k^i. \tag{7.18}$$

We let $f_i = g_*$. From the construction (7.17) is clear and (7.16) for i follows from (7.16) for k together with condition (i).

This completes the inductive construction. The case $i = \alpha$ yields f_α which is the required order isomorphism of a_α which maps the pair x, y to the pair z, w.

□

Theorem 7.3. *For a CHLOTS F and $(i,j) \in \Omega \times \Omega$ define the CHLOTS*

$$F_{(i,j)} \quad = \quad (\widehat{a_i(F)})_{\omega^j}. \tag{7.19}$$

$F_{(0,0)} \cong F$ and if $(i,j) < (\tilde{i}, \tilde{j})$ in $\Omega \times \Omega$, then $F_{(\tilde{i},\tilde{j})}$ is bigger than $F_{(i,j)}$. In particular, $F_{(\tilde{i},\tilde{j})}$ is not homeomorphic to $F_{(i,j)}$.

Proof. Recall that as j varies through the countable ordinals, ω^j varies through the countable, tail-like ordinals and $j < \tilde{j}$ iff $\omega^j < \omega^{\tilde{j}}$. Also, $\omega^0 = 1$. For any countable, tail-like ordinal α F_α defined by (6.1) is a CHLOTS and $F_1 \cong F$. So by Theorem 7.2, $F_{(i,j)}$ is a CHLOTS for all $(i,j) \in \Omega \times \Omega$.

If $i = \tilde{i}$ and $\tilde{F} = \widehat{a_i(F)}$, then $j < \tilde{j}$ implies $\tilde{F}_\beta$ is bigger than $\tilde{F}_\alpha$ with $\beta = \omega^{\tilde{j}}$ and $\alpha = \omega^j$ by Theorem 4.12. Thus, $F_{(\tilde{i},\tilde{j})}$ is bigger than $F_{(i,j)}$.

Now suppose that $i < \tilde{i}$ so that $i + 1 \leq \tilde{i}$.

First, $F_{(\tilde{i},\tilde{j})}$ is at least as big as $F_{(\tilde{i},0)}$ and so by using the projections of Theorem 7.2 we see that $F_{(\tilde{i},0)}$ is as big as $F_{(i+1,0)} = \widehat{a(F_{(i,0)})}$. By Proposition 7.1(d), $F_{(i+1,0)}$ is bigger than $F_{(i,j)}$. Thus, $F_{(\tilde{i},\tilde{j})}$ is bigger than $F_{(i,j)}$.

That $F_{(i,j)}$ and $F_{(\tilde{i},\tilde{j})}$ are not homeomorphic follows just as in Theorem 4.12 because $F'_{(i,j)}$ injects into $F_{(\tilde{i},\tilde{j})}$.

□

For the associated Cantor spaces which are the compactifications of the AS doubles we have the following extension of Theorem 4.14.

Corollary 7.4. *For a CHLOTS F and $(i,j),(\tilde{i},\tilde{j}) \in \Omega \times \Omega$ if $(i,j) < (\tilde{i},\tilde{j})$ in $\Omega \times \Omega$, then $C(F_{(\tilde{i},\tilde{j})})$ is bigger than $C(F_{(i,j)})$. In particular, $C(F_{(\tilde{i},\tilde{j})})$ is not isomorphic to $C(F_{(i,j)})$.*

Proof. If $i = \tilde{i}$ this follows directly from in Theorem 4.14.

If $i < \tilde{i}$, then by Theorem 7.3 $F_{(\tilde{i},\tilde{j})}$ is bigger than $F_{(i,j+1)}$ and so $C(F_{(\tilde{i},\tilde{j})})$ is at least as big as $C(F_{(i,j+1)})$. By Theorem 4.14 again $C(F_{(i,j+1)})$ is bigger than $C(F_{(i,j)})$.

□

We call a LOTS X $\mathbb{R}$-*bounded* if it admits an order injection into $\mathbb{R}_\delta$ for some countable ordinal δ.

Corollary 7.5. *For a CHLOTS F and $(i,j) \in \Omega \times \Omega$ with $0 < i$, the CHLOTS $F_{(i,j)}$ is not $\mathbb{R}$-bounded.*

Proof. $F_{(i,j)}$ is at least as large as $\widehat{a(F)} = F_{(1,0)}$ which is larger than F_α for every countable α and F_α is at least as large as $\mathbb{R}_\alpha$.

□

8. The Tree Characterization of a CHLOTS

8.1. **A Tree for a LOTS and the IHLOTS Tower.** We begin with a version of the partition tree for a LOTS, described in [19] and in [7].

Throughout this section, all intervals in a LOTS X will be assumed nonempty. A singleton is an *improper* closed interval and so an interval with at least two points is a *proper* interval. If I is a proper interval with endpoints $a < b$ then we let $I^\circ = (a,b)$.

For intervals I_1, I_2 in a LOTS we will write

$$I_1 < I_2 \quad \Longleftrightarrow \quad I_1 \neq I_2 \text{ and } c_1 \leq c_2 \text{ for all } c_1 \in I_1, c_2 \in I_2, \tag{8.1}$$

and so $I_1 \cap I_2$ is either empty or consists of a single common endpoint.

Recall that we use $\#X$ for the cardinality of a set X.

Theorem 8.1. *If X is a connected, first countable, bounded LOTS X, there is an Ω bounded subtree T of the simple tree on $\mathbb{Z}, \Omega$ whose branch space is order isomorphic to a dense subset of X. In particular, $\#X \leq 2^{\aleph_0}$.*

Proof. We let m, M be the minimum and maximum of X, respectively, so that $X = [m, M]$. If X is a singleton, then the subtree consisting of the root of the simple tree has branch space isomorphic to X and so we may assume that $m < M$.

For $\alpha \leq \Omega, s \in \mathbb{Z}^\alpha$ we will associate a closed interval $I_s = [a^s, b^s]$ in X. We will call s *proper* when the interval I_s is proper and so $a^s < b^s$.

The construction will satisfy the following properties.

(i) $I_\emptyset = [m, M]$.
(ii) If $\beta < \alpha \leq \Omega$ and $s \in \mathbb{Z}^\alpha$, then $I_s \subset I_{s|\beta}$ and if $s|\beta$ is proper, then $I_s \subset I^\circ_{s|\beta}$.
(iii) If $\alpha \leq \Omega$ is a limit ordinal and $s \in \mathbb{Z}^\alpha$, then $I_s = \bigcap_{\beta<\alpha} I_{s|\beta}$.
(iv) If $s_1 < s_2$ in $\mathbb{Z}^\alpha$, then $I_{s_1} < I_{s_2}$.

If $s \in X^\alpha$ is proper with $I_s = [a^s, b^s]$, then in I_s we choose a $\pm$cofinal embedding of $z : \mathbb{Z} \to (a^s, b^s)$. If $s' \in \mathbb{Z}^{\alpha+1}$ with $s'|\alpha = s$ and $s'(\alpha) = n$, then we let $I_{s'} = [z_n, z_{n+1}]$. The set of successors $S_s \cong \mathbb{Z}$ and if $s'' \in S_s$ has $s''|\alpha = s$ and $s''(\alpha) = n + 1$, then $b^{s'} = a^{s''}$. It follows that if s is proper, then all of its successors are proper. Observe that $I_{s'} \subset I^\circ_s$.

If $s \in X^\alpha$ is improper, then we let $I_{s'} = I_s$ for all $s' \in S_s$.

The construction is completed by using (iii) for limit ordinals.

Conditions (i) - (iv) are easy to check from the inductive construction.

The subtree $T = \{s : s \text{ is proper }\}$, i.e. $s \in T$ iff I_s is a proper interval. It is clear from (i) and (ii) that T is a nonempty subtree of the simple tree.

We can identify the branch space $X(T)$ with

$$\{x \in \mathbb{Z}^\alpha : x|\beta \in T \text{ for all } \beta < \alpha, \text{ and } x \notin T\}. \tag{8.2}$$

That is,

$$x \in X(T) \;\Leftrightarrow\; I_{x|\beta} \text{ is proper for all } \beta < \alpha \text{ and } I_x \text{ is not proper.} \tag{8.3}$$

Since every successor of a proper element is proper, it follows that $h(x)$ is a limit ordinal for all $x \in X(T)$.

For $x \in X(T)$ we let $f(x) = a^x \in X$ so that I_x is the singleton $\{a^x\}$. Furthermore, $\beta \mapsto a^{x|\beta}$ is an embedding of the ordinal $h(x)$ into X with limit (= sup) a^x and $\beta \mapsto b^{x|\beta}$ is an embedding of $h(x)^*$ into X with limit (= inf) a^x. Because X is first countable, Ω does not inject into X. It follows that $h(x)$ is a countable ordinal. Hence, T is Ω bounded.

Observe that if $s \in T^\alpha$ and $\beta < \alpha$, then $I_s \subset I^\circ_{s|\beta}$ and so $a^{s|\beta} < a^s < b^s < b^{s|\beta}$. Also if $s_1 < s_2 \in T^\alpha$, then by (iv) $I^\circ_{s_1} < I^\circ_{s_2}$ in X.

It follows that $x_1 < x_2$ in $X(T)$ implies $a^{x_1} < a^{x_2}$ and so f is an order injection from $X(T)$ into X.

For α a countable ordinal, we let $C_\alpha = \{a^s : s \in T^\alpha\} \cup \{b^s : s \in T^\alpha\}$ and $C = \bigcup_{\alpha<\Omega} C_\alpha$ so that C is the set of endpoints of the proper intervals. Because we are taking the union over the countable ordinals, we have $\#C \leq 2^{\aleph_0}$.

By induction on α we see that for each $\alpha \leq \Omega$ the set X is partitioned into three subsets:

$$\bigcup_{\beta\leq\alpha} C_\beta, \quad \{a^x : x \in X(T) \text{ with height} \leq \alpha\}, \quad \bigcup_{s\in T^\alpha} I_s^\circ. \tag{8.4}$$

So with $\alpha = \Omega$ we obtain X is the disjoint union of C and $f(X(T))$. Since every a^x is a limit of elements of C, it follows that C is dense in X. Because every element of X is thus a limit of a sequence in C it follows that $\#X \leq 2^{\aleph_0}$.

If $s \in T^\alpha$ then s is contained in some branch x and so $a^s < a^x < b^s$. If $s < s_1 \in T$ with $a^s < a^{s_1}$, then $a^s < b^s \leq a^{s_1}$ and so $a^s < a^x < a^{s_1}$. It follows that $f(X(T))$ is dense in X and so is order dense because X is connected. It follows that $f : X(T) \to X$ is an order embedding onto a dense subset.

$\square$

The cardinality result is well-known. See, e.g. [6] Section 4.

We will require a height estimate for a special case.

Theorem 8.2. *If α is a positive ordinal, then $\mathbb{R}_\alpha \cong \widehat{X(T)}$ with T a tree of $\mathbb{Z}$ type with height $h(T) \leq \omega \cdot \alpha$.*

Proof. With $J = [-1, +1] \subset \mathbb{R}$ we can replace $\mathbb{R}_\alpha$ by the isomorph $\{x \in J^\alpha : -1 < x_0 < +1\}$, ie. the interval $(-1+, +1-) \subset J^\alpha$. We let X be the closed interval $[-1+, +1-] \subset J^\alpha$, which is isomorphic to the two point compactification of $\mathbb{R}_\alpha$.

For $a < b$ in X let $\epsilon = min\{i : a_i \neq b_i\}$ so that $a_i = b_i$ for all $i < \epsilon$ and $a_\epsilon < b_\epsilon$. We define the *midpoint* of the interval $[a, b]$ to be the point c with

$$c_i \ = \begin{cases} a_i = b_i \ \text{ for } i < \epsilon, \\ \frac{1}{2}(a_\epsilon + b_\epsilon) \ \text{ for } i = \epsilon, \\ \ 0 \quad \text{for } \epsilon < i < \alpha. \end{cases} \tag{8.5}$$

Notice that on $J \subset \mathbb{R}$ algebraic notions and length are defined.

Now we apply the above construction to X with the proviso that if $I_s = [a^s, b^s]$ is a proper interval, then the $\pm$cofinal sequence $z : \mathbb{Z} \to$

(a^s, b^s) maps $0 \in \mathbb{Z}$ to the midpoint of $[a^s, b^s]$. This then implies for all $n \in \mathbb{Z}$

$$(z_{n+1})_\epsilon - (z_n)_\epsilon \ \leq \ \frac{1}{2}(b^s_\epsilon - a^s_\epsilon). \tag{8.6}$$

We now show, by induction, that for all β with $0 < \beta \leq \alpha$

$$s \in \mathbb{Z}^{\omega\cdot\beta} \quad \Longrightarrow \quad a^s_i = b^s_i \ \text{ for all } \ i < \beta. \tag{8.7}$$

Case 1 $\beta = \gamma + 1$:

Let $s \in \mathbb{Z}^{\omega\cdot\beta}$. The induction hypothesis applied to $s0 = s|\omega\cdot\gamma$ implies that $a^{s0}_i = b^{s0}_i$ for all $i < \gamma$. Since, $[a^s, b^s] \subset [a^{s0}, b^{s0}]$, $a^{s0}_\gamma = b^{s0}_\gamma$ implies $a^s_\gamma = b^s_\gamma$ and so $a^s_i = b^s_i$ for all $i < \beta$.

Assume, instead, that $a^{s0}_\gamma < b^{s0}_\gamma$. Let $sn = s|(\omega\cdot\gamma + n)$ for $n < \omega$. It follows from (8.6) $b^{sn}_\gamma - a^{sn}_\gamma \leq \frac{1}{2^n}(b^{s0}_\gamma - a^{s0}_\gamma)$.

Because $\omega\cdot\beta = \omega\cdot\gamma + \omega$, $I_s = \bigcap_n I_{sn}$ and it follows that $b^s_\gamma = a^s_\gamma$. Hence, again $a^s_i = b^s_i$ for all $i < \beta$.

Case 2 β is a limit ordinal:

If $\gamma < \beta$, then with $s' = s|\omega\cdot\gamma$ we have $I_s \subset I_{s'}$. So the induction hypothesis applied to s' implies that $b^s_i = a^s_i$ for all $i < \gamma$. Since β is a limit ordinal and $\gamma < \beta$ is arbitrary, it follows that $a^s_i = b^s_i$ for all $i < \beta$.

From (8.7) applied with $\beta = \alpha$ we see that for all $s \in \mathbb{Z}^{\omega\cdot\alpha}$, $a^s_i = b^s_i$ for all $i < \alpha$. Since $a^s, b^s \in J^\alpha$ this means $a^s = b^s$. Hence, the interval I_s is improper.

Thus, the tree T consisting of those s with I_s proper is a subtree of the simple tree on $\mathbb{Z}, \omega\cdot\alpha$ and so has height at most $\omega\cdot\alpha$.

$\square$

Corollary 8.3. *If α is a positive ordinal and X is an unbounded LOTS, then there exists an order injection from R_α into the completion $\widehat{X^{\omega\cdot\alpha}}$.*

Proof. We can identify $X^{\omega\cdot\alpha}$ with the branch space on the simple tree on $X, \omega\cdot\alpha$. Since X is unbounded and $\omega\cdot\alpha$ is a limit ordinal, $X^{\omega\cdot\alpha}$ is order dense by Proposition 5.4. In particular, $\mathbb{Z}^{\omega\cdot\alpha}$ is order dense.

By Theorem 8.2 R_α is the completion of the branch space of a tree T which is a subtree of the simple tree on on $\mathbb{Z}, \omega\cdot\alpha$. From the inclusion of T into the simple tree we obtain an order injection from the branch space $X(T)$ into $\mathbb{Z}^{\omega\cdot\alpha}$ from Proposition 6.5. By Proposition 2.10 the extension to the completions is injective. That is, we obtain an order injection from $\mathbb{R}_\alpha$ to $\widehat{\mathbb{Z}^{\omega\cdot\alpha}}$.

Since X is unbounded, there is an order injection from $\mathbb{Z}$ to X. From it we obtain an order injection from $\mathbb{Z}^{\omega\cdot\alpha}$ to $X^{\omega\cdot\alpha}$ which extends to an injection between the completions.

Composing, we obtain the required order injection.

□

Corollary 8.4. *Assume that X, X_1 are HLOTS and that X_1 admits an order injection into $\mathbb{R}_\delta$ for some countable ordinal δ. If β is a positive ordinal and α is a countable, tail-like ordinal such that $\alpha > \omega \cdot \delta \cdot \beta$, then there does not exist an order injection of the completions from $\widehat{X_\alpha}$ into $\widehat{(X_1)_\beta}$ nor an order injection of the Cantor Spaces from $C(\widehat{X_\alpha})$ into $C(\widehat{(X_1)_\beta})$.*

Proof. The injection of X_1 into $\mathbb{R}_\delta$ induces an injection from $\widehat{(X_1)_\beta}$ to $(\mathbb{R}_\delta)_\beta$ which is isomorphic to $\mathbb{R}_{\delta\cdot\beta}$ by (4.21). So there is an injection from $C(\widehat{(X_1)_\beta})$ to $C(\mathbb{R}_{\delta\cdot\beta})$ which in turn injects into $\mathbb{R}_{(\delta\cdot\beta)+1}$.

Since α is tail-like and $\beta, \delta > 0$, $\alpha > \omega\cdot\delta\cdot\beta$ implies $\alpha > \omega\cdot\delta\cdot\beta+\omega = \omega\cdot(\delta\cdot\beta+1)$. Hence, there is an order injection from $\widehat{X^{\omega\cdot(\delta\cdot\beta+1)}}$ into $\widehat{X^\alpha}$.

By Proposition 6.3, $\widehat{X^\alpha} \cong \widehat{X_\alpha}$.

Hence, from an order injection from $\widehat{X_\alpha}$ to $\widehat{(X_1)_\beta}$ we would obtain an order injection from $\widehat{X^{\omega\cdot(\delta\cdot\beta+1)}}$ into $\mathbb{R}_{\delta\cdot\beta}$.

By Corollary 8.3 there exists an order injection from $\mathbb{R}_{\delta\cdot\beta+1}$ into $\widehat{X^{\omega\cdot(\delta\cdot\beta+1)}}$.

The composition would contradict Theorem 4.12.

The order injection from $\mathbb{R}_{\delta\cdot\beta+1}$ into $\widehat{X^{\omega\cdot(\delta\cdot\beta+1)}}$ induces an order injection from $C(\mathbb{R}_{\delta\cdot\beta+1})$ into $C(\widehat{X^{\omega\cdot(\delta\cdot\beta+1)}})$.

As above, from an order injection from $C(\widehat{X_\alpha})$ to $C(\widehat{(X_1)_\beta})$ we would obtain an order injection from $C(\widehat{X^{\omega\cdot(\delta\cdot\beta+1)}})$ into $\mathbb{R}_{(\delta\cdot\beta)+1}$.

Since $\mathbb{R}'_{\delta\cdot\beta+1} \subset C(\mathbb{R}_{\delta\cdot\beta+1})$, the composition would yield an injection from $\mathbb{R}'_{\delta\cdot\beta+1}$ to $\mathbb{R}_{\delta\cdot\beta+1}$. This contradicts Corollary 4.11 which says that $\mathbb{R}_{\delta\cdot\beta+1}$ is order simple.

□

Recall that a LOTS X is $\mathbb{R}$-bounded if it admits an order injection into $\mathbb{R}_\delta$ for some countable ordinal δ. Of course, it then injects into any $\mathbb{R}_\gamma$ with $\delta \le \gamma < \Omega$.

Proposition 8.5. *Assume X is an $\mathbb{R}$-bounded LOTS.*

(a) *For any countable ordinal α, X^α and X_α are $\mathbb{R}$-bounded LOTS.*
(b) *If X is order dense, then its completion is $\mathbb{R}$-bounded.*
(c) *If T is a tree of X type $h(T) < \Omega$, then $X(T)$ is $\mathbb{R}$-bounded.*

Proof. On the one hand, $\mathbb{R}_\delta \subset \mathbb{R}^\delta$. On the other, $\mathbb{R} \cong J^\circ$ implies that $\mathbb{R}^\delta$ injects into $\mathbb{R}_\delta$, i.e. they have the same size. So X injects into $\mathbb{R}_\delta$ iff it injects into $\mathbb{R}^\delta$.

(a): If X injects into $\mathbb{R}^\delta$, then X^α injects into $(\mathbb{R}^\delta)^\alpha \cong \mathbb{R}^{\delta \cdot \alpha}$ (see Proposition 4.7(d)). So X^α is $\mathbb{R}$-bounded when X is and α is countable. Since $X_\alpha \subset X^\alpha$ it is $\mathbb{R}$-bounded as well.

(b): If X is an order dense LOTS, and $j : X \to \mathbb{R}_\delta$ is an order injection, then $\hat{j}$ is an order injection on the completion by Proposition 2.10.

(c): By Proposition 6.5 $X(T)$ injects into X^α if $h(T) = \alpha$. So by (a), $X(T)$ is $\mathbb{R}$-bounded. $\square$

In particular, Proposition 8.5 implies that if T is a tree of $\mathbb{Z}$ type with height a countable limit ordinal, then $X(T)$ and its completion are $\mathbb{R}$-bounded. By sharpening the proof of Theorem 8.2 we obtain the following converse.

Theorem 8.6. *If X is a connected LOTS which admits an order injection $f : X \to \mathbb{R}_\alpha$ with α a positive ordinal, then $X \cong \widehat{X(T)}$ with T a tree of $\mathbb{Z}$ type with height $h(T) \le \omega \cdot \alpha$.*

Proof. With $J = [-1, +1] \subset \mathbb{R}$ we can, as before, replace $\mathbb{R}_\alpha$ by the isomorph $\{x \in J^\alpha : -1 < x_0 < +1\}$, ie. the interval $(-1+, +1-) \subset J^\alpha$. We let Z be the closed interval $[-1+, +1-] \subset J^\alpha$, which is isomorphic to the two point compactification of $\mathbb{R}_\alpha$. Assume that $f : X \to Z$ is an order injection. Let $\pi_i : Z \to J$ be the projection to the i coordinate with $i < \alpha$.

For $a < b$ in X let $\epsilon = min\{i : f(a)_i \ne f(b)_i\}$ so that $f(a)_i = f(b)_i$ for all $i < \epsilon$ and $f(a)_\epsilon < f(b)_\epsilon$. We again have that $\pi_\epsilon \circ f([a, b]) \subset J \subset \mathbb{R}$ and so again algebraic notions and length are defined there. Notice that the order preserving map $\pi_\epsilon \circ f$ need not be continuous and so its image on $[a, b]$ need not be connected. So we have to consider some cases.

Let $t_1 = f(a)_\epsilon + \frac{1}{3}(f(b)_\epsilon - f(a)_\epsilon)$ and $t_2 = f(a)_\epsilon + \frac{2}{3}(f(b)_\epsilon - f(a)_\epsilon)$.

Case 1 (midpoint case): If $\pi_\epsilon \circ f([a, b]) \cap [t_1, t_2] \ne \emptyset$, then let $c \in (a, b)$ such that $f(c)_\epsilon \in [t_1, t_2]$. Because $\pi_\epsilon \circ f$ is order preserving we have that $[a_1, b_1] \subset [a, c]$ or $[a_1, b_1] \subset [c, b]$ implies $f(b_1)_\epsilon - f(a_1)_\epsilon \leq \frac{2}{3}(f(b)_\epsilon - f(a)_\epsilon)$.

Case 2 (edge cases): The left edge case applies when $\pi_\epsilon \circ f((a, b)) \cap (f(a)_\epsilon, t_1) = \emptyset$ whereas the right edge case applies when $\pi_\epsilon \circ f((a, b)) \cap (t_2, f(b)_\epsilon) = \emptyset$. For either edge case, $[a_1, b_1] \subset (a, b)$ implies $f(b_1)_\epsilon - f(a_1)_\epsilon \leq \frac{2}{3}(f(b)_\epsilon - f(a)_\epsilon)$.

Case 3 (boundary cases): If neither Case 1 nor Case 2 applies, then $A_1 = [a, b] \cap (\pi_\epsilon \circ f)^{-1}[f(a)_\epsilon, t_1)$ is a proper convex set which contain a and $A_2 = [a, b] \cap (\pi_\epsilon \circ f)^{-1}(t_2, f(b)_\epsilon]$ is a proper convex set which contains b and their union is $[a, b]$ since Case 1 does not apply. Because X is connected, there exists a unique $c \in (a, b)$ such that $[a, c) \subset A_1$ and $(c, b] \subset A_2$. If $[a_1, b_1] \subset [a, c)$ or $[a_1, b_1] \subset (c, b]$, then $f(b_1)_\epsilon - f(a_1)_\epsilon \leq \frac{2}{3}(f(b)_\epsilon - f(a)_\epsilon)$. The inequality also holds if $b_1 = c$ with $c \in A_1$ and if $a_1 = c$ with $c \in A_2$.

Finally, if $c \in A_1$, then the inequality does not hold for $[c, b_1]$, but now the interval $[c, b_1]$ is itself a left edge case. Similarly, if $c \in A_2$, then the inequality does not hold for $[a_1, c]$, but the interval $[a_1, c]$ is a right edge case.

As we did for Theorem 8.2, we apply the construction from the proof of Theorem 8.1to X with the proviso that if $I_s = [a^s, b^s]$ is a proper interval, then the $\pm$cofinal sequence $z : \mathbb{Z} \to (a^s, b^s)$ maps $0 \in \mathbb{Z}$ to the choice $c \in (a^s, b^s)$ when either Case 1 or Case 3 applies. This then implies for all $n \in \mathbb{Z}$

$$(8.8) \qquad (z_{n+1})_\epsilon - (z_n)_\epsilon \ \leq \ \frac{2}{3}(b^s_\epsilon - a^s_\epsilon),$$

except that in the boundary Case 3 with $n = 0$, we have for $c = z_0 \in A_1$, the interval $[z_0, z_1]$ is a left edge case and for $c = z_0 \in A_2$, $[z_{-1}, z_0]$ is a right edge case.

Now we proceed as before showing, by induction, that for all β with $0 < \beta \leq \alpha$

$$(8.9) \qquad s \in \mathbb{Z}^{\omega \cdot \beta} \quad \Longrightarrow \quad f(a^s)_i = f(b^s)_i \ \text{ for all } \ i < \beta.$$

Case 1 $\beta = \gamma + 1$:

Let $s \in \mathbb{Z}^{\omega \cdot \beta}$. The induction hypothesis applied to $s0 = s|\omega \cdot \gamma$ implies that $f(a^{s0})_i = f(b^{s0})_i$ for all $i < \gamma$. Since, $[a^s, b^s] \subset [a^{s0}, b^{s0}]$, $f(a^{s0})_\gamma = f(b^{s0})_\gamma$ implies $f(a^s)_\gamma = f(b^s)_\gamma$ and so $a^s_i = b^s_i$ for all $i < \beta$.

Assume, instead, that $f(a^{s0})_\gamma < f(b^{s0})_\gamma$ so that $\epsilon = \gamma$ for the interval $[a^{s0}, b^{s0}]$. Let $sn = s|(\omega \cdot \gamma + n)$ for $n < \omega$. At worst, every other step shrinks length by a factor of 2/3. This suffices to show, as before, that $\omega \cdot \beta = \omega \cdot \gamma + \omega$, $I_s = \bigcap_n I_{sn}$ implies $f(b^s)_\gamma = f(a^s)_\gamma$. Hence, again $f(a^s)_i = f(b^s)_i$ for all $i < \beta$.

Case 2 β is a limit ordinal:

If $\gamma < \beta$, then with $s' = s|\omega \cdot \gamma$ we have $I_s \subset I_{s'}$. So the induction hypothesis applied to s' implies that $f(b^s)_i = f(a^s)_i$ for all $i < \gamma$. Since β is a limit ordinal and $\gamma < \beta$ is arbitrary, it follows that $f(a^s)_i = f(b^s)_i$ for all $i < \beta$.

From (8.9) applied with $\beta = \alpha$ we see that for all $s \in \mathbb{Z}^{\omega \cdot \alpha}$, $f(a^s)_i = f(b^s)_i$ for all $i < \alpha$. Since $f(a^s), f(b^s) \in J^\alpha$ this means $f(a^s) = f(b^s)$. Because f is injective, $a^s = b^s$. Hence, the interval I_s is improper.

Thus, the tree T consisting of those s with I_s proper is a subtree of the simple tree on $\mathbb{Z}, \omega \cdot \alpha$ and so has height at most $\omega \cdot \alpha$.

□

Theorem 8.7. *If X is an $\mathbb{R}$-bounded IHLOTS, then the tower of CHLOTS $\widehat{X_{\omega^\gamma}}$ is nondecreasing in size and is strictly increasing in size for sufficiently large γ and the tower of CHLOTS Cantor Spaces $C(\widehat{X_{\omega^\gamma}})$ is nondecreasing in size and is strictly increasing in size for sufficiently large γ.*

To be precise, if X injects into $\mathbb{R}_{\omega^{\gamma_0}}$, then $\widehat{X_{\omega^{\gamma_1}}}$ is strictly bigger than $\widehat{X_{\omega^{\gamma_2}}}$ and $C(\widehat{X_{\omega^{\gamma_1}}})$ is strictly bigger than $C(\widehat{X_{\omega^{\gamma_2}}})$ when $\gamma_1 > 1+\gamma_0+\gamma_2$.

Proof. The precise estimate is clear from Corollary 8.4. In particular, if γ_3 is the smallest tail-like ordinal larger than γ_0, then $1+\gamma_0+\gamma_2 = \gamma_2$ when $\gamma_2 \geq \gamma_3$. That is, beyond γ_3 the sequences are strictly increasing in size.

□

By Corollary 7.5 there exist CHLOTS which are not $\mathbb{R}$-bounded, e.g. $\widehat{a(\mathbb{R})}$.

Now with X a CHLOTS we want to sharpen the tree construction from Theorem 8.1 so that we obtain T as an additive tree.

We begin by choosing two points labeled $-1 < 1 \in X$ and apply the construction to $I = [-1, 1]$. The image $f(X(T))$ of the branch space of the tree constructed in Theorem 8.1 is dense in $(-1, 1)$ which is isomorphic to X.

In order to obtain an additive tree we will have to choose the $\pm$cofinal embeddings of $\mathbb{Z}$ in a coherent way. We will inductively define $t_s : I \to I_s$ which is the constant map when I_s is improper and is an isomorphism when I_s is proper.

8.2. **The Alphabet Construction.** Regard a set A as an *alphabet* and for every $a \in A$ there is defined a map $t_a : I \to I_a$ with I_a an interval contained in I°. If I_a is a proper interval, then t_a is an isomorphism. If I_a is improper, then t_a is the constant map. In either case, t_a is continuous. In addition, we let $I_\emptyset = I$ with $t_\emptyset$ the identity 1_I. We call $A_+ = \{a \in A : I_a \text{ is a proper interval } \} \subset A$ the *proper alphabet* .

We will extend these to A^k for $k < \omega$ and to A^ω, i.e. the spaces of finite and infinite sequences.

A *word* $w \in A^k$ is a finite sequence $a_0 a_2 \dots a_{k-1}$ in A with $\emptyset$ the empty word of length 0. For a word $w \in A^k$ with $k \geq 1$, we define the map $t_w = t_{a_0} \circ t_{a_1} \circ \dots t_{a_{k-1}}$ from I onto its image denoted I_w. We call the word *proper* when all of its letters lie in the proper alphabet. In that case, t_w is an isomorphism. If the interval I_{a_i} is improper, then t_w is the constant map onto the singleton $I_w = (t_{a_0} \circ \dots \circ t_{a_{i-1}})(I_{a_i})$.

If w is the concatenation $w_1 w_2$ and w_1 is proper, then $I_w \subset I^\circ_{w_1}$. Notice that for proper words w_1, w_2, w, the composition $t_{w_1 w} \circ (t_{w_2 w})^{-1} : I_{w_2 w} \to I_{w_1 w}$ is an isomorphism which is the restriction of the isomorphism $t_{w_1} \circ (t_{w_2})^{-1} : I_{w_2} \to I_{w_1}$.

We now consider the space of infinite sequences A^ω. Let τ denote the *shift map* on A^ω with $\tau(s)_i = s_{i+1}$.

For $s \in A^\omega$ we define the associated interval $I_s = \bigcap_n \ I_{s|n}$ with $s|n$ equal to the word $s_0 \dots s_{n-1}$. Because a continuous map commutes with the decreasing intersection of compacta, we have for a word w and $z \in A^\omega$

$$I_{wz} = t_w(I_z). \tag{8.10}$$

Hence, I_{wz} is proper if and only if I_z is proper and the word w is proper.

When I_s is improper, we let $t_s : I \to I_s$ be the constant map. In particular, this applies to any sequence $s \in A^\omega \setminus A^\omega_+$.

Now we restrict attention to sequences in the proper alphabet A_+.

We call two sequences $s_1, s_2 \in A^\omega_+$ *end-equivalent* if there exist $i_1, i_2 \in \omega$ such that $\tau^{i_1}(s_1) = \tau^{i_2}(s_2)$, or, equivalently, when there exist proper words w_1, w_2 and $z \in A^\omega_+$ such that $s_1 = w_1 z, s_2 = w_2 z$. We call them *level end-equivalent* when, in addition, i_1 and i_2 can be chosen equal, or, equivalently, the words w_1 and w_2 have the same length. We call the end-equivalence class of s the *class* of s. The class of s is subdivided into *level classes*.

If for a sequence s the interval I_s is proper, then by (8.10) I_{s_1} is proper for every sequence s_1 end-equivalent to s. We will call the class *proper* when it consists of sequences with proper associated intervals.

A sequence s is *eventually periodic* if there exist $i, p \in \mathbb{N}$ with $p > 0$ such that $\tau^i(s) = \tau^{i+p}(s)$ and so $\tau^i(s) = \tau^{i+np}(s)$ for all $n \in \mathbb{N}$. The minimum such p is called the *period* of s. Clearly if s_1 is end-equivalent to s and s is eventually periodic then s_1 is eventually periodic with the same period. We call a class *periodic* if it consists of eventually periodic sequences. Otherwise we call the class *nonperiodic*.

If s is not eventually periodic then the sequences $\tau^i(s)$ are all distinct. In fact, all lie in different level classes. So if s_1, s_2 are in a nonperiodic class and $k_1, k_2, j_1, j_2 \geq 0$, then

$$(8.11) \quad \tau^{k_1}(s_1) = \tau^{k_2}(s_2) \text{ and } \tau^{k_1+j_1}(s_1) = \tau^{k_2+j_2}(s_2) \implies \quad j_1 = j_2.$$

When a class is proper, we will choose a representative element s for the class and choose $t_s : I \to I_s$ an isomorphism for this representative.

Case 1 Nonperiodic Proper Class: For a nonperiodic proper class let $t_s : I \to I_s$ be the chosen isomorphism for the chosen representative.

If s_1 is end-equivalent to s then we let k be the minimum such that $\tau^k(s) = \tau^{k_1}(s_1)$ for some k_1 and then choose k_1 minimum. Thus, there are unique words w, w_1 of length k, k_1 and $z \in A_+^{\mathbb{N}}$ such that $s = wz$, $s_1 = w_1 z$. By (8.10) $t_w|I_z : I_z \to I_s$ and $t_{w_1}|I_z : I_z \to I_{s_1}$ are isomorphisms. We define $t_{s_1} = t_{w_1} \circ (t_w)^{-1} \circ t_s$.

If $s = w'z'$, $s_1 = w_1'z'$ for finite words w', w_1' and $z' \in A_+^{\omega}$ then by minimality there exists a word u such that $w' = wu$ and $z = uz'$. So by (8.11) it follows that $w_1' = w_1 u$. Again $t_w \circ t_u = t_{w'}$ restricts to an isomorphism from $I_{z'}$ to I_s and $t_{w_1} \circ t_u = t_{w_1'}$ restricts to an isomorphism from $I_{z'}$ to I_{s_1}. So we have

$$(8.12) \ t_{w_1'} \circ (t_{w'})^{-1} \circ t_s = t_{w_1} \circ t_u \circ (t_w \circ t_u)^{-1} \circ t_s = t_{w_1} \circ (t_w)^{-1} \circ t_s = t_{s_1}.$$

So if $s_2 = w_2 \ s_1 = w_2 w_1 z$, then s_2 is end-equivalent to s_1 and

$$(8.13) \qquad t_{s_2} = t_{w_2 w_1} \circ (t_w)^{-1} \circ t_s = t_{w_2} \circ t_{s_1}.$$

It follows that

$$(8.14) \ s_1 = w_1 z, \ s_2 = w_2 z \quad \implies \quad t_{s_1} = t_{w_1} \circ t_z = t_{w_1} \circ (t_{w_2})^{-1} \circ t_{s_2}.$$

Case 2 Periodic Proper Class: If s is eventually periodic with period p, and so for some i $\tau^i(s) = \tau^{i+p}(s)$ then there exists a finite word $e = e_0e_1 \dots e_{p-1}$ such that $\tau^i(s) = e\tau^{i+p}(s)$. We let $\bar{e} \in A_+^\omega$ be the periodic element in the class with $\bar{e}_i = e_j$ if i is congruent to j mod p. Since $\bar{e} = e\bar{e}$ we have $t_e(I_{\bar{e}}) = I_{\bar{e}}$. That is, t_e restricts to an automorphism of $I_{\bar{e}}$ and so, in particular, fixes the endpoints of $I_{\bar{e}}$.

While the period p is uniquely associated with all members of the class, the minimum block e is not unique when $p > 1$. For $i = 1, \dots, p-1$ cyclic permutations $e_i \dots e_{p-1}e_0 \dots e_{i-1}$ are blocks of length p with $\overline{e_i \dots e_{p-1}e_0 \dots e_{i-1}}$ end-equivalent to $\bar{e}$. To be precise, $e_i \dots e_{p-1}\bar{e} = \overline{e_i \dots e_{p-1}e_0 \dots e_{i-1}}$.

The periodic class of period p consists of p level classes, each containing one of the periodic elements $\overline{e_i \dots e_{p-1}e_0 \dots e_{i-1}}$.

Assume that the periodic class is proper. We fix a minimum block e and use $\bar{e}$ as the representative of the class. Then we choose the isomorphism $t_{\bar{e}} : I \to I_{\bar{e}}$. We choose $\overline{e_i \dots e_{p-1}e_0 \dots e_{i-1}}$ as the representative of its level class and define $t_{\overline{e_i \dots e_{p-1}e_0 \dots e_{i-1}}} = t_{e_i \dots e_{p-1}} \circ t_{\bar{e}}$.

Now we operate in each level class separately using its unique periodic element as representative. We look at the level class containing $\bar{e}$.

A sequence s_1 is in the level class of $\bar{e}$ if and only if there exists a word w of length $|w| = np$ for some $n \in \omega$ such that $s_1 = w\bar{e}$. We choose w to be the unique such word with n minimum and define $t_{s_1} = t_w \circ (t_e)^{-n} \circ t_{\bar{e}}$, with $(t_e)^{-n}$ the n-fold iterate of $(t_e)^{-1}$ (= identity when $n = 0$).

If $s_1 = w'\bar{e}$ then $|w'| = n'p$ for some $n' \in \omega$. It follows that $n' = n+k$ for some $k \in \omega$ and $w' = w(e)^k$.

$$t_{w'} \circ (t_e)^{-n'} \circ t_{\bar{e}} = t_w \circ t_e^k \circ (t_e)^{-(n+k)} \circ t_{\bar{e}} = t_{s_1}. \tag{8.15}$$

So we have

$$\begin{aligned} s_1 = w_1\bar{e}, s_2 = w_2\bar{e}, \quad \text{and} \quad |w_1| = np = |w_2| \quad &\Longrightarrow \\ t_{s_1} = t_{w_1} \circ (t_e)^{-n} \circ t_{\bar{e}} = t_{w_1} \circ t_{w_2}^{-1} \circ t_{s_2}. & \end{aligned} \tag{8.16}$$

This completes the Alphabet Construction.

Example 8.8. *Proper intervals and fixed points.*

If, in the Alphabet Construction, e is a finite word with $\bar{e}$ the associated periodic sequence and $I_{\bar{e}} = [a, b]$ interval, then t_e is an automorphism of $[a, b]$ and so the endpoints are fixed points of t_e. They are distinct if the interval is proper.

The set of fixed points $\{x \in I : t_e(x) = x\}$ is closed with minimum m and maximum M. If $m < M$, then $t_e(I)$ is an interval which contains m and M and so $[m, M] \subset t_e(I)$ and so by induction $[m, M] \subset t_e^k(I)$ for all $k \in \omega$. It follows that $[a, b] = [m.M]$. Thus, $\bar{e}$ is improper if and only if t_e has a unique fixed point. Finally, if $[a, b] \subset [a_1, b_1]^\circ$ and $[a_1, b_1] \subset I^\circ$, then there exist isomorphisms $[-1, a] \to [a_1, a]$ and $[b, 1] \to [b, b_1]$. Combining these with the identity on $[a, b]$ we can define $t_e : I \to [a_1, b_1]$ which fixes $[a, b]$.

8.3. **The Additive Tree for a CHLOTS.** We now proceed with our inductive construction of the mappings t_s. As part of the construction we will prove the following:

Composition Property Let $s_1, s_2 \in \mathbb{Z}^\alpha$. If β is an ordinal with $\beta < \alpha$ such that $\tau_\beta(s_1) = \tau_\beta(s_2)$, and $s_1|\beta$ and $s_2|\beta$ are proper, then s_1 is proper if and only if s_2 is proper and in that case then

$$t_{s_1} \;=\; (t_{s_1|\beta}) \circ (t_{s_2|\beta})^{-1} \circ t_{s_2}. \tag{8.17}$$

If an ordinal α is a sum of ordinals $\alpha = \alpha_1 + \alpha_2 + \dots \alpha_k$, we will write $\sigma_i = \alpha_1 + \dots \alpha_i$ for $i = 1, \dots k$ and for $s \in \mathbb{Z}^\alpha$ we will write

$$s_1 = s|\alpha_1, s_2 = \tau_{\alpha_1}(s)|\alpha_2, s_3 = \tau_{\sigma_2}(s)|\alpha_3, \dots, s_k = \tau_{\sigma_{k-1}}(s). \tag{8.18}$$

So that $s = s_1 + s_2 + \dots s_k$ in the simple tree.

Step 1 $\alpha \leq \omega$:

We first let $t_\emptyset$ be the identity on I.

We then choose a $\pm$cofinal embedding of $\mathbb{Z}$ into $(-1, 1)$ and for each $n \in \mathbb{Z}$ we choose an isomorphism $t_n : I \to [z_n, z_{n+1}]$.

We apply the Alphabet Construction with $A = \mathbb{Z}$ and using t_n for $n \in \mathbb{Z}$. So in this case, the entire alphabet is proper and from the Alphabet Construction we obtain $t_s : I \to I_s$ for every $s \in \mathbb{Z}^\alpha$ with $\alpha \leq \omega$. Because a class is either entirely proper or entirely improper, the Composition Property follows from (8.14) in the nonperiodic case and from (8.16) in the periodic case.

Step 2 $\omega^\gamma < \alpha < \omega^{\gamma+1}$ with $\gamma \geq 1$:

For α between ω^γ and $\omega^{\gamma+1}$ Cantor Normal Form is $\alpha = \alpha_1 + \alpha_2 + \dots \alpha_k$ with $\omega^\gamma = \alpha_1 \geq \alpha_2 \geq \dots \alpha_k$ tail-like. As in (8.18) for $s \in \mathbb{Z}^\alpha$ we have the decomposition $s = s_1 + s_2 + \dots s_k$.

We use the induction hypothesis to define

$$t_s = t_{s_1} \circ t_{s_2} \circ \dots t_{s_k} : I \to I_s \tag{8.19}$$

with I_s the image of t_s.

Thus, s is proper if and only if each s_i is proper and the composition is then an isomorphism. If, instead, i is the smallest index such that s_i is improper, then I_s is the singleton $t_{s_1} \circ t_{s_2} \circ \dots t_{s_{i-1}}(I_{s_i})$ and the constant map t_s is the composition $t_{s_1} \circ t_{s_2} \circ \dots t_{s_i}$.

For the Composition Property let $s_1, s_2 \in \mathbb{Z}^\alpha$ and for $\epsilon = 1, 2$ we decompose $s_\epsilon = s_{\epsilon 1} + s_{\epsilon 2} + \dots s_{\epsilon k}$. Assume $\beta < \alpha$ such that $\tau_\beta(s_1) = \tau_\beta(s_2)$ and $s_\epsilon | \beta$ is proper for $\epsilon = 1, 2$.

Let $i \leq k$ be the minimum such that $\beta < \sigma_i$ and let $\beta_1 = \beta \setminus \sigma_{i-1}$. If $i = 1$, then $\sigma_0 = 0$ by convention and $\beta_1 = \beta$.

Since $\tau_\beta(s_1) = \tau_\beta(s_2)$ we have $s_{1j} = s_{2j}$ for $j = i+1, \dots k$ and for such j we will write s_j for $s_{1j} = s_{2j}$. In addition, $\tau_{\beta_1}(s_{1i}) = \tau_{\beta_1}(s_{2i})$.

So from (8.19) we obtain (since $\beta_1 < \alpha_i$):

$$\begin{aligned} t_{s_\epsilon | \sigma^i} &= t_{s_{\epsilon 1}} \circ \dots t_{s_{\epsilon i}}, \\ t_{s_\epsilon | \beta} &= t_{s_{\epsilon 1}} \circ \dots t_{s_{\epsilon (i-1)}} \circ t_{s_{\epsilon i} | \beta_1}, \\ t_{s_\epsilon} &= t_{s_\epsilon | \sigma^i} \circ t_{s_{i+1}} \circ \dots t_{s_k}. \end{aligned} \tag{8.20}$$

Because $s_1 | \beta$ and $s_2 | \beta$ are proper, the maps $t_{s_{\epsilon 1}}, t_{s_{\epsilon(i-1)}}, t_{s_{\epsilon i} | \beta_1}$ are isomorphisms.

By the Composition Property for $s_{1i}, s_{2i} \in \mathbb{Z}^{\alpha_i}$ we have that s_{1i} is improper if and only if s_{2i} is improper in which case s_1 and s_2 are both improper. In addition if any of $s_{i+1}, \dots, s_k$ are improper, then both s_1 and s_2 are improper.

Assume, instead, that both s_{1i} and s_{2i} as well as $s_{i+1}, \dots, s_k$ are proper. The Composition Property for s_{1i}, s_{2i} implies

$$t_{s_{1i}} = (t_{s_{1i} | \beta_1}) \circ (t_{s_{2i} | \beta_1})^{-1} \circ t_{s_{2i}}.$$

From this and (8.20) it follows that

$$t_{s_1 | \sigma^i} = (t_{s_1 | \beta}) \circ (t_{s_2 | \beta})^{-1} \circ t_{s_2 | \sigma^i}.$$

and we compose with $t_{s_{i+1}} \circ \dots t_{s_k}$ to obtain the Composition Property for s_1 and s_2.

If $\alpha' = \alpha + 1$ is the successor of α, then the Cantor Normal Form for α' is $\alpha' = \alpha_1 + \alpha_2 + \dots \alpha_k + \alpha_{k+1}$ with $\alpha_{k+1} = 1$. Recall that 1 is the unique tail-like ordinal which is not a limit ordinal. If $s' \in \mathbb{Z}^{\alpha'}$ with $s' | \alpha = s$, then $t_{s'} = t_s \circ t_n$ where $n = s'(\alpha)$. Thus, in both Step 1 and Step 2, we are using the $\pm$cofinal map $t_s \circ z : \mathbb{Z} \to I_s^\circ$ for I_s, where $z : \mathbb{Z} \to I$ is the $\pm$cofinal map with which we began. so that, inductively, the intervals I_s are those obtained in the construction of Theorem 8.1. As before, each successor s' is proper when s is.

Step 3 $\alpha = \omega^{\gamma+1}$:

We have $\mathbb{Z}^{\omega^{\gamma+1}} = \mathbb{Z}^{\omega^{\gamma}\cdot\omega}$ which we identify with $(\mathbb{Z}^{\omega^{\gamma}})^{\omega}$ as in (4.20). That is, we regard an element of $\mathbb{Z}^{\omega^{\gamma+1}}$ as a sequence of elements of the alphabet $A = \mathbb{Z}^{\omega^{\gamma}}$. The proper alphabet consists of the proper elements of $\mathbb{Z}^{\omega^{\gamma}}$.

We now apply the Alphabet Construction to define $t_s : I \to I_s$ for all $s \in \mathbb{Z}^{\omega^{\gamma+1}}$. In particular, from the Alphabet Construction we obtain (iii) of the construction for Theorem 8.1.

For the Composition Property, let $s_1, s_2 \in \mathbb{Z}^{\omega^{\gamma+1}}$ and β be an ordinal less than $\omega^{\gamma+1}$. Assume that $\tau_\beta(s_1) = \tau_\beta(s_2)$ and that $s_1|\beta$ and $s_2|\beta$ are proper.

Let k be the minimum in ω such that $\beta \leq \omega^{\gamma}\cdot k$. Regarding s_1 and s_2 as sequences, then $\tau_\beta(s_1) = \tau_\beta(s_2)$ implies that $s_1 = w_1 z, s_2 = w_2 z$ with $z \in \mathbb{Z}^{\omega^{\gamma+1}}$ and w_1, w_2 words of length k in the alphabet, or equivalently elements of $\mathbb{Z}^{\omega^{\gamma}\cdot k}$. Furthermore, $s_\epsilon|\beta = w_\epsilon|\beta$ for $\epsilon = 1, 2$ and we are assuming that these are proper.

From the Composition Property applied to $w_1, w_2 \in \mathbb{Z}^{\omega^{\gamma}\cdot k}$ we have that w_1 is improper if and only if w_2 is improper in which case both s_1 and s_2 are improper. In addition if any of the $\mathbb{Z}^{\omega^{\gamma}}$ terms of the sequence z is improper, then both s_1 and s_2 are improper.

Assume, instead, that all of the terms of the sequences s_1 and s_2 lie in the proper alphabet. The Composition Property applied to w_1, w_2 then implies

$$t_{w_1} = t_{w_1|\beta} \circ t^{-1}_{w_2|\beta} \circ t_{w_2}. \tag{8.21}$$

Since s_1 and s_2 are end-equivalent sequences in the proper alphabet, we can apply the Alphabet Construction results. From (8.14), or in the eventually periodic case from (8.16), we obtain from (8.21)

$$\begin{aligned} t_{s_1} = t_{w_1} \circ t^{-1}_{w_2} \circ t_{s_2} = \\ t_{w_1|\beta} \circ t^{-1}_{w_2|\beta} \circ t_{w_2} \circ t^{-1}_{w_2} \circ t_{s_2} \\ = t_{w_1|\beta} \circ t^{-1}_{w_2|\beta} \circ t_{s_2}, \\ = t_{s_1|\beta} \circ t^{-1}_{s_2|\beta} \circ t_{s_2}, \end{aligned} \tag{8.22}$$

proving the Composition Property for s_1 and s_2.

Step 4 $\alpha = \omega^{\gamma}$ with γ a countable limit ordinal :

For $s \in \mathbb{Z}^{\omega^\gamma}$ if $s|\alpha$ is improper for some $\alpha < \omega^\gamma$ then s is improper with a constant t_s. We call s *limit proper* if, instead, $s|\alpha$ is proper for all $\alpha < \omega^\gamma$.

Call two elements $s_1, s_2 \in \mathbb{Z}^{\omega^\gamma}$ *end-equivalent* if there exists $\beta < \omega^\gamma$ such that $\tau_\beta(s_1) = \tau_\beta(s_2)$.

We consider an end-equivalence class of limit proper elements. From the Composition Property, it follows that for every α with $\beta < \alpha < \omega^\gamma$, $t_{s_1|\alpha} = (t_{s_1|\beta}) \circ (t_{s_2|\beta})^{-1} \circ t_{s_2|\alpha}$ and so $I_{s_1|\alpha} = (t_{s_1|\beta}) \circ (t_{s_2|\beta})^{-1}(I_{s_2|\alpha})$. Intersecting as $\alpha \to \omega^\gamma$ we obtain

$$I_{s_1} = (t_{s_1|\beta}) \circ (t_{s_2|\beta})^{-1}(I_{s_2}) \tag{8.23}$$

In particular, s_1 is improper if and only if s_2 is improper.

Assume, instead, that the end-equivalence class consists of proper elements. We choose a representative s and an isomorphism $t_s : I \to I_s$. For s_1 end-equivalent to s let β_1 be the smallest ordinal such that $\tau_{\beta_1}(s) = \tau_{\beta_1}(s_1)$. We define $t_{s_1} = (t_{s_1|\beta_1}) \circ (t_{s|\beta_1})^{-1} \circ t_s$.

If $\beta_2 > \beta_1$ then $\tau_{\beta_2}(s) = \tau_{\beta_2}(s_1)$ and by the Composition Property for $s_1|\beta_2$ and $s_2|\beta_2$, $t_{s_1|\beta_2} = (t_{s_1|\beta_1}) \circ (t_{s|\beta_1})^{-1} \circ t_{s|\beta_2}$. That is, $(t_{s_1|\beta_2}) \circ (t_{s|\beta_2})^{-1}$ is a restriction of $(t_{s_1|\beta_1}) \circ (t_{s|\beta_1})^{-1}$ and so

$$t_{s_1} = (t_{s_1|\beta_2}) \circ (t_{s|\beta_2})^{-1} \circ t_s. \tag{8.24}$$

Finally, for the Composition Property let $s_1, s_2 \in \omega^\gamma$ and $\beta < \omega^\gamma$ be an ordinal such that $\tau_\beta(s_1) = \tau_\beta(s_2)$ with both $s_1|\beta$ and $s_2|\beta$ proper.

If neither s_1 nor s_2 is limit proper then both I_{s_1} and I_{s_2} are improper. Now assume that s_1 is limit proper. If $\alpha \leq \beta$ then $s_2|\alpha = (s_2|\beta)|\alpha$ is proper. If $\beta < \alpha$, then by the Composition Property applied to $s_1|\alpha$ and $s_2|\alpha$, $s_2|\alpha$ is proper because $s_1|\alpha$ is proper. So we may assume that both s_1 and s_2 are limit proper. They are clearly end-equivalent. If the end-equivalence class consists of improper elements, then both s_1 and s_2 are improper.

Assume, instead, that the class consists of proper elements and let s be the representative of the class.

We can choose $\beta_1 > \beta$ so that $\tau_{\beta_1}(s_1) = \tau_{\beta_1}(s_2) = \tau_{\beta_1}(s)$.

$$\begin{gathered} t_{s_1} = (t_{s_1|\beta_1}) \circ (t_{s|\beta_1})^{-1} \circ t_s = \\ (t_{s_1|\beta_1}) \circ (t_{s|\beta_1})^{-1} \circ (t_{s|\beta_1}) \circ (t_{s_2|\beta_1})^{-1} \circ t_{s_2} = \\ (t_{s_1|\beta_1}) \circ (t_{s_2|\beta_1})^{-1} \circ t_{s_2} = (t_{s_1|\beta}) \circ (t_{s_2|\beta})^{-1} \circ t_{s_2}, \end{gathered} \tag{8.25}$$

completing the proof of the Composition Property for s_1 and s_2.

Step 5 $\alpha = \Omega$:

Because the tree T is Ω bounded, every $s \in \mathbb{Z}^\Omega$ is improper and so t_s is the constant map onto I_s.

This completes the inductive construction of the tree T.

Theorem 8.9. *The tree T is an Ω bounded additive tree of $\mathbb{Z}$ type.*

Proof. We saw in Theorem 8.1 that the tree is Ω bounded.

We must show that if $s \in \mathbb{Z}^{\alpha}$ and $\alpha = \alpha_1 + \alpha_2$, then s is proper if and only if both $s|\alpha_1$ and $\tau_{\alpha_1}(s)$ are proper. Since s proper implies $s|\alpha_1$ is proper, we assume that $s_1 = s|\alpha_1$ is proper and prove that $s_2 = \tau_{\alpha_1}(s)$ is proper if and only if $s = s_1 + s_2$ is proper.

First we reduce to the case when α_1 is tail-like. That is, assume the result is true when α_1 is tail-like and in general write $\alpha_1 = \alpha_{11} + \alpha_{12} + \dots \alpha_{1k}$ in Cantor Normal Form and so write $s_1 = s_{11} + s_{12} + \dots s_{1k}$.

Because s_1 is proper, the definition (8.19) implies that $s_{11}, s_{12}, \dots s_{1k}$ are all proper and so are the partial sums. We then complete the proof by induction on k with $k = 1$ the assumed, tail-like, case. We have

$$s = s_1 + s_2 = (s_{11} + \dots s_{1(k-1)}) + (s_{1k} + s_2). \tag{8.26}$$

By the inductive hypothesis, s is proper if and only if $s_{1k} + s_2$ is proper and so, by the initial case, if and only if s_2 is proper.

Now we prove the result assuming that α_1 is tail-like and so equals some ω^{γ_1}. Now we proceed by induction on α_2.

Step 1 $\alpha_2 < \omega^{\gamma_1 + 1} = (\alpha_1)^{\omega}$:

Write $\alpha_2 = \alpha_{21} + \alpha_{22} + \dots \alpha_{2\ell}$ in Cantor Normal Form and so write $s_2 = s_{21} + s_{22} + \dots s_{2\ell}$.

The assumption $\alpha_2 < (\alpha_1)^{\omega}$ implies that $\alpha_1 \geq \alpha_{21}$ and so $\alpha_1 + \alpha_{21} + \alpha_{22} + \dots \alpha_{2\ell}$ is Cantor Normal Form for $\alpha = \alpha_1 + \alpha_2$. Hence, (8.19) implies that s is proper if and only if s_1 and $s_{21}, s_{22}, \dots s_{2\ell}$ are all proper. Furthermore, s_2 is proper if and only if $s_{21}, s_{22}, \dots s_{2\ell}$ are all proper. Since s_1 is assumed proper, it follows that s is proper if and only if s_2 is proper, as required.

Step 2 $\alpha_2 = \omega^{\gamma+1}$ with $\gamma_1 \leq \gamma$:

As before we identify $\mathbb{Z}^{\omega^{\gamma+1}} = \mathbb{Z}^{\omega^{\gamma}\cdot\omega}$ with $(\mathbb{Z}^{\omega^{\gamma}})^{\omega}$ and so regard $\mathbb{Z}^{\omega^{\gamma+1}}$ as sequences on the alphabet $\mathbb{Z}^{\omega^{\gamma}}$ with the proper alphabet consisting of the proper elements of $\mathbb{Z}^{\omega^{\gamma}}$. Now we split into two subcases.

Step 2a $\gamma_1 = \gamma$:

In this case, s_1 is a letter in the proper alphabet which we label w, $s_2 = z$ is a sequence in the alphabet and $s = wz$. If any of the terms in z is not proper, then both s and s_2 are improper.

Assume instead that z is a sequence in the proper alphabet, and so wz is a sequence in the proper alphabet as well. Because wz and z are end-equivalent sequences, it follows from the Alphabet Construction that s_2 is proper if and only if s is.

Step 2b $\gamma_1 < \gamma$:

This time we write $s_2 = wz$ for the sequence in the alphabet where we use w to label the first term. So $w \in \mathbb{Z}^{\omega^\gamma}$ and since $\alpha_1 < \omega^\gamma$, $w_1 = s_1 + w$ is an element of $\mathbb{Z}^{\omega^\gamma}$ as well, with $s = w_1 z$.

By the inductive hypothesis, w is improper if and only w_1 is improper in which case both s and s_2 are improper. In addition, if any term in z is improper, then both s and s_2 are improper.

Assume, instead, that both s and s_2 are sequences in the Proper Alphabet. Because wz and $w_1 z$ are end-equivalent sequences, it follows from the Alphabet Construction that s_2 is proper if and only if s is.

Step 3 $\omega^\gamma < \alpha_2 < \omega^{\gamma+1}$ with $\gamma_1 < \gamma$:

We return to Cantor Normal Form $\alpha_2 = \alpha_{21} + \alpha_{22} + \dots \alpha_{2\ell}$ with $\alpha_{21} = \omega^\gamma$ and we write $s_2 = s_{21} + s_{22} + \dots s_{2\ell}$. Again s_2 is proper if and only if $s_{21}, s_{22}, \dots s_{2\ell}$ are all proper.

By inductive hypothesis, s_{21} is proper if and only if $s_1 + s_{21}$ is proper. But $\alpha_1 + \alpha_{21} = \alpha_{21}$ because $\alpha_1 < \alpha_{21}$ and α_{21} is tail-like. Hence, $\alpha_1 + \alpha_2 = \alpha_{21} + \alpha_{22} + \dots \alpha_{2\ell}$ in Cantor Normal Form with $s_1 + s_2 = (s_1 + s_{21}) + s_{22} + \dots s_{2\ell}$. Thus, $s = s_1 + s_2$ if proper if and only if $(s_1 + s_{21}), s_{22}, \dots s_{2\ell}$ are all proper. It follows that s_2 is proper if and only if s is proper.

Step 4 $\alpha_2 = \omega^\gamma$ with γ a limit ordinal such that $\gamma_1 < \gamma$:

Because γ is a limit ordinal larger than γ_1, it is larger than $\gamma_1 + 1$. So $\beta = \omega^{\gamma_1 + 1}$ is a tail-like ordinal with $\alpha_1 < \beta < \alpha_2$. Hence $\alpha_1 + \beta = \beta$ and $\alpha_1 + \alpha_2 = \alpha_2$.

Because $\alpha_1 + \alpha_2 = \alpha_2$, $s, s_2 \in \mathbb{Z}^{\omega^\gamma}$.

Because $\alpha_1 + \beta = \beta$, $\tau_\beta(s) = \tau_\beta(s_2)$ and $s|\beta = s_1 + (s_2|\beta)$.

By inductive hypothesis, $s|\beta$ is proper if and only if $s_2|\beta$ is proper. If these are improper then both s and s_2 are improper.

Assume, instead, that both $s|\beta$ and $s_2|\beta$ are proper.

For any β_1 with $\beta < \beta_1 < \alpha_2$, the Composition Property for $s|\beta_1$ and $s_2|\beta_1$ implies that $s|\beta_1$ is proper if and only if $s_2|\beta_1$ is proper. If these are improper then both s and s_2 are improper.

Assume, instead, that both $s|\beta_1$ and $s_2|\beta_1$ are proper for all $\beta_1 < \alpha_2$. That is, s and s_2 are end-equivalent limit proper elements of $\mathbb{Z}^{\omega^\gamma}$. From Step 4 of the above construction of the tree, it follows that s is proper if and only if s_2 is proper.

□

Example 8.10. *Trees for* $X = \mathbb{R}$*.*

If F is a countable unbounded LOTS, like $\mathbb{Z}$ or $\mathbb{Q}$, then F^ω can be regarded as the branch space of the simple tree on F, ω. By Proposition 5.4 it is order dense. We can also think of F as an alphabet and F^ω as the space of sequences on F. Any end equivalence class is a countable dense set and so F^ω is separable. By Proposition 2.15(b) the compactification $\bullet\widehat{F^\omega}\bullet$ is isomorphic to the unit interval in $\mathbb{R}$ and so $\widehat{F^\omega} \cong \mathbb{R}$. It follows that F^ω is order isomorphic to a dense subset of $\mathbb{R}$.

In the $F = \mathbb{Z}$ case we let

$$z_n = \begin{cases} -1 + 2^n & \text{for} \quad n < 0, \\ \ 1 - 2^{-n} & \text{for} \quad n \geq 0. \end{cases} \tag{8.27}$$

This defines a $\pm$cofinal embedding of $\mathbb{Z}$ into $I = (-1, 1) \subset \mathbb{R}$. Let t_n from I onto $I_n = [z_n, z_{n+1}]$ be the restriction of the affine map on $\mathbb{R}$ given by

$$t_n(z) = \frac{1}{2}[(z_{n+1} + z_n) + z \cdot (z_{n+1} - z_n)]. \tag{8.28}$$

So the derivative t'_n (equals the slope) is bounded by $1/4$ for all n. If $w \in \mathbb{Z}^k$, then the length in $\mathbb{R}$ of the interval I_w is bounded by 4^{-k}. It follows that for every $s \in \mathbb{Z}^\omega$, I_s is improper. Hence, T is the simple tree on $\mathbb{Z}, \omega$ and $X(T) = \mathbb{Z}^\omega$.

Because there are countably many finite words, C is a countable dense subset of $[-1, 1]$, including -1 and 1 and I is the disjoint union of $X(T)$ and C. Since C is order-isomorphic to $\mathbb{Q} \cap I$, it follows that $X(T) \cong \mathbb{Z}^\omega$ is order-isomorphic to the set of irrationals in I and so to the set of irrationals in $\mathbb{R}$.

If $F = \mathbb{N}$, the map $z : \mathbb{N} \to [0, 1)$ defined by the restriction of the map in (8.27) is a cofinal embedding into $\tilde{I} = [0, 1)$. Now we apply the Alphabet Construction with $A = \mathbb{N}$ and $\tilde{t}_n$ given by

$$\tilde{t}_n(z) = z_n + z \cdot (z_{n+1} - z_n). \tag{8.29}$$

For every finite word w let $\tilde{I}_w = \tilde{t}_w(\tilde{I})$. For each $k < \omega$, $[0, 1)$ is the disjoint union of $\{\tilde{I}_w : w \in \mathbb{N}^k\}$. As above for every $s \in \mathbb{N}^\omega$, $\tilde{I}_s = \bigcup_{k<\omega} \tilde{I}_{s|k}$ is a singleton and so the branch space of the tree can be identified with $\mathbb{N}^\omega$. This time the map $\tilde{f} : X(T) \to \tilde{I}$ given by $x \mapsto a^x$

is surjective as well as injective. Thus, $\mathbb{N}^\omega$ is order isomorphic with $[0,1) \subset \mathbb{R}$.

8.4. **Subsets of $\mathbb{Z}^\omega$.** We regard $\mathbb{Z}^\omega$ on the one hand as the branch space of the simple tree on $\mathbb{Z}, \omega$ and on the other as the space of sequences on the alphabet $\mathbb{Z}$.

We first characterize the subsets W of $\mathbb{Z}^\omega$ which can occur as the set L_ω of ω level vertices of some additive $\mathbb{Z}$ tree. Notice that $L_\omega \neq \emptyset$ if and only if $h(T) > \omega$.

Proposition 8.11. *A subset $W \subset \mathbb{Z}^\omega$ is equal to the set L_ω of level ω vertices in some additive tree T of $\mathbb{Z}$ type with $h(T) > \omega$ if and only if W is saturated by the end equivalence relation. That is, if $s_1, s_2 \in \mathbb{Z}^\omega$ are end equivalent, then $s_1 \in W$ if and only if $s_2 \in W$.*

Proof. If T is an additive tree on $\mathbb{Z}$, then for $k < \omega$, $L_k = \mathbb{Z}^k$. That is, every finite sequence on $\mathbb{Z}$ of length k is a vertex of T with order k. So if $p_1, p_2 \in \mathbb{Z}^{\omega+1}$ so that $o(p_1) = o(p_2) = \omega$, then the associated sequences in $\mathbb{Z}^\omega$ are end equivalent if and only if there exist $q_1, q_2 \in T$ with $o(q_1), o(q_2) < \omega$ and $p \in \mathbb{Z}^{\omega+1}$ such that $p_1 = q_1 + p, p_2 = q_2 + p$. Since $q_1, q_2 \in T$, additivity implies that $p_1 \in T \Leftrightarrow p \in T \Leftrightarrow p_2 \in T$. Hence, $W = L_\omega$ implies that the set is saturated by the end equivalence relation.

For the converse, assume that W is saturated by the end equivalence relation. We apply the inductive construction of Theorem 6.10. Observe first that the simple tree on $\mathbb{Z}, \omega$ is an additive tree with height the limit ordinal ω. Because W is saturated, it is translation invariant in the sense of (6.32). So the construction of the Theorem 6.10 allows us to build an additive tree of $\mathbb{Z}$ type with height ω^2 with $L_\omega = \{p + x : o(p) < \omega, x \in W\}$. Because W is saturated, this set is equal to W.

$\square$

Remark. Recall the shift map τ on $\mathbb{Z}^\omega$. A set $W \subset \mathbb{Z}^\omega$ is invariant in the sense of (6.32) precisely when it is τ invariant, i.e. $s \in W \implies \tau(s) \in W$. The set is saturated by the end equivalence relation if and only if $s \in W \iff \tau(s) \in W$.

An additive tree on the transitive LOTS $\mathbb{Z}$ is homogeneous by Proposition 5.19. It follows that an end-equivalence saturated subset $W \subset \mathbb{Z}^\omega$

is always transitive. In fact we will prove below that it is doubly transitive. We begin by describing certain order preserving maps on $\mathbb{Z}^\omega$.

The Tree Automorphisms:

Because the only order preserving bijections of $\mathbb{Z}$ are translations, and any tree automorphism f maps S_x bijectively onto $S_{f(x)}$, we can construct an arbitrary tree automorphism of $\mathbb{Z}^\omega$ as follows. Choose for each finite word w (including the empty word) $m(w) \in \mathbb{Z}$. Define the associated automorphism by

$$f(x)_i = x_i + m(x_0 \dots x_{i-1}), \tag{8.30}$$

so that $f(x)_0 = x_0 + m(\emptyset)$.

Notice that, using pointwise addition, $\mathbb{Z}^\omega$ is a group. Translation by an element of the group is the special case when $|w_1| = |w_2| \implies m(w_1) = m(w_2)$. Note that for the group $\mathbb{Z}^\omega$ the level end equivalence class of $\bar{0}$ (equals the end equivalence class of $\bar{0}$) is a subgroup and the level end equivalence classes are exactly the cosets for this subgroup. Hence, the set of level end equivalence classes has a group structure induced by pointwise addition. Also it follows that any two level end equivalence classes are isomorphic via a translation map.

Proposition 8.12. *Let $p, q \in \mathbb{Z}^\omega$. There exists a tree automorphism f of $\mathbb{Z}^\omega$ such that $f(p) = q$ and such that for every $x \neq p$ in $\mathbb{Z}^\omega$, x and $f(x)$ are level end equivalent.*

Proof. Given $p \in \mathbb{Z}^\omega$ we choose $m(p_0 \dots p_{i-1}) = q_i - p_i$ for $i = 0, 1, \dots$ and $m(w) = 0$ for all other finite words. Clearly, $f(p) = q$. If $x \neq p$, then let k be the equality level for x and p. That is, $x_i = p_i$ for all $i < k$ and $x_k \neq p_k$. We then have

$$f(x)_i = \begin{cases} q_i & \text{for } i < k, \\ x_k - p_k + q_k & \text{for } i = k, \\ x_i & \text{for } i > k. \end{cases} \tag{8.31}$$

Thus, x and $f(x)$ are level end equivalent.

$\square$

Corollary 8.13. *If $W \subset \mathbb{Z}^\omega$ is saturated by level end equivalence (and so, a fortiori, if it is saturated by end equivalence), then the group of tree automorphisms on $\mathbb{Z}^\omega$ acts transitively on W.*

Proof. If $p, q \in W$ then by Proposition 8.12 there exists a tree automorphism f such that $f(p) = q$ and such that x and $f(x)$ are level end equivalent for all $x \neq p$. It follows that for all x, $x \in W$ if and only if $f(x) \in W$.

$\square$

Notice that if $x, y, z \in \mathbb{Z}^\omega$ with $x_0 = 0, y_0 = 1$ and $z_0 = 2$, then there does not exist a tree automorphism f such that $f(x) = x$ and $f(y) = z$. So for double transitivity of a subset W of $\mathbb{Z}^\omega$ we cannot hope to use tree automorphisms. We require more general order preserving bijections which do not preserve the tree structure.

The Reproduction Isomorphisms:

For a finite word w of length k, $a_w(z) = wz$ defines the canonical order isomorphism from $\mathbb{Z}^\omega$ onto $\{x \in \mathbb{Z}^\omega : x_i = w_i \text{ for } i = 0, \dots k-1\}$. That is, it is the canonical isomorphism from T to T_w where T is the simple tree on $\mathbb{Z}, \omega$.

Furthermore, z and $a_w(z)$ are end equivalent in $\mathbb{Z}^\omega$. So if $W \subset \mathbb{Z}^\omega$ is end equivalence saturated, then a_w is an isomorphism from W onto the subset we denote wW, the copy of W with foot w.

The Lift Maps:

For $x \in \mathbb{Z}^\omega$ with $x \neq \bar{0}$ let $k^*(x) \in \omega$ such that $x_{k^*} \neq 0$ and $x_i = 0$ for all $i < k^*$. That is, k^* is the equality level for the pair $x, \bar{0}$.

The *lift maps* are defined as follows:

$$\ell_+(x) \;=\; \begin{cases} 0x & \text{if } x_{k^*} < 0, \\ x & \text{otherwise.} \end{cases}$$

(8.32)

$$\ell_-(x) \;=\; \begin{cases} 0x & \text{if } x_{k^*} > 0, \\ x & \text{otherwise.} \end{cases} \quad .$$

Let $(\mathbb{Z}^\omega)_+ = \{x \in \mathbb{Z}^\omega : x_0 \geq 0\}, (\mathbb{Z}^\omega)_- = \{x \in \mathbb{Z}^\omega : x_0 \leq 0\}$, and for $W \subset \mathbb{Z}^\omega$ we let $W_\pm = W \cap (\mathbb{Z}^\omega)_\pm$.

Proposition 8.14. *ℓ_+ is an order isomorphism from $\mathbb{Z}^\omega$ onto $(\mathbb{Z}^\omega)_+$ and ℓ_- is an order isomorphism from $\mathbb{Z}^\omega$ onto $(\mathbb{Z}^\omega)_-$. Furthermore, x, $\ell_+(x)$ and $\ell_-(x)$ are end equivalent for all $x \in \mathbb{Z}^\omega$.*

If $W \subset \mathbb{Z}^\omega$ is end equivalence saturated, then ℓ_+ and ℓ_- restrict to isomorphisms from W to W_+ and W_-, respectively.

Proof. We will do the proof for ℓ_+.

It is clear that ℓ_+ is a bijection from $\mathbb{Z}^\omega$ to $(\mathbb{Z}^\omega)_+$ and that x and $\ell_+(x)$ are end equivalent. We must prove that ℓ_+ is order preserving.

We say that x is *fixed* when $\ell_+(x) = x$, and that x is *lifted* when $\ell_+(x) = 0x$. So x is fixed when $x = \bar{0}$ or $x_{k^*} > 0$ and is lifted when $x_{k^*} < 0$.

Assume $x < y$. We say that x and y are on the same side when they are both fixed or both lifted. In that case it is clear that $\ell_+(x) < \ell_+(y)$.

Let k be the equality level for x and y. That is, $x_i = y_i$ for $i < k$ and $x_k < y_k$. If for some $i < k$, $x_i = y_i \neq 0$, then $k^*(x) = k^*(y)$ and $x_{k^*} = y_{k^*}$. So in that case, x and y are on the same side.

Assume now that $x_i = y_i = 0$ for $i < k$. If $x_k < y_k < 0$, then $k^*(x) = k^*(y) = k$ and x and y are on the same side.

If $y_k > 0$, then y is fixed. If x and y are not on the same side then x is lifted. Hence, $\ell_+(x)_k = 0 < y_k = \ell_+(y)_k$ and so $\ell_+(x) < \ell_+(y)$.

If $x_k < y_k = 0$, then x is lifted. If x and y are not on the same side, then y is fixed. So either $y = \bar{0}$ or $k^*(y) > k$ and $y_{k^*} > 0$. In either case we have $\ell_+(x)_{k+1} = x_k < 0 \leq y_{k+1} = \ell(y)_{k+1}$. So in this case as well, $\ell_+(x) < \ell_+(y)$.

This completes the proof that ℓ_+ is an order map.

Since $x, \ell_+(x)$ and $\ell_-(x)$ are all end equivalent, it follows that $\ell_\pm$ restricts to an isomorphism of W onto $W_\pm$.

□

Theorem 8.15. *If W is an end equivalence saturated subset of $\mathbb{Z}^\omega$, then W is doubly transitive and so is an IHLOTS with completion $\mathbb{R}$.*

Proof. Using the reproduction isomorphisms we express various subsets of $\mathbb{Z}^\omega$ as *patterns* of copies of W. The decomposition $W = \bigcup_{i \in \mathbb{Z}} iW$ expresses W as a $\mathbb{Z}$ pattern of copies of W, i.e. shows that it is isomorphic to the lexicographic product $\mathbb{Z} \times W$. Similarly, $W_+ = \bigcup_{i \in \mathbb{N}} iW$ and $W_- = \bigcup_{i \in -\mathbb{N}} iW$ express W_+ as an $\mathbb{N}$ pattern and W_- as an $\mathbb{N}^*$ pattern.

The isomorphism $\ell_+ : W \to W_+$ shows that W is isomorphic to an $\mathbb{N}$ pattern of copies of W. Similarly, ℓ_- shows that W is isomorphic to an $\mathbb{N}^*$ pattern of copies of W.

By replacing W by a translate given by $x \mapsto x - y + \overline{(-1)}$ with $y \in W$, we may assume that $\overline{(-1)} \in W$. Since W is transitive, double transitivity follows from transitivity of the interval $\{x \in W : \overline{(-1)} < x\}$, see Proposition 3.2 (d)(v).

If $(-1)^k$ is the word w of length k with $w_i = -1$ for $i = 0, \ldots, k-1$ and so $(-1)^0$ is the empty word, then we have the decomposition $\{x \in$

$W : \overline{(-1)} < x\} = \bigcup_{k\in\mathbb{N}}(-1)^k W_+$ with $(-1)^{k+1}W_+$ preceding $(-1)^k W_+$. Each $(-1)^k W_+$ is isomorphic to W and so $\{x \in W : \overline{(-1)} < x\}$ is isomorphic to an $\mathbb{N}^*$ pattern of copies of W which is, in turn, isomorphic to W. Hence, $\{x \in W : \overline{(-1)} < x\}$ is transitive and so W is doubly transitive.

For an alternative direct proof, we can instead begin with $x < y \in W$. If the equality level is k then there is a word w of length k and $x', z' \in W$ such that $x = wx', y = wy'$ and $x'_0 < y'_0$. The interval (x, y) in W is contained in wW and is isomorphic via a_w to the interval (x', y') in W. So we may assume that $x_0 < y_0$.

For $k = 1, 2, \ldots$ let $W_{k+} = \{z \in W : x_i = z_i$ for $i = 0, \ldots, k-1$ and $x_k < z_k\}$ so that $W_{k+} \cong W_+ \cong W$ and let $W_{k-} = \{z \in W : y_i = z_i$ for $i = 0, \ldots, k-1$ and $y_k > z_k\}$ so that $W_{k-} \cong W_- \cong W$. For $x_0 < n < y_0$ let $W_{n0} = nW$.

This expresses the interval (x, y) in W as a $\mathbb{Z}$ pattern of copies of W and so it is isomorphic to W. This directly shows that W is weakly homogeneous. It follows from Proposition 3.3 that W is doubly transitive.

By Proposition 2.11 W is of countable type because $\mathbb{Z}^\omega$ is. It then follows from Proposition 3.8 that W is a HLOTS. Since it is a dense subset of the IHLOTS $\mathbb{Z}^\omega$ it is an IHLOTS with completion $\mathbb{R}$.

□

Proposition 8.16. *If X is a proper subset of $\mathbb{R}$ which is invariant under the group of affine maps $x \mapsto q + 2^n \cdot x$ with q varying over $\mathbb{Q}$ and n varying over $\mathbb{Z}$, then X is an IHLOTS. If $\mathbb{Q} \cap X = \emptyset$, then X is isomorphic to an end equivalence saturated subset W of $\mathbb{Z}^\omega$.*

Proof. We apply the construction of Example 8.10.

Observe that t_n is the restriction of an element of the group for all $n \in \mathbb{Z}$. The map $x \mapsto t_n(x - n)$ for $n \leq x \leq n + 1$ defines an isomorphism from X onto $X \cap (-1, 1)$. Invariance implies that X is dense in $\mathbb{R}$ and so has completion $\mathbb{R}$.

First assume that $\mathbb{Q} \cap X = \emptyset$. Let $W = \{s \in \mathbb{Z}^\omega : I_s \subset X\}$. That is, the point in the singleton set I_s lies in X. If w is a finite word, then $t_w(I_s) = I_{ws}$. So the invariance assumption implies that $s \in W \;\Leftrightarrow\; ws \in W$. That is, W is end equivalence saturated. From the definition of the maps t_n it is clear that the set of endpoints C is contained in $\mathbb{Q}$. Because X is disjoint from $\mathbb{Q}$ it follows that for every $x \in X \cap (0, 1)$ there is a unique $s \in W$ such that $I_s = \{x\}$. Thus, $W \cong X \cap (0, 1) \cong X$. Hence, X is an IHLOTS by Theorem 8.15.

If $\mathbb{Q}$ meets X then by invariance, $\mathbb{Q} \subset X$ and so the above result applies to the complement $\mathbb{R} \setminus X$. By Proposition 3.8 X is an IHLOTS because it complement is an IHLOTS.

□

8.5. **The Tree Characterizations.**

Theorem 8.17. *For a LOTS X, the following are equivalent.*

(i) *X is a CHLOTS.*
(ii) *There exists an additive, Ω bounded tree T with $S_0 = \mathbb{Z}$ and $h(T) \geq \omega$, and such that the completion of the branch space $X(T)$ is isomorphic to X.*
(iii) *There exists an additive, Ω bounded tree T with $S_0 = Y$, an IHLOTS with completion $\mathbb{R}$, such that the completion of the branch space $X(T)$ is isomorphic to X.*
(iv) *There exists a reproductive, Ω bounded tree T with S_0 an HLOTS, and such that the completion of the branch space $X(T)$ is isomorphic to X.*

Proof. (i) $\Rightarrow$ (ii): This is Theorem 8.9 applied to our construction above. Note that we require $h(T) \geq \omega$ because with $h(T) = 1$ the simple tree on $\mathbb{Z}$ has $\mathbb{Z}$ as branch space.

(ii) $\Rightarrow$ (iii): If $h(T) = \omega$, then T is the simple tree on $\mathbb{Z}, \omega$ and so $X \cong \mathbb{R}$. As in Example 8.10 we can get $\mathbb{R}$ as the completion of the simple tree on $\mathbb{Q}, \omega$, proving (iii) in this case. We can also use the simple tree on $\mathbb{Q}, 1$.

When $h(T) > \omega$ and so $h(T) \geq \omega^2$, we use the Omega Thinning Construction. By Proposition 6.28 the result is an additive tree of type L_ω. By Proposition 8.11 $W = L_\omega$ is end equivalence saturated and so by Theorem 8.15 W is an IHLOTS with completion $\mathbb{R}$. Finally, Corollary 6.25 implies that the $X(\omega T)$ has the same completion as $X(T)$ and the latter is assumed to be isomorphic to X.

(iii) $\Rightarrow$ (iv): An additive tree is reproductive.

(iv) $\Rightarrow$ (i): Theorem 5.23. □

Remark. For each of the trees described in (ii)-(iv), if α is a countable, tail-like ordinal, then the truncation T^α is a tree of the same sort, e.g. additive or reproductive, and, in addition, with $h(T^\alpha) = \alpha$. It follows from the theorem together with Proposition 8.5 that each $\widehat{X(T^\alpha)}$ is an

$\mathbb{R}$-bounded CHLOTS (In (iv) we must assume that S_0 is $\mathbb{R}$-bounded).

We have seen that if T is an Ω bounded, additive tree of $\mathbb{Z}$ type, then the branch space $X(T)$ is order dense and transitive and it is a dense subset of its completion which is the same as the completion of its Omega thinned tree ωT. Furthermore, $X(\omega T)$ is an IHLOTS and so the completion is a CHLOTS. We do not know whether $X(T)$ itself, though transitive and order dense, is necessarily doubly transitive and so is a HLOTS.

From the theorem we see that via the inductive construction of Theorem 6.10 we can obtain every CHLOTS. We review the construction using alphabet language.

For γ a positive countable ordinal, let $T(\gamma)$ be an additive tree of height ω^γ with $\tilde{L}(\gamma) = \tilde{L}_{\omega^\gamma}$ the set of branches of height ω^γ. By additivity, we have, for $T = T(\gamma), \alpha = \omega^\gamma, \tilde{L} = \tilde{L}(\gamma)$:

$$T \ = \ \{x|\beta : x \in \tilde{L}, \beta < \alpha\}. \tag{8.33}$$

We select $A(\gamma) \subset \tilde{L}(\gamma)$ which is $\pm$ *invariant* in the sense that with $A = A(\gamma), \alpha = \omega^\gamma, T = T(\gamma)$

$$\begin{aligned} x \in A \ \text{ and } \ \beta < \alpha \qquad &\Longrightarrow \qquad \tau_\beta(x) \in A, \\ x \in A \ \text{ and } \ p \in T \qquad &\Longrightarrow \qquad p + x \in A. \end{aligned} \tag{8.34}$$

If we say that $x_1, x_2 \in \tilde{L}(\gamma)$ are *end-equivalent* when there exist $\beta_1, \beta_2 < \omega^\gamma$ such that $\tau_{\beta_1}(x_1) = \tau_{\beta_2}(x_1)$, then $A(\gamma)$ is $\pm$ invariant exactly when it is saturated by the end-equivalence relation.

On the one hand, we let $A(\gamma)$ be the set of vertices of $T(\gamma + 1)$ of order ω^γ. On the other hand, we regard $A(\gamma)$ as an alphabet and let $\tilde{L}(\gamma + 1)$ be all of the elements of $X^{\omega^{\gamma+1}} = X^{\omega^\gamma \cdot \omega}$ which correspond to infinite sequences on the alphabet $A(\gamma)$. Then $T = T(\gamma + 1)$ is defined by (8.33) with $\alpha = \omega^{\gamma+1}, \tilde{L} = \tilde{L}(\gamma + 1)$. It follows from $\pm$ invariance of $A(\gamma)$ that $T(\gamma + 1)$ is an additive tree of height $\omega^{\gamma+1}$ and with $\tilde{L}(\gamma + 1)$ the set of branches of height $\omega^{\gamma+1}$. Furthermore, $T(\gamma) = T(\gamma + 1)^\alpha$ with $\alpha = \omega^\gamma$.

A subset $A(\gamma + 1) \subset \tilde{L}(\gamma + 1)$ is $\pm$ invariant exactly when it corresponds to a set of $A(\gamma)$ sequences which is saturated by the end-equivalence relation on sequences, i.e. the two versions of end-equivalence agree.

Let γ be a limit ordinal. Assume that the additive trees $T(\delta)$ have been defined for all $\delta < \gamma$ with $T(\delta_1) = T(\delta)^\alpha$ with $\alpha = \omega^{\delta_1}$ whenever

$\delta_1 < \delta < \gamma$. Define

$$T(\gamma) \ = \ \bigcup_{\delta < \gamma} T(\delta), \tag{8.35}$$

with $\tilde{L}(\gamma)$ the branches of height ω^γ.

The process stops at a countable ordinal γ if we choose $A(\gamma) = \emptyset$. Otherwise, the process continues to $\gamma = \Omega = \omega^\Omega$ and $A(\Omega) = \emptyset$.

Having defined $T(\gamma), A(\gamma)$ for all positive γ with $\omega^\gamma \leq h(T)$, the branch space is given by

$$X(T) \ = \ \bigcup \{\tilde{L}(\gamma) \setminus A(\gamma) : \omega^\gamma \leq h(T)\}. \tag{8.36}$$

The resulting tree is Ω bounded when $\tilde{L}(\Omega) = \emptyset$. As we saw when we considered the height function in Section 6, we can assure an Ω bounded result, by beginning with R an arbitrary Ω bounded subtree of the simple tree of height Ω and only continuing the construction as long as $T(\gamma)$ remains a subset of R.

8.6. **Trees of Convex Sets.** A convex set J in an order dense LOTS X is *proper* when it has more than one point and so is infinite. In particular, a nonempty open convex set is proper.

Two convex sets J_1, J_2 in X *overlap* when $J_1 \cap J_2$ is proper. If two convex sets do not overlap, then either they are disjoint or their intersection is a singleton consisting of a common endpoint.

We write $J_1 \prec J_2$ when there exist $a, b \in J_1^\circ$ such that $a < c < b$ for all $c \in \overline{J_2}$ and so

$$\overline{J_2} \ \subset \ (a,b) \ \subset \ [a,b] \ \subset \ J_1^\circ. \tag{8.37}$$

Here $\overline{J}$ and J° are the closure and interior, respectively, in the LOTS X.

For $A \subset X$ we let $[A]$ denote the convex closure of A, i.e. the smallest closed, convex set which contains A.

Lemma 8.18. *Let J, J_1, J_2 be proper convex subsets of X an order dense LOTS.*

(a) *The set $\overline{J} \setminus J^\circ$ contains at most two points. The interior J° is dense in J and so is itself a proper open convex set.*
(b) *J_1 overlaps J_2 if and only if $J_1 \cap J_2^\circ \neq \emptyset$.*
(c) *$J_1 \prec J_2$ if and only if there exist $a_2 < a_1 < b_1 < b_2 \in J_1$ such that $a_1 < c < b_1$ for all $c \in J_2$.*
(d) *If X is connected, then $J_1 \prec J_2$ if and only if $\overline{J_2} \subset J_1^\circ$.*

Proof. (a): Let $a \in \overline{J}$.

Case 1: $((-\infty, a) \cap J \neq \emptyset$ and $(a, \infty) \cap J \neq \emptyset.)$ There exist $b_1, c_1 \in J$ with $b_1 < a < c_1$. Choose $b \in (b_1, a), c \in (a, c_1)$ and we have $a, b, c \in J^\circ$ with $a \in (b, c)$.

Case 2: $((-\infty, a) \cap J \neq \emptyset$ and $(a, \infty) \cap J = \emptyset.)$ There exists $b_1 \in J$ with $b_1 < a$. Choose $b \in (b_1, a)$ and we have $[b, a) \subset J^\circ$ and $a = supJ$.

Case 3: $((-\infty, a) \cap J = \emptyset$ and $(a, \infty) \cap J \neq \emptyset.)$ There exists c such that $(a, c] \subset J^\circ$ and $a = infJ$.

Since J is proper, one of these cases applies for every $a \in \overline{J}$. So the only points of $\overline{J} \setminus J^\circ$ are the supremum and infimum of J if either of these exists. So J° is infinite and dense in J.

(b): If $J_1 \cap J_2$ is proper, then by (a) $(J_1 \cap J_2)^\circ$ is infinite. If $J_1 \cap J_2^\circ \neq \emptyset$, then because J_1° is dense in J_1, $J_1^\circ \cap J_2^\circ$ is a nonempty open convex set and so it is infinite.

(c): If $a_2 < a_1 < b_1 < b_2 \in J_1$ with $a_1 < c < b_1$ for all $c \in J_2$, then $\overline{J_2} \subset [a_1, b_1]$ and $[a_1, b_1] \subset (a_2, b_2) \subset J_1^\circ$. If $a \in (a_2, a_1)$ and $b \in (b_1, b_2)$, then a and b satisfy (8.37).

Conversely, if a and b satisfy (8.37), then $[b, \infty) \cap J_1^\circ$ is nonempty and so by (b) it contains a nonempty open interval. Similarly for $(-\infty, a] \cap J_1^\circ$. Since these sets are infinite, we can choose the required a_1, a_2, b_1, b_2.

(d): If X is connected, then with $c_1 = infJ_2, c_2 = supJ_2$, $\overline{J_2} = [c_1, c_2]$. Similarly, $J_1^\circ = (b_1, b_2)$. If $\overline{J_2} \subset J_1^\circ$, then we can choose $a \in (b_1, c_1), b \in (c_2, b_2)$, and then a and b satisfy (8.37).

$\square$

Let $i : X_1 \to X_2$ be an order injection with X_1 order dense and X_2 connected. For J a bounded, proper convex set in X_1 let $[i](J) = [i(J)]$, the convex closure of the image of J. Thus, $[i](J)$ is the closed, bounded, proper interval in X_2 with endpoints the infimum and supremum in X_2 of $i(J)$. In particular, if $a < b \in X_1$, then $[i]([a, b]) = [i(a), i(b)]$.

Lemma 8.19. *Let J_1, J_2 be bounded, proper convex subsets of X_1 an order dense LOTS and $i : X_1 \to X_2$ be an order injection with X_2 connected.*

(a) *J_1 and J_2 overlap in X_1 if and only if $[i](J_1)$ and $[i](J_2)$ overlap in X_2.*

(b) *$J_1 \prec J_2$ in X_1 if and only if $[i](J_1) \prec [i](J_2)$ in X_2.*

Proof. Let $[i](J_1) = [a_1, b_1]$ and $[i](J_2) = [a_2, b_2]$.

(a): $[i](J_1 \cap J_2) \subset [i](J_1) \cap [i](J_2) = [max(a_1, a_2), min(b_1, b_2)]$. So if J_1 and J_2 overlap, then $[i](J_1) \cap [i](J_2)$ is proper. On the other hand, if $[i](J_1)$ and $[i](J_2)$ overlap, then $max(a_1, a_2) < min(b_1, b_2)$. Choose $c \in X_2$ between them. For $\epsilon = 1, 2$, there exist $i(u_\epsilon) \in (a_\epsilon, c) \cap i(J_\epsilon), i(v_\epsilon) \in (c, b_\epsilon) \cap i(J_\epsilon)$ by definition of the sup and inf. Let $u = max(u_1, u_2), v = min(v_1, v_2)$. So $u, v \in [u_\epsilon, v_\epsilon]$ for $\epsilon = 1, 2$, Thus, $[u, v] \subset J_1 \cap J_2$.

(b): $[i](J_1) \prec [i](J_2)$ if and only if $a_1 < a_2 < b_2 < b_1$. We can choose $u_1, u_2, v_1, v_2 \in J_1$ with $a_1 < i(u_2) < i(u_1) < a_2$ and $b_2 < i(v_1) < i(v_2) < b_1$. From Lemma 8.18 (c) it follows that $J_1 \prec J_2$.

Conversely, if $u_2 < u_1 < v_1 < v_2 \in J_1$ such that $u_1 < c < v_1$ for all $c \in J_2$, then $[i](J_2) \subset [i(u_1), i(v_1)] \subset (i(u_2), v(v_2)) \subset [i](J_1)^\circ$ and so $[i](J_1) \prec [i](J_2)$.

$\square$

Remark. If i is the inclusion of X_1 into its completion $X_2 = \widehat{X_1}$, then $[i](J)$ is the closure $\overline{J}$ of J in X_2.

Definition 8.20. *A collection $\mathcal{T}$ of proper convex subsets of an order dense LOTS X is a* tree of convex sets *in X when it satisfies:*

(i) *$X \in \mathcal{T}$ with $J \in \mathcal{T}$ bounded when $J \neq X$.*
(ii) *If J_1 and J_2 are distinct elements of $\mathcal{T}$, then J_1 overlaps J_2 if and only if either $J_1 \prec J_2$ or $J_2 \prec J_1$.*
(iii) *With respect to the ordering $\prec$, $\mathcal{T}$ is a (not necessarily semi-normal) tree.*

The tree $\mathcal{T}$ has X as its root and every other vertex J of $\mathcal{T}$ is bounded by (8.37) since $X \prec J$. We do not assume that (ii) of Definition 5.1 holds, i.e. $J \in \mathcal{T}$ may have a single successor and we do not assume that condition (iii) of Definition 5.1 holds.

For example, for the tree T constructed in the proof of Theorem 8.1, the collection $\{I_s : s \in T\}$ is a tree of convex sets in the connected LOTS $[m, M]$.

Proposition 8.21. *If $\mathcal{T}_1$ is a tree of convex sets in an order dense LOTS X_1 and $i : X_1 \to X_2$ is an order injection with X_2 connected, then $\mathcal{T}_2$ is a tree of closed intervals in X_2 and $f : \mathcal{T}_1 \to \mathcal{T}_2$ is a tree isomorphism with*

$$f(J) = \begin{cases} X_2 \text{ if } J = X_1, \\ [i](J) \text{ if } o(J) > 0, \end{cases} \tag{8.38}$$

and $\mathcal{T}_2$ the image of f.

Proof. This is clear from Lemma 8.19.

□

Theorem 8.22. *If T is a normal tree of unbounded type with $h(T)$ a limit ordinal, then $\mathfrak{T} = \{j_p(X(T_p)) : p \in T\}$ is a tree of clopen convex sets in the branch space $X(T)$. The map j defined by $j(p) = j_p(X(T_p))$ is a tree isomorphism from T to $\mathfrak{T}$. In addition, $\hat{\mathfrak{T}} = \{\overline{j_p((T_p))} : p \in T\}$ is a tree of closed intervals in the completion $\widehat{X(T)}$, with the map f given by $f(j_p(X(T_p))) = \overline{j_p(X(T_p))}$ a tree isomorphism from $\mathfrak{T}$ to $\hat{\mathfrak{T}}$.*

Proof. By Proposition 5.4 the branch space $X(T)$ is order complete. By Proposition 5.5 (c) each $j_p(X(T_p))$ is a clopen convex set in $X(T)$ with completion the open interval $\hat{j}_p(\widehat{X(T_p)})$ in $\widehat{X(T)}$. Since T is of unbounded type, each $j_p(X(T_p))$ is bounded for $p \neq 0$. By the Remark following Lemma 8.19, the common closure of $j_p(X(T_p))$ and $\hat{j}_p(\widehat{X(T_p)})$ in $\widehat{X(T)}$ is $[i](j_p(X(T_p)))$ where i is the inclusion of $X(T)$ into its completion.

We check (i)-(iii) for $\mathfrak{T}$.

For the root 0, $T_0 = T$ and $j_0(X(T_0)) = X(T)$, verifying (i).

If p_1 and p_2 are distinct vertices of T, then $p_1 \prec p_2$ implies $T_{p_2} \subset T_{p_1}$ and there exist $q_1 < q_2 < q_3 \in S_{p_1}$ with either $q_2 = p_2$ or $q_2 \prec p_2$. Let a, b be branches through q_1 and q_3. Then (8.37) is satisfied showing that $j_{p_1}(X(T_{p_1})) \prec j_{p_2}(X(T_{p_2}))$ since the convex sets are clopen.

On the other hand, if $j_{p_1}(X(T_{p_1})) \cap j_{p_2}(X(T_{p_2}))$ is nonempty and so there exists a branch which contains both p_1 and p_2, then either $p_1 \prec p_2$ or $p_2 \prec p_1$.

This implies (ii) and shows that $j : T \to \mathfrak{T}$ is an order isomorphism. Hence, $\mathfrak{T}$ is a tree, verifying (iii).

Finally, the completion results follow from Proposition 8.21.

□

Now we apply these results to show that the branch space of an Aronszajn tree is not $\mathbb{R}$-bounded.

Theorem 8.23. *Let α be a countable ordinal and let $\mathfrak{T}$ be a tree of convex sets in the connected LOTS $\mathbb{R}_\alpha$. If every level of $\mathfrak{T}$ is countable, then $\mathfrak{T}$ is countable.*

Proof. By applying Proposition 8.21 to i equal the identity map on $\mathbb{R}_\alpha$ we can assume that $\mathfrak{T}$ is a tree of closed intervals in $\mathbb{R}_\alpha$. So for $J \in \mathfrak{T}$ with $o(J) > 0$, we have $J = [a(J), b(J)]$ with $a(J) < b(J) \in \mathbb{R}_\alpha$.

Since every level is countable, it suffices to prove that the height $h(\mathcal{T})$ is countable.

If x is a branch of $\mathcal{T}$, then $J \mapsto a(J)$ is an embedding of the ordinal $h(x)$ into $\mathbb{R}_\alpha$. Since α is countable, $\mathbb{R}_\alpha$ is first countable and so Ω cannot be injected into $\mathbb{R}_\alpha$. It follows that $h(x)$ is countable.

For $J \in \mathcal{T}$, let $\epsilon(J)$ be the equality level for $a(J)$ and $b(J)$ so that $a(J)_i = b(J)_i$ for $i < \epsilon$ and $a(J)_\epsilon < b(J)_\epsilon$ with $\epsilon = \epsilon(J)$. Hence, $\epsilon(J) < \alpha$.

We let $r(J) \in \mathbb{R}_\epsilon$ be the common restriction of $a(J)$ and $b(J)$ to ϵ and we call $r(J)$ the *stem* of J. We write $span(J) = [a(J)_\epsilon, b(J)_\epsilon]$ so that $span(J)^\circ = (a(J)_\epsilon, b(J)_\epsilon)$. These are proper intervals in $\mathbb{R}$. Notice that if $q \in span(J)^\circ$, then any $c \in \mathbb{R}_\alpha$ with $c_i = a(J)_i = b(J)_i$ for $i < \epsilon$ and $c_\epsilon = q$ satisfies $a(J) < c < b(J)$ and so $c \in J^\circ$. For example, $q+ \in J^\circ$.

Now we prove by induction that for every $\beta < \alpha$, there exists $\xi_\beta < \Omega$ such that $o(J) > \xi_\beta$ implies $\epsilon(J) > \beta$.

This is trivial for $\beta = 0$.

Define ρ_β to be the smallest ordinal greater than ξ_γ for all $\gamma < \beta$. Thus,

$$\rho_\beta \ = \ sup\ \{\xi_\gamma : \gamma < \beta\} + 1 \tag{8.39}$$

Now suppose $r \in \mathbb{R}_\beta$ and there exists $J \in \mathcal{T}$ such that $stem(J) = r$. Let J be an element of $\mathcal{T}$ with minimum order such that $stem(J) = r$. If $J_1 \prec J$ in $\mathcal{T}$, then $a(J_1) < a(J) < b(J) < b(J_1)$ and so $\epsilon(J_1) \leq \epsilon(J)$. Furthermore, $\epsilon(J_1) = \epsilon(J)$ would imply $stem(J_1) = stem(J)$ violating the minimality condition on J. Hence, $\gamma = \epsilon(J_1) < \epsilon(J) = \beta$. From the inductive hypothesis it follows that $o(J_1) \leq \xi_\gamma$. Therefore,

$$o(J) \ \leq \ sup\,\{o(J_1) : J_1 \prec J\} + 1 \ \leq \ sup\,\{\xi_\gamma : \gamma < \beta\} + 1 \ = \ \rho_\beta. \tag{8.40}$$

Because each level of $\mathcal{T}$ is countable, there are only countably many $J \in \mathcal{T}$ with $o(J) \leq \rho_\beta$ and so the set $A_\beta = \{r \in \mathbb{R}_\beta : r = r(J)$ for some $J \in \mathcal{T}\}$ is countable.

Claim: For each $r \in A_\beta$ there are only countably many $J \in \mathcal{T}$ such that $r(J) = r$.

Suppose instead that for some $r \in A_\beta$ there are uncountably many such J. It follows that for some $q \in \mathbb{Q}$ there is an uncountable $\mathcal{X} \subset \mathcal{T}$ such that for $J \in \mathcal{X}$ $r(J) = r$ and $q \in span(J)^\circ$. So $q+ \in J^\circ$ for all $J \in \mathcal{X}$. Thus, any two members of $\mathcal{X}$ overlap and so by condition (ii) of Definition 8.20, any two are $\prec$ comparable. It follows that $\mathcal{X}$ is contained in an uncountable branch of $\mathcal{T}$ which is, as we have seen above, impossible. This proves the Claim.

Now let $\xi_\beta \geq \rho_\beta$ and $\xi_\beta \geq o(J)$ for all J such that $r(J) = r$ for some $r \in A_\beta$.

If $\epsilon(J) = \gamma < \beta$, then $o(J) \leq \xi_\gamma < \rho_\beta \leq \xi_\beta$.

If $\epsilon(J) = \beta$, then $r(J) \in A_\beta$ and so $o(J) \leq \xi_\beta$.

This completes the inductive step.

The height of $\mathcal{T}$ is bounded by $sup\{\xi_\beta : \beta < \alpha\}$ and so is countable. $\square$

Corollary 8.24. *If T is an Aronszajn tree of unbounded type, then $X(T)$ is not $\mathbb{R}$-bounded.*

Proof. Assume that T is a normal tree of unbounded type with height a limit ordinal and that every level of T is countable. Suppose that $i : X(T) \to \mathbb{R}_\delta$ is an order injection with δ countable.

By Theorem 8.22, there is a tree $\mathcal{T}_1$ of convex sets in $X(T)$ which is isomorphic with T itself. By Proposition 8.21 the injection i induces an isomorphism of $\mathcal{T}_1$ onto a tree $\mathcal{T}_2$ of closed intervals in $\mathbb{R}_\delta$. By Theorem 8.23 the tree $\mathcal{T}_2$ is countable since every level is countable. This in turn implies that T is countable and so with countable height.

However, an Aronszajn tree is uncountable with height Ω. $\square$

9. HLOTS in $\mathbb{R}$

9.1. **Comparisons Along the Tower.** We begin with a pair of useful isomorphisms.

Recall that if X is an IHLOTS with completion $F = \hat{X}$, then by Proposition 3.8(d), the complement $F \setminus X$ is an IHLOTS with the same completion. Also if α is an infinite, tail-like ordinal then by Proposition 6.3, X^α is order isomorphic to a dense subset of X_α. If, in addition, α is countable, then X^α and X_α are HLOTS by Corollary 6.2 and Theorem 4.2.

Lemma 9.1. *Let X be a IHLOTS with completion the CHLOTS F and let α be a countably infinite, tail-like ordinal. The complement $\widehat{X^\alpha} \setminus X^\alpha$ is an IHLOTS and the IHLOTS $(\widehat{X^\alpha} \setminus X^\alpha)^\alpha$ has completion isomorphic to F_α.*

Proof. Let $Y = F \setminus X$ be the complementary IHLOTS. Assume that the interval $J = [-1, +1] \subset F$ has been chosen so that $-1, +1 \in Y$.

Let $J(X) = X \cap J = X \cap J^{\circ}$ and $J(Y) = Y \cap J$. Since X is a HLOTS, $X \cong J(X)$ and so $X^{\alpha} \cong J(X)^{\alpha}$. By applying Proposition 5.9 to the simple tree, it is easy to see that the complement of $J(X)^{\alpha}$ in its completion can be identified with

$$(9.1)\qquad \begin{aligned} Z \quad = \quad & \{z \in J^{\beta+1} : \text{for some } \beta < \alpha \\ & \text{with } z(j) \in J(X) \text{ for all } j < \beta \text{ and } z(\beta) \in J(Y)\} \end{aligned}$$

except that Z includes the endpoints $m, M \in J^1$ with $m(0) = -1$ and $M(0) = +1$. With $Z^{\circ} = Z \setminus \{m, M\}$ we apply Proposition 6.3 to the HLOTS Z°, to see that the completion of $(Z^{\circ})^{\alpha}$ is the same as the completion of $(Z^{\circ})_{\alpha}$. Since the distinguished closed bounded interval in Z° is isomorphic to Z we can identify $(Z^{\circ})_{\alpha}$ with

$$(9.2)\qquad \tilde{Z} \quad = \quad \{x \in Z^{\alpha} : x(0) \in Z^{\circ}\}.$$

We will show that $\tilde{Z}$ is order isomorphic to

$$(9.3)\qquad \begin{aligned} D \quad &= \quad \{x \in J^{\alpha} : x(0) \neq \pm 1 \text{ and } \gamma(x) \cong \alpha\} \\ \text{where} \quad \gamma(x) \quad &= \quad \{i \in \alpha : x(i) \in J(Y)\}. \end{aligned}$$

Since α is tail-like, D includes every $x \in J^{\alpha}$ such that $x(0) \neq \pm 1$ and $x(i) \in J(Y)$ for sufficiently large $i \in \alpha$. So D is dense in $\{x \in J^{\alpha} : x(0) \neq \pm 1\} \cong F_{\alpha}$.

Given $x \in D$ let $\tau_x : \alpha \to \gamma(x)$ be the (unique) order isomorphism which exists by definition of D.

Now we use a variation of the construction of (6.55).

If $\tau : \alpha \to \alpha$ is an order injection, then we define $\tilde{\tau}(0) = 0$ and for $0 < i < \alpha$:

$$(9.4)\qquad \begin{aligned} \tilde{\tau}(i) = sup\{\tau(j) + 1 : j < i\} &= min\{k : k > \tau(j) \text{ for all } j < i\}, \\ \text{So that } \ \tilde{\tau}(i+1) &= \tau(i) + 1, \text{ and} \\ \tilde{\tau}(i) = sup\{\tau(j) : j < i\} & \text{ for } i \text{ a limit ordinal.} \end{aligned}$$

By induction, $\tau(i) \geq i$ and so the image of τ is cofinal in α. From Proposition 2.15(c) follows that $\{[\tilde{\tau}(i), \tau(i)] = [\tilde{\tau}(i), \tilde{\tau}(i+1)) : i < \alpha\}$ is a α indexed partition of α by closed intervals. Following (2.7) we can identify α with the associated sum:

$$(9.5)\qquad \sum_{i<\alpha} [\tilde{\tau}(i), \tilde{\tau}(i+1)) \ \cong \ \alpha.$$

For each $i \in \alpha$ identify the interval $[\tilde{\tau_x}(i), \tilde{\tau_x}(i+1)) \subset \alpha$ with the ordinal $\beta(x,i) + 1 = \tilde{\tau_x}(i+1) \setminus \tilde{\tau_x}(i)$ which has its order type. The restriction of x to this interval yields an element $z(i) \in Z$ and $i \mapsto z(i)$ defines the element of $\tilde{Z}$ which we associate with x. The procedure is order preserving and reversible and so defines the required isomorphism.

□

Lemma 9.2. *Let F be a CHLOTS and $\{f_n : F \to F \quad \text{with } n \in \omega \setminus 1\}$ be a sequence of order surjections. Let Y be the connected LOTS which is the inverse limit of the special inverse system indexed by ω, defined by the sequence $\{f_n\}$. Thus, $x \in \prod_{n\in\omega} F$ is in Y if and only if $f_{n+1}(x_{n+1}) = x_n$ for all $n \in \omega$.*

Let α be a tail-like ordinal with $\alpha \geq \omega^\omega$. With J the distinguished subinterval in F we let $J_0 = J$ and inductively define $J_{n+1} = (f_{n+1})^{-1}(J_n)$. The points of Y whose n^{th} coordinate lies in J_n for all $n \in \omega$ define a compact interval J_Y of the connected LOTS Y and we use it to define the product space Y_α. Then:

$$Y_\alpha \quad \cong \quad F_\alpha. \tag{9.6}$$

Proof. A point $z \in Y_\alpha$ is indexed by the order space product $\alpha \times \omega$ which is order isomorphic to the ordinal product $\omega \cdot \alpha$,

$$z(i,n) \in X(i,n) \quad = \quad \begin{cases} F & \text{if } i = 0 \\ J_n & \text{if } 0 < i < \alpha. \end{cases} \tag{9.7}$$

$$\text{with} \quad f_{n+1}(z(i,n+1)) \; = \; z(i,n) \qquad \text{for } (i,n) \in \alpha \times \omega.$$

For $(i,n) > (0,0)$ in $\alpha \times \omega$ we let $z(<(i,n))$ denote the projection of the point to the subproduct $\Pi\{X(j,m) : (j,m) < (i,n)\}$ and let

(9.8)
$$I(z(<(i,n))) \; = \; \{w(i,n) : w \in Y_\alpha \text{with } w(<(i,n)) = z(<(i,n))\}.$$

$$\text{So that} \quad I(z(<(i,n))) = \begin{cases} J & \text{if } n = 0 \\ (f_n)^{-1}(z(i,n-1)) & \text{if } n > 0. \end{cases}$$

Thus, each $I(z(<(i,n)))$ is a closed bounded interval in $X(i,n)$.

Now let

$$\gamma(z) \quad = \quad \{(0,0)\} \cup \{(i,n) : I(z(<(i,n))) \text{ is nontrivial}\}. \tag{9.9}$$

By (9.8) $(i,0) \in \gamma(z)$ for all $z \in Y_\alpha$. It follows that, identifying $\gamma(z)$ with the ordinal which is its order type,

$$\alpha \quad \leq \quad \gamma(z) \quad \leq \quad \omega \cdot \alpha \quad = \quad \alpha, \tag{9.10}$$

with the latter equation from $\alpha = \omega^\rho$ with $\rho \geq \omega$ so that $1 + \rho = \rho$. Let

$$\tau_z : \alpha \to \alpha \times \omega \tag{9.11}$$

denote the unique order isomorphism onto $\gamma(z)$.

Notice that $z(< (i,n)) = w(< (i,n))$ implies $\gamma(z)$ and $\gamma(w)$ agree through (i,n) and

$$\tau_z(\beta) = \tau_w(\beta) \qquad \text{when} \quad \tau_z(\beta) \leq (i,n). \tag{9.12}$$

For $(i,n) \in \gamma(z)$ choose $g_{z(<(i,n))} : I(z(< (i,n))) \to J$ an order isomorphism which exists because F is a CHLOTS.

We now define for $z \in Y_\alpha$ the associated point $Q(z) \in F_\alpha$

$$Q(z)_\beta \quad = \quad g_{z(<(i,n))}(z(i,n)) \qquad \text{where } (i,n) \ = \ \tau_z(\beta). \tag{9.13}$$

If $z < w$ in Y_α and (i,n) is the smallest coordinate where $z(i,n) \neq w(i,n)$, then $z(i,n) < w(i,n)$ and since both of these points are in $I(z(< (i,n))) = I(w(< (i,n)))$ it follows that $(i,n) \in \gamma(z) \cap \gamma(w)$. With $(i,n) = \tau_z(\beta) = \tau_w(\beta)$ we have $Q(z)_\epsilon = Q(w)_\epsilon$ for all $\epsilon < \beta$ and $Q(z)_\beta < Q_w(\beta)$. Thus, Q is an order injection.

Conversely, given $x \in F_\alpha$ we inductively define the associated point $z \in Y_\alpha$ and the order injection τ_z. Begin with $z(0,0) = x_0$ and $\tau_z(0) = (0,0)$. Now for $0 < \beta < \alpha$ we will define $\tau_z(\beta) = (i,n)$ and $z(k,m) \in X(k,m)$ for all $(k,m) \leq (i,n)$ so that

$$\begin{aligned} (k,m) \ &= \ \tau_z(\epsilon) \quad \text{for some } 0 < \epsilon \leq \beta \\ \iff \qquad & I(z(< (k,m))) \text{ is nontrivial.} \end{aligned} \tag{9.14}$$

Now assume that the definitions have been completed for all $\epsilon < \beta$.

Case 1: If β is a limit ordinal, then let

$$\begin{aligned} \tau_z(\beta) \quad &= \quad sup\ \{\tau_z(\epsilon) : \epsilon < \beta\} = (i,0) \\ z(i,0) \quad &= \quad (g_{z(<(i,0))})^{-1}(x_\beta). \end{aligned} \tag{9.15}$$

Notice that the only limit elements of $\alpha \times \omega$ are of the form $(i,0)$.

Case 2: If $\beta = \epsilon + 1$ and $\tau_z(\epsilon) = (k,m)$, then by finite induction we define for $n = 0, 1, ...$

$$\{z(k,m+n+1)\} \ = \ (f_{m+n+1})^{-1}(z(k,m+n)) \ = \ I(z(< (k,m+n+1))) \tag{9.16}$$

if the preimage is a singleton. There are two possibilities.

If this procedure stops for some finite n, then we define

$$\begin{aligned} \tau_z(\beta) \quad &= \quad (k, m+n+1) \\ z(k,m+n+1) \quad &= \quad (g_{z(<(k,m+n+1))})^{-1}(x_\beta). \end{aligned} \tag{9.17}$$

If this procedure continues for all finite n, then we define

$$\begin{aligned} \tau_z(\beta) \quad &= \quad (k+1, 0) \\ z(k+1,0) \quad &= \quad (g_{z(<(k+1,0))})^{-1}(x_\beta). \end{aligned} \tag{9.18}$$

Notice that in both (9.17) and (9.18) $0 < \beta$ implies $x_\beta \in J$.

The order injection $\beta \mapsto \tau_z(\beta)$ from α into $\alpha \times \omega \cong \alpha$ has a cofinal image and so we obtain a point $z \in Y_\alpha$. Clearly, $Q(z) = x$ and so Q is the required order isomorphism.

□

For the special case where all of the maps f_n are equal to a fixed map f we can define on Y the *shift automorphism* and its inverse f_* by

$$\tau(x)_n \; = \; x_{n+1}, \qquad f_*(x)_n = f(x_n) \tag{9.19}$$

From (2.26) it follows that these are order isomorphisms.

Now we apply these preliminary results. Recall that the size of X lies between X_1 and X_2 if X_1 injects into X and X injects into X_2.

Theorem 9.3. *Assume that F_1 and F_2 are CHLOTS*

(a) *If F_1 and F_2 have the same size and if α is a tail-like ordinal with $\alpha \geq \omega^\omega$, then*

$$(F_1)_\alpha \quad \cong \quad (F_2)_\alpha. \tag{9.20}$$

(b) *If for some countable ordinal β, the size of F_1 lies between F_2 and $(F_2)_\beta$ and if α is a sufficiently large countable tail-like ordinal, then the isomorphism of (9.20) holds. Specifically, for ordinals i, j*

$$\begin{aligned} &\textit{if} \quad \beta \cdot \omega^i \; = \; \omega^i \qquad \textit{and} \qquad j \geq (i + \omega) \\ &\textit{then} \quad \alpha \; = \; \omega^j \; \textit{implies} \quad (F_1)_\alpha \cong (F_2)_\alpha. \end{aligned} \tag{9.21}$$

Proof. (a): By Corollary 4.5 there exist continuous order surjections $g_{21} : F_1 \to F_2$ and $g_{12} : F_2 \to F_1$. Let $g_1 = g_{12} \circ g_{21} : F_1 \to F_1$ and $g_2 = g_{21} \circ g_{12} : F_2 \to F_2$. Let Y_1 be the inverse limit of the special inverse system indexed by ω with $f_n = g_1$ for all $n \in \omega$ and similarly for Y_2. By Lemma 9.2,

$$(F_1)_\alpha \; \cong \; (Y_1)_\alpha \qquad \text{and} \qquad (F_2)_\alpha \; \cong \; (Y_2)_\alpha. \tag{9.22}$$

Now define the order surjection $\tilde{g}_{21} : Y_1 \to Y_2$ to be a copy of g_{21} on each coordinate, and similarly define $\tilde{g}_{12} : Y_2 \to Y_1$. The compositions each way are the shift automorphisms on the inverse limits. So we have $Y_1 \cong Y_2$. Hence, $(Y_1)_\alpha \cong (Y_2)_\alpha$ which completes the required chain of isomorphisms.

(b): As in (2.12) $\beta = \omega^{k_1} + ... + \omega^{k_N}$ with $k_1 \geq ... \geq k_N$. If i is a tail-like ordinal with $i > k_1$, then

$$\omega^i \ \leq \ \beta \cdot \omega^i \ \leq \ \omega^{k_1+1} \cdot \omega^i \ = \ \omega^{k_1+1+i} \ = \ \omega^i. \tag{9.23}$$

Now from Proposition 4.7(c),(d) we have, with $\gamma = \omega^i$ that

$$\begin{aligned} ((F_2)_\beta)_\gamma &\text{ is at least as big as } (F_1)_\gamma, \\ (F_1)_\gamma &\text{ is at least as big as } (F_2)_\gamma, \\ \text{and} \quad ((F_2)_\beta)_\gamma &\cong \ (F_2)_\gamma \end{aligned} \tag{9.24}$$

since $\beta \cdot \gamma = \gamma$. Thus, $(F_1)_\gamma$ and $(F_2)_\gamma$ have the same size.

Now let $\tilde{j} = j \setminus i$ so that $i + \tilde{j} = j$. By assumption $\tilde{j} \geq \omega$ and so $\tilde{\alpha} = \omega^{\tilde{j}} \geq \omega^\omega$. So by part (a) we have

$$((F_2)_\gamma)_{\tilde{\alpha}} \quad \cong \quad ((F_1)_\gamma)_{\tilde{\alpha}}. \tag{9.25}$$

Since $\alpha = \gamma \cdot \tilde{\alpha}$ the result follows from Proposition 4.7(d) again.

□

Remark. The interest in part (a) of the theorem comes from the fact that there exists CHLOTS F_1 and F_2 which are of the same size, i.e. each can be order injected into the other, but which are not order isomorphic, e.g. see Proposition 9.9 below.

Recall that a LOTS X is $\mathbb{R}$-bounded when it admits an order injection into $\mathbb{R}_\delta$ for some countable ordinal δ. Proposition 8.5 implies that if X is an $\mathbb{R}$-bounded HLOTS, then for any countable tail-like ordinal β, $\widehat{X_\beta}$ is an $\mathbb{R}$-bounded CHLOTS.

Corollary 9.4. *If X is an $\mathbb{R}$-bounded CHLOTS, then for α a sufficiently large countable tail-like ordinal $X_\alpha \cong \mathbb{R}_\alpha$.*

Proof. $\mathbb{R}$ injects into any CHLOTS and so if X is an $\mathbb{R}$-bounded CHLOTS, it has size between that of $\mathbb{R}$ and that of $\mathbb{R}_\delta$ for some countable δ. The result follows from Theorem 9.3 (b).

□

If X is an IHLOTS with completion F, then the size of the completion of X_ω lies between F and F_ω. The first conjecture might be that it is order isomorphic with F_ω but we will now see that this is rarely true. We will require some preliminary study of the completion of the elements of the tower over X.

Let X be an IHLOTS with completion the CHLOTS F and let Y be the complementary IHLOTS, i.e. $Y = F \setminus X$. Assume $-1 < +1$ in X

and that J is the interval $[-1,+1] \subset F$. Let α be a limit ordinal. We can regard X_α as the branch space of the subtree T of the simple tree on X, α with $S_0 = X$ but $S_p \cong J \cap X$ instead of all of X. When we apply Proposition 5.9 we can identify the completion with the branch space of the tree completion $\hat{T}$. So we will use

$$\begin{aligned} \widehat{X_\alpha} \quad &= \quad X_\alpha \,\cup\, Y^1 \,\cup \\ &\bigcup_{0<\beta<\alpha} \{x \in F^{\beta+1} : x(i) \in \begin{cases} X & \text{for } i = 0, \\ J \cap X & \text{for } 0 < i < \beta, \\ J \cap Y & \text{for } i = \beta \end{cases} \}. \end{aligned} \tag{9.26}$$

With these identifications the canonical projections for $0 < \beta \le \alpha$

$$\begin{aligned} \pi^\alpha_\beta &: X_\alpha \to X_\beta, \\ \hat{\pi}^\alpha_\beta &: \widehat{X_\alpha} \to \widehat{X_\beta} \end{aligned} \tag{9.27}$$

are the obvious restriction maps. With $\beta = 1$ we identify the completion with F and so define $\hat{\pi}^\alpha : \widehat{X_\alpha} \to F$ which we write as $\hat{\pi}$ when the subscript is unambiguous. We use it to define, for $y \in F$

$$P(y) \quad = \quad \{z \in \widehat{X_\alpha} : z(0) = y\} \;=\; (\hat{\pi})^{-1}(y). \tag{9.28}$$

Since $\hat{\pi}$ is a continuous order surjection, $P(y)$ is a nonempty compact interval in $\widehat{X_\alpha}$ for all $y \in F$.

We carry over the tree concept of *height*, defining for $x \in \widehat{X_\alpha}$

$$h(x) \;=\; \begin{cases} \alpha & \text{for } x \in X_\alpha \\ \beta + 1 & \text{for } x \in F^{\beta+1}. \end{cases} \tag{9.29}$$

We identify Y with the elements of height 1 by $x \mapsto x(0)$, so that $Y \subset \widehat{X_\alpha}$.

If $h(x) \ge \epsilon$, then we will write $x|\epsilon$ for $\hat{\pi}^\alpha_\epsilon(x)$, i.e. the restriction of the map x to the subset ϵ of its domain. Clearly,

$$h(x) > \epsilon \qquad \Longleftrightarrow \qquad \hat{\pi}^\alpha_\epsilon(x) \in X_\epsilon. \tag{9.30}$$

Now suppose $0 < \epsilon < \alpha$ and $w \in X_\epsilon$. We define the compact subinterval $J_w \subset \widehat{X_\alpha}$ and the map $\hat{\pi}_w : J_w \to J$ by

$$\begin{aligned} J_w \quad &= \quad (\hat{\pi}^\alpha_\epsilon)^{-1}(w) \;=\; \{z \in \widehat{X_\alpha} : z|\epsilon = w\} \;=\; [w-, w+], \\ \hat{\pi}_w(z) \quad &= \quad z(\epsilon). \end{aligned} \tag{9.31}$$

It is clear from description (9.26) that $\hat{\pi}_w$ is an order surjection for all $w \in X_\epsilon$. In particular, J_w is nontrivial. For $w \in X_\epsilon, y \in J$ we extend definition (9.28)

$$P_w(y) \quad = \quad \{z \in J_w : z(\epsilon) = y\} \;=\; (\hat{\pi}_w)^{-1}(y). \tag{9.32}$$

As before each $P_w(y)$ is a nonempty compact subinterval of $J_w \subset \widehat{X_\alpha}$.

Now assume that $I = [x, y]$ is a nontrivial, closed interval in $\widehat{X_\alpha}$, so that $x < y$. We denote by $\epsilon(I)$ the equality level of the pair x, y, i.e. $min\ \{i : x_i \neq y_i\}$. With $\epsilon = \epsilon(I)$ we have, for all $z \in I$

$$\begin{gathered} h(z) > \epsilon, \\ x|\epsilon \ = \ z|\epsilon \ = \ y|\epsilon, \qquad \text{and} \\ x(\epsilon) \ \leq \ z(\epsilon) \ \leq \ y(\epsilon), \end{gathered} \tag{9.33}$$

with at least one of the latter inequalities strict. We call the common element of F^ϵ the *stem of* I. From (9.30) we have

$$stem(I) \ \in \ X_{\epsilon(I)}. \tag{9.34}$$

Finally, we define

$$\begin{aligned} span(I) \quad &= \quad [x(\epsilon), y(\epsilon)] \\ span^\circ(I) \quad &= \quad (x(\epsilon), y(\epsilon)). \end{aligned} \tag{9.35}$$

So that $span(I)$ is a nontrivial, compact subinterval of F and is contained in J when $\epsilon > 0$. Its interior $span^\circ(I)$ is nonempty.

The most important special case occurs when $\epsilon(I) = 0$ in which case $stem(I) = \emptyset$. Clearly, with $I = [x, y]$

$$\begin{gathered} \epsilon(I) = 0 \quad \Longleftrightarrow \quad x(0) < y(0) \\ \text{in which case} \\ span(I) \quad = \quad [x(0), y(0)] \quad = \quad \hat{\pi}(I) \end{gathered} \tag{9.36}$$

Notice that (9.33) implies that

$$Y \cap I \ \neq \ \emptyset \qquad \Rightarrow \qquad \epsilon(I) = 0. \tag{9.37}$$

Lemma 9.5. *Let I, I_1, I_2 be nontrivial, closed subintervals of $\widehat{X_\alpha}$.*

(a) *Given $y \in F$ the compact interval $P(y)$ is trivial, i.e. is a singleton, if and only if $y \in Y$. Given $\epsilon > 0$, $w \in X_\epsilon$ and $y \in J$ the compact interval $P_w(y)$ is trivial, i.e. is a singleton, if and only if $y \in Y \cap J$.*

(b) *If $z \in \widehat{X_\alpha}$ with $\hat{\pi}^\alpha_{\epsilon(I)}(z) = stem(I)$, then $h(z) > \epsilon(I)$. If, in addition, $z(\epsilon(I)) \in span^\circ(I)$, then $z \in I$.*

(c) *If I_1 and I_2 are disjoint subintervals such that $\epsilon(I_1) = \epsilon(I_2)$ and $stem(I_1) = stem(I_2)$, then $span(I_1)$ and $span(I_2)$ are non-overlapping subintervals of F, i.e.*

$$span^\circ(I_1) \ \cap \ span^\circ(I_2) \quad = \quad \emptyset \tag{9.38}$$

Proof. (a): This is obvious from the identification (9.26).

(b): Let $I = [x, y]$. By (9.34) and (9.30) $\hat{\pi}^{\alpha}_{\epsilon(I)}(z) = stem(I)$ implies $h(z) > \epsilon(I)$ and $x|\epsilon = z|\epsilon = y|\epsilon$. If, in addition, $z \in span^{\circ}(I)$, then $x(\epsilon) < z(\epsilon) < y(\epsilon)$ and so $z \in (x, y)$.

(c): Obvious from (a) and (b).

□

Theorem 9.6. *Let X be an IHLOTS with completion the CHLOTS F and let Y be the complementary IHLOTS $Y = F \setminus X$. If for any pair α and β of countable limit ordinals the completion $\widehat{X_\alpha}$ is order isomorphic to F_β, then X is a first category subset of F. That is, Y contains a dense G_δ subset of F.*

Proof. Assume that $f : \widehat{X_\alpha} \to F_\beta$ is an order surjection. By Proposition 2.3(a) f is continuous and topologically proper. We will show that if Y does not contain a particular dense G_δ set which we will construct, then f is not injective.

Using the projection $\pi^{\beta}_i : F_\beta \to F_i$ for $0 < i < \beta$ we define, for each $y \in Y \subset \widehat{X_\alpha}$

$$Q(y, i) \quad = \quad (\pi^{\beta}_i \circ f)^{-1}(\pi^{\beta}_i(f(y))) \;=\; f^{-1}(J_{f(y)|i}), \tag{9.39}$$

where the latter equation uses a definition analogous to (9.31). For any $w \in F_i$, $J_w = (\pi^{\beta}_i)^{-1}(w)$ is a nontrivial, compact interval in F_β and so $Q(y, i)$ is a nontrivial, compact interval in $\widehat{X_\alpha}$.

Clearly, for $y_1, y_2 \in Y$ and $i < \beta$

$$\begin{aligned} Q(y_1, i) \cap Q(y_2, i) \;\neq\; \emptyset \quad &\Rightarrow \\ f(y_1)|i = f(y_2)|i \quad &\Rightarrow \\ Q(y_1, i) \;=\; Q(y_2, i).& \end{aligned} \tag{9.40}$$

Because $y \in Q(y, i)$, (9.37) implies $\epsilon(Q(y, i)) = 0$ for all $y \in Y$ and $0 < i < \beta$. It follows from (9.36), (9.40) and Lemma 9.5(c) that distinct members of the set of intervals

$$\mathcal{Q}_i \quad = \quad \{span(Q(y, i)) : y \in Y\} \;=\; \{\pi(Q(y, i)) : y \in Y\} \tag{9.41}$$

are non-overlapping. Since $y \in span(Q(y, i))$ and Y is dense in X, $\mathcal{Q}_i$ has a dense union for each positive $i < \beta$. As each member of $\mathcal{Q}_i$ is nontrivial the open set

$$O_i \quad = \quad \bigcup \{span^{\circ}(Q(y, i)) : y \in Y\} \tag{9.42}$$

is dense in F for each positive $i < \beta$.

Since F is locally compact and β is countable, the Baire Category Theorem implies that $D = \cap O_i$ is a dense G_δ subset of F. If there exists $t \in X \cap D$ then by Lemma 9.5(a) the interval $P(t)$ is nontrivial. We will show that f is constant on $P = P(t)$. It suffices to show that $\pi_i^\beta \circ f$ is constant on P for each $i < \beta$.

Since $t \in O_i$ there exists $y \in Y$ such that $t \in span^\circ(Q(y,i))$. With $I = Q(y,i), \epsilon(I) = 0$, and $stem(I) = \emptyset$. If $x \in P$, then $x(\epsilon(I)) = \hat{\pi}(x) = t \in span^\circ(I)$. So Lemma 9.5(b) implies that $x \in I$. That is, $P \subset Q(y,i)$, and so $\pi_i^\beta(f(x)) = \pi_i^\beta(f(y))$ for all $x \in P$.

Thus, f can be injective only when $D \subset Y$.

□

9.2. **IHLOTS in $\mathbb{R}$.** Now we consider examples which are constructed from IHLOTS in $\mathbb{R}$. First we will see that there are many such.

Proposition 9.7. *Let $\mathcal{G}$ be the countable group of positive,affine transformations of $\mathbb{R}$ with rational coefficients, i.e. of the form $t \mapsto at + b$ with $a, b \in \mathbb{Q}$ and $a > 0$. If X is a nonempty, proper subset of $\mathbb{R}$ which is invariant with respect to the action of $\mathcal{G}$, then X is an IHLOTS. In particular, any proper subfield of $\mathbb{R}$ is an IHLOTS as is its complement.*

Proof. Assume first that X contains some rational number and so that $\mathbb{Q} \subset X$. Apply Lemma 5.22 with $W = \mathbb{Q}$ to see that X is a HLOTS. If $\mathbb{Q} \cap X = \emptyset$, then the same result show that $\mathbb{R} \setminus X$ is a HLOTS and so by Proposition 3.8(d) X itself is a HLOTS. Since X and its complement are clearly dense they are both IHLOTS with completion $\mathbb{R}$. A subfield of $\mathbb{R}$ contains $\mathbb{Q}$ and so is invariant under $\mathcal{G}$.

Notice that this is a special case of Proposition 8.16.

□

We will use a bit of classical topology. A topological space X is called a *Polish space* when it is a s second countable space which admits a complete metric. so, of course, the complete metric space $\mathbb{R}$ is Polish. It is a classical result of Alexandroff and Hausdorff, see [14] page 208, that a G_δ subset of a complete metric space admits a complete metric. Hence, a G_δ subset of a Polish space is Polish.

A *Cantor set* is a non-empty, zero-dimensional, compact, perfect metric space. Any Cantor set is homeomorphic to the classical Cantor Set in $[0, 1]$.

Lemma 9.8. *Let X be a nonempty Polish space.*

(a) *If X has no isolated points, i.e. X is* perfect*, then X contains a Cantor set and so is uncountable.*
(b) *If $X_0 = \cup\{U : U$ is a countable, open subset of $X\}$, then the open set X_0 is a countable Polish space. The complementary closed set, $X_1 = X \setminus X_0$, is a Polish space with no isolated points.*
(c) *The space X is perfect if and only if every nonempty open subset is uncountable.*

Proof. (a) Choose a complete metric and use the usual dichotomy procedure. With $A_0 = X$, define for each word $x \in \{-1, +1\}^{n+1}$ a closed set A_x with a nonempty interior, of diameter at most 2^{-n} such that $A_{x,\pm 1} \subset Int\ A_x$. From the Cantor Intersection Theorem we obtain a topological embedding of $\{-1, +1\}^\omega$ into X.

(b) Because X has a countable base it follows that X_0 is countable. If $x \in X_1$, then any neighborhood U of x in X is uncountable and so $U \cap X_1 = U \setminus X_0$ is uncountable. Any G_δ subset of a Polish space is Polish.

(c) If X is a perfect Polish space, then every nonempty open subset is a perfect Polish space which therefore contains a Cantor set by (a). Thus, every nonempty open subset is uncountable. The converse is obvious.

□

Proposition 9.9. *Let X be an IHLOTS in $\mathbb{R}$ and let α be an infinite tail-like ordinal. If X contains a Cantor set, then X_α, its completion $\widehat{X_\alpha}$ and $\mathbb{R}_\alpha$ all have the same size. If, in addition, X is not a first category subset of $\mathbb{R}$, then no two of these HLOTS are homeomorphic and so not order isomorphic. In particular, if X is the IHLOTS of irrational numbers, i.e. $X = \mathbb{I}$, then $\widehat{X_\alpha}$ and $\mathbb{R}_\alpha$ are CHLOTS of the same size which are not homeomorphic and so not order isomorphic.*

Proof. X_α is a subset of $\mathbb{R}_\alpha$ and so by Proposition 4.7(b) $\widehat{X_\alpha}$ injects into $\mathbb{R}_\alpha$. We can assume that the Cantor set C is contained in the distinguished interval J of X. The order isomorphism between $\mathbb{Q}$ and the set of left end-points in C, excluding $max\ C$, shows that $\mathbb{Q}$ injects into C and so by Proposition 4.7(b) again $\mathbb{R}$ injects into C. In fact, if $\tilde{C}$ is the classical Cantor Set C with the right end-points and the min removed, then the usual Cantor map from C onto $[0, 1]$ restricts to an order isomorphism from $\tilde{C}$ onto $(0, 1)$.

By Proposition 4.7(c) $\mathbb{R}_\alpha \subset \mathbb{R}^\alpha$ injects into $C^\alpha \subset X_\alpha$.

If X is not of first category, then by Theorem 9.6 the CHLOTS $\widehat{X_\alpha}$ and $\mathbb{R}_\alpha$ are not order isomorphic. If $h : \widehat{X_\alpha} \to \mathbb{R}_\alpha$ were a homeomorphism, then by Lemma 3.1 it be either an order isomorphism or an order* isomorphism. Since $\mathbb{R}$ is symmetric, an order isomorphism would then exist.

The IHLOTS X_α has dense holes and so is not locally compact as are $\widehat{X_\alpha}$ and $\mathbb{R}_\alpha$. So it is not homeomorphic to either of them.

The set of irrationals is not of first category by the Baire Category Theorem and it contains a Cantor set by Lemma 9.8(a).

□

We call a subset A of a Polish space X a *Mycielski set* if it is a countably infinite union of Cantor sets in X, see, e.g. [2].

Lemma 9.10. *Let X be a Polish space.*

(a) *A countable union of Mycielski sets in X is a Mycielski set.*
(b) *The nonempty intersection of a Mycielski set and an open set is a Mycielski set when it is nonempty.*
(c) *If A is a Mycielski set and B is a countable subset of the closure of A in X, then $A \cup B$ is a Mycielski set.*

Proof. (a): Obvious.

(b): If $A = \bigcup_n C_n$ with C_n a Cantor set, and U is open, then $A \cap U = \bigcup_n (C_n \cap U)$. Since $C_n \cap U$ is open in C_n it is, when nonempty, the countable union of nonempty clopen subsets of C_n each of which is a Cantor set.

(c): By (a) it suffices to show that $A \cup \{x\}$ is a Mycielski set when x is a limit point of A. Let U_n be the open ball of radius 2^{-n} centered at x. By (b), $A \cap U_n$ is a Mycielski set and since $x \in \overline{A}$ each is nonempty. If C_n is a Cantor set in $A \cap U_n$, then $C_x = \{x\} \cup (\bigcup_n C_n)$ is a Cantor set since it is closed, zero-dimensional and without isolated points. So $A \cup \{x\} = A \cup C_x$ is a Mycielski set.

□

Remark. If A is a Mycielski set and D is a countable set, then $A \setminus D$ need not be a Mycielski set. If X is a Polish space and D, dense in X, is a countable union of closed nowhere dense subsets of X, then by the Baire Category Theorem $X \setminus D$ is a G_δ subset but not an F_σ subset of X. In particular, if A is a Cantor set and D is a countable, dense subset of A, then $A \setminus D$ is not σ-compact and so is not a Mycielski set.

Theorem 9.11. *Let X be a dense, proper subset of $\mathbb{R}$ and let $Y = \mathbb{R} \setminus X$.*

(a) *The following conditions are equivalent.*
   (1) *X is a Mycielski set in $\mathbb{R}$.*
   (2) *Y is a dense subset of $\mathbb{R}$, X is an F_σ subset and every nonempty open subset of X is uncountable.*
   (3) *X is an uncountable F_σ subset of $\mathbb{R}$ which is a HLOTS.*
   (4) *Y is a dense subset of $\mathbb{R}$ which is order isomorphic to $\mathbb{Q}^\omega$.*

(b) *If X_1 is a dense Mycielski subset of $\mathbb{R}$, then there exists $f \in H_+(\mathbb{R})$ such that $f(X) = f(X_1)$.*

Proof. (a) (2) $\Rightarrow$ (1): Let $X = \bigcup_n A_n$ with each A_n closed in $\mathbb{R}$. Let $\mathcal{B}$ be a countable base for $\mathbb{R}$. For each n and $U \in \mathcal{B}$ we use Lemma 9.8 to decompose $U \cap A_n = B(U,n) \cup C(U,n)$ with $B(U,n)$ countable and $C(U,n)$ perfect or empty. The closure $\bar{C}(U,n) \subset A_n$ is perfect and is nowhere dense because Y is dense. Thus, each nonempty $\bar{C}(U,n)$ is a Cantor set. For each U $U \cap X$ is uncountable and so some $\bar{C}(U,n)$ is nonempty. Hence, $\tilde{X} = \bigcup_{U,n}\{\bar{C}(U,n)\}$ is a dense Mycielski set. Hence, $X = \tilde{X} \cup (\bigcup_{U,n} B(U,n))$ is a Mycielski set by Lemma 9.10(c).

(1) $\Rightarrow$ (4): If C is a Cantor set in $\mathbb{R}$ with $min = a$ and $max = b$, then we will call the components of the open set $[a,b] \setminus C$ the *complementary intervals for C*. The LOTS of complementary intervals for C is order isomorphic with $\mathbb{Q}$.

We will repeatedly use the following:

Fact If $\epsilon > 0$ and $a < b \in X$, then there exists a Cantor set $C \subset X$ with $a = min\ C, b = max\ C$, and such that the diameter of each complementary subinterval is less than ϵ. We will call such a C an *X Cantor set for $[a,b]$ with mesh less than ϵ.*

Proof. : Choose $f : \mathbb{Z} \to (a,b)$ an order injection with image $\pm$cofinal and such that for all i $f(i+1) - f(i) < \epsilon/2$. For each i use Lemma 9.10(b) to choose a Cantor set $C(i) \subset X \cap (f(i), f(i+1))$. Let $C = \{a,b\} \cup (\bigcup_i C(i))$.

□

Now write X as the countable union of Cantor sets $C(n)$ and proceed inductively.

Because X is dense we can choose an order injection $f : \mathbb{Z} \to X$ with image $\pm$cofinal in $\mathbb{R}$ and $f(i+1) - f(i) < 1$ for all i. For each i choose an X Cantor set for $[f(i), f(i+1)]$ which contains $[f(i), f(i+1)] \cap C(0)$. Let $A(0)$ be the closed set which is their union. Choose an order isomorphism $q \mapsto J(q)$ from $\mathbb{Q}$ to the set of intervals complementary to $A(0)$ in $\mathbb{R}$. We will call a closed subset A of $\mathbb{R}$ an *extended Cantor* *set* if $A \cap [f(i), f(i+1)]$ is a Cantor set for all $i \in \mathbb{Z}$. If A is a perfect, closed subset of $\mathbb{R}$ with $A(0) \subset A \subset X$ then A is an extended Cantor set because Y is dense.

Assume that we have defined for $i = 0, ..., n$ an extended Cantor set $A(i)$ and an order isomorphism from $\mathbb{Q}^i$ to the set of complementary intervals for $A(i)$ such that for $i = 1, ..., n$ and $q_0...q_i \in \mathbb{Q}^i$

$$A(i-1) \cup C(i) \subset A(i) \qquad \text{and} \qquad \overline{J(q_0...q_i)} \subset J(q_0...q_{i-1}). \tag{9.43}$$

Furthermore, the complementary intervals for $A(i)$ have diameter at most 2^{-i+1}

For the next step, choose for each $A(n)$-complementary interval $J(q_0...q_n) = (a, b)$ an X Cantor set for $[a, b]$ which contains $[a, b] \cap C(n+1)$ and which has mesh less than 2^{-n}. Choose an order isomorphism $q \mapsto J(q_0...q_n q)$ from $\mathbb{Q}$ to the set of complementary intervals. Let $A(n+1)$ be the union of $A(n)$ together with these newly constructed Cantor sets.

From the construction, $X = \bigcup_n A(n)$ and for $x \in \mathbb{Q}^\omega$ the intersection

$$\bigcap \{J(x(0)...x(n)) : n \in \omega\} \quad = \quad \bigcap \{\overline{J(x(0)...x(n))} : n \in \omega\} \tag{9.44}$$

is a single point of Y. If we denote this point $g(x)$, then $g : \mathbb{Q}^\omega \to Y$ is an order isomorphism.

(4) $\Rightarrow$ (3) and (b): By Corollary 6.2 $\mathbb{Q}^\omega$ is a HLOTS. From (4) it follows that Y is a HLOTS and since Y is dense, it has completion $\mathbb{R}$. By Proposition 3.8(d), its complement X is a HLOTS as well.

If X_1 as well as X satisfy condition (4), then there is an order isomorphism between the complements Y_1 and Y because both are isomorphic to $\mathbb{Q}^\omega$. The extension to the completion $\mathbb{R}$ restricts to an isomorphism between X_1 and X.

Since (1) implies (4) we can start with any dense Mycielski set X_1. The isomorphism shows that X is an uncountable F_σ.

(3) $\Rightarrow$ (2): Since X is dense it has completion $\mathbb{R}$ and the complementary IHLOTS Y is dense as well because X is a proper subset of $\mathbb{R}$. Any open interval in X is order isomorphic to X itself and so is uncountable.

$\square$

Beginning with any Mycielski set or Cantor set in $\mathbb{R}$ we can close it up under the action of the rational affine group described in Proposition 9.7. The resulting union is a dense Mycielski set and it is an IHLOTS by Proposition 9.7. Once we know that any two dense Mycielski sets in $\mathbb{R}$ are order isomorphic, as in Theorem 9.11 above, it becomes clear that they are all IHLOTS.

Corollary 9.12. *If X is a dense, Mycielski subset of $\mathbb{R}$, then X is an IHLOTS and the completion $\widehat{X_\omega}$ is order isomorphic to $\mathbb{R}_\omega$.*

Proof. By Theorem 9.11 $\mathbb{Q}^\omega$ is isomorphic to Y, the complement of X in $\mathbb{R}$ and with completion $\mathbb{R}$. That is, $\widehat{\mathbb{Q}^\omega} \setminus \mathbb{Q}^\omega \cong X$.

Lemma 9.1 implies that $(\widehat{\mathbb{Q}^\omega} \setminus \mathbb{Q}^\omega)^\omega$ has completion isomorphic to $\mathbb{R}_\omega$. So X^ω has completion isomorphic to $\mathbb{R}_\omega$. By Proposition 6.3 X^ω and X_ω have isomorphic completions.

□

We saw in Proposition 8.5 that a tree of $\mathbb{Q}$ type with countable height has an $\mathbb{R}$-bounded branch space. In Corollary 8.24 we saw that an Aronszajn tree does not have an $\mathbb{R}$-bounded branch space. Nonetheless, it can happen that a tree with height Ω has an $\mathbb{R}$-bounded branch space.

Example 9.13. *There exists an Ω-bounded normal tree of $\mathbb{Q}$ type and of height Ω whose branch space is $\mathbb{R}$-bounded.*

Proof. Let A be an Aronszajn tree of $\mathbb{Q}$ type, see Corollary 6.22. If α is a countable limit ordinal then by Proposition 5.12 (a), the branch space $X(A^\alpha)$ has completion isomorphic to $\mathbb{R}$ and so we can choose i_α an order injection from $X(A^\alpha)$ to $(-1,1) = J^\circ$.

Let $T^{\omega+1}$ be the simple tree on $\mathbb{Q}, \omega+1$ so that $L_\omega = \mathbb{Q}^\omega$.

Choose a surjection $p \to \alpha(p)$ from the uncountable set $\mathbb{Q}^\omega$ to the set of infinite limit ordinals less than Ω.

We construct T so that for each $p \in \mathbb{Q}^\omega$, $T_p \cong A^{\alpha(p)}$.

It follows that $X(T)$ is isomorphic to the sum $\sum_{p\in\mathbb{Q}^\omega} X(A^{\alpha(p)})$. Using the sum map $\sum_{p\in\mathbb{Q}^\omega} i_{\alpha(p)}$ we obtain an injection from $X(T)$ into $\mathbb{Q}^\omega \times J$. From Theorem 9.11 we can embed $\mathbb{Q}^\omega$ in $\mathbb{R}$ and so obtain an order injection from $X(T)$ into $\mathbb{R} \times J = \mathbb{R}_2$ which injects into $\mathbb{R}_\omega$.

It is clear that T has branches of arbitrarily large countable height, but no uncountable branches. Thus, T is Ω-bounded and with height Ω.

□

9.3. **The Hart-van Mill Construction.** We conclude by describing the results of Hart and van Mill.

Let F be a perfect Polish space, i.e. one with no isolated points, and so, by Lemma 9.8(c), every nonempty open subset of F is uncountable. Following Hart and van Mill, we call $Y \subset F$ a *Bernstein subset*, hereafter a *B set*, if Y meets every Cantor set in F, or, equivalently, if its complement $X = F \setminus Y$ does not contain a Cantor set. (If, as in [1], we consider the Furstenberg family generated by the Cantor sets of F, then the Bernstein sets are the members of the dual family.) We call Y a *Bi-Bernstein subset*, hereafter a *BB set*, if both Y and its complement X are Bernstein subsets.

We denote by $\mathbf{c}$ the cardinal number of $\mathbb{R}$ and so of every Cantor set. Recall that a G_δ subset of $\mathbb{R}$ is a Polish space and so, if it is uncountable, it contains a Cantor set by Lemma 9.8.

Lemma 9.14. *Let F be a perfect Polish space.*

- (a) *If U is an open subset of F and $x \in U$, then there exists a Cantor set C such that $x \in C$ and $C \subset U$.*
- (b) *Assume that Y is a B set in F. If A is any uncountable, G_δ subset of F, then $Y \cap A$ has cardinality $\mathbf{c}$. In particular, if A is a dense G_δ subset of F, then $Y \cap A$ is dense in F.*
- (c) *If Y is a BB set in F, then the complement $F \setminus Y$ is a BB set in F.*
- (d) *If Y_1, Y_2 are BB sets in F and $Y_1 \subset Y \subset Y_2$, then Y is a BB set in F.*

Proof. (a): Let $U_1, U_2, ...$ be a sequence of open subsets of U, each containing x and with diameter tending to zero. By Lemma 9.8(a) each U_n contains a Cantor set C_n. Let $C = \{x\} \cup (\bigcup\{C_n\})$.

(b): Let C be a Cantor subset of A. There exists a homeomorphism $s : C \times C \to C$ where $C \times C$ has the usual product topology, ignoring the order structure. For each $x \in C$, $s(C \times \{x\})$ is a Cantor set which meets Y. As x varies over C we obtain a pairwise disjoint family of cardinality $\mathbf{c}$ which consists of nonempty subsets of $Y \cap A$. In particular, if A is a dense G_δ and U is a nonempty open set, then Y meets $A \cap U$.

(c) and (d) are obvious.

$\square$

Of course, for us it is the special order results which are of importance. For $A \subset \mathbb{R}$ we define a subset of the AS double $\mathbb{R}'$

$$\mathbb{R} \vee A' \quad = \quad \mathbb{R} \times \{-1\} \ \cup \ A \times \{+1\}. \tag{9.45}$$

The gap pairs of $\mathbb{R} \vee A'$ are $x^- < x^+$ for $x \in A$.

Lemma 9.15. *Let X be a LOTS, Y be a B set in $\mathbb{R}$ and $A \subset \mathbb{R}$ disjoint from Y.*

- (a) *If $f : Y \to X$ is a order map, then the image $f(Y)$ is countable or has cardinality $\mathbf{c}$.*

(b) *If $f : \mathbb{R} \vee A' \to X$ is an order map such that $f(Y \times \{-1\})$ is countable, then $A_f = \{t \in A : f(t^-) < f(t^+)\}$ is countable and the image $f(\mathbb{R} \vee A')$ is countable.*

Proof. (a): Let $B = \{x \in X : f^{-1}(x)$ contains more than one point$\}$. For $x \in B$, $f^{-1}(x)$ is a nontrivial interval in Y and so B is countable because Y is separable. For each $x \in B$ let $I(x)$ be the smallest closed interval in $\mathbb{R}$ which contains $f^{-1}(x)$, i.e. the convex hull of the closure in $\mathbb{R}$. Let E_1 be the countable set of endpoints of the intervals $I(x)$ and let E be the complement in $\mathbb{R}$ of union of the intervals $I(x)$. Thus, E is a G_δ subset of $\mathbb{R}$. If E is countable, then $f(Y) = B \cup f(E_1 \cap Y) \cup f(E \cap Y)$ is countable. If E is uncountable, then by Lemma 9.14(b) $E \cap Y$ has cardinality $\mathbf{c}$. Since f is injective on $E \cap Y$ the image has cardinality $\mathbf{c}$.

(b): Identify $\mathbb{R} \setminus A$ with the subset $(\mathbb{R} \setminus A) \times \{-1\} \subset \mathbb{R} \vee A'$ so that $Y \subset \mathbb{R} \vee A'$. For each $x \in f(Y)$ $Y \cap f^{-1}(x)$ is a nonempty interval in Y. Since Y is dense in $\mathbb{R}$, the closure in $\mathbb{R}$, $\overline{Y \cap f^{-1}(x)}$, is a nonempty interval in $\mathbb{R}$ and its $\mathbb{R}$ interior is mapped by f to x. Notice that here we use that f is order preserving rather than continuity of f, which is not assumed.

Let $F = \bigcup\{\overline{Y \cap f^{-1}(x)} : x \in f(Y)\}$, where the closure is again taken in $\mathbb{R}$. Thus, F is a countable union of closed intervals in $\mathbb{R}$. Let F_1 be the countable collection of endpoints of these intervals and let $F_2 = \mathbb{R} \setminus F$ which is countable because it is a G_δ set disjoint from the B set Y. If $t \in \overline{Y \cap f^{-1}(x)} \setminus F_1$ for some $x \in f(Y)$, then $f(t^-) = x$ and if, in addition, $t \in A$, then $f(t^+) = x$. Hence, $A_f \cup \{t \in \mathbb{R} : f(t^-) \notin f(Y)\} \cup \{t \in A : f(t^+) \notin f(Y)\} \subset F_1 \cup F_2$ and so is countable. Thus, A_f and the image of f are countable.

$\square$

Example 9.16. *Products $\mathbb{Z} \times X$ and $\mathbb{Q} \times X$ with X an IHLOTS.*

If X is any IHLOTS, we can choose an order injection $z : \mathbb{Z} \to \hat{X} \setminus X$ which is $\pm$cofinal in $\hat{X}$. Using this we see that $\mathbb{Z} \times X \cong X$.

On the other hand, if X is an IHLOTS dense in $\mathbb{R}$, then $\mathbb{Q} \times X$ is isomorphic to $\tilde{X} = ([0,1] \setminus C) \cap X$ with C the Cantor Set in $[0,1]$.

If X is a dense Mycielski set, then $([0,1] \setminus C) \cap X$ is a Mycielski set dense in $[0,1] \setminus C$ and hence in $(0,1)$. It then follows from Theorem 9.11 that $X \cong \tilde{X} \cong \mathbb{Q} \times X$ and so $\mathbb{Q} \times X$ is an IHLOTS.

On the other hand, if X is a BB-set, and the pair $a < b$ in X is contained in a component of $[0,1] \setminus C$, then the interval $(a,b) \cap \tilde{X}$ is a

BB-set in the real interval (a,b). If a and b lie in different components of $[0,1]\setminus C$, then the intersection of C and the real interval (a,b) contains a Cantor set. Hence, $(a,b)\cap\tilde{X}$ is not a BB-set in (a,b). It follows that $\mathbb{Q}\times X\cong\tilde{X}$ is not doubly transitive.

Definition 9.17. *Let $V\subset\mathbb{R}$ and $\mathcal{H}$ be a nonempty set of subsets of $\mathbb{R}$. We say that $\mathcal{H}$ is a* Hart-van Mill collection with base set V *when the following conditions hold.*

(i) $\mathbb{Q}\ \subset\ V$.

(ii) *Each $Y\in\mathcal{H}\cup\{V\}$ is a BB set, that is $\mathcal{G}$ invariant, where $\mathcal{G}$ is the countable group of positive, affine transformations of $\mathbb{R}$ with rational coefficients.*

(iii) *The elements of $\mathcal{H}\cup\{V\}$ are pairwise disjoint.*

(iv) *If $Y\in\mathcal{H}$, then $-Y=\{-x:x\in Y\}$ is an element of $\mathcal{H}$ distinct from Y.*

(v) *If $f:\mathbb{R}\to\mathbb{R}$ is an order map and $Y\in\mathcal{H}$ is such that the cardinality of $f(Y)\setminus Y$ is $\mathbf{c}$, then the cardinality of $f(Y)\cap V$ is $\mathbf{c}$.*

By replacing V by $V\cup -V$ we may assume that $V=-V$.

Remark. If $f:Y\to\mathbb{R}$ is an order map with $Y\in\mathcal{H}$, then we can extend f to $\mathbb{R}$ by defining $\hat{f}(t)=sup\ f((-\infty,t]\cap Y)$. The order map $\hat{f}$ on $\mathbb{R}$ extends f and so $\hat{f}(Y)=f(Y)$. So we can apply condition (v) even when f is only defined on Y.

If $Y\in\mathcal{H}$, then we let $X(Y)=\mathbb{R}\setminus Y$. More generally, if $\mathcal{J}$ is a nonempty subset of $\mathcal{H}$ we call $X(\mathcal{J})=\mathbb{R}\setminus\cup\mathcal{J}$ *the associated IHLOTS* for $\mathcal{J}$. By Proposition 9.7 each $X(\mathcal{J})$ is an IHLOTS containing $\mathbb{Q}$ with completion $\mathbb{R}$. In addition, $V\subset X(\mathcal{J})\subset\mathbb{R}\setminus Y$ for $Y\in\mathcal{J}$ implies that $X(\mathcal{J})$ is a BB set by Lemma 9.14 (c) and (d). We use $[-1,+1]$ as the distinguished interval in each $X(\mathcal{J})$.

We will do some preliminary setup work which will be used twice. It extends the notation of the proof of Theorem 9.6.

Assume that $\mathcal{J},\mathcal{J}_1$ are subsets of a Hart-van Mill collection $\mathcal{H}$ with base set V, that $Y\in\mathcal{J}\setminus\mathcal{J}_1$ and that α,β are infinite ordinals. Let $X=X(\mathcal{J}),X_1=X(\mathcal{J}_1)$ be the associated IHLOTS so that

$$Y\quad\subset\quad X_1\setminus X\qquad\text{and}\qquad V\quad\subset\quad X\cap X_1. \tag{9.46}$$

Assume that $f:\widehat{X_\alpha}\to\widehat{(X_1)_\beta}$ is a continuous order map.

Since Y is contained in the complement of X, $Y \subset \widehat{X_\alpha}$ consisting of elements of height 1. Define for each $i < \beta$

$$(9.47) \qquad Y_i \quad = \quad \begin{cases} Y & \text{for } i = 0 \\ \{y \in Y : \hat{\pi}_i^\beta(f(y)) \in (X_1)_i\} & \text{for } i > 0 \end{cases}$$

When $0 < i$ and $y \in Y_i$ let

$$(9.48) \qquad Q(y,i) \quad = \quad (\hat{\pi}_i^\beta \circ f)^{-1}(\hat{\pi}_i^\beta(f(y))) = f^{-1}(J_{f(y)|i}) \quad \subset \widehat{X_\alpha},$$

where the latter equation uses definition (9.31) above, which we now recall:

For $w \in (X_1)_i$, we define the compact subinterval $J_w \subset \widehat{(X_1)_\beta}$ and the map $\hat{\pi}_w : J_w \to J$ by

$$(9.49) \qquad \begin{aligned} J_w \quad &= \quad (\hat{\pi}_\iota^\beta)^{-1}(w) \ = \ \{z \in \widehat{(X_1)_\beta} : z|i = w\} \ = \ [w-, w+]. \\ \hat{\pi}_w(z) \quad &= \quad z(i). \end{aligned}$$

As in proof of Theorem 9.6, for $y_1, y_2 \in Y_i$

$$(9.50) \qquad \begin{aligned} Q(y_1,i) \cap Q(y_2,i) \ &\neq \ \emptyset \quad \Rightarrow \\ f(y_1)|i = f(y_2)|i \quad &\Rightarrow \\ Q(y_1,i) \ &= \ Q(y_2,i). \end{aligned}$$

Because $y \in Q(y,i)$, (9.37) implies $\epsilon(Q(y,i)) = 0$ for all $y \in Y_i$ and $0 < i < \beta$. It follows from (9.36), (9.50) and Lemma 9.5(c) that distinct members of the set of intervals

$$(9.51) \qquad \mathcal{Q}_i \quad = \quad \{span(Q(y,i)) : y \in Y_i\} \ = \ \{\hat{\pi}(Q(y,i)) : y \in Y_i\}$$

are non-overlapping. Since $y \in span(Q(y,i))$, we have

$$(9.52) \qquad \begin{aligned} Y_i \cup O_i \ &\subset \ \bigcup \mathcal{Q}_i \qquad \text{with} \\ O_i \quad &= \quad \bigcup \{span^\circ(Q(y,i)) : y \in Y_i\} \end{aligned}$$

an open subset of $\mathbb{R}$.

Lemma 9.18. *Let $0 < i < \beta$.*

(a) $Y \cap O_i \subset Y_i$.
(b) *For all $y \in Y_i$ the closed interval $Q(y,i)$ is nontrivial.*
(c) *The set $\mathcal{Q}_i$ is countable.*
(d) *O_i is a dense subset of $\bigcup \mathcal{Q}_i$.*
(e) *Y_i is dense in $\mathbb{R}$ if and only if O_i is.*
(f) *The image projection $\hat{\pi}(f(Y))$ is a countable subset of $\mathbb{R}$.*
(g) *If there exists D a dense subset of $\widehat{X_\alpha}$ such that $\hat{\pi}(f(D))$ is a countable subset of $\mathbb{R}$, then $\hat{\pi} \circ f$ is constant on $\widehat{X_\alpha}$.*

Proof. (a): If $t \in Y \cap span^{\circ}Q(y,i)$, then by Lemma 9.5(b) $\hat{\pi}^{-1}(t) = P(t) \subset Q(y,i)$. In particular, by (9.48) $f(t) \in J_{f(y)|i}$ and so by (9.49) $f(t)|i = f(y)|i \in (X_1)_i$ which says that $t \in Y_i$.

(b): Because f is continuous and $\widehat{X_\alpha}$ is connected, its image $F = f(\widehat{X_\alpha})$ is connected and so is convex in $\widehat{(X_1)_\beta}$. Let $y \in Y_i$ so that $f(y)|i \in (X_1)_i$.

Case 1: Assume that there exist $a, b \in F$ such that $\hat{\pi}^\beta_i(a) < f(y)|i < \hat{\pi}^\beta_i(b)$. As defined by (9.49) the compact interval $J_{f(y)|i} = (\hat{\pi}^\beta_i)^{-1}(f(y)|i)$ is nontrivial. It is entirely contained in the interval (a,b) and so in the convex set F. Hence, the preimage $Q(y,i)$ is nontrivial.

Case 2: Assume $f(y)|i \ = \ max\ \hat{\pi}^\beta_i(F)$. Let $t \in Y \cap [y, \infty)$ which is an infinite set because Y is unbounded. Since $\hat{\pi}^\beta_i \circ f$ is an order map $f(y)|i = \hat{\pi}^\beta_i(f(t))$ and so $t \in Q(y,i)$. By a similar argument $Q(y,i)$ is nontrivial when $f(y)|i \ = \ min\ \hat{\pi}^\beta_i(F)$.

(c): The intervals in $\mathcal{Q}_i$ are non-overlapping and by (b) they are nontrivial. Since $\mathbb{R}$ is separable, the set of intervals is countable.

(d): The interior of a nontrivial interval is dense in the interval.

(e): Since Y is a BB set it is dense in $\mathbb{R}$ and so $Y \cap O_i$ is dense in O_i. From part (a) it follows that if O_i is dense in $\mathbb{R}$, then Y_i is. On the other hand, $Y_i \subset \bigcup \mathcal{Q}_i$ and so the union is dense in $\mathbb{R}$ when Y_i is. From part (d) it then follows that O_i is dense in $\mathbb{R}$.

(f): Define the order map $\tilde{f} = \hat{\pi} \circ f : Y \to \mathbb{R}$. Notice that $Y_1 = (\tilde{f})^{-1}(X_1)$ and for $y \in Y_1 \ \ Q(y,1) = (\tilde{f})^{-1}(\tilde{f}(y))$. By (c) the set $\mathcal{Q}_1$ is countable and so $X_1 \cap \tilde{f}(Y)$ is countable. By (9.46) this set contains $(Y \cup V) \cap \tilde{f}(Y)$. By condition (v) of the Hart-van Mill collection, adjusted by the remark after Definition 9.17, it follows that $\tilde{f}(Y) \setminus Y$ has cardinality less than $\mathbf{c}$. Since $Y \cap \tilde{f}(Y)$ is countable, the image $\tilde{f}(Y)$ has cardinality less than $\mathbf{c}$. So by Lemma 9.15(a), $\tilde{f}(Y)$ is countable.

(g): Assume that $a, b \in D$ with $\hat{\pi}(f(a)) < \hat{\pi}(f(b))$. We can choose a point $x \in (X_1 \cap (\hat{\pi}(f(a)), \hat{\pi}(f(b))) \setminus \hat{\pi}(f(D))$ since the image is countable. The open interval $(x-, x+) \subset \widehat{(X_1)_\beta}$ is disjoint from $f(D)$. Because f is continuous and D is dense, $(x-, x+)$ is disjoint from $f(\widehat{X_\alpha})$ which is connected. This contradicts $f(a) < x- < x+ < f(b)$. Hence, $\hat{\pi} \circ f$ is constant on D and so on $\widehat{X_\alpha}$.

$\square$

Remark. It is, of course, part (f) which really uses the Hart-van Mill properties.

In preparation for what follows we define $\mathcal{F}$ to be the set of order maps from $\mathbb{R}$ to $\mathbb{R}$, a pointwise closed semigroup of real functions.

Lemma 9.19. *(a) Any $f \in \mathcal{F}$ has only countably many discontinuities.*
(b) The set $\mathcal{F}$ has cardinality $\mathbf{c}$.

Proof. Let D be a dense subset of $\mathbb{R}$. For an order map $g : D \to \mathbb{R}$, define $g_+, g_- \in \mathcal{F}$ by

$$
\begin{aligned}
g_-(t) &= sup\{g(d) : d \in (-\infty, t) \cap D\}, \\
g_+(t) &= inf\{g(d) : d \in (t, \infty) \cap D\},
\end{aligned}
\tag{9.53}
$$

for all $t \in \mathbb{R}$.

For any $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{F}$ if and only if $g = f|D$ is an order map and $g_- \leq f \leq g_+$. Furthermore, f is discontinuous at t if and only if $g_-(t) < g_+(t)$. By separability the family of nonempty intervals $(g_-(t), g_+(t))$ is countable and so f has at most countably many discontinuities. If f is continuous, then it is uniquely determined by g. If not, then there are $\mathbf{c}$ choices of f between g_- and g_+.

If D is countable, then there are $\mathbf{c}$ maps g from D to $\mathbb{R}$.

It follows that the cardinality of $\mathcal{F}$ is $\mathbf{c}$.

□

The amazing result of Hart and van Mill (1985) is the following.

Theorem 9.20. (a) *There exists a Hart-van Mill collection $\mathcal{H}$ of cardinality* $\mathbf{c}$.

(b) *Let $\mathcal{H}$ be a Hart-van Mill collection with base set V. Let $\mathcal{J}$ and $\mathcal{J}_1$ be subsets of $\mathcal{H}$ with associated complementary IHLOTS X and X_1, respectively. If $\mathcal{J} \subset \mathcal{J}_1$, then $X_1 \subset X$ and so for every ordinal α, $\widehat{(X_1)_\alpha}$ injects into $\widehat{X_\alpha}$. If $\mathcal{J}$ is not a subset of $\mathcal{J}_1$, then $\widehat{(X_1)_\omega}$ does not inject into $\widehat{X_\omega}$. So if neither $\mathcal{J}$ nor $\mathcal{J}_1$ includes the other, then the CHLOTS $\widehat{X_\omega}$ and $\widehat{(X_1)_\omega}$ are not comparable with respect to size, i.e. neither injects into the other. In particular, if for some $Y \in \mathcal{J}$ we have $-Y \notin \mathcal{J}$, then the CHLOTS $\widehat{X_\omega}$ is not even comparable in size with its reverse CHLOTS.*

Proof. (a): Let $\tilde{\mathcal{G}}$ denote the group of nonconstant affine transformations of $\mathbb{R}$ with rational coefficients, i.e. of the form $t \mapsto at + b$ with $a \neq 0$ and a, b rational. It contains $\mathcal{G}$ as a subgroup of index two. If

$t \in \mathbb{I}$, the set of irrationals, then $g \mapsto g(t)$ is an injective map from $\tilde{\mathcal{G}}$ to $\mathbb{R}$. In particular, if $t \in \mathbb{I}$, then the orbit sets $\mathcal{G}t$ and $-\mathcal{G}t = \mathcal{G}(-t)$ are disjoint.

For $f \in \mathcal{F}$ let $S(f, \mathcal{A}) = \{t \in \mathbb{R} : f(t) \not\in \mathcal{A}t\}$ for $\mathcal{A} = \mathcal{G}$ or $\tilde{\mathcal{G}}$. Hart and van Mill call f *singular* if the set $f(S(f, \mathcal{G}))$ has cardinality $\mathbf{c}$. Observe that for each $g \in \mathcal{G}$ the equation

$$f(t) \quad = \quad -g(t) \tag{9.54}$$

has at most one solution since f is an order map and $-g$ is a decreasing function. It follows that the set $S(f, \mathcal{G}) \setminus S(f, \tilde{\mathcal{G}})$ is countable. So if f is singular the set $f(S(f, \tilde{\mathcal{G}}))$ has cardinality $\mathbf{c}$.

Let $\mathcal{F}'$ denote the set of singular functions in $\mathcal{F}$. For example all translations by elements of $\mathbb{I}$ lie in $\mathcal{F}'$. So from Lemma 9.19 it follows that the cardinality of $\mathcal{F}'$ is $\mathbf{c}$.

For each $f \in \mathcal{F}'$ choose a set $K(f) \subset S(f, \tilde{\mathcal{G}})$ such that $f|K(f)$ is injective and $f(K(f)) = f(S(f, \tilde{\mathcal{G}}))$. That is, for each $z \in f(S(f, \tilde{\mathcal{G}}))$, we choose one point of $(f|S(f, \tilde{\mathcal{G}}))^{-1}(z)$.

Let $\mathbf{c}$ denote the cardinal, i.e. the first ordinal with cardinality $\mathbf{c}$ and let $\{f_{ij} : i, j < \mathbf{c}\}$ be a listing of $\mathcal{F}'$ so that each function occurs $\mathbf{c}$ times in each row and let $\{C_{ij} : i, j < \mathbf{c}\}$ be a similar listing of the Cantor sets in $\mathbb{R}$.

We will find points $x(ij, 0, \pm), x(ij, 1, \pm), y(ij, 0, \pm), y(ij, 1, \pm)$ in $\mathbb{I}$ such that:

(1) $x(ij, 0, +),\ -x(ij, 0, -)\ \in K(f_{ij})$ and
$y(ij, 0, +) = f(x(ij, 0, +)), \quad y(ij, 0, -) = f(-x(ij, 0, -))$
so that $y(ij, 0, \pm) \in f(K(f_{ij})) = S(f_{ij}, \tilde{\mathcal{G}})$.
(2) $x(ij, 1, +),\ -x(ij, 1, -),\ y(ij, 1, \pm) \in C_{ij}$.
(3) If $(i, j, \alpha, \epsilon) \neq (i', j', \alpha', \epsilon')$
with $i, j, i', j' \in \mathbf{c}; \alpha, \alpha' \in 2, \epsilon, \epsilon' \in \{+, -\}$,
then the four orbit sets $\tilde{\mathcal{G}}(x(ij, \alpha, \epsilon)), \tilde{\mathcal{G}}(x(i'j', \alpha', \epsilon'))$,
$\tilde{\mathcal{G}}(y(ij, \alpha, \epsilon)), \tilde{\mathcal{G}}(y(i'j', \alpha', \epsilon'))$ are pairwise disjoint.

To construct these points we choose a bijection from the index set $\mathbf{c} \times \mathbf{c}$ to $\mathbf{c}$ itself and so well-order the index set with order type the cardinal $\mathbf{c}$. For index ij we let $H(ij)$ be the set of rationals together with the $\tilde{\mathcal{G}}$ orbit of all of the x and y points with index $i'j'$ preceding ij. Thus, $H(ij)$ has cardinality less than $\mathbf{c}$. (Notice that for this reason the lexicographic ordering on the product won't work for our purposes).

Since C_{ij} has cardinality $\mathbf{c}$ we can choose $x(ij, 1, +), -x(ij, 1, -)$, $y(ij, 1, \pm) \in C_{ij} \setminus H(ij)$ with distinct $\tilde{\mathcal{G}}$ orbits and adjoin these four orbits to $H(ij)$ to define $H(ij, 1)$. Since f_{ij} is singular the set $K(f_{ij})$ is defined and has cardinality $\mathbf{c}$ and on it f_{ij} is injective. So we can choose $x(ij,0,+) \in K(f_{ij}) \setminus (H(ij,1) \cup (f_{ij})^{-1}(H(ij,1))$. By definition the $\tilde{\mathcal{G}}$ orbits of $x(ij,0,+)$ and $y(ij,0,+) = f(x(ij,0,+))$ are distinct. Adjoin these two orbits to $H(ij,1)$ to define $H(ij,2)$. Finally, choose $-x(ij,0,-) \in K(f_{ij}) \setminus (H(ij,2) \cup (f_{ij})^{-1}(H(ij,2))$ and let $y(ij,0,-) = f(-x(ij,0,-))$. Notice that $x(ij,0,-)$ is in the $\tilde{\mathcal{G}}$ orbit of $-x(ij,0,-)$ but not in the $\mathcal{G}$ orbit since the point is irrational.

The members of family $\mathcal{H} = \{Y_i^{\epsilon} : i \in \mathbf{c}, \, \epsilon \in \{+,-,\}\}$ and the base set V are given by

$$
\begin{aligned}
Y_i^{+} &= \mathcal{G} \cdot \{x(ij,\alpha,\epsilon) : j \in \mathbf{c}, \alpha \in 2, \epsilon \in \{+,-\}\} \quad \text{for } i \in \mathbf{c}, \\
Y_i^{-} &= \mathcal{G} \cdot \{-x(ij,\alpha,\epsilon) : j \in \mathbf{c}, \alpha \in 2, \epsilon \in \{+,-\}\} \quad \text{for } i \in \mathbf{c}, \\
V &= \mathcal{G} \cdot (\{1\} \cup \{y(ij,\alpha,\epsilon) : i,j \in \mathbf{c}, \alpha \in 2, \epsilon \in \{+,-\}\}).
\end{aligned} \tag{9.55}
$$

Clearly, $Y_i^- = -Y_i^+$ and by condition (3) the indexed family $\mathcal{H} \cup \{V\}$ is pairwise disjoint. By condition (2) each member is a B set and so by disjointness each is a BB set.

Finally, suppose that $f \in \mathcal{F}$ and that for some $Y \in \mathcal{H}$ the set $f(Y) \setminus Y$ has cardinality $\mathbf{c}$. Since Y is $\mathcal{G}$ invariant, $\{t \in Y : f(t) \notin Y\} \subset S(f,\mathcal{G})$. Hence, f is singular and so for each $i \in \mathbf{c}$ the set $Z(f,i) = \{j \in \mathbf{c} : f = f_{ij}\}$ has cardinality $\mathbf{c}$. If $Y = Y_i^{\pm}$, then $\{y(ij,0,\pm) : j \in Z(f,i)\}$ is a subset of $V \cap f(Y)$ of cardinality $\mathbf{c}$. Thus, $\mathcal{H}$ is a Hart-van Mill collection with base set V.

(b): If $\mathcal{J} \subset \mathcal{J}_1$, then by definition $X_1 \subset X$. So for every ordinal α, $\widehat{(X_1)_\alpha}$ injects into $\widehat{X_\alpha}$ by Proposition 4.7(b),(c).

Now assume that $Y \in \mathcal{J} \setminus \mathcal{J}_1$.

Assuming that $f : \widehat{X_\omega} \to \widehat{(X_1)_\omega}$ is a continuous order map, we will show, following Hart and van Mill, that f is a constant. By Proposition 4.4 this exactly says that $\widehat{(X_1)_\omega}$ does not inject into $\widehat{X_\omega}$. We will apply the preliminaries leading up to Lemma 9.18 with $\alpha = \beta = \omega$.

It suffices to show that each coordinate function f_n is a constant. We begin with $f_0 = \hat{\pi} \circ f : \widehat{X_\omega} \to \mathbb{R}$.

For $0 < n < \omega$ let

$$A_n = \{w \in X_n : f_0(w-) < f_0(w+)\}. \tag{9.56}$$

Observe that π^n_m maps A_n into A_m if $0 < m \le n$, i.e. $w \in A_n$ implies $w|m \in A_m$, because $(w|m)- \le w- < w+ \le (w|m)+$.

Since Y is dense in $\mathbb{R}$ it is easy to see from (9.26) that the closure $\bar{Y}$ in $\widehat{X_\omega}$ satisfies

$$\bar{Y} \cong \mathbb{R} \vee X' \tag{9.57}$$

in the notation of (9.45).

From Lemma 9.18(f), $f_0(Y)$ is countable and so from Lemma 9.15(b) $f_0(\overline{Y})$ and A_1 are countable sets.

Now for $0 < n < \omega$ and $w \in X_n$ let

$$Y(w) \quad = \quad \{z \in J_w : z(n) \in Y \cap J\}. \tag{9.58}$$

Recall that J_w is the interval $[w-, w+]$ in $\widehat{X_\omega}$. We define the retraction $r : \mathbb{R} \to J$ by mapping $(-\infty, -1]$ to -1 and $[+1, \infty)$ to $+1$. Then define the order surjection $r_w : \widehat{X_\omega} \to J_w$ by

$$r_w(z)_i \quad = \quad \begin{cases} w_i & \text{for } i < n \\ r(z_0) & \text{for } i = n \\ z_{i-n} & \text{for } i > n. \end{cases} \tag{9.59}$$

Apply Lemma 9.18(f) and then Lemma 9.15(b) to $f_0 \circ r_w$ and conclude that $f_0(\overline{Y(w)})$ and $A_{n+1} \cap J_w$ are countable sets. Since A_{n+1} projects to A_n it follows that $A_{n+1} = \bigcup\{A_{n+1} \cap J_w : w \in A_n\}$ and so by induction A_n is countable for all $0 < n < \omega$. Define

$$\begin{aligned} Y(\omega) \quad &= \quad Y \cup \bigcup\{Y(w) : w \in X_n \text{ for some } 0 < n < \omega\} \\ Z \quad &= \quad \overline{Y} \cup \bigcup\{\overline{Y(w)} : w \in A_n \text{ for some } 0 < n < \omega\} \end{aligned} \tag{9.60}$$

where the closures are taken in $\widehat{X_\omega}$.

The image $f_0(Z)$ is countable. We now show that $f_0(Y(\omega))$ is a subset of $f_0(Z)$ and so it is countable as well.

We show that $f_0(x) \in f_0(Z)$ if $x \in Y(w)$ with $w \in X_n$. If $w \in A_n$, then $x \in Z$. Let $m = min\{i : 0 < i \leq n \text{ and } w|i \not\in A_i\}$. If $i = 1$, then $(w|i)-, (w|i)+ \in \overline{Y}$ and if $i > 1$, then $w|(i-1) \in A_{i-1}$ and $(w|i)-, (w|i)+ \in \overline{Y(w|i-1)}$. Furthermore, f_0 is constant on the interval $[(w|i)-, (w|i)+]$ which contains x. Thus, $f_0(x) = f_0((w|i)-) \in f(Z)$.

Because $Y(\omega)$ is dense in $\widehat{X_\omega}$ it follows from Lemma 9.18(g) that f_0 is constant.

We complete the proof by using induction to show that f_n is constant for all n. If f_i is a constant for all $i < n$ then define $\tilde{f} : \widehat{X_\omega} \to \widehat{(X_1)_\omega}$ by $\tilde{f}(x)_j = f(x)_{j+n}$, i.e. just forget the first n coordinates. Because f is constant on the first n coordinates, this is a continuous order map. By the above initial step result the $\tilde{f}_0 = f_n$ is constant.

□

Remark. For every IHLOTS $X \subset \mathbb{R}$ the CHLOTS $F = \widehat{X_\omega}$ has size between $\mathbb{R}$ and $\mathbb{R}_\omega$. It follows from Theorem 9.3(b) that with α a tail-like ordinal with $\alpha \geq \omega^{\omega \cdot 2}$, then

$$F_\alpha \quad \cong \quad \mathbb{R}_\alpha. \tag{9.61}$$

In particular, for $F = \widehat{X(Y)_\omega}$ with $Y \in \mathcal{H}$, F is not symmetric but F_α is symmetric.

Use the Axiom of Choice to select a subset $\mathcal{H}_+$ of $\mathcal{H}$ so that for all $Y \in \mathcal{H}$ exactly one member of the pair $\{Y, -Y\}$ lies in $\mathcal{H}_+$ and let $\mathcal{H}_-$ be the complement. For every $\mathcal{A} \subset \mathcal{H}_+$ define $\mathcal{J}(\mathcal{A}) = \mathcal{A} \cup (\mathcal{H}_- \setminus \{-Y : Y \in \mathcal{A}\})$. If $\mathcal{A}_1 \neq \mathcal{A}_2$, then neither of the two sets $\mathcal{J}(\mathcal{A}_1), \mathcal{J}(\mathcal{A}_2)$ contains the other. So from the Hart-van Mill Theorem 9.20 we obtain a family of CHLOTS of cardinality $2^{\mathbf{c}}$ each with size between $\mathbb{R}$ and $\mathbb{R}_\omega$ no two of which are comparable with respect to size.

Our final result combines the arguments of Hart and van Mill with Theorem 9.6.

Theorem 9.21. *Assume that $\mathcal{H}$ is a Hart-van Mill collection of subsets of $\mathbb{R}$. For distinct, nonempty subsets $\mathcal{J}, \mathcal{J}_1$ of $\mathcal{H}$ let $X = X(\mathcal{J})$ and $X_1 = X(\mathcal{J}_1)$ be the associated IHLOTS. If α and β are countable limit ordinals, then $\widehat{X_\alpha}$ is not order isomorphic to $\widehat{(X_1)_\beta}$. In particular, if for some $Y \in \mathcal{J}$ we have $-Y \notin \mathcal{J}$, then $\widehat{X_\alpha}$ is not symmetric.*

Proof. Since the two subsets are distinct, we can assume that $Y \in \mathcal{J} \backslash \mathcal{J}_1$.

We will assume that $f : \widehat{X_\alpha} \to \widehat{(X_1)_\beta}$ is an order isomorphism and derive a contradiction by showing that f is not injective. As before we apply the preliminaries leading up to Lemma 9.18.

We prove by induction on $i < \beta$ that the open set O_i is dense in $\mathbb{R}$. By Lemma 9.18(e) this is equivalent to Y_i being dense in $\mathbb{R}$.

Case 1: For the initial step, $i = 1$, we apply Lemma 9.18(f) to see that $\hat{\pi}(f(Y))$ is a countable subset of $\mathbb{R}$. On $\mathbb{R} \setminus X_1$ the map $\hat{\pi}$ is injective and we have assumed that f is injective. Hence, $B_\emptyset = \{y \in Y : f(y)(0) \notin X_1\}$ is countable. By the Baire Category Theorem and Lemma 9.14(b), $Y_1 = Y \setminus B_\emptyset$ is dense in $\mathbb{R}$.

For the case $i = j+1$ with $j > 0$ we first fix y in the set Y_j and prove that $Y_i \cap span^\circ Q(y,j)$ is dense in $span^\circ Q(y,j)$. It will, then follow that Y_i is dense in O_j which is dense in $\mathbb{R}$ by inductive hypothesis. Thus, Y_i is dense in $\mathbb{R}$ in this case.

To analyze $Y_{j+1} \cap span^\circ Q(y,j)$ we use a variation of the initial argument. Let $r : \widehat{X_\alpha} \to Q(y,j)$ be the canonical retraction. That is, if $Q(y,j) = [a,b]$ map $(-\infty, a]$ to a and $[b.\infty)$ to b. Define the order injection $s_w : J_{f(y)|j} \to \widehat{(X_1)_{\beta \backslash j}}$ by forgetting the coordinates in j. We apply Lemma 9.18(f) to $s_w \circ f \circ r$. The analogue of $\hat{\pi} \circ f$ becomes, in this case, $\hat{\pi}_{f(y)|j} \circ f \circ r$. Here $\hat{\pi}_{f(y)|j} = \hat{\pi}_w$ of (9.49) with $w = f(y)|j$.

It follows that $\hat{\pi}_{f(y)|j}(f(Y \cap span^\circ Q(y,j)))$ is a countable subset of $J = [-1,+1]$. On $\{z \in J_{f(y)|j} : z(j) \notin X_1\}$ the map $\hat{\pi}_{f(y)|j}$ is injective and we have assumed that f is injective. Hence, $B_{f(y)|j} = \{t \in Y \cap span^\circ Q(y,j) : f(t)(j) \notin X_1\}$ is countable. For all $t \in Y \cap spanQ(y,j)$ we have $f(t)|j = f(y)|j$. Hence, $Y_{j+1} \cap span^\circ Q(y,j) = Y \cap (span^\circ Q(y,j) \setminus B_{f(y)|j})$. As before the Baire Category Theorem and Lemma 9.14(b) imply that $Y_{j+1} \cap span^\circ Q(y,j)$ is dense in $span^\circ Q(y,j)$.

Case 2: When i is a limit ordinal Lemma 9.18(a) implies that

$$Y_i \quad = \quad \bigcap\{Y_j : j < i\} \quad \supset \quad (\bigcap\{O_j : j < i\}) \cap Y. \tag{9.62}$$

Because β is countable, $\bigcap\{O_j : j < i\}$ is a G_δ set which is dense by induction hypothesis and so the intersection with Y is dense by Lemma 9.14(b). Thus, Y_i is dense in this case as well.

Having completed the induction we see, as in Case 1, that $(\bigcap\{O_j : j < \beta\}) \cap X$ is dense in $\mathbb{R}$. As in the final portion of the proof of Theorem 9.6, if y is in this set, then $P(y)$ is nontrivial and f is constant on $P(y)$. Hence, f is not injective, contradicting our initial assumption. □

It follows from Corollary 9.4 that the tower $\{F_{\omega^\gamma}\}$ above each $F = \widehat{X(\mathcal{J})_\alpha}$ coincides at a sufficiently high level with the tower above $\mathbb{R}$.

On the other hand, from Corollary 8.4 we immediately obtain the following.

Theorem 9.22. *Assume that $\mathcal{H}$ is a Hart-van Mill collection of subsets of $\mathbb{R}$. For not necessarily distinct, nonempty subsets $\mathcal{J}, \mathcal{J}_1$ of $\mathcal{H}$ let $X = X(\mathcal{J})$ and $X_1 = X(\mathcal{J}_1)$ be the associated IHLOTS. If α, β are infinite tail-like ordinals with $\alpha > \omega \cdot \beta$, then there does not exist an order injection from $\widehat{(X_1)_\alpha}$ into $\widehat{(X_2)_\beta}$.*

Corollary 9.23. *Assume that $\mathcal{H}$ is a Hart-van Mill collection of subsets of $\mathbb{R}$. For a subset $\mathcal{J}$ of $\mathcal{H}$ let $X = X(\mathcal{J})$. The transfinite sequence of CHLOTS $\{\widehat{(X)_{\omega^\gamma}},\ 0 < \gamma < \Omega\}$ and the transfinite sequence of CHLOTS Cantor Spaces $\{C(\widehat{(X)_{\omega^\gamma}}),\ 0 < \gamma < \Omega\}$ are nondecreasing* *in size. . Furthermore, if* $\gamma_1 > 1 + \gamma_2$*, then* $\widehat{(X)_{\omega^{\gamma_1}}}$ *is strictly bigger in size than* $\widehat{(X)_{\omega^{\gamma_2}}}$ *and* $C(\widehat{(X)_{\omega^{\gamma_1}}})$ *is strictly bigger in size than* $C(\widehat{(X)_{\omega^{\gamma_2}}})$*.*

Proof. This follows from Theorem 8.7 with $\delta = 1$ and so with $\gamma_0 = 0$. □

Remark: If $\gamma_2 \geq \gamma_3 = \omega$, then $1 + \gamma_2 = \gamma_2$. Hence, after the ω^ω level, the members of the towers are strictly increasing in size.

Using a Hart-van Mill collection of cardinality $\mathbf{c}$ we obtain a collection of cardinality $2^{\mathbf{c}}$ consisting of CHLOTS with size between $\mathbb{R}$ and $\mathbb{R}_\omega$ each forming the base of a tower of CHLOTS of height Ω. The separate towers do not intersect by Theorem 9.21. These towers are disjoint from the tower $\{\mathbb{R}_\alpha\}$ over $\mathbb{R}$ as well by Theorem 9.6. The corresponding Cantor Space towers do not intersect either by (4.37).

10. **Zero Dimensional LOTS**

If J is a nonempty, bounded, clopen convex subset of a complete LOTS X, then $y- = sup\ J$ is an element of J and $y+ = inf\ (y-, \infty) \notin J$. So $y- < y+$ is a gap pair in X. Similarly, there is a gap pair $x- < x+$ with $x- \notin J$ and $x+ =\in J$. Hence, J is the clopen interval $[x+, y-]$. It follows that X is zero-dimensional, i.e. the clopen convex sets form a base for X, if and only if X is *gap pair dense* that is, for every $x < y \in X$ there exists a gap pair $z- < z+ \in X$ with $x \leq z- < z+ \leq y$.

Assume that X is a perfect, complete LOTS, i.e. it has no isolated points. We obtain the quotient space F by identifying each gap pair with a single point. That is, we use the equivalence relation

$$\{(x,y) \in X \times X : x = y \ \text{ or } \{x,y\} \ \text{ is a gap pair in } \ X\}. \tag{10.1}$$

On F there is a unique ordering so that $\pi : X \to F$ is an order surjection which is continuous and topologically proper by Proposition 2.3(a). F is complete because X is and if $\pi(x) < \pi(y)$ in F, then $\{x, y\}$ is not a gap pair and so the interval (x, y) is infinite. It follows that F is order dense and so is connected. Let $A \subset F$ consist of the classes of the gap pairs. We call F the *connected quotient* of X and A the *gap pair set* . If F has extrema, we let F° denote F with the max and min removed if they exist.

Conversely, if F is a connected LOTS and $A \subset F$ we extend 9.45 by defining a subset of the AS double F'

$$F \vee A' \quad = \quad F \times \{-1\} \ \cup \ A \times \{+1\}. \tag{10.2}$$

The gap pairs of $F \vee A'$ are $a- < a+$ for $a \in A$. It is clear that we can identify F with the connected quotient of $F \vee A'$ and A with its gap pair set. On the other hand, if X is a perfect, complete LOTS with connected quotient F and gap pair set A, then X is isomorphic to $F \vee A'$.

In the case of the AS double F' itself the gap pair set A is all of F.

Proposition 10.1. *Let X be a perfect, complete LOTS with connected quotient F and gap pair set A.*

(a) *X is compact or first countable if and only if F satisfies the corresponding property.*
(b) *The LOTS F° is unbounded.*
(c) *X is gap pair dense if and only if A is dense in F. In that case the inclusion of the dense set A into F is an embedding and A is unbounded. The induced map from the completion $\hat{A}$ to F is an isomorphism onto F°.*
(d) *If f is an order automorphism of X, then there is a unique order automorphism g of F such that $\pi \circ f = g \circ \pi$. The induced automorphism satisfies $g(A) = A$. Conversely, if g is an order automorphism of F such that $g(A) = A$, then it is induced from a unique automorphism f of X.*

Proof. (a): The compactness result is clear because π is topologically proper. Every point of F (except the maximum if any) is the limit of a decreasing sequence if and only if every point of X which is not a left end-point is the limit of a decreasing sequence. Similarly for increasing sequences, and these two observations yield the first countability equivalence.

(b): No extreme point of X can be part of a gap pair because X has no isolated points. So if we remove the max and min, if any, from X and F, the resulting LOTS are unbounded.

(c): The density equivalence is clear and the embedding result follows from Proposition 2.3(b).

An extreme point of X cannot be part of a gap pair because X has no isolated points. It follows that the dense set A is contained in F° and so is unbounded. Because F° is connected and unbounded, it is the completion of the dense subset A.

(d): An automorphism f on X maps gap pairs to gap pairs and so induces g. Conversely, given g, f can be regarded as the restriction to $F \vee A'$ of the automorphism g' of F'.

$\square$

Definition 10.2. *Let X be a perfect, complete, zero-dimensional LOTS. We say that X satisfies the* clopen interval condition *when any two bounded clopen intervals are isomorphic.*

Theorem 10.3. *Let X be a perfect, complete, zero-dimensional LOTS.*

(a) *X satisfies the clopen interval condition if and only if A is doubly transitive.*
(b) *If X is first countable and σ-bounded, then it satisfies the clopen interval condition if and only if A is a HLOTS in which case F° is a CHLOTS. In addition, X is then topologically homogeneous.*
(c) *If X is topologically homogeneous, then it is first countable.*

Proof. (a): Let $a < b, c < d$ in the gap pair set A. Because A is unbounded we can choose $e_1 < min(a,c) < max(b,d) < e_2$ in A.

Assume X satisfies the clopen interval condition. Combine isomorphisms

(10.3)
$$[e_1+, a-] \cong [e_1+, c-], [a+, b-] \cong [c+, d-], [b+, e_2-] \cong [d+, e_2-]$$

with the identity on $(-\infty, e_1-] \cup [e_2+, \infty)$ to obtain an automorphism f of X. By Proposition 10.1(d) f induces g on F which preserves A and maps $[a,b]$ to $[c,d]$. Hence, A is doubly transitive.

Conversely, assume that A is doubly transitive. There exists an automorphism h of A which maps $[a,b]$ in A to $[c,d]$ in A. The completion $\hat{h}$ is an automorphism of F° which extends to an automorphism g of F such that $g(A) = A$. By Proposition 10.1(d) again g is induced by an automorphism f on X which clearly maps $[a+, b-]$ to $[c+, d-]$.

(b): If X is first countable and σ-bounded as well as complete, then it is of countable type by Proposition 2.11(d) and so F and A are of countable type by Proposition 2.11 (f) and (g). So A is an IHLOTS if and only if it is doubly transitive by Proposition 3.8(a) in which case the completion F° is a CHLOTS by Proposition 3.8(c).

The topological homogeneity follows from the clopen interval condition with various cases. We provide just a sample.

Suppose $a \in A, b \in F^\circ \setminus A$. Let $e_1, e_2 \in A$ with $e_1 < a, b < e_2$. Choose decreasing sequences $\{a_n\}$ in $A \cap (a, e_2)$ converging to a, $\{b_n\}$ in $A \cap (b, e_2)$ converging to b and an increasing sequence $\{c_n\}$ in $A \cap (e_1, b)$ converging to b. Define

- $J_0 = [e_1+, a-]$, $J_1 = [a_1+, e_2-]$ and
  $J_k = [a_k+, a_{k-1}-]$ for $k \geq 2$.
- $K_0 = [e_1+, c_1-]$ and $K_{2k} = [c_k+, c_{k+1}-]$ for $k \geq 1$.
- $K_1 = [b_1+, e_2-]$ and $K_{2k+1} = [b_{k+1}+, b_k-]$ for $k \geq 1$.

Choose isomorphisms $J_k \cong K_k$ for $k \geq 0$. Together these extend to a homeomorphism of $[e_1+, e_2-]$ which maps $a+$ to b. Use the identity on the complementary set.

(c): A bounded sequence of distinct points in X converges to a point by completeness. Topological homogeneity then implies that every point $x \in X$ is the limit of some sequence in $X \setminus \{x\}$. If $x = a-$ is a left endpoint, then such a sequence must consist of points below x. By going to a subsequence we can assume the sequence $\{y_n\}$ converging to x is increasing. Thus, $\{(y_n, a+)\}$ is a neighborhood base for x. By topological homogeneity it follows that every point has a countable neighborhood base, i.e. X is first countable.

□

Remark: The topological homogeneity argument in (b) is due to Maurice [16].

Corollary 10.4. *If A is any doubly transitive LOTS with completion $F = \hat{A}$, then $X = F \vee A'$ is a perfect, complete, zero-dimensional LOTS which satisfies the clopen interval condition. Furthermore, if $a < b \in A$, then the interval $[a+, b-]$, clopen in X, is a perfect, compact, zero-dimensional LOTS which satisfies the clopen interval condition.*

Proof. This is clear from Theorem 10.3 (a).

□

We turn now to trees of type $2 = \{0, 1\}$. Any tree T of type 2 is isomorphic with a subtree of the simple tree on $2, \alpha$ where $\alpha = h(T)$ and so we will restrict attention to such subtrees so that $p \in T$ with $o(p) = \beta$ is an element of 2^β. In particular, if T is normal with $h(T)$ a limit ordinal, then the height of every branch is a limit ordinal and as in 6.29 we can identify the branch space $X(T)$ with

$$\text{(10.4)}\qquad \begin{aligned}\{x \in 2^\beta : \beta \text{ is an infinite limit ordinal}, x \notin T,\\ \text{and } x|\epsilon \in T \text{ for all } \epsilon < \beta\}.\end{aligned}$$

For $x, y \in X(T)$ we have $x < y$ if there exists $\epsilon = \epsilon(x, y) < h(x), h(y)$ such that $x_i = y_i$ for all $i < \epsilon$, $x_\epsilon = 0$ and $y_\epsilon = 1$.

Lemma 10.5. *Let T be a normal tree of* 2 *type with height a limit ordinal. The pair $x < y \in X(T)$ is a gap pair if and only if there exists $\epsilon < h(x), h(y)$ such that*

$$\text{(10.5)}\qquad \begin{aligned} x_i = y_i \text{ for all } i < \epsilon, \quad x_\epsilon = 0, y_\epsilon = 1, \text{ and}\\ x_j = 1 \text{ for all } \epsilon < j < h(x), \quad y_j = 0 \text{ for all } \epsilon < j < h(y).\end{aligned}$$

Proof. It is clear that if x, y satisfy (10.5), then the interval (x, y) is empty. On the other hand, if for some $\epsilon < j < h(x)$ $x_j = 0$, then there exists $z \in X(T)$ with $z_i = x_i$ for $i < j$ and $z_j = 1$. So $x < z < y$. Similarly, if for some $\epsilon < j < h(y)$ $y_j = 1$ there exists z with $x < z < y$. □

We say that a branch x *eventually equals* 0 if there exists $\beta < h(x)$ such that $x_i = 0$ for all $i \geq \beta$. In that case, we let $\beta^*(x)$ be the minimum of such ordinals β. Similarly, we say that x *eventually equals* 1 if there exists $\beta < h(x)$ such that $x_i = 1$ for all $i \geq \beta$. In that case, we let $\beta^*(x)$ be the minimum of such ordinals β. From Lemma 10.5 we see that $x < y$ is a gap pair if and only if x eventually equals 1, y eventually equals 0 and $\beta^*(x) = \beta^*(y) = \epsilon + 1$. Thus, x is a member of a gap pair if and only if it eventually equals 0 or eventually equals 1 and, in addition, $\beta^*(x)$ is a successor ordinal.

In particular, if we define $\bar{0}, \bar{1} \in 2^{h(T)}$ by $\bar{0}_i = 0, \bar{1}_i = 1$ for all $i < h(T)$, then by normality and (10.4) there are unique ordinals $\gamma_0, \gamma_1 \leq h(T)$ such that $\bar{0}|\gamma_0, \bar{1}|\gamma_1 \in X(T)$. Somewhat abusively, we will denote these branches $\bar{0}$ and $\bar{1}$ so that $h(\bar{0}) = \gamma_0, h(\bar{1}) = \gamma_1$. It is clear that $\bar{0}$ is the minimum element of $X(T)$ and $\bar{1}$ is the maximum.

Theorem 10.6. *If T is a normal tree of type* 2 *with height a limit ordinal, then the branch space $X(T)$ is a perfect, compact, zero-dimensional LOTS. If T is Ω-bounded, then $X(T)$ is first countable.*

Proof. By Proposition 5.7 $X(T)$ is complete. Since 2 is bounded, $X(T)$ is bounded and so is compact. By Proposition 5.12 it is of countable type, and so is first countable, if T is Ω-bounded. From (10.5) it is clear that no left end-point is a right end-point and so there are no isolated points.

Now suppose that $x < z < y$ in $X(T)$. With $\epsilon = \epsilon(x,y)$ we have that $z_i = x_i = y_i$ for all $i < \epsilon$. Suppose that $z_\epsilon = 0$. Since $x < z$, there exists k with $\epsilon < k < h(x), h(z)$ such that $x_k = 0$ and $z_k = 1$. Let $w \in 2^{h(T)}$ with $w_i = z_i$ for all $i \leq k$ and $w_j = 0$ for all $j > k$. By normality and (10.4) there is a limit ordinal γ with $k < \gamma$ such that $w|\gamma \in X(T)$. Thus, $w|\gamma$ is a right end-point which lies between x and y as does its associated left end-point. We proceed similarly if $z_\epsilon = 1$.

Thus, $X(T)$ is gap pair dense and so is zero-dimensional.

□

Corollary 10.7. *If α is a limit ordinal, then 2^α is a perfect, compact, zero-dimensional LOTS. If α is countable, then 2^α is first countable.*

Proof. Apply Theorem 10.6 to the $2, \alpha$ simple tree.

□

Lemma 10.8. *Let T be an additive tree of type 2 with height the tail-like ordinal α. Assume that the branches $\bar{0}, \bar{1} \in X(T)$ have height α.*

(i) *If $x \in X(T)$ eventually equals 0, then there is an isomorphism from $[x, \bar{1}]$ to $X(T) = [\bar{0}, \bar{1}]$.*
(ii) *If $x \in X(T)$ eventually equals 1, then there is an isomorphism from $[\bar{0}, x]$ to $X(T) = [\bar{0}, \bar{1}]$.*

Proof. Observe first that additivity implies that every branch which eventually equals 0 or eventually equals 1 has height α.

Assume x eventually equals 0 and $\beta = \beta^*(x)$. Let $K = \{\beta\} \cup \{k : x_k = 0 \text{ and } k < \beta\}$ and let r be an isomorphism from the well-ordered set $K \subset \alpha$ onto an ordinal $\gamma + 1 \leq \beta + 1 < \alpha$, so that $r(\beta) = \gamma$.

Define $p^\beta = x|\beta$ and for $k \in K$ with $k < \beta$, $p^k_i = x_i$ for $i < k$ and $p^k_k = 1$. Thus, $p^k \in T$ for all $k \in K$ with $o(p^\beta) = \beta$, $o(p^k) = k + 1$ for $k < \beta$. Notice that if $\beta = \gamma + 1$, then $x_\gamma = 1$ by definition of β^*.

Define $q^\beta = \bar{0}|r(\beta)$ and for $k \in K$ with $k < \beta$, $q^k_i = 0$ for $i < r(k)$ and $q^k_{r(k)} = 1$. Notice that $r(\beta) < \alpha = h(\bar{0})$. This is where we need $\alpha = h(\bar{0})$. If, for example, it happened that $h(\bar{0}) < r(\beta)$, then we could not define all the elements q^k.

We show that

$$[x, \bar{1}] = \bigcup_{k \in K} X(T_{p^k}), \quad \text{and} \quad [\bar{0}, \bar{1}] = \bigcup_{k \in K} X(T_{q^k}). \tag{10.6}$$

Note first that $x \in X(T_{p^\beta})$. Now assume $x < y$, and let $\epsilon = \epsilon(x,y)$.

If $\epsilon \geq \beta$, then again $y \in X(T_{p^\beta})$. If $\epsilon < \beta$, then $x_\epsilon = 0$ and $y_\epsilon = 1$. So $\epsilon \in K$ and $y \in T_{p^\epsilon}$.

Next $\bar{0} \in X(T_{q^\beta})$. Now assume $\bar{0} < y$ and let $\epsilon = \epsilon(\bar{0}, y)$.

If $\epsilon \geq r(\beta)$, then again $y \in X(T_{q^\beta})$. If $\epsilon < r(\beta)$, then $y_\epsilon = 1$ and $\epsilon = r(k)$ for some $k \in K$ with $k < \beta$. So $y \in X(T_{q^k})$.

Each of the unions in (10.6) is disjoint. The isomorphism is defined, using additivity, by:

$$p^k + z \mapsto q^k + z \quad \text{for all} \quad k \in K, z \in X(T). \tag{10.7}$$

The proof of (ii) is completely analogous.

□

Theorem 10.9. *If T is an additive tree of type 2 with height the tail-like ordinal α and the branches $\bar{0}, \bar{1} \in X(T)$ have height α, then $X(T)$ is a compact, perfect, zero-dimensional LOTS which satisfies the clopen interval condition.*

Proof. Assume $x < y \in X(T)$ with x eventually equal to 0 and y eventually equal to 1. We prove that the interval $[x, y]$ is isomorphic to $X(T)$. This includes the case when $[x, y]$ is a clopen interval.

Let $\epsilon = \epsilon(x, y)$ so that $x_\epsilon = 0$ and $y_\epsilon = 1$. Let $a = x|(\epsilon + 1), b = y|(\epsilon + 1)$.

The truncations $\tau_{\epsilon+1}(x), \tau_{\epsilon+1}(y)$ are, respectively, eventually equal to 0 and to 1. By Lemma 10.8 there are isomorphisms $f_0 : [\tau_{\epsilon+1}(x), \bar{1}] \to [\bar{0}, \bar{1}]$ and $f_1 : [\bar{0}, \tau_{\epsilon+1}y)] \to [\bar{0}, \bar{1}]$.

Let $0, 1$ denote the elements of level 1 of the tree. Define $f : [x, y] \to X(T)$ by

$$f(z) = \begin{cases} 0 + f_0(\tau_{\epsilon+1}(z)) \ if \ z_\epsilon = 0, \\ 1 + f_1(\tau_{\epsilon+1}(z)) \ if \ z_\epsilon = 1. \end{cases} \tag{10.8}$$

Notice that for $z \in [x, y]$ if $z_\epsilon = 0$, then $z|(\epsilon + 1) = a$ and if $z_\epsilon = 1$ then $z|(\epsilon + 1) = b$. The result follows because T is the disjoint union of T_0 and T_1.

□

As a corollary we obtain the following extension of a theorem of Maurice [15], who proved the result when α is countable.

Corollary 10.10. *If α is an infinite tail-like ordinal, then 2^α is a compact, perfect, symmetric, zero-dimensional LOTS which satisfies the clopen interval condition.*

Proof. Apply Theorem 10.9 to the $2, \alpha$ simple tree. Interchanging 0 and 1, we obtain an order* automorphism of 2^α and so it is symmetric. □

In general if α is a limit ordinal, let $\pi_\alpha : 2^\alpha \to F_\alpha$ be the projection to the connected quotient of the zero-dimensional LOTS 2^α. As in Corollary 10.10 2^α is symmetric and so the quotient F_α is symmetric as well.

Theorem 10.11. *If $\alpha > \beta$ are limit ordinals, then 2^α is bigger than 2^β and F_α is bigger than F_β. In particular, 2^α and 2^β are not isomorphic. The connected quotients F_α and F_β are not homeomorphic.*

Proof. From the order surjection π_α we obtain an order injection $i_\alpha : F_\alpha \to 2^\alpha$. Explicitly we map each $a \in A$ to the left end-point $a-$ of the pair. This induces the injection $i'_\alpha : F'_\alpha \to (2^\alpha)' = 2^{\alpha+1}$.

Assume that $h : 2^\beta \to 2^\alpha$ is an order injection. If $a < b$ in 2^β and $h(a) < h(b)$ is a gap pair in 2^α, then $a < b$ is a gap pair in 2^β. Otherwise, there exists c with $a < c < b$ and so $h(a) < h(c) < h(b)$. It follows that the order map $\pi_\alpha \circ h \circ i_\beta : F_\beta \to F_\alpha$ is injective.

Now choose $z \in 2^{\alpha \setminus \beta}$ with z not eventually equal to 0 or eventually equal to 1. Let $z_0 = 0 + z, z_1 = 1 + z$, with $0, 1$ here regarded as elements of order 1 in the simple tree. Since $\alpha > \beta$, $\alpha \setminus \beta$ is a limit ordinal and so $z_0, z_1 \in 2^{\alpha \setminus \beta}$ with neither eventually equal to 0 or 1.

Define the order injection $f : (2^\beta)' \to 2^\alpha$ by

$$f(a-) \;=\; a + z_0, \quad \text{and} \quad f(a+) = a + z_1. \tag{10.9}$$

By the choice of z it follows that the image of f is disjoint from all of the gap pairs in 2^α. Hence, $\pi_\alpha \circ f : (2^\beta)' \to F_\alpha$ is injective.

Because 2^β and F_β are compact LOTS both are order simple by Corollary 4.11.

From the projection $\pi^\alpha_\beta : 2^\alpha \to 2^\beta$ we obtain an order injection $h : 2^\beta \to 2^\alpha$ and as noted above, $\pi_\alpha \circ h \circ i_\beta : F_\beta \to F_\alpha$ is an order injection. Thus, 2^α is at least as big as 2^β and F_α is at least as big as F_β.

If there were an order injection $q : 2^\alpha \to 2^\beta$, then $q \circ f : (2^\beta)' \to 2^\beta$ would be an order injection, violating the order simplicity of 2^β.

If there were an order injection $q : F_\alpha \to F_\beta$, then $q \circ (\pi_\alpha \circ f) \circ i'_\beta : (F_\beta)' \to F_\beta$ would be an order injection, violating the order simplicity of F_β.

In particular, we see that 2^α is not isomorphic to 2^β and F_α not isomorphic to F_β. By Lemma 3.1 a homeomorphism between connected

LOTS is either an order isomorphism or an order* isomorphism. Since F_α is symmetric, an order* isomorphism would yield an order isomorphism. Thus F_α not homeomorphic to F_β.

$\square$

Remarks: For α and β countable, Maurice [15] proved that 2^α is not isomorphic to 2^β.

The zero-dimensional LOTS 2^ω is isomorphic to the Cantor Set and the connected quotient F_ω is isomorphic to the unit interval in $\mathbb{R}$. Hence, 2^ω embeds in F_ω and, in particular, 2^ω and F_ω have the same size.

11. Appendix: Treybig's Homogeneity Theorem

In this Appendix we present the proof of Treybig's Homogeneity Theorem 3.5.

Let X be a LOTS and $H_+(X)$ be the group of order automorphisms.

Lemma 11.1. *If $f_1, f_2 \in H_+(X)$ and for some $a \in X$, $f_1(a) = f_2(a)$ then $f_3 \in H_+(X)$ where* $f_3(x) = \begin{cases} f_1(x) \text{ for } x \le a, \\ f_2(x) \text{ for } x \ge a. \end{cases}$

Proof. Let $g = f_2^{-1} \circ f_1 \in H_+$ so that $g(a) = a$. Hence $x \ge a \iff g(x) \ge a$. So g' defined to be g on $(-\infty, a]$ and the identity on $[a, \infty)$ is in H_+ and $f_3 = f_2 \circ g'$.

$\square$

Lemma 11.2. *If $f_1, f_2 \in H_+(X)$ and for some $a \in X$, $f_1(a) > f_2(a)$, then there exists $f_3 \in H_+(X)$ which equals f_1 on an open interval containing a and such that $f_3(x) \ge f_2(x)$ for all x.*

Proof. As before let $g = f_2^{-1} \circ f_1 \in H_+$ so that $g(a) > a$. Inductively, $k \mapsto g^k(a)$ defines an increasing bi-infinite sequence for $k \in \mathbb{Z}$ and $g(x) > x$ for all $x \in J = \bigcup_{k \in \mathbb{Z}} [g^k(a), g^{k+1}(a)] = \bigcup_{k \in \mathbb{Z}} (g^{k-1}(a), g^{k+1}(a))$. So J is an open convex set which contains a and $g(J) = J$. Points above J are mapped above J and the points below J are mapped below J. So g' defined to be g on J and the identity otherwise is in H_+. Let $f_3 = f_2 \circ g'$.

$\square$

Lemma 11.3. *Assume X is complete. If there exists a sequence $\{f_n\} \subset H_+$ which converges pointwise to the identity on X and such that $f_n(x) < x$ for all $x \in X$ and $n \in \mathbb{N}$, then for each $u \in X$ $\{f_n^k(u) : n \in \mathbb{N}, k \in \mathbb{Z}\}$ is dense in X. In particular, X is separable.*

Proof. If $\{f_n^k(u)\}$ were bounded below for some n then the infimum would be a fixed point of f_n, contradicting the assumption that $f_n(x) < x$ for all x. Similarly, the sequence is not bounded above for any n.

Let $a < b \in X$. By pointwise convergence there exists $n \in \mathbb{N}$ such that $a < f_n(b) < b$.

Because the orbits are unbounded the set $K = \{f_n^j(u) : j \in \mathbb{Z}$ and $f_n^j(u) < b\}$ is nonempty. Let z be its supremum so that $z \leq b$. There exists j such that $f_n(z) < f_n^j(u) \leq z$. If $z \leq a$ then $z < f_n^{j-1}(u) \leq f_n^{-1}(z) \leq f_n^{-1}(a) < b$. That is, $f_n^{j-1}(u) \in K$ violating the definition of z.

Since $a < z$ there exists $f_n^k(u) \in K$ with $a < f_n^k(u)$. That is, $f_n^k(u)$ is in the open interval (a, b). Density follows.

□

Remark: Completeness is needed. Let $X = \mathbb{R} \times \mathbb{R}$ which is not complete and not separable. Let $f_n(x_1, x_2) = (x_1, x_2 - \frac{1}{n})$.

For the transitive LOTS $\mathbb{Z}$ for each pair $a, b \in \mathbb{Z}$ there is a unique element $f \in H_+(\mathbb{Z})$ such that $f(a) = b$. This *translational uniqueness* does not occur for any nontrivial connected LOTS.

Proposition 11.4. *If X is a nontrivial connected, transitive LOTS, then it is not true that for every $a, b \in X$ there is a unique element $f \in H_+(X)$ such that $f(a) = b$.*

Proof. Assume instead that translational uniqueness holds for X. This implies that if $f, g \in H_+$ and $f(a) = g(a)$ for some a then $f(x) = g(x)$ for all x. It follows that if $f(a) > g(a)$, then $f(x) > g(x)$ for all x, because if $f(b) \leq g(b)$ for some b then $f(c) = g(c)$ for some c between a and b, because X is connected.

Let $\{a_n\}$ be an increasing sequence in X converging to a. Let $f_n \in H_+$ with $f_n(a) = a_n$. Hence, for all x $f_n(x) < f_{n+1}(x) < x = 1(x)$. Where $1 \in H_+$ is the identity map.

Claim: For all x $\{f_n(x)\}$ converges to x.

If not, then there is some b such that $c = sup\{f_n(b)\} < b$. Fix $g \in H_+$ with $g(b) = c$. Since for all n, $f_n(b) < c = g(b) < 1(b)$, it follows that $f_n(x) < g(x) < x$ for all x. Applied with $x = a$ this contradicts the convergence of $\{a_n = f_n(a)\}$ to a.

Thus, $\{f_n\}$ converges to 1 pointwise.

From Lemma 11.3 it follows that X is separable and so, since it is connected and unbounded, it is isomorphic to $\mathbb{R}$. However, translational uniqueness clearly does not hold for $\mathbb{R}$.

□

Corollary 11.5. *If X is a nontrivial connected, transitive LOTS, then one of the following holds.*

(i) *For all $a \in X$ there exists $f \in H_+(X)$ and $b \in X$ with $a < b$ such that $f(x) = x$ for all $x \leq a$ and $f(x) < x$ for all $x \in (a, b]$.*

(ii) *For all $a \in X$ there exists $f \in H_+(X)$ and $b \in X$ with $b < a$ such that $f(x) = x$ for all $x \geq a$ and $f(x) > x$ for all $x \in [b, a)$.*

Proof. From Proposition 11.4 there exist $f_1, f_2 \in H_+$ and $s, t \in X$ such that $f_1(t) = f_2(t)$ and $f_1(s) > f_2(s)$. Letting $g = f_2^{-1} \circ f_1$ we have $g(t) = t$ and $g(s) > s$.

Assume $t < s$. Let J be the connected component containing s of the open set $\{x : g(x) > x\}$ and let t_1 be its infimum, i.e. its left endpoint. Hence, $t \leq t_1$ and $g(t_1) = t_1$. Thus, $g(x) > x$ for all $x \in (t_1, s] \subset J$.

For any $a \in X$, let $h \in H_+$ such that $h(a) = t_1$ and let $r = h^{-1} \circ g \circ h$ so that $r(a) = a$ and $r(x) > x$ for all $x \in (a, b]$ with $b = h^{-1}(s)$. We can apply Lemma 11.1 to replace r by r' which agrees with r on $[a, \infty)$ and with the identity on $(-\infty, a]$.

Let $f = r'^{-1}$.

This is case (i). We similarly obtain (ii) if $s < t$.

□

Lemma 11.6. *Let X be a nontrivial connected, transitive LOTS and let $a, b, c \in X$ with $a < b$. For all $x > c$ there exists $y \in (c, x)$ and $f \in H_+$ mapping $[a, b]$ to $[c, y]$.*

Proof. Let U be the set of $x > c$ such that $[a, b]$ can be mapped onto $[c, x]$ by a member of H_+ and let V be the set of $x < c$ such that $[a, b]$ can be similarly mapped onto $[x, c]$. Observe that by transitivity U and V are nonempty. Let $u = inf\ U$ and $v = sup\ V$. Our goal is to prove $u = c$. Suppose instead that $c < u$.

Case 1: Assume that (i) of Corollary 11.5 holds. Choose $w_1 \in X, g \in H_+$ so that $c < w_1 < u$ and $g(x) = x$ for $x \in (-\infty, w_1]$ and $g(x) < x$ for $x \in (w_1, w_2]$ with $w_1 < w_2 < u$. Choose $w \in (w_1, w_2)$ and $k \in H_+$ such that $k(w) = u$. By Lemma 11.2 we can choose k

so that $k(x) \geq x$ for all $x \in X$. Define $u_2 = k(w_2) > k(w) = u$ and $u_1 = k(w_1) \geq w_1 > c$. Hence, $c < u_1 < u < u_2$. Let $f = k \circ g \circ k^{-1}$ so that $f(x) = x$ for $x \in (-\infty, u_1]$ and $f(x) < x$ for $x \in (u_1, u_2]$. It follows that for any $y \in (u_1, u_2]$ the sequence $f^n(y)$ is decreasing with limit u_1.

There exists $x_0 \in (u, u_2)$ and $f_0 \in H_+$ which maps $[a, b]$ onto $[c, x_0]$. Clearly, $f^n \circ f_0$ maps $[a, b]$ onto $[c, f^n(x_0)]$. Because the sequence $\{f^n(x_0)\}$ converges to u_1, it follows that for sufficiently large n, $f^n(x_0) < u$. This contradicts the definition of u.

Case 2: Assume that (ii) of Corollary 11.5 holds. From an argument similar to the one for Case 1, it follows that $v = c$. Choose $u_1 \in (c, u)$ and $g \in H_+$ so that $u_1 = g(c)$. Hence, $g^{-1}(c) < c$. Because $v = c$, there exists $h \in H_+$ mapping $[a, b]$ onto $[z, c]$ with z a point of $(g^{-1}(c), c)$. Let $u_2 = g(z)$ so that $c < u_2 < u_1 < u$. $g \circ h$ maps $[a, b]$ onto $[u_2, u_1]$. Let $k \in H_+$ with $k(a) = c$. By definition of u, $k(b) \geq u$. Since $k(a) < (g \circ h)(a)$ and $k(b) > (g \circ h)(b)$, there exists $t \in (a, b)$ such that $k(t) = (g \circ h)(t)$. Apply Lemma 11.1 to define f to equal k on $(-\infty, t]$ and equal to $g \circ h$ on $[t, \infty)$. Since f maps $[a, b]$ onto $[c, u_1]$ we again contradict the definition of u.

□

Now we complete the proof of Treybig's Theorem.

Theorem 11.7. *If X is a nontrivial connected, transitive LOTS, then X is doubly transitive.*

Proof. Given $a < b$ and $c < d$. Choose $g \in H_+$ with $g(b) = d$.

Case 1: $g(a) \leq c$. There exists h mapping $[a, b]$ to $[c, e]$ with $c < e < d$. Because $g(a) \leq c = h(a)$ and $g(b) = d > e = h(b)$, there exists $t \in [a, b]$ such that $g(t) = h(t)$. If $f = h$ on $(-\infty, t]$ and $f = g$ on $[t, \infty)$ then f maps $[a, b]$ to $[c, d]$.

Case 2: $g(a) > c$. There exists h^{-1} mapping $[c, d]$ onto $[a, e]$ with $a < e < b$. So h maps $[a, e]$ onto $[c, d]$. Because $h(a) = c < g(a)$ and $h(e) = d = g(b) > g(e)$, there exists $t \in (a, e)$ such that $h(t) = g(t)$. For f use h on $(-\infty, t]$ and g on $[t, \infty)$ to map $[a, b]$ to $[c, d]$.

□

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