Volume growth for infinite graphs and translation surfaces

P. Colognese and M. Pollicott[*]

Warwick University

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1 Introduction

We begin by recalling the definition of volume entropy for compact Riemannian manifolds due to Manning [7]. Let M be a compact manifold with Riemannian metric ρ and universal cover $\widetilde{M}$ equipped with the lifted metric $\widetilde{\rho}$. Fix a point $c \in \widetilde{M}$ and consider a ball $B(c, R)$ of radius $R > 0$ centred at c.

Definition 1.1. *The volume entropy of M is defined by*

$$h = h(M) := \lim_{R \to \infty} \frac{1}{R} \log \mathrm{Vol}_{\widetilde{\rho}}(B(c, R)),$$

where $\mathrm{Vol}_{\widetilde{\rho}}$ denotes the Riemannian volume on $\widetilde{M}$ with respect to $\widetilde{\rho}$.

For manifolds (M, ρ) of non-positive curvature this coincides with the topological entropy h of the associated geodesic flow [7]. In the case of manifolds with negative sectional curvature, Margulis [9] showed in his thesis that there is a simple asymptotic formula: There exists $C > 0$ such that

$$\lim_{R \to +\infty} \frac{\mathrm{Vol}_{\widetilde{\rho}}(B(c, R))}{e^{hR}} = C.$$

A closely related result in [9] gave an asymptotic formula for the number $\Pi(x, R)$ of geodesic arcs starting and finishing at a given point x of length at most R: There exists $D > 0$ such that

$$\lim_{R \to +\infty} \frac{\Pi(x, R)}{e^{hR}} = D.$$

A related notion of volume entropy was considered for directed, finite, connected, non-cyclic graphs without terminal vertices by Lim in [6]. In this note we extend Lim's definition of volume entropy to suitable infinite graphs and show the analogue of Margulis' result in

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[*]Department of Mathematics, Warwick University, Coventry, CV4 7AL, UK

this context (Theorem 2.4). As an application we show a version of Margulis' theorem for the natural analogue of volume growth for translation surfaces (Theorem 6.11).

This note originated as a summer MPhil project of the first author. It may have been possible to apply the transfer operator methods in [15], but instead we employ a more direct and elementary approach.

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2   Infinite Graphs

In this section we will introduce the types of graphs we shall we working with as well as basic definitions which will be used throughout the paper.

Let $\mathcal{G}$ be a non-empty connected oriented graph. Let $\mathcal{V} = \mathcal{V}(\mathcal{G})$ and $\mathcal{E} = \mathcal{E}(\mathcal{G})$ be the vertex and oriented edge sets respectively. For every edge e, let $i(e)$ and $t(e)$ denote the initial and the terminal vertex of e, respectively. We can define a length distance d on $\mathcal{G}$ by introducing a length function $\ell : \mathcal{E} \to \mathbb{R}$ which assigns a positive real number $\ell(e)$ to each edge $e \in \mathcal{E}$.

Example 2.1 (Infinite Graph). *Consider a graph $\mathcal{G}$ formed from one vertex and a countably infinite number of edges.*
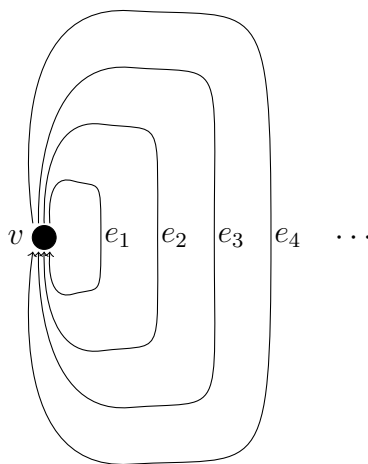

Figure 1:   A single vertex $\mathcal{V} = \{v\}$ and infinitely many edges $\mathcal{E} = \{e_n\}_{n=1}^{\infty}$.

A *path* in $\mathcal{G}$ corresponds to a sequence of edges $p = e_1 \ldots e_n$ for which $t(e_j) = i(e_{j+1})$, for $1 \leq j < n$ and we denote its length by $\ell(p) = \sum_{j=1}^{n} \ell(e_j)$.

Let $\mathcal{P}_{\mathcal{G}}(x, R) = \{p = e_1 \ldots e_n : i(e_1) = x, \ell(p) \leq R\}$ denote the set of all such paths of length at most R starting at $x \in \mathcal{V}(\mathcal{G})$. We denote its cardinality by $N_{\mathcal{G}}(x, R) = \#\mathcal{P}_{\mathcal{G}}(x, R)$.

Definition 2.2. *We define the volume entropy of $(\mathcal{G}, \ell, x)$ as*

$$h(\mathcal{G}, \ell, x) = \limsup_{R \to +\infty} \frac{1}{R} \log N_{\mathcal{G}}(x, R).$$

However, we need to make further assumptions on the length function ℓ for $h(\mathcal{G}, \ell, x)$ to be finite. To see this, consider the graph $\mathcal{G}$ in Example 2.1 which has a single vertex and an infinite number of edges, and assume that the lengths don't tend to infinity. Then for R sufficiently large, $N_{\mathcal{G}}(x, R) = \infty$ and thus $h(\mathcal{G}, \ell, x) = \infty$.

We summarise below the properties of the graph that are needed in the proof.

Graph Hypotheses. Henceforth, we shall consider graphs with finite vertex set $\mathcal{V}$ and a countable edge set $\mathcal{E}$. Furthermore we require that $\mathcal{E}$ and the associated length function satisfy the following properties:

(H1) For all $\sigma > 0$ we have $\sum_{e \in \mathcal{E}} e^{-\sigma \ell(e)} < \infty$;

(H2) For all edges $e, e' \in \mathcal{E}$ there exists a path in $\mathcal{G}$ which starts with e and ends with e'[1]; and

(H3) There does not exist a $d > 0$ such that

$$\{\ell(c) : c \text{ is a closed path}\} \subset d\mathbb{N}.$$

Under the above hypotheses, the volume entropy $h = h(\mathcal{G}, \ell, x)$ does not depend on the choice of base point x.

Lemma 2.3. *If the graph $\mathcal{G}$ satisfies (H1) and (H2) then $0 < h < \infty$.*

Proof. By assumption (H2), and the pigeonhole principle applied to $\mathcal{V}$, there exist a path connecting the base point x to some vertex v and two closed paths, c_1 and c_2, which pass through v. By considering all possible concatenations of these closed paths it is clear that there exists $b > 0$ such that $N_{\mathcal{G}}(x, R) \geq 2^{\lfloor R/b \rfloor}$ for all $R > 0$ and hence $h \geq \frac{\log 2}{b} > 0$.

To see that h is finite we can formally write

$$\sum_{p \in \mathcal{P}_{\mathcal{G}}(x, R)} e^{-\sigma \ell(p)} \leq \sum_{n=1}^{\infty} \left(\sum_{e \in \mathcal{E}} e^{-\sigma \ell(e)} \right)^n, \tag{2.1}$$

for $\sigma > 0$, where the Right Hand Side involves all possible sums of edge lengths. Using (H1) one can see that for $\sigma = \sigma_0$ sufficiently large $\sum_{e \in \mathcal{E}} e^{-\sigma \ell(e)} < 1$ and thus the geometric series on the Right Hand Side of (2.1) converges. In particular, since h is easily seen to be the absicssa of convergence of the series on the Left Hand Side of (2.1) we see that $h \leq \sigma_0 < +\infty$, as required. $\square$

Our main result for $\mathcal{G}$ is the following asymptotic for the growth of paths.

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[1] A slightly weaker assumption would be to require that for a sufficiently large finite subset $\mathcal{E}_0 \subset \mathcal{E}$, for every $e, e' \in \mathcal{E}_0$ there exists a path in $\mathcal{G}$ which starts with e and ends with e'

Theorem 2.4. *If the graph $\mathcal{G}$ satisfies (H1),(H2) and (H3) then there exists a constant $C > 0$ such that $N_{\mathcal{G}}(x, R) \sim Ce^{hR}$, i.e.,*

$$\lim_{R \to +\infty} \frac{N_{\mathcal{G}}(x, R)}{e^{hR}} = C.$$

The proof follows the lines of the classical proof of the prime number theorem. In particular, it is based on the use of a Tauberian theorem (in Section 5). This, in turn, depends on the properties of the complex function $\eta_{\mathcal{G}}(z)$, the Laplace transform of $N_{\mathcal{G}}(x, R)$ (defined in Section 4). The function $\eta_{\mathcal{G}}(z)$ is analysed using matrices introduced in the next section. In the special case of finite graphs, the asymptotic in Theorem 2.4 could be easily deduced using ideas in [13] for finite matrices.

Remark 2.5. Without hypothesis (H3) this theorem may not hold. For example, even in the case of finite graphs, if we consider the graph $\mathcal{G}$ with a single vertex and two edges of length 1, then $N_{\mathcal{G}}(x, R) = 2^{\lfloor R \rfloor}$ for all $R > 0$. In this case, the limit in Theorem 2.4 does not converge.

3   Countable Matrices

In this section consider a graph $\mathcal{G}$ and length function ℓ which satisfy hypotheses (H1)-(H3). Let us order the edge set $\mathcal{E} = (e_a)_{a \in \mathbb{N}}$ by non-decreasing length and write $\ell(a) := \ell(e_a)$, $a \in \mathbb{N}$.

Definition 3.1. *We can associate to $\mathcal{G}$ the infinite matrix M_0 defined by*

$$M_0(a, b) = \begin{cases} 1 & \text{if } t(a) = i(b), \\ 0 & \text{otherwise.} \end{cases}$$

For each $z \in \mathbb{C}$ we define the matrix M_z by $M_z(a, b) = M_0(a, b)e^{-z\ell(b)}$ for $a, b \in \mathcal{E}$.

Let $P(n, a, b)$ denote the set of paths in $\mathcal{G}$ consisting of n edges, starting with edge e_a and ending with edge e_b. It then follows from formal matrix multiplication that for any $n \geq 1$, we can write the $(a, b)^{th}$ entry of the nth power of the matrix as:

$$M_z^n(a, b) = e^{z\ell(a)} \sum_{p \in P(n+1,a,b)} e^{-z\ell(p)}, \tag{3.1}$$

which will be finite by hypothesis (H1).

Given a matrix $L = (L(a, b))_{a,b=1}^{\infty}$ with $\sup_a \sum_b |L(a, b)| < +\infty$ we can associate to L a bounded linear operator $\widehat{L} : \ell^{\infty}(\mathbb{C}) \to \ell^{\infty}(\mathbb{C})$ by

$$\widehat{L}(\underline{u}) = \left(\sum_{b=1}^{\infty} L(a, b)u_b \right)_{a=1}^{\infty} \text{ where } \underline{u} = (u_b)_{b=1}^{\infty} \in \ell^{\infty}(\mathbb{C}).$$

In particular, by hypothesis (H1), when $Re(z) > 0$ we can associate to M_z a bounded operator $\widehat{M_z} : \ell^\infty(\mathbb{C}) \to \ell^\infty(\mathbb{C})$ by

$$\widehat{M_z}(\underline{u}) = \Big(\sum_{b=1}^\infty M_z(a,b) u_b \Big)_{a=1}^\infty.$$

To proceed, we would like to understand the domain of meromorphicity of the linear operator $(I - \widehat{M_z})^{-1} : \ell^\infty(\mathbb{C}) \to \ell^\infty(\mathbb{C})$, where I denotes the identity operator. To this end, we shall make use of an idea by Hofbauer and Keller in [4], where they observe that the invertibility of certain operators of the above form depends only on the determinant of an associated finite matrix.

Fix $\epsilon > 0$ and, for convenience, assume also $h > \epsilon$. Given $k \geq 1$, we can truncate the matrix M_z to the $k \times k$ matrix $A_z = (M_z(i,j))_{i,j=1}^k$. Then we can then write

$$M_z = \begin{pmatrix} A_z & B_z \\ C_z & D_z \end{pmatrix}$$

where, in particular, $D_z = (M_z(i+k, j+k))_{i,j=1}^\infty$. Again, we can interpret $I - \widehat{D_z}$ as a bounded linear operator on $\ell^\infty(\mathbb{C})$ and write $(I - \widehat{D_z})^{-1} = \sum_{m=0}^\infty \widehat{D_z}^m$ if the operator $\widehat{D_z}$ has norm $\|\widehat{D_z}\| < 1$. In particular, this is true when $Re(z) \geq \epsilon$ for k sufficiently large, since by (H1) we have

$$\|\widehat{D_z}\| \leq \sup_{n \in \mathbb{N}} \sum_{m=1}^\infty |D_z(n,m)| \leq \sum_{m=1}^\infty e^{-Re(z)\ell(m+k)} \leq \sum_{m=1}^\infty e^{-\epsilon\ell(m+k)} < 1. \tag{3.2}$$

Writing $\ell^\infty(\mathbb{C})$ as the corresponding direct sum of two subspaces, we can then easily verify that

$$I - \widehat{M_z} = \begin{pmatrix} I - \widehat{A_z} - \widehat{B_z}(I - \widehat{D_z})^{-1}\widehat{C_z} & -\widehat{B_z}(I - \widehat{D_z})^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\widehat{C_z} & I - \widehat{D_z} \end{pmatrix}. \tag{3.3}$$

Let us denote the $k \times k$ matrix $W_z := A_z + B_z(I - D_z)^{-1}C_z$, where each entry is given by a convergent series. By (3.3), whenever $\det(I - W_z) \neq 0$ then we see that $I - \widehat{M_z}$ is invertible, with inverse

$$(I - \widehat{M_z})^{-1} = \begin{pmatrix} I & 0 \\ (I - \widehat{D_z})^{-1}\widehat{C_z} & (I - \widehat{D_z})^{-1} \end{pmatrix} \begin{pmatrix} (I - \widehat{W_z})^{-1} & (I - \widehat{W_z})^{-1}\widehat{B_z}(I - \widehat{D_z})^{-1} \\ 0 & I \end{pmatrix}. \tag{3.4}$$

This leads to the following result.

Lemma 3.2. *The operator $(I - \widehat{M_z})^{-1}$ has an analytic extension to $Re(z) > 0$ except when* $\det(I - W_z) = 0$.

Proof. This follows from the identity (3.4) and since the $\epsilon > 0$ chosen in the above construction can be chosen arbitrarily small. $\square$

4   Complex functions

We can now introduce a complex function whose analytic properties will be useful in deriving our asymptotic estimates for $N_{\mathcal{G}}(x, R)$. Fix $x \in \mathcal{V}$.

Definition 4.1. *We can formally define the complex function*

$$\eta_{\mathcal{G}}(z) = \int_0^\infty e^{-zR} dN_{\mathcal{G}}(x, R) = \sum_{p \in P(x)} e^{-z\ell(p)}, \quad z \in \mathbb{C},$$

where $P(x) = \{p = e_1 \cdots e_n : n \geq 0, i(e_1) = x\}$ is the set of paths in $\mathcal{G}$ starting at x.

We first observe that $\eta_{\mathcal{G}}(z)$ converges to an analytic function for $Re(z) > h$, by virtue of Definition 2.2. In order to construct a meromorphic extension of $\eta_{\mathcal{G}}(z)$ we shall relate $\eta_{\mathcal{G}}(z)$ to the matrix M_z. For $Re(z) > 0$, we define:

(a) $\underline{w}(z) = (\chi_{\mathcal{E}_x}(e_j) e^{-z\ell(j)})_{j=1}^\infty \in \ell^1(\mathbb{C})$ where $\chi_{\mathcal{E}_x}$ denotes the characteristic function of the set $\mathcal{E}_x = \{e \in \mathcal{E} : i(e) = x\}$ of edges whose initial vertex is x; and

(b) $\underline{1} = (1)_{j=1}^\infty \in \ell^\infty(\mathbb{C})$ is the vector all of whose entries are equal to 1,

then we can formally rewrite $\eta_{\mathcal{G}}(z)$ as

$$\begin{aligned}
\eta_{\mathcal{G}}(z) = \sum_{p \in P(x)} e^{-z\ell(p)} &= \underline{w}(z) \cdot \Big(\sum_{n=0}^\infty \widehat{M_z}^n \Big) \underline{1} \\
&= \underline{w}(z) \cdot \Big(I - \widehat{M_z} \Big)^{-1} \underline{1},
\end{aligned} \tag{4.1}$$

where $w \cdot v = \sum_{j=1}^\infty w_j v_j$ for $w \in \ell^1(\mathbb{C})$ and $v \in \ell^\infty(\mathbb{C})$. Observe that for $Re(z) > 0$ we have $\underline{w}(z) \in \ell^1(\mathbb{C})$ by (H1). In particular, by Lemma 3.2 the expression in (4.1) extends to $Re(z) > 0$, and the locations of the poles are given by those z such that the finite rank operator $(I - W_z)$ is not invertible. Moreover, we can easily write

$$\eta_{\mathcal{G}}(z) = \frac{\phi(z)}{\det(I - W_z)} \tag{4.2}$$

where $\phi(z)$ is holomorphic on $Re(z) > 0$.

Proposition 4.2. $\eta_{\mathcal{G}}(z)$ *has a meromorphic extension to $Re(z) > 0$.*

Proof. Observe that $\det(I - W_z)$ is the sum of a countable number of holomorphic functions which uniformly converge on any compact domain in $Re(z) > 0$ and hence $\det(I - W_z)$ is holomorphic. The result follows from the identity (4.2). $\square$

Let $\epsilon < h$. By (3.2) we can choose k large enough such that $(I - \widehat{D_z})$ is invertible, on the half plane $Re(z) \geq \epsilon$. Recall that a non-negative $n \times n$ matrix M is *irreducible* if for all i, j satisfying $1 \leq i, j \leq n$ there exists a natural number m such that $(M^m)_{i,j} > 0$.

Lemma 4.3. *Let $\sigma > 0$. Then W_σ is a non-negative irreducible matrix. Furthermore, W_σ has a simple maximal positive eigenvalue $\rho(\sigma) = \rho(W_\sigma)$, which depends analytically on σ and satisfies $\rho'(\sigma) < 0$.*

Proof. Recall that $W_\sigma = A_\sigma + B_\sigma(I - D_\sigma)^{-1}C_\sigma$, which by construction is a non-negative matrix. We can also deduce that the matrix W_σ is irreducible. To see this, note that by assumption (H2), for all $1 \le i, j \le k$, there exists some path of length n starting with edge e_i and ending with edge e_j. Such a path can be broken up into sub-paths of two types. The first type consists of those paths that stay completely within $\{e_1, \cdots, e_k\}$, and the second type which consists of those paths that initially enter the complement $\mathcal{E} - \{e_1, \cdots, e_k\}$ and finally leave at their end. Note that $W_\sigma^n(i, j)$ is a sum including powers of $A_\sigma(i, j)$ (corresponding to sub-paths of the first type) and $B_\sigma(I - D_\sigma)^{-1}C_\sigma$ (corresponding to sub-paths of the second type), where the powers are less than or equal to n. Hence $W_\sigma^n(i, j) > 0$.

We can now apply the Perron-Frobenius theorem (see [3]) to deduce that the maximal positive eigenvalue $\rho(\sigma) > 0$ for W_σ exists and that W_σ has associated positive left and right eigenvectors $u(\sigma)$ and $v(\sigma)$ (which we normalise so that $u(\sigma)v(\sigma) = 1$). By differentiating the eigenvalue equations for $u(\sigma)$ and $v(\sigma)$, one can show that

$$\rho'(\sigma) = u(\sigma)^T W_\sigma' v(\sigma) < 0,$$

where W_σ' is the matrix with entries $W_\sigma'(i, j) = \frac{dW_x(i,j)}{dx}(\sigma) < 0$ for all i, j (see [14] for a similar argument). $\qquad\square$

Proposition 4.4. *h is a simple pole of $\eta_{\mathcal{G}}(z)$.*

Proof. For z in a neighbourhood of h, we denote by $\rho(z)$ the perturbed eigenvalue of W_z. We can write $\det(I - W_z) = (1 - \rho(z))\Pi_{i=2}^k(1 - \lambda_i(z))$, where the $\lambda_i(z)$ denote the other eigenvalues of W_z. Since the $\lambda_i(z)$ are bounded away from 1 for z near h (by the Perron-Frobenius theorem and standard perturbation theory), $\phi(h) \ne 0$ and $\rho'(h) \ne 0$ (by Lemma 4.3), we can conclude that $(z - h)\eta_{\mathcal{G}}(z)$ converges to a non-zero constant, as z tends to h. $\qquad\square$

Proposition 4.5. *$\eta_{\mathcal{G}}(z)$ has no poles other than h on the line $Re(z) = h$.*

Proof. Suppose for a contraction that there exists another pole at $h + it$ ($t \ne 0$). Let c be any closed path and choose an integer $k_c > k$ such that the edges of c have index smaller than k_c. Then construct the $k_c \times k_c$ matrices W_z. From equation (4.2) we see that $\det(I - W_{h+it}) = 0$, and thus 1 is an eigenvalue for W_{h+it} and W_h. Furthermore, we can see that $\rho(W_h) = 1$ since otherwise $\eta_{\mathcal{G}}(z)$ has a pole at $c > h$, contradicting Definition 2.2.

Next observe that $|W_{h+it}(a, b)| \le W_h(a, b)$ for all $1 \le a, b \le k$. Since $\rho(W_{h+it}) \ge 1 = \rho(W_h)$, we can apply Wielandt's theorem (see [3]) which allows us to conclude that $\rho(W_{h+it}) = \rho(W_h) = 1$ and that there exists a diagonal matrix D, whose non-zero entries have unit modulus such that $W_{h+it} = DW_hD^{-1}$, and thus for all n we have $W_{h+it}^n = DW_h^nD^{-1}$.

Suppose that the closed path c begins with some edge e_a and consists of n edges. One can check that $W_{h+it}^n(a, a) = W_h^n(a, a)$ (since $W_{h+it}^n = DW_h^nD^{-1}$) and that $e^{(h+it)\ell(c)}$ is one of the terms in the left hand sum. However, this can only hold if t is such that $\ell(c)t = 2\pi m_c$ for some non-zero integer m_c. As c was arbitrary, the above construction implies that for all closed paths c, $\ell(c) \in d\mathbb{N}$ with $d = 2\pi/t$ which contradicts (H3). $\qquad\square$

5   Proof of Theorem 2.4

We can complete the proof using a similar approach to Parry in [13], where he considered only finite matrices. In particular, we will use the following formulation of the Ikehara–Wiener Tauberian theorem [2] applied to our counting function, $N_{\mathcal{G}}(x, R)$.

Theorem 5.1 (Ikehara–Wiener Tauberian theorem). *Let $A : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic, non-decreasing function and formally denote $\eta(z) := \int_0^\infty e^{-zR} dA(R)$, for $z \in \mathbb{C}$. Then suppose that $\eta(z)$ has the following properties:*

1. *there exists some $a > 0$ such that $\eta(z)$ is analytic on $Re(z) > a$;*

2. *$\eta(z)$ has a meromorphic extension to a neighbourhood of the half-plane $Re(z) \geq a$;*

3. *a is a simple pole for $\eta(z)$, i.e., $C = \lim_{\epsilon \searrow 0}(z - a)\eta(z) > 0$; and*

4. *the extension of $\eta(z)$ has no poles on the line $Re(z) = a$ other than a.*

Then $A(R) \sim Ce^{aR}$ as $R \to +\infty$.

From the results in the previous section we see that the $\eta_{\mathcal{G}}(s)$ satisfies the assumptions of Theorem 5.1 with $a = h$ and so we have proved Theorem 2.4.

6   Translation surfaces

In this section we will consider a definition of volume entropy for translation surfaces and prove asymptotic results using the work developed in the previous sections.

Definition 6.1. *A translation surface X is a compact surface with a flat metric except at a finite set $\Sigma = \{x_1, \ldots, x_n\}$ of singular points with cone angles $2\pi(k(x_i)+1)$, where $k(x_i) \in \mathbb{N}$, for $i = 1, \ldots, n$.*

A path which does not pass through singularities is a locally distance minimizing geodesic if it is a straight line segment. This includes geodesics which start and end at singularities, known as *saddle connections*. We will consider oriented saddle connections.

Geodesics can change direction if they go through a singular point, and a pair of line segments ending and beginning, respectively, at a singular point form a geodesic if the angle between them is at least π. Thus a locally distance minimising geodesic (of length R) on a translation surface X with singularity set Σ, is a curve $\gamma : [0, R] \to X$ satisfying the following conditions:

- There exist $0 \leq t_1 < \ldots < t_n \leq R$, where $n \geq 0$, such that $\gamma(t_i) \in \Sigma$;

- For $t_i < t < t_{i+1}$ $\gamma(t) \in X \backslash \Sigma$;

- $\gamma : (t_i, t_{i+1}) \to X \backslash \Sigma$ is a geodesic segment (possibly a saddle connection);

- The smallest angle between $\gamma|_{(t_{i-1}, t_i)}$ and $\gamma|_{(t_i, t_{i+1})}$ is at least π (cf. [1], Lemma 2.1).

Let $\mathcal{S} = \{s_1, s_2, ...\}$ be the set of oriented saddle connections ordered by non-decreasing lengths.

Definition 6.2. *We define a saddle connection path $p = (s_{i_1}, ..., s_{i_n})$ to be a finite string of oriented saddle collections $s_{i_1}, ..., s_{i_n}$ which form a geodesic path.*

We denote by $\ell(p) = \ell(s_1) + \ell(s_2) + \cdots + \ell(s_n)$ the sum of the lengths of the constituent saddle connections. We let $i(p), t(p) \in \Sigma$ denote the initial and terminal singularities, respectively, of the saddle connection path p.

Example 6.3 (Square tiled surfaces [16]). *We can consider the square-tiled surfaces by identifying opposite sides of arrangements of a finite number of copies of the unit square (Figure 2). The values of the lengths of the saddle connections are of the form $\{\sqrt{n^2 + m^2} : (n, m) \in \mathbb{Z}^2 - (0,0)\ co\text{-}prime\}$.*
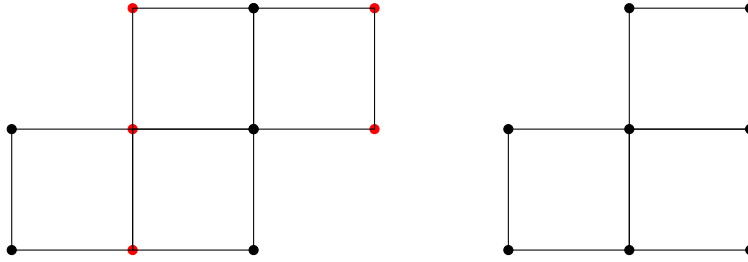

Figure 2: (i) A square tiled surface formed from four tiles; (ii) a square tiled surface formed from three tiles.

We now turn our attention to defining a notion of volume entropy for translation surfaces in terms of the growth of the volume of a ball as its radius tends to infinity. By analogy with the definition of volume entropy for Riemannian manifolds (Definition 1.1) we can consider the rate of growth of balls in the universal cover $\widetilde{X}$ of X.

Definition 6.4. *Let $\tilde{x} \in \widetilde{X}$ and consider a ball $B(\tilde{x}, R) \subset \widetilde{X}$ of radius $R > 0$ with centre $\tilde{x}$. We define the volume entropy of X to be*

$$h = h(X) := \limsup_{R \to +\infty} \frac{1}{R} \log \mathrm{Vol}_{\widetilde{X}}(B(\tilde{x}, R))$$

where $\mathrm{Vol}_{\widetilde{X}}$ denotes the natural volume on $\widetilde{X}$.

Definition 6.4 is closely related to the definition of Dankwart [1], which was formulated in terms of orbital counting. As in the case of the definitions of volume entropy for Riemannian manifolds and finite metric graphs, h is independent of the choice of $\tilde{x}$. For convenience, we can take $\tilde{x}$ to be the lift of a singularity $x \in \Sigma$.

It is useful to interpret this definition in terms of X rather than $\widetilde{X}$. To this end we have the following definition.

Definition 6.5. *Let $m_R(y)$ be number of distinct geodesic arcs in X from x to y of length at most R.*

We can now rewrite $\mathrm{Vol}_{\widetilde{X}}(B(\tilde{x}, R)) = \int_X m_R(y)d\mathrm{Vol}_X(y)$ (see Figure 3). For economy of notation we will write $V(x, R) := \mathrm{Vol}_{\widetilde{X}}(B(\tilde{x}, R))$.
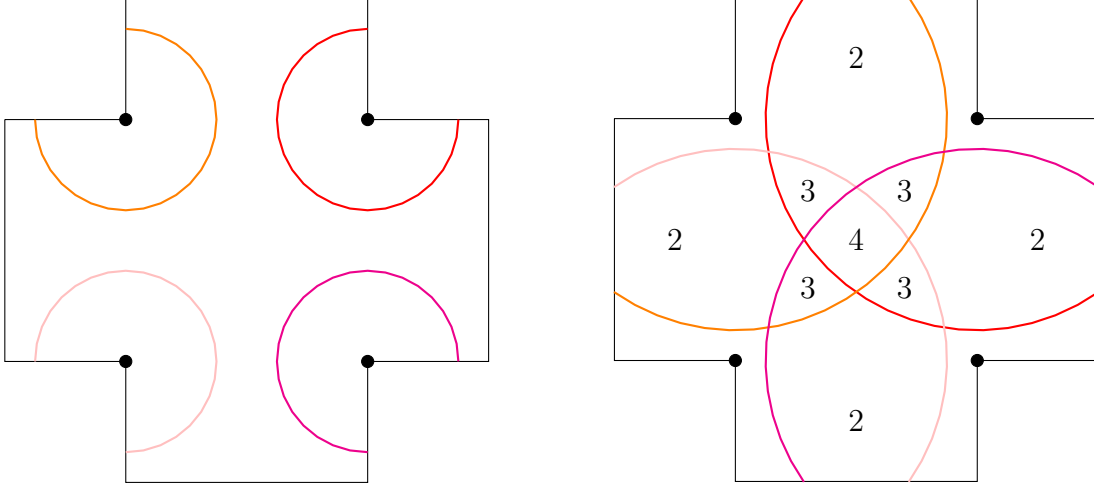

Figure 3: (i) A small ball centred at a singularity; (ii) As the radius R increases the ball overlaps with itself (and the values of the multiplicity function $m_R(\cdot)$ are indicated).

Let $x \in \Sigma$ be a singularity, then we define

$$\pi(x, R) := \{p : i(p) = x \text{ and } l(p) \leq R\}$$

to be the number of saddle connection paths starting at x of length less than or equal to R.

Lemma 6.6. *Let X be a translation surface and fix a singularity $x \in \Sigma$ and let $2\pi(k(x)+1)$ be the cone angle of x. Then for $R > 0$,*

$$V(x, R) = (k(x) + 1)\pi R^2 + \sum_{p \in \pi(x,R)} k(t(p))\pi(R - \ell(p))^2,$$

where the singularity at the end of path p has cone angle $2\pi(k(t(p)) + 1)$.

Proof. The volume contributed by the geodesics starting from x which do not pass through a singularity is given by $(k(x) + 1)\pi R^2$, where $2\pi(k(x) + 1)$ is the cone angle at x. On the other hand, the contribution to the volume by those geodesics γ which pass through one or more singularities comes when the geodesic leaves its last singularity at time $\ell(p) < R$, say. It can exit in one of $2\pi k(p)$ directions. Then the total volume of such γ is given by $k(t(p))\pi(R - (\ell(p))^2$. $\qquad\square$

We shall now prove asymptotic results for translation surfaces using the analysis developed for infinite graphs.
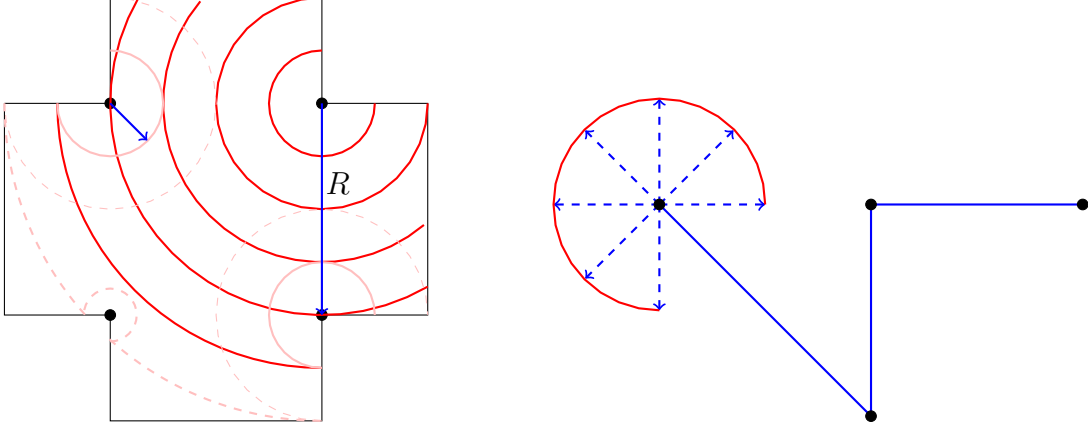
Figure 4: (i) The radii of $B(x, R)$ are concatenations of saddle connections followed by a radial line segment from a singularity; (ii) A heuristic figure illustrating that the boundary of $B(x, R)$ will consist of the union of circular arcs centred on singularities reached via concatenations of saddle connections

Definition 6.7. *We can associate to X the countable matrix M_0, indexed by $\mathcal{S}$, defined by*

$$M_0(s, s') = \begin{cases} 1 & \text{if } ss' \text{ form a saddle connection path,} \\ 0 & \text{otherwise.} \end{cases}$$

For each $z \in \mathbb{C}$ we define the matrix M_z by $M_z(s, s') = M_0(s, s')e^{-z\ell(s')}$ for $s, s' \in \mathcal{S}$.

In order that the matrices have the same properties that served us well for graphs, we require specific features of a translation surface.

Translation Hypotheses. Henceforth, we shall consider translation surfaces whose countable set of saddle connections is denoted by $\mathcal{S}$. Moreover, we require that $\mathcal{S}$ and the lengths of the saddle connections satisfy the following properties:

(T1) For all $\sigma > 0$ we have $\sum_{s \in \mathcal{S}} e^{-\sigma\ell(s)} < +\infty$;

(T2) For any directed saddle connections $s, s' \in \mathcal{S}$ there exists a saddle connection path beginning with s and ending with s'; and

(T3) There does not exist a $d > 0$ such that

$$\{\ell(c) : c \text{ is a closed saddle connection path}\} \subset d\mathbb{N}.$$

We claim that the above hypotheses hold for all translation surfaces.

Property (T1) follows from the lower bound in following result (see [10] and [11]).

Proposition 6.8. *Let X be a translation surface and let $N(X, L)$ denote the number of saddle connections on X of length less than or equal to L. Then there exists constants $0 < c_1 < c_2 < \infty$ such that*
$$c_1 L^2 \leq N(X, L) \leq c_2 L^2,$$
for L sufficiently large.

To see that Hypotheses (T2) and (T3) hold for all translation surfaces, we require the following result in [1] which we restate for our purposes here.

Proposition 6.9. *Let X be a translation surface. If $s, s' \in \mathcal{S}$ are oriented saddle connections then there exists a saddle connection path which starts with s and ends with s'.*

Hypothesis (T2) follows immediately from this fact.

To show Hypothesis (T3) holds for all surfaces we first note that if the lengths of all closed geodesics were an integer multiple of some constant d, then the length of every saddle connection would be an integer multiple of $d/2$. To see this, let s be any saddle connection on X. If $i(s) = t(s)$ then s is a closed geodesic and so we are done. If $i(s) \neq t(s)$ then by Proposition 6.9, there exists a closed saddle connection path c_i such that c_i passes through $i(s)$ and that $\bar{s} c_i s$ forms a saddle connection path (where $\bar{s}$ is the saddle connection s with reversed orientation). Similarly, there exists a closed saddle connection path c_t which starts and ends at $t(s)$, such that $s c_t \bar{s}$ forms a saddle connection path. Note that the concatenation $s c_t \bar{s} c_i$ is also a closed saddle connection path of length $2\ell(s) + \ell(c_t) + \ell(c_i)$ and so by Hypothesis (T3), $\ell(s) \in (d/2)\mathbb{N}$. Let us now assume for a contradiction that (T3) does not hold and, in particular, the above property holds for the saddle connection lengths.

Using results in [12], X contains an embedded cylinder C (the product of a circle with an interval) whose boundaries consist of a single saddle connection or multiple parallel saddle connections. We now aim to construct a countable family of triangles using this cylinder (Figure 5). Fix two singularities x and y, one from each boundary. Let b denote the union
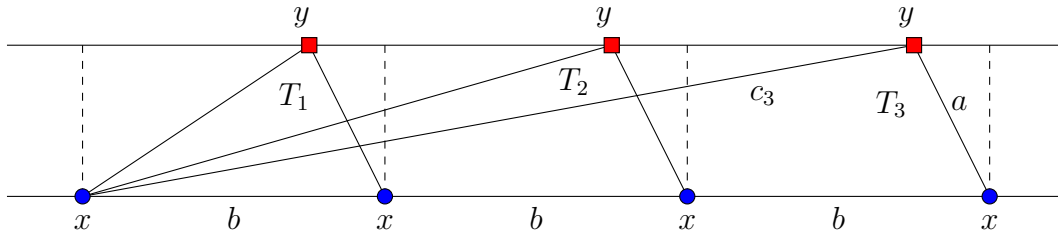

Figure 5: Three copies of a cylinder on X with two singularities on separate boundaries represented by circles and squares. The corresponding triangles T_1, T_2 and T_3 are also drawn.

of saddle connections which form the boundary of the cylinder connecting x to itself. Let a be one of the saddle connection connecting x to y across the cylinder such that the angle between a and b is acute. Then consider the unique saddle connection c_n connecting x to y which is defined to be the third side in a triangle T_n whose other edges are b concatenated with itself n times and a. By hypothesis each edge has length which is an integer multiple

of $d/2$. However, by elementary Euclidean geometry we can show that this cannot hold for all sufficiently large n, giving the required contradiction.

To derive an asymptotic estimate for $V(x, R)$ we can associate the complex function

$$\eta_X(z) = \int_0^\infty e^{-zR} dV(x, R).$$

Let $\mathcal{P}(x) := \{p : i(p) = x\}$ denote the set of saddle connection paths starting at x. We can rewrite $\eta_X(z)$ as follows:

$$
\begin{aligned}
\eta_X(z) &= \int_0^\infty e^{-zR} \left(\frac{d}{dR} \mathrm{Vol}(B(x, R)) \right) dR \\
&= 2\pi(k(x) + 1) \int_0^\infty e^{-zR} R dR + 2\pi \sum_{p \in \mathcal{P}(x)} k(t(p)) \int_{\ell(p)}^\infty e^{-zR}(R - \ell(p)) dR \\
&= 2\pi(k(x) + 1) \int_0^\infty e^{-zR} R dR + 2\pi \sum_{p \in \mathcal{P}(x)} k(t(p)) e^{-z\ell(p)} \int_0^\infty e^{-zR} R dR \qquad (6.1) \\
&= \frac{2\pi}{z^2}(k(x) + 1) + \frac{2\pi}{z^2} \sum_{p \in \mathcal{P}(x)} k(t(p)) e^{-z\ell(p)} \\
&= \frac{2\pi}{z^2}(k(x) + 1) + \frac{2\pi}{z^2} \underline{v}(z) \cdot (I - \widehat{M_z})^{-1} \underline{u},
\end{aligned}
$$

where $\underline{u} = (k(t(s_j)))_{j=1}^\infty \in \ell^\infty(\mathbb{C})$ and $\underline{v}(z) = (\chi_{\mathcal{E}_x}(s_j) e^{-z\ell(s_j)})_{j=1}^\infty \in \ell^1(\mathbb{C})$, where $\chi_{\mathcal{E}_x}$ denotes the characteristic function of the set $\mathcal{E}_x = \{s \in \mathcal{S} : i(s) = x\}$ of saddle connections starting from the singularity $x \in \Sigma$.

Lemma 6.10. *The function $\eta_X(z)$ is analytic for $Re(z) > h$ and has a meromorphic extension to $Re(z) > 0$. Moreover $\eta_X(z)$ has a simple pole at $z = h$ and no other poles on $Re(z) = h$.*

Proof. We can apply the analysis of $(I - \widehat{M_z})^{-1}$ in Section 4 to (6.1), where we use hypotheses (T1)-(T3) in place of (H1)-(H3). $\qquad\square$

We can now apply Theorem 5.1 to deduce the following.

Theorem 6.11. *There exists a $C > 0$ such that $V(x, R) \sim Ce^{hR}$ as $R \to +\infty$, i.e.,*

$$\lim_{R \to +\infty} \frac{V(x, R)}{e^{hR}} = C.$$

Typically $C = C(x)$ will depend on the choice of x.

There is a closely related result for counting the number of geodesic arcs $N_X(x, y, R)$ starting at $x \in \Sigma$ and finishing at $y \in \Sigma$.

Proposition 6.12. *There exists a $D > 0$ such that $N_X(x, y, R) \sim De^{hR}$ as $R \to +\infty$, i.e.,*

$$\lim_{R \to +\infty} \frac{N_X(x, y, R)}{e^{hR}} = D.$$

Proof. The proof simply requires replacing the function $\eta_X(z)$ by the function

$$\eta_N(z) = \int_0^\infty e^{-zR} dN_X(x, R) = \underline{v}(z) \cdot (I - \widehat{M_z})^{-1} \underline{w},$$

where $\underline{w} = (\chi_{\mathcal{F}}(s_i))_{i=1}^\infty$, with $\chi_{\mathcal{F}}(s)$ denoting the characteristic function for the set $\mathcal{F} = \{s \in \mathcal{S} : t(s_i) = y\}$ of saddle connections ending at the singularity y and $u(z)$ was defined after equation (6.1). Again the properties of $(I - \widehat{M_z})^{-1}$ allow one to apply Theorem 5.1 to deduce the result. □

Remark 6.13. We conclude with some final remarks.

1. It is not necessary for the ball in Theorem 6.11 to be centered at a singularity. Let $y \in X - \Sigma$ and let G be the set of geodesics g, from y to a singularity, such that g has length $\ell(g)$. Order G by non-decreasing lengths. We define a matrix P where

   (a) the rows are indexed by such geodesics g and the columns are indexed by the oriented saddle connections s;

   (b) the non-zero entries correspond to pairs g, s such that:

      i. The singularity $t(g)$ at the end of g is the same as that $i(s)$ at the start of the saddle connection s; and

      ii. The geodesic g and saddle connection s have an angle of at least π between them.

   (c) The non-zero entries are $P_z(g, s) = e^{-z\ell(s)}$.

   One can then modify the complex function to $\eta_X(z) = \frac{2\pi}{z^2}(k(x) + 1) + \frac{2\pi}{z^2} \underline{v}_p(z) \cdot \widehat{P_z}(I - \widehat{M_z})^{-1} \underline{u}$, where $\underline{v}_p(z) = (e^{-z\ell(g)})_{g \in G} \in \ell^1(\mathbb{C})$ and then continue the proof as in Theorem 6.11.

2. Theorem 6.11 also follows as a corollary of Theorem 6.12 by using a simple approximation argument. In particular, this shows that $C = D \int_0^\infty e^{-u} u^2 du$.

3. Let $L(x, R)$ be the total circumference of a circle centred at x and whose radius is a geodesic of length R. The same approach as in the proof of Theorem 6.11 (or an approximation argument as in item 2 would give an asymptotic formula of the form $L(x, R) \sim E e^{hR}$, as $R \to +\infty$.

4. Eskin and Rafi have announced a closely related asymptotic result to Theorem 6.11 for closed geodesics on X. By studying zeta functions $\zeta_X(z)$ instead of eta functions $\eta_X(z)$ they show that the number of closed geodesics of length at most $R > 0$ is asymptotic to e^{hR}/hR as $R \to +\infty$.

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