

Crossing-symmetric Twist Field Correlators and Entanglement Negativity in Minimal CFTs

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ABSTRACT: We study conformal twist field four-point functions on a \mathbb{Z}_N orbifold. We examine in detail the case $N = 3$ and analyze theories obtained by replicated N -times a minimal model with central charge $c < 1$. A fastly convergent expansion of the twist field correlation function in terms of sphere conformal blocks with central charge Nc is obtained by exploiting covering map techniques. We discuss extensive applications of the formalism to the entanglement of two disjoint intervals in CFT, in particular we propose a conformal block expansion for the partially transposed reduced density matrix. Finally, we refine the bounds on the structure constants of unitary CFTs determined previously by the genus two modular bootstrap.

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1 Introduction

The conformal bootstrap approach in two-dimensions [1] exploits the infinite-dimensional Virasoro algebra together with the associativity of the Operator Product Expansion (OPE) in the attempt to classify all possible Conformal Field Theories (CFTs). In short [2], the associativity of the conformal algebra leads to functional relations between its special

functions dubbed conformal blocks, which can be solved, under certain assumptions on the spectrum of the theory, either analytically or numerically.

In the last thirty years, the bootstrap technique allowed calculating exactly or with great precision correlation functions of many two-dimensional critical statistical models [1]. It also led to the solution of two-dimensional quantum gravity, the Liouville theory [3]. Despite these successes, there are still critical phenomena in two dimensions for which a complete bootstrap solution is lacking. We can mention for instance geometrical phase transitions such as percolation [4], recently analyzed in deep in [5, 6], or disordered systems [7, 8]. In particular, the latter are described by coupling copies of the original pure CFT to a relevant field, and then flow to a new fixed point whose properties are largely unknown, see [9–11].

Besides their applications to disordered systems, CFTs on a replicated space-time geometry are familiar also in the context of quantum information. Partition functions of CFTs on Riemann surfaces with \mathbb{Z}_N symmetry can be interpreted as powers of reduced density matrices for subsystems embedded into an extended quantum state, either pure or mixed [12, 13]. From the reduced density matrix, one extracts entanglement measures such as Rényi entropies [14–16] or the logarithmic negativity [17–19]. Finally, the study of CFTs with cyclic or permutational symmetry has provided an explicit verification of the AdS/CFT correspondence [20]. A possible approach to CFTs defined on a \mathbb{Z}_N (or S_N) symmetric space-time exploits their mapping [21] to the orbifold theory $\text{CFT}^{\otimes N}/\mathbb{Z}_N$. Instead of considering a single theory on a Riemann surface with \mathbb{Z}_N symmetry and central charge c , one studies the N -fold tensor product $\text{CFT}^{\otimes N}/\mathbb{Z}_N$ with central charge Nc on the sphere. The multivaluedness of the correlators, when their argument encircles a ramification point of the Riemann surface, is implemented by local fields called twist fields [22, 23]. Higher genus partition functions are then mapped to multipoint correlators of twist fields.

In this paper, we will analyze the CFT partition function on a one-parameter ($0 < x < 1$) family of genus $N - 1$ Riemann surfaces, specified in Sec. 2 and already considered in [24]. Due to the \mathbb{Z}_N symmetry of the space-time, the CFT partition function can be expressed in terms of the four-point correlator of the \mathbb{Z}_N twist field on the sphere. Modular invariance of the partition function is implemented by the transformation $x \mapsto 1 - x$ and reduces to the crossing symmetry of the twist field four-point function, see also [25]. For $N = 2$ and $N = 3$, the study of the crossing symmetry equations restricts respectively the operator content and the OPE coefficients of the original conformal theory [26–29], with central charge c , on the sphere.

Already for \mathbb{Z}_3 -symmetric Riemann surfaces with genus two, the determination of crossing symmetric twist field correlation functions is challenging. For instance, they are not known explicitly even for the simplest CFTs such as the minimal models [1] with $c < 1$. On the one hand, if the seed CFT has a finite number of OPE channels, i.e. is rational, the twist field correlators expand over a finite number of special functions dubbed orbifold conformal blocks. On the other hand, differently from what happens on the sphere, the latter cannot be calculated efficiently through a recursion relation [28, 30]. In practice, one sets up a combinatorial expansion of the orbifold conformal blocks, which produces a slowly convergent power series about $x = 0$. By truncating the power series in x , the

crossing symmetry properties of the twist field correlator, and the modular invariance of the higher genus partition function, are poorly reproduced. Here, we revisit the problem by focussing on the minimal models. We propose a systematic expansion of the \mathbb{Z}_3 -orbifold conformal blocks that allows building crossing symmetric twist field four-point functions with significantly better accuracy than previous attempts [31, 32]. An alternative approach which involves the formulation of differential equations satisfied by the partition functions has been put forward in [33]. Applications to entanglement measures and the modular bootstrap will be also investigated.

The outline of the paper is as follows. In Sec. 2, we introduce our notations for a CFT on a Riemann surface with \mathbb{Z}_N symmetry and briefly review the orbifold construction. The orbifold conformal blocks at $N = 2, 3$ and their small- x expansion, obtained through the covering map [21] procedure, are discussed in Sec. 3. It will be further presented a regularization prescription for the singularities in the power series that are produced, for $c < 1$, by the states with zero norm. The contribution of the $\text{CFT}^{\otimes N}$ descendants to the small- x expansion reorganizes [24] into conformal blocks of primary fields with respect to a Virasoro algebra with central charge Nc . Once this decomposition is found, we show in Sec. 4, how the convergence of the orbifold conformal blocks can be improved [31] by using the elliptic recursion formula [34]. Refs. [31, 32] wrote down analogous expansions in terms of Virasoro conformal blocks with central charge Nc but they only took into account the orbifold primary contributions.

We first apply the formalism to entanglement measures in tripartite systems at zero temperature. In particular, we calculate in Sec. 5 power series representations of the trace of the third power of the reduced density matrix and its partial transpose for two disjoint intervals. The two traces are distinguished field theoretically by the presence of fields with non-zero conformal spins. Prior calculations of the partial transpose of the reduced density matrix, which enters in the so-called logarithmic negativity, were performed for free fermions [19, 35–42] or in the large central charge limit [43–45]. Our approach reproduces the free fermion results [35]. Finally, in Sec. 5, we show that for $N = 3$, our expansion of the orbifold conformal blocks can incrementally improve the OPE fusion coefficients bounds found in [28]. Moreover, we demonstrate that, within our scheme, the numerical bootstrap approach to the genus two twist field correlation functions converges. These new results and observations can pave the way to the solution of a long-standing problem: the determination via a bootstrap approach of the mutual information and logarithmic negativities in interacting CFTs.

2 CFT Partition Functions on \mathbb{Z}_N -symmetric Riemann Surfaces

Let us consider a CFT denoted by \mathcal{C} , with central charge c , defined on a Riemann surface $\Sigma_g(x)$ of genus g . In the following, we will refer to \mathcal{C} as the seed theory. We will restrict to the family of Riemann surfaces $\Sigma_g(x)$ of genus $g = N - 1$ described by the complex algebraic curve

$$w^N = \frac{z(z-1)}{z-x}, \quad (2.1)$$

which has branch points of order N at $z_b = 0, x, 1$, and ∞ . Although Eq. (2.1) can be extended to complex values of x , in the following, we will always assume x to be a real variable, $x = \bar{x}$, with $0 < x < 1$. Eq. (2.1) can be interpreted [46] as a N -sheeted cover of the compactified complex plane (Riemann sphere) $\mathbb{C} \cup \{\infty\}$ with coordinate z . Furthermore, the surfaces $\Sigma_g(x)$ possess a \mathbb{Z}_N symmetry since Eq. (2.1) is invariant under the transformation $w \mapsto e^{\frac{2\pi i k}{N}} w$, with $k = 0, 1, \dots, N-1$. This transformation amounts to a cyclic permutation of the N sheets of the surface, where each sheet is labelled by the choice of the branch of the N -th root in Eq. (2.1).

The CFT partition function $\mathcal{Z}_g(x)$ on $\Sigma_g(x)$ depends on the choice of the metric within the same conformal class. We choose then a flat metric. The partition function with a flat metric everywhere on the surface can be derived from the orbifold $\mathcal{C}^{\otimes N}/\mathbb{Z}_N$ [21]. In this theory, there exist \mathbb{Z}_N twist and anti-twist fields σ_N and $\tilde{\sigma}_N$, which are spinless primary fields of conformal dimension [22, 23]

$$h_{\sigma_N} = \bar{h}_{\sigma_N} = \frac{c}{24} \left(N - \frac{1}{N} \right). \quad (2.2)$$

When inserted on the complex plane at the branch points of the algebraic curve in Eq. (2.1), they implement the multivaluedness of correlation functions under the analytic continuation $(z - z_b) \mapsto (z - z_b)e^{2\pi i}$. One then finds that

$$\mathcal{Z}_g(x) = e^{cS_{\text{anom.}}(x)} \langle \sigma_N(\infty) \tilde{\sigma}_N(1) \sigma_N(x, \bar{x}) \tilde{\sigma}_N(0) \rangle, \quad (2.3)$$

where the prefactor $e^{cS_{\text{anom.}}(x)}$ in Eq. (2.3) is the Weyl anomaly which can be explicitly calculated [21]. It takes into account that in the orbifold approach the metric employed to determine the partition function on $\Sigma_g(x)$ is a flat metric on each sheet of the surface but with conical singularities at the location of the twist fields. Consider, for instance, the case $N = 2$, for which, under the Abel-Jacobi map [47], $\Sigma_1(x)$ is conformally equivalent to a flat torus of modulus

$$\tau(x) = i \frac{K(1-x)}{K(x)}, \quad (2.4)$$

where $K(x)$ is the complete elliptic integral of first kind [48]. By evaluating the conformal anomaly in Eq. (2.3), one has [21]

$$\mathcal{Z}_1(x) = |2^8 x(1-x)|^{c/12} \langle \sigma_2(\infty) \tilde{\sigma}_2(1) \sigma_2(x, \bar{x}) \tilde{\sigma}_2(0) \rangle. \quad (2.5)$$

The partition function $\mathcal{Z}_g(x)$ is invariant under modular transformations [26, 49, 50]. For the class of surfaces $\Sigma_g(x)$, the moduli space is one-dimensional and modular invariance implies the crossing symmetry of the twist field four-point correlation function [27]

$$\langle \sigma_N(\infty) \tilde{\sigma}_N(1) \sigma_N(1-x, 1-\bar{x}) \tilde{\sigma}_N(0) \rangle = \langle \sigma_N(\infty) \tilde{\sigma}_N(1) \sigma_N(x, \bar{x}) \tilde{\sigma}_N(0) \rangle. \quad (2.6)$$

For example, if we consider a torus with modulus τ that of Eq. (2.4), then the modular transformation $\tau \mapsto -1/\tau$ implies $x \mapsto 1-x$. An analogous observation holds for $\Sigma_2(x)$, as discussed in [27]. Eq. (2.6) can be actually extended analytically to complex values of x , see for instance Sec. 5.2.

3 Orbifold Conformal Blocks

The twist field four-point correlator in Eq. (2.3) as a function of $x \in (0, 1)$ can be analytically continued to $z \in \mathbb{C}$. In the orbifold $\mathcal{C}^{\otimes N}/\mathbb{Z}_N$, it admits the following decomposition

$$\langle \sigma_N(\infty) \tilde{\sigma}_N(1) \sigma_N(z, \bar{z}) \tilde{\sigma}_N(0) \rangle = \sum_{\mathbf{h}, \bar{\mathbf{h}}} D_{\mathbf{h}, \bar{\mathbf{h}}} \mathcal{G}_{c, \mathbf{h}}^{(N)}(z) \mathcal{G}_{c, \bar{\mathbf{h}}}^{(N)}(\bar{z}), \quad (3.1)$$

where $\mathbf{h} \equiv \{h_1, \dots, h_N\}$ and h_j is the conformal dimension of a primary field of the seed theory. The functions $\mathcal{G}_{c, \mathbf{h}}^{(N)}(z)$, defined below, will be termed orbifold conformal blocks. They are normalized such that for small $|z|$

$$\mathcal{G}_{c, \mathbf{h}}^{(N)}(z) = z^{|\mathbf{h}| - 2h_{\sigma_N}} [1 + O(z)], \quad (3.2)$$

where $|\mathbf{h}| = \sum_j h_j$. In this section and Sec. 4, we will show how to extract systematically, by means of the orbifold conformal algebra, higher order terms in the expansion about $z = 0$ in Eq. (3.2). The case $N = 3$ has been discussed in [28] from which some notations are borrowed. The structure constants $D_{\mathbf{h}, \bar{\mathbf{h}}}$ in Eq. (3.1) are not algebraically determined and instead characterize the specific bootstrap solution under consideration.

We focus first on the holomorphic sector of the seed CFT. We denote by $\phi_h(z)$ the holomorphic primary field with conformal dimension h and by $\phi_h^M(z)$ one of its descendants. The descendants are labelled by the partition $M \equiv \{m_1, \dots, m_q\}$, $1 \leq m_1 \leq m_2 \leq \dots \leq m_q$, of the positive integer $|M| = \sum_j m_j$. In terms of the Virasoro generators $L_{-m}(z)$, defined in Eq. (A.1), the holomorphic field $\phi_h^M(z)$ is then

$$\phi_h^M(z) = L_{-M}(z) \phi_h(z), \quad \text{where } L_{-M}(z) = L_{-m_1}(z) \dots L_{-m_q}(z). \quad (3.3)$$

The field $\phi_h^M(z)$ has conformal dimension $h + |M|$.

We will employ the field-state correspondence $|\phi_h\rangle \equiv \lim_{z \rightarrow 0} \phi_h(z)|0\rangle$, with $|0\rangle$ the vacuum in \mathcal{C} , and the Virasoro scalar product [51]. The latter can be defined by constructing the dual Hilbert space through the identification $\langle \phi_h| \equiv \lim_{z \rightarrow \infty} z^{2h} \langle 0| \phi_h(z)$, where $\langle 0|$ is the dual of the vacuum state. Furthermore, we denote by G_{M_1, M_2}^h the matrix of scalar products

$$G_{M_1, M_2}^h = \langle \phi_h^{M_1} | \phi_h^{M_2} \rangle. \quad (3.4)$$

If not stated otherwise, in what follows we consider irreducible Verma module representations [2], i.e. there are not descendant states with vanishing norm, referred here as null vectors. The role of null vectors will be analyzed in more detail in Sec. 3.4. The fields that enter the correlation functions are specified by gluing the holomorphic and anti-holomorphic sectors of \mathcal{C} and are tensor products of the form

$$\phi_h^M(z) \phi_{\bar{h}}^{\bar{M}}(\bar{z}). \quad (3.5)$$

In all applications, we will actually restrict ourselves to the case of diagonal seed CFTs, i.e. theories that have a diagonal partition function on the torus [49]. In this case, in Eq. (3.5), only couplings between holomorphic and antiholomorphic fields with the same

labels $h = \bar{h}$ are possible. However, in this section we prefer to maintain the discussion more general and include the possibility of non-diagonal couplings. In the tensor product $\mathcal{C}^{\otimes N}$, an holomorphic primary $\phi_{\mathbf{h}}$, labelled by a set \mathbf{h} of conformal dimensions, is the tensor product of holomorphic primary fields of the seed theory

$$\phi_{\mathbf{h}}(z) = \phi_{h_1}(z) \otimes \cdots \otimes \phi_{h_N}(z), \quad (3.6)$$

and has conformal dimension $|\mathbf{h}|$. If $\mathbf{M} \equiv \{M_1, \dots, M_N\}$ stands for a collection of N partitions of the positive integers $|M_1|, \dots, |M_N|$, then the descendants of $\phi_{\mathbf{h}}$ will be indicated by $\phi_{\mathbf{h}}^{\mathbf{M}}(z)$,

$$\phi_{\mathbf{h}}^{\mathbf{M}}(z) = \phi_{h_1}^{M_1}(z) \otimes \cdots \otimes \phi_{h_N}^{M_N}(z). \quad (3.7)$$

To complete the fields in Eq. (3.7) to a basis for a representation of the tensor product of the Virasoro algebra, we shall allow M_j to be the empty set, with the convention that $\phi_{h_j}^{\emptyset} \equiv \phi_{h_j}$. The corresponding scalar product matrix $\mathbf{G}_{\mathbf{M}\mathbf{N}}^{\mathbf{h}}$, of size $\prod_{j=1}^N |M_j|^2 \times \prod_{j=1}^N |N_j|^2$, is defined from the scalar product in \mathcal{C} as

$$\mathbf{G}_{\mathbf{M}\mathbf{N}}^{\mathbf{h}} = \langle \phi_{\mathbf{h}}^{\mathbf{M}} | \phi_{\mathbf{h}}^{\mathbf{N}} \rangle = \prod_{j=1}^N G_{M_j, N_j}^{h_j}. \quad (3.8)$$

The construction of the anti-holomorphic sector is, under the replacement $z \rightarrow \bar{z}$, the same as the one presented above and the fields in $\mathcal{C}^{\otimes N}$ are then the tensor products $\phi_{\mathbf{h}}^{\mathbf{M}}(z) \phi_{\bar{\mathbf{h}}}^{\bar{\mathbf{M}}}(\bar{z})$. The expansion about $z = 0$ of $\mathcal{G}_{c, \mathbf{h}}^{(N)}(z)$ can be determined by inserting the resolution of the identity in the representation \mathbf{h} . By using the basis in Eq. (3.7), i.e. including among the elements of \mathbf{M} and \mathbf{N} also the empty set, it follows that

$$\mathcal{G}_{c, \mathbf{h}}^{(N)}(z) = z^{|\mathbf{h}| - 2h_{\sigma_N}} \sum_{\mathbf{M}, \mathbf{N}} z^{|\mathbf{M}|} \tilde{\rho}_{\mathbf{M}}^{\mathbf{h}} [\mathbf{G}_{\mathbf{M}, \mathbf{N}}^{\mathbf{h}}]^{-1} \rho_{\mathbf{N}}^{\mathbf{h}}, \quad (3.9)$$

where $\rho_{\mathbf{N}}^{\mathbf{h}}$ and $\tilde{\rho}_{\mathbf{M}}^{\mathbf{h}}$ are matrix elements between descendant fields. In terms of the orbifold structure constants,

$$C_{\mathbf{h}, \bar{\mathbf{h}}} \equiv \langle \phi_{\mathbf{h}} \phi_{\bar{\mathbf{h}}} | \sigma_N(1) | \tilde{\sigma}_N \rangle, \quad \tilde{C}_{\mathbf{h}, \bar{\mathbf{h}}} \equiv \langle \phi_{\mathbf{h}} \phi_{\bar{\mathbf{h}}} | \tilde{\sigma}_N(1) | \sigma_N \rangle, \quad (3.10)$$

one has

$$\rho_{\mathbf{M}}^{\mathbf{h}} = \frac{\langle \phi_{\mathbf{h}}^{\mathbf{M}} \phi_{\bar{\mathbf{h}}} | \sigma_N(1) | \tilde{\sigma}_N \rangle}{C_{\mathbf{h}, \bar{\mathbf{h}}}}, \quad \tilde{\rho}_{\mathbf{M}}^{\mathbf{h}} = \frac{\langle \phi_{\mathbf{h}}^{\mathbf{M}} \phi_{\bar{\mathbf{h}}} | \tilde{\sigma}_N(1) | \sigma_N \rangle}{\tilde{C}_{\mathbf{h}, \bar{\mathbf{h}}}}. \quad (3.11)$$

The matrix elements $\rho_{\mathbf{M}}^{\mathbf{h}}$ and $\tilde{\rho}_{\mathbf{M}}^{\mathbf{h}}$ are entirely fixed by the holomorphic part of the orbifold conformal algebra. Even if it is not manifest in their expressions in Eq. (3.11), they are complex rational functions of the dimensions h_j and the central charge c . We will show how to compute them in the next section. Due to the symmetry properties of the cyclic twist and anti-twist fields, $\tilde{\rho}_{\mathbf{M}}^{\mathbf{h}}$ is the complex conjugate of $\rho_{\mathbf{M}}^{\mathbf{h}}$.

The structure constants in Eq. (3.10), on the other hand, encode the way the holomorphic and anti-holomorphic sector are glued to build the twist correlation function of the model under consideration. Plugging Eq. (3.9) into Eq. (3.1), one concludes that

$$D_{\mathbf{h}, \bar{\mathbf{h}}} = C_{\mathbf{h}, \bar{\mathbf{h}}} \tilde{C}_{\mathbf{h}, \bar{\mathbf{h}}}. \quad (3.12)$$

3.1 The Computation of the Orbifold Three-point Functions

The computation of the twist field four-point function boils down to determine the orbifold three-point functions, see Eq. (3.11),

$$\rho_M^h C_{\mathbf{h}, \bar{\mathbf{h}}} = \langle \phi_{\mathbf{h}}^M \phi_{\bar{\mathbf{h}}} | \sigma_N(1) | \tilde{\sigma}_N \rangle. \quad (3.13)$$

These quantities can be calculated by considering a N -to-one conformal map $t \mapsto z(t)$ with branch points at $z_b = \{0, 1\}$ such that, near these points, it behaves as $z - z_b \sim (t - t_b)^N$. That is, the t -surface must be a N -sheeted cover with genus zero, a Riemann sphere, of the complex plane with a branch cut along $z \in (0, 1)$. Moreover, the point $z = \infty$, where the holomorphic field $\phi_{\mathbf{h}}^M$ in Eq. (3.13) is inserted, must be mapped to N different points $t_\infty = \{t_1, \dots, t_N\}$ in the covering space. Let us denote by $\hat{\phi}_{h_j}^{M_j}(t_j)$ the images of the field $\phi_{h_j}^{M_j}(\infty)$ under the covering map $t \mapsto z(t)$, that is

$$\hat{\phi}_{h_j}^{M_j}(t_j) = \left(\frac{dz(t)}{dt} \right)^{-h_j} \Big|_{t=t_j} \mathcal{L}_{-M_j}(t_j) \phi_{h_j}(t_j), \quad (3.14)$$

where $\mathcal{L}_{-M_j}(t_j)$ is the pullback of the Virasoro operator $L_{-M_j}(z = \infty)$, see Eq. (A.6). The pullback is a linear combination of Virasoro generators acting at the point t_j in the t -plane, as we discuss in detail in Appendix A. The holomorphic part of the three-point function of the $\mathcal{C}^{\otimes N}/\mathbb{Z}_N$ orbifold in Eq. (3.13) is then equal to the N -point function of the seed theory \mathcal{C} on a sphere

$$\langle \phi_{\mathbf{h}}^M(\infty) \sigma_N(1) \tilde{\sigma}_N(0) \rangle = \langle \hat{\phi}_{h_1}^{M_1}(t_1) \cdots \hat{\phi}_{h_N}^{M_N}(t_N) \rangle. \quad (3.15)$$

3.2 Case $N = 2$: The Torus

We now illustrate the method discussed in the previous section when $N = 2$. In this case, the orbifold three-point function of Eq. (3.13) can be calculated by considering the two-to-one map

$$z(t) = \frac{(t+1)^2}{4t}, \quad (3.16)$$

which transforms $t_\infty = \{0, \infty\}$ into $z = \infty$ and has branch points of order two at $z_b = \{0, 1\}$. By applying Eq. (3.14) and Eq. (3.15), the three-point functions in Eq. (3.13) reduce to scalar products. In particular, one has, see also Eq. (3.10),

$$C_{\mathbf{h}, \bar{\mathbf{h}}} = \langle \hat{\phi}_{h_1} \hat{\phi}_{\bar{h}_1} | \hat{\phi}_{h_2} \hat{\phi}_{\bar{h}_2} \rangle = \delta_{h_1, h_2} \delta_{\bar{h}_1, \bar{h}_2} 2^{-4(h_1 + \bar{h}_1)}, \quad (3.17)$$

while ρ^{h_1, h_2} is the symmetric $|M_1| \times |M_1|$ matrix (descendant fields at different levels are orthogonal)

$$\rho_{M_1, M_2}^{h_1, h_2} = \langle \mathcal{L}_{-M_1} \phi_{h_1} | \mathcal{L}_{-M_2} \phi_{h_2} \rangle \delta_{h_1, h_2}. \quad (3.18)$$

The explicit expression of the pullback \mathcal{L}_{-n} under the map of Eq. (3.16) is written in Eq. (A.8) of Appendix A. The matrix ρ^{h_1, h_2} in Eq. (3.18) can be easily calculated by exploiting the Virasoro algebra commutation relations. Moreover, in the case $N = 2$, the twist and anti-twist fields are identified ($\sigma_2 = \tilde{\sigma}_2$) and, from Eqs. (3.10) and (3.11), we conclude that $\tilde{C}_{\mathbf{h}, \bar{\mathbf{h}}} = C_{\mathbf{h}, \bar{\mathbf{h}}}$ and $\tilde{\rho}_{M_1, M_2}^{h_1, h_2} = \rho_{M_1, M_2}^{h_1, h_2}$.

Finally, by substituting Eq.(3.18) in Eq.(3.9), we arrive at a power series expansion of the $N = 2$ orbifold conformal block

$$\mathcal{G}_{c,\{h_1,h_2\}}^{(2)}(z) = \delta_{h_1,h_2} z^{h_1+h_2-\frac{c}{8}} \sum_{M_1,M_2} \sum_{N_1,N_2} z^{|M_1|+|M_2|} \rho_{M_1,M_2}^{h_1,h_2} \prod_{j=1}^2 [G_{M_j,N_j}^{h_j}]^{-1} \rho_{N_1,N_2}^{h_1,h_2}. \quad (3.19)$$

By recalling that the matrices G^{h_j} and ρ^{h_1,h_2} are symmetric, the coefficients of the combinatorial expansion in Eq. (3.19) can be organized as matrix products, presented in the diagram of Fig. 1. This observation helps with the organization of the bookkeeping of the states.

Eq. (3.19) shows that the number of $N = 2$ orbifold conformal blocks is the same as the number of conformal families of the seed theory \mathcal{C} . This conclusion is of course consistent with the well known construction of the CFT partition function on a flat torus as a sesquilinear form of the irreducible Virasoro characters $\chi_{c,h}(\tau(x))$ [49, 50],

$$\mathcal{Z}_1(x) = \sum_{h,\bar{h}} n_{h,\bar{h}} \chi_{c,h}(\tau(x)) \chi_{c,\bar{h}}(\bar{\tau}(x)). \quad (3.20)$$

We can relate the conformal block in Eq. (3.19) and the Virasoro character by considering theories with a diagonal partition function on the torus. In this case, the multiplicities in Eq. (3.20) are $n_{h,\bar{h}} = \delta_{h,\bar{h}}$ and only couplings between holomorphic and anti-holomorphic fields belonging to the same Virasoro algebra representation are allowed. Eq. (3.17) must be now supplemented by the constraint $\mathbf{h} = \bar{\mathbf{h}}$ and we then derive from Eq. (3.12)

$$D_{\mathbf{h},\bar{\mathbf{h}}} = \delta_{h_1,h_2} \delta_{\bar{h}_1,\bar{h}_2} \delta_{\mathbf{h},\bar{\mathbf{h}}} 2^{-16h_1}. \quad (3.21)$$

By substituting Eq. (3.21) into Eq. (3.1) and then comparing Eq. (2.5) with Eq. (3.20), one eventually identifies [24]

$$\mathcal{G}_{c,\{h,h\}}^{(2)}(x) = 2^{8h-c/3} [x(1-x)]^{-c/24} \chi_{c,h}(\tau(x)), \quad (3.22)$$

which we will also use in Sec. 3.4.

3.3 Case $N = 3$: \mathbb{Z}_3 -symmetric Riemann Surfaces with Genus Two

For $N = 3$, the orbifold three-point function in Eq. (3.13) can be computed by introducing, for instance, the three-to-one map [28]

$$z(t) = \frac{(t+\omega)^3}{3\omega(1-\omega)t(t-1)}, \quad \omega = e^{\frac{2\pi i}{3}}. \quad (3.23)$$

This transformation has branch points of order three at $z_b = \{0, 1\}$ and maps the points $t_\infty = \{0, 1, \infty\}$ in the t -surface into $z = \infty$. The t -surface, which has again the topology of a sphere, is then a triple covering of the complex plane with a cut along $z \in (0, 1)$. By recalling Eq. (3.10) and applying Eqs. (3.14) and (3.15) one has

$$C_{\mathbf{h},\bar{\mathbf{h}}} = \langle \hat{\phi}_{h_1} \hat{\phi}_{\bar{h}_1} | \hat{\phi}_{h_2}(1) \hat{\phi}_{\bar{h}_2}(1) | \hat{\phi}_{h_3} \hat{\phi}_{\bar{h}_3} \rangle = [3\omega(1-\omega)]^{-h_1-h_2-h_3} [3\omega^2(1-\omega^2)]^{-\bar{h}_1-\bar{h}_2-\bar{h}_3} C_{\mathbf{h},\bar{\mathbf{h}}}^{\text{seed}}. \quad (3.24)$$

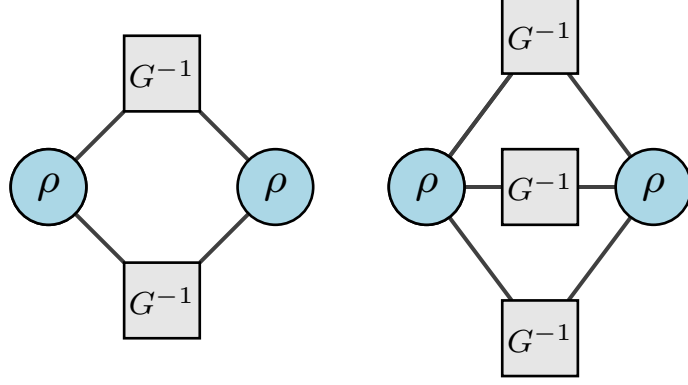


Figure 1. On the left: Matrix product representation of the genus one conformal block in Eq. (3.19). On the right: Matrix product representation of the genus two conformal block in Eq. (3.29).

where the structure constants $C_{\mathbf{h},\bar{\mathbf{h}}}^{\text{seed}}$ are calculated in the seed CFT, namely

$$C_{\mathbf{h},\bar{\mathbf{h}}}^{\text{seed}} = \langle \phi_{h_1} \phi_{\bar{h}_1} | \phi_{h_2}(1) \phi_{\bar{h}_2}(1) | \phi_{h_3} \phi_{\bar{h}_3} \rangle. \quad (3.25)$$

Analogously from Eq. (3.11), it follows

$$\rho_{M_1, M_2, M_3}^{h_1, h_2, h_3} = \frac{\langle \mathcal{L}_{-M_1} \phi_{h_1} | \mathcal{L}_{-M_2}(1) \phi_{h_2}(1) | \mathcal{L}_{-M_3} \phi_{h_3} \rangle}{\langle \phi_{h_1} | \phi_{h_2}(1) | \phi_{h_3} \rangle}. \quad (3.26)$$

The pullback $\mathcal{L}_{-n}(t_\infty)$ of the Virasoro descendant $L_{-n}(z = \infty)$ under the conformal map in Eq. (3.23) are obtained, as in the case $N = 2$, by using Eq. (A.6), see also Eq. (A.10). The elements of Eq. (3.26) can be calculated by implementing in a computer algebra system the Ward identities of Appendix A. Finally, the structure constants $\tilde{C}_{\mathbf{h},\bar{\mathbf{h}}}$ and matrix elements $\tilde{\rho}_{M_1, M_2, M_3}^{h_1, h_2, h_3}$, which follow by exchanging twist with anti-twist fields in Eqs. (3.10) and (3.11), are determined by considering the complex conjugate of the map of Eq. (3.23). One finds

$$\tilde{C}_{\mathbf{h},\bar{\mathbf{h}}} = [3\omega^2(1 - \omega^2)]^{-h_1 - h_2 - h_3} [3\omega(1 - \omega)]^{-\bar{h}_1 - \bar{h}_2 - \bar{h}_3} C_{\mathbf{h},\bar{\mathbf{h}}}^{\text{seed}}, \quad (3.27)$$

and

$$\tilde{\rho}_{M_1, M_2, M_3}^{h_1, h_2, h_3} = [\rho_{M_1, M_2, M_3}^{h_1, h_2, h_3}]^*. \quad (3.28)$$

The resulting expressions for the tensor $\rho_{M_1, M_2, M_3}^{h_1, h_2, h_3}$ in Eq. (3.26) can be then eventually plugged into Eq. (3.9) to get the expansion

$$\mathcal{G}_{c, \{h_1, h_2, h_3\}}^{(3)}(z) = z^{\sum_{j=1}^3 h_j - \frac{2c}{9}} \sum_{\{M_j\}, \{N_j\}} z^{\sum_{j=1}^3 |M_j|} \rho_{M_1, M_2, M_3}^{h_1, h_2, h_3} \prod_{j=1}^3 [G_{M_j, N_j}^{h_j}]^{-1} [\rho_{N_1, N_2, N_3}^{h_1, h_2, h_3}]^*, \quad (3.29)$$

which is also illustrated pictorially in Fig. 1. The conformal block $\mathcal{G}_{c, \{h_1, h_2, h_3\}}^{(3)}$ in Eq. (3.29) is manifestly symmetric under permutations of h_1 , h_2 and h_3 , consistently with the \mathbb{Z}_3 (in fact S_3) symmetry of the orbifold CFT. From Eq. (3.24), it follows that the $N = 3$ orbifold conformal blocks are in one-to-one correspondence [27, 28] with the non-zero structure constants of \mathcal{C} . Indeed, if we insert Eqs. (3.24) and (3.27) into Eq. (3.12), we conclude that, see also Eq. (3.1),

$$D_{\mathbf{h},\bar{\mathbf{h}}} = 27^{-|\mathbf{h}| - |\bar{\mathbf{h}}|} (C_{\mathbf{h},\bar{\mathbf{h}}}^{\text{seed}})^2. \quad (3.30)$$

3.4 Minimal Model Orbifold Conformal Blocks

We consider now a seed theory which is built upon irreducible representations of the Virasoro algebra, labelled by c and h , constructed by removing all the null vectors [2]. A prominent example of such CFTs are the minimal models [1] which have central charge

$$c = c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}, \quad (3.31)$$

being p and q positive coprime integers. The minimal models are rational theories, that is they contain a finite number of primaries $\phi_{h_{r,s}^{p,q}}$ with conformal dimensions

$$h_{r,s}^{p,q} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}, \quad 1 \leq r \leq q - 1, 1 \leq s \leq p - 1, \quad (3.32)$$

whose OPE algebra closes. We will denote by $\mathbf{h}_{\text{deg}} \equiv \{h_{r_j, s_j}^{p,q}\}_{j=1}^N$ a set of N conformal dimension as in Eq. (3.32), the orbifold conformal block for a minimal model is then the function $\mathcal{G}_{c_{p,q}, \mathbf{h}_{\text{deg}}}^{(N)}(z)$.

Due to the chain of resonances $h_{r,s}^{p,q} = h_{q-r, p-s}^{p,q} = h_{q+r, p+s}^{p,q} = \dots$, the Verma module labelled by $c_{p,q}$ and $h_{r,s}^{p,q}$ has an infinite series of null vectors at the levels $rs, (q-r)(p-s), (q+r)(p+s), \dots$. As already emphasized, these states are absent from the spectrum of \mathcal{C} . Consequently, when calculating the combinatorial expansion in Eq. (3.9) in a minimal model, one has to choose a basis of descendants for each representation $h_{r_j, s_j}^{p,q}$ which is free of null vectors. Even if there are closed expressions for the null vectors, the construction of such a basis becomes cumbersome at higher levels, due to the resonances mentioned above. We refer to [52] where this particular issue is discussed in more detail. Here we will follow an alternative path, inspired by the AGT approach to the minimal models [53–55], see also [56].

The analytic properties of $\mathcal{G}_{c, \mathbf{h}_{\text{deg}}}^{(N)}(z)$ for $N = 2$ and $N = 3$ as a function of c can be understood by examining Eqs. (3.19) and (3.29) respectively. If there is a null vector at a certain level in the Verma module with conformal dimension $h_{r_j, s_j}^{p,q}$, the matrix elements of $[G^{h_{r_j, s_j}^{p,q}}]^{-1}$ have a simple pole. More precisely, they are order $O((c - c_{p,q})^{-1})$ since the null vector is orthogonal to all the states belonging to that level. At the same time, each factor of ρ produces a single zero, namely is order $O(c - c_{p,q})$. Indeed, for $N = 2$, the elements of ρ , see Eq. (3.18), are scalar products and therefore vanish when evaluated with a null vector. For $N = 3$, the vanishing of the tensor $\rho^{h_{r_1, s_1}, h_{r_2, s_2}, h_{r_3, s_3}}$ is instead a consequence of the fusion rules of the minimal model, see Eq. (3.26). For eliminating the null vector contribution from Eqs. (3.19) and (3.29), it is enough, if $\mathbf{h} \neq \mathbf{0}$ where $\mathbf{0} \equiv \{0, \dots, 0\}$, to regularize the central charge differently in the ρ and G factors. In particular, we take

$$\rho^{\mathbf{h}_{\text{deg}}}(c) \rightarrow \rho^{\mathbf{h}_{\text{deg}}}(c_{p,q} + \varepsilon^2), \quad G^{h_{r_j, s_j}^{p,q}}(c) \rightarrow G^{h_{r_j, s_j}^{p,q}}(c_{p,q} + \varepsilon), \quad (3.33)$$

with $\varepsilon > 0$, in such a way that the null vectors contributions in Eq. (3.9) are now zero in the limit $\varepsilon \rightarrow 0$. By recalling Eq. (3.22), it is possible to check that at $N = 2$, the power series about $z = 0$ of the regularized conformal block reproduces the analogous expansion of the character $\chi_{c_{p,q}, h_{r,s}^{p,q}}(\tau(z))$ in the minimal model [51].

The case $\mathbf{h} = \mathbf{0}$ is slightly different since the Verma module of the identity contains a null vector already at level one for any value of the central charge. Hence, we can follow a similar strategy but modifying the dependence of the matrices G^h and ρ^h on the conformal dimensions rather than the central charge. We replace then

$$\rho^{\mathbf{h}=\mathbf{0}}(c) \rightarrow \rho^{\mathbf{h}=\{\delta^2, \dots, \delta^2\}}(c), \quad G^{h=0}(c) \rightarrow G^{h=\delta}(c), \quad (3.34)$$

with $\delta > 0$ in Eq. (3.9). The limit $\delta \rightarrow 0$ defines the regularized conformal block $\mathcal{G}_{c,\mathbf{0}}^{(N)}(z)$ for $N = 2, 3$. In particular, it provides, for $N = 2$ upon using Eq. (3.22), the character of the irreducible representation of the identity. When $N = 3$, note that, if the identity representation is present in only one of the replicas, for instance $h_1 = 0$, then, according to Eq. (3.24), we shall have the same conformal family in the other two replicas, i.e. $h_2 = h_3$. In order to find the correct conformal block, in this case, one must identify first $h_2 = h_3$, while keeping $h_1 \neq 0$, and then perform the limit $h_1 \rightarrow 0$.

Let us see how the regularization procedure in Eqs. (3.33, 3.34) works in a particular example: the Ising CFT, which corresponds to $p = 4$ and $q = 3$, namely $c_{4,3} = 1/2$ in Eq. (3.31). For instance, for $h_{1,2} = 1/16$, the first few terms of the regularized expansion of Eq. (3.19) about $z = 0$ are

$$\mathcal{G}_{\frac{1}{2}, \{\frac{1}{16}, \frac{1}{16}\}}^{(2)}(z) = z^{1/16} \left[1 + \frac{z}{16} + \frac{17z^2}{512} + \frac{187z^3}{8192} + \frac{9163z^4}{524288} + O(z^5) \right]. \quad (3.35)$$

The coefficient of the $O(z^4)$ term would be different if the regularization scheme in Eq. (3.33) was not implemented. Indeed the Verma module with conformal dimension $h = 1/16$ at $c = 1/2$ possesses a null vector at level two. If this vector appeared on both replicas of the seed theory, i.e. in the sector $|M_1| = |M_2| = 2$ of Eq. (3.19), it would modify the coefficient of the $O(z^4)$ term in the conformal block. After implementing the regularization scheme in Eq. (3.33), its contribution is instead of order $O(\varepsilon^2)$ and drops in the limit $\varepsilon \rightarrow 0$.

As a second example, we analyze the regularized $N = 3$ orbifold conformal block for $c = 1/2$, $h_1 = h_2 = 1/16$ and $h_3 = 0$; the result of the expansion in Eq. (3.29) is

$$\mathcal{G}_{\frac{1}{2}, \{\frac{1}{16}, \frac{1}{16}, 0\}}^{(3)}(z) = z^{1/72} \left[1 + \frac{7}{108}z + \frac{1595}{46656}z^2 + \frac{118405}{5038848}z^3 + \frac{26160455}{1451188224}z^4 + O(z^5) \right]. \quad (3.36)$$

Again, when the null vector at level two in the Verma module with $c = 1/2$ and $h = 1/16$ pops up in the sum of Eq. (3.29) simultaneously in two replicas, it alters the coefficient of the $O(z^4)$ term. This spurious contribution is again of order $O(\varepsilon^2)$ once we apply the regularization of Eq. (3.33), and it is therefore removed in the limit $\varepsilon \rightarrow 0$.

4 Orbifold Conformal Blocks in terms of Sphere Conformal Blocks

Eq. (3.9) provides a small $|z|$ expansion of the orbifold conformal block of the type

$$\mathcal{G}_{c,\mathbf{h}}^{(N)}(z) = z^{|\mathbf{h}| - 2h_{\sigma_N}} \sum_{j=0}^{\infty} a_j z^j. \quad (4.1)$$

However, the computation of the coefficients a_j quickly becomes impossible to accomplish and one has to approximate $\mathcal{G}_{c,\mathbf{h}}^{(N)}(z)$ by truncating the previous series at some value L ,

$$\mathcal{G}_{c,\mathbf{h}}^{(N)}(z) \sim z^{|\mathbf{h}|-2h_{\sigma_N}} \sum_{j=0}^L a_j z^j. \quad (4.2)$$

In the case $N = 3$, for instance, we are able to reach $L = 6$.

A crucial point here is that the convergence of the series in Eq. (3.9) is slow close to $|z| = 1$. This, in turn, implies that, if the conformal block $\mathcal{G}_{c,\mathbf{h}}^{(N)}(z)$ is approximated by the truncated sum of Eq. (4.2), one misses the global properties of the twist field four-point function and, in particular, the crossing symmetry of Eq. (2.6). In [28], this problem was tackled for $N = 3$ by using a transformation from the z -plane to the pillow frame introduced in [57]. Ref. [28] derived then a series expansion in terms of the elliptic nome $q(z) = e^{i\pi\tau(z)}$, with $\tau(z)$ defined in Eq. (2.4), of the form

$$\mathcal{G}_{c,\mathbf{h}}^{(N)}(z) \sim \Lambda(q(z)) \sum_{l=0}^L A_l q(z)^l. \quad (4.3)$$

The explicit expressions of the function $\Lambda(q)$ and of the coefficient A_l are given in Sec. 5.2, see Eq. (5.20). This expansion drastically improves the convergence properties of the twist field correlation function near $|z| = 1$.

In this section, we will reobtain the expansion in Eq (4.3) from a different perspective and show how it could be further improved. As we discuss in detail in Appendix B, the orbifold algebra admits as a sub-algebra a Virasoro algebra with central charge Nc [58], generated by the symmetric stress energy tensor in Eq. (B.1). Then it is natural to expand the orbifold conformal blocks $\mathcal{G}_{c,\mathbf{h}}^{(N)}(z)$ as a linear combination of Virasoro sphere conformal blocks with central charge Nc . More specifically,

$$\mathcal{G}_{c,\mathbf{h}}^{(N)}(z) = \sum_{l=0}^{\infty} \alpha_l^{\mathbf{h}} \mathcal{F}_{Nc,|\mathbf{h}|+l}(z), \quad (4.4)$$

where we have denoted by $\mathcal{F}_{Nc,h}(z)$ the Virasoro sphere conformal blocks with the four external dimensions fixed to h_{σ_N} and with internal channel of dimension h . The conformal block $\mathcal{F}_{Nc,|\mathbf{h}|+l}(z)$ resums the contribution of the conformal family generated by a field with conformal dimension $|\mathbf{h}| + l$, primary with respect to the orbifold Virasoro sub-algebra of Eq. (B.4). The coefficients $\alpha_l^{\mathbf{h}}$ can be thought as Clebsch-Gordan coefficients for a decomposition of a N -fold tensor product of Virasoro algebra representations into a direct sum of irreducible representations with central charge Nc . A more detailed discussion concerning the meaning of Eq. (4.4) as well as the algebraic nature of the coefficients $\alpha_l^{\mathbf{h}}$ can be found in Section 4.1.

Now observe that, if we truncate Eq. (4.4)

$$\mathcal{G}_{c,\mathbf{h}}^{(N)}(z) \sim \sum_{l=0}^L \alpha_l^{\mathbf{h}} \mathcal{F}_{Nc,|\mathbf{h}|+l}(z), \quad (4.5)$$

then the coefficients $\alpha_l^{\mathbf{h}}$ can be easily derived by expanding $\mathcal{F}_{Nc,|\mathbf{h}|+l}(z)$ in power series and identifying order by order the sums of Eqs. (4.2) and (4.5). Eq. (4.5) is already a better approximation than Eq. (4.2), as each $\mathcal{F}_{Nc,|\mathbf{h}|+l}(z)$ can be computed until orders much larger than $z^{|\mathbf{h}|-2h_{\sigma_N}+L}$. In other words, this means that we can take into account the contribution of certain very low descendants of the orbifold algebra that were previously inaccessible.

The second step is to express the $\mathcal{F}_{Nc,h}(z)$ as an expansion in terms of the elliptic nome $q(z)$ by using the elliptic Zamolodchikov recursion relation [34],

$$\mathcal{F}_{Nc,h}(z) = f(h, Nc, z) H(h, Nc, q(z)), \quad (4.6)$$

where

$$f(h, Nc, z) = [16q(z)]^{h-\frac{Nc-1}{24}} [z(1-z)]^{\frac{Nc-1}{24}-2h_{\sigma_N}} \vartheta_3(q(z))^{\frac{Nc-1}{2}-16h_{\sigma_N}}, \quad (4.7)$$

$$H(h, Nc, q) = 1 + \sum_{j=1}^{\infty} a_j(h, Nc) q^{2j}. \quad (4.8)$$

The coefficients $a_j(h, Nc)$ can be computed recursively to very large values of j , see Ref. [34].

Finally, if the elliptic recursion is truncated at level $L' > L$ and then combined with Eq. (4.5), we get the following approximation for $\mathcal{G}_{c,\mathbf{h}}^{(N)}(z)$,

$$\mathcal{G}_{c,\mathbf{h}}^{(N)}(z) \sim \Lambda(q) \left(\sum_{l=0}^L A_l q(z)^l + \sum_{l=L+1}^{L+L'} A'_l q(z)^l \right), \quad (4.9)$$

that provides sub-leading corrections to Eq. (4.3). We distinguish between coefficients A_l and A'_l because the former take into account the contribution from all the states at level $l < L$. The q^l terms with $l > L$ come from the contribution of the descendants of the primaries at the levels $l < L$ in the orbifold sub-algebra. Therefore, although both Eqs. (4.3) and (4.9) approximate the conformal block $\mathcal{G}_{c,\mathbf{h}}^{(N)}(z)$ with an error $o(q(z)^L)$, Eq. (4.9) gives a better approximation than Eq. (4.3).

4.1 Orbifold Conformal Algebra and the Coefficients $\alpha_l^{\mathbf{h}}$

To illustrate the main idea behind Eq. (4.4), it is sufficient to consider the first level contribution to $\mathcal{G}_{c,\mathbf{h}}^{(N)}(z)$ for the case $N = 3$. If we explicitly calculate the first order term in the expansion of Eq. (3.29), we have that

$$\mathcal{G}_{c,\mathbf{h}}^{(3)}(z) = z^{|\mathbf{h}|-2h_{\sigma_N}} \left[1 + \left(\frac{|\mathbf{h}|}{2} + \frac{1}{54} \left(\frac{(h_1 - h_2)^2}{h_3} + \frac{(h_1 - h_3)^2}{h_2} + \frac{(h_2 - h_3)^2}{h_1} \right) \right) z + O(z^2) \right], \quad (4.10)$$

where we recall that here $\mathbf{h} = \{h_1, h_2, h_3\}$ and $|\mathbf{h}| = h_1 + h_2 + h_3$. The conformal block $\mathcal{G}_{c,\mathbf{h}}^{(3)}(z)$ is associated with the field $\phi_{\mathbf{h}} = \phi_{h_1} \otimes \phi_{h_2} \otimes \phi_{h_3}$ and its descendants. In particular, the coefficient of the $z^{|\mathbf{h}|-2h_{\sigma_N}+1}$ term comes from the contribution of three descendants,

$$|v_1\rangle = L_{-1}\phi_{h_1} \otimes \phi_{h_2} \otimes \phi_{h_3}, \quad |v_2\rangle = \phi_{h_1} \otimes L_{-1}\phi_{h_2} \otimes \phi_{h_3}, \quad |v_3\rangle = \phi_{h_1} \otimes \phi_{h_2} \otimes L_{-1}\phi_{h_3}. \quad (4.11)$$

The symmetric linear combination of the three states above corresponds to the descendant of the orbifold Virasoro sub-algebra of Eq. (B.4)

$$\mathbf{L}_{-1}\phi_{\mathbf{h}} = |v_1\rangle + |v_2\rangle + |v_3\rangle. \quad (4.12)$$

This descendant contributes with the term with coefficient $|\mathbf{h}|/2$ to the expansion of Eq. (4.9). This contribution is taken into account by the first Virasoro sphere conformal block in the expansion of Eq. (4.4), that is

$$\mathcal{F}_{3c,|\mathbf{h}|}(z) = z^{|\mathbf{h}|-2h_{\sigma_N}} \left[1 + \frac{|\mathbf{h}|}{2}z + O(z^2) \right]. \quad (4.13)$$

Let us consider now the orthogonal complement to the span of the vector $\mathbf{L}_{-1}\phi_{\mathbf{h}}$ in Eq. (B.4). This is a two dimensional real vector space that contains the states

$$|\Psi\rangle = \mu|v_1\rangle + \lambda|v_2\rangle - \frac{\mu h_1 + \lambda h_2}{h_3}|v_3\rangle, \quad (4.14)$$

for $\lambda, \mu \in \mathbb{R}$. One can check that any vector in the orthogonal complement is a primary of the $3c$ Virasoro sub-algebra of Eq. (B.4), i.e. $\mathbf{L}_n|\Psi\rangle = 0$ for all $n > 0$ and $\mathbf{L}_0|\Psi\rangle = (h_1 + h_2 + h_3 + 1)|\Psi\rangle$. The contribution of these states and their symmetric descendants $\mathbf{L}_{-M}|\Psi\rangle$ to the conformal block $\mathcal{G}_{c,\mathbf{h}}^{(3)}(z)$ is given by

$$\alpha_1^{\mathbf{h}} \mathcal{F}_{3c,|\mathbf{h}|+1}(z), \quad (4.15)$$

where

$$\alpha_1^{\mathbf{h}} = \frac{1}{54} \left[\frac{(h_1 - h_2)^2}{h_3} + \frac{(h_1 - h_3)^2}{h_2} + \frac{(h_2 - h_3)^2}{h_1} \right]. \quad (4.16)$$

The coefficient $\alpha_1^{\mathbf{h}}$ can be determined by choosing an orthogonal basis $\{|\Psi_1\rangle, |\Psi_2\rangle\}$ for the subspace in Eq. (4.14). Then one can find the result in Eq.(4.16) from the expression

$$\alpha_1^{\mathbf{h}} = \frac{1}{D_{\mathbf{h},\bar{\mathbf{h}}}} \left[\frac{\langle \Psi_1 \phi_{\bar{\mathbf{h}}} | \sigma_3(1) | \tilde{\sigma}_3 \rangle \langle \Psi_1 \phi_{\bar{\mathbf{h}}} | \tilde{\sigma}_3(1) | \sigma_3 \rangle}{\langle \Psi_1 | \Psi_1 \rangle} + \frac{\langle \Psi_2 \phi_{\bar{\mathbf{h}}} | \sigma_3(1) | \tilde{\sigma}_3 \rangle \langle \Psi_2 \phi_{\bar{\mathbf{h}}} | \tilde{\sigma}_3(1) | \sigma_3 \rangle}{\langle \Psi_2 | \Psi_2 \rangle} \right] \quad (4.17)$$

where $D_{\mathbf{h},\bar{\mathbf{h}}}$ is related to the structure constant as in Eq. (3.12). As we already pointed out for the matrix elements ρ and $\tilde{\rho}$, defined in Eq. (3.11), the coefficients $\alpha_1^{\mathbf{h}}$ are algebraic in nature, despite the holomorphic and anti-holomorphic fields appear in the equation above. This can be seen from the fact that they are rational functions of the central charge c and of the holomorphic dimensions h_j .

The states $\mathbf{L}_{-M}\phi_{\mathbf{h}}$ and $\mathbf{L}_{-M}|\Psi_{1,2}\rangle$ do not span the full space of descendants of $\phi_{\mathbf{h}}$ for levels larger than one. This is the reason why other Virasoro conformal blocks appear and one finally gets the expansion of Eq. (4.4). In fact, the previous analysis may be extended to higher levels and generalized to any number of copies N . If we denote by $\mathfrak{p}(l)$ the number of partitions of the integer l , then there are $\mathfrak{p}(l)$ symmetric descendants $\mathbf{L}_{-M}\phi_{\mathbf{h}}$ at level l . However, for an arbitrary number of replicas N and assuming that there are no null vectors,

it is easy to check that at level l the total number $\mathcal{N}_{N,l}$ of linearly independent descendants of $\phi_{\mathbf{h}}$ is

$$\mathcal{N}_{N,l} = \sum_{\substack{Y, |Y|=l \\ |j_Y| \leq N}} \frac{N!}{(N - |j_Y|)! \prod_{i=1}^{|j_Y|} d_Y(i)!} \prod_{j=1}^{|j_Y|} \mathbf{p}(i_Y(j)), \quad (4.18)$$

where Y denotes a partition of l . If we consider the Young tableau associated to Y , then $|i_Y|$ and $|j_Y|$ denote its number of columns and rows respectively, $i_Y(j)$ is the number of columns in the row j and $d_Y(i)$ is the number of rows with i columns.

Therefore, at level l , we shall find a set of $\mathcal{M}_{N,l}$ fields $|\Psi_j\rangle$, $j = 1, \dots, \mathcal{M}_{N,l}$, orthogonal to all the $\mathbf{p}(l)$ symmetric descendants at level l and such that $\mathbf{L}_0|\Psi_j\rangle = (|\mathbf{h}| + l)|\Psi_j\rangle$, $\mathbf{L}_n|\Psi_j\rangle = 0$ for all $n > 0$. If there are no null vectors, the numbers $\mathcal{M}_{N,l}$ are defined by the recursion relation

$$\mathcal{M}_{N,l} = \mathcal{N}_{N,l} - \sum_{m=0}^{l-1} \mathcal{M}_{N,m} \mathbf{p}(l-m). \quad (4.19)$$

The contribution of these new primaries fields, which all have conformal dimension $|\mathbf{h}| + l$, and their descendants $\mathbf{L}_{-M}|\Psi_j\rangle$ is included into the sphere conformal block $\mathcal{F}_{N_c, |\mathbf{h}|+l}(z)$ in the expansion of the orbifold conformal block, see Eq. (4.4). The corresponding structure constant $\alpha_l^{\mathbf{h}}$ is given by

$$\alpha_l^{\mathbf{h}} = \frac{1}{D_{\mathbf{h}, \bar{\mathbf{h}}}} \sum_{j=1}^{\mathcal{M}_{N,l}} \frac{\langle \Psi_j \phi_{\bar{\mathbf{h}}} | \sigma_N(1) | \tilde{\sigma}_N \rangle \langle \Psi_j \phi_{\bar{\mathbf{h}}} | \tilde{\sigma}_N(1) | \sigma_N \rangle}{\langle \Psi_j | \Psi_j \rangle}, \quad (4.20)$$

and can be checked explicitly. The primary fields at different levels, as well as their descendants, are orthogonal since they belong to different representations of the Virasoro sub-algebra of Eq. (B.4).

5 Applications

We now pass to discuss applications of the formalism developed in Sec. 3 and Sec. 4. In Sec. 5.1, we will focus on the determination of the Rényi entropies and the third power of the partial transpose of the reduced density matrix for two disjoint intervals in a minimal CFT. We further discuss how the expansion in Eq. (4.9) can be used to improve the bounds on the CFT structure constants analyzed in [28], see Sec. 5.2 and Sec. 5.3.

5.1 Entanglement of Two Disjoint Intervals: Rényi Entropies and the Partially Transposed Density Matrix

Consider a critical one-dimensional quantum model in the ground state. Let us suppose that the system is divided into two spatial regions A and B . The entanglement between these two subsystems can be characterized by the moments $\text{Tr} \rho_A^N$ of the reduced density matrix ρ_A , defined in Eq. (C.1) of Appendix C. We will further assume that the subsystem A consists of two disjoint regions A_1 and A_2 , i.e. $A = A_1 \cup A_2$. Then one can also study the entanglement between A_1 and A_2 , which can be quantified by the moments $\text{Tr}(\rho_A^{T_2})^N$ of the partial transpose of ρ_A , which we will denote by $\rho_A^{T_2}$ and defined in Eq. (C.5).

Now we suppose that the universal properties of the quantum model at the critical point are determined by the CFT \mathcal{C} . Without loss of generality, we can take as regions A_1 and A_2 the intervals $A_1 = (0, x)$ and $A_2 = (1, \infty)$, with $0 < x < 1$. Then the moments $\text{Tr} \rho_A^N$ and $\text{Tr}(\rho_A^{T_2})^N$ are equal to [15, 16]

$$\text{Tr} \rho_A^N = K_N \langle \sigma_N(\infty) \tilde{\sigma}_N(1) \sigma_N(x) \tilde{\sigma}_N(0) \rangle, \quad (5.1)$$

and [18, 19]

$$\text{Tr}(\rho_A^{T_2})^N = K_N \langle \tilde{\sigma}_N(\infty) \sigma_N(1) \sigma_N(x) \tilde{\sigma}_N(0) \rangle, \quad (5.2)$$

where K_N is a non-universal constant. Note that the partial transposition crucially exchanges the twist and anti-twist operators at the points $z = 1$ and ∞ . The crossing transformation $z \mapsto 1 - z$ maps subsystem $A = (0, x) \cup (1, \infty)$ into its complement $B = (-\infty, 0) \cup (1 - x, 1)$. Thus the crossing invariance of the four-point correlation function $\langle \sigma_N(\infty) \tilde{\sigma}_N(1) \sigma_N(x) \tilde{\sigma}_N(0) \rangle$ implies that $\text{Tr} \rho_A^N = \text{Tr} \rho_B^N$ and, therefore, the well-known property that for a pure state the entanglement entropies of a subsystem and its complementary are equal. This conclusion is no longer true in general for the moments $\text{Tr}(\rho_A^{T_2})^N$ and, in fact, the four-point function $\langle \tilde{\sigma}_N(\infty) \sigma_N(1) \sigma_N(x) \tilde{\sigma}_N(0) \rangle$ is not crossing invariant. The case $N = 2$ is peculiar, since the twist and anti-twist fields are identified, $\sigma_2 = \tilde{\sigma}_2$, and therefore $\text{Tr} \rho_A^2 = \text{Tr}(\rho_A^{T_2})^2$.

The correlation functions of Eqs. (5.1) and (5.2) are related by the conformal transformation

$$y(z) = \frac{z}{z-1}, \quad (5.3)$$

which maps the points $(0, x, 1, \infty)$ into $(0, x/(x-1), \infty, 1)$. We can rewrite then

$$\langle \tilde{\sigma}_N(\infty) \sigma_N(1) \sigma_N(x) \tilde{\sigma}_N(0) \rangle = (1-x)^{-4h_{\sigma_N}} \langle \sigma_N(\infty) \tilde{\sigma}_N(1) \sigma_N(x/(x-1)) \tilde{\sigma}_N(0) \rangle, \quad (5.4)$$

and therefore Eq. (5.2) can be recast in the form

$$\text{Tr}(\rho_A^{T_2})^N = K_N (1-x)^{-4h_{\sigma_N}} \langle \sigma_N(\infty) \tilde{\sigma}_N(1) \sigma_N(x/(x-1)) \tilde{\sigma}_N(0) \rangle. \quad (5.5)$$

Thus, while for determining $\text{Tr} \rho_A^N$ it is enough to calculate the twist field four-point correlator in the interval $0 < x < 1$, the computation of $\text{Tr}(\rho_A^{T_2})^N$ requires to extend this function to the domain $x/(x-1) < 0$. The moments of the reduced density matrix can be calculated directly from Eqs. (3.1) and (4.4), and we have

$$\text{Tr} \rho_A^N = K_N \sum_{\mathbf{h}, \bar{\mathbf{h}}} \sum_{l, l'} \tilde{D}_{\mathbf{h}, \bar{\mathbf{h}}}^{l, l'} \mathcal{F}_{Nc, |\mathbf{h}|+l}(x) \mathcal{F}_{Nc, |\bar{\mathbf{h}}|+l'}(x) \quad (5.6)$$

with $\tilde{D}_{\mathbf{h}, \bar{\mathbf{h}}}^{l, l'} = D_{\mathbf{h}, \bar{\mathbf{h}}} \alpha_l^{\mathbf{h}} \alpha_{l'}^{\bar{\mathbf{h}}}$. As previously emphasized, in order to obtain an analogous expansion for the moments of the partially transposed density matrix one must determine the analytic continuation of the twist field four-point function along the negative real axis. Such analytic continuation can be straightforwardly calculated from the expression of the orbifold conformal blocks in terms of the elliptic nome $q(x) = e^{i\pi\tau(x)}$, see Eq. (4.9). Indeed, one can easily prove that

$$\tau\left(\frac{x}{x-1}\right) = \tau(x) + 1, \quad \text{and} \quad q\left(\frac{x}{x-1}\right) = e^{i\pi} q(x). \quad (5.7)$$

In other words, $\text{Tr}(\rho_A^{T_2})^N$ can be again computed from Eqs. (3.1) and (4.4) but evaluating now the Virasoro conformal blocks at $e^{i\pi}q(x)$,

$$\text{Tr}(\rho_A^{T_2})^N = K_N(1-x)^{-4h_{\sigma_N}} \sum_{\mathbf{h}, \bar{\mathbf{h}}} \sum_{l, l'} \tilde{D}_{\mathbf{h}, \bar{\mathbf{h}}}^{l, l'} \mathcal{F}_{Nc, |\mathbf{h}|+l}(e^{i\pi}q) \mathcal{F}_{Nc, |\bar{\mathbf{h}}|+l'}(e^{-i\pi}q). \quad (5.8)$$

Note that, for convenience, in Eq. (5.8) and some more equations below, we have traded x for the elliptic nome q in the argument of $\mathcal{F}_{Nc, h}$. Observe that, for $N = 2$, Eq. (5.7) corresponds to perform the modular transformation $\tau(x) \mapsto \tau(x) + 1$ on the modulus of the torus $\Sigma_1(x)$. Then $\mathcal{Z}_1(x)$ and $\mathcal{Z}_1(x/(x-1))$ are the partition functions of \mathcal{C} on a flat torus of moduli $\tau(x)$ and $\tau(x) + 1$ respectively. Taking into account Eqs. (5.1) and (5.5), the invariance of the partition function on the torus under modular transformations, $\mathcal{Z}_1(x) = \mathcal{Z}_1(x/(x-1))$, implies the identity $\text{Tr} \rho_A^2 = \text{Tr}(\rho_A^{T_2})^2$ anticipated earlier.

We can recast Eq. (5.8) in a simpler and instructive form. By recalling the Zamolodchikov recursion for the sphere conformal blocks and applying the identities of the elliptic functions

$$\vartheta_2(e^{\pm i\pi}q) = e^{\pm i\pi/4}\vartheta_2(q), \quad \vartheta_3(e^{\pm i\pi}q) = \vartheta_4(q), \quad \vartheta_4(e^{\pm i\pi}q) = \vartheta_3(q), \quad (5.9)$$

and

$$x = \left(\frac{\vartheta_2(q)}{\vartheta_3(q)} \right)^4, \quad 1-x = \left(\frac{\vartheta_4(q)}{\vartheta_3(q)} \right)^4, \quad (5.10)$$

one can show that

$$\mathcal{F}_{Nc, h}(e^{\pm i\pi}q) = e^{\pm i\pi(h-2h_{\sigma_N})}(1-x)^{2h_{\sigma_N}} \mathcal{F}_{Nc, h}(q). \quad (5.11)$$

By plugging the last equality into Eq. (5.8), the conformal block expansion of the moments $\text{Tr}(\rho_A^{T_2})^N$ can be eventually rewritten as

$$\text{Tr}(\rho_A^{T_2})^N = K_N \sum_{\mathbf{h}, \bar{\mathbf{h}}} \sum_{l, l'} e^{i\pi(|\mathbf{h}|-|\bar{\mathbf{h}}|+l-l')} \tilde{D}_{\mathbf{h}, \bar{\mathbf{h}}}^{l, l'} \mathcal{F}_{Nc, |\mathbf{h}|+l}(x) \mathcal{F}_{Nc, |\bar{\mathbf{h}}|+l'}(x). \quad (5.12)$$

By comparing Eq. (5.12) above with Eq. (5.6), we conclude that the moments $\text{Tr}(\rho_A^{T_2})^N$ admit the same Virasoro conformal block decomposition as $\text{Tr} \rho_A^N$, but with the structure constants multiplied by a spin dependent phase. We can then state the main result of this section as

$$\text{Tr} \rho_A^N - \text{Tr}(\rho_A^{T_2})^N = \text{sum over the channels with conformal spin } |\mathbf{h}|-|\bar{\mathbf{h}}|+l-l' \neq 2k, \quad k \in \mathbb{Z}. \quad (5.13)$$

Note that, in the case $N = 2$, due to the identity $\text{Tr} \rho_A^2 = \text{Tr}(\rho_A^{T_2})^2$, the channels indicated above cannot appear in the conformal block decompositions of $\text{Tr} \rho_A^2$ and $\text{Tr}(\rho_A^{T_2})^2$.

To the best of our knowledge, for $c \leq 1$, analytic expressions of $\text{Tr} \rho_A^N$ and $\text{Tr}(\rho_A^{T_2})^N$ have only been calculated for free theories; namely the compactified massless boson, the massless Dirac and Majorana fermions [15, 16, 18, 19, 35, 59–61]. The latter corresponds to the Ising CFT, which is a minimal model with $c = 1/2$, and we shall focus on it first.

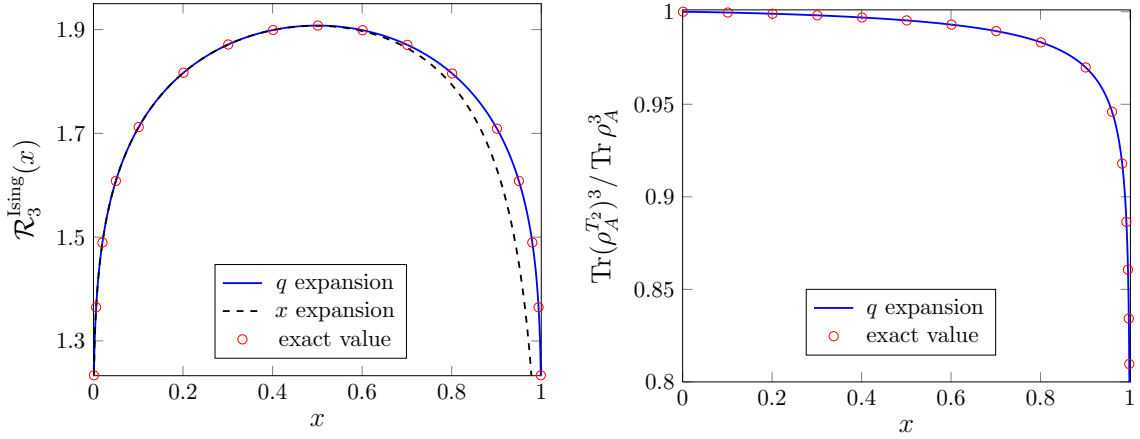


Figure 2. Analysis of the moments $\text{Tr } \rho_A^N$ and $\text{Tr}(\rho_A^{T_2})^N$ for $N = 3$ in the free Majorana fermion (Ising CFT). In the left panel, we consider the function $\mathcal{R}_3(x)$ defined in Eq. (5.14). The dots correspond to the exact values calculated using Eq. (E.7). The continuous line has been obtained by approximating, in Eq. (5.14), $\text{Tr } \rho_A^3$ with its expansion in sphere conformal blocks, see Eq. (5.6) and Appendix E, truncated at level $L = 6$. The dashed curve has been determined instead by decomposing $\text{Tr } \rho_A^3$ in terms of $N = 3$ orbifold conformal blocks, see Eqs. (5.1) and (E.1), and approximating these functions with their power series about $x = 0$ in Eq. (3.29) up to $L = 6$. In the right panel, we study the ratio $\text{Tr}(\rho_A^{T_2})^3 / \text{Tr } \rho_A^3$. The dots are the exact values computed from Eqs. (5.14) and (5.15) by substituting Eq. (E.7) for the function \mathcal{R}_3 . The continuous line is the result if we approximate $\text{Tr } \rho_A^3$ and $\text{Tr}(\rho_A^{T_2})^3$ with their expansions in terms of sphere conformal blocks up to level $L = 6$. In all the cases, the sphere conformal blocks have been calculated using the elliptic recursion of Eq. (4.6), truncated at level $L' = 8$.

For comparing with the previous literature, let us rewrite the moments $\text{Tr } \rho_A^N$ and $\text{Tr}(\rho_A^{T_2})^N$ in the form [15]

$$\text{Tr } \rho_A^N = K_N x^{-4h_{\sigma_N}} (1-x)^{-4h_{\sigma_N}} \mathcal{R}_N(x) \quad (5.14)$$

and [19]

$$\text{Tr}(\rho_A^{T_2})^N = K_N x^{-4h_{\sigma_N}} (1-x)^{4h_{\sigma_N}} \mathcal{R}_N\left(\frac{x}{x-1}\right). \quad (5.15)$$

In Refs. [16, 35], an exact expression for the function $\mathcal{R}_N(z)$, $z \in \mathbb{C}$, was found in the Ising CFT, which is also reported in Eq. (E.7) of Appendix E. We have checked that, for $N = 2, 3$, Eq. (E.7) is exactly reproduced in the intervals $0 < x < 1$ and $x/(x-1) < 0$ by the conformal block expansions of Eqs. (5.6) and (5.12) respectively. In Appendix E, we report the explicit conformal block decomposition, with the values of the structure constants, of the twist field four-point function in the Ising CFT for $N = 3$ up to fifth order.

In Fig. 2 left, we plot the function $\mathcal{R}_3^{\text{Ising}}(x)$, see Eq. (5.14). The continuous line has been drawn by employing the regularized conformal block expansion of Eq. (5.6) for $\text{Tr } \rho_A^3$ and the dots are the exact values given by Eq. (E.7). The agreement between both results is excellent. Note that, due to the crossing symmetry of the twist field four-point function and, therefore, of $\text{Tr } \rho_A^N$, $\mathcal{R}_N(x)$ must satisfy in general that $\mathcal{R}_N(x) = \mathcal{R}_N(1-x)$. On the other hand, if we decompose $\text{Tr } \rho_A^3$ in $N = 3$ orbifold conformal blocks and we expand them

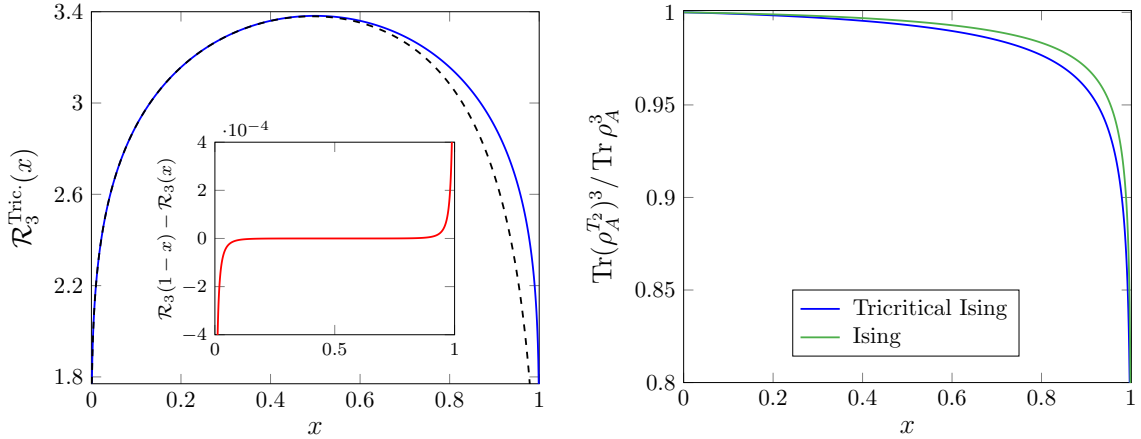


Figure 3. Analysis of the moments $\text{Tr} \rho_A^N$ and $\text{Tr}(\rho_A^{T_2})^N$ for $N = 3$ in the Tricritical Ising CFT. In the left panel, we plot the function $\mathcal{R}_3(x)$, introduced in Eq. (5.14), by replacing $\text{Tr} \rho_A^3$ with its expansion in sphere conformal blocks, Eq. (5.6), which we truncate at level $L = 6$. On the other hand, the dashed line has been computed by decomposing $\text{Tr} \rho_A^3$ in $N = 3$ orbifold conformal blocks and substituting their small x representation in Eq. (3.29). In the inset of left panel, we check the crossing symmetry of the continuous blue line. In the right panel, we study the ratio $\text{Tr}(\rho_A^{T_2})^3 / \text{Tr} \rho_A^3$. The curve has been obtained by applying the sphere conformal block expansions for $\text{Tr} \rho_A^3$ and $\text{Tr}(\rho_A^{T_2})^3$ of Eqs. (5.6) and (5.12), up to level $L = 6$. As a comparison, we also plot the result presented in Fig. 2 for the Ising CFT. The sphere conformal blocks have been computed until level $L' = 8$ using the elliptic recursion of Eq. (4.6).

using the small x representation given in Eq. (3.29), then the result is the dashed curve, which clearly does not display this symmetry for x close to one. In Fig. 2 right, we plot the quotient $\text{Tr}(\rho_A^{T_2})^3 / \text{Tr} \rho_A^3$. The solid curve has been obtained by calculating the conformal block expansions of Eq. (5.6) and (5.12) for $\text{Tr} \rho_A^3$ and $\text{Tr}(\rho_A^{T_2})^3$ while the dots represents the ratio of Eqs. (5.14) and (5.15), computed by applying Eq. (E.7). Again the conformal block expansion matches with the previously known results.

In Refs. [31, 32], the function $\mathcal{R}_3(x)$ for the Ising CFT was already studied by expanding the twist field four-point function in sphere conformal blocks and employing the Zamolodchikov elliptic recursion to speed up its convergence rate. However, in our formalism, the mentioned works only took into account in Eq. (5.6) contributions from the Virasoro sub-algebra primaries at the level $l = 0$. As explained in Sec. 4, the inclusion of the orbifold descendants in Eq. (3.29) permits to keep track instead of all the Virasoro sub-algebra primaries up to level $l = 6$ in Eq. (4.5). By reaching higher levels in the conformal block expansion, one obviously gets a much better cross-symmetric approximation for the twist field correlator, cf. Fig. 2 of Ref. [32]. Moreover, a complete understanding of the conformal block expansion of the twist field four-point function allowed us to characterize field theoretically, see Eq. (5.12), the difference between the two partial traces $\text{Tr}(\rho_A^{T_2})^3$ and $\text{Tr} \rho_A^3$.

Finally, as a further application, in Fig. 3, we plot the results for the moments $\text{Tr} \rho_A^3$ and $\text{Tr}(\rho_A^{T_2})^3$ in the Tricritical Ising CFT with $c = 7/10$ (see [62]). The Tricritical Ising CFT

appears at the quantum critical point of a device for quantum computation known as the golden chain [63]. It is an interacting CFT for which, especially concerning entanglement of disjoint intervals, there are no previous results available in the literature. In Fig. 3 left, we plot the function $\mathcal{R}_3(x)$ for this theory, extracted from the corresponding conformal block decomposition of $\text{Tr } \rho_A^3$. In Appendix F, one can find the explicit conformal block expansion, as well as the values of the structure constants, of the twist field four-point function for the Tricritical Ising CFT when $N = 3$ up to fifth order. In the inset, we check the crossing invariance of $\mathcal{R}_3^{\text{Tric.}}(x)$ using the sphere conformal block expansion. We observe that the function is symmetric under $x \mapsto 1 - x$ for almost all $x \in (0, 1)$. In the same plot, we also included the curve (the dashed line) that one determines by decomposing $\text{Tr } \rho_A^3$ in $N = 3$ orbifold conformal blocks and employing the small x representation of Eq. (3.29). As it is clear from the plot, the result fails to be cross-symmetric. In Fig. 3 right, we plot the ratio of moments $\text{Tr}(\rho_A^{T_2})^3 / \text{Tr } \rho_A^3$ determined from the conformal block decomposition of Eqs. (5.6) and (5.12). Here we also compare with the results found for the Ising CFT, the green line.

5.2 Constraints on the CFT Structure Constants

In Ref. [28], the crossing symmetry of the twist field four-point function for the case $N = 3$ was employed to extract some non-trivial constraints on the structure constants of the seed theory \mathcal{C} . For this purpose, the authors mapped the orbifold conformal block $\mathcal{G}_{c,h}^{(3)}(z)$ from the z -plane to the 3-fold pillow frame, which makes apparent some positivity properties of the block for unitary theories. As we already mentioned in Sec. 4, this transformation produces a series for the orbifold conformal block in terms of the elliptic nome $q(z)$ of the form of Eq. (4.3). In this section, we will see how to rederive such constraints on the structure constants from the expansion of the orbifold conformal blocks in terms of the ones on the sphere, studied in Sec. 4. This expansion results in a power series in $q(z)$ of the form of Eq. (4.9). Therefore, as we will see, it provides slightly tighter constraints on the structure constants than the expansion used in Ref. [28].

First, as in Ref. [28], we must recast the Zamolodchikov recursion relation, Eq. (4.6), for the Virasoro conformal blocks $\mathcal{F}_{Nc,h}(z)$ in order to make manifest certain positivity properties that we will need later. In Ref. [57], the conformal block $\mathcal{F}_{Nc,h}(z)$, which is defined on the sphere, is mapped to the pillow (the quotient of a flat torus by \mathbb{Z}_2 , which is topologically equivalent to a sphere with four conical singularities). The important result for us is that the transformed blocks $\tilde{\mathcal{F}}_{Nc,h}(z)$ read

$$\tilde{\mathcal{F}}_{Nc,h}(q) = \vartheta_3(q)^{16h_{\sigma_N} - \frac{Nc}{2}} [z(1-z)]^{2h_{\sigma_N} - \frac{Nc}{24}} \mathcal{F}_{Nc,h}(z). \quad (5.16)$$

If $\tilde{\mathcal{F}}_{Nc,h}(z)$ is interpreted as a sum over the states on the pillow, then it admits the expansion

$$\tilde{\mathcal{F}}_{Nc,h}(q) = q^{h - \frac{Nc}{24}} \sum_{n=0}^{\infty} \tilde{a}_n(h, Nc) q^n, \quad (5.17)$$

where the coefficients $\tilde{a}_n(h, Nc)$ are sums of scalar products between the descendant states at level n . Consequently, they are non-negative, $\tilde{a}_n(h, Nc) \geq 0$, for unitary theories. On

the other hand, if the elliptic recursion of Eq. (4.6) is applied in Eq. (5.16), one concludes that

$$\tilde{\mathcal{F}}_{Nc,h}(q) = (16q)^{h-\frac{Nc}{24}} \prod_{k=1}^{\infty} (1-q^{2k})^{-\frac{1}{2}} H(h, Nc, q). \quad (5.18)$$

This identity implies that the coefficients $\tilde{a}_n(h, Nc)$ are strictly related to the coefficients $a_j(h, Nc)$, which in general are not positive definite, of the expansion of $H(h, Nc, q)$, see Eq. (4.8). In conclusion, we can rewrite the Virasoro conformal blocks in terms of the (positive definite) coefficients $\tilde{a}_n(h, Nc)$,

$$\mathcal{F}_{Nc,h}(z) = q^{h-\frac{Nc}{24}} [z(1-z)]^{\frac{Nc}{24}-2h_{\sigma N}} \vartheta_3(q)^{\frac{Nc}{2}-16h_{\sigma N}} \sum_{n=0}^{\infty} \tilde{a}_n(h, Nc) q^n. \quad (5.19)$$

If we now truncate at order $L' > L$ the power series of Eq. (5.19) and insert it into the truncated expansion of $\mathcal{G}_{c,h}(z)$ in terms of $\mathcal{F}_{Nc,|\mathbf{h}|+l}(z)$ of Eq. (4.5), we get the following approximation for the orbifold conformal blocks, cf. Eqs. (4.3) and (4.9),

$$\mathcal{G}_{c,h}^{(N)}(z) \sim q^{|\mathbf{h}|-\frac{Nc}{24}} [z(1-z)]^{\frac{Nc}{24}-2h_{\sigma N}} \vartheta_3(q)^{\frac{Nc}{2}-16h_{\sigma N}} \left(\sum_{l=0}^L A_l q^l + \sum_{l=L+1}^{L+L'} A'_l q^l \right). \quad (5.20)$$

The coefficients A_l and A'_l are sums of terms of the form $\alpha_m^{\mathbf{h}} \tilde{a}_n(|\mathbf{h}| + m, Nc)$, as one can easily conclude from the combination of Eqs. (4.5) and (5.19). The structure constants $\alpha_m^{\mathbf{h}}$ are, by their definition in Eq. (4.20), non-negative in unitary theories. This implies that both A_l and A'_l are also non-negative for any l . If we only consider the term with coefficients A_l , the approximation of Eq. (5.20) reduces to the one used in Ref. [28] for the case $N = 3$. As we already remarked in Sec. 4, the expansion in Virasoro conformal blocks of Eq. (4.5), together with the elliptic recursion of Eq. (4.6), allows to further incorporate the contribution of some of the descendant states at higher levels, the term with coefficients A'_l in Eq. (5.20), improving the approximation of Ref. [28].

We can now apply the result of Eq. (5.20) to derive some constraints on the structure constants of the seed theory \mathcal{C} . Using the decomposition of Eq. (3.1) of the twist field correlation function in terms of orbifold conformal blocks, we can rewrite the crossing symmetry condition of Eq. (2.6) in the form

$$\sum_{\mathbf{h}, \bar{\mathbf{h}}} D_{\mathbf{h}, \bar{\mathbf{h}}} \left[\mathcal{G}_{c,\mathbf{h}}^{(N)}(z) \mathcal{G}_{c,\bar{\mathbf{h}}}^{(N)}(\bar{z}) - \mathcal{G}_{c,\mathbf{h}}^{(N)}(1-z) \mathcal{G}_{c,\bar{\mathbf{h}}}^{(N)}(1-\bar{z}) \right] = 0. \quad (5.21)$$

In the rest of this section, we will restrict to the case $N = 3$, for which the coefficients $D_{\mathbf{h}, \bar{\mathbf{h}}}$ are, according to Eq. (3.30), proportional to the square of the structure constants $C_{\mathbf{h}, \bar{\mathbf{h}}}^{\text{seed}}$ of the seed theory \mathcal{C} . If, as in the usual numerical bootstrap approach [64, 65], we act on Eq. (5.21) with a linear functional

$$\gamma(f) = \sum_{n,m} \gamma_{n,m} \partial_z^n \partial_{\bar{z}}^m f(z, \bar{z}) \Big|_{z=\bar{z}=\frac{1}{2}}, \quad (5.22)$$

where $\gamma_{n,m}$ are real coefficients and $f(z, \bar{z})$ an arbitrary function, then we can derive a set of linear equations for $(C_{\mathbf{h}, \bar{\mathbf{h}}}^{\text{seed}})^2$. To compare with the results of Ref. [28], let us take the

linear functional that only contains the first derivative, $\gamma \equiv \partial_z|_{z=\bar{z}=\frac{1}{2}}$. If we apply it to Eq. (5.21) for $N = 3$, we find the condition

$$\sum_{\mathbf{h}, \bar{\mathbf{h}}} 27^{-|\mathbf{h}| - |\bar{\mathbf{h}}|} (C_{\mathbf{h}, \bar{\mathbf{h}}}^{\text{seed}})^2 \mathcal{G}_{\bar{\mathbf{h}}}^{(3)}(1/2) \partial_z \mathcal{G}_{\mathbf{h}}^{(3)}(z) \Big|_{z=\frac{1}{2}} = 0. \quad (5.23)$$

For unitary theories, $(C_{\mathbf{h}, \bar{\mathbf{h}}}^{\text{seed}})^2$ are positive and, in order Eq. (5.23) to be satisfied, the factor $\mathcal{G}_{\bar{\mathbf{h}}}^{(3)}(z) \partial_z \mathcal{G}_{\mathbf{h}}^{(3)}(z) \Big|_{z=1/2}$ must be negative on a domain \mathcal{D} of the space of triplets of conformal dimensions $\{(h_1, h_2, h_3) \in \mathbb{R}^3 \mid h_1, h_2, h_3 \geq 0\}$, and non-negative otherwise. The points (h_1, h_2, h_3) where this factor vanishes are the boundary of the domain \mathcal{D} and typically form a compact surface. Thus Eq. (5.23) implies that the structure constants corresponding to points outside \mathcal{D} are bounded by those associated to points inside it.

The condition of being a point in the boundary of the domain \mathcal{D} can be rewritten as

$$W_c(h_1, h_2, h_3) = 0, \quad \text{with} \quad W_c(h_1, h_2, h_3) \equiv \partial_z \log \mathcal{G}_{\mathbf{h}}^{(3)}(z) \Big|_{z=\frac{1}{2}}. \quad (5.24)$$

If we now apply the approximation found in Eq. (5.20) for $\mathcal{G}_{\mathbf{h}}^{(3)}(z)$, then we find

$$W_c^{(L, L')}(h_1, h_2, h_3) = \frac{\pi^2}{K(\frac{1}{2})^2} \left[h_1 + h_2 + h_3 - \left(\frac{1}{8} + \frac{5}{72\pi} \right) c + \frac{\sum_{l=1}^L l A_l e^{-\pi l} + \sum_{l=L+1}^{L+L'} l A'_l e^{-\pi l}}{\sum_{l=0}^L A_l e^{-\pi l} + \sum_{l=L+1}^{L+L'} A'_l e^{-\pi l}} \right]. \quad (5.25)$$

Observe that it is at this point that the positivity of the coefficients A_l and A'_l for unitary theories previously discussed plays the crucial role, since it implies that the last term in Eq. (5.25) is positive too. This means that the domain $\mathcal{D}_{L, L'}$ of points (h_1, h_2, h_3) for which $W_c^{(L, L')}(h_1, h_2, h_3) < 0$ shrinks as L and L' increase and it converges to the domain \mathcal{D} , $\mathcal{D}_{L, L'} \rightarrow \mathcal{D}$, in the limit $L, L' \rightarrow \infty$. If in the last term we only include the sum over the coefficients A_l , we recover the approximation considered in Ref. [28] (cf. Eq. (3.8) of that reference). By including the contribution of some of the descendants of the sub-algebra primary fields at the levels $l = 1, \dots, L$, we improve the convergence of Eq. (5.25) and produce a slightly smaller domain $\mathcal{D}_{L, L'}$, as Fig. 4 shows. In any case, the convergence with the domain \mathcal{D} is very fast due to the exponential decay of those terms in Eq. (5.25).

5.3 Bootstrapping Genus Two Partition Functions

In this final section, we implement a numerical bootstrap approach based on Virasoro sphere conformal blocks to determine numerically the structure constants $D_{\mathbf{h}, \bar{\mathbf{h}}}$ of Eq. (3.1) for $N = 2$ and $N = 3$. For minimal seed theories and $N = 2, 3$, these constants are known, see Eq. (3.21) and Eq. (3.30). However, the motivation of this section is twofold. First, we show that, even in the simplest case, the stability of the numerical procedure is greatly sensitive to the symmetry $x \mapsto 1 - x$ of the orbifold conformal blocks. As a matter of fact, the numerical outcomes start to converge only if one uses the approximation of Eq. (4.9). Secondly, we hope that the results in this section could serve as a guide to set up a numerical bootstrap scheme for higher genus ($N > 3$) or non-minimal theories.

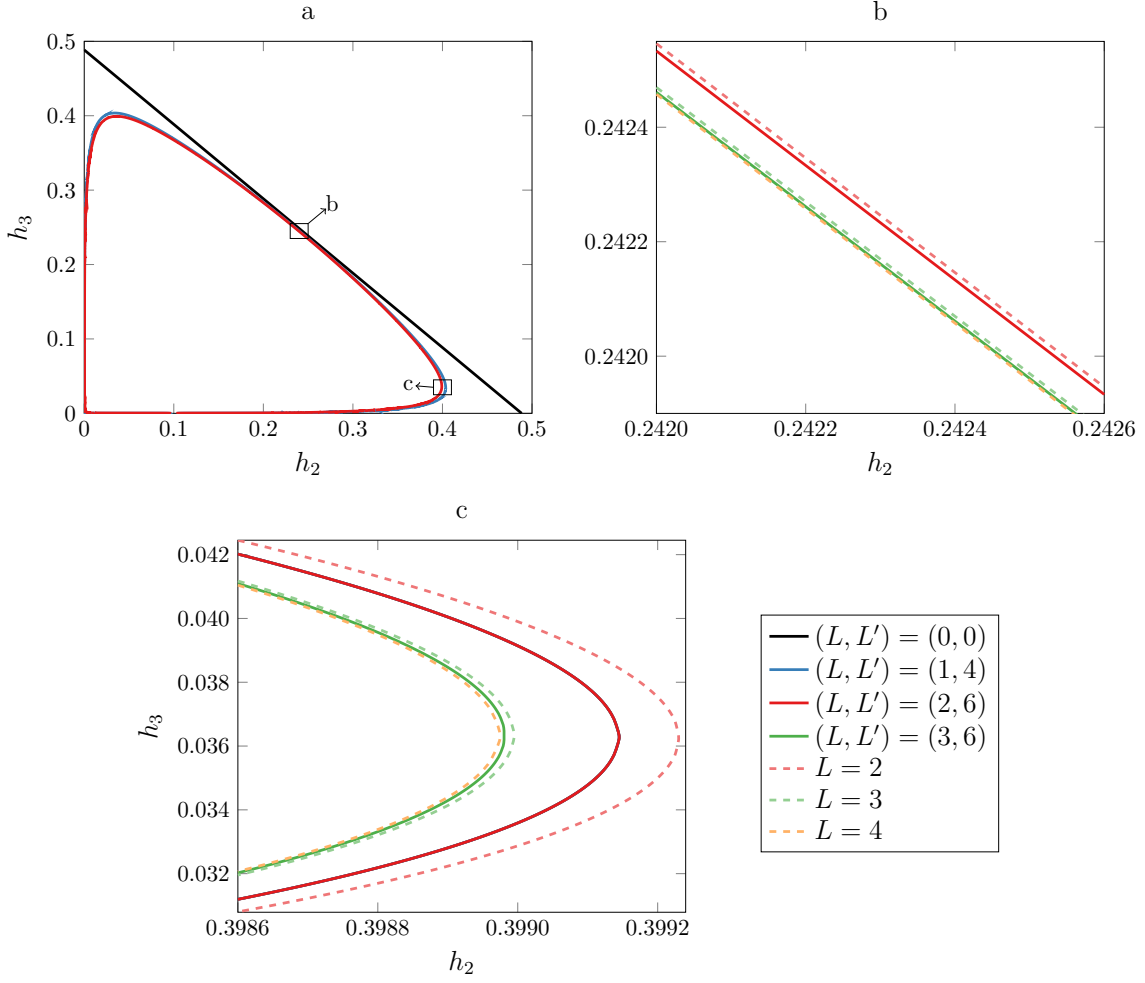


Figure 4. The continuous lines represent the points in the (h_2, h_3) plane that satisfy the equation $W_c^{(L, L')}(h_1, h_2, h_3) = 0$ for $c = 4$ and $h_1 = 0.1$ and several values of the truncation levels L, L' , see Eq. (5.25). The dashed lines are the points that satisfy the same equation but removing from $W_c^{(L, L')}(h_1, h_2, h_3)$ the sums with coefficients A'_l . This is the approximation considered in Ref. [28]. The colouring of the dashed lines corresponds to take different upper bounds L in the sums with coefficients A_l . The panels b and c are the magnification of the regions indicated by a box in the plot a.

As we already discussed in Sec. 5.2, the combination of the decomposition of Eq. (3.1) and the crossing symmetry condition in Eq. (2.6) leads to Eq. (5.21). If we now normalize to one the structure constant of the channel $h_1 = h_2 = \dots = h_N = 0$, that is $D_{\mathbf{0}, \mathbf{0}} = 1$, then Eq. (5.21) can be rewritten as

$$\sum_{\substack{\mathbf{h}, \bar{\mathbf{h}} \\ (\mathbf{h}, \bar{\mathbf{h}}) \neq (\mathbf{0}, \mathbf{0})}} D_{\mathbf{h}, \bar{\mathbf{h}}} \left[\mathcal{G}_{c, \mathbf{h}}^{(N)}(z) \mathcal{G}_{c, \bar{\mathbf{h}}}^{(N)}(\bar{z}) - \mathcal{G}_{c, \mathbf{h}}^{(N)}(1-z) \mathcal{G}_{c, \bar{\mathbf{h}}}^{(N)}(1-\bar{z}) \right] \\ = \mathcal{G}_{c, \mathbf{0}}^{(N)}(1-z) \mathcal{G}_{c, \mathbf{0}}^{(N)}(1-\bar{z}) - \mathcal{G}_{c, \mathbf{0}}^{(N)}(z) \mathcal{G}_{c, \mathbf{0}}^{(N)}(\bar{z}). \quad (5.26)$$

For the minimal models introduced in Sec. 3.4, the number of channels in Eq. (5.26) is finite. For instance, for $N = 2$, it corresponds to the number of conformal families of \mathcal{C} and for $N = 3$, is the number of non-zero structure constants. Therefore, for a given point z , the crossing symmetry relation is a linear equation in N_c (the number of channels except $(\mathbf{0}, \mathbf{0})$) unknowns $D_{\mathbf{h}, \bar{\mathbf{h}}}$. Note that this procedure boils down to choose a different linear functional to be applied to the crossing equation than the one used in Eq. (5.22).

As done in [66], we can draw uniformly N_c random points $\{z_j\}$ in the square $[1/2 - \kappa, 1/2 + \kappa] \times [-i\kappa, i\kappa]$. Let us also require that each point is separated from the real axis and each other point by a distance

$$\delta = \frac{\kappa}{\sqrt{N_c}} \quad (5.27)$$

where κ is an arbitrary positive number (which we will fix later). By imposing Eq. (5.26) at each $z = z_j$, one obtains a linear system with N_c unknowns and N_c equations. By truncating the expansion in Eq. (4.9) of the conformal blocks $\mathcal{G}_{c, \mathbf{h}}^{(N)}(z)$ at given L, L' , it is possible to calculate a set of structure constants $D_{\mathbf{h}, \bar{\mathbf{h}}}(L, L')$ for any random realization of the points $\{z_j\}$. If the bootstrap converges, we expect that the variance of the solutions $D_{\mathbf{h}, \bar{\mathbf{h}}}(L, L')$ will be small and that $D_{\mathbf{h}, \bar{\mathbf{h}}}(L, L') \rightarrow D_{\mathbf{h}, \bar{\mathbf{h}}}$, for $L, L' \rightarrow \infty$.

As a first benchmark of the method, we have considered the Tricritical Ising CFT with $c = 7/10$ on a torus ($N = 2$). In Appendix D, we gather the operator content of this model. It belongs to the diagonal series of the minimal models, namely its partition function \mathcal{Z}_1 on the torus, see Eq. (3.20), is

$$\mathcal{Z}_1(x) = \sum_h |\chi_{c, h}(\tau(x))|^2, \quad (5.28)$$

for $c = 7/10$ and the sum running over the conformal dimensions in Eq. (D.4). We first used the regularization prescription explained in Section 3.4 to compute the combinatorial expansion of Eq. (3.19) until level $L = 6$. From it, we obtained the approximation for the $N = 2$ orbifold conformal blocks of Eq. (4.9), truncated at $L, L' = 6$. In Table 1, we have considered 100 different sets of random points $\{z_j\}$ and we have calculated the mean and the coefficient of variance (the standard deviation divided by the mean) of the values for the structure constants derived by solving Eq. (5.26) for each sample of random points. One can see that the numerical bootstrap converges.

We have then applied the numerical bootstrap to determine the $N = 3$ twist field correlation function for the following models: The Ising CFT ($c = 1/2$), the Lee-Yang CFT ($c = -22/5$), and the Gaffnian CFT ($c = -3/5$) [67, 68]. Note that the last two theories are non-unitary, see Refs. [33, 69] for related studies. All of them belong to the diagonal series of minimal models and, therefore, only pairings between holomorphic and antiholomorphic primaries with the same conformal dimensions ($\mathbf{h} = \bar{\mathbf{h}}$) are possible. From Eq. (3.30) and Eq. (5.26) we can extract numerically the seed CFT structure constant as $(C_{\mathbf{h}, \mathbf{h}}^{\text{seed}})^2 = 27^{2|\mathbf{h}|} D_{\mathbf{h}, \mathbf{h}}$. In Appendix D, we remind the field content and the fusion rules of the minimal models under consideration.

Table 2 shows the results of the $N = 3$ numerical bootstrap for these three CFTs: Ising (top), Lee-Yang (center) and the Gaffnian (bottom). We consider again 100 different

h	$D_{h,h}$	
	numerical bootstrap	exact (Eq. (3.21))
σ	0.65975046 (8.8×10^{-5})	0.65975396
σ'	0.00781211 (8.0×10^{-4})	0.00781250
ε	0.32987828 (6.4×10^{-5})	0.32987698
ε'	0.00128874 (1.9×10^{-3})	0.00128858
ε''	5.92566×10^{-8} (2.7×10^{-1})	5.96046×10^{-8}

Table 1. Results of the numerical bootstrap for the Tricritical Ising CFT on the torus ($N = 2$). The first column indicates the channel labelled by the corresponding primary field, see Eq. (D.4). The second column corresponds to the mean value for each structure constant calculated after considering 100 different sets of random points with $\kappa = 0.22$. We have truncated the expansion in sphere conformal blocks and the elliptic recursion at level $L, L' = 6$. In brackets, the coefficient of variation.

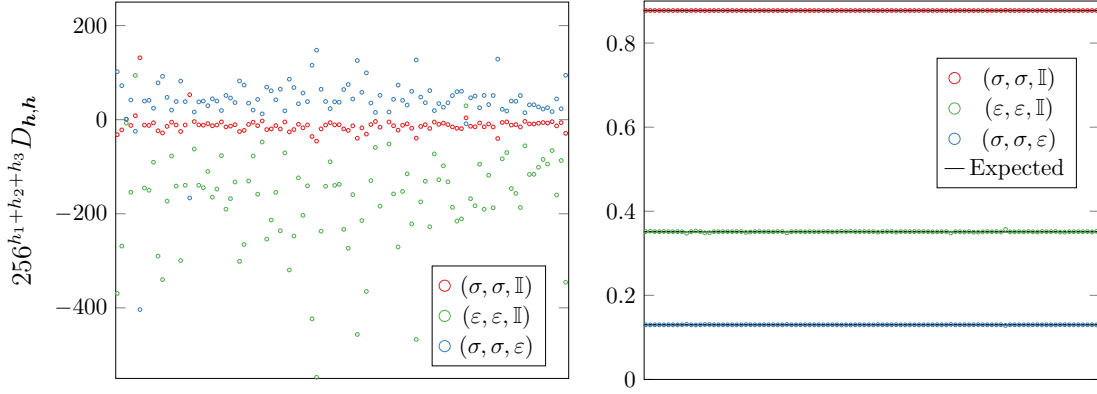


Figure 5. We plot the values obtained for the $N = 3$ structure constants $D_{h,h}$ (rescaled by a factor for representation purposes) in the Ising CFT by solving the cross-symmetry Eq. (5.26) for 100 different sets of random points, choosing $\kappa = 0.22$, and considering two different approximations for the orbifold conformal blocks. On the left, we have taken the small z expansion derived for them in Eq. (3.29), truncated at order $L = 6$. On the right, we have expanded them in sphere conformal blocks, as in Eq. (4.5), until $L = 6$ and we have applied the elliptic recursion of Eq. (4.6) to compute the latter up to level $L' = 6$. The solid lines correspond to the values expected for the structure constants.

sets $\{z_j\}$ of random points. In the second column of each chart, we write the mean and the coefficient of variance computed for $(C_{h,h}^{\text{seed}})^2$ after solving Eq. (5.26) with each set of random points. The third column collects the exact results for the square of the seed structure constants [70]. The agreement between the two values is excellent.

Finally, in Figure 5, we compare the numerical solutions of Eq. (5.26) obtained by first replacing the orbifold conformal blocks with their power series expansion about $z = 0$ (on the left) and then by their representation in terms of sphere conformal blocks (on the right). The results are shown for the Ising CFT. One can see that in the first case the bootstrap approach cannot predict the correct structure constants. This ultimately can be traced back to the $z \mapsto 1 - z$ asymmetry that is visible in Figure 2.

$$c = 1/2$$

(h_1, h_2, h_3)	$(C_{\mathbf{h},\mathbf{h}}^{\text{seed}})^2$	
	numerical bootstrap	exact
$(\sigma, \sigma, \mathbb{I})$	1.00001359 (9.7×10^{-5})	1
$(\varepsilon, \varepsilon, \mathbb{I})$	1.00032363 (2.8×10^{-3})	1
$(\sigma, \sigma, \varepsilon)$	0.24993371 (2.0×10^{-3})	0.25

$$c = -22/5$$

(h_1, h_2, h_3)	$(C_{\mathbf{h},\mathbf{h}}^{\text{seed}})^2$	
	numerical bootstrap	exact
$(\varphi, \varphi, \mathbb{I})$	0.99999804 (8.9×10^{-6})	1
$(\varphi, \varphi, \varphi)$	-3.65310941 (8.5×10^{-6})	-3.65311624

$$c = -3/5$$

(h_1, h_2, h_3)	$(C_{\mathbf{h},\mathbf{h}}^{\text{seed}})^2$	
	numerical bootstrap	exact
$(\sigma, \sigma, \mathbb{I})$	0.99981087 (4.1×10^{-4})	1
$(\varepsilon, \varepsilon, \mathbb{I})$	0.99983570 (1.0×10^{-3})	1
(ψ, ψ, \mathbb{I})	0.99629757 (2.1×10^{-2})	1
$(\sigma, \sigma, \varepsilon)$	-0.27381762 (4.1×10^{-4})	-0.27373889
$(\varepsilon, \varepsilon, \varepsilon)$	-4.37665844 (2.2×10^{-3})	-4.37982231
$(\sigma, \varepsilon, \psi)$	0.24859929 (1.1×10^{-2})	0.25

Table 2. Results of the numerical bootstrap for the Ising (top), Lee-Yang (center), and the Gaffnian (bottom) CFTs on a genus two \mathbb{Z}_3 -symmetric Riemann surface ($N = 3$). For each minimal model, the second column collects the mean value determined for each structure constant after performing the bootstrap with 100 different sets of random points $\{z_j\}$, fixing $\kappa = 0.22$. The sphere conformal block expansion and the elliptic recursion were truncated at level $L, L' = 6$. In the brackets we write the coefficient of variation. The third column contains the exact value of $(C_{\mathbf{h},\mathbf{h}}^{\text{seed}})^2$, calculated in [70] for the minimal models.

6 Conclusions

In this paper we analyzed conformal four-point twist field correlation functions in the \mathbb{Z}_N orbifold. Such correlators can be also interpreted as CFT partition functions on Riemann surfaces of genus $N - 1$, with a cyclic symmetry. We focused in particular on seed theories \mathcal{C} which belong to the series of the minimal models with $c < 1$. From a CFT perspective, we extended the results in [28] in two directions. Firstly, we provided a regularization scheme for the combinatorial expansion of the orbifold conformal blocks that gets rid of all the null vectors when $c < 1$. Then, we proposed a method to systematically expand the genus $N - 1$ conformal blocks into sphere conformal blocks of central charge Nc . The latter are more suitable for applications, since they can be calculated, by the recursion formula [34],

as power series in the elliptic variable q .

The sphere conformal blocks, which appear in the decomposition, can be identified order by order in q only by comparison with the combinatorial expansion of the orbifold conformal blocks that incorporates the descendant states in the replicated theory $\mathcal{C}^{\otimes N}$. The inclusion of the descendant contributions, and their evaluation through conformal mappings, represents a substantial improvement to the previous power series in q of the twist field correlators [31, 32].

We examined in detail the case $N = 3$, i.e. genus two Riemann surfaces, and discussed extensive applications of the formalism. In particular, we calculated Rényi entropies for two disjoint intervals in minimal CFTs, reproducing with great accuracy the available results for the free Majorana fermion and providing new ones for interacting theories. We also showed how the partially transposed reduced density matrix for two disjoint intervals could be calculated in a conformal block expansion and applied this result to the Tricritical Ising CFT. The representation in terms of sphere conformal blocks of the twist field correlators was also employed to refine the bound on the structure constants of unitary theories found in [28].

There are a couple of possible future directions that are worth to be mentioned. First, it would be useful to investigate whether our formalism can be extended to arbitrary values of the genus $g = N - 1$. This task involves the determination of the descendant N -point function in Eq. (3.15) for a rational CFT. We expect the latter to be analytic in g at least for a free compactified boson, thus allowing to recover the results for the entanglement entropy and negativity discussed in [15, 19, 42, 59, 61] from a different route. Also it will be important to understand if a recursive formula for higher genus conformal blocks, such as that put forward in [30], can be effectively implemented when $c < 1$, due to the additional null vector resonances.

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A Transformation Properties of Virasoro Descendants

Consider a CFT on the compactified complex plane (Riemann sphere) with coordinate z . The Virasoro generators are defined by their action on the fields,

$$L_{-n}(z)\phi_h^M(z) = \oint_{C_z} \frac{dz'}{2\pi i} (z' - z)^{-n+1} T(z') \phi_h^M(z), \quad (\text{A.1})$$

where C_z is a closed contour containing z and $T(z)$ is the stress energy tensor. From the current-current OPE,

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + \text{regular terms}, \quad (\text{A.2})$$

one obtains the Virasoro algebra

$$[L_n(0), L_m(0)]\phi_h^M(0) = \left[(n-m)L_{n+m}(0) + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \right] \phi_h^M(0). \quad (\text{A.3})$$

The Virasoro generators acting on a field inserted at $z = \infty$ are given by

$$L_{-n}(z = \infty) = - \oint_{C_\infty} \frac{dz}{2\pi i} z^{n+1} T(z). \quad (\text{A.4})$$

Let us see how these operators transform under a conformal map $t \mapsto z(t)$. By recalling the transformation of the stress energy tensor when we apply it,

$$T(z) \mapsto \left(\frac{dz}{dt} \right)^{-2} \left[T(t) - \frac{c}{12} \{z(t), t\} \right], \quad (\text{A.5})$$

where $\{z(t), t\}$ is the Schwarzian derivative, the Virasoro descendant in Eq. (A.4) is then transformed into the linear combination of descendants acting on the point t_∞ in the t -surface whose image is $z(t_\infty) = \infty$

$$\mathcal{L}_{-n}(t_\infty) = - \oint_{C_{t_\infty}} \frac{dt}{2\pi i} \left(\frac{dz}{dt} \right)^{-1} [z(t)]^{n+1} \left[T(t) - \frac{c}{12} \{z(t), t\} \right], \quad (\text{A.6})$$

where C_{t_∞} is a contour encircling the point t_∞ , and $T(t)$ can be expressed as

$$T(t) = \sum_{m \in \mathbb{Z}} (t - t_\infty)^{-m-2} L_m(t = t_\infty). \quad (\text{A.7})$$

The results in Sec. 3.2 can be derived by evaluating Eq. (A.6) with the map given in Eq. (3.16). In particular, we can write down the expansion of the Virasoro descendants in Eq. (A.6) by applying the residue theorem; if $n \geq 1$ one has

$$\mathcal{L}_{-n}(t_\infty) = \sum_{m \geq -n} a_{nm} L_m(t = t_\infty) + \frac{c}{32}(n-1), \quad t_\infty = \{0, \infty\}, \quad (\text{A.8})$$

with

$$a_{nm} = \frac{1}{4^n} \left[2^{2n+1} - \binom{2n+1}{m+n+1} {}_2F_1(1, m-n, m+n+2, -1) \right]. \quad (\text{A.9})$$

In the case $N = 3$, studied in Sec. 3.3, the orbifold three-point function of Eq. (3.24) is calculated by applying the conformal map of Eq. (3.23). The pullbacks $\mathcal{L}_{-n}(t_\infty)$ of the Virasoro descendants $L_{-n}(z = \infty)$ under this map can be also derived by using Eq. (A.6). We have, for $n \geq 1$,

$$\mathcal{L}_{-n}(t_\infty) = \sum_{m \geq n} a_{nm}^{t_\infty} L_m(t = t_\infty) + \frac{c}{27}(n-1), \quad t_\infty = \{0, 1, \infty\} \quad (\text{A.10})$$

where the coefficients $a_{nm}^{t\infty}$ can be determined in closed form from the residue theorem, see also Eq. (2.9) of Ref. [28].

The three-point correlations on the sphere that appear in the expansion in Eq. (3.29) of the $N = 3$ orbifold conformal block can be computed recursively by employing the following Ward identities [71],

$$\langle L_{-n}\phi_{h_1}^{M_1}|\phi_{h_2}^{M_2}(1)|\phi_{h_3}^{M_3}\rangle = \langle \phi_{h_1}^{M_1}|\phi_{h_2}^{M_2}(1)|L_n\phi_{h_3}^{M_3}\rangle + \sum_{m=1}^n \binom{n+1}{m+1} \langle \phi_{h_1}^{M_1}|L_m\phi_{h_2}^{M_2}(1)|\phi_{h_3}^{M_3}\rangle, \quad (\text{A.11})$$

$$\langle \phi_{h_1}^{M_1}|L_{-n}\phi_{h_2}^{M_2}(1)|\phi_{h_3}^{M_3}\rangle = \sum_{m=0}^{\infty} \binom{n+m-2}{m} \left[\langle L_{m+n}\phi_{h_1}^{M_1}|\phi_{h_2}^{M_2}(1)|\phi_{h_3}^{M_3}\rangle + (-1)^n \langle \phi_{h_1}^{M_1}|\phi_{h_2}^{M_2}(1)|L_{m-1}\phi_{h_3}^{M_3}\rangle \right], \quad (\text{A.12})$$

and

$$\langle \phi_{h_1}|\phi_{h_2}(1)|L_{-n}\phi_{h_3}\rangle = \langle L_{-n}\phi_{h_3}|\phi_{h_2}(1)|\phi_{h_1}\rangle. \quad (\text{A.13})$$

B Orbifold Virasoro Sub-algebra

In each sheet of the orbifold theory $\mathcal{C}^{\otimes N}/\mathbb{Z}_N$, we can consider a copy $\mathbf{T}^{(j)}(z)$, $j = 1, \dots, N$, of the stress-energy tensor of the seed theory \mathcal{C} . Then the stress-energy tensor of the orbifold theory $\mathbf{T}(z)$ is

$$\mathbf{T}(z) = \sum_{j=1}^N \mathbf{T}^{(j)}(z). \quad (\text{B.1})$$

It generates transformations affecting all the sheets in the same way. Its Fourier modes,

$$\mathbf{L}_n = \oint_{C_0} \frac{dz}{2\pi i} z^{n+1} \mathbf{T}(z), \quad (\text{B.2})$$

where C_0 is a contour encircling the point $z = 0$, are symmetric under the exchange of sheets,

$$\mathbf{L}_n = \sum_{j=1}^N \mathbf{L}_n^{(j)}, \quad \mathbf{L}_n^{(j)} = \mathbb{I} \otimes \dots \overset{j}{\otimes} L_n \otimes \dots \otimes \mathbb{I}, \quad (\text{B.3})$$

and form a Virasoro algebra,

$$[\mathbf{L}_n, \mathbf{L}_m] = (n-m)\mathbf{L}_{n+m} + \frac{Nc}{12}n(n^2-1)\delta_{n+m,0}, \quad (\text{B.4})$$

with central charge Nc .

C Entanglement Entropy and Logarithmic Negativity

Let us consider a generic quantum system that can be divided into two spatial regions, which we call A and B , such that the total Hilbert space factorizes as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We suppose that the system is in a pure state $|\Psi\rangle \in \mathcal{H}$. Hence the state of subsystem A is described by the reduced density matrix

$$\rho_A = \text{Tr}_{\mathcal{H}_B} |\Psi\rangle\langle\Psi|, \quad (\text{C.1})$$

with $\text{Tr}_{\mathcal{H}_B}$ denoting the partial trace in the space \mathcal{H}_B . The entanglement between regions A and B can be analysed using the moments of the reduced density matrix, $\text{Tr} \rho_A^N$. In particular, the entanglement entropy

$$S_A = -\text{Tr}(\rho_A \log \rho_A) \quad (\text{C.2})$$

can be calculated from the Rényi entanglement entropies

$$S_A^{(N)} = \frac{1}{1-N} \log \text{Tr} \rho_A^N \quad (\text{C.3})$$

by exploiting the so-called replica trick [12, 13]

$$S_A = \lim_{N \rightarrow 1^+} S_A^{(N)} = - \lim_{N \rightarrow 1^+} \frac{\partial}{\partial N} \text{Tr} \rho_A^N. \quad (\text{C.4})$$

If subsystem A consists of two disjoint regions A_1 and A_2 , such that $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$, then one can consider the partial transpose $\rho_A^{T_2}$ of ρ_A . If $\{|e_j^{(l)}\rangle\}$ denotes a basis of the space \mathcal{H}_{A_l} , then the matrix elements of $\rho_A^{T_2}$ are defined as

$$\langle e_{j_1}^{(1)} e_{k_1}^{(2)} | \rho_A^{T_2} | e_{j_2}^{(1)} e_{k_2}^{(2)} \rangle = \langle e_{j_1}^{(1)} e_{k_2}^{(2)} | \rho_A | e_{j_2}^{(1)} e_{k_1}^{(2)} \rangle. \quad (\text{C.5})$$

The moments $\text{Tr}(\rho_A^{T_2})^N$ encode the entanglement between regions A_1 and A_2 . A particular measure of the entanglement between these two regions is for instance the logarithmic negativity [17],

$$\mathcal{E} = \log \text{Tr} |\rho_A^{T_2}|, \quad (\text{C.6})$$

which can also be calculated through the replica trick [18, 19]

$$\mathcal{E} = \lim_{n_e \rightarrow 1^+} \log \text{Tr}(\rho_A^{T_2})^{n_e}, \quad (\text{C.7})$$

by taking the analytic continuation of the moments $\text{Tr}(\rho_A^{T_2})^N$ with even exponent $N = n_e$.

D Operator Content and Fusion Rules in the Minimal Models considered

In this Appendix, we recollect the list of primary fields and fusion rules for the conformal minimal models under consideration in the paper. The general expression for the central charge $c_{p,q}$ and the conformal dimension of the primaries of these models are respectively given in Eqs. (3.31) and (3.32).

- Lee-Yang CFT: $(p, q) = (5, 2)$, $c_{5,2} = -22/5$

ϕ_h	h
φ	$-1/5$

 $\varphi \times \varphi = \mathbb{I} + \varphi$
(D.1)

- Ising CFT: $(p, q) = (4, 3)$, $c_{4,3} = 1/2$

ϕ_h	h
σ	$1/16$
ε	$1/2$

 $\begin{aligned} \sigma \times \sigma &= \mathbb{I} + \varepsilon \\ \varepsilon \times \varepsilon &= \mathbb{I} \\ \sigma \times \varepsilon &= \sigma \end{aligned}$
(D.2)

- Gaffnian CFT: $(p, q) = (5, 3)$, $c_{5,3} = -3/5$

ϕ_h	h
σ	$-1/20$
ψ	$3/4$
ε	$1/5$

 $\begin{aligned} \sigma \times \sigma &= \mathbb{I} + \varepsilon & \sigma \times \varepsilon &= \sigma + \psi \\ \varepsilon \times \varepsilon &= \mathbb{I} + \varepsilon & \sigma \times \psi &= \varepsilon \\ \psi \times \psi &= \mathbb{I} & \varepsilon \times \psi &= \sigma \end{aligned}$
(D.3)

- Tricritical Ising CFT: $(p, q) = (5, 4)$, $c_{5,4} = 7/10$

ϕ_h	h
σ	$3/80$
σ'	$7/16$
ε	$1/10$
ε'	$3/5$
ε''	$3/2$

 $\begin{aligned} \varepsilon \times \varepsilon &= \mathbb{I} + \varepsilon' & \sigma \times \sigma &= \mathbb{I} + \varepsilon + \varepsilon' + \varepsilon'' \\ \varepsilon' \times \varepsilon' &= \mathbb{I} + \varepsilon' & \sigma' \times \sigma' &= \mathbb{I} + \varepsilon'' \\ \varepsilon \times \varepsilon' &= \varepsilon + \varepsilon'' & \sigma \times \sigma' &= \varepsilon + \varepsilon' \\ \varepsilon \times \sigma' &= \sigma & \varepsilon' \times \sigma' &= \sigma \\ \varepsilon \times \sigma &= \sigma' + \sigma & \varepsilon' \times \sigma &= \sigma' + \sigma \end{aligned}$
(D.4)

E $N = 3$ Orbifold Conformal Blocks for the Ising CFT

For the Ising CFT, according to its fusion rules in Eq. (D.2), the conformal block decomposition of Eq. (3.1) of the \mathbb{Z}_3 twist field four-point function takes the form

$$\begin{aligned} \langle \sigma_3(\infty) \tilde{\sigma}_3(1) \sigma_3(z, \bar{z}) \tilde{\sigma}_3(0) \rangle &= \left| \mathcal{G}_{\frac{1}{2}, \{0,0,0\}}^{(3)}(z) \right|^2 + 3D_{\sigma\sigma\mathbb{I}} \left| \mathcal{G}_{\frac{1}{2}, \{\frac{1}{16}, \frac{1}{16}, 0\}}^{(3)}(z) \right|^2 \\ &\quad + 3D_{\varepsilon\varepsilon\mathbb{I}} \left| \mathcal{G}_{\frac{1}{2}, \{\frac{1}{2}, \frac{1}{2}, 0\}}^{(3)}(z) \right|^2 + 3D_{\sigma\sigma\varepsilon} \left| \mathcal{G}_{\frac{1}{2}, \{\frac{1}{16}, \frac{1}{16}, \frac{1}{2}\}}^{(3)}(z) \right|^2, \end{aligned} \quad (\text{E.1})$$

where

$$D_{\sigma\sigma\mathbb{I}} = \frac{1}{3^{3/4}}, \quad D_{\varepsilon\varepsilon\mathbb{I}} = \frac{1}{729}, \quad D_{\sigma\sigma\varepsilon} = \frac{1}{4 \cdot 3^{15/4}}. \quad (\text{E.2})$$

The $N = 3$ orbifold conformal blocks above have the following expansions in terms of Virasoro sphere conformal blocks up to level $L = 5$, cf. Eq. (4.5),

$$\mathcal{G}_{\frac{1}{2}, \{0,0,0\}}^{(3)}(z) \sim \mathcal{F}_{\frac{3}{2}, 0}(z) + \frac{49}{10451673} \mathcal{F}_{\frac{3}{2}, 4}(z) + \frac{2}{4782969} \mathcal{F}_{\frac{3}{2}, 5}(z), \quad (\text{E.3})$$

$$\begin{aligned} \mathcal{G}_{\frac{1}{2}, \{\frac{1}{16}, \frac{1}{16}, 0\}}^{(3)}(z) &\sim \mathcal{F}_{\frac{3}{2}, \frac{1}{8}}(z) + \frac{1}{432} \mathcal{F}_{\frac{3}{2}, \frac{9}{8}}(z) + \frac{2209}{2612736} \mathcal{F}_{\frac{3}{2}, \frac{17}{8}}(z) + \frac{590597}{5965996032} \mathcal{F}_{\frac{3}{2}, \frac{25}{8}}(z) \\ &+ \frac{61593283775}{9375929753665536} \mathcal{F}_{\frac{3}{2}, \frac{33}{8}}(z) + \frac{13237693484267}{24583253711470460928} \mathcal{F}_{\frac{3}{2}, \frac{41}{8}}(z), \quad (\text{E.4}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{1}{2}, \{\frac{1}{2}, \frac{1}{2}, 0\}}^{(3)}(z) &\sim \mathcal{F}_{\frac{3}{2}, 1}(z) + \frac{1}{54} \mathcal{F}_{\frac{3}{2}, 2}(z) + \frac{1}{3402} \mathcal{F}_{\frac{3}{2}, 3}(z) + \frac{2401}{8109396} \mathcal{F}_{\frac{3}{2}, 4}(z) \\ &+ \frac{6245}{1499726502} \mathcal{F}_{\frac{3}{2}, 5}(z) + \frac{539}{12033950004} \mathcal{F}_{\frac{3}{2}, 6}(z), \quad (\text{E.5}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_{\frac{1}{2}, \{\frac{1}{16}, \frac{1}{16}, \frac{1}{2}\}}^{(3)}(z) &\sim \mathcal{F}_{\frac{3}{2}, \frac{5}{8}}(z) + \frac{49}{432} \mathcal{F}_{\frac{3}{2}, \frac{13}{8}}(z) + \frac{637}{124416} \mathcal{F}_{\frac{3}{2}, \frac{21}{8}}(z) + \frac{176647}{1988665344} \mathcal{F}_{\frac{3}{2}, \frac{29}{8}}(z) \\ &+ \frac{6395744863}{1086928370270208} \mathcal{F}_{\frac{3}{2}, \frac{37}{8}}(z) + \frac{528656973059}{299112879787868160} \mathcal{F}_{\frac{3}{2}, \frac{45}{8}}(z). \quad (\text{E.6}) \end{aligned}$$

For the Ising CFT, the function $\mathcal{R}_N(z)$ that appears in Eqs. (5.14) and (5.15) is of the form [16, 35]

$$\mathcal{R}_N^{\text{Ising}}(z) = \frac{1}{2^{N-1} |\Theta[\frac{0}{0}](\Omega(z))|} \sum_{\boldsymbol{\varepsilon}, \boldsymbol{\delta}} |\Theta[\frac{\boldsymbol{\varepsilon}}{\boldsymbol{\delta}}](\Omega(z))|, \quad z \in \mathbb{C}, \quad (\text{E.7})$$

where

$$\Theta[\frac{\boldsymbol{\varepsilon}}{\boldsymbol{\delta}}](\Omega) = \sum_{\boldsymbol{m} \in \mathbb{Z}^{N-1}} e^{i\pi(\boldsymbol{m}+\boldsymbol{\varepsilon})^t \cdot \Omega(\boldsymbol{m}+\boldsymbol{\varepsilon}) + 2\pi i(\boldsymbol{m}+\boldsymbol{\varepsilon})^t \cdot \boldsymbol{\delta}}.$$

The characteristics of the Theta function above are half-integer vectors, $\boldsymbol{\varepsilon}, \boldsymbol{\delta} \in (\mathbb{Z}/2)^{N-1}$. The sum in Eq. (E.7) runs over all the vectors $\boldsymbol{\varepsilon}, \boldsymbol{\delta}$ with components $\varepsilon_j, \delta_j \in \{0, 1/2\}$, and $\Omega(z)$ is the symmetric $(N-1) \times (N-1)$ matrix

$$\Omega_{rs}(z) = \frac{2i}{N} \sum_{k=1}^{N-1} \sin\left(\frac{\pi k}{N}\right) \cos\left[\frac{2\pi k}{N}(r-s)\right] \beta_{k/N}(z), \quad (\text{E.8})$$

in which

$$\beta_{k/N}(z) = \frac{{}_2F_1(k/N, 1 - k/N, 1; 1 - z)}{{}_2F_1(k/N, 1 - k/N, 1; z)}.$$

F $N = 3$ Orbifold Conformal Blocks for the Tricritical Ising CFT

If we take into account the fusion rules of Eq. (D.4), the decomposition of the \mathbb{Z}_3 twist field four-point correlation function in the Tricritical Ising CFT reads

$$\begin{aligned}
\langle \sigma_3(\infty) \tilde{\sigma}_3(1) \sigma_3(z, \bar{z}) \tilde{\sigma}_3(0) \rangle = & \left| \mathcal{G}_{\frac{7}{10}, \{0,0,0\}}^{(3)}(z) \right|^2 + 3D_{\sigma\sigma\mathbb{I}} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{3}{80}, 0\}}^{(3)}(z) \right|^2 \\
& + 3D_{\sigma'\sigma'\mathbb{I}} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{7}{16}, \frac{7}{16}, 0\}}^{(3)}(z) \right|^2 + 3D_{\varepsilon\varepsilon\mathbb{I}} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{1}{10}, \frac{1}{10}, 0\}}^{(3)}(z) \right|^2 \\
& + 3D_{\varepsilon'\varepsilon'\mathbb{I}} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{5}, \frac{3}{5}, 0\}}^{(3)}(z) \right|^2 + 3D_{\varepsilon''\varepsilon''\mathbb{I}} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{2}, \frac{3}{2}, 0\}}^{(3)}(z) \right|^2 \\
& + 3D_{\sigma\sigma\varepsilon} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{3}{80}, \frac{1}{10}\}}^{(3)}(z) \right|^2 + 3D_{\sigma\sigma\varepsilon'} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{3}{80}, \frac{3}{5}\}}^{(3)}(z) \right|^2 \\
& + 3D_{\sigma\sigma\varepsilon''} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{3}{80}, \frac{3}{2}\}}^{(3)}(z) \right|^2 + 6D_{\sigma\sigma'\varepsilon} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{7}{16}, \frac{1}{10}\}}^{(3)}(z) \right|^2 \\
& + 6D_{\sigma\sigma'\varepsilon'} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{7}{16}, \frac{3}{5}\}}^{(3)}(z) \right|^2 + 3D_{\sigma'\sigma'\varepsilon''} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{7}{16}, \frac{7}{16}, \frac{3}{2}\}}^{(3)}(z) \right|^2 \\
& + 3D_{\varepsilon\varepsilon\varepsilon'} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{1}{10}, \frac{1}{10}, \frac{3}{5}\}}^{(3)}(z) \right|^2 + 6D_{\varepsilon\varepsilon'\varepsilon''} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{1}{10}, \frac{3}{5}, \frac{3}{2}\}}^{(3)}(z) \right|^2 \\
& + D_{\varepsilon'\varepsilon'\varepsilon'} \left| \mathcal{G}_{\frac{7}{10}, \{\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\}}^{(3)}(z) \right|^2 \tag{F.1}
\end{aligned}$$

The structure constants D_{h_1, h_2, h_3} above are determined, according to Eq. (3.30), by the OPE coefficients of the Tricritical Ising CFT, whose analytic values can be found for instance in Ref. [62].

The expansions in sphere conformal blocks until level $L = 5$ of the $N = 3$ orbifold conformal blocks that appear in the previous expression read

$$\mathcal{G}_{\frac{7}{10}, \{0,0,0\}}^{(3)}(z) \sim \mathcal{F}_{\frac{21}{10}, 0}(z) + \frac{17}{3838185} \mathcal{F}_{\frac{21}{10}, 4}(z) + \frac{2}{4782969} \mathcal{F}_{\frac{21}{10}, 5}(z), \tag{F.2}$$

$$\begin{aligned}
\mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{3}{80}, 0\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{3}{40}}(z) + \frac{1}{720} \mathcal{F}_{\frac{21}{10}, \frac{43}{40}}(z) + \frac{3265267}{2547417600} \mathcal{F}_{\frac{21}{10}, \frac{83}{40}}(z) + \frac{3665197}{50512896000} \mathcal{F}_{\frac{21}{10}, \frac{123}{40}}(z) \\
& + \frac{155865980497261283}{18705929999865937920000} \mathcal{F}_{\frac{21}{10}, \frac{163}{40}}(z) + \frac{462436566571594180410731}{659386782637573125832704000000} \mathcal{F}_{\frac{21}{10}, \frac{203}{40}}(z), \tag{F.3}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{\frac{7}{10}, \{\frac{7}{16}, \frac{7}{16}, 0\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{7}{8}}(z) + \frac{7}{432} \mathcal{F}_{\frac{21}{10}, \frac{15}{8}}(z) + \frac{125}{6594048} \mathcal{F}_{\frac{21}{10}, \frac{23}{8}}(z) + \frac{7557625}{34022301696} \mathcal{F}_{\frac{21}{10}, \frac{23}{8}}(z) \\
& + \frac{136592048239}{41822362293239808} \mathcal{F}_{\frac{21}{10}, \frac{39}{8}}(z) + \frac{3691268343947}{18937730481342382080} \mathcal{F}_{\frac{21}{10}, \frac{47}{8}}(z), \tag{F.4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{\frac{7}{10}, \{\frac{1}{10}, \frac{1}{10}, 0\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{1}{5}}(z) + \frac{1}{270} \mathcal{F}_{\frac{21}{10}, \frac{6}{5}}(z) + \frac{4489}{6718950} \mathcal{F}_{\frac{21}{10}, \frac{11}{5}}(z) + \frac{734651}{6957940500} \mathcal{F}_{\frac{21}{10}, \frac{16}{5}}(z) \\
& + \frac{33710613028}{7001301644623125} \mathcal{F}_{\frac{21}{10}, \frac{21}{5}}(z) + \frac{561570243431821}{1229479741472989781250} \mathcal{F}_{\frac{21}{10}, \frac{26}{5}}(z), \tag{F.5}
\end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{7}{10}, \{\frac{3}{5}, \frac{3}{5}, 0\}}^{(3)} \sim & \mathcal{F}_{\frac{21}{10}, \frac{6}{5}}(z) + \frac{1}{45} \mathcal{F}_{\frac{21}{10}, \frac{11}{5}}(z) + \frac{12664}{12885075} \mathcal{F}_{\frac{21}{10}, \frac{16}{5}}(z) + \frac{75304}{304266375} \mathcal{F}_{\frac{21}{10}, \frac{21}{5}}(z) \\ & + \frac{5224025025746}{910725734424436875} \mathcal{F}_{\frac{21}{10}, \frac{26}{5}}(z) + \frac{41871765632}{94232186890734375} \mathcal{F}_{\frac{21}{10}, \frac{31}{5}}(z), \quad (\text{F.6}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{7}{10}, \{\frac{3}{2}, \frac{3}{2}, 0\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, 3}(z) + \frac{1}{18} \mathcal{F}_{\frac{21}{10}, 4}(z) + \frac{230}{22113} \mathcal{F}_{\frac{21}{10}, 5}(z) + \frac{100}{443961} \mathcal{F}_{\frac{21}{10}, 6}(z) \\ & + \frac{96002975}{10064284515558} \mathcal{F}_{\frac{21}{10}, 7}(z) + \frac{58456305937}{17581104541057428} \mathcal{F}_{\frac{21}{10}, 8}(z), \quad (\text{F.7}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{3}{80}, \frac{1}{10}\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{7}{40}}(z) + \frac{5}{1296} \mathcal{F}_{\frac{21}{10}, \frac{47}{40}}(z) + \frac{475}{373248} \mathcal{F}_{\frac{21}{10}, \frac{87}{40}}(z) + \frac{14170687}{97122115584} \mathcal{F}_{\frac{21}{10}, \frac{127}{40}}(z) \\ & + \frac{71253834712735}{4997056637667115008} \mathcal{F}_{\frac{21}{10}, \frac{167}{40}}(z) + \frac{86091919475921327}{95239084395063869964288} \mathcal{F}_{\frac{21}{10}, \frac{207}{40}}(z), \quad (\text{F.8}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{3}{80}, \frac{3}{5}\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{27}{40}}(z) + \frac{5}{16} \mathcal{F}_{\frac{21}{10}, \frac{67}{40}}(z) + \frac{2476157}{118070784} \mathcal{F}_{\frac{21}{10}, \frac{107}{40}}(z) + \frac{555083831}{1041791901696} \mathcal{F}_{\frac{21}{10}, \frac{147}{40}}(z) \\ & + \frac{35552447512576825}{476153434940080914432} \mathcal{F}_{\frac{21}{10}, \frac{187}{40}}(z) + \frac{152443525738528231}{16655797786474708992000} \mathcal{F}_{\frac{21}{10}, \frac{227}{40}}(z), \quad (\text{F.9}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{3}{80}, \frac{3}{5}\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{63}{40}}(z) + \frac{169}{80} \mathcal{F}_{\frac{21}{10}, \frac{103}{40}}(z) + \frac{4956437831}{16444684800} \mathcal{F}_{\frac{21}{10}, \frac{143}{40}}(z) + \frac{40879507267951}{2130203225088000} \mathcal{F}_{\frac{21}{10}, \frac{183}{40}}(z) \\ & + \frac{48838420003812575817893}{41322611540300358942720000} \mathcal{F}_{\frac{21}{10}, \frac{223}{40}}(z) + \frac{853096471556262783844477}{9087626895163066810368000000} \mathcal{F}_{\frac{21}{10}, \frac{263}{40}}(z), \quad (\text{F.10}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{7}{16}, \frac{1}{10}\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{23}{40}}(z) + \frac{1301}{15120} \mathcal{F}_{\frac{21}{10}, \frac{63}{40}}(z) + \frac{3195223}{653990400} \mathcal{F}_{\frac{21}{10}, \frac{103}{40}}(z) + \frac{111395056711}{745930902528000} \mathcal{F}_{\frac{21}{10}, \frac{143}{40}}(z) \\ & + \frac{69649784481358065419}{11050452065922572943360000} \mathcal{F}_{\frac{21}{10}, \frac{183}{40}}(z) + \frac{4967477167662960708160673}{3123989432446707136069632000000} \mathcal{F}_{\frac{21}{10}, \frac{223}{40}}(z), \quad (\text{F.11}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{7}{10}, \{\frac{3}{80}, \frac{7}{16}, \frac{3}{5}\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{43}{40}}(z) + \frac{1423}{45360} \mathcal{F}_{\frac{21}{10}, \frac{83}{40}}(z) + \frac{237113}{210470400} \mathcal{F}_{\frac{21}{10}, \frac{123}{40}}(z) + \frac{3550509597637}{9426871673856000} \mathcal{F}_{\frac{21}{10}, \frac{163}{40}}(z) \\ & + \frac{134673192184141631833}{6783814636189023928320000} \mathcal{F}_{\frac{21}{10}, \frac{203}{40}}(z) + \frac{17146721811911780522868659}{15403567241995839166611456000000} \mathcal{F}_{\frac{21}{10}, \frac{243}{40}}(z), \quad (\text{F.12}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{\frac{7}{10}, \{\frac{7}{16}, \frac{7}{16}, \frac{3}{5}\}}^{(3)}(z) \sim & \mathcal{F}_{\frac{21}{10}, \frac{19}{8}}(z) + \frac{289}{3024} \mathcal{F}_{\frac{21}{10}, \frac{27}{8}}(z) + \frac{113339}{101380608} \mathcal{F}_{\frac{21}{10}, \frac{35}{8}}(z) + \frac{795728725}{5786102439936} \mathcal{F}_{\frac{21}{10}, \frac{43}{8}}(z) \\ & + \frac{36813418825}{914736180363264} \mathcal{F}_{\frac{21}{10}, \frac{51}{8}}(z) + \frac{5346115131945399913}{1885729815637402501251072} \mathcal{F}_{\frac{21}{10}, \frac{59}{8}}(z), \quad (\text{F.13}) \end{aligned}$$

$$\mathcal{G}_{\frac{7}{10}, \{\frac{1}{10}, \frac{1}{10}, \frac{3}{5}\}}^{(3)}(z) \sim \mathcal{F}_{\frac{21}{10}, \frac{4}{5}}(z) + \frac{5}{54} \mathcal{F}_{\frac{21}{10}, \frac{9}{5}}(z) + \frac{25}{9828} \mathcal{F}_{\frac{21}{10}, \frac{14}{5}}(z) + \frac{733}{9893988} \mathcal{F}_{\frac{21}{10}, \frac{19}{5}}(z) + \frac{27915235}{2181879918072} \mathcal{F}_{\frac{21}{10}, \frac{24}{5}}(z) + \frac{178775929427}{123348562272120978} \mathcal{F}_{\frac{21}{10}, \frac{29}{5}}(z), \quad (\text{F.14})$$

$$\mathcal{G}_{\frac{7}{10}, \{\frac{3}{10}, \frac{3}{5}, \frac{3}{2}\}}^{(3)}(z) \sim \mathcal{F}_{\frac{21}{10}, \frac{11}{5}}(z) + \frac{173}{810} \mathcal{F}_{\frac{21}{10}, \frac{16}{5}}(z) + \frac{753719}{114945075} \mathcal{F}_{\frac{21}{10}, \frac{21}{5}}(z) + \frac{108761449}{205479038250} \mathcal{F}_{\frac{21}{10}, \frac{26}{5}}(z) + \frac{409782125417}{6102655912923750} \mathcal{F}_{\frac{21}{10}, \frac{31}{5}}(z) + \frac{105239895304351091}{24755684834277326812500} \mathcal{F}_{\frac{21}{10}, \frac{36}{5}}(z), \quad (\text{F.15})$$

and

$$\mathcal{G}_{\frac{7}{10}, \{\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\}}^{(3)}(z) \sim \mathcal{F}_{\frac{21}{10}, \frac{9}{5}}(z) + \frac{961}{458055} \mathcal{F}_{\frac{21}{10}, \frac{19}{5}}(z) + \frac{355}{341172} \mathcal{F}_{\frac{21}{10}, \frac{24}{5}}(z) + \frac{46601998346}{761410878222969} \mathcal{F}_{\frac{21}{10}, \frac{29}{5}}(z) + \frac{819218}{544871045511} \mathcal{F}_{\frac{21}{10}, \frac{34}{5}}(z). \quad (\text{F.16})$$

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