

ENTROPY POINTS AND APPLICATIONS FOR FREE SEMIGROUP ACTIONS

FAGNER B. RODRIGUES*, THOMAS JACOBUS, AND MARCUS V. SILVA

ABSTRACT. The aim of this manuscript is to study some local properties of the topological entropy of a free semigroup action. In order to do that we focus on the set of entropy points of a free semigroup action, show that this set carries the full entropy of the system (which, with respect to the chaoticity of the system, gives a fundamental relevance to such set) and obtain many interesting properties of such set. Our results are inspired by the ones presented in [23].

1. INTRODUCTION

The concept of entropy, introduced into the realm of dynamical systems more than fifty years ago, has become an important ingredient in the characterization of the complexity of dynamical systems.

In the classical setting, $f : (X, d) \rightarrow (X, d)$ is a continuous map acting on a compact metric space and the topological entropy, denoted by $h_{\text{top}}(f)$, counts, in exponential scale, the number of distinguishable orbits. It is an important tool to describe the chaotic behaviour of f . In the end of the last century Bufetov [8] presented a definition of topological entropy of a free semigroup action G generated by a finite set of continuous maps $G_1 = \{id, f_1, \dots, f_p\}$ acting on a compact metric space X and the fixed random walk $\eta_p = \left(\frac{1}{p}, \dots, \frac{1}{p}\right)^{\mathbb{N}}$. Moreover, the author related it with the topological entropy of the shift map $\sigma : \Sigma_p^+ \rightarrow \Sigma_p^+$ and the induced continuous skew product $\mathcal{F}_G : \Sigma_p^+ \times X \rightarrow \Sigma_p^+ \times X$. The actions of semigroups and their dynamic aspects have been extensively studied in recent years, see for example [1, 2, 3, 4, 6, 12, 13, 14, 17, 20]. Recently, in [9, 10, 11], a more general definition of topological entropy based in the one introduced in [8] was exploited. In that context the random walk considered in G is any probability measure in Σ_p^+ .

In [23] the authors explored the concept of entropy point (point for which the topological entropy is positive for any neighborhood that contains it) for a single dynamic. Through the results presented in this work, it was possible to see the importance of such a notion for a better understanding of the local behavior of the dynamics, its topological entropy and also its metric entropy, since they showed that the support of ergodic measures is contained in the set of entropy points and that the metric entropy of an ergodic measure represents a lower bound for the topological entropy of the Borelian sets, among other important results, as the ones about the entropy function (see Definition 2.3).

The main objective of this work is, through the definition of topological entropy of [8], to obtain important properties that help to describe the local behavior of the action and its topological entropy. Our results are motivated by those presented in

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*Corresponding author e-mail: fagnerbernardini@gmail.com.

[23] and, as in [23], we show that the set of entropy points of the free semigroup action contains all the topological entropy of the system; given a measure of probability in the phase space, under certain conditions, the metric entropy of the action with respect to that measure represents a lower bound for the topological entropy of any Borelian set; there is always a closed enumerable set with total entropy, such set has at most one accumulation point and, in the case of the existence of a limit point, the entropy function evaluated at that point coincides with the entropy of the action.

1.1. Setting. Given a finite set of continuous maps $g_i : X \rightarrow X$, $i \in \mathcal{P} = \{1, 2, \dots, p\}$, $p \geq 1$, and the finitely generated semigroup (G, \circ) with the finite set of generators $G_1 = \{id, g_1, g_2, \dots, g_p\}$, we write $G = \bigcup_{n \in \mathbb{N}_0} G_n$ where $G_0 = \{id\}$ and $\underline{g} \in G_n$ if and only if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$, with $g_{i_j} \in G_1$ (for notational simplicity's sake we will use $g_j g_i$ instead of the composition $g_j \circ g_i$). A semigroup can have multiple generating sets. We will assume that the generator set G_1 is minimal, meaning that no function g_j , for $j = 1, \dots, p$, can be expressed as a composition of the remaining generators. We shall consider different concatenations instead of the elements in G they create. One way to interpret this statement is to consider the itinerary map $\iota : \mathbb{F}_p \rightarrow G$ given by $\underline{i} = i_n \dots i_1 \mapsto \underline{g}_{\underline{i}} := g_{i_n} \dots g_{i_1}$, where \mathbb{F}_p is the free semigroup with p generators, and to regard concatenations on G as images by ι of paths on \mathbb{F}_p . Set $G_1^* = G_1 \setminus \{id\}$ and, for every $n \geq 1$, let G_n^* denote the space of concatenations of n elements in G_1^* . To summon each element \underline{g} of G_n^* , we will write $|\underline{g}| = n$ instead of $\underline{g} \in G_n^*$. In G , one consider the semigroup operation of concatenation defined as usual: if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$ and $\underline{h} = h_{i_m} \dots h_{i_2} h_{i_1}$, where $n = |\underline{g}|$ and $m = |\underline{h}|$, then $\underline{g}\underline{h} = g_{i_n} \dots g_{i_2} g_{i_1} h_{i_m} \dots h_{i_2} h_{i_1} \in G_{m+n}^*$. The finitely generated semigroup G induces an *action* in X , say

$$\begin{aligned} \mathbb{S} : G \times X &\rightarrow X \\ (g, x) &\mapsto g(x). \end{aligned}$$

We say that \mathbb{S} is a *semigroup action* if, for any $\underline{g}, \underline{h} \in G$ and every $x \in X$, we have $\mathbb{S}(\underline{g}\underline{h}, x) = \mathbb{S}(\underline{g}, \mathbb{S}(\underline{h}, x))$. The action \mathbb{S} is continuous if the map $\underline{g} : X \rightarrow X$ is continuous for any $\underline{g} \in G$.

2. PRELIMINARIES AND MAIN RESULTS

Consider a finitely generated free semigroup (G, G_1) acting on a compact metric space X . Let $K \subset X$ be a compact set. Given $\underline{g} = g_{i_n} \dots g_{i_1} \in G_n$, we say a set $E \subset K$ is $(\underline{g}, \varepsilon)$ -*separated set* if $d_{\underline{g}}(x_1, x_2) > \varepsilon$ for any distinct $x_1, x_2 \in E$. When no confusion is possible with the notation for the concatenation of semigroup elements, the maximum cardinality of a $(\underline{g}, \varepsilon)$ -separated sets of K will be denoted by $s(K, \underline{g}, \varepsilon)$. We say that $F \subset K$ is a $(\underline{g}, \varepsilon)$ -*spanning set* if given $x \in K$ there exists $y \in F$ so that if $d_{\underline{g}}(x, y) < \varepsilon$. When no confusion is possible with the notation for the concatenation of semigroup elements, the minimum cardinality of a $(\underline{g}, \varepsilon)$ -spanning sets of K will be denoted by $b(K, \underline{g}, \varepsilon)$. We now recall the notion of topological entropy introduced by Bufetov [8].

Definition 2.1. Given a compact set $K \subset X$, we define

$$h_{top}(K, \mathbb{S}) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log S_n(K, \mathbb{S}, \varepsilon), \quad (1)$$

where

$$S_n(K, \mathbb{S}, \varepsilon) = \frac{1}{p^n} \sum_{\underline{g} \in G_n^*} s(K, \underline{g}, \varepsilon), \quad (2)$$

where the sum is taken over all concatenation \underline{g} of n -elements of $G_1 \setminus \{id\}$ and $p = |G_1 \setminus \{id\}|$. The *topological entropy* $h_{top}(K, \mathbb{S})$ is defined for $E = X$.

With Definition 1 we can talk about entropy points:

- (1) We say that $x_0 \in X$ is an *entropy point* x_0 if for any closed neighbourhood K of x_0 we have

$$h_{top}(K, \mathbb{S}) > 0.$$

- (2) We say that $x_0 \in X$ is a *full entropy point* x_0 if for any closed neighbourhood K of x_0 we have

$$h_{top}(K, \mathbb{S}) = h_{top}(X, \mathbb{S})$$

holds.

Entropy points are those for which local neighborhoods reflect the complexity of the entire dynamical system.

In [16], as in the case of a single dynamics, the authors proved that the set of full entropy points is not empty. More precisely, it holds the following:

Theorem 2.2. *Let $\mathbb{S} : G \times X \rightarrow X$ be a finitely generated free semigroup action. Then $E_p^f(X, \mathbb{S}) \neq \emptyset$.*

Although not entirely related, the dynamics of semigroup actions has a strong connection with skew-products. Indeed, if X is a compact metric space and one considers a finite set of continuous maps $g_i : X \rightarrow X$, $i \in \{1, 2, \dots, p\}$, $p \geq 1$ consider the step skew-product

$$\begin{aligned} \mathcal{F}_G : \Sigma_p^+ \times X &\rightarrow \Sigma_p^+ \times X \\ (\omega, x) &\mapsto (\sigma(\omega), g_{\omega_1}(x)) \end{aligned} \quad (3)$$

where $\omega = (\omega_1, \omega_2, \dots)$ is an element of the full unilateral space of sequences $\Sigma_p^+ = \{1, 2, \dots, p\}^{\mathbb{N}}$ and σ denotes the shift map on Σ_p^+ . With this notation we will write $\mathcal{F}_G^n(\omega, x) = (\sigma^n(\omega), f_\omega^n(x))$ for every $n \geq 1$.

2.1. metric entropy and entropy points. In what follows we denote by $\mathcal{M}(X)$ the set of measures on X , by $\mathcal{M}_1(X)$ the set of probability measures on X and for a given continuous map $f : X \rightarrow X$ we denote by $\mathcal{M}_{inv}(X)$ the set of f -invariant probability measures in X . In [11] the authors defined the metric entropy of a free semigroup action \mathbb{S} with respect $\nu \in \mathcal{M}_1(X)$ as

$$h_\nu(\mathbb{S}) = \sup_{\mu \in \Pi(\nu, \sigma)} h_\mu(\mathcal{F}_G) - \log p,$$

where $\Pi(\nu, \sigma)$ is the subset of $\mathcal{M}_1(\Sigma_p^+ \times X)$ which are \mathcal{F}_G -invariant, $(\pi_{\Sigma_p^+})_* \mu$ is σ -invariant and $(\pi_X)_* \mu = \nu$. They proved that it satisfies a variational principle given by

$$h_{top}(\mathbb{S}) = \sup_{\{\nu \in \mathcal{M}(X) : \Pi(\nu, \sigma) \neq \emptyset\}} h_\nu(\mathbb{S}, \eta_{\underline{a}}).$$

Denote by $E_p(X, \mathbb{S})$ the set of entropy points of \mathbb{S} , by $E_p^f(X, \mathbb{S})$ the set of all full entropy points of \mathbb{S} . We also denote by $E_p(\Sigma_p^+ \times X, \mathcal{F}_G)$ the set of entropy points of \mathcal{F}_G and by $E_p^f(\Sigma_p^+ \times X, \mathcal{F}_G)$ the set of full entropy points of \mathcal{F}_G .

Our first result shows that the metric entropy and the topological entropy of a free semigroup action are concentrated on the the set of entropy points.

Theorem A. *Let $\mathbb{S} : G \times X \rightarrow X$ be a finitely generated free semigroup action. Then*

- i. Let $\nu \in \mathcal{M}_1(X)$ so that $\Pi(\sigma, \nu)_{erg} \neq \emptyset$, then $\text{supp}(\nu) \subset E_p(X, \mathbb{S})$;
- ii. $h_{top}(E_p(X, \mathbb{S})) = h_{top}(X, \mathbb{S})$.

As proved in [23], it is possible to get an upper bound for the metric of $\nu \in \mathcal{M}_1(X)$ for which $\Pi(\sigma, \nu) \neq \emptyset$ in terms of the local information of the topological entropy.

Theorem B. *Let $\mathbb{S} : G \times X \rightarrow X$ be a finitely generated free semigroup action, d a metric on X $\nu \in \mathcal{M}_1(X)$ so that $\Pi(\sigma, \nu)_{erg} \neq \emptyset$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \{S_d(K, \mathbb{S}, \varepsilon) : K \in \mathcal{B}_X \text{ with } \nu(K) > 0\} \geq h_\nu(\mathbb{S}).$$

In particular, for any $K \in \mathcal{B}_X$, $h_{top}(K, \mathbb{S}) \geq h_\nu(\mathbb{S})$.

2.2. Entropy function. For each $\varepsilon > 0$ and $x \in X$, define

$$h_d(x, \varepsilon) = \inf\{B_d(K, \mathbb{S}, \varepsilon) : K \text{ is compact neighbourhood of } x\},$$

where the subscribed d is to emphasize the metric d and

$$B_d(K, \mathbb{S}, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{\underline{g} \in G_n^*} b_d(K, \underline{g}, \varepsilon) \right),$$

and $b_d(K, \underline{g}, \varepsilon)$ denotes the minimum cardinality of a $(\underline{g}, \varepsilon)$ -spanning set in the metric d . As $h_d(x, \varepsilon)$ increases as ε decreases to zero, it is well defined the following

$$h_d(x) = \lim_{\varepsilon \rightarrow 0^+} h_d(x, \varepsilon) \quad (4)$$

and it is less or equal to $h_{top}(X, \mathbb{S})$. In fact, it depends only on the topology of X and we can denote by $h_{top}(x)$.

Definition 2.3. Let $\mathbb{S} : G \times X \rightarrow X$ be a continuous finitely generated free semigroup action. The function $h_{top} : X \rightarrow [0, h_{top}(X, \mathbb{S})]$, $x \mapsto h_{top}(x)$ is called the entropy function of \mathbb{S} .

Since $B_d(K, \mathbb{S}, \varepsilon) \leq S_d(K, \mathbb{S}, \varepsilon) \leq B_d(K, \mathbb{S}, \varepsilon/2)$, we have

$$h_{top}(x) = \lim_{\varepsilon \rightarrow 0} \inf\{S_d(K, \mathbb{S}, \varepsilon) : K \text{ is compact neighbourhood of } x\}.$$

The next theorem gives a lower bound, depending on the metric entropy, for the entropy function in each point. Moreover, it is proved that, as one may hope, the supremum of the entropy function is given by the topological entropy.

Theorem C. *Let $\mathbb{S} : G \times X \rightarrow X$ be a continuous finitely generated free semigroup action.*

- i. $h_{top}(x) \geq \sup\{h_\nu(\mathbb{S}) : \nu \in \mathcal{M}_1(X), \Pi(\sigma, \nu)_{erg} \neq \emptyset \text{ and } x \in \text{supp}(\nu)\}$.
- ii. For each $\varepsilon > 0$, $h_d(\cdot, \varepsilon)$ is upper semi continuous, consequently, h is Borel measurable.
- iii. If $K \subset X$ is closed, then $\sup_{x \in K} h_{top}(x) \geq h_{top}(K, \mathbb{S})$. In particular,

$$\sup_{x \in X} h_{top}(x) = h_{top}(X, \mathbb{S}).$$

- iv. If G_1 is a finite set of homeomorphisms on X then h is \mathbb{S} -invariant and for $\nu \in \mathcal{M}_1(X)$ satisfying $\Pi(\sigma, \nu)_{erg} \neq \emptyset$,

$$\int_X h_{top}(x) d\nu \geq h_\nu(\mathbb{S}).$$

Our last result shows that the topological entropy of a free semigroup action may be computed in terms of countable sets.

Theorem D. *Let $\mathbb{S} : G \times X \rightarrow X$ be a continuous finitely generated free semigroup action. Then there exists a countable closed subset $K \subset X$ such that $h_{top}(K, \mathbb{S}) = h_{top}(X, \mathbb{S})$. Moreover, K can be chosen such that the set of limit points of K has at most one limit point, and K has a unique limit point if and only if there is $x \in X$ with $h_{top}(x) = h_{top}(X, \mathbb{S})$.*

3. PROOFS

Before we prove the main results we present some important observations on the characterization of the set of entropy points of a free semigroup action.

Proposition 3.1. *Let $\omega = \omega_1 \omega_2 \dots \in \Sigma_p^+$. Then $h_{top}([\omega_1 \dots \omega_\ell] \times K, \mathcal{F}_G) = h_{top}(K, \mathbb{S}) + \log p$.*

Proof. Let n be a positive integer and consider $N = p^n$. There are N distinct words of length n in \mathbb{F}_p^+ . Denote these words by $\theta^1, \dots, \theta^N$. Let $(\theta(i))_{i=1}^N \subset \Sigma_p^+$ be a sequence such that $\theta(i)|_{[1, n]} = \theta^i$. We notice that, for $0 < \varepsilon < \frac{1}{2}$, the sequence $(\omega_1 \dots \omega_\ell \theta(i))_{i=1}^N \subset [\omega_1 \dots \omega_\ell]$ is (n, ε, σ) -separated subset of $[\omega_1 \dots \omega_\ell]$.

Let $\theta^i = \theta_1^i \dots \theta_n^i$ and $\underline{g}^{(i)} = g_{\theta_1^i} \dots g_{\theta_n^i} g_{\omega_\ell} \dots g_{\omega_1}$. Set $Z_i = Z(\underline{g}^{(i)}, \varepsilon, K)$, where $\underline{g}^{(i)}$ is the element in G given by the c and let the points $x_1^i, \dots, x_{Z_i}^i$ form a $(\underline{g}^{(i)}, \varepsilon)$ -separated subset of K . Then the points

$$(\omega_1 \dots \omega_\ell \theta(i), x_j^i) \in [\omega_1 \dots \omega_\ell] \times K, \quad i = 1, \dots, N \quad j = 1, \dots, Z_i,$$

form a $(n, \varepsilon, \mathcal{F}_G)$ -separated subset of $[\omega_1 \dots \omega_\ell] \times K$. It implies that

$$S([\omega_1 \dots \omega_\ell] \times K, \mathcal{F}_G, n, \varepsilon) \geq \sum_{i=1}^N s(K, \underline{g}^{(i)}, \varepsilon),$$

and then $h_{top}([\omega_1 \dots \omega_\ell] \times K, \mathcal{F}_G) \geq h_{top}(K, \mathbb{S}) + \log p$.

For the converse inequality, consider $\varepsilon > 0$ and take $C(\varepsilon)$ an arbitrary positive integer such that $p^{-C(\varepsilon)} < \frac{\varepsilon}{100}$. We notice that there exist $N = p^{n+2C(\varepsilon)}$ distinct words of length $n + 2C(\varepsilon) + \ell$. Denote these words by $\omega^1, \dots, \omega^N$.

Let $(\omega(i))_{i=1}^N \subset \Sigma_p^+$ be an arbitrary sequence satisfying $\omega(i)|_{[1, n+2C(\varepsilon)+\ell]} = \omega^i$. This sequence forms a (n, ε, σ) -spanning set of $[\omega_1 \dots \omega_\ell]$. Denote $\theta^i = \omega(i)|_{[0, n+\ell]}$, by $\underline{g}^{(i)}$ the concatenation associated to θ^i and $B_i = B(\underline{g}^{(i)}, \varepsilon, K)$ and assume that the points $x_1^i, \dots, x_{B_i}^i$ form a $(\underline{g}^{(i)}, \varepsilon)$ -spanning subset of K . It follows that the points $(\omega(i), x_j^i)$ with $i = 1, \dots, N$ and $j = 1, \dots, B_i$, form a $(\underline{g}^{(i)}, n, \varepsilon)$ -spanning set of K . In that case, there exists a positive constant $T = T(\varepsilon)$ depending only on ε so that

$$B([\omega_1 \dots \omega_\ell] \times K, \mathcal{F}_G, n + \ell, \varepsilon) \leq \sum_{i=1}^{n+2C(\varepsilon)+\ell} b(K, \mathbb{S}, \underline{g}^{(i)}, \varepsilon) \leq T(\varepsilon) \sum_{|\underline{g}|=n+\ell} b(K, \mathbb{S}, \underline{g}, \varepsilon).$$

From the last inequalities we obtain

$$\begin{aligned} h_{top}([\omega_1 \dots \omega_\ell] \times K, \mathcal{F}_G) \\ \leq h_{top}(K, \mathbb{S}) + \log p. \end{aligned}$$

It concludes the proof. □

As an immediate consequence of Proposition 3.1 we have the following.

Corollary 3.2. *Under the above notations we have that $E_p(\Sigma_p^+ \times X, \mathcal{F}_G) = \Sigma_p^+ \times X$.*

Proof. Given $(\omega, x) \in \Sigma_p^+ \times X$ and F closed neighbourhood containing (ω, x) , by the definition of the product topology of $\Sigma_p^+ \times X$, there exist closed subsets $[\omega_1 \dots \omega_\ell] \subset \Sigma_p^+$ and $K \subset X$, so that $[\omega_1 \dots \omega_\ell] \times K \subset F$. As $h_{top}(F, \mathcal{F}) \geq h_{top}([\omega_1 \dots \omega_\ell] \times K)$ and, by Proposition 3.1, $h_{top}([\omega_1 \dots \omega_\ell] \times K) \geq \log p$. \square

If one denotes by π_X the canonical projection of $\Sigma_p^+ \times X$ on the first coordinate, the next corollary guarantees that if $(\omega, x) \in E_p^f(\Sigma_p^+ \times X, \mathcal{F}_G)$ then $\pi_X(\omega, x) \in E_p^f(X, \mathbb{S})$.

Corollary 3.3. *Let $(\omega, x) \in \Sigma_p^+ \times X$ be a full entropy point for \mathcal{F}_G . Then $x \in X$ is a full entropy point for \mathbb{S} .*

Proof. Assume that $\omega = \omega_1 \omega_2 \dots$ and consider $[\omega_1 \dots \omega_\ell] \times K$ compact neighbourhood of (ω, x) . By Proposition 3.1

$$h_{top}(X, \mathbb{S}) + \log p = h_{top}(\Sigma_p^+ \times X, \mathcal{F}_G) = h_{top}([\omega_1 \dots \omega_\ell] \times K, \mathcal{F}_G) = h_{top}(K, \mathbb{S}) + \log p.$$

\square

Another important consequence of Proposition 3.1 is the following,

Corollary 3.4. *Let $\mathbb{S} : G \times X \rightarrow X$ be a finitely generated free semigroup action. Then $E_p^f(X, \mathbb{S}) \neq \emptyset$.*

The fact of the semigroup action having positive entropy when restricted to a subset of X is related presence of entropy points in such subset, as the next shows.

Proposition 3.5. *Let $\mathbb{S} : G \times X \rightarrow X$ be a finitely generated free semigroup action and K be a closed subset of X .*

- i. *If $h_{top}(K, \mathbb{S}) > 0$, then $K \cap E_p(X, \mathbb{S}) \neq \emptyset$;*
- ii. *If $h_{top}(K, \mathbb{S}) = h_{top}(X, \mathbb{S})$, then $K \cap E_p^f(X, \mathbb{S}) \neq \emptyset$.*

Proof. We start proving (i). Cover K by finitely many closed balls $B_1^1, \dots, B_{\ell_1}^1$ with diameter at most 1. We know that $h_{top}(K, \mathbb{S}) = \max_j h_{top}(B_{j_1}^1, \mathbb{S})$ and it implies the existence of j_1 so that $h_{top}(K, \mathbb{S}) = h_{top}(B_{j_1}^1, \mathbb{S})$. Cover $B_{j_1}^1$ by finitely many closed balls $B_1^2, \dots, B_{\ell_2}^2$ with diameter at most $\frac{1}{2}$. By the same reasoning, there exists j_2 for which $h_{top}(K, \mathbb{S}) = h_{top}(B_{j_1}^1, \mathbb{S}) = h_{top}(B_{j_2}^2, \mathbb{S})$. By induction, for each $n \geq 2$, there is a closed ball $B_{j_n}^n \subset B_{j_{n-1}}^{n-1}$ with diameter at most $\frac{1}{n}$ such that $h_{top}(K, \mathbb{S}) = h_{top}(B_{j_n}^n, \mathbb{S})$. Set x the unique point in the intersection of the closed balls $B_{j_n}^n$. By definition, x is an entropy point for \mathbb{S} .

To prove (ii) we notice that, since $h_{top}(K, \mathbb{S}) = h_{top}(X, \mathbb{S})$, then $h_{top}(\Sigma_p^+ \times K, \mathcal{F}_G) = h_{top}(\Sigma_p^+ \times X, \mathcal{F}_G)$. By [23, Theorem 3.5] there exists $(\omega, x) \in \Sigma_p^+ \times K \cap E_p^f(\Sigma_p^+ \times X, \mathcal{F}_G)$. By Corollary 3.3 we have that $\pi_X(\omega, x) \in K \cap E_p^f(X, \mathbb{S})$, and it proves (ii). \square

3.1. Proof of Theorem A. Before we start the proof we recall the definition of Katok's entropy of a probability measure in X , which was extended for the semigroup setting in [11].

3.1.1. Katok's entropy.

Definition 3.6. Given a Borel probability measure ν on X , $\delta \in (0, 1)$ and $\varepsilon > 0$, define

$$h_\nu^K(\mathbb{S}, \varepsilon, \delta) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log S_\nu(X, \mathbb{S}, n, \varepsilon, \delta), \quad (5)$$

where

$$S_\nu(X, \mathbb{S}, n, \varepsilon, \delta) = \frac{1}{p^n} \sum_{\underline{g} \in G_n^*} s_\nu(\underline{g}, \varepsilon, \delta), \quad (6)$$

and for $\underline{g} \in G_n^*$

$$s_\nu(\underline{g}, \varepsilon, \delta) = \inf_{\{E \subseteq X : \nu(E) > 1-\delta\}} s(E, \underline{g}, \varepsilon)$$

and $s(\underline{g}, \varepsilon, E)$ denotes the maximal cardinality of the $(\underline{g}, \varepsilon)$ -separated subsets of E .

The *entropy of the semigroup action \mathbb{S} with respect to ν* is defined by

$$h_\nu^K(\mathbb{S}) = \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} h_\nu^K(\mathbb{S}, \varepsilon, \delta) \quad (7)$$

Observe that the previous limit is well defined due to the monotonicity of the function

$$(\varepsilon, \delta) \mapsto \frac{1}{n} \log S_\nu(X, \mathbb{S}, n, \varepsilon, \delta)$$

on the unknowns ε and δ . Moreover, if the set of generators is $G_1 = \{Id, f\}$, we recover the notion proposed by Katok for a single dynamics f .

By [11, Theorem C] we have that

$$h_{top}(X, \mathbb{S}) = \sup_{\nu \in \mathcal{M}_1(X)} h_\nu^K(\mathbb{S}).$$

Let us proceed to the proof of Theorem A. We recall that for $\nu \in \mathcal{M}_1(X)$, we denote the set of ergodic measures in $\Pi(\sigma, \nu)$ by $\Pi(\sigma, \nu)_{erg}$.

Since $h_\nu(\mathbb{S}) > 0$, by the definition of the metric entropy, we have that there exists $\mu \in \Pi(\sigma, \nu)_{erg}$ for which $h_\mu(\mathcal{F}_G) > \log p$. If $x \in \text{supp}(\nu)$, for any closed neighbourhood N_x of x , $\nu(N_x) = \mu(\Sigma_p^+ \times N_x) > 0$. By [23, Theorem 3.7] we have that

$$h_{top}(N_x, \mathbb{S}) + \log p = h_{top}(\Sigma_p^+ \times N_x, \mathcal{F}_G) \geq h_\mu(\mathcal{F}_G) \log p,$$

which implies that $h_{top}(N_x, \mathbb{S}) > 0$ and then, $x \in E_p(X, \mathbb{S})$, concluding the proof item (i).

For (ii) we notice that

$$h_{top}(X, \mathbb{S}) = \sup_{\{\nu \in \mathcal{M}_1(X) : \Pi(\sigma, \nu)_{erg} \neq \emptyset\}} h_\nu(\mathbb{S}).$$

By (i) we have that the supremum is taken on the probability measures on X with support contained in $E_p(X, \mathbb{S})$. By [11, Theorem C] we have that

$$h_\nu(\mathbb{S}) \leq h_\nu^K(\mathbb{S}),$$

and it implies that $h_{top}(X, \mathbb{S}) = h_{top}(E_p(X, \mathbb{S}), \mathbb{S})$, since $\text{supp}(\nu) \subset E_p(X, \mathbb{S})$.

3.2. Proof of Theorem B. Let $0 < \delta < 1$ fixed. Let $\varepsilon > 0$ and $K \in \mathcal{B}_X$ satisfy $\nu(K) > 0$. As $\Pi(\sigma, \nu)_{erg} \neq \emptyset$, for $\mu \in \Pi(\sigma, \nu)_{erg}$, by the ergodicity of μ , there exists $m(K) \in \mathbb{N}$ such that $\mu\left(\bigcup_{i=0}^{m(K)} \mathcal{F}_G^i(\Sigma_p^+ \times K)\right) \geq 1 - \delta$.

Claim. If $K^{(m)} := \pi_X\left(\bigcup_{i=0}^{m(K)} \mathcal{F}_G^i(\Sigma_p^+ \times K)\right)$, then $S_d(K, \mathbb{S}, \varepsilon) = S_d(K^{(m)}, \mathbb{S}, \varepsilon)$.

Proof of the Claim. First of all we notice that $\nu(K^{(m)}) \geq 1 - \delta$. Fix $n \in \mathbb{N}$ and take $\underline{g} \in G_n^*$. Let $E_{\underline{g}} \subset K^{(m)}$ be a $(\underline{g}, \varepsilon)$ -separated set. In particular there exists $0 \leq i_0(n) \leq m$ so that

$$\left| E_{\underline{g}} \cap \pi_X(\mathcal{F}_G(\Sigma_p^+ \times K)) \right| \geq \frac{|E_{\underline{g}}|}{m+1}. \quad (8)$$

If $\underline{g} = g_{j_n} \dots g_{j_1}$, we have that $\underline{g}_{i_0(n)}^{-1}(E) \cap K$ is a $(\underline{g}, \varepsilon)$ -separated set in K . Taking $\tilde{\underline{g}} = \underline{g}_{i_0(n)}$, we have that $\underline{g}_{i_0(n)}^{-1}(E)$ is $(\tilde{\underline{g}}, \varepsilon)$ -separated, with $\tilde{\underline{g}} \in G_{n+m}$. By (8)

$$s(K, \tilde{\underline{g}}, \varepsilon) \geq \frac{1}{m+1} s(K^{(m)}, \underline{g}, \varepsilon) \geq \frac{1}{m+1} s(K, \underline{g}, \varepsilon).$$

Denote by \tilde{G}_{m+n}^* the subset of \tilde{g} obtained in the previous construction. It shows that

$$\begin{aligned} S_{m+n}(K, \mathbb{S}, \varepsilon) &= \frac{1}{p^{m+n}} \sum_{\underline{g} \in G_{m+n}^*} s(K, \underline{g}, \varepsilon) \\ &\geq \frac{1}{p^{m+n}} \frac{1}{m+1} \sum_{\tilde{g} \in \tilde{G}_{m+n}^*} s(K, \tilde{g}, \varepsilon) \\ &\geq \frac{1}{p^{m+n}} \frac{1}{m+1} \sum_{\underline{g} \in G_n^*} s(K^{(m)}, \underline{g}, \varepsilon) \\ &\geq \frac{1}{p^{m+n}} \frac{1}{m+1} \sum_{\underline{g} \in G_n^*} s(K, \underline{g}, \varepsilon), \end{aligned}$$

and it implies

$$S_{m+n}(K, \mathbb{S}, \varepsilon) \geq \frac{1}{p^m} \frac{1}{m+1} S_n(K^{(m)}, \mathbb{S}, \varepsilon) \geq \frac{1}{p^m} \frac{1}{m+1} S_n(K, \mathbb{S}, \varepsilon),$$

which guarantees $S(K, \mathbb{S}, \varepsilon) = S_d(K^{(m)}, \mathbb{S}, \varepsilon)$, and this proves the claim. Let us proceed to finalize the proof of the theorem. As, for any $\varepsilon > 0$, $n \in \mathbb{N}$ and $R \subset X$

$$B_n(R, \mathbb{S}, \varepsilon) \leq S_n(R, \mathbb{S}, \varepsilon),$$

we have

$$\begin{aligned} &\inf\{S_d(K, \mathbb{S}, \varepsilon) : K \in \mathcal{B}_X \text{ with } \nu(K) > 0\} \\ &= \inf\{S_d(K^{(m)}, \mathbb{S}, \varepsilon) : K \in \mathcal{B}_X \text{ with } \nu(K) > 0\} \\ &\geq \inf\left\{\limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{p^n} B(K^{(m)}, \mathbb{S}, \varepsilon) : K \in \mathcal{B}_X \text{ with } \nu(K) > 0\right\} \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{n} \inf\left\{\log \frac{1}{p^n} B_n(K^{(m)}, \mathbb{S}, \varepsilon) : K \in \mathcal{B}_X \text{ with } \nu(K) > 0\right\} \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{p^n} B_\nu(\mathbb{S}, n, \varepsilon). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ and noticing that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{p^n} B_\nu(\mathbb{S}, n, \varepsilon) \geq h_\nu(\mathbb{S}),$$

we conclude the proof.

3.3. Proof of Theorem C. We notice that (i) is an immediate consequence of Theorem B.

For (ii), let $\varepsilon > 0$. For $r \in \mathbb{R}$, if $h_{top}(x_0, \varepsilon) < r$, then $B(K, \mathbb{S}, \varepsilon) < r$, for some closed neighbourhood K of x_0 . In particular, $h_{top}(x, \varepsilon) < r$ for each $x \in K$. So, $h_{top}(\cdot, \varepsilon)$ is upper semi continuous. As we can obtain h as the limit of $\{h_{top}(x, \frac{1}{n})\}_{n \in \mathbb{N}}$, sequence of upper semi continuous maps, it is a Borel map.

(iii). Cover K by finitely many closed balls $B_1^1, \dots, B_{\ell_1}^1$ with diameter at most 1. We know that $h_{top}(K, \mathbb{S}) = \max_j h_{top}(B_{j_1}, \mathbb{S})$ and it implies the existence of j_1 so that $B_d(K, \mathbb{S}, \varepsilon) = B_d(B_{j_1} \cap K, \mathbb{S}, \varepsilon)$. Cover B_{j_1} by finitely many closed balls $B_1^2, \dots, B_{\ell_2}^2$ with diameter at most $\frac{1}{2}$. By the same reasoning, there exists j_2 for which $B_d(K, \mathbb{S}, \varepsilon) = B_d(B_{j_2} \cap K, \mathbb{S}, \varepsilon)$. By induction, for each $n \geq 2$, there is a closed ball $B_{j_n}^n \subset B_{j_{n-1}}^n$ with diameter at most $\frac{1}{n}$ such that $B_d(K, \mathbb{S}, \varepsilon) = B_d(B_{j_n} \cap K, \mathbb{S}, \varepsilon)$. Set x_0 the unique point in the intersection of the closed balls $B_{j_n}^n$, which belongs to

K , since K is closed. If K' is any closed neighbourhood of x_0 , for $n \in \mathbb{N}$ sufficiently large, $B_{j_n} \cap K \subset K'$, which implies

$$B_d(K', \mathbb{S}, \varepsilon) \geq B_d(B_{j_n} \cap K, \mathbb{S}, \varepsilon) = B_d(B_{j_n} \cap K, \mathbb{S}, \varepsilon)$$

and

$$h_{top}(x_0) \geq h_d(x_0, \varepsilon) \geq B_d(B_{j_n} \cap K, \mathbb{S}, \varepsilon).$$

Hence, letting $\varepsilon \rightarrow 0^+$,

$$\sup_{x \in K} h_{top}(x) \geq h_{top}(K, \mathbb{S}).$$

For the last part of Theorem C we need of the following proposition, which is a consequence the definition of the entropy function of \mathbb{S} and \mathcal{F}_G .

Proposition 3.7. *Let $\mathbb{S} : G \times X \rightarrow X$ be a finitely generated free semigroup action. For $x \in X$,*

$$\sup_{\omega \in \Sigma_p^+} h_{D \times d}((\omega, x), \cdot) \leq h_d(x) + \log p.$$

Proof. Fix $\varepsilon > 0$. Let $\omega \in \Sigma_p^+$ and $V \subset \Sigma_p^+ \times X$ be a closed neighbourhood of (ω, x) . Since we consider the product topology in $\Sigma_p^+ \times X$, there exist $\Sigma \subset \Sigma_p^+$ and $Z \subset X$ so that $(\omega, x) \in \Sigma \times Z \subset \Sigma_p^+ \times X$. Let $Z = Z(\varepsilon)$ so that, for $\delta > 0$, $h_d(x, \varepsilon) \leq B_d(Z, \mathbb{S}, x, \varepsilon) \leq h_d(x, \varepsilon) + \delta$. It follows that

$$\begin{aligned} h_{D \times d}((\omega, x), \varepsilon) &\leq B_{D \times d}(\Sigma \times Z, \mathcal{F}_G, (\omega, x), \varepsilon) \\ &\leq B_{D \times d}(\Sigma_p^+ \times Z, \mathcal{F}_G, (\omega, x), \varepsilon) \\ &\leq B_d(Z, \mathbb{S}, x, \varepsilon) + \log p \\ &\leq h_d(x, \varepsilon) + \log p + \delta. \end{aligned}$$

As δ was taken arbitrary, $h_{D \times d}((\omega, x), \varepsilon) \leq h_d(x, \varepsilon) + \log p$, which implies

$$\sup_{\omega \in \Sigma_p^+} h_{D \times d}((\omega, x), \cdot) \leq h_d(x) + \log p$$

and finishes the proof. \square

Assuming that G_1 is a finite set of homeomorphisms of X we have that \mathcal{F}_G is a homeomorphism. By [23, Theorem 4.5], $h_{top}(\cdot, \cdot)$ is \mathcal{F}_G -invariant and if $\alpha \in \mathcal{M}_{erg}(\Sigma_p^+ \times X)$ then

$$\int_{\Sigma_p^+ \times X} h_{top}(\omega, x) d\alpha \geq h_\alpha(\mathcal{F}_G),$$

For $\nu = (\pi_X)_* \alpha$ we obtain, by Proposition 3.7,

$$\begin{aligned} h_\nu(\mathbb{S}) &= \sup_{\mu \in \Pi(\sigma, \nu)_{erg}} h_\mu - \log p \\ &\leq \sup_{\mu \in \Pi(\sigma, \nu)_{erg}} \int_{\Sigma_p^+ \times X} h_{top}(\omega, x) d\mu - \log p \\ &\leq \sup_{\mu \in \Pi(\sigma, \nu)_{erg}} \int_{\Sigma_p^+ \times X} h_{top}(x) d\mu - \log p \\ &= \int_{\Sigma_p^+ \times X} h_{top}(x) d\nu. \end{aligned}$$

3.4. Proof of Theorem D. We will get the proof of the theorem as a consequence of the following proposition.

Proposition 3.8. *Let $\mathbb{S} : G \times X \rightarrow X$ be a finitely generated free semigroup action and h its entropy function.*

- i. *If K is a countable closed subset with a unique limit point x_0 , then $h_{top}(x_0) \geq h_{top}(K, \mathbb{S})$.*
- ii. *Let $x_0 \in X$. Then there exists a countable closed subset $K \subset X$ such that $x_0 \in K$ is its unique limit point in X and $h_{top}(x_0) = h_{top}(K, \mathbb{S})$.*

Proof. Let d be a metric on X . To simplify the notation we drop the subscribed d in the definition of h . We start with (i). Fix $\varepsilon > 0$. By hypotheses, for any closed neighbourhood Z of x_0 , $K \setminus Z$ is a finite set and it implies $B(Z, \mathbb{S}, \varepsilon) \geq B(K, \mathbb{S}, \varepsilon)$. By the definition of h we obtain $h_{top}(x_0) \geq h_d(x_0, \varepsilon) \geq B(K, \mathbb{S}, \varepsilon)$. Letting $\varepsilon \rightarrow 0$ we get the conclusion.

For (ii) we assume $h_{top}(x_0) < \infty$. For each $n \in \mathbb{N}$ we set

$$K_n = \left\{ x \in X : d(x, x_0) \leq \frac{1}{n} \right\}.$$

For each $m \in \mathbb{N}$ choose ε_m such that

$$h_{top}(x_0) - \frac{1}{m} < \inf_{n \in \mathbb{N}} S(K_n, \mathbb{S}, \varepsilon_m) = \inf_{n \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log S_k(K_n, \mathbb{S}, \varepsilon_m)}{k}.$$

So, there exists an increasing sequence $\{k_{n,m}\}_{n \in \mathbb{N}} \subset \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$\frac{1}{p^{k_{n,m}}} \sum_{\underline{g} \in G_{k_{n,m}}^*} S(K_n, \underline{g}, \varepsilon_m) = S_{k_{n,m}}(K_n, \mathbb{S}, \varepsilon_m) \geq e^{k_{n,m}(h_{top}(x_0) - \frac{1}{m})}.$$

Now take $\underline{g} \in G_{k_{n,m}}^*$ and denote by $E_{\underline{g},(m,n)}$ a $(\underline{g}, \varepsilon_m)$ -separated set in K_n of maximum cardinality. Define

$$E_{m,n} = \bigcup_{\underline{g} \in G_{k_{n,m}}^*} E_{\underline{g},(m,n)} \cup \{x_0\} \text{ and } K = \bigcup_{m \geq 1} \bigcup_{n \geq m} E_{m,n}.$$

If V is a neighbourhood of x_0 , by the definition of the sets K_n , we have that $K_{n_0} \subset V$ for n_0 large enough. So,

$$K \setminus V \subset K \setminus K_{n_0} \subset \bigcup_{m=1}^{n_0-1} \bigcup_{n=m}^{n_0-1} E_{m,n}.$$

As each $E_{m,n}$ is a finite set, $K \setminus V$ is finite and it guarantees that x_0 is the unique possible limit point of K in X .

Let us prove that $h_{top}(x_0) = h_{top}(K, \mathbb{S})$. To do that fix $m \in \mathbb{N}$. For $n \geq m$, take $\underline{g} \in G_{k_{n,m}}^*$, and notice that $E_{\underline{g},(m,n)}$ is a $(\underline{g}, \varepsilon_m)$ -separated set in K . It gives

$$\frac{1}{p^{k_{n,m}}} \sum_{\underline{g} \in G_{k_{n,m}}^*} s(K, \underline{g}, \varepsilon_m) \geq \frac{1}{p^{k_{n,m}}} \sum_{\underline{g} \in G_{k_{n,m}}^*} |E_{\underline{g},(m,n)}| \geq e^{k_{n,m}(h_{top}(x_0) - \frac{1}{m})}.$$

In such case we obtain

$$\begin{aligned}
h_{top}(K, \mathbb{S}) &\geq S(K, \mathbb{S}, \varepsilon_m) = \limsup_{k \rightarrow \infty} \frac{\log S_k(K, \mathbb{S}, \varepsilon_m)}{k} \\
&\geq \limsup_{n \rightarrow \infty} \frac{\log S_{k_{n,m}}(K, \mathbb{S}, \varepsilon_m)}{k_{n,m}} \\
&\geq \limsup_{n \rightarrow \infty} \frac{\log e^{k_{n,m}(h_{top}(x_0) - \frac{1}{m})}}{k_{n,m}} \\
&= h_{top}(x_0) - \frac{1}{m}.
\end{aligned}$$

As the equality holds for all $m \in \mathbb{N}$ we obtain $h_{top}(K, \mathbb{S}) \geq h_{top}(x_0)$. By (i) we get the desired equality in (ii). \square

Let us proceed to the proof of Theorem D. For any $\varepsilon > 0$ and any $\varepsilon > 0$ we consider $B_\varepsilon(x)$ the open ball of radius $\varepsilon > 0$ and center x . As X is compact, by Theorem C, there exists $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that

$$\lim_{n \rightarrow \infty} x_n = x_0 \text{ and } \lim_{n \rightarrow \infty} h_{top}(x_n) = h_{top}(X, \mathbb{S}).$$

Let $\{r_n\}_{n \in \mathbb{N}}$ be any given sequence of positive real numbers which converges to 0. Applying Proposition 3.8, given $n \in \mathbb{N}$ it is possible to take a countable closed subset K_n such that $h_{top}(K_n, \mathbb{S}) = h_{top}(x_n)$ and x_n is its unique limit point in X . Moreover, $K_n \setminus B_{r_n}(x_n)$ is a finite subset and, under such observation, without loss of generality, we assume $K_n \subset B_{r_n}(x_n)$.

Define $K = \{x_0\} \cup \bigcup_{n \in \mathbb{N}} K_n$. As each K_n is a closed countable set, K is a countable closed subset of X and the set of limit points of K in X is just $\{x_0\} \cup \{x_n : n \in \mathbb{N}\}$, as $x_n \rightarrow x_0$ and $r_n \rightarrow 0$ when $n \rightarrow \infty$. Finally

$$h_{top}(K, \mathbb{S}) \geq h_{top}(K_n, \mathbb{S}) = h_{top}(x_n) \Rightarrow h_{top}(K, \mathbb{S}) \geq \lim_{n \rightarrow \infty} h_{top}(x_n) = h_{top}(X, \mathbb{S}),$$

which ends the proof by taking K as the desired set, since $h_{top}(K, \mathbb{S}) \leq h_{top}(X, \mathbb{S})$.

4. EXAMPLES

Our first example consider a free semigroup action where the generating set is given by expanding maps.

Example 4.1. We say that a C^1 -local diffeomorphism $f : M \rightarrow M$ on a compact Riemannian manifold is an *expanding map* if there are constants $C > 0$ and $0 < \lambda < 1$ such that $\|(Df^n(x))^{-1}\| \leq C\lambda^n$ for every $n \geq 1$ and $x \in X$. In [16] the authors proved that if $G_1 = \{g_1, g_2, \dots, g_k\}$ is a finite set of expanding maps acting on M and G is the generated semigroup by G_1 then every point $x \in M$ is a full entropy point for the free semigroup action \mathbb{S} .

The next example shows that it is possible to get a free semigroup action where the fixed generating set is not given by expanding maps, but the set of full entropy points is still the hole phase space.

Example 4.2. For any $\beta > 0$, consider the interval map $f_\beta : [0, 1] \rightarrow [0, 1]$ given by

$$f_\beta(x) = \begin{cases} x(1 + (2x)^\beta) & , \text{if } x \in [0, \frac{1}{2}] \\ 2x - 1 & , \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

also known as Maneville-Pomeau map. Although f_β is not continuous it induces a continuous and topologically mixing circle map \tilde{f}_β taking $\mathbb{S}^1 = [0, 1]/\sim$ with the identification $0 \sim 1$. Let G be the semigroup generated by $G_1 = \{id, \tilde{f}_\beta, R_\alpha\}$

where R_α is the rotation of angle α . Clearly no element of G_1 is an expanding map. Again, by [16] we have that every $x \in \mathbb{S}^1$ is a full entropy point.

In the two previous examples we have that the set of entropy points is the hole phase space. In the next we present a semigroup action where the set of entropy points is different of the ambient space.

Example 4.3. Let X_1 and X_2 be compact metric spaces. For $i \in \{1, 2\}$ take $f_i : X_i \rightarrow X_i$ and $g_i : X_i \rightarrow X_i$, $i \in \{1, 2\}$, continuous maps. Then define $X = X_1 \cup X_2$ and

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X_1, \\ f_2(x), & \text{if } x \in X_2 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} g_1(x), & \text{if } x \in X_1, \\ g_2(x), & \text{if } x \in X_2 \end{cases}.$$

If $f_2 = g_2 = id_{X_2}$ we have that the topological entropy of the semigroup action generated by $G_1 = \{id_X, f, g\}$ coincides with the topological entropy of the semigroup generated by $H_1 = \{id_{X_1}, f_1, g_1\}$. In particular, $E_p(X, \mathbb{S}) \subset X_1$.

In Proposition 3.5 we see that if the topological entropy of a closed subset is positive then this set contains an entropy point. In what follows we are going to show that the converse, in general, is not true.

Example 4.4. Consider

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Define

$$A = \begin{pmatrix} C & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} I & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

matrices in $M_5(\mathbb{R})$. Then we have that $AB = BA$. These matrices induces non transitive linear endomorphisms on the torus $\mathbb{T}^5 = \mathbb{R}^5/\mathbb{Z}^5$. Moreover, since for any $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{T}^5$ and $m, n \in \mathbb{N}$

$$A^n B^m(x) = (C^n(x_1, x_2), C^m(x_3, x_4), 0),$$

the action given by the semigroup generated by $G_1 = \{id_{\mathbb{T}^5}, A, B\}$ does not admit a point with dense orbit. Another important consequence of the last inequality is that $h_{top}(\pi(\{(0, 0, 0, 0, x) : x \in \mathbb{R}\}, \mathbb{S})) = 0$, where $\pi : \mathbb{R}^5 \rightarrow \mathbb{R}^5/\mathbb{Z}^5$ is the canonical projection. By the other hand, given $z \in \pi(\{(0, 0, 0, 0, x) : x \in \mathbb{R}\})$, we have that $h_{top}(z) > 0$, i.e., $z \in E_p(\mathbb{T}^5, \mathbb{S})$.

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DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL, BRAZIL.
 Email address: fagnerbernardini@gmail.com

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL, BRAZIL.
 Email address: jacobus.math@gmail.com

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL, BRAZIL.
 Email address: marcus423@gmail.com