ARITHMETIC OF SOME REAL TRIQUADRATIC FIELDS; UNITS AND 2-CLASS GROUPS

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ABSTRACT. In this paper, we compute the unit groups and the 2-class numbers of the Fröhlich's triquadratic fields $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$, where p and q are two prime numbers such that ($p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 4$) or ($p \equiv 5$ or $3 \pmod 8$ and $q \equiv 3 \pmod 4$). Furthermore, we determine some families of the fields $\mathbb{K}$ whose 2-class groups are trivial or cyclic non trivial, and some other families with 2-class groups isomorphic to the Klein group.

1. Introduction

A Fröhlich multiquadratic field of degree 2^n is a real multiquadratic field of the form $F_n = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, ..., \sqrt{p_n})$, where the p_i's are prime numbers. These fields are of major interest in class field theory and genus theory of quadratic and biquadratic fields. Their study has a long history, and here we shall quote some works which are related to the subject of this paper. In the best of our knowledge, when $n \geq 3$, all the facts that we have about the class groups of these fields concern the cyclicity of their 2-class groups and the parity of their class numbers. For example, in [10], Fröhlich showed that if more than four finite primes are ramified in a finite extension $K/\mathbb{Q}$, then the class number of K is even and therefore F_n, with $n \geq 5$ has an even class number. The parity of the class number of the quadratic field (i.e. F_1) can be determined using genus theory. The biquadratic field (i.e. F_2) was studied by Fröhlich [10], Conner and Hurrelbrink [9] and Kučera [13]. The parity of the class numbers of Fröhlich fields of degree 8 (i.e. F_3) was studied by Bulant [6] who used the method of Kučera which is based on circular units. Furthermore, the authors of [15] determined a list of the fields F_3 with $p_i \equiv 3 \pmod 4$ whose 2-class groups are cyclic non trivial. Finally, the parity of the class number of F_4, was investigated in [14]. We believe that after this list of interesting works, it is time to go further and discover more and different arithmetical properties of these fields.

In the present paper, we provide the unit groups and the 2-class numbers of the Fröhlich field $F_3 := \mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$, where p and q are two prime numbers such that ($p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 4$) or ($p \equiv 5$ or $3 \pmod 8$ and $q \equiv 3 \pmod 4$). Furthermore, we shall give some families of $\mathbb{K}$ with 2-class

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groups of type $(2,2)$. Note that the reason behind choosing this form comes from our expertise and previous studies which showed the importance of these fields in the study of many problems of class field theory and genus theory related to biquadratic and triquadratic fields [8, 7]. Note also that the fields $\mathbb{K}$ represent the first step of the cyclotomic $\mathbb{Z}_2$-extension of the fields $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ and our results may also be very useful for studying some problems related to Iwasawa theory on biquadratic and triquadratic fields (see [7, Theorem 3.6] for a direct example of such applications).

The plan of this paper is as follows; In Sec. 2, we collect some preliminary results which we shall use later. In Sec. 3, we provide unit groups and 2-class numbers of the Fröhlich fields $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Therein we give some families whose 2-class groups are trivial or cyclic non trivial. In the last section, we provide some families of Fröhlich fields whose 2-class groups are of type $(2,2)$.

Notations

Let k be a number field. We shall use the following notations for the rest of this paper:

- ⋆ $h_2(k)$: The 2-class number of k,
- ⋆ E_k: The unit group of k,
- ⋆ $q(k) = (E_k : \prod_i E_{k_i})$ is the unit index of k, if k is multiquadratic, where k_i are the quadratic subfields of k,
- ⋆ $h(k)$: The class number of k,
- ⋆ $h_2(d)$: The 2-class number of a quadratic field $\mathbb{Q}(\sqrt{d})$,
- ⋆ ε_d: The fundamental unit of a real quadratic field $\mathbb{Q}(\sqrt{d})$,
- ⋆ $N(\varepsilon_d)$: The norm of ε_d,
- ⋆ τ_i: Defined in Page 3,
- ⋆ k_i: Defined in Page 3,
- ⋆ u: Defined in Lemma 2.2.

2. **Preparations**

Let us start this section by recalling the method given in [16], that describes a fundamental system of units of a real multiquadratic field K_0. Let σ_1 and σ_2 be two distinct elements of order 2 of the Galois group of $K_0/\mathbb{Q}$. Let K_1, K_2 and K_3 be the three subextensions of K_0 invariant by σ_1, σ_2 and $\sigma_3 = \sigma_1\sigma_2$, respectively. Let ε denote a unit of K_0. Then

$$\varepsilon^2 = \varepsilon\varepsilon^{\sigma_1}\varepsilon\varepsilon^{\sigma_2}(\varepsilon^{\sigma_1}\varepsilon^{\sigma_2})^{-1},$$

and we have, $\varepsilon\varepsilon^{\sigma_1} \in E_{K_1}$, $\varepsilon\varepsilon^{\sigma_2} \in E_{K_2}$ and $\varepsilon^{\sigma_1}\varepsilon^{\sigma_2} \in E_{K_3}$. It follows that the unit group of K_0 is generated by the elements of E_{K_1}, E_{K_2} and E_{K_3}, and the square roots of elements of $E_{K_1}E_{K_2}E_{K_3}$ which are perfect squares in K_0.

This method is very useful for computing a fundamental system of units of a real biquadratic number field, however, in the case of a real triquadratic number field the problem of the determination of the unit group becomes very difficult and demands some specific computations and eliminations, as what we will see in the next section. We shall consider the field $\mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$, where p and q are two distinct prime numbers. Thus, we have the following diagram:

$$\begin{tikzcd}
 & \mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q}) & \\
k_1=\mathbb{Q}(\sqrt{2},\sqrt{p}) \arrow[ur] & k_2=\mathbb{Q}(\sqrt{2},\sqrt{q}) \arrow[u] & k_3=\mathbb{Q}(\sqrt{2},\sqrt{pq}) \arrow[ul] \\
 & \mathbb{Q}(\sqrt{2}) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

FIGURE 1. Intermediate fields of $\mathbb{K}/\mathbb{Q}(\sqrt{2})$

Let τ_1, τ_2 and τ_3 be the elements of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ defined by

$$\begin{array}{lll}
\tau_1(\sqrt{2})=-\sqrt{2}, & \tau_1(\sqrt{p})=\sqrt{p}, & \tau_1(\sqrt{q})=\sqrt{q},\\
\tau_2(\sqrt{2})=\sqrt{2}, & \tau_2(\sqrt{p})=-\sqrt{p}, & \tau_2(\sqrt{q})=\sqrt{q},\\
\tau_3(\sqrt{2})=\sqrt{2}, & \tau_3(\sqrt{p})=\sqrt{p}, & \tau_3(\sqrt{q})=-\sqrt{q}.
\end{array}$$

Note that $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})=\langle\tau_1,\tau_2,\tau_3\rangle$ and the subfields k_1, k_2 and k_3 are fixed by $\langle\tau_3\rangle$, $\langle\tau_2\rangle$ and $\langle\tau_2\tau_3\rangle$ respectively. Therefore, a fundamental system of units of $\mathbb{K}$ consists of seven units chosen from those of k_1, k_2 and k_3, and from the square roots of the elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{K}$. We have the following lemmas:

Lemma 2.1 ([1], Lemma 5)**.** *Let $d>1$ be a square-free integer and $\varepsilon_d=x+y\sqrt{d}$, where x, y are integers or semi-integers. If $N(\varepsilon_d)=1$, then $2(x+1)$, $2(x-1)$, $2d(x+1)$ and $2d(x-1)$ are not squares in $\mathbb{Q}$.*

With the above notations, we have:

Lemma 2.2. *Let $p\equiv 1 \pmod 8$ be a prime number. Put $\varepsilon_{2p}=\beta+\alpha\sqrt{2p}$ with $\beta,\alpha\in\mathbb{Z}$. If $N(\varepsilon_{2p})=1$, then $\sqrt{\varepsilon_{2p}}=\frac{1}{\sqrt{2}}(\alpha_1+\alpha_2\sqrt{2p})$, for some integers α_1,α_2 such that $\alpha=\alpha_1\alpha_2$. It follows that:*

σ	$1+\tau_2$	$1+\tau_1\tau_2$	$1+\tau_1\tau_3$	$1+\tau_2\tau_3$	$1+\tau_1$
$\sqrt{\varepsilon_{2p}}^{\sigma}$	$(-1)^u$	$-\varepsilon_{2p}$	$(-1)^{u+1}$	$(-1)^u$	$(-1)^{u+1}$

(1)

for some u in $\{0,1\}$ such that $\frac{1}{2}(\alpha_1^2-2p\alpha_2^2)=(-1)^u$.

Proof. The reader can check it easily. $\square$

Lemma 2.3 ([3], Theorem 6)**.** *Let $p\equiv 1 \pmod 4$ be a prime number. We have*

1. *If* $N(\varepsilon_{2p}) = -1$, *then* $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ *is a fundamental system of units of* $k_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p})$.
2. *If* $N(\varepsilon_{2p}) = 1$, *then* $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2p}}\}$ *is a fundamental system of units of* $k_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p})$.

Lemma 2.4. *Let* $p \equiv 1 \pmod 8$ *and* $q \equiv 3 \pmod 8$ *be two primes such that* $\left(\dfrac{p}{q}\right) = -1$.

1. *Let* x *and* y *be two integers such that* $\varepsilon_{2pq} = x + y\sqrt{2pq}$. *Then we have*
   i. $(x-1)$ *is a square in* $\mathbb{N}$,
   ii. $\sqrt{2\varepsilon_{2pq}} = y_1 + y_2\sqrt{2pq}$ *and* $2 = -y_1^2 + 2pqy_2^2$, *for some integers* y_1 *and* y_2.
2. *Let* v *and* w *be two integers such that* $\varepsilon_{pq} = v + w\sqrt{pq}$. *Then we have*
   i. $(v-1)$ *is a square in* $\mathbb{N}$,
   ii. $\sqrt{2\varepsilon_{pq}} = w_1 + w_2\sqrt{pq}$ *and* $2 = -w_1^2 + pqw_2^2$, *for some integers* w_1 *and* w_2.

Proof. It is known that $N(\varepsilon_{2pq}) = 1$. Then, by the unique factorization in $\mathbb{Z}$ and Lemma 2.1 there exist some integers y_1 and y_2 ($y = y_1y_2$) such that

$$(1): \begin{cases} x \pm 1 = y_1^2 \\ x \mp 1 = 2pqy_2^2, \end{cases} \qquad (2): \begin{cases} x \pm 1 = py_1^2 \\ x \mp 1 = 2qy_2^2, \end{cases} \qquad \text{or} \qquad (3): \begin{cases} x \pm 1 = 2py_1^2 \\ x \mp 1 = qy_2^2, \end{cases}$$

$*$ System (2) can not occur since it implies $-1 = \left(\frac{2qy_1^2}{p}\right) = \left(\frac{x\mp 1}{p}\right) = \left(\frac{x\pm 1 \mp 2}{p}\right) = \left(\frac{\mp 2}{p}\right) = \left(\frac{2}{p}\right) = 1$, which is absurd.

$*$ We similarly show that System (3) and $\begin{cases} x+1 = y_1^2 \\ x-1 = 2pqy_2^2 \end{cases}$ can not occur.

Therefore $\begin{cases} x-1 = y_1^2 \\ x+1 = 2pqy_2^2 \end{cases}$ which gives the first item. The proof of the second item is analogous. □

Lemma 2.5. *Let* $q \equiv 3 \pmod 8$ *be a prime number. We have*

1. *Let* c *and* d *be two integers such that* $\varepsilon_{2q} = c + d\sqrt{2q}$. *Then we have*
   i. $c - 1$ *is a square in* $\mathbb{N}$,
   ii. $\sqrt{2\varepsilon_{2q}} = d_1 + d_2\sqrt{2q}$ *and* $2 = -d_1^2 + 2qd_2^2$, *for some integers* d_1 *and* d_2.
2. *Let* α *and* β *be two integers such that* $\varepsilon_q = \alpha + \beta\sqrt{q}$. *Then we have*
   i. $\alpha - 1$ *is a square in* $\mathbb{N}$,
   ii. $\sqrt{2\varepsilon_q} = \beta_1 + \beta_2\sqrt{q}$ *and* $2 = -\beta_1^2 + q\beta_2^2$, *for some integers* β_1 *and* β_2.

Furthermore, for any prime number $p \equiv 1 \pmod 4$ *we have:*

ε	ε_2	ε_p	$\sqrt{\varepsilon_q}$	$\sqrt{\varepsilon_{2q}}$
$\varepsilon^{1+\tau_1}$	-1	ε_p^2	$-\varepsilon_q$	1
$\varepsilon^{1+\tau_2}$	ε_2^2	-1	ε_q	ε_{2q}
$\varepsilon^{1+\tau_3}$	ε_2^2	ε_p^2	-1	-1
$\varepsilon^{1+\tau_1\tau_2}$	-1	-1	$-\varepsilon_q$	1
$\varepsilon^{1+\tau_1\tau_3}$	-1	ε_p^2	1	$-\varepsilon_{2q}$
$\varepsilon^{1+\tau_2\tau_3}$	ε_2^2	-1	-1	-1

TABLE 1. Norm maps on units

Proof. Similar to that of Lemma 2.4. □

Lemma 2.6. *Let $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$ be two primes such that* $\left(\dfrac{p}{q}\right) = 1$.

1. *Let x and y be two integers such that $\varepsilon_{2pq} = x + y\sqrt{2pq}$. Then*
   i. *$(x-1)$, $p(x-1)$ or $2p(x+1)$ is a square in $\mathbb{N}$,*
   ii. *Furthermore, we have*
      a) *If $(x-1)$, then $\sqrt{2\varepsilon_{2pq}} = y_1 + y_2\sqrt{2pq}$ and $2 = -y_1^2 + 2pqy_2^2$.*
      b) *If $p(x-1)$, then $\sqrt{2\varepsilon_{2pq}} = y_1\sqrt{p} + y_2\sqrt{2q}$ and $2 = -py_1^2 + 2qy_2^2$.*
      c) *If $2p(x+1)$, then $\sqrt{2\varepsilon_{2pq}} = y_1\sqrt{2p} + y_2\sqrt{q}$ and $2 = 2py_1^2 - qy_2^2$.*

   Where y_1 and y_2 are two integers such that $y = y_1y_2$.
2. *Let v and w be two integers such that $\varepsilon_{pq} = v + w\sqrt{pq}$. Then we have*
   i. *$(v-1)$, $p(v-1)$ or $2p(v+1)$ is a square in $\mathbb{N}$,*
   ii. *Furthermore, we have*
      a) *If $(v-1)$, then $\sqrt{2\varepsilon_{pq}} = w_1 + w_2\sqrt{pq}$ and $2 = -w_1^2 + pqw_2^2$.*
      b) *If $p(v-1)$, then $\sqrt{2\varepsilon_{pq}} = w_1\sqrt{p} + w_2\sqrt{q}$ and $2 = -pw_1^2 + qw_2^2$.*
      c) *If $2p(v+1)$, then $\sqrt{\varepsilon_{pq}} = w_1\sqrt{p} + w_2\sqrt{q}$ and $1 = pw_1^2 - qw_2^2$.*

   Where w_1 and w_2 are two integers such that $w = w_1w_2$ in a) and b), and $w = 2w_1w_2$ in c).

Proof. We proceed as in the proof of 2.4. □

Now we recall the following lemmas:

Lemma 2.7 ([12])**.** *Let K be a multiquadratic number field of degree 2^n, $n \in \mathbb{N}$, and k_i the $s = 2^n - 1$ quadratic subfields of K. Then*

$$h(K) = \frac{1}{2^v}(E_K : \prod_{i=1}^{s} E_{k_i}) \prod_{i=1}^{s} h(k_i),$$

with

$$v = \begin{cases} n(2^{n-1}-1); & \text{if } K \text{ is real,} \\ (n-1)(2^{n-2}-1) + 2^{n-1} - 1 & \text{if } K \text{ is imaginary.} \end{cases}$$

Lemma 2.8. *Let $q \equiv 3 \pmod 4$ and $p \equiv 1 \pmod 4$ be two distinct primes. Then*

1. *By* [9, Corollary 18.4], *we have $h_2(p) = h_2(q) = h_2(2q) = h_2(2) = h_2(-2) = h_2(-q) = h_2(-1) = 1$.*
2. *If $\left(\dfrac{p}{q}\right) = -1$, then $h_2(pq) = h_2(2pq) = h_2(-pq) = 2$, else $h_2(pq)$, $h_2(2pq)$ and $h_2(-pq)$ are divisible by 4 (cf.* [9, Corollaries 19.6 and 19.7]*).*
3. *If $q \equiv 3 \pmod 8$, then $h_2(-2q) = 2$ (cf.* [9, Corollary 19.6]*).*

3. Unit groups of real triquadratic number fields and their 2-class numbers

Keep the notations in the above section. In this section we shall compute the unit groups and the 2-class numbers of the Fröhlich fields $\mathbb{K}$.

3.1. The case: $p \equiv 1 \pmod 8$, $q \equiv 3 \pmod 4$ and $\left(\dfrac{p}{q}\right) = -1$.

We firstly need to state the next two lemmas that will be very useful to prove our first main result:

Theorem 3.1. *Let $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$ be two primes such that $\left(\dfrac{p}{q}\right) = -1$. Put $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Then*

1. *If $N(\varepsilon_{2p}) = -1$, we have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}, \sqrt{\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}}\rangle$$
   - *The 2-class group of $\mathbb{K}$ is cyclic of order $\frac{1}{2}h_2(2p)$.*
2. *If $N(\varepsilon_{2p}) = 1$, we have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}}, \sqrt{\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}}\rangle,$$
   where $a \in \{0, 1\}$ such that $a \equiv 1 + u \pmod 2$.
   - *The 2-class group of $\mathbb{K}$ is cyclic of order $h_2(2p)$.*

Proof. We shall use the preparations exposed in Page 3. Therefore, we need the unit groups of the intermediate fields k_1, k_2 and k_3.

1. Assume that $N(\varepsilon_{2p}) = -1$. By Lemma 2.3, $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. One can easily deduce from Lemmas 2.5 and 2.4 that $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,
$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$.
Remark that the question now becomes about the solvability in $\mathbb{K}$ of the equation: $\xi^2 - \varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g = 0$. Assuming that this equation has solutions in $\mathbb{K}$, we shall firstly use norm maps from $\mathbb{K}$ to its subextensions to eliminate the forms with do not occur.
➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2} = 1+\tau_2$. Using Lemma 2.4 we get $\sqrt{\varepsilon_{2pq}}^{1+\tau_2} = (\frac{1}{\sqrt{2}}(y_1+y_2\sqrt{2pq}))\times\tau_2(\frac{1}{\sqrt{2}}(y_1+y_2\sqrt{2pq})) = (\frac{1}{\sqrt{2}}(y_1+y_2\sqrt{2pq}))\times(\frac{1}{\sqrt{2}}(y_1-y_2\sqrt{2pq})) = \frac{1}{2}(y_1^2-2pqy_2) = \frac{1}{2}(-2) = -1$. Similarly we have $\sqrt{\varepsilon_{pq}}^{1+\tau_2} = -1$. So by Table 1, we have:

$$\begin{aligned}N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot(-1)^e\cdot(-1)^f\cdot(-1)^{gs}\varepsilon_2^g\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+e+f+gs}\varepsilon_2^g.\end{aligned}$$

for some $s\in\{0,1\}$. Thus, $b+e+f+gs\equiv 0 \pmod 2$. Since ε_2 is not a square in k_2, then $g=0$. Therefore $b+e+f\equiv 0 \pmod 2$ and

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_5} = 1+\tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. We have $\sqrt{\varepsilon_{pq}}^{1+\tau_1\tau_2} = 1$ and $\sqrt{\varepsilon_{2pq}}^{1+\tau_1\tau_2} = -\varepsilon_{2pq}$. Then, by Table 1, we get:

$$\begin{aligned}N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot(-1)^c\cdot\varepsilon_q^c\cdot 1\cdot 1\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\ &= (-1)^{a+b+c+f}\varepsilon_q^c\cdot\varepsilon_{2pq}^f.\end{aligned}$$

Thus $a+b+c+f=0 \pmod 2$. By Lemmas 2.5 and 2.4, none of ε_q and ε_{2pq} is a square in k_5. Then $f=c$. Thus, $a=b$. Therefore,

$$\xi^2 = \varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}^f\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f,$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6} = 1+\tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. We have $\sqrt{\varepsilon_{pq}}^{1+\tau_1\tau_3} = 1$ and $\sqrt{\varepsilon_{2pq}}^{1+\tau_1\tau_3} = -\varepsilon_{2pq}$. Then, by Table 1, we get:

$$\begin{aligned}N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2a}\cdot 1\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\ &= \varepsilon_p^{2a}(-1)^{a+d+f}\varepsilon_{2q}^d\varepsilon_{2pq}^f.\end{aligned}$$

Thus $a+d+f=0 \pmod 2$. Again by Lemmas 2.5 and 2.4, none of ε_{2q} and ε_{2pq} is a square in k_6. Then $d=f$. Therefore $a=0$ and

$$\xi^2 = \sqrt{\varepsilon_q}^f\sqrt{\varepsilon_{2q}}^f\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_3}=1+\tau_2\tau_3$, with $k_3=\mathbb{Q}(\sqrt{2},\sqrt{pq})$. Note that $\sqrt{\varepsilon_{pq}}^{1+\tau_2\tau_3}=\varepsilon_{pq}$ and $\sqrt{\varepsilon_{2pq}}^{1+\tau_2\tau_3}=\varepsilon_{2pq}$. Then, by Table 1, we have:

$$\begin{aligned} N_{\mathbb{K}/k_3}(\xi^2) &= (-1)^f\cdot(-1)^f\cdot\varepsilon_{pq}^e\cdot\varepsilon_{2pq}^f\\ &= \varepsilon_{pq}^e\cdot\varepsilon_{2pq}^f. \end{aligned}$$

By Lemma 2.4, both ε_{pq} and ε_{2pq} are squares in k_3. So we deduce nothing.
➠ Let us apply the norm $N_{\mathbb{K}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. Note that $\sqrt{\varepsilon_{pq}}^{1+\tau_1}=-\varepsilon_{pq}$ and $\sqrt{\varepsilon_{2pq}}^{1+\tau_1}=1$. Then, by Table 1, we have:

$$\begin{aligned} N_{\mathbb{K}/k_4}(\xi^2) &= (-1)^f\cdot\varepsilon_q^f\cdot 1\cdot(-1)^e\cdot\varepsilon_{pq}^e\cdot 1\\ &= (-1)^{f+e}\varepsilon_q^f\varepsilon_{pq}^e. \end{aligned}$$

Thus, $f=e$ and

$$\xi^2=\sqrt{\varepsilon_q}^f\sqrt{\varepsilon_{2q}}^f\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_{2pq}}^f.$$

Let us show that the square root of $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ is an element of $\mathbb{K}$. Note that one can easily check that the 2-class group of $k_5=\mathbb{Q}(\sqrt{2p},\sqrt{q})$ is cyclic and by Lemmas 2.7 and 2.8, we have $h_2(k_5)=\frac{1}{4}q(k_5)h_2(2p)h_2(q)h_2(2pq)=\frac{1}{2}q(k_5)h_2(2p)$. Using Lemmas 2.4 and 2.5, we show that $q(k_5)=2$. Thus $h_2(k_5)=h_2(2p)$. Since $\mathbb{K}/k_5$ is an unramified quadratic extension, then

$$h_2(\mathbb{K})=\frac{1}{2}\cdot h_2(k_5)=\frac{1}{2}\cdot h_2(2p). \tag{2}$$

Assume by absurd that $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ is not a square in $\mathbb{K}$. Then $q(\mathbb{K})=2^5$. By Lemma 2.7, we have:

$$\begin{aligned} h_2(\mathbb{K}) &= \frac{1}{2^9}q(\mathbb{K})h_2(2)h_2(p)h_2(q)h_2(2p)h_2(2q)h_2(pq)h_2(2pq) \qquad (3)\\ &= \frac{1}{2^9}\cdot 2^5\cdot 1\cdot 1\cdot 1\cdot h_2(2p)\cdot 1\cdot 2\cdot 2=\frac{1}{4}\cdot h_2(2p). \end{aligned}$$

Which is a contradiction with (2). Therefore $f=1$ and $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ is a square in $\mathbb{K}$. Hence, we have

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt{\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}}\rangle.$$

2. Let us now prove the second item. Assume that $N(\varepsilon_{2p})=1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_{2p}}\}$ is a fundamental system of units of k_1 and one can easily deduce from Lemmas 2.5 and 2.4 that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. So we have

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}}\rangle.$$

Put

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$. Assume that ξ belongs to $\mathbb{K}$. We shall proceed as above, by using the norm maps from $\mathbb{K}$ to its subextensions. Note that these norms are already computed in the proof of the first item, and we shall use (1) for the norms of $\sqrt{\varepsilon_{2p}}$. Let u be the integer defined in Lemma 2.2.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2} = 1 + \tau_2$. We have

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a} \cdot (-1)^b \cdot \varepsilon_q^c \cdot \varepsilon_{2q}^d \cdot (-1)^e \cdot (-1)^f \cdot (-1)^{gu} \\ &= \varepsilon_2^{2a} \varepsilon_q^c \varepsilon_{2q}^d \cdot (-1)^{b+e+f+gu}. \end{aligned}$$

Thus, $b + e + f + gu \equiv 0 \pmod 2$.

➠ Let us apply the norm map $N_{\mathbb{K}/k_5} = 1 + \tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a \cdot (-1)^b \cdot (-1)^c \cdot \varepsilon_q^c \cdot 1 \cdot 1 \cdot (-1)^f \cdot \varepsilon_{2pq}^f \cdot (-1)^g \cdot \varepsilon_{2p}^g \\ &= (-1)^{a+b+c+f+g} \cdot \varepsilon_q^c \varepsilon_{2pq}^f \varepsilon_{2p}^g. \end{aligned}$$

Thus, $a + b + c + f + g \equiv 0 \pmod 2$ and $c + f + g \equiv 0 \pmod 2$. Therefore, $a = b$ and

$$\xi^2 = \varepsilon_2^a \varepsilon_p^a \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^e \sqrt{\varepsilon_{2pq}}^f \sqrt{\varepsilon_{2p}}^g.$$

with $c + f + g \equiv 0 \pmod 2$.

➠ Let us apply the norm map $N_{\mathbb{K}/k_6} = 1 + \tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a \cdot \varepsilon_p^{2a} \cdot 1 \cdot (-1)^d \cdot \varepsilon_{2q}^d \cdot 1 \cdot (-1)^f \cdot \varepsilon_{2pq}^f \cdot (-1)^{gu} \cdot (-1)^g \\ &= \varepsilon_p^{2a} \cdot (-1)^{a+d+f+ug+g} \cdot \varepsilon_{2q}^d \varepsilon_{2pq}^f. \end{aligned}$$

Thus, $a + d + f + ug + g \equiv 0 \pmod 2$ and $d = f$. Therefore, $a + ug + g \equiv 0 \pmod 2$ and

$$\xi^2 = \varepsilon_2^a \varepsilon_p^a \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^f \sqrt{\varepsilon_{pq}}^e \sqrt{\varepsilon_{2pq}}^f \sqrt{\varepsilon_{2p}}^g.$$

➠ Let us apply the norm map $N_{\mathbb{K}/k_3} = 1 + \tau_2\tau_3$, with $k_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_3}(\xi^2) &= \varepsilon_2^{2a} \cdot (-1)^a \cdot (-1)^c \cdot (-1)^f \cdot \varepsilon_{pq}^e \cdot \varepsilon_{2pq}^f \cdot (-1)^{gu} \\ &= \varepsilon_2^{2a} \varepsilon_{pq}^e \varepsilon_{2pq}^f \cdot (-1)^{a+c+f+ug}. \end{aligned}$$

Thus, $a + c + f + ug \equiv 0 \pmod 2$. Therefore, from these discussions, it follows that we have:

$$a + e + f + ug \equiv 0 \pmod 2 \tag{4}$$
$$c + f + g \equiv 0 \pmod 2 \tag{5}$$
$$a + ug + g \equiv 0 \pmod 2 \tag{6}$$
$$a + c + f + ug \equiv 0 \pmod 2 \tag{7}$$

From (4), (6) and (5), we deduce that $e=c$. Thus

$$\xi^2=\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^f\sqrt{\varepsilon_{pq}}^c\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g.$$

On the other hand, as above, we show that the 2-class group of k_5 is cyclic and that we have:

$$\begin{aligned} h_2(\mathbb{K})=\frac{1}{2}\cdot h_2(k_5) &= \frac{1}{2}\cdot\frac{1}{4}q(k_5)h_2(2p)h_2(q)h_2(2pq)\\ &= \frac{1}{2}\cdot\frac{1}{4}\cdot 4\cdot h_2(2p)\cdot 1\cdot 2\\ &= h_2(2p), \end{aligned} \tag{8}$$

and by class number formula (Lemma 2.7), we have

$$\begin{aligned} h_2(\mathbb{K}) &= \frac{1}{2^9}q(\mathbb{K})h_2(2)h_2(p)h_2(q)h_2(2p)h_2(2q)h_2(pq)h_2(2pq)\\ &= \frac{1}{2^9}\cdot q(\mathbb{K})\cdot 1\cdot 1\cdot 1\cdot h_2(2p)\cdot 1\cdot 2\cdot 2=\frac{1}{2^7}\cdot q(\mathbb{K})\cdot h_2(2p). \end{aligned}$$

Therefore, $q(\mathbb{K})=2^7$.

Assume that each solution has $g=0$, then by (6) $a=0$. So by (5) and (4) $f=c=e$. Therefore, $\xi^2=\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^c\sqrt{\varepsilon_{pq}}^c\sqrt{\varepsilon_{2pq}}^c$. Thus, $q(\mathbb{K})=2^5$ or 2^6. Which is absurd. This implies that there must be a solution having $g=1$. So by (5), $c\neq f$, and by (6) $a\equiv 1+u \pmod 2$. Finally, we have

$$\xi^2=\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{pq}}^c\sqrt{\varepsilon_{2p}} \text{ or } \varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}^f\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}},$$

Since $q(\mathbb{K})=2^7$, then both of $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}$ and $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ are squares in $\mathbb{K}$, where $a\equiv 1+u \pmod 2$ and u is defined in Lemma 2.2. Which completes the proof.

□

3.2. The case: $p\equiv 1 \pmod 8$, $q\equiv 3 \pmod 8$ and $\left(\frac{p}{q}\right)=1$.

For the sake that the reader could follow the proofs of this section, we suggest to start by reading carefully the proof of Theorem 3.1 which is exposed some helpful details.

The following table summarizes very useful computations which we shall use frequently.

ε	Conditions	$\varepsilon^{1+\tau_2}$	$\varepsilon^{1+\tau_1\tau_2}$	$\varepsilon^{1+\tau_1\tau_3}$	$\varepsilon^{1+\tau_2\tau_3}$	$\varepsilon^{1+\tau_1}$
$\sqrt{\varepsilon_{2pq}}$	$(x-1)$ is a square in $\mathbb{N}$	-1	$-\varepsilon_{2pq}$	$-\varepsilon_{2pq}$	ε_{2pq}	1
	$p(x-1)$ is a square in $\mathbb{N}$	1	ε_{2pq}	$-\varepsilon_{2pq}$	$-\varepsilon_{2pq}$	1
	$2p(x+1)$ is a square in $\mathbb{N}$	-1	$-\varepsilon_{2pq}$	ε_{2pq}	$-\varepsilon_{2pq}$	1
$\sqrt{\varepsilon_{pq}}$	$(v-1)$ is a square in $\mathbb{N}$	-1	1	1	ε_{pq}	$-\varepsilon_{pq}$
	$p(v-1)$ is a square in $\mathbb{N}$	1	-1	1	$-\varepsilon_{pq}$	$-\varepsilon_{pq}$
	$2p(v+1)$ is a square in $\mathbb{N}$	-1	-1	1	$-\varepsilon_{pq}$	ε_{pq}

TABLE 2. Norms maps on units

Let $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right) = 1$. Then, by Lemmas 2.7 and 2.7, we have:

$$h_2(\mathbb{K}) = \frac{1}{2^9} q(\mathbb{K}) \cdot h_2(2p) \cdot h_2(pq) \cdot h_2(2pq). \tag{9}$$

Remark 3.2. Notice that by Lemma 2.6 there are nine possibilities which will be covered case by case by the following Theorems 3.3-3.11.

Theorem 3.3. *Let $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right) = 1$. Put $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Assume furthermore that $x-1$ and $v-1$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 2.6.*

1. *If $N(\varepsilon_{2p}) = -1$, we have*
   - *The unit group of $\mathbb{K}$ is :*

     $E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}, \sqrt{\sqrt{\varepsilon_q}^a \sqrt{\varepsilon_{2q}}^a \sqrt{\varepsilon_{pq}}^a \sqrt{\varepsilon_{2pq}}^{1+b}} \rangle$

     where $a, b \in \{0, 1\}$ such that $a \neq b$ and $a = 1$ if and only if $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ is a square in $\mathbb{K}$.
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^{4-a}} h_2(2p) h_2(pq) h_2(2pq)$.*

2. *Assume that* $N(\varepsilon_{2p}) = 1$ *and define* $a \in \{0, 1\}$ *to satisfy* $a \equiv 1 + u \pmod 2$. *Then we have*
   - *The unit group of* $\mathbb{K}$ *is :*

$$E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2^{ar'}\varepsilon_p^{ar'}\sqrt{\varepsilon_q}^{r'}\sqrt{\varepsilon_{pq}}^{r'}\sqrt{\varepsilon_{2p}}^{1+s'}}, \sqrt{\varepsilon_2^{ar}\varepsilon_p^{ar}\sqrt{\varepsilon_{2q}}^{r}\sqrt{\varepsilon_{2pq}}^{1+s}\sqrt{\varepsilon_{2p}}^{r}} \rangle$$

   where $r, r', s, s' \in \{0, 1\}$ *such that* $r \neq s$ *(resp.* $r' \neq s'$*) and* $r = 1$ *(resp.* $r' = 1$*) if and only if* $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ *(resp.* $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}$*) is a square in* $\mathbb{K}$.
   - *The* 2*-class number of* $\mathbb{K}$ *equals* $\frac{1}{2^{4-r-r'}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. 1. Assume that $N(\varepsilon_{2p}) = -1$. By Lemma 2.3, $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. One can easily deduce from Lemmas 2.5 and 2.4 that $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}} \rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$. As $x - 1$ and $v - 1$ are squares in $\mathbb{N}$, then clearly with the same computations as in the proof of Theorem 3.1, we get :

$$\xi^2 = \sqrt{\varepsilon_q}^f\sqrt{\varepsilon_{2q}}^f\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_{2pq}}^f.$$

So the first item.

2. The same computations in the second part of the proof of Theorem 3.1 give the second item.

The part concerning the 2-class number follows from the above discussions and (9). ∎

Theorem 3.4. *Let* $p \equiv 1 \pmod 8$ *and* $q \equiv 3 \pmod 8$ *be two primes such that* $\left(\frac{p}{q}\right) = 1$. *Put* $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. *Assume furthermore that* $x - 1$ *and* $p(v - 1)$ *are squares in* $\mathbb{N}$, *where* x *and* v *are defined in Lemma 2.6. Then*

1. *Assume that* $N(\varepsilon_{2p}) = -1$. *We have*
   - *The unit group of* $\mathbb{K}$ *is :*

$$E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}} \rangle.$$

   - *The* 2*-class number of* $\mathbb{K}$ *equals* $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.
2. *Assume that* $N(\varepsilon_{2p}) = 1$ *and let* $a \in \{0, 1\}$ *such that* $a \equiv 1 + u \pmod 2$. *We have*

- *The unit group of* $\mathbb{K}$ *is :*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{a\alpha}\sqrt{\varepsilon_{2q}}^{\alpha}\sqrt{\varepsilon_{2pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

where $\alpha,\gamma\in\{0,1\}$ *such that* $\alpha\neq\gamma$ *and* $\alpha=1$ *if and only if* $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ *is a square in* $\mathbb{K}$.

- *The* 2*-class number of* $\mathbb{K}$ *equals* $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. We shall make use of (1) and Tables 1 and 2.

1. Assume that $N(\varepsilon_{2p})=-1$. Note that $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. So we have:

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2}=(-1)^v\varepsilon_2$, for some $v\in\{0,1\}$. Then, by Tables 1 and 2, we get:

$$\begin{aligned}N_{\mathbb{K}/k_2}(\xi^2)&=\varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gv}\varepsilon_2^g\\&=\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gv}\varepsilon_2^g.\end{aligned}$$

Thus, $b+f+gv\equiv 0\pmod 2$ and $g=0$. Therefore, $b=f$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^f\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$.Then, by Tables 1 and 2, we get:

$$\begin{aligned}N_{\mathbb{K}/k_5}(\xi^2)&=(-1)^a\cdot(-1)^f\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\&=(-1)^{a+d}\varepsilon_q^d\cdot\varepsilon_{2pq}^f.\end{aligned}$$

Thus $a=d=f$ and

$$\xi^2=\varepsilon_2^f\varepsilon_p^f\varepsilon_{pq}^c\sqrt{\varepsilon_q}^f\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By Tables 1 and 2, we get:

$$\begin{aligned}N_{\mathbb{K}/k_6}(\xi^2)&=(-1)^f\cdot\varepsilon_p^{2f}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\&=\varepsilon_p^{2a}(-1)^e\varepsilon_{2q}^e\varepsilon_{2pq}^f.\end{aligned}$$

Thus $e=f=0$. Hence

$$\xi^2=\varepsilon_{pq}^c.$$

By Lemma 2.6 ε_{pq} is a square in $\mathbb{K}$, therefore

$$E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

2. Assume that $N(\varepsilon_{2p}) = 1$. Note that $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2} = 1 + \tau_2$. By Tables 1 and 2, we have

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gu} \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gu}. \end{aligned}$$

Thus, $b + f + gu \equiv 0 \pmod 2$.

➠ Let us apply the norm map $N_{\mathbb{K}/k_5} = 1 + \tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. By Tables 1 and 2, we have

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot 1\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g \\ &= (-1)^{a+b+d+f+g}\varepsilon_{2q}^d\varepsilon_{2pq}^f\varepsilon_{2p}^g. \end{aligned}$$

Thus, $a + b + d + f + g \equiv 0 \pmod 2$ and $d + f + g \equiv 0 \pmod 2$. Therefore $a = b$ and

$$\xi^2 = \varepsilon_2^a\varepsilon_p^a\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6} = 1 + \tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. By Tables 1 and 2, we have

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2a}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^{gu+g} \\ &= \varepsilon_p^{2a}(-1)^{a+e+f+g+ug}\varepsilon_{2q}^e\varepsilon_{2pq}^f. \end{aligned}$$

Thus, $a + e + f + g + ug \equiv 0 \pmod 2$ and $e = f$. Therefore, $a + g + ug \equiv 0 \pmod 2$ and

$$\xi^2 = \varepsilon_2^a\varepsilon_p^a\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^e\sqrt{\varepsilon_{2p}}^g.$$

➠ Let us apply the norm map $N_{\mathbb{K}/k_3} = 1 + \tau_2\tau_3$, with $k_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$. By Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_3}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^a\cdot\varepsilon_{pq}^{2c}\cdot(-1)^d\cdot(-1)^e\cdot\varepsilon_{2pq}^e\cdot(-1)^{gu} \\ &= \varepsilon_2^{2a}\varepsilon_{2pq}^e\cdot(-1)^{a+d+e+gu}. \end{aligned}$$

Thus, $a + d + e + gu \equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. So, by Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_4}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2a}\cdot\varepsilon_{pq}^{2a}\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot 1\cdot(-1)^{gu+g}\\ &= \varepsilon_2^{2a}\varepsilon_{pq}^{2a}\cdot(-1)^{a+d+g+gu}\varepsilon_q^d. \end{aligned}$$

Thus $d=0$ and so $a+e+gu\equiv 0 \pmod 2$. Since $a+g+ug\equiv 0 \pmod 2$, we have $e=g$. Since ε_{pq} is a square in $\mathbb{K}$, we can disregard it in the form of ξ^2. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^e\sqrt{\varepsilon_{2p}}^e,$$

with $a+e+eu\equiv 0 \pmod 2$. So the result (cf. (9)).

□

Theorem 3.5. *Let $p\equiv 1 \pmod 8$ and $q\equiv 3 \pmod 8$ be two primes such that $\left(\dfrac{p}{q}\right)=1$. Put $\mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $x-1$ and $2p(v+1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 2.6.*

1. *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*
2. *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u \pmod 2$. We have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{a\alpha}\sqrt{\varepsilon_{2q}}^{\alpha}\sqrt{\varepsilon_{2pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$
   where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ is a square in $\mathbb{K}$.
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1. Assume that $N(\varepsilon_{2p})=-1$. Note that $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. So we have:

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. We have

$\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2} = (-1)^{v'}\varepsilon_2$, for some $v' \in \{0,1\}$. Then, by Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gv'}\varepsilon_2^g \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gv'}\varepsilon_2^g. \end{aligned}$$

Thus, $b+f+gv' \equiv 0 \pmod 2$ and $g=0$. Then $b=f$ and

$$\xi^2 = \varepsilon_2^a\varepsilon_p^f\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_5} = 1+\tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q},\sqrt{2p})$. By Tables 1 and 2, we have:

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^f\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot(-1)^f\cdot\varepsilon_{2pq}^f \\ &= (-1)^{a+d}\varepsilon_q^d\cdot\varepsilon_{2pq}^f. \end{aligned}$$

Thus $a=d=f$ and

$$\xi^2 = \varepsilon_2^f\varepsilon_p^f\varepsilon_{pq}^c\sqrt{\varepsilon_q}^f\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6} = 1+\tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p},\sqrt{2q})$. Then, by Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^f\cdot\varepsilon_p^{2f}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot\varepsilon_{2pq}^f \\ &= \varepsilon_p^{2a}(-1)^e\varepsilon_{2q}^e\varepsilon_{2pq}^f. \end{aligned}$$

Thus $e=f=0$. Hence

$$\xi^2 = \varepsilon_{pq}^c.$$

By Lemma 2.6 ε_{pq} is a square in $\mathbb{K}$, therefore

$$E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

The rest of the first item is direct from Lemma 2.7.

2. Assume that $N(\varepsilon_{2p}) = 1$. Note that $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2} = 1+\tau_2$. We have

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gu} \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gu}. \end{aligned}$$

Thus, $b+f+gu \equiv 0 \pmod 2$.

➠ Let us apply the norm map $N_{\mathbb{K}/k_5} = 1+\tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a \cdot (-1)^b \cdot 1 \cdot (-1)^d \cdot \varepsilon_{2q}^d \cdot 1 \cdot (-1)^f \cdot \varepsilon_{2pq}^f \cdot (-1)^g \cdot \varepsilon_{2p}^g \\ &= (-1)^{a+b+d+f+g} \varepsilon_{2q}^d \varepsilon_{2pq}^f \varepsilon_{2p}^g. \end{aligned}$$

Thus, $a+b+d+f+g \equiv 0 \pmod 2$ and $d+f+g \equiv 0 \pmod 2$. Therefore $a = b$ and

$$\xi^2 = \varepsilon_2^a \varepsilon_p^a \varepsilon_{pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{2pq}}^f \sqrt{\varepsilon_{2p}}^g.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6} = 1+\tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a \cdot \varepsilon_p^{2a} \cdot 1 \cdot 1 \cdot (-1)^e \cdot \varepsilon_{2q}^e \cdot (-1)^f \cdot \varepsilon_{2pq}^f \cdot (-1)^{gu+g} \\ &= \varepsilon_p^{2a} (-1)^{a+e+f+g+ug} \varepsilon_{2q}^e \varepsilon_{2pq}^f. \end{aligned}$$

Thus, $a+e+f+g+ug \equiv 0 \pmod 2$ and $e = f$. Therefore, $a+g+ug \equiv 0 \pmod 2$ and

$$\xi^2 = \varepsilon_2^a \varepsilon_p^a \varepsilon_{pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{2pq}}^e \sqrt{\varepsilon_{2p}}^g.$$

➠ Let us apply the norm map $N_{\mathbb{K}/k_3} = 1+\tau_2\tau_3$, with $k_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$. So, by Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_3}(\xi^2) &= \varepsilon_2^{2a} \cdot (-1)^a \cdot \varepsilon_{pq}^{2c} \cdot (-1)^d \cdot (-1)^e \cdot \varepsilon_{2pq}^e \cdot (-1)^{gu} \\ &= \varepsilon_2^{2a} \varepsilon_{2pq}^e \cdot (-1)^{a+d+e+gu}. \end{aligned}$$

Thus, $a+d+e+gu \equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_4} = 1+\tau_1$, with $k_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. So, by Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_4}(\xi^2) &= (-1)^a \cdot \varepsilon_p^{2a} \cdot \varepsilon_{pq}^{2a} \cdot (-1)^d \cdot \varepsilon_q^d \cdot 1 \cdot 1 \cdot (-1)^{gu+g} \\ &= \varepsilon_2^{2a} \varepsilon_{pq}^{2a} \cdot (-1)^{a+d+g+gu} \varepsilon_q^d. \end{aligned}$$

Thus $d = 0$ and so $a+e+gu \equiv 0 \pmod 2$. As $a+g+ug \equiv 0 \pmod 2$, we have $e = g$. Since ε_{pq} is a square in $\mathbb{K}$ (Lemma 2.6), we can disregard it in the form of ξ^2. Therefore,

$$\xi^2 = \varepsilon_2^a \varepsilon_p^a \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{2pq}}^e \sqrt{\varepsilon_{2p}}^e,$$

with $a+e+eu \equiv 0 \pmod 2$. So the result (cf. (9)).

□

Theorem 3.6. *Let $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right) = 1$. Put $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Assume furthermore that $p(x-1)$ and $(v-1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 2.6. Then*

1. *Assume that $N(\varepsilon_{2p}) = -1$. We have*

- *The unit group of* $\mathbb{K}$ *is :*

$$E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

- *The* 2*-class number of* $\mathbb{K}$ *equals* $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.

2. *Assume that* $N(\varepsilon_{2p}) = 1$ *and let* $a \in \{0,1\}$ *such that* $a \equiv 1+u \pmod 2$. *We have*
   - *The unit group of* $\mathbb{K}$ *is :*

$$E_{\mathbb{K}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{a\alpha}\sqrt{\varepsilon_q}^{\alpha}\sqrt{\varepsilon_{pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

   where $\alpha, \gamma \in \{0,1\}$ *such that* $\alpha \neq \gamma$ *and* $\alpha = 1$ *if and only if* $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}$ *is a square in* $\mathbb{K}$.
   - *The* 2*-class number of* $\mathbb{K}$ *equals* $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. 1. Assume that $N(\varepsilon_{2p}) = -1$. Note that $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2, \varepsilon_{2pq}, \sqrt{\varepsilon_{pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{2pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\varepsilon_{2pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2} = 1+\tau_2$. By Tables 1 and 2, we have:

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gu}\varepsilon_2^g \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gu}\varepsilon_2^g. \end{aligned}$$

Thus, $b+f+gu \equiv 0 \pmod 2$ and $g = 0$. Then $b = f$ and

$$\xi^2 = \varepsilon_2^a\varepsilon_p^f\varepsilon_{2pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_5} = 1+\tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. Then, by Tables 1 and 2, we have:

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^f\cdot\varepsilon_{2pq}^{2c}\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot 1 \\ &= \varepsilon_{2pq}^{2c}(-1)^{a+d}\varepsilon_q^d. \end{aligned}$$

Thus $a = d = 0$ and

$$\xi^2 = \varepsilon_p^f\varepsilon_{2pq}^c\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By Tables 1 and 2, we have:

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= \varepsilon_p^{2f}\cdot\varepsilon_{2pq}^{2c}\cdot(-1)^e\cdot\varepsilon_{2q}^{e}\cdot 1\\ &= \varepsilon_p^{2a}\varepsilon_{2pq}^{2c}(-1)^e\varepsilon_{2q}^{e}. \end{aligned}$$

Therefore $e=0$ and

$$\xi^2=\varepsilon_p^{f}\varepsilon_{2pq}^{c}\sqrt{\varepsilon_{pq}}^{f}.$$

➠ Let us apply the norm map $N_{\mathbb{K}/k_3}=1+\tau_2\tau_3$, with $k_3=\mathbb{Q}(\sqrt{2},\sqrt{pq})$. By Tables 1 and 2, we get:

$$N_{\mathbb{K}/k_3}(\xi^2) = (-1)^f\cdot\varepsilon_{2pq}^{2c}\cdot\varepsilon_{pq}^{2f}=\varepsilon_{2pq}^{2c}\varepsilon_{pq}^{2f}(-1)^f.$$

Thus, $f=0$ and

$$\xi^2=\varepsilon_{2pq}^{c}.$$

Since ε_{2pq} is a square in $\mathbb{K}$, then we have the first item.

2. Assume that $N(\varepsilon_{2p})=1$. In this case we have: $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{2pq},\sqrt{\varepsilon_{pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. Thus,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{2pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2=\varepsilon_2^{a}\varepsilon_p^{b}\varepsilon_{2pq}^{c}\sqrt{\varepsilon_q}^{d}\sqrt{\varepsilon_{2q}}^{e}\sqrt{\varepsilon_{pq}}^{f}\sqrt{\varepsilon_{2p}}^{g},$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. We have

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^{d}\cdot\varepsilon_{2q}^{e}\cdot(-1)^f\cdot(-1)^{gu}\\ &= \varepsilon_2^{2a}\varepsilon_q^{c}\varepsilon_{2q}^{d}\cdot(-1)^{b+f+gu}. \end{aligned}$$

Thus, $b+f+gu\equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot\varepsilon_{2pq}^{2c}\cdot(-1)^d\cdot\varepsilon_q^{d}\cdot 1\cdot 1\cdot(-1)^g\cdot\varepsilon_{2p}^{g}\\ &= \varepsilon_{2pq}^{2c}(-1)^{a+b+d+g}\varepsilon_q^{d}\varepsilon_{2p}^{g}. \end{aligned}$$

Thus, $a+b+d+g\equiv 0 \pmod 2$ and $d=g$. So $a=b$. Therefore,

$$\xi^2=\varepsilon_2^{a}\varepsilon_p^{a}\varepsilon_{2pq}^{c}\sqrt{\varepsilon_q}^{g}\sqrt{\varepsilon_{2q}}^{e}\sqrt{\varepsilon_{pq}}^{f}\sqrt{\varepsilon_{2p}}^{g}.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2a}\cdot\varepsilon_{2pq}^{2c}\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^{e}\cdot 1\cdot(-1)^{gu+g}\\ &= \varepsilon_p^{2a}\varepsilon_{2pq}^{2c}(-1)^{a+e+g+ug}\varepsilon_{2q}^{e}. \end{aligned}$$

Thus, $a+e+g+ug\equiv 0 \pmod 2$ and $e=0$. Therefore, $a+g+ug\equiv 0 \pmod 2$. Since $a+f+gu\equiv 0 \pmod 2$, we have $f=g$. As ε_{2pq} is a square in $\mathbb{K}$ then we may put:

$$\xi^2=\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}^g\sqrt{\varepsilon_{pq}}^g\sqrt{\varepsilon_{2p}}^g,$$

where $a+g+ug\equiv 0 \pmod 2$. Which gives the result (cf. (9)).

□

Theorem 3.7. *Let $p\equiv 1 \pmod 8$ and $q\equiv 3 \pmod 8$ be two primes such that $\left(\dfrac{p}{q}\right)=1$. Put $\mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $p(x-1)$ and $p(v-1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 2.6. Then*

1. *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*
2. *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u \pmod 2$. We have*
   - *The unit group of $\mathbb{K}$ is :*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_q}^{\alpha}\sqrt{\varepsilon_{2q}}^{\alpha}\sqrt{\varepsilon_{pq}}^{\alpha}\sqrt{\varepsilon_{2pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

   where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ is a square in $\mathbb{K}$.
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1. Assume that $N(\varepsilon_{2p})=-1$. Note that $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. We have Then, by Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot 1\cdot(-1)^{gu}\varepsilon_2^g\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}\varepsilon_2^g. \end{aligned}$$

Thus, $b+gu\equiv 0 \pmod 2$ and $g=0$. Therefore, $b=0$ and

$$\xi^2=\varepsilon_2^a\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_5} = 1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q}, \sqrt{2p})$. By Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\ &= \varepsilon_{2pq}^f(-1)^{a+d+f}\varepsilon_q^d. \end{aligned}$$

Thus, $a+d+f\equiv 0 \pmod 2$ and $d=0$. Therefore, $a=f$ and

$$\xi^2=\varepsilon_2^a\varepsilon_{pq}^c\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^a.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6} = 1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p}, \sqrt{2q})$. Then, by Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot(-1)^a\cdot\varepsilon_{2pq}^a\\ &= (-1)^e\varepsilon_{2q}^e\varepsilon_{2pq}^a. \end{aligned}$$

Thus $e=a=0$. Therefore $e=0$ and

$$\xi^2=\varepsilon_{pq}^c.$$

Since by the above Lemma ε_{pq} is a square in $\mathbb{K}$ so we have the first item.

2. Assume that $N(\varepsilon_{2p})=1$. We have $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2p}}\}$, is a fundamental system of units of k_1, and $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. So we have

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}, \sqrt{\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2} = 1+\tau_2$. We have:

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot 1\cdot(-1)^{gu}\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}. \end{aligned}$$

Thus, $b+gu\equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_5} = 1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q}, \sqrt{2p})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g.\\ &= \varepsilon_{2pq}^f(-1)^{a+b+d+f+g}\varepsilon_q^d\varepsilon_{2p}^g. \end{aligned}$$

Thus, $a+b+d+f+g\equiv 0 \pmod 2$ and $d=g$. Therefore, $a+b+f\equiv 0 \pmod 2$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^d.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6} = 1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^{ud+d}\\ &= (-1)^{a+e+f+ug+g}\varepsilon_{2q}^e\varepsilon_{2pq}^f. \end{aligned}$$

Thus, $a+e+f+ud+d\equiv 0 \pmod 2$ and $e=f$. Then, $a+ud+d\equiv 0 \pmod 2$. Since $b+du\equiv 0 \pmod 2$ and $a+b+f\equiv 0 \pmod 2$, we have $f=d$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^{ud}\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^d.$$

Since by the above Lemma ε_{pq} is a square in $\mathbb{K}$. then we can put

$$\xi^2=\varepsilon_2^a\varepsilon_p^{ud}\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^d.$$

where $a+ud+d\equiv 0 \pmod 2$. Which completes the proof.

□

Theorem 3.8. *Let $p\equiv 1 \pmod 8$ and $q\equiv 3 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Put $\mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $p(x-1)$ and $2p(v+1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 2.6. Then*

1. *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{K}$ is :*

   $$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt{\varepsilon_p^\alpha\sqrt{\varepsilon_{2q}}^{1+\gamma}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^\alpha}\rangle.$$

   where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_p\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$ is a square in $\mathbb{K}$
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*
2. *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u \pmod 2$. We have*
   - *The unit group of $\mathbb{K}$ is :*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_q}^\alpha\sqrt{\varepsilon_{2q}}^\alpha\sqrt{\varepsilon_{pq}}^\alpha\sqrt{\varepsilon_{2pq}}^\alpha\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

   where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ is a square in $\mathbb{K}$.
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1. Assume that $N(\varepsilon_{2p})=-1$. Note that $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$.

➟ Let us start by applying the norm map $N_{\mathbb{K}/k_2} = 1 + \tau_2$. By Tables 1 and 2 we have

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gu}\varepsilon_2^g \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gu}\varepsilon_2^g. \end{aligned}$$

Thus, $b + f + gu \equiv 0 \pmod 2$ and $g = 0$. Therefore, $b = f$ and

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^b.$$

➟ Let us apply the norm $N_{\mathbb{K}/k_5} = 1 + \tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. Tables 1 and 2 give:

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot(-1)^b\cdot\varepsilon_{2pq}^b \\ &= \varepsilon_{2pq}^b(-1)^{a+d}\varepsilon_q^d. \end{aligned}$$

Thus, $a = d = 0$. Therefore,

$$\xi^2 = \varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^b.$$

➟ Let us apply the norm $N_{\mathbb{K}/k_6} = 1 + \tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. So by Tables 1 and 2 we have

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= \varepsilon_{2p}^{2b}\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot(-1)^b\cdot\varepsilon_{2pq}^b \\ &= \varepsilon_{2p}^{2b}(-1)^{e+b}\varepsilon_{2q}^e\varepsilon_{2pq}^b. \end{aligned}$$

Thus $b = e$. Therefore

$$\xi^2 = \varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_{2q}}^b\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^b.$$

By applying the other norms we deduce nothing new. Since by the above Lemma ε_{pq} is a square in $\mathbb{K}$, we have the first item.

2. Assume that $N(\varepsilon_{2p}) = 1$. We have $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2p}}\}$ is a fundamental system of units of k_1, and $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. So we have

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}, \sqrt{\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. We have:

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot 1\cdot(-1)^{gu}\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}. \end{aligned}$$

Thus, $b+gu\equiv 0\pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g.\\ &= \varepsilon_{2pq}^f(-1)^{a+b+d+f+g}\varepsilon_q^d\varepsilon_{2p}^g. \end{aligned}$$

Thus, $a+b+d+f+g\equiv 0\pmod 2$ and $d=g$. Therefore, $a+b+f\equiv 0\pmod 2$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^d.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^{ud+d}\\ &= \varepsilon_p^{2b}(-1)^{a+e+f+ud+d}\varepsilon_{2q}^e\varepsilon_{2pq}^f. \end{aligned}$$

Thus, $a+e+f+ud+d\equiv 0\pmod 2$ and $e=f$. Then, $a+ud+d\equiv 0\pmod 2$. Since $b+du\equiv 0\pmod 2$ and $a+b+f\equiv 0\pmod 2$, we have $f=d$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^{ud}\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^d.$$

Since by the above Lemma ε_{pq} is a square in $\mathbb{K}$. then we can put

$$\xi^2=\varepsilon_2^a\varepsilon_p^{ud}\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^d.$$

where $a+ud+d\equiv 0\pmod 2$. Which completes the proof.

□

Theorem 3.9. *Let $p\equiv 1\pmod 8$ and $q\equiv 3\pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Put $\mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $2p(x+1)$ and $(v-1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 2.6. Then*

1. *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{K}$ is :*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*
2. *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u\pmod 2$. We have*

- *The unit group of* $\mathbb{K}$ *is :*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{a\alpha}\sqrt{\varepsilon_q}^{\alpha}\sqrt{\varepsilon_{pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

  where $\alpha,\gamma\in\{0,1\}$ *such that* $\alpha\neq\gamma$ *and* $\alpha=1$ *if and only if* $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}$ *is a square in* $\mathbb{K}$.
- *The* 2*-class number of* $\mathbb{K}$ *equals* $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. 1. Assume that $N(\varepsilon_{2p})=-1$. Note that $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{2pq},\sqrt{\varepsilon_{pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{2pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. So, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{2pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. By Tables 1 and 2 we have

$$\begin{array}{rcl} N_{\mathbb{K}/k_2}(\xi^2) &=& \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gu}\varepsilon_2^g\\ &=& \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gu}\varepsilon_2^g.\end{array}$$

Thus, $b+f+gu\equiv 0 \pmod 2$ and $g=0$. Therefore, $b=f$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{2pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}}^b.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. Tables 1 and 2 give:

$$\begin{array}{rcl} N_{\mathbb{K}/k_5}(\xi^2) &=& (-1)^a\cdot\varepsilon_p^{2b}\cdot\varepsilon_{2pq}^{2c}\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot 1\\ &=& \varepsilon_p^{2b}\varepsilon_{2pq}^{2c}(-1)^{a+d}\varepsilon_q^d.\end{array}$$

Thus, $a=d=0$. Therefore,

$$\xi^2=\varepsilon_p^b\varepsilon_{2pq}^c\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}}^b.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. So by Tables 1 and 2 we have

$$\begin{array}{rcl} N_{\mathbb{K}/k_6}(\xi^2) &=& \varepsilon_p^{2b}\cdot\varepsilon_{2pq}^{2c}\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot 1\\ &=& \varepsilon_p^{2b}\varepsilon_{2pq}^{2c}(-1)^e\varepsilon_{2q}^e.\end{array}$$

So $e=0$ and therefore,

$$\xi^2=\varepsilon_p^b\varepsilon_{2pq}^c\sqrt{\varepsilon_{pq}}^b.$$

➠ Let us apply the norm map $N_{\mathbb{K}/k_3} = 1 + \tau_2\tau_3$, with $k_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$. So, by Tables 1 and 2, we get:

$$\begin{aligned} N_{\mathbb{K}/k_3}(\xi^2) &= (-1)^b \cdot \varepsilon_{2pq}^{2c} \cdot \varepsilon_{pq}^{e} \\ &= \varepsilon_{2pq}^{2c} \cdot (-1)^b \varepsilon_{pq}^{b}. \end{aligned}$$

Thus, $b = 0$ and so $\xi^2 = \varepsilon_{2pq}^{c}$. Since by Lemma 2.6, ε_{2pq} is a square in $\mathbb{K}$, then we have the result.

2. Assume that $N(\varepsilon_{2p}) = 1$. In this case we have

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{2pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2p}} \rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. So, we can assume that

$$\xi^2 = \varepsilon_2^a \varepsilon_p^b \varepsilon_{2pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{pq}}^f \sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2} = 1 + \tau_2$. We have:

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a} \cdot (-1)^b \cdot 1 \cdot \varepsilon_q^d \cdot \varepsilon_{2q}^e \cdot (-1)^f \cdot (-1)^{gu} \\ &= \varepsilon_2^{2a} \varepsilon_q^c \varepsilon_{2q}^d \cdot (-1)^{b+f+gu}. \end{aligned}$$

So we have $b + f + gu \equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_5} = 1 + \tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_5}(\xi^2) &= (-1)^a \cdot (-1)^b \cdot \varepsilon_{2pq} \cdot (-1)^d \cdot \varepsilon_q^d \cdot 1 \cdot 1 \cdot (-1)^g \cdot \varepsilon_{2p}^g. \\ &= \varepsilon_{2pq}^f (-1)^{a+b+d+g} \varepsilon_q^d \varepsilon_{2p}^g. \end{aligned}$$

Thus, $a + b + d + g \equiv 0 \pmod 2$ and $d = g$. Therefore, $a = b$ and

$$\xi^2 = \varepsilon_2^a \varepsilon_p^a \varepsilon_{2pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{pq}}^f \sqrt{\varepsilon_{2p}}^d,$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6} = 1 + \tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. We have

$$\begin{aligned} N_{\mathbb{K}/k_6}(\xi^2) &= (-1)^a \cdot \varepsilon_p^{2a} \cdot \varepsilon_{2pq}^{2c} \cdot 1 \cdot (-1)^e \cdot \varepsilon_{2q}^e \cdot (-1)^{ud+d} \\ &= \varepsilon_p^{2a} \varepsilon_{2pq}^{2c} (-1)^{a+e+ud+d} \varepsilon_{2q}^e. \end{aligned}$$

Thus, $a+e+ud+d \equiv 0 \pmod 2$ and $e = 0$. Then, $a+ud+d \equiv 0 \pmod 2$. As the above discussions imply $a + f + du \equiv 0 \pmod 2$, then $f = d$. Therefore;

$$\xi^2 = \varepsilon_2^a \varepsilon_p^a \varepsilon_{2pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{pq}}^d \sqrt{\varepsilon_{2p}}^d,$$

Since by Lemma 2.6, ε_{2pq} is a square in $\mathbb{K}$, then we can put

$$\xi^2 = \varepsilon_2^a \varepsilon_p^a \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{pq}}^d \sqrt{\varepsilon_{2p}}^d,$$

where $a + ud + d \equiv 0 \pmod 2$. Which completes the proof.

$\square$

Theorem 3.10. *Let $p\equiv 1 \pmod 8$ and $q\equiv 3\pmod 8$ be two primes such that $\left(\dfrac{p}{q}\right)=1$. Put $\mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $2p(x+1)$ and $p(v-1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 2.6. Then*

1. *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt{\varepsilon_p^{\alpha}\sqrt{\varepsilon_q}^{1+\gamma}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}}\rangle,$$
   where α, $\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_p\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$ is a square in $\mathbb{K}$
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*
2. *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u \pmod 2$. We have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}},\sqrt{\varepsilon_p^{\alpha}\sqrt{\varepsilon_q}^{1+\gamma}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}}\rangle$$
   where α, $\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_p\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$ is a square in $\mathbb{K}$.
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1. Assume that $N(\varepsilon_{2p})=-1$. Note that $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. By Tables 1 and 2 we have

$$\begin{aligned} N_{\mathbb{K}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gu}\varepsilon_2^g\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gu}\varepsilon_2^g.\end{aligned}$$

Thus, $b+f+gu\equiv 0 \pmod 2$ and $g=0$. Therefore, $b=f$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^b,$$

➠ Let us apply the norm $N_{\mathbb{K}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. Tables 1 and 2 give:

$$\begin{array}{rcl} N_{\mathbb{K}/k_5}(\xi^2) & = & (-1)^a\cdot(-1)^b\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot\varepsilon_{2pq}^b\\ & = & (-1)^{a+b+d}\varepsilon_q^d\varepsilon_{2pq}^b. \end{array}$$

Thus, $a+b+d=0 \pmod 8$ et $d=b$. Therefore, $a=0$ and

$$\xi^2=\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^b\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^b.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. So by Tables 1 and 2 we have

$$\begin{array}{rcl} N_{\mathbb{K}/k_6}(\xi^2) & = & \varepsilon_p^{2b}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot\varepsilon_{2pq}^b\\ & = & \varepsilon_p^{2b}\varepsilon_{2pq}^b(-1)^e\varepsilon_{2q}^e. \end{array}$$

So $e=0$ and therefore,

$$\xi^2=\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^b\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^b.$$

As ε_{pq} is a square in $\mathbb{K}$, then we can put $\xi^2=\varepsilon_p^b\sqrt{\varepsilon_q}^b\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^b$. By applying $1+\tau_2\tau_3$, $1+\tau_3$ and $1+\tau_1$, we deduce thing new. So the first item.

2. In this case we have:

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. We have:

$$\begin{array}{rcl} N_{\mathbb{K}/k_2}(\xi^2) & = & \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot(-1)^f\cdot(-1)^{gu}\\ & = & \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gu}. \end{array}$$

Thus, $b+f+gu\equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. We have

$$\begin{array}{rcl} N_{\mathbb{K}/k_5}(\xi^2) & = & (-1)^a\cdot(-1)^b\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g.\\ & = & (-1)^{a+b+d+g}\varepsilon_q^d\varepsilon_{2pq}^f\varepsilon_{2p}^g. \end{array}$$

Thus, $a+b+d+g\equiv 0 \pmod 2$ and $d+f+g\equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. We have

$$\begin{array}{rcl} N_{\mathbb{K}/k_6}(\xi^2) & = & (-1)^a\cdot\varepsilon_p^{2b}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot\varepsilon_{2pq}^f\cdot(-1)^{ug+g}\\ & = & \varepsilon_p^{2b}\varepsilon_{2pq}^f(-1)^{a+e+ug+g}\varepsilon_{2q}^e. \end{array}$$

Thus, $a+e+ug+g\equiv 0 \pmod 2$ and $e=0$. Then, $a+ug+g\equiv 0 \pmod 2$. Since $b+f+gu\equiv 0 \pmod 2$ and $a+b+d+g\equiv b+d+ug\equiv 0 \pmod 2$, this implies that $f=d$. The equality $d+f+g\equiv 0 \pmod 2$ gives $g=0$. Thus, $a=0$ and $b=f$. Therefore,

$$\xi^2=\varepsilon_p^f\varepsilon_{pq}^c\sqrt{\varepsilon_q}^f\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

As ε_{pq} is a square in $\mathbb{K}$, then we can put $\xi^2=\varepsilon_p^f\sqrt{\varepsilon_q}^f\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f$.

□

Theorem 3.11. *Let $p\equiv 1 \pmod 8$ and $q\equiv 3 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Put $\mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $2p(x+1)$ and $2p(v+1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 2.6. Then*

1. *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*
2. *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u \pmod 2$. We have*
   - *The unit group of $\mathbb{K}$ is :*
   $$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$
   where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ is a square in $\mathbb{K}$.
   - *The 2-class number of $\mathbb{K}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1. Assume that $N(\varepsilon_{2p})=-1$. Note that $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$, is a fundamental system of units of k_1. Using Lemma 2.6, we show that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Therefore, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. By Tables 1 and 2 we have

$$\begin{aligned}N_{\mathbb{K}/k_2}(\xi^2)&=\varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot 1\cdot(-1)^{gu}\varepsilon_2^g\\&=\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+f+gu}\varepsilon_2^g.\end{aligned}$$

Thus, $b+gu\equiv 0 \pmod 2$ and $g=0$. Therefore, $b=0$ and

$$\xi^2=\varepsilon_2^a\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f,$$

➠ Let us apply the norm $N_{\mathbb{K}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. Tables 1 and 2 give:

$$\begin{array}{rcl} N_{\mathbb{K}/k_5}(\xi^2) &=& (-1)^a\cdot\varepsilon_p^{2b}\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot\varepsilon_{2pq}^f\\ &=& \varepsilon_p^{2b}(-1)^{a+d}\varepsilon_q^d\varepsilon_{2pq}^f. \end{array}$$

Thus, $a=d=f$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_{pq}^c\sqrt{\varepsilon_q}^a\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^a.$$

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. So by Tables 1 and 2 we have

$$\begin{array}{rcl} N_{\mathbb{K}/k_5}(\xi^2) &=& (-1)^a\cdot\varepsilon_p^{2b}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot\varepsilon_{2pq}^a\\ &=& \varepsilon_p^{2b}\varepsilon_{2pq}^a(-1)^{e+b}\varepsilon_{2q}^e. \end{array}$$

So $a=e=0$. Thus,

$$\xi^2=\varepsilon_{pq}^c.$$

So the first item.

2. In this case we have:

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\varepsilon_{pq},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}}\rangle.$$

Let ξ be an element of $\mathbb{K}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Thus, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\varepsilon_{pq}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{K}/k_2}=1+\tau_2$. We have:

$$\begin{array}{rcl} N_{\mathbb{K}/k_2}(\xi^2) &=& \varepsilon_2^{2a}\cdot(-1)^b\cdot 1\cdot\varepsilon_q^d\cdot\varepsilon_{2q}^e\cdot 1\cdot(-1)^{gu}\\ &=& \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}. \end{array}$$

Thus, $b+gu\equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. We have

$$\begin{array}{rcl} N_{\mathbb{K}/k_5}(\xi^2) &=& (-1)^a\cdot(-1)^b\cdot 1\cdot(-1)^d\cdot\varepsilon_q^d\cdot 1\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g.\\ &=& (-1)^{a+b+d+g}\varepsilon_q^d\varepsilon_{2pq}^f\varepsilon_{2p}^g. \end{array}$$

Thus, $a+b+d+g\equiv 0 \pmod 2$ and $d+f+g\equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{K}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. We have

$$\begin{array}{rcl} N_{\mathbb{K}/k_6}(\xi^2) &=& (-1)^a\cdot\varepsilon_p^{2b}\cdot 1\cdot 1\cdot(-1)^e\cdot\varepsilon_{2q}^e\cdot\varepsilon_{2pq}^f\cdot(-1)^{ug+g}\\ &=& \varepsilon_p^{2b}\varepsilon_{2pq}^f(-1)^{a+e+ug+g}\varepsilon_{2q}^e. \end{array}$$

Thus, $a+e+ug+g\equiv 0 \pmod 2$ and $e=0$. Then, $a+ug+g\equiv 0 \pmod 2$. Since $b+gu\equiv 0 \pmod 2$ and $a+ug+g\equiv 0 \pmod 2$, this implies that

$a+b+g\equiv 0 \pmod 2$. As $a+b+d+g\equiv 0 \pmod 2$ (resp. $d+f+g\equiv 0 \pmod 2$), then $d=0$ (resp. $f=g$). Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^{gu}\varepsilon_{pq}^c\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^g\sqrt{\varepsilon_{2p}}^g,$$

with $a+ug+g\equiv 0 \pmod 2$. By applying the other norms we deduce nothing new. So we have the second item.

□

3.3. **The cases:** $p\equiv 5$ **or** $3 \pmod 8$ **and** $q\equiv 3 \pmod 4$**.**

The following theorem provide some families with odd class number and explicit unit groups.

Theorem 3.12. *Let p and q be two primes. Put $\mathbb{K}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Then*

1. *If $p\equiv 3 \pmod 8$, $q\equiv 7 \pmod 8$ and $\left(\frac{p}{q}\right)=1$,*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_p},\sqrt{\varepsilon_{2pq}},\sqrt[4]{\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}},\sqrt[4]{\varepsilon_2^2\varepsilon_q\varepsilon_{2q}\varepsilon_p\varepsilon_{2p}}\,\rangle.$$

2. *If $p\equiv 3 \pmod 8$, $q\equiv 7 \pmod 8$ and $\left(\frac{p}{q}\right)=-1$,*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2p}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt[4]{\varepsilon_q\varepsilon_p\varepsilon_{2p}\varepsilon_{pq}\varepsilon_{2pq}},\sqrt[4]{\varepsilon_2^2\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}}\,\rangle.$$

3. *If $p\equiv 5 \pmod 8$, $q\equiv 7 \pmod 8$ and $\left(\frac{p}{q}\right)=1$,*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt[4]{\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}}\rangle.$$

4. *If $p\equiv 5 \pmod 8$, $q\equiv 7 \pmod 8$ and $\left(\frac{p}{q}\right)=-1$,*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt[4]{\varepsilon_2^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}\rangle.$$

5. *If $p\equiv 5 \pmod 8$, $q\equiv 3 \pmod 8$ and $\left(\frac{p}{q}\right)=1$, then*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt[4]{\varepsilon_p^2\varepsilon_{2q}\varepsilon_{pq}\varepsilon_{2pq}}\rangle.$$

6. *If $p\equiv 5 \pmod 8$, $q\equiv 3 \pmod 8$ and $\left(\frac{p}{q}\right)=-1$,*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt[4]{\varepsilon_2^2\varepsilon_p^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}\rangle.$$

7. *If $p\equiv q\equiv 3 \pmod 8$, then*

$$E_{\mathbb{K}}=\langle -1,\varepsilon_2,\sqrt{\varepsilon_p},\sqrt{\varepsilon_{2p}},\sqrt{\varepsilon_q},\sqrt{\varepsilon_{pq}},\sqrt[4]{\varepsilon_p\varepsilon_q\varepsilon_{2pq}},\sqrt[4]{\varepsilon_{2p}\varepsilon_{2q}\varepsilon_{2pq}}\rangle.$$

Furthermore, in all the above cases the class number of $\mathbb{K}$ is odd.

Proof. The points 4, 5 and 6 are proved in [8] and [7] respectively. The proof of the rest demands very long computations as above, however we suggest to the reader proceed as in the proof of Theorem 3.1 or [7, Theorem 2.5] to construct a detailed prove. □

4. Some families of Fröhlich triquadratic fields whose 2-class groups are of type $(2,2)$

Now we can give some families of real triquadratic number fields whose 2-class groups are of type $(2,2)$.

Theorem 4.1. *Let $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$ be two primes such that $\left(\dfrac{p}{q}\right) = 1$. Put $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Then the 2-class group of $\mathbb{K}$ is of type $(2,2)$ in the following cases:*

1. *$h_2(pq) = h_2(2pq) = 2 \cdot h_2(2p) = 4$ and $(x-1)$ or $(v-1)$ is not square in $\mathbb{N}$.*
2. *$N(\varepsilon_{2p}) = -1$, $h_2(pq) = h_2(2pq) = h_2(2p) = 4$ and one of the following conditions is satisfied:*
   a. *$(x-1)$ and $2p(v+1)$ are squares in $\mathbb{N}$.*
   b. *$(x-1)$ and $p(v-1)$ are squares in $\mathbb{N}$.*
   c. *$p(x-1)$ and $(v-1)$ are squares in $\mathbb{N}$.*
   d. *$p(x-1)$ and $p(v-1)$ are squares in $\mathbb{N}$.*
   e. *$2p(x+1)$ and $(v-1)$ are squares in $\mathbb{N}$.*
   f. *$2p(x+1)$ and $2p(v+1)$ are squares in $\mathbb{N}$.*

where x and v are defined in Lemma 2.6.

Proof. We shall prove the first item and the reader can similarly prove the second one. So assume that we are in conditions of the first item. Note that by [13, Theorem 2], we have $N(\varepsilon_{2p}) = 1$. Using Lemmas 2.7 and 2.6, we get $h_2(k_3) = \frac{1}{4}q(k_3) \cdot 4 \cdot 4 = 8$, where $k_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$.Therefore, as by [2] the 2-rank of the class group of k_3 equals 2, the 2-class group of k_3 is of type $(2,4)$. By the second items of Theorems 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10 and 3.11, we have $h_2(\mathbb{K}) = \frac{1}{2^{4-\alpha}} h_2(2p)h_2(pq)h_2(2pq)$, for some $\alpha \in \{0,1\}$. If we assume that $\alpha = 0$, then $h_2(\mathbb{K}) = \frac{1}{2^4} \cdot 2 \cdot 4 \cdot 4 = 2$. This implies that the 2-class group of $\mathbb{K}$ is cyclic, but this is impossible by class field theory and the fact that $\mathbb{K}/k_5$ is an unramified quadratic extension. Therefore, $h_2(\mathbb{K}) = 4$. Now, let us show that the 2-class group of $\mathbb{K}$ is not cyclic. By [4, Theorem 4.1, (iii)], k_3 admits three quadratic extensions $\mathbb{K}_1 = \mathbb{K}$ and the conjugate extensions $\mathbb{K}_2 := k_3(\sqrt{\alpha_1^*})$ and $\mathbb{K}_3$, all contained in $\mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{\alpha_1^*})$ which is an unramified extension of k_3 of degree 4, where the α_1^* is (the element attached to p) defined in [4, Theorem 4.1, (iii)]. So by the group theoretic properties given in [5, p. 110] the 2-class group of $\mathbb{K}$ is not cyclic. Hence the 2-class group of $\mathbb{K}$ is of type $(2,2)$. □

References

[1] A. Azizi, Sur la capitulation des 2-classes d'idéaux de $\Bbbk = \mathbb{Q}(\sqrt{2pq}, i)$, où $p \equiv -q \equiv 1 \pmod 4$, Acta. Arith. 94 (2000), 383-399.

[2] A. Azizi et A. Mouhib, Sur le rang du 2-groupe de classes de $\mathbb{Q}(\sqrt{m}, \sqrt{d})$ où $m = 2$ ou un premier $p \equiv 1 \pmod 4$. Trans. Amer. Math. Soc., 353 (2001), 2741–2752.

[3] A. Azizi et M. Talbi, Capitulation des 2-classes d'idéaux de certains corps biquadratiques cycliques, Acta Arith., 127 (2007), 231-248.

[4] S. Bae and Q. Yue, Hilbert genus fields of real biquadratic fields. Ramanujan J 24, 161–181 (2011).

[5] E. Benjamin and C. Snyder, Classification of metabelian 2-groups G with $Gab = (2, 2n)$, $n \geq 2$, and rank $d(G') = 2$; Applications to real quadratic number fields, J. Pure Appl. Algebra, 223 (2019), 108-130.

[6] M. Bulant, On the parity of the class number of the field $\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$, J. Number Theory 68 (1998) 72-86.

[7] M. M. Chems-Eddin, Unit groups of some multiquadratic number fields and 2-class groups. Period. Math. Hung. 84 (2022), 235–249. https://doi.org/10.1007/s10998-021-00402-0

[8] M. M. Chems-Eddin, A. Zekhnini and A. Azizi, *Units and 2-class field towers of some multiquadratic number fields*, Turk J. Math., 44 (2020), 1466-1483.

[9] P. E. Conner and J. Hurrelbrink, Class number parity, Ser. Pure. Math. 8, World Scientific, 1988.

[10] A. Fröhlich, Central Extensions, Galois Groups and Ideal Class Groups of Number Fields, Contemp. Math., Vol. 24, Amer. Math. Soc., 1983.

[11] P. Kaplan, Sur le 2-groupe des classes d'idéaux des corps quadratiques, J. Reine. Angew. Math., 283/284 (1976), 313-363.

[12] S. Kuroda, Über die Klassenzahlen algebraischer Zahlkörper, Nagoya Math. J., 1 (1950), 1–10.

[13] R. Kučera, On the parity of the class number of a biquadratic field, *J. Number Theory* **52** (1995), 43–52.

[14] A. Mouhib, On the parity of the class number of multiquadratic number fields, J. Number Theory 129 (2009), 1205-1211.

[15] A. Mouhib and A. Movahhedi, Sur le 2-groupe de classes des corps mutiquadratiques réels, J. Théor. Nr. Bordx 17 (2005), 619-641. 1205-1211.

[16] H. Wada, On the class number and the unit group of certain algebraic number fields. J. Fac. Sci. Univ. Tokyo, 13 (1966), 201–209.

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