STOCHASTIC QUANTIZATION OF THE Φ^3_3-MODEL

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Abstract. We study the construction of the Φ^3_3-measure and complete the program on the (non-)construction of the focusing Gibbs measures, initiated by Lebowitz, Rose, and Speer (1988). This problem turns out to be critical, exhibiting the following phase transition. In the weakly nonlinear regime, we prove normalizability of the Φ^3_3-measure and show that it is singular with respect to the massive Gaussian free field. Moreover, we show that there exists a shifted measure with respect to which the Φ^3_3-measure is absolutely continuous. In the strongly nonlinear regime, by further developing the machinery introduced by the authors, we establish non-normalizability of the Φ^3_3-measure. Due to the singularity of the Φ^3_3-measure with respect to the massive Gaussian free field, this non-normalizability part poses a particular challenge as compared to our previous works. In order to overcome this issue, we first construct a σ-finite version of the Φ^3_3-measure and show that this measure is not normalizable. Furthermore, we prove that the truncated Φ^3_3-measures have no weak limit in a natural space, even up to a subsequence.

We also study the dynamical problem for the canonical stochastic quantization of the Φ^3_3-measure, namely, the three-dimensional stochastic damped nonlinear wave equation with a quadratic nonlinearity forced by an additive space-time white noise (= the hyperbolic Φ^3_3-model). By adapting the paracontrolled approach, in particular from the works by Gubinelli, Koch, and the first author (2018) and by the authors (2020), we prove almost sure global well-posedness of the hyperbolic Φ^3_3-model and invariance of the Gibbs measure in the weakly nonlinear regime. In the globalization part, we introduce a new, conceptually simple and straightforward approach, where we directly work with the (truncated) Gibbs measure, using the Boué-Dupuis variational formula and ideas from theory of optimal transport.

Contents

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1. Introduction

1.1. **Overview.** In this paper, we study the Φ^3_3-measure on the three-dimensional torus on $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, formally written as

$$d\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx\right) d\mu(u), \tag{1.1}$$

and its associated stochastic quantization. Here, μ is the massive Gaussian free field on $\mathbb{T}^3$ and the coupling constant $\sigma \in \mathbb{R} \setminus \{0\}$ measures the strength of the cubic interaction. The associated energy functional for the Φ^3_3-measure ρ in (1.1) is given by

$$E(u) = \frac{1}{2}\int_{\mathbb{T}^3} |\langle \nabla \rangle u|^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx, \tag{1.2}$$

where $\langle \nabla \rangle = \sqrt{1-\Delta}$. Since u^3 is not sign definite, the sign of σ does not play any role and, in particular, the problem is not defocusing even if $\sigma < 0$.

Our main goal in this paper is to study the construction of the Φ^3_3-measure and its associated dynamics, following the program on the (non-)construction of focusing[1] Gibbs measures, initiated by Lebowitz, Rose, and Speer [43]. Let us first go over the known results. In the seminal work [43], Lebowitz, Rose, and Speer studied the one-dimensional case and constructed the one-dimensional focusing Gibbs measures[2] in the L^2-(sub)critical setting

[1]By "focusing", we also mean the non-defocusing (non-repulsive) case, such as the cubic interaction appearing in (1.1), such that the interaction potential (for example, $\frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx$ in (1.1)) is unbounded from above.

[2]As pointed out by Carlen, Fröhlich, and Lebowitz [16, p. 315], there is in fact an error in the Gibbs measure construction in [43], which was amended by Bourgain [8] (for $2 < p < 6$ with any $K > 0$ and $p = 6$ with

(i.e. $2 < p \le 6$) with an L^2-cutoff:

$$d\rho(u) = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}}|u|^2dx\le K\}}\exp\bigg(\frac{1}{p}\int_{\mathbb{T}}|u|^p dx\bigg)d\mu(u) \tag{1.3}$$

or with a taming by the L^2-norm:

$$d\rho(u) = Z^{-1}\exp\bigg(\frac{1}{p}\int_{\mathbb{T}}|u|^p dx - A\Big(\int_{\mathbb{T}}u^2dx\Big)^q\bigg)d\mu(u) \tag{1.4}$$

for some appropriate $q = q(p)$, where μ denotes the periodic Wiener measure on $\mathbb{T}$. See Remark 2.1 in [43]. Here, the parameter $A > 0$ denotes the so-called (generalized) chemical potential and the expression (1.4) is referred to as the generalized grand-canonical Gibbs measure. See also the work by Carlen, Fröhlich, and Lebowitz [16] for a further discussion, where they describe the details of the construction of the generalized grand-canonical Gibbs measure in (1.4) in the L^2-subcritical setting ($2 < p < 6$). In [43], Lebowitz, Rose, and Speer also proved non-normalizability of the focusing Gibbs measure ρ in (1.3):

$$\mathbb{E}_\mu\bigg[\mathbf{1}_{\{\int_{\mathbb{T}}|u|^2dx\le K\}}\exp\bigg(\frac{1}{p}\int_{\mathbb{T}}|u|^p dx\bigg)\bigg] = \infty$$

in (i) the L^2-supercritical case ($p > 6$) for any $K > 0$ and (ii) the L^2-critical case ($p > 6$), provided that $K > \|Q\|^2_{L^2(\mathbb{R})}$, where Q is the (unique[3]) optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ such that $\|Q\|^6_{L^6(\mathbb{R})} = 3\|Q'\|^2_{L^2(\mathbb{R})}$. In a recent work [61], the first and third authors with Sosoe proved that the focusing L^2-critical Gibbs measure ρ in (1.3) (with $p = 6$) is indeed constructible at the optimal mass threshold $K = \|Q\|^2_{L^2(\mathbb{R})}$, thus answering an open question posed by Lebowitz, Rose, and Speer [43] and completing the program in the one-dimensional case.

In the two-dimensional setting, Brydges and Slade [15] continued the study on the focusing Gibbs measures and showed that with the quartic interaction ($p = 4$), the focusing Gibbs measure ρ in (1.3) (even with proper renormalization on the potential energy $\frac{1}{4}\int_{\mathbb{T}^2}|u|^4dx$ and on the L^2-cutoff) is not normalizable as a probability measure. See also [60] for an alternative proof. In view of

$$\mathbf{1}_{\{|\cdot|\le K\}}(x) \le \exp\big(-A|x|^\gamma\big)\exp\big(AK^\gamma\big) \tag{1.5}$$

for any $K > 0$, $\gamma > 0$, and $A > 0$, this non-normalizability result of the focusing Gibbs measure on $\mathbb{T}^2$ with the quartic interaction ($p = 4$) also applies to the generalized grand-canonical Gibbs measure in (1.4). Furthermore, the same non-normalizability applies for higher order interaction (for an integer $p \ge 5$).

In [9], Bourgain reported Jaffe's construction of a Φ^3_2-measure endowed with a Wick-ordered L^2-cutoff:

$$d\rho = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}^2}:u^2:\,dx\le K\}}e^{\frac{1}{3}\int_{\mathbb{T}^2}:u^3:\,dx}d\mu(u),$$

where $:u^2:$ and $:u^3:$ denote the Wick powers of u, and μ denotes the massive Gaussian free field on $\mathbb{T}^2$. See also [60]. We point out that such a Gibbs measure with a (Wick-ordered) L^2-cutoff is not suitable for stochastic quantization in the heat and wave settings due to the

$0 < K \ll 1$) and the first and third authors with Sosoe [61] (for $p = 6$ and $K \le \|Q\|^2_{L^2(\mathbb{R})}$). See [61] for a further discussion.

[3]Up to the symmetries.

lack of the L^2-conservation. In [9], Bourgain instead constructed the following generalized grand-canonical formulation of the Φ^3_2-measure:

$$d\rho(u) = Z^{-1} e^{\frac{1}{3}\int_{\mathbb{T}^2} :u^3:\, dx - A\left(\int_{\mathbb{T}^2} :u^2:\, dx\right)^2} d\mu(u)$$

for sufficiently large $A > 0$. See [63, 34, 52, 36] for the associated (stochastic) nonlinear wave dynamics.

In this paper, we consider the three-dimensional case and complete the focusing Gibbs measure construction program initiated by Lebowitz, Rose, and Speer [43]. More precisely, we consider the following generalized grand-canonical formulation of the Φ^3_3-measure (namely, with a taming by the Wick-ordered L^2-norm):

$$d\rho(u) = Z^{-1} \exp\left(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, dx - A\left|\int_{\mathbb{T}^3} :u^2:\, dx\right|^\gamma\right) d\mu(u) \tag{1.6}$$

for suitable $A, \gamma > 0$. We now state our first main result in a somewhat formal manner. See Theorem 1.8 for the precise statement.

Theorem 1.1. *The following phase transition holds for the Φ^3_3-measure in* (1.6).

(i) (weakly nonlinear regime). *Let $0 < |\sigma| \ll 1$ and $\gamma = 3$. Then, by introducing a further renormalization, the Φ^3_3-measure ρ in* (1.6) *exists as a probability measure, provided that $A = A(\sigma) > 0$ is sufficiently large. In this case, the resulting Φ^3_3-measure ρ and the massive Gaussian free field μ on $\mathbb{T}^3$ are mutually singular.*

(ii) (strongly nonlinear regime). *When $|\sigma| \gg 1$, the Φ^3_3-measure in* (1.6) *is not normalizable for any $A > 0$ and $\gamma > 0$. Furthermore, the truncated Φ^3_3-measures ρ_N (see* (1.25) *below) do not have a weak limit, as measures on $\mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$, even up to a subsequence.*

Theorem 1.1 shows that the Φ^3_3-model is critical in terms of the measure construction. In the case of a higher order focusing interaction on $\mathbb{T}^3$ (replacing $:u^3:$ by $:u^p:$ in (1.6) for an integer $p \geq 4$ with $\sigma > 0$ when p is even), or the Φ^3_4-model on the four-dimensional torus $\mathbb{T}^4$, the focusing nonlinear interaction gets only worse and thus we expect that the same approach would yield non-normalizability. Hence, in view of the previous results [43, 8, 15, 61, 60], Theorem 1.1 completes the focusing Gibbs measure construction program, thus answering an open question posed by Lebowitz, Rose, and Speer (see "Extension to higher dimensions" in [43, Section 5]). See also our companion paper [53], where we completed the program on the (non-)construction of the focusing Hartree Gibbs measures in the three-dimensional setting. See Remark 1.3 for a further discussion.

We point out that in the weakly nonlinear regime, the Φ^3_3-measure ρ is constructed only as a weak limit of the truncated Φ^3_3-measures. Moreover, we prove that there exists a shifted measure with respect to which the Φ^3_3-measure is absolutely continuous; see Appendix A. As for the non-normalizability result in Theorem 1.1 (ii), our proof is based on a refined version of the machinery introduced by the authors [53] and the first and third authors with Seong [60], which was in turn inspired by the work of the third author and Weber [74] on the non-construction of the Gibbs measure for the focusing cubic nonlinear Schrödinger equation (NLS) on the real line, giving an alternative proof of Rider's result [66]. We, however, point out that there is an additional difficulty in proving Theorem 1.1 (ii) due to the singularity of the Φ^3_3-measure with respect to the base massive Gaussian free field μ. (Note that the focusing Gibbs measures considered in [53, 60] are equivalent to the base Gaussian measures.) In order to overcome this difficulty, we first introduce a reference measure[4] ν_δ and construct a σ-finite version of the Φ^3_3-measure (expressed in terms of the reference measure ν_δ). We then show that this σ-finite version of the Φ^3_3-measure is not normalizable. See Section 4.

Remark 1.2. (i) As the name suggests, the Φ^3_3-measure is of interest from the point of view of constructive quantum field theory. In the defocusing case ($\sigma < 0$) with a quartic interaction (u^4 in place of u^3), the measure ρ in (1.1) corresponds to the well-studied Φ^4_3-measure. The construction of the Φ^4_3-measure is one of the early achievements in constructive quantum field theory. For an overview of the constructive program, see the introductions in [1, 32].

(ii) In the one- and two-dimensional cases, the non-normalizability of the focusing Gibbs measures emerges in the L^2-critical case ($p = 6$ when $d = 1$ and $p = 4$ when $d = 2$), suggesting its close relation to the finite time blowup phenomena of the associated focusing NLS. See [61] for a further discussion. In the three-dimensional case, it is interesting to note that the Φ^3_3-model is L^2-*subcritical* and yet we have the non-normalizability (in the strongly nonlinear regime). Thus, the non-normalizability of the Φ^3_3-measure is not related to a blowup phenomenon. Note that, unlike the focusing Φ^6_1- and Φ^4_2-models which make sense in the complex-valued setting, the Φ^3_3-model makes sense only in the real-valued setting. It seems of interest to investigate a possible relation to the following Gagliardo-Nirenberg inequality:

$$\int_{\mathbb{R}^3} |u(x)|^3 dx \lesssim \|u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \|u\|_{\dot{H}^1(\mathbb{R}^3)}^{\frac{3}{2}}.$$

(iii) Consider a Φ^3_3-measure with a Wick-ordered L^2-cutoff:[5]

$$d\rho(u) = Z^{-1} \mathbf{1}_{\{|\int_{\mathbb{T}^3} :u^2: dx| \le K\}} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, dx\bigg) d\mu(u). \tag{1.7}$$

Then, an analogue of Theorem 1.1 holds for the Φ^3_3-measure in (1.7). In view of (1.5), Theorem 1.1 implies normalizability of the Φ^3_3-measure in (1.7) (with a further renormalization) in the weakly nonlinear regime ($0 < |\sigma| \ll 1$). On the other hand, in the strongly nonlinear regime ($|\sigma| \gg 1$), a modification of the proof of Theorem 1.12 (ii) (see also [53, 60]) yields non-normalizability of the Φ^3_3-measure in (1.7) for any $K > 0$.

Remark 1.3. In [11], Bourgain studied the invariant Gibbs dynamics for the focusing Hartree NLS on $\mathbb{T}^3$ (with $\sigma > 0$):

$$i\partial_t u + (1 - \Delta)u - \sigma(V * |u|^2)u = 0, \tag{1.8}$$

where $V = \langle \nabla \rangle^{-\beta}$ is the Bessel potential of order $\beta > 0$. In [11], Bourgain first constructed the focusing Gibbs measure with a Hartree-type interaction (for complex-valued u), endowed with a Wick-ordered L^2-cutoff:

$$d\rho(u) = Z^{-1} \mathbf{1}_{\{\int_{\mathbb{T}^3} :|u|^2: dx \le K\}} \, e^{\frac{\sigma}{4}\int_{\mathbb{T}^3} (V * :|u|^2:) :|u|^2: dx} d\mu(u)$$

[4]This reference measure is introduced as a tamed version of the Φ^3_3-measure and is not to be confused with the shifted measure mentioned above. See Proposition 4.1.

[5]With a slight modification, one may also consider ρ in (1.7) with a slightly different cutoff $\mathbf{1}_{\{\int_{\mathbb{T}^3} :u^2: dx \le K\}}$, i.e. without an absolute value, and prove the same (non-)normalizability results. See Remark 5.10 in [53].

for $\beta > 2$ and then constructed the invariant Gibbs dynamics for the associated dynamical problem.[6] In [53], we continued the study of the focusing Hartree Φ^4_3-measure in the generalized grand-canonical formulation (with $\sigma > 0$):

$$d\rho(u) = Z^{-1} \exp\bigg(\frac{\sigma}{4}\int_{\mathbb{T}^3} (V* :u^2:) :u^2: \, dx - A\bigg|\int_{\mathbb{T}^3} :u^2: \, dx\bigg|^\gamma\bigg) d\mu(u) \tag{1.9}$$

and established a phase transition in two respects (i) the focusing Hartree Φ^4_3-measure ρ in (1.9) is constructible for $\beta > 2$, while it is not for $\beta < 2$ and (ii) when $\beta = 2$, the focusing Hartree Φ^4_3-measure is constructible for $0 < \sigma \ll 1$, while it is not for $\sigma \gg 1$. See [53] for the precise statements. These results in [53] in particular show the critical nature of the focusing Hartree Φ^4_3-model when $\beta = 2$. In the same work, we also constructed the invariant Gibbs dynamics for the associated (canonical) stochastic quantization equation. See also [53, 12, 13] for the defocusing case ($\sigma < 0$). Note that when $\beta = 0$, the defocusing Hartree Φ^4_3-measure reduces to the usual Φ^4_3-measure.

In terms of scaling, the focusing Hartree Φ^4_3-model with $\beta = 2$ corresponds to the Φ^3_3-model and as such, they share some common features. For example, they are both critical with a phase transition, depending on the size of the coupling constant σ. At the same time, however, there are some differences. While the focusing Hartree Φ^4_3-measure with $\beta = 2$ is absolutely continuous with respect to the base massive Gaussian free field μ, the Φ^3_3-measure studied in this paper is singular with respect to the base massive Gaussian free field μ. As mentioned above, this singularity of the Φ^3_3-measure causes an additional difficulty in proving non-normalizability in the strongly nonlinear regime $|\sigma| \gg 1$.

Next, we discuss the dynamical problem associated with the Φ^3_3-measure constructed in Theorem 1.1. In the following, we consider the canonical stochastic quantization equation [65, 67] for the Φ^3_3-measure in (1.6) (with $\gamma = 3$). More precisely, we study the following stochastic damped nonlinear wave equation (SdNLW) with a quadratic nonlinearity, posed on $\mathbb{T}^3$:

$$\partial_t^2 u + \partial_t u + (1-\Delta)u - \sigma u^2 = \sqrt{2}\xi, \qquad (x,t) \in \mathbb{T}^3 \times \mathbb{R}_+, \tag{1.10}$$

where $\sigma \in \mathbb{R} \setminus \{0\}$, u is an unknown function, and ξ denotes a (Gaussian) space-time white noise on $\mathbb{T}^3 \times \mathbb{R}_+$ with the space-time covariance given by

$$\mathbb{E}\big[\xi(x_1,t_1)\xi(x_2,t_2)\big] = \delta(x_1 - x_2)\delta(t_1 - t_2).$$

In this introduction, we keep our discussion at a formal level and do not worry about various renormalizations required to give a proper meaning to the equation (1.10).

With $\vec{u} = (u, \partial_t u)$, define the energy $\mathcal{E}(\vec{u})$ by

$$\begin{aligned}\mathcal{E}(\vec{u}) &= E(u) + \frac{1}{2}\int_{\mathbb{T}^3} (\partial_t u)^2 dx \\ &= \frac{1}{2}\int_{\mathbb{T}^3} |\langle\nabla\rangle u|^2 dx + \frac{1}{2}\int_{\mathbb{T}^3} (\partial_t u)^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx,\end{aligned}$$

[6] By combining the construction of the focusing Hartree Gibbs measure in the critical case ($\beta = 2$) with $0 < \sigma \ll 1$ in [53] and the well-posedness result in [21], this result on the focusing Hartree NLS (1.8) by Bourgain [11] can be extended to the critical case $\beta = 2$ (in the weakly nonlinear regime $0 < \sigma \ll 1$).

where $E(u)$ is as in (1.2). This is precisely the energy (= Hamiltonian) of the (deterministic) nonlinear wave equation (NLW) on $\mathbb{T}^3$ with a quadratic nonlinearity:

$$\partial_t^2 u + (1-\Delta)u - \sigma u^2 = 0. \tag{1.11}$$

Then, by letting $v = \partial_t u$, we can write (1.10) as the first order system:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{E}}{\partial v} \\ -\frac{\partial \mathcal{E}}{\partial u} \end{pmatrix} + \begin{pmatrix} 0 \\ -v + \sqrt{2}\xi \end{pmatrix},$$

which shows that the SdNLW dynamics (1.10) is given as a superposition of the deterministic NLW dynamics (1.11) and the Ornstein-Uhlenbeck dynamics for $v = \partial_t u$:

$$\partial_t v = -v + \sqrt{2}\xi.$$

Now, consider the Gibbs measure $\vec{\rho}$, formally given by

$$\begin{aligned} d\vec{\rho}(\vec{u}) &= Z^{-1} e^{-\mathcal{E}(\vec{u})} d\vec{u} = d\rho \otimes d\mu_0(\vec{u}) \\ &= Z^{-1} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx\bigg) d(\mu \otimes \mu_0)(u,v), \end{aligned} \tag{1.12}$$

where ρ is the Φ^3_3-measure in (1.1) and μ_0 denotes the white noise measure; see (1.15). See Remark 1.13 for the precise definition of the Gibbs measure $\vec{\rho}$. Then, the observation above shows that $\vec{\rho}$ is expected to be invariant under the dynamics of the quadratic SdNLW (1.10). Indeed, from the stochastic quantization point of view, the equation (1.10) is the so-called canonical stochastic quantization equation (namely, the Hamiltonian stochastic quantization) for the Φ^3_3-measure; see [67]. For this reason, it is natural to refer to (1.10) as the *hyperbolic* Φ^3_3*-model*.

Let us now state our main dynamical result in a somewhat formal manner. See Theorem 1.15 for the precise statement.

Theorem 1.4. *Let $\gamma = 3$ and $0 < |\sigma| \ll 1$. Suppose that $A = A(\sigma) > 0$ is sufficiently large as in Theorem 1.1 (i). Then, the hyperbolic Φ^3_3-model* (1.10) *on the three-dimensional torus $\mathbb{T}^3$ (with a proper renormalization) is almost surely globally well-posed with respect to the random initial data distributed by the (renormalized) Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ in* (1.12)*. Furthermore, the Gibbs measure $\vec{\rho}$ is invariant under the resulting dynamics.*

In view of the critical nature of the Φ^3_3-measure, Theorem 1.4 is sharp in the sense that almost sure global well-posedness does not extend to SdNLW with a focusing nonlinearity of a higher order. The construction of the Φ^3_3-measure in Theorem 1.1 requires us to introduce several renormalizations together with the taming by the Wick-ordered L^2-norm. This introduces modifications to the equation (1.10). See Subsection 1.3 and Sections 5 and 6 for the precise formulation of the problem.

Over the last five years, stochastic nonlinear wave equations (SNLW) in the singular setting have been studied extensively in various settings:[7]

$$\partial_t^2 u + \partial_t u + (1-\Delta)u + \mathcal{N}(u) = \xi \tag{1.13}$$

for a power-type nonlinearity [34, 35, 36, 22, 23, 58, 52, 51, 72, 53, 13, 64] and for trigonometric and exponential nonlinearities [56, 59, 57]. We also mention the works [63, 55, 54, 13]

[7]Some of the works mentioned below are on SNLW without damping.

on nonlinear wave equations with rough random initial data. In [35], by combining the paracontrolled calculus, originally introduced in the parabolic setting [33, 17, 47], with the multilinear harmonic analytic approach, more traditional in studying dispersive equations, Gubinelli, Koch, and the first author studied the quadratic SNLW (1.10) (without the damping). The paracontrolled approach in the wave setting was also used in our previous work [53] and was further developed by Bringmann [13]. In order to prove local well-posedness of the hyperbolic Φ^3_3-model (1.10), we also follow the paracontrolled approach, in particular combining the analysis in [35, 53]. See Section 5. As for the globalization part, a naive approach would be to apply Bourgain's invariant measure argument [8, 10]. However, due to the singularity of the Φ^3_3-measure ρ with respect to the base massive Gaussian free field μ (and the fact that the truncated Φ^3_3-measure ρ_N converges to ρ only weakly), there is an additional difficulty to overcome for the hyperbolic Φ^3_3-model. Hence, Bourgain's invariant measure argument is not directly applicable. In the context of the defocusing Hartree cubic NLW on $\mathbb{T}^3$, Bringmann [13] encountered a similar difficulty and developed a new globalization argument. While it is possible to adapt Bringmann's analysis to our current setting, we instead introduce a new alternative argument, which is conceptually simple and straightforward. In particular, we extensively use the variational approach and also use ideas from theory of optimal transport to directly estimate a probability with respect to the limiting Gibbs measure $\vec{\rho}$ (in particular, without going through shifted measures as in [13]). See Subsection 1.3 and Section 6 for details.

Remark 1.5. A slight modification of our proof of Theorem 1.4 yields the corresponding results (namely, almost sure global well-posedness and invariance of the associated Gibbs measure) for the (deterministic) quadratic NLW (1.11) on $\mathbb{T}^3$ in the weakly nonlinear regime.

Remark 1.6. We point out that an analogue of Theorem 1.4 also holds for the parabolic Φ^3_3-model, namely, the stochastic nonlinear heat equation with a quadratic nonlinearity:

$$\partial_t u + (1-\Delta)u - \sigma u^2 = \sqrt{2}\xi, \qquad (x,t) \in \mathbb{T}^3 \times \mathbb{R}_+. \tag{1.14}$$

Thanks to the strong smoothing of the heat propagator, the well-posedness of (1.14) follows from elementary analysis based on the first order expansion (also known as the Da Prato-Debussche trick [19]). See for example [24]. While there is an extra term coming from the taming by the Wick-ordered L^2-norm (see, for example, (1.33) in the hyperbolic case), this term does not cause any issue in the parabolic setting.

Remark 1.7. In [71], the third author introduced a new approach to establish unique ergodicity of Gibbs measures for stochastic dispersive/hyperbolic equations. This was further developed in [73] to prove ergodicity of the hyperbolic Φ^4_2-model, namely (1.13) on $\mathbb{T}^2$ with $\mathcal{N}(u) = u^3$. See also [27] by the third author and Forlano on the asymptotic Feller property of the invariant Gibbs dynamics for the cubic SNLW on $\mathbb{T}^2$ with a slightly smoothed noise. The ergodic property of the hyperbolic Φ^3_3-model is a challenging problem, in particular due to its non-defocusing nature.

1.2. **Construction of the Φ^3_3-measure.** In this subsection, we describe a renormalization procedure and also a taming by the Wick-ordered L^2-norm required to construct the Φ^3_3-measure in (1.6) and make a precise statement (Theorem 1.8). For this purpose, we first fix some notations. Given $s \in \mathbb{R}$, let μ_s denote a Gaussian measure with the Cameron-Martin space $H^s(\mathbb{T}^3)$, formally defined by

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^3} e^{-\frac{1}{2}\langle n \rangle^{2s} |\widehat{u}(n)|^2} d\widehat{u}(n), \tag{1.15}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. When $s = 1$, the Gaussian measure μ_s corresponds to the massive Gaussian free field, while it corresponds to the white noise measure μ_0 when $s = 0$. For simplicity, we set

$$\mu = \mu_1 \qquad \text{and} \qquad \vec{\mu} = \mu \otimes \mu_0. \tag{1.16}$$

Define the index sets Λ and Λ_0 by

$$\Lambda = \bigcup_{j=0}^{2} \mathbb{Z}^j \times \mathbb{N} \times \{0\}^{2-j} \qquad \text{and} \qquad \Lambda_0 = \Lambda \cup \{(0,0,0)\} \tag{1.17}$$

such that $\mathbb{Z}^3 = \Lambda \cup (-\Lambda) \cup \{(0,0,0)\}$. Then, let $\{g_n\}_{n \in \Lambda_0}$ and $\{h_n\}_{n \in \Lambda_0}$ be sequences of mutually independent standard complex-valued[8] Gaussian random variables and set $g_{-n} := \overline{g_n}$ and $h_{-n} := \overline{h_n}$ for $n \in \Lambda_0$. Moreover, we assume that $\{g_n\}_{n \in \Lambda_0}$ and $\{h_n\}_{n \in \Lambda_0}$ are independent from the space-time white noise ξ in (1.10). We now define random distributions $u = u^\omega$ and $v = v^\omega$ by the following Gaussian Fourier series:[9]

$$u^\omega = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle} e_n \qquad \text{and} \qquad v^\omega = \sum_{n \in \mathbb{Z}^3} h_n(\omega) e_n, \tag{1.18}$$

where $e_n = e^{in\cdot x}$. Denoting by $\mathrm{Law}(X)$ the law of a random variable X (with respect to the underlying probability measure $\mathbb{P}$), we then have

$$\mathrm{Law}(u, v) = \vec{\mu} = \mu \otimes \mu_0$$

for (u, v) in (1.18). Note that $\mathrm{Law}(u, v) = \vec{\mu}$ is supported on

$$\mathcal{H}^s(\mathbb{T}^3) := H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$$

for $s < -\frac{1}{2}$ but not for $s \geq -\frac{1}{2}$ (and more generally in $W^{s,p}(\mathbb{T}^3) \times W^{s-1,p}(\mathbb{T}^3)$ for any $1 \leq p \leq \infty$ and $s < -\frac{1}{2}$).

We now consider the Φ^3_3-measure formally given by (1.1). Since u in the support of the massive Gaussian free field μ is merely a distribution, the cubic potential energy in (1.1) is not well defined and thus a proper renormalization is required to give a meaning to the potential energy. In order to explain the renormalization process, we first study the regularized model.

Given $N \in \mathbb{N}$, we denote by $\pi_N = \pi_N^{\mathrm{cube}}$ the frequency projector onto the (spatial) frequencies $\{n = (n_1, n_2, n_3) \in \mathbb{Z}^3 : \max_{j=1,2,3} |n_j| \leq N\}$, defined by

$$\pi_N f = \pi_N^{\mathrm{cube}} f = \sum_{n \in \mathbb{Z}^3} \chi_N(n) \widehat{f}(n) e_n, \tag{1.19}$$

associated with a Fourier multiplier $\chi_N = \chi_N^{\mathrm{cube}}$:

$$\chi_N(n) = \chi_N^{\mathrm{cube}}(n) = \mathbf{1}_Q\big(N^{-1} n\big), \tag{1.20}$$

[8]This means that $g_0, h_0 \sim \mathcal{N}_{\mathbb{R}}(0,1)$ and $\operatorname{Re} g_n, \operatorname{Im} g_n, \operatorname{Re} h_n, \operatorname{Im} h_n \sim \mathcal{N}_{\mathbb{R}}(0, \frac{1}{2})$ for $n \neq 0$.

[9]By convention, we endow $\mathbb{T}^3$ with the normalized Lebesgue measure $dx_{\mathbb{T}^3} = (2\pi)^{-3} dx$.

where Q denotes the cube of side length 2 in $\mathbb{R}^3$ centered at the origin:

$$Q = \big\{ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \max_{j=1,2,3} |\xi_j| \le 1 \big\}. \tag{1.21}$$

It turns out that, due to the critical nature of the Φ^3_3-measure, a choice of frequency projectors makes a difference. See Remark 1.9 and Subsection 1.4 below for discussions on different frequency projectors. In comparing different frequency projectors, we refer to $\pi_N = \pi_N^{\text{cube}}$ in (1.19) as the cube frequency projector in the following.

Let u be as in (1.18) and set $u_N = \pi_N u$. For each fixed $x \in \mathbb{T}^3$, $u_N(x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\sigma_N = \mathbb{E}\big[u_N^2(x)\big] = \sum_{n \in \mathbb{Z}^3} \frac{\chi_N^2(n)}{\langle n \rangle^2} \sim N \longrightarrow \infty, \tag{1.22}$$

as $N \to \infty$. Note that σ_N is independent of $x \in \mathbb{T}^3$ due to the stationarity of μ. We define the Wick powers $:\!u_N^2\!:$ and $:\!u_N^3\!:$ by setting

$$:\!u_N^2\!: = H_2(u_N; \sigma_N) = u_N^2 - \sigma_N \quad \text{and} \quad :\!u_N^3\!: = H_3(u_N; \sigma_N) = u_N^3 - 3\sigma_N u_N,$$

where $H_k(x, \sigma)$ denotes the Hermite polynomial of degree k with variance parameter σ defined by the generating function:

$$e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma).$$

This suggests us to consider the following renormalized potential energy:

$$R_N(u) = -\frac{\sigma}{3} \int_{\mathbb{T}^3} :\!u_N^3\!: dx + A \bigg| \int_{\mathbb{T}^3} :\!u_N^2\!: dx \bigg|^\gamma. \tag{1.23}$$

As in the case of the Φ^4_3-measure in [3], the renormalized potential energy $R_N(u)$ in (1.23) is divergent (as $N \to \infty$) and thus we need to introduce a further renormalization. This leads to the following renormalized potential energy:

$$R_N^\diamond(u) = R_N(u) + \alpha_N, \tag{1.24}$$

where α_N is a diverging constant (as $N \to \infty$) defined in (3.14) below. Finally, we define the truncated (renormalized) Φ^3_3-measure ρ_N by

$$d\rho_N(u) = Z_N^{-1} e^{-R_N^\diamond(u)} d\mu(u), \tag{1.25}$$

where the partition function Z_N is given by

$$Z_N = \int e^{-R_N^\diamond(u)} d\mu(u). \tag{1.26}$$

Then, we have the following construction and non-normalizability of the Φ^3_3-measure. Due to the singularity of the Φ^3_3-measure with respect to the base Gaussian measure $\vec{\mu}$, we need to state our non-normalizability result in a careful manner. Compare this with [53, Theorem 1.15] and [60, Theorem 1.3]. See the beginning of Section 4 for a further discussion.

Theorem 1.8. *There exist $\sigma_1 \ge \sigma_0 > 0$ such that the following statements hold.*

(i) (weakly nonlinear regime). *Let $0 < |\sigma| < \sigma_0$. Then, by choosing $\gamma = 3$ and $A = A(\sigma) > 0$ sufficiently large, we have the uniform exponential integrability of the density*:

$$\sup_{N\in\mathbb{N}} Z_N = \sup_{N\in\mathbb{N}} \Big\| e^{-R_N^\diamond(u)} \Big\|_{L^1(\mu)} < \infty \tag{1.27}$$

and the truncated Φ^3_3-measure ρ_N in (1.25) *converges weakly to a unique limit ρ, formally given by*[10]

$$d\rho(u) = Z^{-1} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, dx - A\bigg|\int_{\mathbb{T}^3} :u^2:\, dx\bigg|^3 - \infty\bigg) d\mu(u). \tag{1.28}$$

In this case, the resulting Φ^3_3-measure ρ and the base massive Gaussian free field μ are mutually singular.

(ii) (strongly nonlinear regime). *Let $|\sigma| > \sigma_1$ and $\gamma \ge 3$. Then, the Φ^3_3-measure is not normalizable in the following sense.*

Fix $\delta > 0$. Given $N \in \mathbb{N}$, let $\nu_{N,\delta}$ be the following tamed version of the truncated Φ^3_3-measure:

$$d\nu_{N,\delta}(u) = Z_{N,\delta}^{-1} \exp\Big(- \delta\|\pi_N u\|^{20}_{B^{-\frac{3}{4}}_{3,\infty}} - R_N^\diamond(u)\Big) d\mu(u). \tag{1.29}$$

Then, $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ converges weakly to some limiting probability measure ν_δ and the following σ-finite version of the Φ^3_3-measure:

$$\begin{aligned} d\overline{\rho}_\delta &= \exp\Big(\delta\|u\|^{20}_{B^{-\frac{3}{4}}_{3,\infty}}\Big) d\nu_\delta \\ &= \lim_{N\to\infty} Z_{N,\delta}^{-1} \exp\Big(\delta\|u\|^{20}_{B^{-\frac{3}{4}}_{3,\infty}}\Big) \exp\Big(- \delta\|\pi_N u\|^{20}_{B^{-\frac{3}{4}}_{3,\infty}} - R_N^\diamond(u)\Big) d\mu(u) \end{aligned}$$

is a well defined measure on $\mathcal{C}^{-100}(\mathbb{T}^3)$. Furthermore, this σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure is not normalizable:

$$\int 1\, d\overline{\rho}_\delta = \infty.$$

Under the same assumption, the sequence $\{\rho_N\}_{N\in\mathbb{N}}$ of the truncated Φ^3_3-measures in (1.25) *does not converge to any weak limit, even up to a subsequence, as measures on the Besov space $B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$.*

In the weakly nonlinear regime, we also prove that the Φ^3_3-measure ρ is absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1)+\sigma\mathfrak{Z}(1)+\mathcal{W}(1))$, where $\mathrm{Law}(Y(1)) = \mu$, $\mathfrak{Z} = \mathfrak{Z}(Y)$ is the limit of the quadratic process $\mathfrak{Z}^N$ defined in (3.11), and the auxiliary quintic process $\mathcal{W} = \mathcal{W}(Y)$ is defined in (A.1). While we do not use this property in this paper, we present the proof in Appendix A for completeness.

As in case of the Φ^4_3-measure in [3], we can prove uniform exponential integrability of the truncated density $e^{-R_N^\diamond(u)}$ in $L^p(\mu)$ only for $p = 1$ due to the second renormalization introduced in (1.24). See also [53, 12] for a similar phenomenon in the case of the defocusing Hartree Φ^4_3-measure. We point out that the renormalized potential energy $R_N^\diamond(u)$ in (1.24)

[10] By hiding α_N in (1.25) into the partition function Z_N, we could also say that the limiting Φ^3_3-measure ρ is formally given by (1.6) (with $\gamma = 3$).

does *not* converge to any limit and neither does the density $e^{-R_N^\diamond(u)}$, which is essentially the source of the singularity of the Φ^3_3-measure with respect to the massive Gaussian free field μ.

As in [53], following the variational approach introduced by Barashkov and Gubinelli [3], we use the Boué-Dupuis variational formula (Lemma 3.1) to prove Theorem 1.8. In fact, we make use of the Boué-Dupuis variational formula in almost every single step of the proof. In proving Theorem 1.8 (i), we first use the variational formula to establish the uniform exponential integrability (1.27) of the truncated density $e^{-R_N^\diamond(u)}$, from which tightness of the truncated Φ^3_3-measure ρ_N in (1.25) follows. See Subsection 3.2. Due to the singularity of the Φ^3_3-measure, we need to apply a change of variables (see (3.12)) in the variational formulation and thus we need to treat the taming part more carefully than that for the focusing Hartree Φ^4_3-measure studied in [53]. See Lemma 3.6 below. This lemma also reflects the critical nature of the Φ^3_3-measure.

In Subsection 3.3, we prove uniqueness of the limiting Φ^3_3-measure. Our main strategy is to follow the approach introduced in our previous work [53] and compare two (arbitrary) subsequences $\rho_{N_{k_1}}$ and $\rho_{N_{k_2}}$, using the variational formula. We point out, however, that, due to the critical nature of the Φ^3_3-measure, our uniqueness argument becomes more involved than that in [53, Subsection 6.3] for the subcritical defocusing Hartree Φ^4_3-measure. In particular, we need to make use of a certain orthogonality property to eliminate a problematic term. See Remark 3.9. See also Subsection 1.4.

In proving the singularity of the Φ^3_3-measure, we once again follow the direct approach introduced in [53], making use of the variational formula. We point out that the proof of the singularity of the Φ^4_3-measure by Barashkov and Gubinelli [4] goes through the shifted measure. On the other hand, as in [53], our proof is based on a direct argument without referring to shifted measures. See Subsection 3.4.

Let us now turn to the strongly nonlinear regime considered in Theorem 1.8 (ii). As mentioned above, due to the singularity of the Φ^3_3-measure, our formulation of the non-normalizability result in Theorem 1.8 (ii) is rather subtle. In the situation where the truncated density $e^{-R_N^\diamond(u)}$ converges to the limiting density (as in [53, 60]), it would suffice to prove

$$\sup_{N\in\mathbb{N}} \mathbb{E}_\mu\Big[e^{-R_N^\diamond(u)}\Big] = \infty, \tag{1.30}$$

since (1.30) would imply that there is no normalization constant which would make the limit of the measure $e^{-R_N^\diamond(u)}d\mu(u)$ into a probability measure. In the current problem, however, the potential energy $R_N^\diamond(u)$ in (1.24) (and the corresponding density $e^{-R_N^\diamond(u)}$) does *not* converge to any limit. Thus, even if we prove a statement of the form (1.30), we may still choose a sequence of constants $\widehat{Z}_N$ such that the measures $\widehat{Z}_N^{-1}e^{-R_N^\diamond(u)}d\mu$ have a weak limit. A similar phenomenon happens for the Φ^4_3-measure, where one needs to introduce the second order renormalization; see [3]. The non-convergence of the truncated Φ^3_3-measures claimed in Theorem 1.8 (ii) tells us that this can not happen for the Φ^3_3-measure. See also Remark 1.10 below.

Our strategy is to first construct a σ-finite version of the Φ^3_3-measure and then prove its non-normalizability. As stated in Theorem 1.8 (ii), we first introduce a tamed version $\nu_{N,\delta}$ of the truncated Φ^3_3-measure, by introducing an appropriate taming function F; see (4.6) below. The first step is to show that this tamed truncated Φ^3_3-measure $\nu_{N,\delta}$ converges weakly to some limit ν_δ (Proposition 4.1). We then define a σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure by setting

$$d\overline{\rho}_\delta = e^{\delta F(u)} d\nu_\delta$$

and prove that $\overline{\rho}_\delta$ is not normalizable (Proposition 4.2). Here, the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure clearly depends on the choice of a taming function F. Our choice is quite natural since the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure is absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$, just like the (normalizable) Φ^3_3-measure in the weakly nonlinear regime discussed above. See Remark A.3.

Once we construct the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure, our argument follows closely the strategy introduced in [53, 60] for establishing non-normalizability, using the Boué-Dupuis variational formula. For this approach, we need to construct a drift achieving the desired divergence, where (the antiderivative of) the drift is designed to look like "$-Y(1)$ + a perturbation", where $\mathrm{Law}(Y(1)) = \mu$; see (4.41) below. Here, the perturbation term is bounded in $L^2(\mathbb{T}^3)$ but has a large L^3-norm, thus having a highly concentrated profile, such as a soliton or a finite time blowup profile. As compared to our previous works [53, 60], there is an additional difficulty in proving the non-normalizability claim in Theorem 1.8 (ii) due to the singularity of the Φ^3_3-measure, which forces us to use a change of variables (see (3.12)) in the variational formulation. See Remark 4.7. The non-convergence of the truncated Φ^3_3-measures ρ_N stated in Theorem 1.8 (ii) follows as a corollary to the non-normalizability of the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure; see Proposition 4.4 and Subsection 4.4. If the Φ^3_3-measure existed as a probability measure in the strongly nonlinear regime, then we would expect its support to be contained in $\mathcal{C}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)$ for any $\varepsilon > 0$, just as in the weakly nonlinear regime (and the Φ^4_3-measure). For this reason, the Besov space $B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$ is a quite natural space to consider. The restriction $\gamma \geq 3$ in Theorem 1.8 (ii) comes from the construction of the tamed version ν_δ of the Φ^3_3-measure; see (4.13) below. For $\gamma < 3$, the taming by the Wick-ordered L^2-norm in (1.6) becomes weaker and thus we expect an analogous non-normalizability result to hold.

Remark 1.9. We prove Theorem 1.8 for the cube frequency projector $\pi_N = \pi_N^{\mathrm{cube}}$ defined in (1.19). If we instead consider the ball frequency projector π_N^{ball} defined in (1.41) below, then our argument for the non-convergence claim in the strongly nonlinear regime (Proposition 4.4) breaks down, while the other claims in Theorem 1.8 remain true for the ball frequency projector π_N^{ball}. If we consider the smooth frequency projector π_N^{smooth} defined in (1.42) below, then our argument for the uniqueness of the limiting Φ^3_3-measure in the weakly nonlinear regime (Proposition 3.8) breaks down. In particular, the latter issue is closely related to the critical nature of the Φ^3_3-model and, while we believe that uniqueness of the limiting Φ^3_3-measure holds even in the case of the smooth frequency projector π_N^{smooth}, it seems non-trivial to prove this claim by a modification of our argument. We point out that the same issue also appears in showing uniqueness of the limit ν_δ of the tamed version $\nu_{N,\delta}$ of the truncated Φ^3_3-measure in (1.29) in the strongly nonlinear regime (Proposition 4.1) and in the dynamical part (Proposition 6.10). See Subsection 1.4 for a further discussion. See also Remarks 3.9 and 4.12.

Remark 1.10. In the strongly nonlinear regime, Theorem 1.8 (ii) tells us that the truncated Φ^3_3-measures ρ_N do not converge weakly to any limit as measures on $B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$. It is, however, possible that the truncated Φ^3_3-measures converges weakly to some limit (say, the Dirac delta measure δ_0 on the trivial function) as measures on some space with a very weak topology, say $\mathcal{C}^{-100}(\mathbb{T}^3)$. Theorem 1.8 (ii) shows that if such weak convergence takes place, it must do so in a very pathological manner.

Remark 1.11. The second renormalization in (1.24) (i.e. the cancellation of the diverging constant α_N) appears only at the level of the measure. The associated equation (see (1.38) below) does not see this additional renormalization.

Remark 1.12. It is of interest to investigate a threshold value $\sigma_* > 0$ such that the construction of the Φ^3_3-measure (Theorem 1.8 (i)) holds for $0 < |\sigma| < \sigma_*$, while the non-normalizability of the Φ^3_3-measure (Theorem 1.8 (ii)) holds for $|\sigma| > \sigma_*$. If such a threshold value σ_* could be determined, it would also be of interest to determine whether the Φ^3_3-measure is normalizable at the threshold $|\sigma| = \sigma_*$. Such a problem, however, requires optimizing all the estimates in the proof of Theorem 1.8 and is out of reach at this point. See a recent work [61] by Sosoe and the first and third authors for such analysis in the one-dimensional case.

Remark 1.13. Consider the truncated Gibbs measure $\vec{\rho}_N = \rho_N \otimes \mu_0$ for the hyperbolic Φ^3_3-model (1.10) with the density:

$$d\vec{\rho}_N(u, v) = Z_N^{-1} e^{-R_N^\diamond(u)} d\vec{\mu}(u, v), \tag{1.31}$$

where $R_N^\diamond(u)$ and $\vec{\mu}$ are as in (1.24) and (1.16), respectively. Since the potential energy $R_N^\diamond(u)$ is independent of the second component v, Theorem 1.8 directly applies to the truncated Gibbs measure $\vec{\rho}_N$. In particular, in the weakly nonlinear regime ($0 < |\sigma| < \sigma_0$), the truncated Gibbs measure $\vec{\rho}_N$ converges weakly to the limiting Gibbs measure

$$\vec{\rho} = \rho \otimes \mu_0, \tag{1.32}$$

where ρ is the limiting Φ^3_3-measure constructed in Theorem 1.8 (i). Moreover, the limiting Gibbs measure $\vec{\rho}$ and the base Gaussian measure $\vec{\mu} = \mu \otimes \mu_0$ are mutually singular.

1.3. **Hyperbolic Φ^3_3-model.** In this subsection, we provide a precise meaning to the hyperbolic Φ^3_3-model (1.10) and make Theorem 1.4 more precise. By considering the Langevin equation for the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ constructed in Remark 1.13, we formally obtain the following quadratic SdNLW (= the hyperbolic Φ^3_3-model):

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u - \sigma \ :u^2: + M(:u^2:)u = \sqrt{2}\xi, \tag{1.33}$$

where M is defined by

$$M(w) = 6A\bigg|\int_{\mathbb{T}^3} w dx\bigg|\int_{\mathbb{T}^3} w dx. \tag{1.34}$$

Here, the term $M(:u^2:)u$ in (1.33) comes from the taming by the Wick-ordered L^2-norm appearing in (1.28). The term $:u^2:$ denotes the Wick renormalization[11] of u^2, formally given by $:u^2: = u^2 - \infty$. Namely, the equation (1.33) is just a formal expression at this point. In the following, we provide the meaning of the process u in (1.33) by a limiting procedure. In Section 5, we use the paracontrolled calculus to give a more precise meaning to (1.33) by rewriting it into a system for three unknowns. See (5.28) below.

[11] In order to give a proper meaning to $:u^2:$, we need to assume a structure on u. We postpone this discussion to Section 5.

Given $N \in \mathbb{N}$, we consider the following quadratic SdNLW with a truncated noise:

$$\partial_t^2 u_N + \partial_t u_N + (1-\Delta)u_N - \sigma :\! u_N^2 \!: + M(:\! u_N^2 \!:)u_N = \sqrt{2}\pi_N \xi, \tag{1.35}$$

where π_N is as in (1.19) and the renormalized nonlinearity is defined by

$$:\! u_N^2 \!: = u_N^2 - \sigma_N \tag{1.36}$$

with σ_N as in (1.22). See also (5.10). In Section 5, we study SdNLW (1.35) with the truncated noise and prove the following local well-posedness statement for the hyperbolic Φ^3_3-model.

Theorem 1.14. *Given $s > \frac{1}{2}$, let $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$. Let $(\phi_0^\omega, \phi_1^\omega)$ be a pair of the Gaussian random distributions with $\mathrm{Law}(\phi_0^\omega, \phi_1^\omega) = \vec{\mu} = \mu \otimes \mu_0$. Then, the solution $(u_N, \partial_t u_N)$ to the quadratic SdNLW (1.35) with the truncated noise and the initial data*

$$(u_N, \partial_t u_N)|_{t=0} = (u_0, u_1) + \pi_N(\phi_0^\omega, \phi_1^\omega) \tag{1.37}$$

converges to a stochastic process $(u, \partial_t u) \in C([0,T]; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ almost surely, where $T = T(\omega)$ is an almost surely positive stopping time.

The limit $(u, \partial_t u)$ formally satisfies the equation (1.33). Here, we took the initial data of the form (1.37) for simplicity of the presentation. A slight modification of the proof yields an analogue of Theorem 1.14 with deterministic initial data $(u_N, \partial_t u_N)|_{t=0} = (u_0, u_1)$. In this case, we need to choose a diverging constant σ_N, depending on t. See [34, 35] for such an argument.

We follow the paracontrolled approach in [35], where the quadratic SNLW on $\mathbb{T}^3$ was studied. However, the additional term M in (1.33) and (1.35) contains an ill-defined product $:\! u^2 \!:$ (or $:\! u_N^2 \!:$ in the limiting sense). In order to treat this term, the analysis in [35] is not sufficient and thus we also need to adapt the paracontrolled analysis in our previous work [53] and rewrite the equation into a system for three unknowns. (Note that in [35], the resulting system was for two unknowns.) We also point out that, unlike [35] (see also [47] in the context of the parabolic Φ^4_3-model), the equation for a less regular, paracontrolled component in our system (see (5.28) below) is nonlinear in the unknowns. We then construct a continuous map from the space of enhanced data sets to solutions. While the proof of Theorem 1.14 follows from a slight modification of the arguments in [35, 53], we present details in Section 5 for readers' convenience.

In order to establish our main goal in the dynamical part of the program (Theorem 1.4), we need to study the hyperbolic Φ^3_3-model with the Gibbs measure initial data. Since the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ in (1.32) and the Gaussian field $\vec{\mu} = \mu \otimes \mu_0$ are mutually singular as shown in Theorem 1.8, it may seem that the local well-posedness in Theorem 1.14 with the Gaussian initial data (plus smoother deterministic initial data) is irrelevant. However, as we see in Section 6, the analysis for proving Theorem 1.14 provides us with a good intuition of the well-posedness problem for the hyperbolic Φ^3_3-model with the Gibbs measure initial data. Furthermore, one of advantages of considering the Gaussian initial data (as in (1.37)) is that it provides a clear reason why σ_N appears in the renormalization in (1.36), since σ_N is nothing but the variance of the first order approximation (= the stochastic convolution defined in (5.4)) to the solution to (1.35); see (5.10). This is the main reason for considering the local-in-time problem with the Gaussian initial data.

Next, we turn our attention to the globalization problem. For this purpose, we need to consider a different approximating equation. Given $N \in \mathbb{N}$, we consider the truncated hyperbolic Φ^3_3-model:

$$\begin{aligned} &\partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N \\ &\quad - \sigma \pi_N\big(:\!(\pi_N u_N)^2\!: \big) + M(:\!(\pi_N u_N)^2\!:) \pi_N u_N = \sqrt{2}\xi, \end{aligned} \tag{1.38}$$

where $:(\pi_N u_N)^2 : = (\pi_N u_N)^2 - \sigma_N$. A slight modification of the proof of Theorem 1.14 yields uniform (in N) local well-posedness of the truncated equation (1.38) (with the same limiting process $(u, \partial_t u)$ as in Theorem 1.14) for the initial data of the form (1.37). By exploiting (formal) invariance of the truncated Gibbs measure $\vec{\rho}_N$ in (1.31),[12] we see that the truncated hyperbolic Φ^3_3-model (1.38) is almost surely globally well-posed with respect to the truncated Gibbs measure $\vec{\rho}_N$ and, moreover, $\vec{\rho}_N$ is invariant under the resulting dynamics; see Lemma 6.4.

We now state almost sure global well-posedness of the hyperbolic Φ^3_3-model.

Theorem 1.15. *Let $0 < |\sigma| < \sigma_0$ and $A = A(\sigma) > 0$ is sufficiently large as in Theorem 1.8 (i). Then, there exists a non-trivial stochastic process $(u, \partial_t u) \in C(\mathbb{R}_+; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ for any $\varepsilon > 0$ such that, given any $T > 0$, the solution $(u_N, \partial_t u_N)$ to the truncated hyperbolic Φ^3_3-model* (1.38) *with the random initial data distributed by the truncated Gibbs measure $\vec{\rho}_N = \rho_N \otimes \mu_0$ in* (1.31) *converges to $(u, \partial_t u)$ in $C([0,T]; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$. Furthermore, we have* $\mathrm{Law}\big((u(t), \partial_t u(t))\big) = \vec{\rho}$ *for any $t \in \mathbb{R}_+$.*

The main difficulty in proving Theorem 1.15 comes from the mutual singularity of the Gibbs measure $\vec{\rho}$ and the base Gaussian measure $\vec{\mu}$ (and the fact that the truncated Gibbs measure $\vec{\rho}_N$ converges to $\vec{\rho}$ only weakly) such that Bourgain's invariant measure argument [8, 10] is not directly applicable. In the context of the defocusing Hartree NLW on $\mathbb{T}^3$, Bringmann [13] encountered the same issue, and introduced a new globalization argument, where a large time stability theory (in the paracontrolled setting) plays a crucial role. Bourgain's invariant measure argument is often described (see [13]) as "the probabilistic version of a deterministic global theory using a (sub-critical) conservation law". In [13], Bringmann considers the quantity $\vec{\rho}_M((u_N, \partial_t u_N)(t) \in A)$, where $(u_N, \partial_t u_N)$ is the solution to the truncated equation with a cutoff parameter N. While such an expression is not conserved for $M \neq N$, it should be close to being constant in time when $M, N \gg 1$. For this reason, he describes his new globalization argument as "the probabilistic version of a deterministic global theory using almost conservation laws". We also point out that Bringmann's analysis relies on the fact that the (truncated) Gibbs measure is absolutely continuous with respect to a shifted measure [53, 12] (as in Appendix A below).

While it is possible to follow Bringmann's approach, we instead introduce a new simple alternative argument to prove almost sure global well-posedness. Our approach consists of the following four steps:

1. We first establish a uniform (in N) exponential integrability of the truncated enhanced data set (see (6.10) below) with respect to the truncated measure (Proposition 6.5). We directly achieve this by combining the variational approach with space-time estimates

[12] This is essentially Bourgain's invariant measure argument [8] applied to the truncated hyperbolic Φ^3_3-model (1.38), whose nonlinear part is finite dimensional.

without any reference to (the truncated version of) the shifted measure constructed in Appendix A.

2. Next, by a slight modification of the local well-posedness argument, we prove a stability result (Proposition 6.8). This is done by a simple contraction argument, with an exponentially decaying weight in time.

3. Then, using the invariance of the truncated Gibbs measure, we establish a uniform (in N) control on the solution to the truncated system (see (6.58) below) with a large probability. The argument relies on a discrete Gronwall argument but is very straightforward.

4. In the last step, we study the convergence property of the distributions of the truncated enhanced data sets, emanating from the truncated Gibbs measures. In particular, we study the Wasserstein-1 distance of such a distribution with the limiting distribution, using ideas from theory of optimal transport (the Kantorovich duality). See Proposition 6.10 below.

Once we establish these four steps, Theorem 1.15 follows in a straightforward manner. We believe that our new globalization argument is very simple, at least at a conceptual level, and is easy to implement. See Section 6 for further details.

Remark 1.16. (i) In this paper, we treated the hyperbolic Φ^3_3-model. In the three-dimensional case, it is possible to consider the defocusing quartic interaction potential, namely the Φ^4_3-measure. This leads to the following hyperbolic Φ^4_3-model on $\mathbb{T}^3$:

$$\partial_t^2 u + \partial_t u + (1-\Delta)u + u^3 = \sqrt{2}\xi. \tag{1.39}$$

Over the last ten years, the parabolic Φ^4_3-model:

$$\partial_t u + (1-\Delta)u + u^3 = \sqrt{2}\xi, \tag{1.40}$$

has been studied extensively by many authors. See [38, 33, 17, 42, 47, 48, 1, 31] and references therein. Up to date, the well-posedness issue of the hyperbolic Φ^4_3-model (1.39) remains as an important open problem.[13] In [64], using Bringmann's analysis [13], Y. Wang, Zine, and the first author recently proved local well-posedness of the cubic stochastic NLW[14] on $\mathbb{T}^3$ with an almost space-time white noise forcing (i.e. replacing ξ by $\langle \nabla \rangle^{-\alpha}\xi$ for any $\alpha > 0$ in (1.39)).

(ii) In the parabolic setting (1.14), there is no issue is applying Bourgain's invariant measure argument in the usual manner since it is possible to prove local well-posedness with deterministic initial data at the regularity of the Φ^3_3-measure. See [39] in the case of the parabolic Φ^4_3-model (1.40).

1.4. **On frequency projectors.** We conclude this introduction by discussing different frequency projectors. Given $N \in \mathbb{N}$, define the ball frequency projector π_N^{ball} onto the frequencies $\{n \in \mathbb{Z}^3 : |n| \le N\}$ by setting

$$\pi_N^{\text{ball}} f = \sum_{n \in \mathbb{Z}^3} \chi_N^{\text{ball}}(n) \widehat{f}(n) e_n, \tag{1.41}$$

[13] In a recent preprint [14], Bringmann, Deng, Nahmod, and Yue resolved this open problem in the case of the Gibbsian initial data with no stochastic forcing.

[14] In [64], the authors considered the undamped SNLW but the same analysis applies to the damped SNLW.

associated with a Fourier multiplier

$$\chi_N^{\mathrm{ball}}(n) = \mathbf{1}_B\big(N^{-1}n\big),$$

where B denotes the unit ball in $\mathbb{R}^3$ centered at the origin:

$$B = \big\{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi| \le 1\big\}.$$

We also define the smooth frequency projector π_N^{smooth} onto the frequencies $\{n \in \mathbb{Z}^3 : |n| \le N\}$ by setting

$$\pi_N^{\mathrm{smooth}} f = \sum_{n\in\mathbb{Z}^3} \chi_N^{\mathrm{smooth}}(n)\widehat{f}(n)e_n, \tag{1.42}$$

associated with a Fourier multiplier

$$\chi_N^{\mathrm{smooth}}(n) = \chi\big(N^{-1}n\big)$$

for some fixed even function $\chi \in C_c^\infty(\mathbb{R}^3; [0,1])$ with $\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^3 : |\xi| \le 1\}$ and $\chi \equiv 1$ on $\{\xi \in \mathbb{R}^3 : |\xi| \le \frac{1}{2}\}$.

In Subsections 1.2 and 1.3, we stated the (non-)construction of the Φ_3^3-measure (Theorem 1.8) and the dynamical results for the hyperbolic Φ_3^3-model (Theorems 1.14 and 1.15), using the cube frequency projector $\pi_N = \pi_N^{\mathrm{cube}}$ defined in (1.19). In comparison with the ball frequency projector π_N^{ball} and the smooth frequency projector π_N^{smooth}, there are two important properties that the cube frequency projector π_N^{cube} possesses simultaneously.

(i) As a composition of (modulated) Hilbert transforms in different coordinate directions, the cube frequency projector π_N^{cube} is uniformly (in N) bounded in $L^p(\mathbb{T}^3)$ for any $1 < p < \infty$.

(ii) The cube frequency projector is indeed a projection, in particular satisfying $(\mathrm{Id} - \pi_N^{\mathrm{cube}})\pi_N^{\mathrm{cube}} = 0$.

We make use of both of these properties in a crucial manner. Note that while the ball frequency projector π_N^{ball} satisfies the property (ii), it is bounded in $L^p(\mathbb{T}^3)$ only for $p = 2$ [26] and thus the property (i) is not satisfied. On the other hand, by Young's inequality, the smooth frequency projector π_N^{smooth} is bounded on $L^p(\mathbb{T}^3)$ for any $1 \le p \le \infty$ but it does not satisfy the property (ii).

Roughly speaking, Theorem 1.8 on the (non-)construction of the Φ_3^3-measure consists of the following five results:

(1) the uniform exponential integrability (1.27) and tightness of the truncated Φ_3^3-measures ρ_N in the weakly nonlinear regime,

(2) uniqueness of the limiting Φ_3^3-measure in the weakly nonlinear regime,

(3) mutual singularity of the Φ_3^3-measure and the base Gaussian free field in the weakly nonlinear regime,

(4) non-normalizability of the Φ_3^3-measure in the strongly nonlinear regime,

(5) non-convergence of the truncated Φ_3^3-measures ρ_N in the strongly nonlinear regime.

Starting with the truncated Φ_3^3-measures ρ_N in (1.25) defined in terms of the cube frequency projector π_N^{cube} in (1.19), we establish (1) - (5) in Sections 3 and 4. In proving (5), the property (i) above plays an important role and thus our argument does not apply to the ball frequency projector π_N^{ball}. See Remark 4.12.

In establishing (2), uniqueness of the limiting Φ_3^3-measure (Proposition 3.8), we crucially make use of the property (ii) to show that a certain problematic term vanishes; see I_2 in (3.72). It turns out that this problematic term reflects the critical nature of the problem, where there is no room to spare, not even logarithmically. In the case of the cube frequency projector π_N^{cube}, the property (ii) allows us to conclude that this term in fact vanishes. In the case of the smooth projector π_N^{smooth}, the property (ii) does not hold and thus we need to show by hand that this problematic term tends to 0. As mentioned above, however, there is no room to spare and it seems rather non-trivial to prove such a convergence result by a modification of our argument. See Remark 3.9. In establishing (4) and (5), we first construct a reference measure ν_δ as a limit of the tamed version $\nu_{N,\delta}$ of the truncated Φ_3^3-measure in (1.29) (Proposition 4.1). With the smooth projector π_N^{smooth}, the same issue also appears in showing uniqueness of the limit ν_δ.

While we believe that Theorem 1.8 holds for both the ball frequency projector π_N^{ball} (in particular (5) above) and the smooth frequency projector π_N^{smooth} (in particular (2) above), we do not pursue these issues further in this paper in order to keep the paper length under control.

Let us now turn to the dynamical part. As for the smooth frequency projector π_N^{smooth}, there is no modification needed for the local well-posedness part. However, as mentioned above, there is no uniqueness of the limiting Φ_3^3-measure in this case. Furthermore, we point out that the proof of Proposition 6.10 also breaks down for the smooth frequency projector π_N^{smooth} since part of the argument relies on the proof of Proposition 3.8; see (6.120). On the other hand, as for the ball frequency projector π_N^{ball}, both Theorems 1.14 and 1.15 hold as they are stated. However, the proof of the local well-posedness part needs to be modified in view of the unboundedness of the ball frequency projector π_N^{ball} in the Strichartz spaces (see (5.47)). Note that this issue can be easily remedied by using the Fourier restriction norm method via the (L^2-based) $X^{s,b}$-spaces as in [63, 13, 64].

2. Notations and basic lemmas

In describing regularities of functions and distributions, we use $\varepsilon > 0$ to denote a small constant. We usually suppress the dependence on such $\varepsilon > 0$ in an estimate. For $a, b > 0$, we use $a \lesssim b$ to mean that there exists $C > 0$ such that $a \leq Cb$. By $a \sim b$, we mean that $a \lesssim b$ and $b \lesssim a$.

In dealing with space-time functions, we use the following short-hand notation $L^q_T L^r_x = L^q([0,T]; L^r(\mathbb{T}^3))$, etc.

2.1. **Sobolev and Besov spaces.**

Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We define the L^2-based Sobolev space $H^s(\mathbb{T}^d)$ by the norm:

$$\|f\|_{H^s} = \|\langle n \rangle^s \widehat{f}(n)\|_{\ell^2_n}.$$

We also define the L^p-based Sobolev space $W^{s,p}(\mathbb{T}^d)$ by the norm:

$$\|f\|_{W^{s,p}} = \big\|\mathcal{F}^{-1}[\langle n \rangle^s \widehat{f}(n)]\big\|_{L^p}.$$

When $p = 2$, we have $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$.

Let $\phi : \mathbb{R} \to [0,1]$ be a smooth bump function supported on $[-\frac{8}{5}, \frac{8}{5}]$ and $\phi \equiv 1$ on $\big[-\frac{5}{4}, \frac{5}{4}\big]$. For $\xi \in \mathbb{R}^d$, we set $\varphi_0(\xi) = \phi(|\xi|)$ and

$$\varphi_j(\xi) = \phi\big(\tfrac{|\xi|}{2^j}\big) - \phi\big(\tfrac{|\xi|}{2^{j-1}}\big) \tag{2.1}$$

for $j \in \mathbb{N}$. Then, for $j \in \mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$, we define the Littlewood-Paley projector $\mathbf{P}_j$ as the Fourier multiplier operator with a symbol φ_j. Note that we have

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$$

for each $\xi \in \mathbb{R}^d$. Thus, we have

$$f = \sum_{j=0}^{\infty} \mathbf{P}_j f.$$

Let us now recall the definition and basic properties of paraproducts introduced by Bony [6]. See [2, 33] for further details. Given two functions f and g on $\mathbb{T}^3$ of regularities s_1 and s_2, we write the product fg as

$$\begin{aligned} fg &= f \mathbin{\olessthan} g + f \mathbin{\oequal} g + f \mathbin{\ogreaterthan} g \\ &:= \sum_{j<k-2} \mathbf{P}_j f \, \mathbf{P}_k g + \sum_{|j-k|\leq 2} \mathbf{P}_j f \, \mathbf{P}_k g + \sum_{k<j-2} \mathbf{P}_j f \, \mathbf{P}_k g. \end{aligned} \tag{2.2}$$

The first term $f \mathbin{\olessthan} g$ (and the third term $f \mathbin{\ogreaterthan} g$) is called the paraproduct of g by f (the paraproduct of f by g, respectively) and it is always well defined as a distribution of regularity $\min(s_2, s_1 + s_2)$. On the other hand, the resonant product $f \mathbin{\oequal} g$ is well defined in general only if $s_1 + s_2 > 0$. See Lemma 2.2 below. In the following, we also use the notation $f \mathbin{\ogreaterequal} g := f \mathbin{\ogreaterthan} g + f \mathbin{\oequal} g$. In studying a nonlinear problem, main difficulty usually arises in making sense of a product. Since paraproducts are always well defined, such a problem comes from a resonant product. In particular, when the sum of regularities is negative, we need to impose an extra structure to make sense of a (seemingly) ill-defined resonant product. See Section 5 for a further discussion on the paracontrolled approach in this direction.

Next, we recall the basic properties of the Besov spaces $B^s_{p,q}(\mathbb{T}^d)$ defined by the norm:

$$\|u\|_{B^s_{p,q}} = \Big\| 2^{sj} \|\mathbf{P}_j u\|_{L^p_x} \Big\|_{\ell^q_j(\mathbb{Z}_{\geq 0})}.$$

We denote the Hölder-Besov space by $\mathcal{C}^s(\mathbb{T}^d) = B^s_{\infty,\infty}(\mathbb{T}^d)$. Note that (i) the parameter s measures differentiability and p measures integrability, (ii) $H^s(\mathbb{T}^d) = B^s_{2,2}(\mathbb{T}^d)$, and (iii) for $s > 0$ and not an integer, $\mathcal{C}^s(\mathbb{T}^d)$ coincides with the classical Hölder spaces $C^s(\mathbb{T}^d)$; see [30].

We recall the basic estimates in Besov spaces. See [2, 37] for example.

Lemma 2.1. *The following estimates hold.*
(i) (interpolation) *Let $s, s_1, s_2 \in \mathbb{R}$ and $p, p_1, p_2 \in (1, \infty)$ such that $s = \theta s_1 + (1-\theta) s_2$ and $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ for some $0 < \theta < 1$. Then, we have*

$$\|u\|_{W^{s,p}} \lesssim \|u\|^{\theta}_{W^{s_1,p_1}} \|u\|^{1-\theta}_{W^{s_2,p_2}}. \tag{2.3}$$

(ii) (immediate embeddings) *Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$. Then, we have*

$$\begin{aligned} \|u\|_{B^{s_1}_{p_1,q_1}} &\lesssim \|u\|_{B^{s_2}_{p_2,q_2}} \qquad \text{for } s_1 \le s_2,\ p_1 \le p_2, \text{ and } q_1 \ge q_2, \\ \|u\|_{B^{s_1}_{p_1,q_1}} &\lesssim \|u\|_{B^{s_2}_{p_1,\infty}} \qquad \text{for } s_1 < s_2, \\ \|u\|_{B^{0}_{p_1,\infty}} &\lesssim \|u\|_{L^{p_1}} \lesssim \|u\|_{B^{0}_{p_1,1}}. \end{aligned} \tag{2.4}$$

(iii) (Besov embedding) *Let $1 \le p_2 \le p_1 \le \infty$, $q \in [1, \infty]$, and $s_2 \ge s_1 + d\big(\frac{1}{p_2} - \frac{1}{p_1}\big)$. Then, we have*

$$\|u\|_{B^{s_1}_{p_1,q}} \lesssim \|u\|_{B^{s_2}_{p_2,q}}.$$

(iv) (duality) *Let $s \in \mathbb{R}$ and $p, p', q, q' \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Then, we have*

$$\bigg| \int_{\mathbb{T}^d} uv \, dx \bigg| \le \|u\|_{B^s_{p,q}} \|v\|_{B^{-s}_{p',q'}}, \tag{2.5}$$

where $\int_{\mathbb{T}^d} uv\, dx$ denotes the duality pairing between $B^s_{p,q}(\mathbb{T}^d)$ and $B^{-s}_{p',q'}(\mathbb{T}^d)$.

(v) (fractional Leibniz rule) *Let $p, p_1, p_2, p_3, p_4 \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}$. Then, for every $s > 0$, we have*

$$\|uv\|_{B^s_{p,q}} \lesssim \|u\|_{B^s_{p_1,q}} \|v\|_{L^{p_2}} + \|u\|_{L^{p_3}} \|v\|_{B^s_{p_4,q}}. \tag{2.6}$$

The interpolation (2.3) follows from the Littlewood-Paley characterization of Sobolev norms via the square function and Hölder's inequality.

Lemma 2.2 (paraproduct and resonant product estimates)**.** *Let $s_1, s_2 \in \mathbb{R}$ and $1 \le p, p_1, p_2, q \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, we have*

$$\|f \mathbin{\otimes} g\|_{B^{s_2}_{p,q}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.7}$$

When $s_1 < 0$, we have

$$\|f \mathbin{\otimes} g\|_{B^{s_1+s_2}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.8}$$

When $s_1 + s_2 > 0$, we have

$$\|f \mathbin{\ominus} g\|_{B^{s_1+s_2}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.9}$$

The product estimates (2.7), (2.8), and (2.9) follow easily from the definition (2.2) of the paraproduct and the resonant product. See [2, 46] for details of the proofs in the non-periodic case (which can be easily extended to the current periodic setting).

We also recall the following product estimate from [34].

Lemma 2.3. *Let $0 \le s \le 1$.*

(i) *Let $1 < p_j, q_j, r < \infty$, $j = 1, 2$ such that $\frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}$. Then, we have*

$$\|\langle \nabla \rangle^s (fg)\|_{L^r(\mathbb{T}^3)} \lesssim \|\langle \nabla \rangle^s f\|_{L^{p_1}(\mathbb{T}^3)} \|g\|_{L^{q_1}(\mathbb{T}^3)} + \|f\|_{L^{p_2}(\mathbb{T}^3)} \|\langle \nabla \rangle^s g\|_{L^{q_2}(\mathbb{T}^3)}.$$

(ii) *Let $1 < p, q, r < \infty$ such that $s \ge 3\big(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\big)$. Then, we have*

$$\|\langle \nabla \rangle^{-s} (fg)\|_{L^r(\mathbb{T}^3)} \lesssim \|\langle \nabla \rangle^{-s} f\|_{L^p(\mathbb{T}^3)} \|\langle \nabla \rangle^s g\|_{L^q(\mathbb{T}^3)}.$$

Note that while Lemma 2.3 (ii) was shown only for $s = 3\big(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\big)$ in [34], the general case $s \ge 3\big(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\big)$ follows the embedding $L^{p_1}(\mathbb{T}^3) \subset L^{p_2}(\mathbb{T}^3)$, $p_1 \ge p_2$.

2.2. **On discrete convolutions.** Next, we recall the following basic lemma on a discrete convolution.

Lemma 2.4. *Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha + \beta > d \qquad \text{and} \qquad \alpha < d.$$

Then, we have

$$\sum_{n = n_1 + n_2} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{-\alpha + \lambda}$$

for any $n \in \mathbb{Z}^d$, where $\lambda = \max(d - \beta, 0)$ when $\beta \neq d$ and $\lambda = \varepsilon$ when $\beta = d$ for any $\varepsilon > 0$.

Lemma 2.4 follows from elementary computations. See, for example, [29, Lemma 4.2] and [48, Lemma 4.1].

2.3. **Tools from stochastic analysis.** We conclude this section by recalling useful lemmas from stochastic analysis. See [68, 50] for basic definitions. Let (H, B, μ) be an abstract Wiener space. Namely, μ is a Gaussian measure on a separable Banach space B with $H \subset B$ as its Cameron-Martin space. Given a complete orthonormal system $\{e_j\}_{j \in \mathbb{N}} \subset B^*$ of $H^* = H$, we define a polynomial chaos of order k to be an element of the form $\prod_{j=1}^\infty H_{k_j}(\langle x, e_j \rangle)$, where $x \in B$, $k_j \neq 0$ for only finitely many j's, $k = \sum_{j=1}^\infty k_j$, H_{k_j} is the Hermite polynomial of degree k_j, and $\langle \cdot, \cdot \rangle = {}_B\langle \cdot, \cdot \rangle_{B^*}$ denotes the B-B^* duality pairing. We then denote the closure of polynomial chaoses of order k under $L^2(B, \mu)$ by $\mathcal{H}_k$. The elements in $\mathcal{H}_k$ are called homogeneous Wiener chaoses of order k. We also set

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j$$

for $k \in \mathbb{N}$.

As a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup due to Nelson [49], we have the following Wiener chaos estimate [69, Theorem I.22]. See also [70, Proposition 2.4].

Lemma 2.5. *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}$$

for any finite $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$.

Lastly, we recall the following orthogonality relation for the Hermite polynomials. See [50, Lemma 1.1.1].

Lemma 2.6. *Let f and g be jointly Gaussian random variables with mean zero and variances σ_f and σ_g. Then, we have*

$$\mathbb{E}\big[H_k(f; \sigma_f) H_\ell(g; \sigma_g)\big] = \delta_{k\ell} k! \big\{\mathbb{E}[fg]\big\}^k,$$

where $H_k(x, \sigma)$ denotes the Hermite polynomial of degree k with variance parameter σ.

3. Construction of the Φ^3_3-measure in the weakly nonlinear regime

In this section, we present the construction of the Φ^3_3-measure in the weakly nonlinear regime (Theorem 1.8 (i)). Our proof is based on the variational approach introduced by Barashkov and Gubinelli [3]. See the Boué-Dupuis variational formula (Lemma 3.1) below. In Subsection 3.1, we briefly go over the setup of the variational formulation for a partition function. In Subsection 3.2, we first establish the uniform exponential integrability (1.27) and then prove tightness of the truncated Φ^3_3-measures ρ_N in (1.25), which implies weak convergence of a subsequence. In Subsection 3.3, we follow the approach introduced in our previous work [53] and prove uniqueness of the limiting Φ^3_3-measure, thus establishing weak convergence of the entire sequence $\{\rho_N\}_{N\in\mathbb{N}}$. Finally, in Subsection 3.4, we show that the Φ^3_3-measure and the base Gaussian free field μ in (1.16) are mutually singular. While our proof of singularity of the Φ^3_3-measure is inspired by the discussion in Section 4 of [4], we directly prove singularity without referring to a shifted measure. In Appendix A, we show that the Φ^3_3-measure is indeed absolutely continuous with respect to the shifted measure $\text{Law}(Y(1)+\sigma\mathfrak{Z}(1)+\mathcal{W}(1))$, where $\text{Law}(Y(1))=\mu$, $\mathfrak{Z}=\mathfrak{Z}(Y)$ is the limit of the quadratic process $\mathfrak{Z}^N$ defined in (3.11), and the auxiliary quintic process $\mathcal{W}=\mathcal{W}(Y)$ is defined in (A.1).

3.1. **Boué-Dupuis variational formula.** Let $W(t)$ be the cylindrical Wiener process on $L^2(\mathbb{T}^3)$ (with respect to the underlying probability measure $\mathbb{P}$):

$$W(t)=\sum_{n\in\mathbb{Z}^3}B_n(t)e_n, \tag{3.1}$$

where $\{B_n\}_{n\in\mathbb{Z}^3}$ is defined by $B_n(t)=\langle\xi,\mathbf{1}_{[0,t]}\cdot e_n\rangle_{x,t}$. Here, $\langle\cdot,\cdot\rangle_{x,t}$ denotes the duality pairing on $\mathbb{T}^3\times\mathbb{R}$. Note that we have, for any $n\in\mathbb{Z}^3$,

$$\text{Var}(B_n(t))=\mathbb{E}\Big[\langle\xi,\mathbf{1}_{[0,t]}\cdot e_n\rangle_{x,t}\overline{\langle\xi,\mathbf{1}_{[0,t]}\cdot e_n\rangle_{x,t}}\Big]=\|\mathbf{1}_{[0,t]}\cdot e_n\|^2_{L^2_{x,t}}=t.$$

As a result, we see that $\{B_n\}_{n\in\Lambda_0}$ is a family of mutually independent complex-valued Brownian motions conditioned so that $B_{-n}=\overline{B_n}$, $n\in\mathbb{Z}^3$.[15] We then define a centered Gaussian process $Y(t)$ by

$$Y(t)=\langle\nabla\rangle^{-1}W(t). \tag{3.2}$$

Then, we have $\text{Law}(Y(1))=\mu$. By setting $Y_N=\pi_NY$, we have $\text{Law}(Y_N(1))=(\pi_N)_\#\mu$. In particular, we have $\mathbb{E}[Y_N(1)^2]=\sigma_N$, where σ_N is as in (1.22).

Next, let $\mathbb{H}_a$ denote the space of drifts, which are the progressively measurable processes belonging to $L^2([0,1];L^2(\mathbb{T}^3))$, $\mathbb{P}$-almost surely. For later use, we also define $\mathbb{H}^1_a$ to be the space of drifts, which are the progressively measurable processes belonging to $L^2([0,1];H^1(\mathbb{T}^3))$, $\mathbb{P}$-almost surely. Namely, we have

$$\mathbb{H}^1_a=\langle\nabla\rangle^{-1}\mathbb{H}_a. \tag{3.3}$$

We now state the Boué-Dupuis variational formula [7, 76]; in particular, see Theorem 7 in [76]. See also Theorem 2 in [3].

[15] In particular, B_0 is a standard real-valued Brownian motion.

Lemma 3.1. *Let $Y(t) = \langle \nabla \rangle^{-1} W(t)$ be as in (3.2). Fix $N \in \mathbb{N}$. Suppose that $F : C^\infty(\mathbb{T}^3) \to \mathbb{R}$ is measurable such that $\mathbb{E}\big[|F(Y_N(1))|^p\big] < \infty$ and $\mathbb{E}\big[|e^{-F(Y_N(1))}|^q\big] < \infty$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$-\log \mathbb{E}\Big[e^{-F(Y_N(1))}\Big] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\bigg[F(Y_N(1) + \pi_N I(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg], \tag{3.4}$$

where $I(\theta)$ is defined by

$$I(\theta)(t) = \int_0^t \langle \nabla \rangle^{-1} \theta(t') dt'. \tag{3.5}$$

Lemma 3.1 plays a fundamental role in almost every step of the argument presented in this section and Section 4.

We state a useful lemma on the pathwise regularity estimates of $:\!Y^k(t)\!:$ and $I(\theta)(1)$.

Lemma 3.2. (i) *For $k = 1, 2$, any finite $p \geq 2$, and $\varepsilon > 0$, $:\!Y_N^k(t)\!:$ converges to $:\!Y^k(t)\!:$ in $L^p(\Omega; \mathcal{C}^{-\frac{k}{2}-\varepsilon}(\mathbb{T}^3))$ and also almost surely in $\mathcal{C}^{-\frac{k}{2}-\varepsilon}(\mathbb{T}^3)$. Moreover, we have*

$$\mathbb{E}\Big[\| :\!Y_N^k(t)\!: \|_{\mathcal{C}^{-\frac{k}{2}-\varepsilon}}^p\Big] \lesssim p^{\frac{k}{2}} < \infty, \tag{3.6}$$

uniformly in $N \in \mathbb{N}$ and $t \in [0, 1]$. We also have

$$\mathbb{E}\Big[\| :\!Y_N^2(t)\!: \|_{H^{-1}}^2\Big] \sim t^2 \log N \tag{3.7}$$

for any $t \in [0, 1]$.

(ii) *For any $N \in \mathbb{N}$, we have*

$$\mathbb{E}\bigg[\int_{\mathbb{T}^3} :\!Y_N^3(1)\!: dx\bigg] = 0.$$

(iii) *For any $\theta \in \mathbb{H}_a$, we have*

$$\|I(\theta)(1)\|_{H^1}^2 \leq \int_0^1 \|\theta(t)\|_{L^2}^2 dt.$$

Proof. The bound (3.6) for $\varepsilon > 0$ follows from the Wiener chaos estimate (Lemma 2.5), Lemma 2.6, and then carrying out summations, using Lemma 2.4. See, for example, [34, 35]. As for (3.7), proceeding as in the proof of Lemma 2.5 in [62] with Lemma 2.6, we have

$$\begin{aligned}
&\mathbb{E}\Big[\| :\!Y_N^2(t)\!: \|_{H^{-1}}^2\Big] \\
&= \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n \rangle^2} \int_{\mathbb{T}^3_x \times \mathbb{T}^3_y} \mathbb{E}\Big[H_2(Y_N(x,t); t\sigma_N) H_2(Y_N(y,t); t\sigma_N)\Big] e_n(y - x) dx dy \\
&= \sum_{n \in \mathbb{Z}^3} \frac{t^2}{\langle n \rangle^2} \sum_{n_1, n_2 \in \mathbb{Z}^3} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1 \rangle^2 \langle n_2 \rangle^2} \int_{\mathbb{T}^3_x \times \mathbb{T}^3_y} e_{n_1+n_2-n}(x - y) dx dy \\
&= \sum_{n \in \mathbb{Z}^3} \frac{t^2}{\langle n \rangle^2} \sum_{n = n_1 + n_2} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1 \rangle^2 \langle n_2 \rangle^2},
\end{aligned} \tag{3.8}$$

where $\chi_N(n_j)$ is as in (1.20). The upper bound in (3.7) follows from applying Lemma 2.4 to (3.8). As for the lower bound, we consider the contribution from $|n| \le \frac{2}{3}N$ and $\frac{1}{4}|n| \le |n_1| \le \frac{1}{2}|n|$ (which implies $|n_2| \sim |n|$ and $|n_j| \le N$, $j = 1, 2$). Then, from (3.8), we obtain

$$\mathbb{E}\Big[\| :Y_N^2(t): \|_{H^{-1}}^2\Big] \gtrsim \sum_{\substack{n\in\mathbb{Z}^3 \\ |n|\le\frac{2}{3}N}} \frac{t^2}{\langle n\rangle^3} \sim t^2 \log N,$$

which proves the lower bound in (3.7). As for (ii), it follows from recalling the definition $:Y_N^3(1):= H_3(Y_N(1);\sigma_N)$ (with σ_N as in (1.22)) and the orthogonality relation of the Hermite polynomials (Lemma 2.6 with $k = 3$ and $\ell = 0$). Lastly, the claim in (iii) follows from Minkowski's integral inequality and Cauchy-Schwarz inequality; see Lemma 4.7 in [37]. $\square$

Remark 3.3. In [37, 57], a slightly different (and weaker) variational formula was used. See also Lemma 1 in [3]. Given a drift $\theta \in \mathbb{H}_a$, we define the measure $\mathbb{Q}_\theta$ whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is given by the following stochastic exponential:

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} = e^{\int_0^1 \langle\theta(t), dW(t)\rangle - \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt},$$

where $\langle\cdot,\cdot\rangle$ stands for the usual inner product on $L^2(\mathbb{T}^3)$. Let $\mathbb{H}_c$ denote the subspace of $\mathbb{H}_a$ consisting of drifts such that $\mathbb{Q}_\theta(\Omega) = 1$. Then, the (weaker) variational formula used in [37, 57] is given by (3.4), where the infimum is taken over $\mathbb{H}_c \subset \mathbb{H}_a$ and we replace Y and $\mathbb{E} = \mathbb{E}_\mathbb{P}$ by $Y_\theta = Y - I(\theta)$ and $\mathbb{E}_{\mathbb{Q}_\theta}$. Here, $\mathbb{E} = \mathbb{E}_\mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}_\theta}$ denote expectations with respect to the underlying probability measure $\mathbb{P}$ and the measure $\mathbb{Q}_\theta$, respectively. In such a formulation, Y_θ and the measure $\mathbb{Q}_\theta$ depend on a drift θ. This, however, is not suitable for our purpose, since we construct a drift θ in (3.4) depending on Y.

3.2. **Uniform exponential integrability and tightness.** In this subsection, we first prove the uniform exponential integrability (1.27) via the Boué-Dupuis variational formula (Lemma 3.1). Then, we establish tightness of the truncated Φ^3_3-measures $\{\rho_N\}_{N\in\mathbb{N}}$.

As in the case of the Φ^4_3-measure studied in [3] (see also Section 6 in [53]), we need to introduce a further renormalization than the standard Wick renormalization (see (1.24)). As a result, the resulting Φ^3_3-measure is singular with respect to the base Gaussian free field μ; see Subsection 3.4. We point out that this extra renormalization appears only at the level of the measure and thus does not affect the dynamical problem, at least locally in time.[16] In the following, we use the following short-hand notations: $Y_N(t) = \pi_N Y(t)$, $\Theta(t) = I(\theta)(t)$, and $\Theta_N(t) = \pi_N\Theta(t)$ with $Y_N = Y_N(1)$ and $\Theta_N = \Theta_N(1)$. We also use $Y = Y(1)$ and $\Theta = \Theta(1)$.

Let us first explain the second renormalization introduced in (1.24). Let R_N be as in (1.23) and set

$$\widetilde{Z}_N = \int e^{-R_N(u)} d\mu(u).$$

By Lemma 3.1, we can express the partition function $\widetilde{Z}_N$ as

$$-\log \widetilde{Z}_N = \inf_{\theta\in\mathbb{H}_a} \mathbb{E}\bigg[R_N(Y+\Theta) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg].$$

[16]As mentioned in Section 1, this singularity of the Φ^3_3-measure causes an additional difficulty for the globalization problem.

By expanding the cubic Wick power, we have

$$\begin{aligned}
-\frac{\sigma}{3}\int_{\mathbb{T}^3} :(Y_N+\Theta_N)^3:\, dx = -\frac{\sigma}{3}\int_{\mathbb{T}^3} :Y_N^3:\, dx - \sigma\int_{\mathbb{T}^3} :Y_N^2:\,\Theta_N dx \\
- \sigma\int_{\mathbb{T}^3} Y_N\Theta_N^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3}\Theta_N^3 dx.
\end{aligned} \tag{3.9}$$

In view of Lemma 3.2, the first term on the right-hand side vanishes under an expectation, while we can estimate the third and fourth terms on the right-hand side of (3.9) (see Lemma 3.5). As we see below, the second term turns out to be divergent (and does not vanish under an expectation). From the Ito product formula, we have

$$\mathbb{E}\bigg[\int_{\mathbb{T}^3} :Y_N^2:\ \Theta_N dx\bigg] = \mathbb{E}\bigg[\int_0^1\int_{\mathbb{T}^3} :Y_N^2(t):\ \dot\Theta_N(t) dx dt\bigg], \tag{3.10}$$

where we have $\dot\Theta_N(t) = \langle\nabla\rangle^{-1}\pi_N\theta(t)$ in view of (3.5). Define $\mathfrak{Z}^N$ with $\mathfrak{Z}^N(0)=0$ by its time derivative:

$$\dot{\mathfrak{Z}}^N(t) = (1-\Delta)^{-1} :Y_N^2(t): \tag{3.11}$$

and set $\mathfrak{Z}_N = \pi_N\mathfrak{Z}^N$. Then, we perform a change of variables:

$$\dot\Upsilon^N(t) = \dot\Theta(t) - \sigma\dot{\mathfrak{Z}}_N(t) \tag{3.12}$$

and set $\Upsilon_N = \pi_N\Upsilon^N$. From (3.10), (3.11), and (3.12), we have

$$\mathbb{E}\bigg[-\sigma\int_{\mathbb{T}^3} :Y_N^2:\,\Theta_N dx + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg] = \frac{1}{2}\mathbb{E}\bigg[\int_0^1 \|\dot\Upsilon^N(t)\|_{H^1_x}^2 dt\bigg] - \alpha_N, \tag{3.13}$$

where the divergent constant α_N is given by

$$\alpha_N = \frac{\sigma^2}{2}\mathbb{E}\bigg[\int_0^1 \|\dot{\mathfrak{Z}}_N(t)\|_{H^1_x}^2 dt\bigg] \longrightarrow \infty, \tag{3.14}$$

as $N\to\infty$. The divergence in (3.14) can be easily seen from the spatial regularity $1-\varepsilon$ of $\dot{\mathfrak{Z}}_N(t) = (1-\Delta)^{-1} :Y_N^2(t):$ (with a uniform bound in $N\in\mathbb{N}$). See Lemma 3.2.

In view of the discussion above, we define $R_N^\diamond$ as in (1.24), which removes the divergent constant α_N in (3.13). Then, from (1.26) and the Boué-Dupuis variational formula (Lemma 3.1), we have

$$-\log Z_N = \inf_{\theta\in\mathbb{H}_a}\mathbb{E}\bigg[R_N^\diamond(Y+\Theta) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg] \tag{3.15}$$

for any $N\in\mathbb{N}$. By setting

$$\mathcal{W}_N(\theta) = \mathbb{E}\bigg[R_N^\diamond(Y+\Theta) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 dt\bigg], \tag{3.16}$$

it follows from (1.23) with $\gamma=3$, (1.24), (3.9), (3.13), and Lemma 3.2 (ii) that

$$\begin{aligned}
\mathcal{W}_N(\theta) = \mathbb{E}\bigg[-\sigma\int_{\mathbb{T}^3} Y_N\Theta_N^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3}\Theta_N^3 dx \\
+ A\bigg|\int_{\mathbb{T}^3}\Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\Big)dx\bigg|^3 + \frac{1}{2}\int_0^1 \|\dot\Upsilon^N(t)\|_{H^1_x}^2 dt\bigg].
\end{aligned} \tag{3.17}$$

We also set

$$\Upsilon_N = \Upsilon_N(1) = \pi_N\Upsilon^N(1) \qquad \text{and} \qquad \mathfrak{Z}_N = \mathfrak{Z}_N(1) = \pi_N\mathfrak{Z}^N(1). \tag{3.18}$$

In view of the change of variables (3.12), we have

$$\Theta_N = \Upsilon_N + \sigma \pi_N \mathfrak{Z}_N =: \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N, \qquad \text{i.e. } \widetilde{\mathfrak{Z}}_N := \pi_N \mathfrak{Z}_N. \tag{3.19}$$

Namely, the original drift θ in (3.15) depends on Y. By the definition (3.11) and (3.18), $\mathfrak{Z}_N$ is determined by Y_N. Hence, in the following, we view $\dot{\Upsilon}^N$ as a drift and study the minimization problem (3.15) by first studying each term in (3.17) (where we now view $\mathcal{W}_N$ as a function of $\dot{\Upsilon}^N$) and then taking an infimum in $\dot{\Upsilon}^N \in \mathbb{H}^1_a$, where $\mathbb{H}^1_a$ is as in (3.3). Our main goal is to show that $\mathcal{W}_N(\dot{\Upsilon}^N)$ in (3.17) is bounded away from $-\infty$, uniformly in $N \in \mathbb{N}$ and $\dot{\Upsilon}^N \in \mathbb{H}^1_a$.

Remark 3.4. In this paper, we work with the cube frequency projector $\pi_N = \pi_N^{\text{cube}}$ defined in (1.19), satisfying $\pi_N^2 = \pi_N$. In view of (3.18) and (3.19), we have $\widetilde{\mathfrak{Z}}_N = \mathfrak{Z}_N$. Nonetheless, we introduce the notation $\widetilde{\mathfrak{Z}}_N$ in (3.19) to indicate the modifications necessary to consider the case of the smooth frequency projector π_N^{smooth} defined in (1.42), which does not satisfy $(\pi_N^{\text{smooth}})^2 = \pi_N^{\text{smooth}}$. This comment applies to the remaining part of the paper.

We first state two lemmas whose proofs are presented at the end of this subsection. While the first lemma is elementary, the second lemma (Lemma 3.6) requires much more careful analysis, reflecting the critical nature of the Φ^3_3-measure.

Lemma 3.5. *Let $A > 0$ and $0 < |\sigma| < 1$. Then, there exist small $\varepsilon > 0$ and a constant $c > 0$ such that, for any $\delta > 0$, there exists $C_\delta > 0$ such that*

$$\bigg| \int_{\mathbb{T}^3} Y_N \Theta_N^2 dx \bigg| \lesssim 1 + C_\delta \|Y_N\|^c_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \delta \|\Upsilon_N\|^6_{L^2} + \delta \|\Upsilon_N\|^2_{H^1} + \|\mathfrak{Z}_N\|^c_{\mathcal{C}^{1-\varepsilon}}, \tag{3.20}$$

$$\bigg| \int_{\mathbb{T}^3} \Theta_N^3 dx \bigg| \lesssim 1 + \|\Upsilon_N\|^6_{L^2} + \|\Upsilon_N\|^2_{H^1} + \|\mathfrak{Z}_N\|^3_{\mathcal{C}^{1-\varepsilon}}, \tag{3.21}$$

and

$$\begin{aligned} A \bigg| \int_{\mathbb{T}^3} \Big(:\!Y_N^2\!: + 2 Y_N \Theta_N + \Theta_N^2 \Big) dx \bigg|^3 &\geq \frac{A}{2} \bigg| \int_{\mathbb{T}^3} \Big(2 Y_N \Upsilon_N + \Upsilon_N^2 \Big) dx \bigg|^3 - \delta \|\Upsilon_N\|^6_{L^2} \\ &\quad - C_{\delta,\sigma} \bigg\{ \bigg| \int_{\mathbb{T}^3} :\!Y_N^2\!: dx \bigg|^3 + \|Y_N\|^6_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|\mathfrak{Z}_N\|^6_{\mathcal{C}^{1-\varepsilon}} \bigg\}, \end{aligned} \tag{3.22}$$

uniformly in $N \in \mathbb{N}$, where $\Theta_N = \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N$ as in (3.19).

The next lemma allows us to control the term $\|\Upsilon_N\|^6_{L^2}$ appearing in Lemma 3.5.

Lemma 3.6. *There exists a non-negative random variable $B(\omega)$ with $\mathbb{E}[B^p] \leq C_p < \infty$ for any finite $p \geq 1$ such that*

$$\|\Upsilon_N\|^6_{L^2} \lesssim \bigg| \int_{\mathbb{T}^3} \Big(2 Y_N \Upsilon_N + \Upsilon_N^2 \Big) dx \bigg|^3 + \|\Upsilon_N\|^2_{H^1} + B(\omega), \tag{3.23}$$

uniformly in $N \in \mathbb{N}$.

By assuming Lemmas 3.5 and 3.6, we now prove the uniform exponential integrability (1.27) and tightness of the truncated Φ^3_3-measures ρ_N.

• **Uniform exponential integrability:** In view of (3.17) and Lemma 3.6, define the positive part $\mathcal{U}_N$ of $\mathcal{W}_N$ by

$$\mathcal{U}_N(\dot{\Upsilon}^N) = \mathbb{E}\bigg[\frac{A}{2}\bigg|\int_{\mathbb{T}^3}\Big(2Y_N\Upsilon_N + \Upsilon_N^2\Big)dx\bigg|^3 + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg]. \tag{3.24}$$

As a corollary to Lemma 3.2 (i) with (3.11), we have, for any finite $p \geq 1$,

$$\mathbb{E}\Big[\|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^p\Big] \leq \int_0^1 \mathbb{E}\Big[\|:Y_N^2(t):\|_{\mathcal{C}^{-1-\varepsilon}}^p\Big]dt \lesssim p < \infty, \tag{3.25}$$

uniformly in $N \in \mathbb{N}$. Then, by applying Lemmas 3.5 and 3.6 to (3.17) together with Lemma 3.2 and (3.25), we obtain

$$\begin{aligned}\mathcal{W}_N(\dot{\Upsilon}^N) &\geq -C_0 + \mathbb{E}\bigg[\Big(\frac{A}{2} - c|\sigma|\Big)\bigg|\int_{\mathbb{T}^3}\Big(2Y_N\Upsilon_N + \Upsilon_N^2\Big)dx\bigg|^3 \\ &\qquad\qquad + \Big(\frac{1}{2} - c|\sigma|\Big)\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg] \\ &\geq -C_0' + \frac{1}{10}\mathcal{U}_N(\dot{\Upsilon}^N),\end{aligned} \tag{3.26}$$

for any $0 < |\sigma| < \sigma_0$, provided $A = A(\sigma_0) > 0$ is sufficiently large. Noting that the estimate (3.26) is uniform in $N \in \mathbb{N}$ and $\dot{\Upsilon}^N \in \mathbb{H}_a^1$, we conclude that

$$\inf_{N\in\mathbb{N}}\inf_{\dot{\Upsilon}^N\in\mathbb{H}_a^1}\mathcal{W}_N(\dot{\Upsilon}^N) \geq \inf_{N\in\mathbb{N}}\inf_{\dot{\Upsilon}^N\in\mathbb{H}_a^1}\bigg\{-C_0' + \frac{1}{10}\mathcal{U}_N(\dot{\Upsilon}^N)\bigg\} \geq -C_0' > -\infty. \tag{3.27}$$

Therefore, the uniform exponential integrability (1.27) follows from (3.15), (3.16), and (3.27).

• **Tightness:** Next, we prove tightness of the truncated Φ_3^3-measures $\{\rho_N\}_{N\in\mathbb{N}}$. Although it follows from a slight modification of the argument in our previous work [53, Subsection 6.2], we present a proof here for readers' convenience.

As a preliminary step, we first prove that Z_N in (1.26) is uniformly bounded away from 0:

$$\inf_{N\in\mathbb{N}} Z_N > 0. \tag{3.28}$$

In view of (3.15) and (3.16), it suffices to establish an upper bound on $\mathcal{W}_N$ in (3.17). By Lemma 2.1 and (3.19), we have

$$\begin{aligned}\bigg|\int_{\mathbb{T}^3} 2Y_N\Theta_N dx\bigg|^3 &\lesssim \|Y_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}^3\|\Theta_N\|_{H^{\frac{1}{2}+2\varepsilon}}^3 \\ &\lesssim 1 + \|Y_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}^c + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^c + \|\Upsilon_N\|_{H^1}^c.\end{aligned}$$

Thus, we have

$$\begin{aligned}A\bigg|\int_{\mathbb{T}^3}\Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\Big)dx\bigg|^3 \\ \lesssim 1 + \|:Y_N^2:\|_{\mathcal{C}^{-1-\varepsilon}}^3 + \|Y_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}^c + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^c + \|\Upsilon_N\|_{H^1}^c.\end{aligned} \tag{3.29}$$

Then, from (3.17), Lemma 3.5, and (3.29) with Lemma 3.2 and (3.25), we obtain

$$\inf_{\dot{\Upsilon}^N\in\mathbb{H}_a^1}\mathcal{W}_N \lesssim 1 + \inf_{\dot{\Upsilon}^N\in\mathbb{H}_a^1}\mathbb{E}\bigg[\bigg(\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg)^c\bigg] \lesssim 1$$

by taking $\dot{\Upsilon}^N \equiv 0$, for example. This proves (3.28).

We now prove tightness of the truncated Φ^3_3-measures. Fix small $\varepsilon > 0$ and let $B_R \subset H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)$ be the closed ball of radius $R > 0$ centered at the origin. Then, by Rellich's compactness lemma, we see that B_R is compact in $H^{-\frac{1}{2}-2\varepsilon}(\mathbb{T}^3)$. In the following, we show that given any small $\delta > 0$, there exists $R = R(\delta) \gg 1$ such that

$$\sup_{N\in\mathbb{N}} \rho_N(B_R^c) < \delta. \tag{3.30}$$

Given $M \gg 1$, let F be a bounded smooth non-negative function such that

$$F(u) = \begin{cases} M, & \text{if } \|u\|_{H^{-\frac{1}{2}-\varepsilon}} \le \frac{R}{2}, \\ 0, & \text{if } \|u\|_{H^{-\frac{1}{2}-\varepsilon}} > R. \end{cases} \tag{3.31}$$

Then, from (3.28), we have

$$\rho_N(B_R^c) \le Z_N^{-1} \int e^{-F(u) - R_N^\diamond(u)} d\mu \lesssim \int e^{-F(u)-R_N^\diamond(u)} d\mu =: \widehat{Z}_N, \tag{3.32}$$

uniformly in $N \gg 1$. Under the change of variables (3.12) (see also (3.13)), define $\widehat{R}_N^\diamond(Y + \Upsilon^N + \sigma\mathfrak{Z}_N)$ by

$$\begin{aligned} \widehat{R}_N^\diamond(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) = -\frac{\sigma}{3}\int_{\mathbb{T}^3} :Y_N^3:\, dx - \sigma \int_{\mathbb{T}^3} Y_N \Theta_N^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} \Theta_N^3 dx \\ + A\bigg| \int_{\mathbb{T}^3} \Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\Big) dx\bigg|^3, \end{aligned} \tag{3.33}$$

where $\Theta_N = \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N$ with $\widetilde{\mathfrak{Z}}_N = \pi_N \mathfrak{Z}_N$ as in (3.19). Then, by (3.32) and the Boué-Dupuis variational formula (Lemma 3.1), we have

$$\begin{aligned} -\log \widehat{Z}_N = \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[& F(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) \\ & + \widehat{R}_N^\diamond(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|^2_{H^1_x} dt\bigg]. \end{aligned} \tag{3.34}$$

Since $Y + \sigma\mathfrak{Z}_N \in \mathcal{H}_{\le 2}$, it follows from Lemma 3.2, (3.25), Chebyshev's inequality, and choosing $R \gg 1$ that

$$\begin{aligned} &\mathbb{P}\Big(\|Y + \Upsilon^N + \sigma\mathfrak{Z}_N\|_{H^{-\frac{1}{2}-\varepsilon}} > \tfrac{R}{2}\Big) \\ &\quad \le \mathbb{P}\Big(\|Y + \sigma\mathfrak{Z}_N\|_{H^{-\frac{1}{2}-\varepsilon}} > \tfrac{R}{4}\Big) + \mathbb{P}\Big(\|\Upsilon^N\|_{H^1} > \tfrac{R}{4}\Big) \\ &\quad \le \frac{1}{2} + \frac{16}{R^2}\mathbb{E}\Big[\|\Upsilon^N\|^2_{H^1_x}\Big], \end{aligned} \tag{3.35}$$

uniformly in $N \in \mathbb{N}$ and $R \gg 1$. Then, from (3.31), (3.35), and Lemma 3.2, we obtain

$$\begin{aligned} \mathbb{E}\Big[F(Y + \Upsilon^N + \sigma\mathfrak{Z}_N)\Big] &\ge M\mathbb{E}\Big[\mathbf{1}_{\{\|Y + \Upsilon^N + \sigma\mathfrak{Z}_N\|_{H^{-\frac{1}{2}-\varepsilon}} \le \frac{R}{2}\}}\Big] \\ &\ge \frac{M}{2} - \frac{16M}{R^2}\mathbb{E}\Big[\|\Upsilon^N\|^2_{H^1_x}\Big] \\ &\ge \frac{M}{2} - \frac{1}{4}\mathbb{E}\bigg[\int_0^1 \|\dot{\Upsilon}^N(t)\|^2_{H^1_x} dt\bigg], \end{aligned} \tag{3.36}$$

where we set $M = \frac{1}{64}R^2$ in the last step. Hence, from (3.34), (3.36), and repeating the computation leading to (3.27) (by possibly making σ_0 smaller), we obtain

$$\begin{aligned} -\log \widehat{Z}_N &\ge \frac{M}{2} + \inf_{\dot{\Upsilon}^N \in \mathbb{H}^1_a} \mathbb{E}\bigg[\widehat{R}^\diamond_N(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) + \frac{1}{4}\int_0^1 \|\dot{\Upsilon}^N(t)\|^2_{H^1_x} dt\bigg] \\ &\ge \frac{M}{4}, \end{aligned} \tag{3.37}$$

uniformly $N \in \mathbb{N}$ and $M = \frac{1}{64}R^2 \gg 1$. Therefore, given any small $\delta > 0$, by choosing $R = R(\delta) \gg 1$ and setting $M = \frac{1}{64}R^2 \gg 1$, the desired bound (3.30) follows from (3.32) and (3.37). This proves tightness of the truncated Φ^3_3-measures $\{\rho_N\}_{N\in\mathbb{N}}$.

We conclude this subsection by presenting the proofs of Lemmas 3.5 and 3.6.

Proof of Lemma 3.5. From (2.5), (2.6), (2.4), and (2.3) in Lemma 2.1 followed by Young's inequality, we have

$$\begin{aligned} \bigg|\int_{\mathbb{T}^3} Y_N \Theta_N^2 dx\bigg| &\lesssim \|Y_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|\Theta_N\|_{H^{\frac{1}{2}+2\varepsilon}} \|\Theta_N\|_{L^2} \\ &\lesssim \|Y_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \Big(\|\Upsilon_N\|_{H^{\frac{1}{2}+2\varepsilon}} \big(\|\Upsilon_N\|_{L^2} + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}\big) + \|\mathfrak{Z}_N\|^2_{\mathcal{C}^{1-\varepsilon}}\Big) \\ &\lesssim \|Y_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \Big(\|\Upsilon_N\|_{L^2}^{\frac{1}{2}-2\varepsilon} \|\Upsilon_N\|_{H^1}^{\frac{1}{2}+2\varepsilon} \big(\|\Upsilon_N\|_{L^2} + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}\big) + \|\mathfrak{Z}_N\|^2_{\mathcal{C}^{1-\varepsilon}}\Big) \\ &\lesssim 1 + C_\delta \|Y_N\|^c_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \delta\|\Upsilon_N\|^6_{L^2} + \delta\|\Upsilon_N\|^2_{H^1} + \|\mathfrak{Z}_N\|^c_{\mathcal{C}^{1-\varepsilon}}, \end{aligned} \tag{3.38}$$

which yields (3.20). As for the second estimate (3.21), it follows from Sobolev's inequality, the interpolation (2.3), and Young's inequality that

$$\bigg|\int_{\mathbb{T}^3} \Upsilon_N^3 dx\bigg| \lesssim \|\Upsilon_N\|^3_{H^{\frac{1}{2}}} \lesssim \|\Upsilon_N\|^{\frac{3}{2}}_{L^2} \|\Upsilon_N\|^{\frac{3}{2}}_{H^1} \lesssim \|\Upsilon_N\|^6_{L^2} + \|\Upsilon_N\|^2_{H^1}, \tag{3.39}$$

while Hölder's inequality with (2.4) shows

$$\bigg|\int_{\mathbb{T}^3} \Upsilon_N^2 \widetilde{\mathfrak{Z}}_N dx\bigg| + \bigg|\int_{\mathbb{T}^3} \Upsilon_N \widetilde{\mathfrak{Z}}_N^2 dx\bigg| + \bigg|\int_{\mathbb{T}^3} \widetilde{\mathfrak{Z}}_N^3 dx\bigg| \lesssim 1 + \|\Upsilon_N\|^6_{L^2} + \|\mathfrak{Z}_N\|^3_{\mathcal{C}^{1-\varepsilon}}.$$

Note that, given any $\gamma > 0$, there exists a constant $C = C(J) > 0$ such that

$$\bigg|\sum_{j=1}^J a_j\bigg|^\gamma \ge \frac{1}{2}|a_1|^\gamma - C\bigg(\sum_{j=2}^J |a_j|^\gamma\bigg) \tag{3.40}$$

for any $a_j \in \mathbb{R}$. See Section 5 in [53]. Then, from (3.40) and Cauchy's inequality, we have

$$
\begin{aligned}
& A\bigg|\int_{\mathbb{T}^3}\Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\Big)dx\bigg|^3 \\
& \geq \frac{A}{2}\bigg|\int_{\mathbb{T}^3}\Big(2Y_N\Upsilon_N + \Upsilon_N^2\Big)dx\bigg|^3 - CA\bigg\{\bigg|\int_{\mathbb{T}^3} :Y_N^2: dx\bigg|^3 + |\sigma|^3\bigg|\int_{\mathbb{T}^3} Y_N\widetilde{\mathfrak{Z}}_N dx\bigg|^3 \\
& \qquad + |\sigma|^3\bigg|\int_{\mathbb{T}^3} \Upsilon_N\widetilde{\mathfrak{Z}}_N dx\bigg|^3 + \sigma^6\bigg|\int_{\mathbb{T}^3}\widetilde{\mathfrak{Z}}_N^2 dx\bigg|^3\bigg\} \\
& \geq \frac{A}{2}\bigg|\int_{\mathbb{T}^3}\Big(2Y_N\Upsilon_N + \Upsilon_N^2\Big)dx\bigg|^3 - \delta\|\Upsilon_N\|_{L^2}^6 \\
& \qquad - C_{\delta,\sigma}\bigg\{\bigg|\int_{\mathbb{T}^3} :Y_N^2: dx\bigg|^3 + \|Y_N\|^6_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|\mathfrak{Z}_N\|^6_{\mathcal{C}^{1-\varepsilon}}\bigg\}.
\end{aligned}
$$

This proves (3.22). This completes the proof of Lemma 3.5. □

Next, we present the proof of Lemma 3.6.

Proof of Lemma 3.6. If we have

$$\|\Upsilon_N\|^2_{L^2} \gg \bigg|\int_{\mathbb{T}^3} Y_N\Upsilon_N dx\bigg|, \tag{3.41}$$

then, we have

$$\|\Upsilon_N\|^6_{L^2} = \bigg(\int_{\mathbb{T}^3}\Upsilon_N^2 dx\bigg)^3 \sim \bigg|\int_{\mathbb{T}^3}\Big(2Y_N\Upsilon_N + \Upsilon_N^2\Big)dx\bigg|^3, \tag{3.42}$$

which shows (3.23). Hence, we assume that

$$\|\Upsilon_N\|^2_{L^2} \lesssim \bigg|\int_{\mathbb{T}^3} Y_N\Upsilon_N dx\bigg| \tag{3.43}$$

in the following.

Given $j \in \mathbb{N}$, define the sharp frequency projections Π_j with a Fourier multiplier $\mathbf{1}_{\{|n|\leq 2\}}$ when $j = 1$ and $\mathbf{1}_{\{2^{j-1}<|n|\leq 2^j\}}$ when $j \geq 2$. We also set $\Pi_{\leq j} = \sum_{k=1}^j \Pi_k$ and $\Pi_{>j} = \mathrm{Id} - \Pi_{\leq j}$. Then, write Υ_N as

$$\Upsilon_N = \sum_{j=1}^\infty \Pi_j\Upsilon_N = \sum_{j=1}^\infty(\lambda_j\Pi_jY_N + w_j), \tag{3.44}$$

where λ_j and w_j are given by

$$\lambda_j := \begin{cases} \frac{\langle\Upsilon_N, \Pi_jY_N\rangle}{\|\Pi_jY_N\|^2_{L^2}}, & \text{if } \|\Pi_jY_N\|_{L^2} \neq 0, \\ 0, & \text{otherwise,}\end{cases} \qquad \text{and} \qquad w_j := \Pi_j\Upsilon_N - \lambda_j\Pi_jY_N. \tag{3.45}$$

By definition, $w_j = \Pi_j w_j$ is orthogonal to $\Pi_j Y_N$ (and also to Y_N) in $L^2(\mathbb{T}^3)$. Thus, we have

$$\|\Upsilon_N\|_{L^2}^2 = \sum_{j=1}^{\infty} \Big(\lambda_j^2 \|\Pi_j Y_N\|_{L^2}^2 + \|w_j\|_{L^2}^2\Big), \tag{3.46}$$

$$\int_{\mathbb{T}^3} Y_N \Upsilon_N dx = \sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2. \tag{3.47}$$

Hence, from (3.43), (3.46), and (3.47), we have

$$\sum_{j=1}^{\infty} \lambda_j^2 \|\Pi_j Y_N\|_{L^2}^2 \lesssim \bigg|\sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg|. \tag{3.48}$$

Fix $j_0 = j_0(\omega) \in \mathbb{N}$ (to be chosen later). By Cauchy-Schwarz's inequality and (3.45), we have

$$\begin{aligned}
\bigg|\sum_{j=j_0+1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg| &\le \bigg(\sum_{j=1}^{\infty} \lambda_j^2 2^{2j} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac12} \bigg(\sum_{j=j_0+1}^{\infty} 2^{-2j} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac12} \\
&\le \bigg(\sum_{j=1}^{\infty} 2^{2j} \|\Pi_j \Upsilon_N\|_{L^2}^2\bigg)^{\frac12} \bigg(\sum_{j=j_0+1}^{\infty} 2^{-2j} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac12} \\
&\sim \|\Upsilon_N\|_{H^1} \|\Pi_{>j_0} Y_N\|_{H^{-1}}.
\end{aligned} \tag{3.49}$$

On the other hand, it follows from Cauchy-Schwarz's inequality, (3.48), and Cauchy's inequality that

$$\begin{aligned}
\bigg|\sum_{j=1}^{j_0} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg| &\le \bigg(\sum_{j=1}^{\infty} \lambda_j^2 \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac12} \bigg(\sum_{j=1}^{j_0} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac12} \\
&\le C\bigg|\sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg|^{\frac12} \bigg(\sum_{j=1}^{j_0} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac12} \\
&\le \frac12 \bigg|\sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg| + C' \|\Pi_{\le j_0} Y_N\|_{L^2}^2.
\end{aligned} \tag{3.50}$$

Hence, from (3.49) and (3.50), we obtain

$$\bigg|\sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg| \lesssim \|\Upsilon_N\|_{H^1} \|\Pi_{>j_0} Y_N\|_{H^{-1}} + \|\Pi_{\le j_0} Y_N\|_{L^2}^2. \tag{3.51}$$

Since Y_N is spatially homogeneous, we have

$$\|\Pi_{>j_0} Y_N\|_{H^{-1}}^2 = \int_{\mathbb{T}^3} :(\langle\nabla\rangle^{-1} \Pi_{>j_0} Y_N)^2 : dx + \mathbb{E}\big[(\langle\nabla\rangle^{-1} \Pi_{>j_0} Y_N)^2\big]. \tag{3.52}$$

Recalling (3.2), we can bound the second term by

$$\widetilde{\sigma}_{j_0} := \mathbb{E}\big[(\langle\nabla\rangle^{-1} \Pi_{>j_0} Y_N)^2\big] = \sum_{\substack{n \in \mathbb{Z}^3 \\ |n| > 2^{j_0}}} \frac{\chi_N^2(n)}{\langle n\rangle^4} \lesssim 2^{-j_0}. \tag{3.53}$$

Let $Z_{N,j_0} = \langle \nabla \rangle^{-1} \Pi_{>j_0} Y_N$. Proceeding as in the proof of Lemma 2.5 in [62] with Lemma 2.6, we have

$$\begin{aligned}
\mathbb{E}\bigg[\bigg(\int_{\mathbb{T}^3} :Z_{N,j_0}^2: dx\bigg)^2\bigg] &= \int_{\mathbb{T}^3_x \times \mathbb{T}^3_y} \mathbb{E}\Big[H_2(Z_{N,j_0}(x); \widetilde{\sigma}_{j_0}) H_2(Z_{N,j_0}(y); \widetilde{\sigma}_{j_0})\Big] dxdy \\
&= 2 \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3 \\ |n_j| > 2^{j_0}}} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1 \rangle^4 \langle n_2 \rangle^4} \int_{\mathbb{T}^3_x \times \mathbb{T}^3_y} e_{n_1+n_2}(x-y) dxdy \\
&= 2 \sum_{\substack{n \in \mathbb{Z}^3 \\ |n| > 2^{j_0}}} \frac{\chi_N^4(n)}{\langle n \rangle^8} \sim 2^{-5j_0}.
\end{aligned} \tag{3.54}$$

Now, define a non-negative random variable $B_1(\omega)$ by

$$B_1(\omega) = \bigg(\sum_{j=1}^\infty 2^{4j}\Big(\int_{\mathbb{T}^3} :Z_{N,j}^2: dx\Big)^2\bigg)^{\frac{1}{2}}. \tag{3.55}$$

By Minkowski's integral inequality, the Wiener chaos estimate (Lemma 2.5), and (3.54), we have

$$\mathbb{E}\big[B_1^p\big] \le p^p \Bigg(\sum_{j=1}^\infty 2^{4j} \bigg\| \int_{\mathbb{T}^3} :Z_{N,j}^2: dx \bigg\|_{L^2(\Omega)}^2\Bigg)^{\frac{p}{2}} \lesssim p^p < \infty \tag{3.56}$$

for any finite $p \ge 2$ (and hence for any finite $p \ge 1$). Hence, from (3.52), (3.53), and (3.55), we obtain

$$\|\Pi_{>j_0} Y_N\|_{H^{-1}}^2 \lesssim 2^{-2j_0} B_1(\omega) + 2^{-j_0}. \tag{3.57}$$

Next, define a non-negative random variable $B_2(\omega)$ by

$$B_2(\omega) = \sum_{j=1}^\infty \bigg| \int_{\mathbb{T}^3} :(\Pi_j Y_N)^2: dx \bigg|.$$

Then, a similar computation shows

$$\begin{aligned}
\|\Pi_{\le j_0} Y_N\|_{L^2}^2 &= \int_{\mathbb{T}^3} :(\Pi_{\le j_0} Y_N)^2: dx + \mathbb{E}\big[(\Pi_{\le j_0} Y_N)^2\big] \\
&\lesssim B_2(\omega) + 2^{j_0}
\end{aligned} \tag{3.58}$$

and $\mathbb{E}\big[B_2^p\big] \le C_p < \infty$ for any finite $p \ge 1$.

Therefore, putting (3.43), (3.47) (3.51), (3.57), and (3.58) together, choosing $2^{j_0} \sim 1 + \|\Upsilon_N\|_{H^1}^{\frac{2}{3}}$, and applying Cauchy's inequality, we obtain

$$\begin{aligned}
\|\Upsilon_N\|_{L^2}^6 &\lesssim \bigg| \int_{\mathbb{T}^3} Y_N \Upsilon_N dx \bigg|^3 = \bigg| \sum_{j=0}^\infty \lambda_j \|\Pi_j Y_N\|_{L^2}^2 \bigg|^3 \\
&\lesssim \Big(2^{-3j_0} B_1(\omega)^{\frac{3}{2}} + 2^{-\frac{3}{2}j_0}\Big) \|\Upsilon_N\|_{H^1}^3 + B_2^3(\omega) + 2^{3j_0} \\
&\lesssim \|\Upsilon_N\|_{H^1}^2 + B_1^3(\omega) + B_2^3(\omega) + 1.
\end{aligned} \tag{3.59}$$

This proves (3.23) in the case (3.43) holds. This concludes the proof of Lemma 3.6. □

Remark 3.7. From the proof of Lemma 3.6 (see (3.41) and (3.59)) with Lemma 3.6, we also have

$$\begin{aligned}\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^3} Y_N \Upsilon_N dx\bigg|^3\bigg] &\lesssim \mathbb{E}\Big[\|\Upsilon_N\|_{L^2}^6 + \|\Upsilon_N\|_{H^1}^2\Big] + 1\\ &\lesssim \mathcal{U}_N + 1,\end{aligned} \tag{3.60}$$

where $\mathcal{U}_N$ is as in (3.24).

3.3. **Uniqueness of the limiting Φ^3_3-measure.** The tightness of the truncated Gibbs measures $\{\rho_N\}_{N\in\mathbb{N}}$, proven in the previous subsection, together with Prokhorov's theorem implies existence of a weakly convergent subsequence. In this subsection, we prove uniqueness of the limiting Φ^3_3-measure, which allows us to conclude the weak convergence of the entire sequence $\{\rho_N\}_{N\in\mathbb{N}}$. While we follow the uniqueness argument in our previous work [53, Subsection 6.3], there are extra terms to control due to the focusing nature of the problem under consideration.

Proposition 3.8. *Let $\{\rho_{N_k^1}\}_{k=1}^\infty$ and $\{\rho_{N_k^2}\}_{k=1}^\infty$ be two weakly convergent subsequences of the truncated Φ^3_3-measures $\{\rho_N\}_{N\in\mathbb{N}}$ defined in* (1.25)*, converging weakly to $\rho^{(1)}$ and $\rho^{(2)}$ as $k\to\infty$, respectively. Then, we have $\rho^{(1)}=\rho^{(2)}$.*

Proof. • **Step 1:** We first show that

$$\lim_{k\to\infty} Z_{N_k^1} = \lim_{k\to\infty} Z_{N_k^2}, \tag{3.61}$$

where Z_N is as in (1.26). By taking a further subsequence, we may assume that $N_k^1 \ge N_k^2$, $k\in\mathbb{N}$. Recall the change of variables (3.12) and let $\widehat{R}^\diamond_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N)$ be as in (3.33). Then, by the Boué-Dupuis variational formula (Lemma 3.1), we have

$$-\log Z_{N_k^j} = \inf_{\dot\Upsilon^{N_k^j}\in\mathbb{H}_a^1} \mathbb{E}\bigg[\widehat{R}^\diamond_{N_k^j}(Y+\Upsilon^{N_k^j}+\sigma\mathfrak{Z}_{N_k^j}) + \frac{1}{2}\int_0^1 \|\dot\Upsilon^{N_k^j}(t)\|_{H_x^1}^2 dt\bigg] \tag{3.62}$$

for $j=1,2$ and $k\in\mathbb{N}$. We point out that Y and $\mathfrak{Z}_N$ do not depend on the drift $\dot\Upsilon^N$ in (3.62).

Given $\delta>0$, let $\underline{\Upsilon}^{N_k^2}$ be an almost optimizer for (3.62) with $j=2$:

$$-\log Z_{N_k^2} \ge \mathbb{E}\bigg[\widehat{R}^\diamond_{N_k^2}(Y+\underline{\Upsilon}^{N_k^2}+\sigma\mathfrak{Z}_{N_k^2}) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}^{N_k^2}(t)\|_{H_x^1}^2 dt\bigg] - \delta. \tag{3.63}$$

By setting $\underline{\Upsilon}_{N_k^2} := \pi_{N_k^2}\underline{\Upsilon}^{N_k^2}$, we have

$$\pi_{N_k^1}\underline{\Upsilon}_{N_k^2} = \underline{\Upsilon}_{N_k^2} \tag{3.64}$$

since $N_k^1 \geq N_k^2$. Then, by choosing $\Upsilon^{N_k^1} = \underline{\Upsilon}_{N_k^2}$, it follows from (3.63) and (3.64) that

$$\begin{aligned}
&-\log Z_{N_k^1} + \log Z_{N_k^2} \\
&\le \inf_{\dot\Upsilon^{N_k^1} \in \mathbb{H}_a^1} \mathbb{E}\bigg[\widehat{R}^\diamond_{N_k^1}(Y + \Upsilon^{N_k^1} + \sigma \mathfrak{Z}_{N_k^1}) + \frac{1}{2}\int_0^1 \|\dot\Upsilon^{N_k^1}(t)\|^2_{H^1_x} dt\bigg] \\
&\qquad - \mathbb{E}\bigg[\widehat{R}^\diamond_{N_k^2}(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}^{N_k^2}(t)\|^2_{H^1_x} dt\bigg] + \delta \\
&\le \mathbb{E}\bigg[\widehat{R}^\diamond_{N_k^1}(Y + \underline{\Upsilon}_{N_k^2} + \sigma \mathfrak{Z}_{N_k^1}) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}_{N_k^2}(t)\|^2_{H^1_x} dt\bigg] \\
&\qquad - \mathbb{E}\bigg[\widehat{R}^\diamond_{N_k^2}(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}^{N_k^2}(t)\|^2_{H^1_x} dt\bigg] + \delta \\
&\le \mathbb{E}\Big[\widehat{R}^\diamond(Y_{N_k^1} + \underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^1}) - \widehat{R}^\diamond(Y_{N_k^2} + \underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^2})\Big] + \delta,
\end{aligned} \tag{3.65}$$

where $\widetilde{\mathfrak{Z}}_{N_k^j} = \pi_{N_k^j} \mathfrak{Z}_{N_k^j}$ is as in (3.19). Here, $\widehat{R}^\diamond$ is defined by

$$\begin{aligned}
\widehat{R}^\diamond(Y + \Upsilon + \sigma \mathfrak{Z}) &= -\sigma \int_{\mathbb{T}^3} Y \Theta^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} \Theta^3 dx \\
&\qquad + A\bigg|\int_{\mathbb{T}^3} \Big(:Y^2: + 2Y\Theta + \Theta^2\Big) dx\bigg|^3,
\end{aligned} \tag{3.66}$$

where $\Theta = \Upsilon + \sigma \mathfrak{Z}$.

We now estimate the right-hand side of (3.65). The main point is that in the difference

$$\mathbb{E}\Big[\widehat{R}^\diamond(Y_{N_k^1} + \underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^1}) - \widehat{R}^\diamond(Y_{N_k^2} + \underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^2})\Big], \tag{3.67}$$

we only have differences in Y-terms and $\mathfrak{Z}$-terms, which allows us to gain a negative power of N_k^2. The contribution from the first term on the right-hand side in (3.66) is given by

$$\begin{aligned}
&-\sigma \mathbb{E}\bigg[\int_{\mathbb{T}^3} (Y_{N_k^1} - Y_{N_k^2}) \underline{\Upsilon}^2_{N_k^2} dx\bigg] \\
&\qquad - \sigma^2 \mathbb{E}\bigg[\int_{\mathbb{T}^3} (Y_{N_k^1} - Y_{N_k^2})(2\underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^1}) \widetilde{\mathfrak{Z}}_{N_k^1} dx\bigg] \\
&\qquad - \sigma^2 \mathbb{E}\bigg[\int_{\mathbb{T}^3} Y_{N_k^2} (\widetilde{\mathfrak{Z}}_{N_k^1} - \widetilde{\mathfrak{Z}}_{N_k^2})(2\underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^1} + \sigma \widetilde{\mathfrak{Z}}_{N_k^2}) dx\bigg].
\end{aligned} \tag{3.68}$$

Let $\mathcal{U}_{N_k^2} = \mathcal{U}_{N_k^2}(\dot{\underline{\Upsilon}}^{N_k^2})$ be as in (3.24) with $\Upsilon_N = \underline{\Upsilon}_{N_k^2}$ and $\Upsilon^N = \underline{\Upsilon}^{N_k^2}$. Then, from Lemmas 3.2 and 3.6, we have

$$\mathbb{E}\Big[\|\underline{\Upsilon}_{N_k^2}\|^2_{H^1} + \|\underline{\Upsilon}_{N_k^2}\|^6_{L^2}\Big] \lesssim 1 + \mathcal{U}_{N_k^2}.$$

Now, proceeding as in (3.38) together with Hölder's inequality in ω and Young's inequality, we bound the first term in (3.68) by

$$
\begin{aligned}
\mathbb{E}\Big[&\|Y_{N_k^1} - Y_{N_k^2}\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|\underline{\Upsilon}_{N_k^2}\|_{L^2}^{\frac{3}{2}-2\varepsilon} \|\underline{\Upsilon}_{N_k^2}\|_{H^1}^{\frac{1}{2}+2\varepsilon}\Big] \\
&\le \|Y_{N_k^1} - Y_{N_k^2}\|_{L_\omega^{\frac{6}{3-4\varepsilon}} \mathcal{C}_x^{-\frac{1}{2}-\varepsilon}} \|\underline{\Upsilon}_{N_k^2}\|_{L_\omega^6 L_x^2}^{\frac{3}{2}-2\varepsilon} \|\underline{\Upsilon}_{N_k^2}\|_{L_\omega^2 H_x^1}^{\frac{1}{2}+2\varepsilon} \\
&\lesssim (N_k^2)^{-a} \|\underline{\Upsilon}_{N_k^2}\|_{L_\omega^6 L_x^2}^{\frac{3}{2}-2\varepsilon} \|\underline{\Upsilon}_{N_k^2}\|_{L_\omega^2 H_x^1}^{\frac{1}{2}+2\varepsilon} \\
&\lesssim (N_k^2)^{-a}\Big(1 + \mathcal{U}_{N_k^2}\Big),
\end{aligned}
\tag{3.69}
$$

where the second inequality follows from a modification of the proof of Lemma 3.2 (i) and noting that the Fourier transform of $Y_{N_k^1} - Y_{N_k^2}$ is supported on the frequencies $\{|n| \gtrsim N_k^2\}$, which allows us to gain a small negative power of N_k^2. Note that the implicit constants in (3.69) depend on $A > 0$ and σ. However, the sizes of A and $|\sigma|$ do not play any role in the subsequent analysis and thus we suppress the dependence on A and σ in the following. The same comment applies to Subsections 3.3 and 3.4.

The second and third terms in (3.68) and the second term on the right-hand side of (3.66) can be handled in a similar manner (with (3.25) to control the $\widetilde{\mathfrak{Z}}_{N_k^j}$-terms). As a result, we can bound the first two terms on the right-hand side of (3.66) by

$$
(N_k^2)^{-a}\Big(C(Y_{N_k^1}, Y_{N_k^2}, \mathfrak{Z}_{N_k^1}, \mathfrak{Z}_{N_k^2}) + \mathcal{U}_{N_k^2}\Big) \lesssim (N_k^2)^{-a}\Big(1 + \mathcal{U}_{N_k^2}\Big) \tag{3.70}
$$

for some small $a > 0$, where $C(Y_{N_k^1}, Y_{N_k^2}, \mathfrak{Z}_{N_k^1}, \mathfrak{Z}_{N_k^2})$ denotes certain high moments of various stochastic terms involving $Y_{N_k^j}$ and $\mathfrak{Z}_{N_k^j}$, $j = 1, 2$, which are bounded by some constant, independent of N_k^j, $j = 1, 2$, in view of Lemma 3.2 and (3.25).

It remains to treat the difference coming from the last term in (3.66). By Young's and Hölder's inequalities, we have

$$
\begin{aligned}
\mathbb{E}\bigg[&\bigg|\int_{\mathbb{T}^3}\Big(:Y_{N_k^1}^2: + 2Y_{N_k^1}(\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1}) + (\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1})^2\Big)dx\bigg|^3 \\
&\quad - \bigg|\int_{\mathbb{T}^3}\Big(:Y_{N_k^2}^2: + 2Y_{N_k^2}(\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2}) + (\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2})^2\Big)dx\bigg|^3\bigg] \\
\lesssim \bigg\{&\bigg\|\int_{\mathbb{T}^3}\Big(:Y_{N_k^1}^2: - :Y_{N_k^2}^2:\Big)dx\bigg\|_{L_\omega^3} + \bigg\|\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})\underline{\Upsilon}_{N_k^2}dx\bigg\|_{L_\omega^3} \\
&+ \bigg\|\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})\widetilde{\mathfrak{Z}}_{N_k^1}dx\bigg\|_{L_\omega^3} + \bigg\|\int_{\mathbb{T}^3}Y_{N_k^2}(\widetilde{\mathfrak{Z}}_{N_k^1} - \widetilde{\mathfrak{Z}}_{N_k^2})dx\bigg\|_{L_\omega^3} \\
&+ \bigg\|\int_{\mathbb{T}^3}(\widetilde{\mathfrak{Z}}_{N_k^1} - \widetilde{\mathfrak{Z}}_{N_k^2})(2\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2})dx\bigg\|_{L_\omega^3}\bigg\} \\
\times \bigg\{&\bigg\|\int_{\mathbb{T}^3}\Big(:Y_{N_k^1}^2: + 2Y_{N_k^1}(\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1}) + (\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1})^2\Big)dx\bigg\|_{L_\omega^3}^2 \\
&+ \bigg\|\int_{\mathbb{T}^3}\Big(:Y_{N_k^2}^2: + 2Y_{N_k^2}(\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2}) + (\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2})^2\Big)dx\bigg\|_{L_\omega^3}^2\bigg\}
\end{aligned}
$$

$$=: \mathrm{I} \times \mathrm{II}. \tag{3.71}$$

We divide I into two groups:

$$\begin{aligned} \mathrm{I} &= \bigg(\mathrm{I} - \bigg\| \int_{\mathbb{T}^3} (Y_{N_k^1} - Y_{N_k^2}) \underline{\Upsilon}_{N_k^2} dx \bigg\|_{L^3_\omega} \bigg) + \bigg\| \int_{\mathbb{T}^3} (Y_{N_k^1} - Y_{N_k^2}) \underline{\Upsilon}_{N_k^2} dx \bigg\|_{L^3_\omega} \\ &=: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{3.72}$$

By the definition (1.19) of the cube frequency projector $\pi_N = \pi_N^{\text{cube}}$, we have

$$\int_{\mathbb{T}^3} (Y_{N_k^1} - Y_{N_k^2}) \underline{\Upsilon}_{N_k^2} dx \int_{\mathbb{T}^3} \pi_{N_k^2} (Y_{N_k^1} - Y_{N_k^2}) \cdot \underline{\Upsilon}_{N_k^2} dx = 0 \tag{3.73}$$

and thus $\mathrm{I}_2 = 0$.

By Lemma 2.1, Hölder's inequality in ω, and Young's inequality, followed by Lemma 3.6 with (3.24), we can estimate I_1 in (3.72) by

$$\begin{aligned} \mathrm{I}_1 &\lesssim \| :Y^2_{N_k^1}: - :Y^2_{N_k^2}: \|_{L^3_\omega \mathcal{C}_x^{-1-\varepsilon}} \\ &\quad + \|Y_{N_k^1} - Y_{N_k^2}\|_{L^6_\omega \mathcal{C}_x^{-\frac{1}{2}-\varepsilon}} \|\widetilde{\mathfrak{Z}}_{N_k^1}\|_{L^6_\omega \mathcal{C}_x^{1-\varepsilon}} + \|Y_{N_k^2}\|_{L^6_\omega \mathcal{C}_x^{-\frac{1}{2}-\varepsilon}} \|\widetilde{\mathfrak{Z}}_{N_k^1} - \widetilde{\mathfrak{Z}}_{N_k^2}\|_{L^6_\omega \mathcal{C}_x^{1-\varepsilon}} \\ &\quad + \|\widetilde{\mathfrak{Z}}_{N_k^1} - \widetilde{\mathfrak{Z}}_{N_k^2}\|_{L^6_\omega \mathcal{C}_x^{1-\varepsilon}} \Big(\|\underline{\Upsilon}_{N_k^2}\|_{L^6_\omega L^2_x} + \|\widetilde{\mathfrak{Z}}_{N_k^1}\|_{L^6_\omega \mathcal{C}_x^{1-\varepsilon}} + \|\widetilde{\mathfrak{Z}}_{N_k^2}\|_{L^6_\omega \mathcal{C}_x^{1-\varepsilon}} \Big) \\ &\lesssim (N_k^2)^{-a} \Big(1 + \mathcal{U}_{N_k^2} \Big)^{\frac{1}{6}}, \end{aligned} \tag{3.74}$$

where we used Lemma 3.2 and (3.25) in bounding the terms involving $Y_{N_k^j}$ and $\widetilde{\mathfrak{Z}}_{N_k^j} = \pi_{N_k^j} \mathfrak{Z}_{N_k^j}$. As for II in (3.71), it follows from (3.60), Lemma 3.6, and (3.24) that

$$\begin{aligned} &\bigg\| \int_{\mathbb{T}^3} \Big(:Y^2_{N_k^j}: + 2 Y_{N_k^j} (\underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^j}) + (\underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^j})^2 \Big) dx \bigg\|_{L^3_\omega} \\ &\lesssim 1 + \bigg\| \int_{\mathbb{T}^3} Y_{N_k^j} \underline{\Upsilon}_{N_k^2} dx \bigg\|_{L^3_\omega} + \|\underline{\Upsilon}_{N_k^2}\|^2_{L^6_\omega L^2_x} \\ &\lesssim 1 + \mathcal{U}_{N_k^2}^{\frac{1}{3}}. \end{aligned} \tag{3.75}$$

From (3.26), (3.16), (3.17), (3.33), and replacing $\underline{\Upsilon}^{N_k^2}$ by 0 in view of (3.62), we have

$$\begin{aligned} \sup_{k \in \mathbb{N}} \mathcal{U}_{N_k^2}(\dot{\underline{\Upsilon}}^{N_k^2}) &\le 10 C_0' + 10 \sup_{k \in \mathbb{N}} \mathbb{E} \bigg[\widehat{R}^\diamond_{N_k^2}(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}) + \frac{1}{2} \int_0^1 \|\dot{\underline{\Upsilon}}^{N_k^2}(t)\|^2_{H^1_x} dt \bigg] \\ &\lesssim 1 + \delta + \sup_{k \in \mathbb{N}} \mathbb{E} \big[\widehat{R}^\diamond_{N_k^2}(Y + 0 + \sigma \mathfrak{Z}_{N_k^2}) \big] \\ &\lesssim 1. \end{aligned} \tag{3.76}$$

Hence, from (3.73), (3.74), (3.75), and (3.76), we obtain that

$$\mathrm{I} \cdot \mathrm{II} \lesssim (N_k^2)^{-a} \longrightarrow 0, \tag{3.77}$$

as $k \to \infty$. Therefore, from (3.70) and (3.77), we conclude that

$$\mathbb{E} \Big[\widehat{R}^\diamond (Y_{N_k^1} + \underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^1}) - \widehat{R}^\diamond (Y_{N_k^2} + \underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^2}) \Big] \longrightarrow 0, \tag{3.78}$$

as $k \to \infty$. Since the choice of $\delta > 0$ was arbitrary, it follows from (3.65) and (3.78) that

$$\lim_{k\to\infty} Z_{N_k^1} \geq \lim_{k\to\infty} Z_{N_k^2}. \tag{3.79}$$

By taking a subsequence of $\{N_k^2\}_{k\in\mathbb{N}}$, still denoted by $\{N_k^2\}_{k\in\mathbb{N}}$, we may assume that $N_k^1 \leq N_k^2$. By repeating the computation above, we then obtain

$$\lim_{k\to\infty} Z_{N_k^1} \leq \lim_{k\to\infty} Z_{N_k^2}. \tag{3.80}$$

Therefore, (3.61) follows from (3.79) and (3.80).

• **Step 2:** Next, we prove $\rho^{(1)} = \rho^{(2)}$. This claim follows from a small modification of Step 1. For this purpose, we need to prove that for every bounded Lipschitz continuous function $F : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$, we have

$$\lim_{k\to\infty} \int \exp(F(u)) d\rho_{N_k^1} \geq \lim_{k\to\infty} \int \exp(F(u)) d\rho_{N_k^2}$$

under the condition $N_k^1 \geq N_k^2$, $k \in \mathbb{N}$ (which can be always satisfied by taking a subsequence of $\{N_k^1\}_{k\in\mathbb{N}}$). In view of (1.26) and (3.61), it suffices to show

$$\begin{aligned} \limsup_{k\to\infty} \bigg[& - \log \Big(\int \exp(F(u) - R^\diamond_{N_k^1}(u)) d\mu \Big) \\ & \qquad + \log \Big(\int \exp(F(u) - R^\diamond_{N_k^2}(u)) d\mu \Big) \bigg] \leq 0. \end{aligned} \tag{3.81}$$

By the Boué-Dupuis variational formula (Lemma 3.1), we have

$$\begin{aligned} & - \log \Big(\int \exp(F(u) - R^\diamond_{N_k^j}(u)) d\mu \Big) \\ & = \inf_{\dot\Upsilon^{N_k^j} \in \mathbb{H}_a^1} \mathbb{E} \bigg[- F(Y + \Upsilon^{N_k^j} + \sigma \mathfrak{Z}_{N_k^j}) \\ & \qquad + \widehat{R}^\diamond_{N_k^j}(Y + \Upsilon^{N_k^j} + \sigma \mathfrak{Z}_{N_k^j}) + \frac{1}{2} \int_0^1 \|\dot\Upsilon^{N_k^j}(t)\|^2_{H^1_x} dt \bigg], \end{aligned} \tag{3.82}$$

where $\widehat{R}^\diamond_{N_k^j}(Y + \Upsilon^{N_k^j} + \sigma \mathfrak{Z}_{N_k^j})$ is as in (3.33). Given $\delta > 0$, let $\underline{\Upsilon}^{N_k^2}$ be an almost optimizer for (3.82) with $j = 2$:

$$\begin{aligned} & - \log \Big(\int \exp(F(u) - R^\diamond_{N_k^2}(u)) d\mu \Big) \\ & \geq \mathbb{E} \bigg[- F(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}) \\ & \qquad + \widehat{R}^\diamond_{N_k^2}(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}) + \frac{1}{2} \int_0^1 \|\dot{\underline{\Upsilon}}^{N_k^2}(t)\|^2_{H^1_x} dt \bigg] - \delta. \end{aligned}$$

Then, by choosing $\Upsilon^{N^1_k} = \underline{\Upsilon}_{N^2_k} = \pi_{N^2_k}\underline{\Upsilon}^{N^2_k}$ and proceeding as in (3.65), we have

$$
\begin{aligned}
&-\log\Big(\int \exp(F(u) - R^\diamond_{N^1_k}(u))d\mu\Big) + \log\Big(\int \exp(F(u) - R^\diamond_{N^2_k}(u))d\mu\Big)\\
&\le \mathbb{E}\bigg[-F(Y + \underline{\Upsilon}_{N^2_k} + \sigma\mathfrak{Z}_{N^1_k})\\
&\qquad + \widehat{R}^\diamond_{N^1_k}(Y + \underline{\Upsilon}_{N^2_k} + \sigma\mathfrak{Z}_{N^1_k}) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}_{N^2_k}(t)\|^2_{H^1_x}dt\bigg]\\
&\quad - \mathbb{E}\bigg[-F(Y + \underline{\Upsilon}^{N^2_k} + \sigma\mathfrak{Z}_{N^2_k})\\
&\qquad + \widehat{R}^\diamond_{N^2_k}(Y + \underline{\Upsilon}^{N^2_k} + \sigma\mathfrak{Z}_{N^2_k}) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}^{N^2_k}(t)\|^2_{H^1_x}dt\bigg] + \delta\\
&\le \mathrm{Lip}(F)\cdot\mathbb{E}\Big[\|\pi^\perp_{N^2_k}\underline{\Upsilon}^{N^2_k} - \sigma(\mathfrak{Z}_{N^1_k} - \mathfrak{Z}_{N^2_k})\|_{\mathcal{C}^{-100}}\Big]\\
&\quad + \mathbb{E}\Big[\widehat{R}^\diamond(Y_{N^1_k} + \underline{\Upsilon}_{N^2_k} + \sigma\widetilde{\mathfrak{Z}}_{N^1_k}) - \widehat{R}^\diamond(Y_{N^2_k} + \underline{\Upsilon}_{N^2_k} + \sigma\widetilde{\mathfrak{Z}}_{N^2_k})\Big] + \delta,
\end{aligned}
\tag{3.83}
$$

where $\pi^\perp_N = \mathrm{Id} - \pi_N$ and $\widehat{R}^\diamond$ is as in (3.66). We can proceed as in Step 1 to show that the second term on the right-hand side of (3.83) satisfies (3.78). Here, we need to use the boundedness of F in showing an analogue of (3.76) in the current context (with an almost optimizer $\underline{\Upsilon}^{N^2_k}$ for (3.82)).

Finally, we estimate the first term on the right-hand side of (3.83). Write

$$
\begin{aligned}
&\mathbb{E}\Big[\|\pi^\perp_{N^2_k}\underline{\Upsilon}^{N^2_k} - \sigma(\mathfrak{Z}_{N^1_k} - \mathfrak{Z}_{N^2_k})\|_{\mathcal{C}^{-100}}\Big]\\
&\qquad \lesssim \mathbb{E}\Big[\|\pi^\perp_{N^2_k}\Upsilon^{N^2_k}\|_{\mathcal{C}^{-100}}\Big] + \mathbb{E}\Big[\|\mathfrak{Z}_{N^1_k} - \mathfrak{Z}_{N^2_k}\|_{\mathcal{C}^{-100}}\Big].
\end{aligned}
$$

A standard computation with (3.11) shows that the second term on the right-hand side tends to 0 as $k \to \infty$. As for the first term, from Lemma 3.2 and (an analogue of) (3.76), we obtain

$$
\mathbb{E}\Big[\|\pi^\perp_{N^2_k}\underline{\Upsilon}^{N^2_k}\|_{\mathcal{C}^{-100}}\Big] \lesssim (N^2_k)^{-a}\|\underline{\Upsilon}^{N^2_k}\|_{L^2_\omega H^1_x} \lesssim (N^2_k)^{-a}\Big(\sup_{k\in\mathbb{N}} \mathcal{U}_{N^2_k}\Big)^{\frac{1}{2}} \longrightarrow 0,
$$

as $k \to \infty$. Since the choice of $\delta > 0$ was arbitrary, we conclude (3.81) and hence $\rho^{(1)} = \rho^{(2)}$. This completes the proof of Proposition 3.8. □

Remark 3.9. In the proof of Proposition 3.8, we used the orthogonality relation (3.73) to conclude that $\mathrm{I}_2 = 0$. While the same orthogonality holds for the ball frequency projector π^{ball}_N in (1.41), such an orthogonality relation is false for the smooth frequency projector π^{smooth}_N in (1.42). As seen from the proof of Lemma 3.6 and the uniform bound (3.76) on $\mathcal{U}_{N^2_k}(\dot{\underline{\Upsilon}}^{N^2_k})$, the quantity I_2 in (3.72) is critical (with respect to the spatial regularity/integrability and also with respect to the ω-integrability). From Remark 3.7 and (3.76), we see that the quantity I_2 is bounded, uniformly in $k \in \mathbb{N}$. In the absence of the orthogonality (3.73), however, we do not know how to show that this term tends to 0 as $k \to \infty$ in the case of the smooth frequency projector π^{smooth}_N. We point out that the same issue also appears in the proofs of Propositions 4.1 and 6.10 in the case of the smooth frequency projector π^{smooth}_N.

3.4. **Singularity of the Φ^3_3-measure.** We conclude this section by proving mutual singularity of the Φ^3_3-measure ρ, constructed in the previous subsections, and the base Gaussian free field μ in (1.16). In Section 4 of [4], Barashkov and Gubinelli proved the singularity of the Φ^4_3-measure by making use of the shifted measure. In the following, we follow our previous work [53] and present a direct proof of singularity of the Φ^3_3-measure without referring to a shifted measure. See also Appendix A, where we construct a shifted measure with respect to which the Φ^3_3-measure is absolutely continuous.

Proposition 3.10. *Let R_N be as in* (1.23) *with $\gamma = 3$, and $\varepsilon > 0$. Then, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that the set*

$$S := \big\{u \in H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) : \lim_{k\to\infty} (\log N_k)^{-\frac{3}{4}} R_{N_k}(u) = 0\big\}$$

satisfies

$$\mu(S) = 1 \qquad \text{but} \qquad \rho(S) = 0. \tag{3.84}$$

In particular, the Φ^3_3-measure ρ and the massive Gaussian free field μ in (1.16) *are mutually singular.*

Proof. From (1.23) with $\gamma = 3$, the Wiener chaos estimate (Lemma 2.5), Lemma 2.6, and Lemma 2.4, we have

$$\begin{aligned}
\|R_N(u)\|^2_{L^2(\mu)} &\lesssim \bigg\| \int_{\mathbb{T}^3} :u_N^3:\, dx \bigg\|^2_{L^2(\mu)} + \bigg\| \int_{\mathbb{T}^3} :u_N^2:\, dx \bigg\|^6_{L^6(\mu)} \\
&\lesssim \bigg\| \int_{\mathbb{T}^3} :u_N^3:\, dx \bigg\|^2_{L^2(\mu)} + \bigg\| \int_{\mathbb{T}^3} :u_N^2:\, dx \bigg\|^6_{L^2(\mu)} \\
&\lesssim \sum_{\substack{n_1+n_2+n_3=0\\ n_j \in NQ}} \langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\langle n_3\rangle^{-2} + \bigg(\sum_{\substack{n_1+n_2=0\\ n_j\in NQ}} \langle n_1\rangle^{-2}\langle n_2\rangle^{-2} \bigg)^3 \\
&\lesssim \sum_{|n_1|, |n-n_1| \lesssim N} \langle n_1\rangle^{-2}\langle n-n_1\rangle^{-1} + 1 \\
&\lesssim \log N,
\end{aligned}$$

where Q denotes the cube of side length 2 in $\mathbb{R}^3$ centered at the origin as in (1.21). Thus, we have

$$\lim_{N\to\infty} (\log N)^{-\frac{3}{4}} \|R_N(u)\|_{L^2(\mu)} \lesssim \lim_{N\to\infty} (\log N)^{-\frac{1}{4}} = 0.$$

Hence, there exists a subsequence such that

$$\lim_{k\to\infty} (\log N_k)^{-\frac{3}{4}} R_{N_k}(u) = 0,$$

almost surely with respect to μ. This proves $\mu(S) = 1$ in (3.84).

Given $k \in \mathbb{N}$, define $G_k(u)$ by

$$G_k(u) = (\log N_k)^{-\frac{3}{4}} R_{N_k}(u). \tag{3.85}$$

In the following, we show that $e^{G_k(u)}$ tends to 0 in $L^1(\rho)$. This will imply that there exists a subsequence of $G_k(u)$ tending to $-\infty$, almost surely with respect to the Φ^3_3-measure ρ, which in turn yields the second claim in (3.84): $\rho(S) = 0$.

Let ϕ be a smooth bump function as in Subsection 2.1. By Fatou's lemma, the weak convergence of ρ_M to ρ, the boundedness of ϕ, and (1.25), we have

$$\begin{aligned}\int e^{G_k(u)} d\rho(u) &\le \liminf_{K\to\infty} \int \phi\bigg(\frac{G_k(u)}{K}\bigg) e^{G_k(u)} d\rho(u)\\ &= \liminf_{K\to\infty} \lim_{M\to\infty} \int \phi\bigg(\frac{G_k(u)}{K}\bigg) e^{G_k(u)} d\rho_M(u)\\ &\le \lim_{M\to\infty} \int e^{G_k(u)} d\rho_M(u) = Z^{-1} \lim_{M\to\infty} \int e^{G_k(u) - R^\diamond_M(u)} d\mu(u)\\ &=: Z^{-1} \lim_{M\to\infty} C_{M,k},\end{aligned} \tag{3.86}$$

provided that $\lim_{M\to\infty} C_{M,k}$ exists. Here, $Z = \lim_{M\to\infty} Z_M$ denotes the partition function for ρ.

Our main goal is to show that the right-hand side of (3.86) tends to 0 as $k\to\infty$. As in the previous subsections, we proceed with the change of variables (3.12):

$$\dot\Upsilon^M(t) = \dot\Theta(t) - \sigma \dot{\mathfrak{Z}}_M(t).$$

Then, by the Boué-Dupuis variational formula (Lemma 3.1) and (3.85), we have

$$\begin{aligned}-\log C_{M,k} &= \inf_{\dot\Upsilon^M \in \mathbb{H}^1_a} \mathbb{E}\bigg[-(\log N_k)^{-\frac34} R_{N_k}(Y + \Upsilon^M + \sigma \mathfrak{Z}_M)\\ &\qquad\qquad + \widehat R^\diamond_M(Y + \Upsilon^M + \sigma\mathfrak{Z}_M) + \frac12 \int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x} dt\bigg]\\ &=: \inf_{\dot\Upsilon^M\in\mathbb{H}^1_a} \widehat{\mathcal{W}}_{M,k}(\dot\Upsilon^M),\end{aligned} \tag{3.87}$$

where $\widehat R^\diamond_N$ is as in (3.33). In the following, we prove that the right-hand side (and hence the left-hand side) of (3.87) diverges to ∞ as $k\to\infty$.

Proceeding as in Subsection 3.2 (see (3.26)), we bound the last two terms on the right-hand side of (3.87) as

$$\mathbb{E}\bigg[\widehat R^\diamond_M(Y + \Upsilon^M + \sigma\mathfrak{Z}_M) + \frac12 \int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x} dt\bigg] \ge -C_0 + \frac{1}{10}\mathcal{U}_M, \tag{3.88}$$

where $\mathcal{U}_M = \mathcal{U}_M(\dot\Upsilon^M)$ is given by (3.24) with $\Upsilon_N = \pi_M \Upsilon^M$ and $\Upsilon^N = \Upsilon^M$:

$$\mathcal{U}_M = \mathbb{E}\bigg[\frac{A}{2}\bigg|\int_{\mathbb{T}^3} \Big(2Y_M \pi_M\Upsilon^M + (\pi_M\Upsilon^M)^2\Big) dx\bigg|^3 + \frac12\int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x} dt\bigg]. \tag{3.89}$$

Next, we study the first term on the right-hand side of (3.87), which gives the main (divergent) contribution. From (1.23) with $\gamma = 3$, we have

$$\begin{aligned}R_{N_k}(Y + \Upsilon^M + \sigma\mathfrak{Z}_M) &= -\frac{\sigma}{3}\int_{\mathbb{T}^3} :Y^3_{N_k}: dx - \sigma \int_{\mathbb{T}^3} :Y^2_{N_k}: \Theta_{N_k} dx\\ &\quad - \sigma \int_{\mathbb{T}^3} Y_{N_k}\Theta^2_{N_k} dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} \Theta^3_{N_k} dx\\ &\quad + A\bigg|\int_{\mathbb{T}^3} \Big(:Y^2_{N_k}: + 2Y_{N_k}\Theta_{N_k} + \Theta^2_{N_k}\Big) dx\bigg|^3\\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}\end{aligned} \tag{3.90}$$

for $N_k \le M$, where Θ_{N_k} is given by

$$\Theta_{N_k} := \pi_{N_k}\Theta = \pi_{N_k}\Upsilon^M + \sigma\pi_{N_k}\mathfrak{Z}_M. \tag{3.91}$$

As we see below, under an expectation, the second term II on the right-hand side of (3.90) (which is precisely the term removed by the second renormalization) gives a divergent contribution; see (3.97) below. From Lemma 3.2, the first term I on the right-hand side of (3.90) gives 0 under an expectation. As for the last three terms, we proceed as in Subsection 3.2 (see also the proof of Proposition 3.8) and obtain

$$\big|\mathbb{E}\big[\mathrm{III} + \mathrm{IV} + \mathrm{V}\big]\big| \lesssim C(Y_{N_k}, \pi_{N_k}\mathfrak{Z}_M) + \mathcal{U}_{N_k} \lesssim 1 + \mathcal{U}_{N_k} \tag{3.92}$$

where $C(Y_{N_k}, \pi_{N_k}\mathfrak{Z}_M)$ denotes certain high moments of various stochastic terms involving Y_{N_k} and $\pi_{N_k}\mathfrak{Z}_M$ and $\mathcal{U}_{N_k} = \mathcal{U}_{N_k}(\partial_t\pi_{N_k}\Upsilon^M)$ is given by (3.24) with $\Upsilon_N = \Upsilon^N = \pi_{N_k}\Upsilon^M$:

$$\mathcal{U}_{N_k} = \mathbb{E}\bigg[\frac{A}{2}\bigg|\int_{\mathbb{T}^3}\Big(2Y_{N_k}\pi_{N_k}\Upsilon^M + (\pi_{N_k}\Upsilon^M)^2\Big)dx\bigg|^3 + \frac{1}{2}\int_0^1 \|\partial_t(\pi_{N_k}\Upsilon^M)(t)\|_{H^1_x}^2 dt\bigg]. \tag{3.93}$$

In view of the smallness of $(\log N_k)^{-\frac{3}{4}}$ in (3.87), the second term in (3.93) can be controlled by the positive terms $\mathcal{U}_M$ in (3.88) (in particular by the second term in (3.89)). As for the first term in (3.93), it follows from (3.60), $\pi_{N_k}\Upsilon^M = \pi_{N_k}\pi_M\Upsilon^M$ for $N_k \le M$, and Lemma 3.6 with (3.89) that

$$\begin{aligned}
\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^3}\Big(2Y_{N_k}\Upsilon^M + (\pi_{N_k}\Upsilon^M)^2\Big)dx\bigg|^3\bigg] &\lesssim \bigg\|\int_{\mathbb{T}^3} Y_{N_k}\pi_{N_k}\Upsilon^M dx\bigg\|_{L^3_\omega}^3 + \|\pi_{N_k}\Upsilon^M\|_{L^6_\omega L^2_x}^6 \\
&\lesssim 1 + \|\pi_M\Upsilon^M\|_{L^6_\omega L^2_x}^6 + \|\Upsilon^M\|_{L^2_\omega H^1_x}^2 \\
&\lesssim 1 + \mathcal{U}_M
\end{aligned}$$

for $N_k \le M$. Hence, $\mathcal{U}_{N_k}$ in (3.93) can be controlled by $\mathcal{U}_M$ in (3.89):

$$\mathcal{U}_{N_k} \lesssim 1 + \mathcal{U}_M. \tag{3.94}$$

Hence, from (3.87), (3.88), (3.90), (3.92), and (3.94), we obtain

$$\widehat{\mathcal{W}}_{M,k}(\dot{\Upsilon}^M) \ge \sigma(\log N_k)^{-\frac{3}{4}}\mathbb{E}\bigg[\int_{\mathbb{T}^3} :Y_{N_k}^2: \Theta_{N_k} dx\bigg] - C_1 + \frac{1}{20}\mathcal{U}_M \tag{3.95}$$

for any $M \ge N_k \gg 1$.

Therefore, it remains to estimate the contribution from the second term on the right-hand side of (3.90). Let us first state a lemma whose proof is presented at the end of this subsection.

Lemma 3.11. *We have*

$$\mathbb{E}\bigg[\int_0^1 \langle \dot{\mathfrak{Z}}_N(t), \dot{\mathfrak{Z}}_M(t)\rangle_{H^1_x} dt\bigg] \sim \log N \tag{3.96}$$

for any $1 \le N \le M$, *where* $\dot{\mathfrak{Z}}_N = \pi_N \dot{\mathfrak{Z}}^N$.

By assuming Lemma 3.11, we complete the proof of Proposition 3.10. By (3.10), (3.11) with $\mathfrak{Z}_{N_k} = \pi_{N_k}\mathfrak{Z}^{N_k}$, (3.91), Lemma 3.11, Cauchy's inequality (with small $\varepsilon_0 > 0$), and Lemma 3.2

(see (3.7)), we have

$$\begin{aligned}\sigma\mathbb{E}\bigg[\int_{\mathbb{T}^3} :Y_{N_k}^2:\Theta_{N_k}dx\bigg] &= \sigma\mathbb{E}\bigg[\int_0^1\int_{\mathbb{T}^3} :Y_{N_k}^2(t):\dot{\Theta}_{N_k}(t)dt\bigg]\\ &= \sigma^2\mathbb{E}\bigg[\int_0^1 \langle \dot{\mathfrak{Z}}_{N_k}(t), \dot{\mathfrak{Z}}_M(t)\rangle_{H^1_x}dt\bigg] + \sigma\mathbb{E}\bigg[\int_0^1 \langle \dot{\mathfrak{Z}}_{N_k}(t), \dot{\Upsilon}^M(t)\rangle_{H^1_x}dt\bigg]\\ &\geq c\log N_k - \varepsilon_0\mathbb{E}\bigg[\int_0^1 \| :Y_{N_k}^2(t): \|^2_{H^{-1}_x}dt\bigg] - C_{\varepsilon_0}\mathbb{E}\bigg[\int_0^1 \|\dot{\Upsilon}^M(t)\|^2_{H^1_x}dt\bigg]\\ &\geq \frac{c}{2}\log N_k - C_{\varepsilon_0}\mathbb{E}\bigg[\int_0^1 \|\dot{\Upsilon}^M(t)\|^2_{H^1_x}dt\bigg]\end{aligned} \tag{3.97}$$

for $M \geq N_k \gg 1$. Thus, putting (3.87), (3.95), and (3.97) together, we have

$$-\log C_{M,k} \geq \inf_{\dot{\Upsilon}^M\in\mathbb{H}^1_a}\Big\{c(\log N_k)^{\frac14} - C_2 + \frac{1}{40}\mathcal{U}_M\Big\} \geq c(\log N_k)^{\frac14} - C_2 \tag{3.98}$$

for any sufficiently large $k \gg 1$ (such that $N_k \gg 1$). Hence, from (3.98), we obtain

$$C_{M,k} \lesssim \exp\Big(-c(\log N_k)^{\frac14}\Big) \tag{3.99}$$

for $M \geq N_k \gg 1$, uniformly in $M \in \mathbb{N}$. Therefore, by taking limits in $M \to \infty$ and then $k \to \infty$, we conclude from (3.86) and (3.99) that

$$\lim_{k\to\infty}\int e^{G_k(u)}d\rho(u) = 0$$

as desired. This completes the proof of Proposition 3.10. □

We conclude this section by presenting the proof of Lemma 3.11.

Proof of Lemma 3.11. For simplicity, we suppress the time dependence in the following. From (3.11), we have

$$\widehat{\dot{\mathfrak{Z}}}_N(n) = \langle n\rangle^{-2}\sum_{\substack{n_1,n_2\in\mathbb{Z}^3\\ n=n_1+n_2\neq 0}}\widehat{Y}_N(n_1)\widehat{Y}_N(n_2) \tag{3.100}$$

for $n \neq 0$. On the other hand, when $n = 0$, it follows from Lemma 2.6 that

$$\mathbb{E}\Big[|\widehat{\dot{\mathfrak{Z}}}_N(0)|^2\Big] = \mathbb{E}\bigg[\Big(\sum_{\substack{n_1\in\mathbb{Z}^3\\ n_1\in NQ}}\big(|\widehat{Y}_N(n_1)|^2 - \langle n_1\rangle^{-2}\big)\Big)^2\bigg] \lesssim \sum_{n_1\in\mathbb{Z}^3}\langle n_1\rangle^{-4} \lesssim 1, \tag{3.101}$$

where Q is as in (1.21). Hence, from (3.100) and (3.101), we have

$$\begin{aligned}\mathbb{E}\bigg[\int_0^1 \langle \dot{\mathfrak{Z}}_N(t), \dot{\mathfrak{Z}}_M(t)\rangle_{H^1_x}dt\bigg] &= \int_0^1 \mathbb{E}\bigg[\sum_{n\in\mathbb{Z}^3}\langle n\rangle^2\widehat{\dot{\mathfrak{Z}}}_N(n,t)\overline{\widehat{\dot{\mathfrak{Z}}}_M(n,t)}\bigg]dt\\ &= \int_0^1 \mathbb{E}\bigg[\sum_{n\in\mathbb{Z}^3\setminus\{0\}}\langle n\rangle^2\widehat{\dot{\mathfrak{Z}}}_N(n,t)\overline{\widehat{\dot{\mathfrak{Z}}}_M(n,t)}\bigg]dt + O(1).\end{aligned}$$

We now proceed as in the proof of (3.7) in Lemma 3.2 (i). By applying (3.11) and Lemma 2.6, and summing over $\big\{|n| \le \frac{2}{3}N,\ \frac{1}{4}|n| \le |n_1| \le \frac{1}{2}|n|\big\}$ (which implies $|n_2| \sim |n|$ and $|n_j| \le N$, $j = 1, 2$), we have

$$
\begin{aligned}
&\mathbb{E}\bigg[\sum_{n\in\mathbb{Z}^3\setminus\{0\}} \langle n\rangle^2 \widehat{\dot{\mathfrak{Z}}}_N(n,t)\overline{\widehat{\dot{\mathfrak{Z}}}_M(n,t)}\bigg]\\
&= \sum_{n\in\mathbb{Z}^3} \frac{\chi_N(n)\chi_M(n)}{\langle n\rangle^2}\\
&\quad\times \int_{\mathbb{T}^3_x\times\mathbb{T}^3_y} \mathbb{E}\Big[H_2(Y_N(x,t);t\sigma_N)H_2(Y_N(y,t);t\sigma_N)\Big]e_n(y-x)dxdy\\
&= \sum_{n\in\mathbb{Z}^3} \frac{t^2\chi_N(n)\chi_M(n)}{\langle n\rangle^2} \sum_{n_1,n_2\in\mathbb{Z}^3} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2} \int_{\mathbb{T}^3_x\times\mathbb{T}^3_y} e_{n_1+n_2-n}(x-y)dxdy\\
&= \sum_{n\in\mathbb{Z}^3} \frac{t^2\chi_N(n)\chi_M(n)}{\langle n\rangle^2} \sum_{n=n_1+n_2} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2} \sim t^2\log N,
\end{aligned}
$$

where $\chi_N(n_j)$ is as in (1.20). By integrating on $[0, 1]$, we obtain the desired bound (3.96). □

4. Non-normalizability in the strongly nonlinear regime

4.1. **Reference measures and the σ-finite Φ^3_3-measure.** In this section, we prove non-normalizability of the Φ^3_3-measure in the strongly nonlinear regime (Theorem 1.8 (ii)). In [53], we introduced a strategy for establishing non-normalizability in the context of the focusing Hartree Φ^4_3-measures on $\mathbb{T}^3$, using the Boué-Dupuis variational formula. We point out that, in [53], the focusing Hartree Φ^4_3-measures were absolutely continuous with respect to the base Gaussian free field μ. Moreover, the truncated potential energy $R_N^{\text{Hartree}}(u)$ and the corresponding density $e^{-R_N^{\text{Hartree}}(u)}$ of the truncated focusing Hartree Φ^4_3-measures formed convergent sequences. In [53], we proved the following version of the non-normalizability of the focusing Hartree Φ^4_3-measure:

$$
\sup_{N\in\mathbb{N}} \mathbb{E}_\mu\Big[e^{-R_N^{\text{Hartree}}(u)}\Big] = \infty. \tag{4.1}
$$

Denoting the limiting density by $e^{-R^{\text{Hartree}}(u)}$, this result says that the σ-finite version of the focusing Hartree Φ^4_3-measure:

$$
e^{-R^{\text{Hartree}}(u)}d\mu(u)
$$

is not normalizable (i.e. there is no normalization constant to make this into a probability measure). See also [60] for an analogous non-normalizability result for the log-correlated focusing Gibbs measures with a quartic interaction potential.

The main new difficulty in our current problem is the singularity of the Φ^3_3-measure. In particular, the potential energy $R_N^\diamond(u)$ in (1.24) (and the corresponding density $e^{-R_N^\diamond(u)}$) does *not* converge to any limit. Hence, even if we prove a non-normalizability statement of the form (4.1), it might still be possible that by choosing a sequence of constants $\widehat{Z}_N$ appropriately, the measure $\widehat{Z}_N^{-1}e^{-R_N^\diamond(u)}d\mu$ has a weak limit. This is precisely the case for the Φ^4_3-measure; see [3]. The non-convergence claim in Theorem 1.8 (ii) for the truncated Φ^3_3-measures (see Proposition 4.4 below) tells us that this is not the case for the Φ^3_3-measure.

In order to overcome this issue, we first construct a reference measure ν_δ as a weak limit of the following tamed version of the truncated Φ^3_3-measure (with $\delta > 0$):

$$d\nu_{N,\delta}(u) = Z_{N,\delta}^{-1} \exp\Big(- \delta F(\pi_N u) - R_N^\diamond(u)\Big) d\mu(u)$$

for some appropriate taming function F; see (4.6). See Proposition 4.1. We also show that $F(u)$, without the frequency projection π_N on u, is well defined almost surely with respect to the limiting reference measure $\nu_\delta = \lim_{N\to\infty} \nu_{N,\delta}$. This allows us to construct a σ-finite version of the Φ^3_3-measure:

$$d\overline{\rho}_\delta = e^{\delta F(u)} d\nu_\delta = \lim_{N\to\infty} Z_{N,\delta}^{-1} e^{\delta F(u)} e^{-\delta F(\pi_N u) - R_N^\diamond(u)} d\mu(u). \tag{4.2}$$

The main point is that while the truncated Φ^3_3-measure ρ_N ($= \nu_{N,\delta}$ with $\delta = 0$) may not be convergent, the tamed version $\nu_{N,\delta}$ of the truncated Φ^3_3-measure converges to the limit ν_δ, thus allowing us to define a σ-finite version of the Φ^3_3-measure. We then show that this σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure in (4.2) is not normalizable in the strongly nonlinear regime. See Proposition 4.2. Furthermore, as a corollary to this non-normalizability result of the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure, we also show that the sequence $\{\rho_N\}_{N\in\mathbb{N}}$ of the truncated Φ^3_3-measures defined in (1.25) does not converge weakly in a natural space[17] $\mathcal{A}(\mathbb{T}^3)$ (see (4.3) below) for the Φ^3_3-measure. See Proposition 4.4.

We first state the construction of the reference measure. Let p_t be the kernel of the heat semigroup $e^{t\Delta}$. Then, define the space $\mathcal{A} = \mathcal{A}(\mathbb{T}^3)$ via the norm:

$$\|u\|_{\mathcal{A}} := \sup_{0<t\le 1} \Big(t^{\frac{3}{8}} \|p_t * u\|_{L^3(\mathbb{T}^3)}\Big). \tag{4.3}$$

Recall from [44, Theorem 5.3][18] (see also [75, (2.41)] and [2, Theorem 2.34]) that

$$\mathcal{A} = B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3). \tag{4.4}$$

In particular, the space $\mathcal{A}$ contains the support of the massive Gaussian free field μ on $\mathbb{T}^3$ and thus we have $\|u\|_{\mathcal{A}} < \infty$, μ-almost surely. See Lemma 4.6 below. In the following, for simplicity of notation, we use $\mathcal{A}$ rather than $B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3)$. Moreover, the notation $\mathcal{A}$ is suitable for our purpose, since we make use of the characterization (4.3) extensively via the Schauder estimate, which we recall now (see for example [59]):

$$\|p_t * u\|_{L^q(\mathbb{T}^3)} \le C_{\alpha,p,q}\, t^{-\frac{\alpha}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|\langle\nabla\rangle^{-\alpha} u\|_{L^p(\mathbb{T}^3)} \tag{4.5}$$

for any $\alpha \ge 0$ and $1 \le p \le q \le \infty$. From the Schauder estimate (4.5) (or directly from (4.4)), we see that $W^{-\frac{3}{4},3}(\mathbb{T}^3) \subset \mathcal{A}$.

Given $N \in \mathbb{N}$, we set $u_N = \pi_N u$. Then, given $\delta > 0$ and $N \in \mathbb{N}$, we define the measure $\nu_{N,\delta}$ by

$$d\nu_{N,\delta}(u) = Z_{N,\delta}^{-1} \exp\Big(- \delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\Big) d\mu(u) \tag{4.6}$$

[17] For example, in the weakly nonlinear regime, the support of the limiting Φ^3_3-measure constructed in Theorem 1.8 (i) is contained in the space $\mathcal{A}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$.

[18] The discussion in [44] is on $\mathbb{R}^d$, but a slight modification yields the corresponding result on $\mathbb{T}^d$.

for $N \in \mathbb{N}$ and $\delta > 0$, where $R_N^\diamond$ is as in (1.24) and

$$Z_{N,\delta} = \int \exp\Big(- \delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\Big) d\mu(u). \tag{4.7}$$

Namely, $\nu_{N,\delta}$ is a tamed version of the truncated Φ^3_3-measure ρ_N in (1.25). We prove that the sequence $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ converges weakly to some limiting probability measure ν_δ.

Proposition 4.1. *Let $\sigma \neq 0$ and $\gamma \geq 3$. Then, given any $\delta > 0$, the sequence of measures $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ defined in* (4.6) *converges weakly to a unique probability measure ν_δ, and similarly $Z_{N,\delta}$ converges to Z_δ. Moreover, $\|u\|_{\mathcal{A}}$ is finite ν_δ-almost surely, and we have*

$$d\nu_\delta(u) = \frac{\exp(-(\delta - \delta')\|u\|_{\mathcal{A}}^{20})}{\int \exp(-(\delta - \delta')\|u\|_{\mathcal{A}}^{20}) d\nu_{\delta'}(u)} d\nu_{\delta'}(u) \tag{4.8}$$

for $\delta > \delta' > 0$.

This proposition allows us to define a σ-finite version of the Φ^3_3-measure by

$$d\overline{\rho}_\delta = e^{\delta \|u\|_{\mathcal{A}}^{20}} d\nu_\delta \tag{4.9}$$

for any $\delta > 0$. At a very *formal* level, $\delta\|u\|_{\mathcal{A}}^{20}$ in the exponent of (4.9) and $-\delta\|u_N\|_{\mathcal{A}}^{20}$ in the exponent of (4.6) cancel each other in the limit as $N \to \infty$, and thus the right-hand side of (4.8) formally looks like $Z_\delta^{-1} \lim_{N\to\infty} e^{-R_N^\diamond(u)} d\mu$. While this discussion is merely formal, it explains why we refer to the measure $\overline{\rho}_\delta$ as a σ-finite version of the Φ^3_3-measure. The identity (4.8) shows how ν_δ's for different values of $\delta > 0$ are related. When $\delta = 0$, the expression $Z_\delta \overline{\rho}_\delta$ would formally correspond to a limit of $e^{-R_N^\diamond(u)} d\mu$, but in order to achieve the weak convergence claimed in Proposition 4.1 and construct a σ-finite version of the Φ^3_3-measure, we need to start with a tamed version (i.e. $\delta > 0$) of the truncated Φ^3_3-measure. For the sake of concreteness, we chose a taming via the $\mathcal{A}$-norm but it is possible to consider a different taming (say, based on some other norm) and obtain the same result.

The next proposition shows that the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure defined in (4.9) is not normalizable in the strongly nonlinear regime.

Proposition 4.2. *Let $\sigma \gg 1$ and $\gamma \geq 3$. Given $\delta > 0$, let ν_δ be the measure constructed in Proposition 4.1 and let $\overline{\rho}_\delta$ be as in* (4.9). *Then, we have*

$$\int 1\, d\overline{\rho}_\delta = \int \exp\Big(\delta \|u\|_{\mathcal{A}}^{20}\Big) d\nu_\delta = \infty. \tag{4.10}$$

Remark 4.3. (i) A slight modification of the computation in Subsection 3.4 combined with the analysis in Subsection 4.2 presented below (Step 1 of the proof of Proposition 4.1) shows that the tamed version ν_δ of the Φ^3_3-measure, constructed in Proposition 4.1, and the massive Gaussian free field μ are mutually singular, just like the Φ^3_3-measure in the weakly nonlinear regime, constructed in Section 3. As a consequence, the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure defined in (4.9) and the massive Gaussian free field μ are mutually singular.

(ii) In Appendix A, we show that the limiting Φ^3_3-measure is absolutely continuous with respect to the shifted measure $\operatorname{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$ in the weakly nonlinear regime. A slight modification of the argument in Appendix A also shows that the tamed version ν_δ of the Φ^3_3-measure constructed in Proposition 4.1 and the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure in (4.9) are also absolutely continuous with respect to the same shifted measure, even in the strongly nonlinear regime. See Remark A.3. This shows that the measure $\overline{\rho}_\delta$ in (4.9) is a quite natural candidate to consider as a σ-finite version of the Φ^3_3-measure.

As a corollary to (the proofs of) Propositions 4.1 and 4.2, we show the following non-convergence result for the truncated Φ^3_3-measure ρ_N in (1.25).

Proposition 4.4. *Let $\sigma \gg 1$, $\gamma \geq 3$, and $\mathcal{A} = \mathcal{A}(\mathbb{T}^3)$ be as in* (4.3)*. Then, the sequence $\{\rho_N\}_{N\in\mathbb{N}}$ of the truncated Φ^3_3-measures defined in* (1.25) *does not converge weakly to any limit as probability measures on $\mathcal{A}$. The same claim holds for any subsequence $\{\rho_{N_k}\}_{k\in\mathbb{N}}$.*

In Subsection 4.2, we present the proof of Proposition 4.1. In Subsection 4.3, we then prove the non-normalizability (Proposition 4.2). Finally, we present the proof of Proposition 4.4 in Subsection 4.4.

4.2. **Construction of the reference measure.** In this subsection, we present the proof of Proposition 4.1 on the construction of the reference measure ν_δ. We first establish several preliminary lemmas.

Lemma 4.5. *Let the $\mathcal{A}$-norm be as in* (4.3)*. Then, we have*

$$\|u\|_{\mathcal{A}} \lesssim \|u\|_{H^{-\frac{1}{4}}}.$$

Proof. This is immediate from the Schauder estimate (4.5). $\square$

Lemma 4.6. *We have $W^{-\frac{3}{4},3}(\mathbb{T}^3) \subset \mathcal{A}$ and thus the quantity $\|u\|_{\mathcal{A}}$ is finite μ-almost surely. Moreover, given any $1 \leq p < \infty$, we have*

$$\mathbb{E}_\mu\Big[\|\pi_N u\|^p_{\mathcal{A}}\Big] \leq C_p < \infty, \tag{4.11}$$

uniformly in $N \in \mathbb{N}\cup\{\infty\}$ with the understanding that $\pi_\infty = \mathrm{Id}$.

Proof. As we already mentioned, the first claim follows from the Schauder estimate (4.5) (or from (4.4)). As for the bound (4.11), from the Schauder estimate (4.5), Minkowski's integral inequality, and the Wiener chaos estimate (Lemma 2.5) with (1.18), we have

$$\begin{aligned}
\mathbb{E}_\mu\Big[\|\pi_N u\|^p_{\mathcal{A}}\Big] &\lesssim \mathbb{E}_\mu\Big[\|u\|^p_{W^{-\frac{3}{4},3}}\Big] \lesssim \Big\|\|\langle\nabla\rangle^{-\frac{3}{4}}u(x)\|_{L^p(\mu)}\Big\|^p_{L^3_x}\\
&\leq p^{\frac{p}{2}}\Big\|\|\langle\nabla\rangle^{-\frac{3}{4}}u(x)\|_{L^2(\mu)}\Big\|^p_{L^3_x}\\
&\leq p^{\frac{p}{2}}\bigg(\sum_{n\in\mathbb{Z}^3}\frac{1}{\langle n\rangle^{\frac{7}{2}}}\bigg)^p < \infty.
\end{aligned}$$

This proves (4.11). $\square$

We now present the proof of Proposition 4.1.

Proof of Proposition 4.1. • **Step 1:** In this first part, we prove that $Z_{N,\delta}$ in (4.7) is uniformly bounded in $N \in \mathbb{N}$. As for the tightness of $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ and the uniqueness of ν_δ claimed in the statement, we can repeat arguments analogous to those in Subsections 3.2 and 3.3 and thus we omit details.

From (4.7) and the Boué-Dupuis variational formula (Lemma 3.1) with the change of variables (3.12), we have

$$\begin{aligned} -\log Z_{N,\delta} = \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[&\delta \|Y_N + \Theta_N\|_{\mathcal{A}}^{20} - \sigma \int_{\mathbb{T}^3} Y_N \Theta_N^2 dx - \frac{\sigma}{3} \int_{\mathbb{T}^3} \Theta_N^3 dx \\ &+ A\bigg| \int_{\mathbb{T}^3} \Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2 \Big) dx \bigg|^\gamma \\ &+ \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt \bigg], \end{aligned} \tag{4.12}$$

where $\Theta_N = \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N$ with $\widetilde{\mathfrak{Z}}_N = \pi_N \mathfrak{Z}_N$ as in (3.19). Our goal is to establish a uniform lower bound on the right-hand side of (4.12). Unlike Subsection 3.2, we do not assume smallness on $|\sigma|$. In this case, a rescue comes from the extra positive term $\delta\|Y_N + \Theta_N\|_{\mathcal{A}}^{20}$ as compared to (3.17).

Given any $0 < c_0 < 1$, it follows from Young's inequality (3.40) with $\gamma \geq 3$ that

$$\bigg| \int_{\mathbb{T}^3} \Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2 \Big) dx \bigg|^\gamma \geq c_0 \bigg| \int_{\mathbb{T}^3} \Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2 \Big) dx \bigg|^3 - C. \tag{4.13}$$

Then, taking an expectation and applying Lemmas 3.5 and 3.6 with Lemma 3.2 and (3.25), we have

$$\mathbb{E}\bigg[A \bigg| \int_{\mathbb{T}^3} \Big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2 \Big) dx \bigg|^\gamma \bigg] \geq C_0 \mathbb{E}\Big[\|\Upsilon_N\|_{L^2}^6 \Big] - C_1 \mathbb{E}\Big[\|\Upsilon_N\|_{H^1}^2 \Big] - C \tag{4.14}$$

for some $C_0 > 0$ and $0 < C_1 \leq \frac{1}{4}$. Hence, it follows from (4.12), (4.14), and Lemma 3.5 together with Lemma 3.2 and (3.25) that there exists $C_2 > 0$ such that

$$\begin{aligned} -\log Z_{N,\delta} \geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[&\delta \|Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20} - \frac{\sigma}{3} \int_{\mathbb{T}^3} (\Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)^3 dx \\ &+ C_2 \|\Upsilon_N\|_{L^2}^6 + C_2 \|\Upsilon_N\|_{H^1}^2 \bigg] - C. \end{aligned} \tag{4.15}$$

By Young's inequality, we have

$$\begin{aligned} \bigg| \int_{\mathbb{T}^3} \Upsilon_N^2 \widetilde{\mathfrak{Z}}_N dx \bigg| + \bigg| \int_{\mathbb{T}^3} \Upsilon_N \widetilde{\mathfrak{Z}}_N^2 dx \bigg| &\leq \|\Upsilon_N\|_{L^2}^2 \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}} + \|\Upsilon_N\|_{L^2} \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^2 \\ &\leq \frac{C_2}{2|\sigma|} \|\Upsilon_N\|_{L^2}^6 + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^c + C_\sigma. \end{aligned} \tag{4.16}$$

Hence, from (4.15) and (4.16) with (3.40) (with $\gamma = 20$) and Lemma 4.6, we obtain

$$\begin{aligned} -\log Z_{N,\delta} \geq \inf_{\dot{\Upsilon}_N \in \mathbb{H}_a^1} \mathbb{E}\bigg[&\frac{\delta}{2} \|\Upsilon_N\|_{\mathcal{A}}^{20} - \frac{|\sigma|}{3} \|\Upsilon_N\|_{L^3}^3 \\ &+ \frac{C_2}{2} \|\Upsilon_N\|_{L^2}^6 + C_2 \|\Upsilon_N\|_{H^1}^2 \bigg] - C. \end{aligned} \tag{4.17}$$

Now, we need to estimate the L^3-norm of Υ_N. From (4.3), Sobolev's inequality, and the mean value theorem: $|1-e^{-t|n|^2}| \lesssim (t|n|^2)^\theta$ for any $0 \le \theta \le 1$, we have

$$\begin{aligned}\|\Upsilon_N\|_{L^3}^3 &\lesssim t^{-\frac{9}{8}}\|\Upsilon_N\|_{\mathcal{A}}^3 + \|\Upsilon_N - p_t * \Upsilon_N\|_{H^{\frac12}}^3 \\ &\lesssim t^{-\frac{9}{8}}\|\Upsilon_N\|_{\mathcal{A}}^3 + t^{\frac34}\|\Upsilon_N\|_{H^1}^3\end{aligned}$$

for $0 < t \le 1$. By choosing $t^{\frac34} \sim \big(1 + \frac{|\sigma|}{C_2}\|\Upsilon_N\|_{H^1}\big)^{-1}$ and applying Young's inequality, we obtain

$$\begin{aligned}|\sigma|\|\Upsilon_N\|_{L^3}^3 &\le C_{C_2,|\sigma|}\|\Upsilon_N\|_{H^1}^{\frac32}\|\Upsilon_N\|_{\mathcal{A}}^3 + \frac{C_2}{4}\|\Upsilon_N\|_{H^1}^2 + 1 \\ &\le C_{C_2,|\sigma|,\delta} + \frac{\delta}{4}\|\Upsilon_N\|_{\mathcal{A}}^{20} + \frac{C_2}{2}\|\Upsilon_N\|_{H^1}^2.\end{aligned} \tag{4.18}$$

Therefore, from (4.17) and (4.18), we conclude that

$$Z_{N,\delta} \le C_\delta < \infty,$$

uniformly in $N \in \mathbb{N}$.

• **Step 2:** Next, we show that $\|u\|_{\mathcal{A}}$ is finite ν_δ-almost surely. Let η be a smooth function with compact support with $\int_{\mathbb{R}^3} |\eta(\xi)|^2 d\xi = 1$ and set

$$\widehat{\rho}(\xi) = \int_{\mathbb{R}^3} \eta(\xi - \xi_1)\overline{\eta(-\xi_1)}d\xi_1.$$

Given $\varepsilon > 0$, define ρ_ε by

$$\rho_\varepsilon(x) = \sum_{n\in\mathbb{Z}^3} \widehat{\rho}(\varepsilon n)e^{in\cdot x}. \tag{4.19}$$

Since the support of $\widehat{\rho}$ is compact, the sum on the right-hand side is over finitely many frequencies. Thus, given any $\varepsilon > 0$, there exists $N_0(\varepsilon) \in \mathbb{N}$ such that

$$\rho_\varepsilon * u = \rho_\varepsilon * u_N \tag{4.20}$$

for any $N \ge N_0(\varepsilon)$. From the Poisson summation formula, we have

$$\rho_\varepsilon(x) = \sum_{n\in\mathbb{Z}^3} \varepsilon^{-3}\big|\mathcal{F}_{\mathbb{R}^3}^{-1}(\eta)(\varepsilon^{-1}x + 2\pi n)\big|^2 \ge 0,$$

where $\mathcal{F}_{\mathbb{R}^3}^{-1}$ denotes the inverse Fourier transform on $\mathbb{R}^3$. Noting that

$$\|\rho_\varepsilon\|_{L^1(\mathbb{T}^3)} = \int_{\mathbb{T}^3} \rho_\varepsilon(x)dx = \widehat{\rho}(0) = \|\eta\|_{L^2(\mathbb{R}^3)}^2 = 1,$$

we have, from Young's inequality, that

$$\|\rho_\varepsilon * u\|_{\mathcal{A}} \le \|u\|_{\mathcal{A}}. \tag{4.21}$$

Moreover, $\{\rho_\varepsilon\}$ defined above is an approximation to the identity on $\mathbb{T}^3$ and thus for any distribution u on $\mathbb{T}^3$, $\rho_\varepsilon * u \to u$ in the $\mathcal{A}$-norm, as $\varepsilon \to 0$.

Let $\delta > \delta' > 0$. By Fatou's lemma, the weak convergence of $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ from Step 1 with (4.20), (4.21), and the definition (4.6) of $\nu_{N,\delta}$, we have

$$\begin{aligned}\int \exp\Big((\delta-\delta')\|u\|_{\mathcal{A}}^{20}\Big)d\nu_\delta &\le \liminf_{\varepsilon\to 0}\int \exp\Big((\delta-\delta')\|\rho_\varepsilon * u\|_{\mathcal{A}}^{20}\Big)d\nu_\delta\\ &= \liminf_{\varepsilon\to 0}\lim_{N\to\infty}\int \exp\Big((\delta-\delta')\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}\Big)d\nu_{N,\delta}\\ &\le \lim_{N\to\infty}\int \exp\Big((\delta-\delta')\|u_N\|_{\mathcal{A}}^{20}\Big)d\nu_{N,\delta}\\ &= \lim_{N\to\infty}\frac{Z_{N,\delta'}}{Z_{N,\delta}}\int 1\, d\nu_{N,\delta'}\\ &= \frac{Z_{\delta'}}{Z_\delta}.\end{aligned}$$

Hence, we have

$$\int \exp\Big((\delta-\delta')\|u\|_{\mathcal{A}}^{20}\Big)d\nu_\delta < \infty$$

for any $\delta > \delta' > 0$. By choosing $\delta' = \frac{\delta}{2}$, we obtain

$$\int \exp\Big(\frac{\delta}{2}\|u\|_{\mathcal{A}}^{20}\Big)d\nu_\delta < \infty,$$

which shows that $\|u\|_{\mathcal{A}}$ is finite almost surely with respect to ν_δ.

• **Step 3:** Finally, we prove the relation (4.8). We first note that it suffices to show that

$$\frac{Z_\delta}{Z_{\delta'}}d\nu_\delta = \exp\Big(-(\delta-\delta')\|u\|_{\mathcal{A}}^{20}\Big)d\nu_{\delta'} \tag{4.22}$$

for any $\delta > \delta' > 0$. In fact, once we have (4.22), by integration, we obtain

$$\frac{Z_\delta}{Z_{\delta'}} = \int \exp\Big(-(\delta-\delta')\|u\|_{\mathcal{A}}^{20}\Big)d\nu_{\delta'} \tag{4.23}$$

and thus (4.8) follows from (4.22) and (4.23).

Let $F : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$ be a bounded Lipschitz function with $F \ge 0$. The dominated convergence theorem, the weak convergence of $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ from Step 1, and (4.6) yield that

$$\begin{aligned}&\frac{Z_\delta}{Z_{\delta'}}\int F(u)d\nu_\delta - \int F(u)\exp\Big(-(\delta-\delta')\|u\|_{\mathcal{A}}^{20}\Big)d\nu_{\delta'}\\ &= \lim_{\varepsilon\to 0}\bigg(\frac{Z_\delta}{Z_{\delta'}}\int F(u)d\nu_\delta - \int F(u)\exp\Big(-(\delta-\delta')\|\rho_\varepsilon * u\|_{\mathcal{A}}^{20}\Big)d\nu_{\delta'}\bigg)\\ &= \lim_{\varepsilon\to 0}\lim_{N\to\infty}\bigg(\frac{Z_{N,\delta}}{Z_{N,\delta'}}\int F(u)d\nu_{N,\delta} - \int F(u)\exp\Big(-(\delta-\delta')\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}\Big)d\nu_{N,\delta'}\bigg)\\ &= \lim_{\varepsilon\to 0}\lim_{N\to\infty}\int F(u)\Big[\exp\Big(-(\delta-\delta')\|u_N\|_{\mathcal{A}}^{20}\Big) - \exp\Big(-(\delta-\delta')\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}\Big)\Big]d\nu_{N,\delta'}.\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
\bigg| \frac{Z_\delta}{Z_{\delta'}} \int F(u) d\nu_\delta - \int F(u) \exp\Big(- (\delta - \delta') \|u\|_{\mathcal{A}}^{20} \Big) d\nu_{\delta'} \bigg| & \\
\lesssim \limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \int \bigg| \exp\Big(- (\delta - \delta') \|u_N\|_{\mathcal{A}}^{20} \Big) & \\
- \exp\Big(- (\delta - \delta') \|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20} \Big) \bigg| d\nu_{N,\delta'}(u) & \\
\lesssim \limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \int \bigg| \exp\Big(- (\delta - \delta') \|\pi_N u^N(\omega)\|_{\mathcal{A}}^{20} \Big) & \\
- \exp\Big(- (\delta - \delta') \|\rho_\varepsilon * \pi_N u^N(\omega)\|_{\mathcal{A}}^{20} \Big) \bigg| d\mathbb{P}(\omega), &
\end{aligned}
\tag{4.24}
$$

where u^N is a random variable with $\mathrm{Law}(u^N) = \nu_{N,\delta'}$. Noting that the integrand is uniformly bounded by 2, it follows from the bounded convergence theorem that the right-hand side of (4.24) tends to 0 once we show that $\|\rho_\varepsilon * \pi_N u^N(\omega) - \pi_N u^N(\omega)\|_{\mathcal{A}}$ tends to 0 in measure (with respect to $\mathbb{P}$). Namely, it suffices to show

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{P}\big(\{ \omega \in \Omega : \|\rho_\varepsilon * \pi_N u^N(\omega) - \pi_N u^N(\omega)\|_{\mathcal{A}} > \alpha \big) & \\
= \lim_{\varepsilon \to 0} \lim_{N \to \infty} \nu_{N,\delta'}\big(\{ \|u_N - \rho_\varepsilon * u_N\|_{\mathcal{A}} > \alpha \} \big) = 0 &
\end{aligned}
$$

for any $\alpha > 0$.

From (4.3) and (4.5), we have

$$
\|u_N - \rho_\varepsilon * u_N\|_{\mathcal{A}} \lesssim \|u_N - \rho_\varepsilon * u_N\|_{W^{-\frac{3}{4},3}} \lesssim \varepsilon^{\frac{1}{8}} \|u_N\|_{W^{-\frac{5}{8},3}}. \tag{4.25}
$$

Hence, from Chebyshev's inequality and (4.25), it suffices to prove

$$
\int \|u_N\|_{W^{-\frac{5}{8},3}} d\nu_{N,\delta'} \lesssim \int \exp\big(\|u_N\|_{W^{-\frac{5}{8},3}} \big) d\nu_{N,\delta'} \le C_{\delta'} < \infty, \tag{4.26}
$$

uniformly in $N \in \mathbb{N}$. We use the variational formulation as in (4.12), and write

$$
\begin{aligned}
& - \log \bigg(\int \exp\big(\|u_N\|_{W^{-\frac{5}{8},3}} \big) d\nu_{N,\delta'} \bigg) \\
& \quad = \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[\delta' \|Y_N + \Theta_N\|_{\mathcal{A}}^{20} - \|Y_N + \Theta_N\|_{W^{-\frac{5}{8},3}} - \sigma \int_{\mathbb{T}^3} Y_N \Theta_N^2 dx \\
& \qquad\qquad - \frac{\sigma}{3} \int_{\mathbb{T}^3} \Theta_N^3 dx + A \bigg| \int_{\mathbb{T}^3} \Big(:\! Y_N^2 \!: + 2 Y_N \Theta_N + \Theta_N^2 \Big) dx \bigg|^\gamma \\
& \qquad\qquad + \frac{1}{2} \int_0^1 \|\dot{\Upsilon}^N(t)\|_{H^1_x}^2 dt \bigg] \\
& \qquad + \log Z_{N,\delta'},
\end{aligned}
$$

where $\Theta_N = \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N$. From Lemma 3.2 and (3.25), we have, for any finite $p \ge 1$,

$$
\mathbb{E}\Big[\|Y_N\|_{W^{-\frac{5}{8},3}}^p + \|\mathfrak{Z}_N\|_{W^{-\frac{5}{8},3}}^p \Big] < \infty, \tag{4.27}
$$

uniformly in $N \in \mathbb{N}$. See also the proof of Lemma 4.6. Then, arguing as in (4.17) and (4.18) with Young's inequality, Sobolev's inequality, and (4.27), we obtain

$$
\begin{aligned}
& -\log\bigg(\int \exp\big(\|u_N\|_{W^{-\frac{5}{8},3}}\big) d\nu_{N,\delta'}\bigg)\\
& \quad \ge \inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\bigg[-\|\Upsilon_N\|_{W^{-\frac{5}{8},3}} + C_0\big(\|\Upsilon_N\|_{L^2}^6 + \|\Upsilon_N\|_{H^1}^2\big) + \frac{\delta'}{4}\|\Upsilon_N\|_{\mathcal{A}}^{20}\bigg] - C_{C_0,\delta'}\\
& \quad \gtrsim -1.
\end{aligned}
$$

This proves (4.26) and hence concludes the proof of Proposition 4.1. $\square$

4.3. **Non-normalizability of the σ-finite measure $\overline{\rho}_\delta$.** In this subsection, we present the proof of Proposition 4.2 on the non-normalizability of the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure defined in (4.9).

Given $\varepsilon > 0$, let ρ_ε be as in (4.19). Then, by (4.21), the weak convergence of $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ (Proposition 4.1), (4.20), and (4.6), we have

$$
\begin{aligned}
& \int \exp\Big(\delta\|u\|_{\mathcal{A}}^{20}\Big) d\nu_\delta \ge \int \exp\Big(\delta\|\rho_\varepsilon * u\|_{\mathcal{A}}^{20}\Big) d\nu_\delta\\
& \quad \ge \limsup_{L\to\infty} \int \exp\Big(\delta \min\big(\|\rho_\varepsilon * u\|_{\mathcal{A}}^{20}, L\big)\Big) d\nu_\delta\\
& \quad = \limsup_{L\to\infty} \lim_{N\to\infty} \int \exp\Big(\delta \min\big(\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}, L\big)\Big) d\nu_{N,\delta}\\
& \quad = \limsup_{L\to\infty} \lim_{N\to\infty} Z_{N,\delta}^{-1} \int \exp\Big(\delta \min\big(\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}, L\big) - \delta\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\Big) d\mu(u).
\end{aligned}
$$

Hence, (4.10) is reduced to showing that

$$
\limsup_{L\to\infty} \lim_{N\to\infty} \mathbb{E}_\mu\Big[\exp\Big(\delta \min\big(\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}, L\big) - \delta\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\Big)\Big] = \infty. \tag{4.28}
$$

Let $Y = Y(1)$ be as in (3.2). By the Boué-Dupuis variational formula (Lemma 3.1) with the change of variables (3.12), we have

$$
\begin{aligned}
& -\log \mathbb{E}\Big[\exp\Big(\delta \min\big(\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}, L\big) - \delta\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\Big)\Big]\\
& \quad = \inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\bigg[-\delta \min\big(\|\rho_\varepsilon * (Y_N + \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20}, L\big) + \delta\|Y_N + \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20}\\
& \qquad\qquad + \widehat{R}_N^\diamond(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot\Upsilon^N(t)\|_{H^1_x}^2 dt\bigg],
\end{aligned}
\tag{4.29}
$$

where $\widehat{R}_N^\diamond$ is as in (3.33) with the third power in the last term replaced by the γth power. With $\Theta_N = \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N$, a slight modification of (3.38) yields

$$
\begin{aligned}
& \bigg|\int_{\mathbb{T}^3} Y_N \Theta_N^2 dx\bigg| = \bigg|\int_{\mathbb{T}^3} Y_N(\Upsilon_N^2 + 2\sigma\Upsilon_N\widetilde{\mathfrak{Z}}_N + \sigma^2\widetilde{\mathfrak{Z}}_N^2) dx\bigg|\\
& \quad \le C_\sigma\Big(1 + \|Y_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}^c + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^c\Big) + \frac{1}{100|\sigma|}\Big(\|\Upsilon_N\|_{L^2}^3 + \|\Upsilon_N\|_{H^1}^2\Big).
\end{aligned}
\tag{4.30}
$$

By Young's inequality, we have

$$\begin{aligned}\bigg|\int_{\mathbb{T}^3}\Theta_N^3dx-\int_{\mathbb{T}^3}\Upsilon_N^3dx\bigg|&=\bigg|\int_{\mathbb{T}^3}\Big(3\sigma\Upsilon_N^2\widetilde{\mathfrak{Z}}_N+3\sigma^2\Upsilon_N\widetilde{\mathfrak{Z}}_N^2+\sigma^3\widetilde{\mathfrak{Z}}_N^3\Big)dx\bigg|\\&\le C_\sigma\|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^3+\frac{1}{100|\sigma|}\|\Upsilon_N\|_{L^2}^3.\end{aligned}\tag{4.31}$$

Then, applying (4.30) and (4.31) with Lemma 3.2 and (3.25) to (4.29), we obtain

$$\begin{aligned}&-\log\mathbb{E}\Big[\exp\Big(\delta\min\big(\|\rho_\varepsilon*u\|_{\mathcal{A}}^{20},L\big)-\delta\|u_N\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)\Big]\\&\le\inf_{\dot\Upsilon^N\in\mathbb{H}_a^1}\mathbb{E}\bigg[-\delta\min\big(\|\rho_\varepsilon*(Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20},L\big)+\delta\|Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20}\\&\qquad-\frac{\sigma}{3}\int_{\mathbb{T}^3}\Upsilon_N^3dx+\|\Upsilon_N\|_{L^2}^3+A\bigg|\int_{\mathbb{T}^3}\Big(:Y_N^2:+2Y_N\Theta_N+\Theta_N^2\Big)dx\bigg|^\gamma\\&\qquad+\frac34\int_0^1\|\dot\Upsilon^N(t)\|_{H_x^1}^2dt\bigg]+C_\sigma,\end{aligned}\tag{4.32}$$

where $\Theta_N=\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N$.

In the following, we show that the right-hand side of (4.32) tends to $-\infty$ as $N,L\to\infty$, provided that $|\sigma|>0$ is sufficiently large. By following the strategy introduced in our previous works [53, 60], we construct a drift $\dot\Upsilon^N$, achieving this goal. The main idea is to construct a drift $\dot\Upsilon^N$ such that Υ^N looks like "$-Y(1)+$ a perturbation" (see (4.41)), where the perturbation term is bounded in $L^2(\mathbb{T}^3)$ but has a large cubic integral (see (4.36) below). While we do not make use of solitons in this paper, one should think of this perturbation as something like a soliton or a finite blowup solution (at a fixed time) with a highly concentrated profile.

Remark 4.7. While our construction of the drift follows that in [53], we need to proceed more carefully in our current problem in handling the first two terms under the expectation in (4.32). If we simply apply (3.40) (with $\gamma=20$) to separate Υ_N from Y_N and $\sigma\widetilde{\mathfrak{Z}}_N$, we end up with an expression like

$$-\delta\min\Big(\frac12\|\rho_\varepsilon*\Upsilon_N\|_{\mathcal{A}}^{20},L\Big)+2\delta\|\Upsilon_N\|_{\mathcal{A}}^{20}$$

such that the coefficients of $\|\rho_\varepsilon*\Upsilon_N\|_{\mathcal{A}}^{20}$ and $\|\Upsilon_N\|_{\mathcal{A}}^{20}$ no longer agree, which causes a serious trouble. We instead need to keep the same coefficient for the first two terms under the expectation in (4.32) and make use of the difference structure. Compare this with the analysis in [53, 60], where no such cancellation was needed.

Fix a parameter $M\gg1$. Let $f:\mathbb{R}^3\to\mathbb{R}$ be a real-valued Schwartz function such that the Fourier transform $\widehat f$ is a smooth even non-negative function supported $\{\frac12<|\xi|\le1\}$ such that $\int_{\mathbb{R}^3}|\widehat f(\xi)|^2d\xi=1$. Define a function f_M on $\mathbb{T}^3$ by

$$f_M(x):=M^{-\frac32}\sum_{n\in\mathbb{Z}^3}\widehat f\Big(\frac{n}{M}\Big)e_n,\tag{4.33}$$

where $\widehat{f}$ denotes the Fourier transform on $\mathbb{R}^3$ defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} f(x)e^{-in\cdot x}dx.$$

Then, a direct calculation shows the following lemma.

Lemma 4.8. *For any $M \in \mathbb{N}$ and $\alpha > 0$, we have*

$$\int_{\mathbb{T}^3} f_M^2 dx = 1 + O(M^{-\alpha}), \tag{4.34}$$

$$\int_{\mathbb{T}^3} (\langle \nabla \rangle^{-1} f_M)^2 dx \lesssim M^{-2}, \tag{4.35}$$

$$\int_{\mathbb{T}^3} |f_M|^3 dx \sim \int_{\mathbb{T}^3} f_M^3 dx \sim M^{\frac{3}{2}}. \tag{4.36}$$

Proof. As for (4.34) and (4.35), see the proof of Lemma 5.13 in [53]. From (4.33) and the fact that $\widehat{f}$ is supported on $\{\frac{1}{2} < |\xi| \le 1\}$, we have

$$\int_{\mathbb{T}^3} f_M^3 dx = M^{-\frac{9}{2}} \sum_{n_1, n_2 \in \mathbb{Z}^3} \widehat{f}\Big(\frac{n_1}{M}\Big)\widehat{f}\Big(\frac{n_2}{M}\Big)\widehat{f}\Big(-\frac{n_1+n_2}{M}\Big) \sim M^{\frac{3}{2}}. \tag{4.37}$$

The bound $\|f_M\|_{L^3}^3 \gtrsim M^{\frac{3}{2}}$ follows from (4.37), while $\|f_M\|_{L^3}^3 \lesssim M^{\frac{3}{2}}$ follows from Hausdorff-Young's inequality. This proves (4.36). □

We define Z_M and α_M by

$$Z_M := \sum_{|n| \le M} \widehat{Y\big(\tfrac{1}{2}\big)}(n)e_n \qquad \text{and} \qquad \alpha_M := \mathbb{E}\big[Z_M^2(x)\big]. \tag{4.38}$$

Note that α_M is independent of $x \in \mathbb{T}^3$ thanks to the spatial translation invariance of Z_M. Then, we have the following lemma. See Lemma 5.14 in [53] for the proof.

Lemma 4.9. *Let $M \gg 1$ and $1 \le p < \infty$. Then, we have*

$$\begin{aligned}
&\alpha_M \sim M, \qquad\qquad (4.39)\\
&\mathbb{E}\bigg[\int_{\mathbb{T}^3} |Z_M|^p dx\bigg] \le C(p)M^{\frac{p}{2}},\\
&\mathbb{E}\bigg[\Big(\int_{\mathbb{T}^3} Z_M^2 dx - \alpha_M\Big)^2\bigg] + \mathbb{E}\bigg[\Big(\int_{\mathbb{T}^3} Y_N Z_M dx - \int_{\mathbb{T}^3} Z_M^2 dx\Big)^2\bigg] \lesssim 1,\\
&\mathbb{E}\bigg[\Big(\int_{\mathbb{T}^3} Y_N f_M dx\Big)^2\bigg] + \mathbb{E}\bigg[\Big(\int_{\mathbb{T}^3} Z_M f_M dx\Big)^2\bigg] \lesssim M^{-2}
\end{aligned}$$

for any $N \ge M$.

We now present the proof of Proposition 4.2.

Proof of Proposition 4.2. As described above, our main goal is to prove (4.28).

Fix $N \in \mathbb{N}$, appearing in (4.32). For $M \gg 1$, we set f_M, Z_M, and α_M as in (4.33) and (4.38). We now choose a drift $\dot{\Upsilon}^N$ for (4.32) by setting

$$\dot{\Upsilon}^N(t) = 2 \cdot \mathbf{1}_{t > \frac{1}{2}} \langle \nabla \rangle \Big(- Z_M + \operatorname{sgn}(\sigma)\sqrt{\alpha_M} f_M \Big), \tag{4.40}$$

where $\operatorname{sgn}(\sigma)$ is the sign of $\sigma \neq 0$. Then, we have

$$\Upsilon^N := I(\dot{\Upsilon}^N)(1) = \int_0^1 \langle \nabla \rangle^{-1} \dot{\Upsilon}^N(t) dt = -Z_M + \operatorname{sgn}(\sigma)\sqrt{\alpha_M} f_M. \tag{4.41}$$

Note that for $N \geq M \geq 1$, we have $\Upsilon_N = \pi_N \Upsilon^N = \Upsilon^N$, since Z_M and f_M are supported on the frequencies $\{|n| \leq M\}$.

Let us first make some preliminary computations. We start with the first two terms under the expectation in (4.32):

$$\begin{aligned}
&-\delta \min\big(\|\rho_\varepsilon * (Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20}, L\big) + \delta \|Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20} \\
&\quad = -\delta \min\big(\|\rho_\varepsilon * (Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20} - \|Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20}, \\
&\qquad\qquad L - \|Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20}\big) \\
&\quad =: -\delta \min(\mathrm{I}, \mathrm{II}).
\end{aligned} \tag{4.42}$$

We first consider II. From Lemma 4.5, (2.3), and Lemma 4.8, we have

$$\|f_M\|_{\mathcal{A}} \lesssim \|f_M\|_{H^{-\frac{1}{4}}} \lesssim \|f_M\|_{L^2}^{\frac{3}{4}} \|f_M\|_{H^{-1}}^{\frac{1}{4}} \lesssim M^{-\frac{1}{4}}. \tag{4.43}$$

From (4.41), (4.39) in Lemma 4.9, and (4.43), we have

$$\begin{aligned}
\mathrm{II} &\geq L - 2\alpha_M^{10} \|f_M\|_{\mathcal{A}}^{20} - C\Big(\|Y_N\|_{\mathcal{A}}^{20} + \|Z_M\|_{\mathcal{A}}^{20} + |\sigma| \|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\Big) \\
&\geq L - C_0 M^5 - C\Big(\|Y_N\|_{\mathcal{A}}^{20} + \|Z_M\|_{\mathcal{A}}^{20} + |\sigma| \|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\Big) \\
&\geq \frac{1}{2} L - C\Big(\|Y_N\|_{\mathcal{A}}^{20} + \|Z_M\|_{\mathcal{A}}^{20} + |\sigma| \|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\Big)
\end{aligned} \tag{4.44}$$

for $L \gg M^5$. Note that the second term on the right-hand side is harmless since it is bounded under an expectation. Next, we turn to I in (4.42). Let δ_0 denote the Dirac delta on $\mathbb{T}^3$. Then, by applying (4.41), Young's inequality, Lemma 4.5, (4.39), and (4.34) in Lemma 4.8 and by choosing $\varepsilon = \varepsilon(M) > 0$ sufficiently small, we have

$$\begin{aligned}
\mathrm{I} &\geq -\Big| \|\rho_\varepsilon * (Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20} - \|Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20} \Big| \\
&\geq -C \|(\rho_\varepsilon - \delta_0) * (Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}} \|Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{19} \\
&\geq -C \alpha_M^{10} \|(\rho_\varepsilon - \delta_0) * f_M\|_{H^{-\frac{1}{4}}}^{20} - C\Big(\|Y_N\|_{\mathcal{A}}^{20} + \|Z_M\|_{\mathcal{A}}^{20} + |\sigma| \|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\Big) \\
&\geq -C \varepsilon^5 M^{10} - C\Big(\|Y_N\|_{\mathcal{A}}^{20} + \|Z_M\|_{\mathcal{A}}^{20} + |\sigma| \|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\Big) \\
&= -C_0 - C\Big(\|Y_N\|_{\mathcal{A}}^{20} + \|Z_M\|_{\mathcal{A}}^{20} + |\sigma| \|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\Big).
\end{aligned} \tag{4.45}$$

Therefore, from (4.42), (4.44), and (4.45) together with (4.38), Lemma 4.6 and (3.25), we obtain

$$\mathbb{E}\Big[-\delta \min(\mathrm{I}, \mathrm{II}) \Big] \leq C(\delta, \sigma). \tag{4.46}$$

Next, we treat the third term under the expectation in (4.32). This term gives the main contribution. From (4.41) and Young's inequality with Lemma 4.8, we have

$$\begin{aligned} & \sigma \int_{\mathbb{T}^3} \Upsilon_N^3 dx - |\sigma| \alpha_M^{\frac{3}{2}} \int_{\mathbb{T}^3} f_M^3 dx \\ & = -\sigma \int_{\mathbb{T}^3} Z_M^3 dx + 3|\sigma| \int_{\mathbb{T}^3} Z_M^2 \sqrt{\alpha_M} f_M dx - 3\sigma \int_{\mathbb{T}^3} Z_M \alpha_M f_M^2 dx \\ & \geq -\eta |\sigma| \alpha_M^{\frac{3}{2}} \int_{\mathbb{T}^3} f_M^3 dx - C_\eta |\sigma| \int_{\mathbb{T}^3} |Z_M|^3 dx \end{aligned} \tag{4.47}$$

for any $0 < \eta < 1$. Then, it follows from (4.47) with $\eta = \frac{1}{2}$ and Lemmas 4.8 and 4.9 that

$$\begin{aligned} \mathbb{E}\bigg[\sigma \int_{\mathbb{T}^3} \Upsilon_N^3 dx\bigg] & \geq (1-\eta)|\sigma| \alpha_M^{\frac{3}{2}} \int_{\mathbb{T}^3} f_M^3 dx - C_\eta |\sigma| \mathbb{E}\bigg[\int_{\mathbb{T}^3} |Z_M|^3 dx\bigg] \\ & \gtrsim |\sigma| M^3 - |\sigma| M^{\frac{3}{2}} \\ & \gtrsim |\sigma| M^3 \end{aligned} \tag{4.48}$$

for $M \gg 1$.

We now treat the fourth and sixth terms under the expectation in (4.32). From (4.41), we have $\Upsilon_N \in \mathcal{H}_{\leq 1}$. Then, by the Wiener chaos estimate (Lemma 2.5) and (4.41) with Lemmas 4.8 and 4.9, we have

$$\mathbb{E}\Big[\|\Upsilon_N\|_{L^2}^3\Big] \lesssim \mathbb{E}\Big[\|\Upsilon_N\|_{L^2}^2\Big]^{\frac{3}{2}} \lesssim M^{\frac{3}{2}}. \tag{4.49}$$

Recall that both $\widehat{Z}_M$ and $\widehat{f}_M$ are supported on $\{|n| \leq M\}$. Then, from (4.40), (4.41), and Lemmas 4.8 and 4.9 as above, we have

$$\mathbb{E}\bigg[\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg] \lesssim M^2 \mathbb{E}\Big[\|\Upsilon^N\|_{L^2}^2\Big] \lesssim M^3. \tag{4.50}$$

We state a lemma which controls the fifth term under the expectation in (4.32). We present the proof of this lemma at the end of this subsection.

Lemma 4.10. *Let $\gamma > 0$. Then, we have*

$$\mathbb{E}\bigg[\Big|\int_{\mathbb{T}^3} :(Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)^2: dx\Big|^\gamma\bigg] \leq C(\sigma, \gamma) < \infty, \tag{4.51}$$

uniformly in $N \geq M \geq 1$.[19]

Therefore, putting (4.32), (4.46), (4.48), (4.49), (4.50), and Lemma 4.10 together, we obtain

$$\begin{aligned} & -\log \mathbb{E}\Big[\exp\Big(\delta \min\big(\|\rho_\varepsilon * u\|_{\mathcal{A}}^{20}, L\big) - \delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\Big)\Big] \\ & \leq -C_1 |\sigma| M^3 + C_2 M^3 + C(\delta, \sigma, \gamma) \end{aligned} \tag{4.52}$$

for some $C_1, C_2 > 0$, provided that $L \gg M^5 \gg 1$ and $\varepsilon = \varepsilon(M) > 0$ sufficiently small. By taking the limits in N and L, we conclude from (4.52) that

$$\begin{aligned} & \limsup_{L \to \infty} \lim_{N \to \infty} \mathbb{E}_\mu\Big[\exp\Big(\delta \min\big(\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}, L\big) - \delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\Big)\Big] \\ & \geq \exp\Big(C_1 |\sigma| M^3 - C_2 M^3 - C_0(\sigma)\Big) \longrightarrow \infty, \end{aligned}$$

[19] Recall from (4.41) that the definition of Υ_N depends on M.

as $M \to \infty$, provided that $|\sigma|$ is sufficiently large. This proves (4.28) and thus we conclude the proof of Proposition 4.2. $\square$

We conclude this subsection by presenting the proof of Lemma 4.10.

Proof of Lemma 4.10. From (3.11) and (3.19), we have

$$\begin{aligned} \int_{\mathbb{T}^3} \Upsilon_N \widetilde{\mathfrak{Z}}_N dx &= \int_0^1 \int_{\mathbb{T}^3} \langle \nabla \rangle^{-\frac{3}{4}} \Upsilon_N \cdot \langle \nabla \rangle^{-\frac{5}{4}} \pi_N^2 (:\! Y_N^2(t) \!:) dx dt \\ &\le \|\Upsilon_N\|_{H^{-\frac{3}{4}}} \int_0^1 \| :\! Y_N^2(t) \!: \|_{H^{-\frac{5}{4}}} dt. \end{aligned} \tag{4.53}$$

As for the first factor, it follows from (4.41), (2.3), (4.39), and Lemma 4.8 that

$$\begin{aligned} \|\Upsilon_N\|_{H^{-\frac{3}{4}}} &\lesssim \|Z_M\|_{H^{-\frac{3}{4}}} + \sqrt{\alpha_M} \|f_M\|_{H^{-\frac{3}{4}}} \\ &\lesssim \|Z_M\|_{H^{-\frac{3}{4}}} + \sqrt{\alpha_M} \|f_M\|_{H^{-1}}^{\frac{3}{4}} \|f_M\|_{L^2}^{\frac{1}{4}} \\ &\lesssim \|Z_M\|_{H^{-\frac{3}{4}}} + M^{-\frac{1}{4}}. \end{aligned} \tag{4.54}$$

Hence, from (4.53), (4.54), (4.38), and Lemma 3.2, we obtain

$$\mathbb{E}\bigg[\Big| \int_{\mathbb{T}^3} \Upsilon_N \widetilde{\mathfrak{Z}}_N dx \Big|^2 \bigg] \lesssim \mathbb{E}\Big[\|\Upsilon_N\|_{H^{-\frac{3}{4}}}^2 \Big] + \mathbb{E}\Big[\| :\! Y_N^2(t) \!: \|_{L^1_t([0,1]; H_x^{-\frac{5}{4}})}^2 \Big] \lesssim 1. \tag{4.55}$$

From (4.41), we have

$$\begin{aligned} & \Upsilon_N^2 + 2 Y_N \Upsilon^N \\ &\quad = Z_M^2 - 2\,\mathrm{sgn}(\sigma) \sqrt{\alpha_M} Z_M f_M + \alpha_M f_M^2 \\ &\quad\quad - 2 Y_N Z_M + 2\,\mathrm{sgn}(\sigma) \sqrt{\alpha_M} Y_N f_M \\ &\quad = (Z_M^2 - \alpha_M) - 2\,\mathrm{sgn}(\sigma) \sqrt{\alpha_M} Z_M f_M + \alpha_M(-1 + f_M^2) + 2\alpha_M \\ &\quad\quad - 2(Y_N Z_M - Z_M^2) - 2(Z_M^2 - \alpha_M) - 2\alpha_M + 2\,\mathrm{sgn}(\sigma) \sqrt{\alpha_M} Y_N f_M \\ &\quad = -(Z_M^2 - \alpha_M) - 2\,\mathrm{sgn}(\sigma) \sqrt{\alpha_M} Z_M f_M + \alpha_M(-1 + f_M^2) \\ &\quad\quad - 2(Y_N Z_M - Z_M^2) + 2\,\mathrm{sgn}(\sigma) \sqrt{\alpha_M} Y_N f_M. \end{aligned} \tag{4.56}$$

Note from (3.11) and (4.41) that $\int_{\mathbb{T}^3} :\!(Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)^2\!: dx \in \mathcal{H}_{\le 4}$. Then, from the Wiener chaos estimate (Lemma 2.5), (4.41), (4.55), (4.56), and Lemmas 3.2 and 4.9 with (4.34), we have

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\int_{\mathbb{T}^3} :(Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N)^2: dx\Big|^\gamma\Big]\\
&\le C(\gamma)\bigg\{\mathbb{E}\Big[\Big|\int_{\mathbb{T}^3} :(Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N)^2: dx\Big|^2\Big]\bigg\}^{\frac{\gamma}{2}}\\
&= C(\gamma)\bigg\{\mathbb{E}\Big[\Big|\int_{\mathbb{T}^3} :Y_N^2: dx+\int_{\mathbb{T}^3}(\Upsilon_N^2+2Y_N\Upsilon_N)dx+\sigma^2\int_{\mathbb{T}^3}\widetilde{\mathfrak{Z}}_N^2 dx\\
&\qquad\qquad +2\sigma\int_{\mathbb{T}^3}\Upsilon_N\widetilde{\mathfrak{Z}}_N dx+2\sigma\int_{\mathbb{T}^3}Y_N\widetilde{\mathfrak{Z}}_N dx\Big|^2\Big]\bigg\}^{\frac{\gamma}{2}}\\
&\le C(\gamma)\bigg\{\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3} :Y_N^2: dx\Big)^2\Big]+\sigma^4\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3}\widetilde{\mathfrak{Z}}_N^2 dx\Big)^2\Big]\\
&\qquad +\sigma^2\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3}\Upsilon_N\widetilde{\mathfrak{Z}}_N dx\Big)^2\Big]+\sigma^2\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3}Y_N\widetilde{\mathfrak{Z}}_N dx\Big)^2\Big]\\
&\qquad +\mathbb{E}\Big[\Big(-\int_{\mathbb{T}^3}Y_N Z_M dx+\int_{\mathbb{T}^3}Z_M^2 dx\Big)^2\Big]\\
&\qquad +\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3}Z_M^2 dx-\alpha_M\Big)^2\Big]+\alpha_M^2\Big(-1+\int_{\mathbb{T}^3}f_M^2 dx\Big)^2\\
&\qquad +\alpha_M\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3}Y_N f_M dx\Big)^2\Big]+\alpha_M\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3}Z_M f_M dx\Big)^2\Big]\bigg\}^{\frac{\gamma}{2}}\\
&\le C(\sigma,\gamma),
\end{aligned}
$$

which yields the bound (4.51). □

4.4. **Non-convergence of the truncated Φ^3_3-measures.** In this subsection, we present the proof of Proposition 4.4 on non-convergence of the truncated Φ^3_3-measures $\{\rho_N\}_{N\in\mathbb{N}}$.

We first define a slightly different tamed version of the truncated Φ^3_3-measure by setting

$$
d\nu^{(N)}_\delta(u)=(Z^{(N)}_\delta)^{-1}\exp\Big(-\delta\|u\|^{20}_{\mathcal{A}}-R^\diamond_N(u)\Big)d\mu(u) \tag{4.57}
$$

for $N\in\mathbb{N}$ and $\delta>0$, where the $\mathcal{A}$-norm and $R^\diamond_N$ are as in (4.3) and (1.24), respectively, and

$$
Z^{(N)}_\delta=\int\exp\Big(-\delta\|u\|^{20}_{\mathcal{A}}-R^\diamond_N(u)\Big)d\mu(u).
$$

As compared to $\nu_{N,\delta}$ in (4.6), there is no frequency cutoff π_N in the taming $-\delta\|u\|^{20}_{\mathcal{A}}$ in (4.57). As a corollary to the proof of Proposition 4.1, we obtain the following convergence result for $\nu^{(N)}_\delta$.

Lemma 4.11. *Let $\delta>0$, Then, as measures on $\mathcal{C}^{-100}(\mathbb{T}^3)$, the sequence of measures $\{\nu^{(N)}_\delta\}_{N\in\mathbb{N}}$ defined in* (4.57) *converges weakly to the limiting measure ν_δ constructed in Proposition 4.1.*

Proof. By the definitions (4.6) and (4.57) of $\nu_{N,\delta}$ and $\nu_\delta^{(N)}$, it suffices to prove

$$\lim_{N\to\infty}\bigg\{\int F(u)\exp\Big(-\delta\|u\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)d\mu(u) \\ -\int F(u)\exp\Big(-\delta\|u_N\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)d\mu(u)\bigg\}=0$$

for any bounded continuous function $F:\mathcal{C}^{-100}(\mathbb{T}^3)\to\mathbb{R}$. In the following, we prove

$$\lim_{N\to\infty}\int\Big|\exp\Big(-\delta\|u\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)-\exp\Big(-\delta\|u_N\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)\Big|d\mu(u)=0. \tag{4.58}$$

By the uniform boundedness of the frequency projector π_N on $\mathcal{A}$, we have

$$\|u_N\|_{\mathcal{A}}\lesssim\|u\|_{\mathcal{A}}, \tag{4.59}$$

uniformly in $N\in\mathbb{N}$. Then, it follows from the mean-value theorem, (4.59), and the Schauder estimate (4.5) that there exists $c_0>0$ such that

$$\begin{aligned}
&\int\Big|\exp\Big(-\delta\|u\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)-\exp\Big(-\delta\|u_N\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)\Big|\,d\mu(u)\\
&\lesssim\delta\int\exp\Big(-\delta\min\big(\|u\|_{\mathcal{A}}^{20},\|u_N\|_{\mathcal{A}}^{20}\big)-R_N^\diamond(u)\Big)\big|\|u\|_{\mathcal{A}}^{20}-\|u_N\|_{\mathcal{A}}^{20}\big|\,d\mu(u)\\
&\lesssim\delta\int\exp\Big(-\delta c_0\|u_N\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)\|u-u_N\|_{\mathcal{A}}\|u\|_{\mathcal{A}}^{19}d\mu(u)\\
&\lesssim\delta\int\exp\Big(-\delta c_0\|u_N\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)N^{-\frac18}\|u\|_{W^{-\frac58,3}}^{20}d\mu(u).
\end{aligned} \tag{4.60}$$

In the last step, we used the following bound:

$$\|u-u_N\|_{\mathcal{A}}\lesssim\|\pi_N^\perp u\|_{W^{-\frac34,3}}\lesssim N^{-\frac18}\|u\|_{W^{-\frac58,3}},$$

which follows from (4.3), (4.5), and the fact that $\pi_N^\perp u=u-u_N$ has the frequency support $\{|n|\gtrsim N\}$. Therefore, by (4.6), Proposition 4.1, and (4.26), we obtain

$$\begin{aligned}
&\limsup_{N\to\infty}\int\Big|\exp\Big(-\delta\|u\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)-\exp\Big(-\delta\|u_N\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)\Big|d\mu(u)\\
&\lesssim\delta\lim_{N\to\infty}\int\exp\Big(-\delta c_0\|u_N\|_{\mathcal{A}}^{20}-R_N^\diamond(u)\Big)N^{-\frac18}\|u\|_{W^{-\frac58,3}}^{20}d\mu(u)\\
&=\delta\lim_{N\to\infty}N^{-\frac18}Z_{N,c_0\delta}\int\|u\|_{W^{-\frac58,3}}^{20}d\nu_{N,c_0\delta}\\
&=0.
\end{aligned}$$

This proves (4.58). □

Remark 4.12. In the penultimate step of (4.60), we used the boundedness of the cube frequency projector $\pi_N=\pi_N^{\text{cube}}$ on $L^3(\mathbb{T}^3)$ and hence this argument does not work for the ball frequency projector π_N^{ball} defined in (1.41).

We conclude this section by presenting the proof of Proposition 4.4.

Proof of Proposition 4.4. Suppose by contradiction that, as probability measures on $\mathcal{A}$, $\{\rho_{N_k}\}_{k\in\mathbb{N}}$ has a weak limit ν_0. Then, given any $\delta > 0$, from Lemma 4.11 with (4.57) and (1.25), we have

$$\begin{aligned} d\nu_\delta &= \lim_{k\to\infty} \frac{\exp\Big(- \delta\|u\|_{\mathcal{A}}^{20} - R^{\diamond}_{N_k}(u)\Big)}{\int \exp\Big(- \delta\|v\|_{\mathcal{A}}^{20} - R^{\diamond}_{N_k}(v)\Big) d\mu(v)} d\mu(u) \\ &= \lim_{k\to\infty} \frac{\exp\big(- \delta\|u\|_{\mathcal{A}}^{20}\big)}{\int \exp\big(- \delta\|v\|_{\mathcal{A}}^{20}\big) d\rho_{N_k}(v)} d\rho_{N_k}(u) \\ &= \frac{\exp\big(- \delta\|u\|_{\mathcal{A}}^{20}\big)}{\int \exp\big(- \delta\|v\|_{\mathcal{A}}^{20}\big) d\nu_0(v)} d\nu_0(u), \end{aligned} \tag{4.61}$$

where the limits are interpreted as weak limits of measures on $\mathcal{C}^{-100}(\mathbb{T}^3)$. Note that, in the last step, we used the weak convergence in $\mathcal{A}$ of the truncated Φ^3_3-measures ρ_{N_k}, since $\exp(-\delta\|u\|_{\mathcal{A}}^{20})$ is continuous on $\mathcal{A}$, but not on $\mathcal{C}^{-100}(\mathbb{T}^3)$. Therefore, from (4.61) and (4.9), we obtain

$$d\nu_0(u) = \bigg(\int \exp\big(- \delta\|v\|_{\mathcal{A}}^{20}\big) d\nu_0(v)\bigg) d\overline{\rho}_\delta(u). \tag{4.62}$$

By assumption, ν_0 is a probability measure on $\mathcal{A}$ and thus $\|u\|_{\mathcal{A}} < \infty$, ν_0-almost surely. By the fact that ν_0 is a probability measure, (4.62), and Proposition 4.2, we obtain

$$\begin{aligned} 1 &= \int 1 \, d\nu_0 \\ &= \int \exp\Big(- \delta\|u\|_{\mathcal{A}}^{20}\Big) d\nu_0(u) \int 1 \, d\overline{\rho}_\delta(u) \\ &= \infty, \end{aligned}$$

which yields a contradiction. Therefore, no subsequence of the truncated Φ^3_3-measures ρ_N has a weak limit as probability measures on $\mathcal{A}$. □

5. Local well-posedness

In this section, we present the proof of Theorem 1.14 on local well-posedness of the (renormalized) hyperbolic Φ^3_3-model (1.33):

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u - \sigma :u^2: + M(:u^2:)u = \sqrt{2}\xi, \tag{5.1}$$

where M is defined as in (1.34). For the local theory, the size of $\sigma \neq 0$ does not play any role and hence we set $\sigma = 1$ in the remaining part of this section. As mentioned in Section 1, local well-posedness of (5.1) follows from a slight modification of the argument in [35, 53]. We, however, point out that the argument in [35] on the quadratic SNLW alone is not sufficient due to the additional term $M(:u^2:)u$, coming from the taming in constructing the Φ^3_3-measure.

5.1. **Paracontrolled approach.** In this subsection, we go over a paracontrolled approach to rewrite the equation (5.1) into a system of three unknowns. While our presentation closely follows those in [35, 53], we present some details for readers' convenience. Proceeding in the spirit of [17, 47, 35, 53], we transform the quadratic SdNLW (5.1) to a system of PDEs. In order to treat the additional term $M(:u^2:)u$ in (5.1), which contains an ill-defined product in $:u^2:$, we follow the approach in our previous work [53] on the focusing Hartree Φ^4_3-model, which leads to the system of three equations; see (5.28) below. Compare this with [17, 47, 35], where the resulting systems consist of two equations. At the end of this subsection, we state a local well-posedness result of the resulting system.

The main difficulty in studying the hyperbolic Φ^3_3-model (5.1) comes from the roughness of the space-time white noise. This is already manifested at the level of the linear equation. Let Ψ denote the stochastic convolution, satisfying the following linear stochastic damped wave equation:

$$\begin{cases} \partial_t^2\Psi + \partial_t\Psi + (1-\Delta)\Psi = \sqrt{2}\xi \\ (\Psi, \partial_t\Psi)|_{t=0} = (\phi_0, \phi_1), \end{cases}$$

where $(\phi_0, \phi_1) = (\phi_0^\omega, \phi_1^\omega)$ is a pair of the Gaussian random distributions with $\text{Law}(\phi_0^\omega, \phi_1^\omega) = \vec{\mu} = \mu \otimes \mu_0$ in (1.16). Define the linear damped wave propagator $\mathcal{D}(t)$ by

$$\mathcal{D}(t) = e^{-\frac{t}{2}} \frac{\sin\Big(t\sqrt{\frac{3}{4} - \Delta}\Big)}{\sqrt{\frac{3}{4} - \Delta}}$$

viewed as a Fourier multiplier operator. By setting

$$[\![n]\!] = \sqrt{\frac{3}{4} + |n|^2}, \tag{5.2}$$

we have

$$\mathcal{D}(t)f = e^{-\frac{t}{2}} \sum_{n\in\mathbb{Z}^3} \frac{\sin(t[\![n]\!])}{[\![n]\!]} \widehat{f}(n) e_n. \tag{5.3}$$

Then, the stochastic convolution Ψ can be expressed as

$$\Psi(t) = S(t)(\phi_0, \phi_1) + \sqrt{2}\int_0^t \mathcal{D}(t-t')dW(t'), \tag{5.4}$$

where $S(t)$ is defined by

$$S(t)(f,g) = \partial_t\mathcal{D}(t)f + \mathcal{D}(t)(f+g) \tag{5.5}$$

and W denotes a cylindrical Wiener process on $L^2(\mathbb{T}^3)$ defined in (3.1). It is easy to see that Ψ almost surely lies in $C(\mathbb{R}_+; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3))$ for any $\varepsilon > 0$; see Lemma 5.4 below. In the following, we use $\varepsilon > 0$ to denote a small positive constant, which can be arbitrarily small.

In the following, we adopt Hairer's convention to denote the stochastic terms by trees; the vertex "$\bullet$" corresponds to the space-time white noise ξ, while the edge denotes the Duhamel integral operator $\mathcal{I}$ given by

$$\mathcal{I}(F)(t) = \int_0^t \mathcal{D}(t-t')F(t')dt' = \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin\Big((t-t')\sqrt{\frac{3}{4}-\Delta}\Big)}{\sqrt{\frac{3}{4}-\Delta}} F(t')dt'. \tag{5.6}$$

With a slight abuse of notation, we set

$$\uparrow := \Psi, \tag{5.7}$$

where Ψ is as in (5.4), with the understanding that $\uparrow$ in (5.7) includes the random linear solution $S(t)(\phi_0, \phi_1)$. As mentioned above, $\uparrow$ has (spatial) regularity[20] $-\frac{1}{2}-$.

Given $N \in \mathbb{N}$, we define the truncated stochastic terms $\uparrow_N$ and Υ_N by

$$\uparrow_N := \pi_N \uparrow \qquad \text{and} \qquad \Upsilon_N := \mathcal{I}(\vee_N) = \int_0^t \mathcal{D}(t-t')\vee_N(t')dt', \tag{5.8}$$

where π_N is the frequency projector defined in (1.19) and $\vee_N$ is the Wick power defined by

$$\vee_N := \uparrow_N^2 - \sigma_N \tag{5.9}$$

with

$$\sigma_N = \mathbb{E}\big[\uparrow_N^2(x,t)\big] = \sum_{n\in\mathbb{Z}^3} \frac{\chi_N^2(n)}{\langle n\rangle^2} \sim N \longrightarrow \infty, \tag{5.10}$$

as $N \to \infty$. Note that σ_N in (5.10) is independent[21] of $(x,t) \in \mathbb{T}^3 \times \mathbb{R}_+$ and agrees with σ_N defined in (1.22). Note that we have $\vee = \lim_{N\to\infty} \vee_N$ in $C([0,T]; W^{-1-,\infty}(\mathbb{T}^3))$ almost surely. See Lemma 5.4.

Next, we define the second order stochastic term Υ:

$$\Upsilon := \mathcal{I}(\vee) = \int_0^t \mathcal{D}(t-t')\vee(t')dt',$$

as a limit of Υ_N defined in (5.8). With a naive regularity counting, with one degree of smoothing from the damped wave Duhamel integral operator $\mathcal{I}$ in (5.6), one may expect that Υ has regularity $0- = 2(-\frac{1}{2}-)+1$. However, by exploiting the multilinear dispersive smoothing effect, Gubinelli, Koch, and the first author showed that there is an extra $\frac{1}{2}$-smoothing for Υ and that Υ has regularity $\frac{1}{2}-$. See Lemma 5.6 below. See also [51, 13, 64] for analogous multilinear dispersive smoothing for the random wave equations. In particular, see [13, 64], where multilinear smoothing has been studied extensively for higher order stochastic objects in the cubic case.

If we proceed with the second order expansion as in [35]:

$$u = \uparrow + \Upsilon + v,$$

the residual term v satisfies the equation of the form:

$$(\partial_t^2 + \partial_t + 1 - \Delta)v = 2v\uparrow + 2\uparrow\Upsilon + \text{other terms}.$$

Inheriting the worse regularity $-\frac{1}{2}-$ of $\uparrow$, the second term $\uparrow\Upsilon$ has regularity $-\frac{1}{2}-$. Hence, we expect v to have regularity at most $\frac{1}{2}- = (-\frac{1}{2}-)+1$. In particular, the product $v\uparrow$ is not well defined since $(\frac{1}{2}-)+(-\frac{1}{2}-) < 0$.

In order to overcome this problem, we now introduce a paracontrolled ansatz as in [47, 35]:

$$u = \uparrow + \Upsilon + X + Y, \tag{5.11}$$

[20] We only discuss spatial regularities of various stochastic objects in this part. Hereafter, we use $a-$ to denote $a-\varepsilon$ for arbitrarily small $\varepsilon > 0$.

[21] This comes from the space-time translation invariance of the truncated stochastic convolution $\uparrow_N$.

where X and Y satisfy

$$\begin{aligned}(\partial_t^2 + \partial_t + 1 - \Delta)X &= 2(X + Y + \Upsilon) \mathbin{\textcircled{\scriptsize $<$}} \uparrow - M(:u^2:)\uparrow, &(5.12)\\ (\partial_t^2 + \partial_t + 1 - \Delta)Y &= (X + Y + \Upsilon)^2 + 2(X + Y + \Upsilon) \mathbin{\textcircled{\scriptsize $\geqslant$}} \uparrow \\ &\quad - M(:u^2:)(X + Y + \Upsilon) &(5.13)\end{aligned}$$

with the understanding that

$$:u^2: = (X + Y + \Upsilon)^2 + 2(X + Y)\uparrow + 2\uparrow\Upsilon + \vee. \tag{5.14}$$

Here, $\mathbin{\textcircled{\scriptsize $\geqslant$}} = \mathbin{\textcircled{\scriptsize $>$}} + \mathbin{\textcircled{\scriptsize $=$}}$. Note that, in the X-equation (5.12), we collected the worst terms from the v-equation, while all the terms in the Y-equation (5.13) are expected to behave better (that is, if the resonant product in (5.13) can be given a meaning). We point out that the problematic term $M(:u^2:)$ appears in *both* equations, unlike the situation in [35].

There are two resonant products in the system (5.12) - (5.13), which do not a priori make sense: $\Upsilon \mathbin{\textcircled{\scriptsize $=$}} \uparrow$ and $X \mathbin{\textcircled{\scriptsize $=$}} \uparrow$. We can use stochastic analysis and multilinear harmonic analysis to give a meaning to the first resonant product:

$$\Upsilon_{\!\uparrow} := \Upsilon \mathbin{\textcircled{\scriptsize $=$}} \uparrow$$

as a distribution of regularity $0- = (\frac{1}{2}-) + (-\frac{1}{2}-)$ (without renormalization). See Lemma 5.7 below. This in particular says that Y has expected regularity $1-$.

In view of Lemma 2.2, the right-hand side of (5.12) has regularity $-\frac{1}{2}-$ (if we pretend that $M(:u^2:)$ makes sense), and thus we expect that X has regularity $\frac{1}{2}-$. In particular, the resonant product $X \mathbin{\textcircled{\scriptsize $=$}} \uparrow$ in the Y-equation is not well defined since the sum of the regularities is negative. In [35], this issue was overcome by substituting the Duhamel formulation of the X-equation into the resonant product $X \mathbin{\textcircled{\scriptsize $=$}} \uparrow$ and then introducing certain paracontrolled operators (see (5.20), (5.21), and (5.23) below). This was possible in [35] since there was no additional term $M(:u^2:)$ in the system, in particular in the X-equation. In our current problem, the problematic resonant product $X \mathbin{\textcircled{\scriptsize $=$}} \uparrow$ also appears in $M(:u^2:)$, in particular, in the X-equation. Thus, a strategy in [47, 35] of substituting the Duhamel formulation of the X-equation into $X \mathbin{\textcircled{\scriptsize $=$}} \uparrow$ would lead to an infinite iteration of such substitutions. We point out that such an infinite iteration of the Duhamel formulation works in certain situations but we choose an alternative approach which is simpler.

The main idea is to follow the strategy in our previous work [53] and introduce a new unknown, representing the problematic resonant product:

$$\text{“}\mathfrak{R} = X \mathbin{\textcircled{\scriptsize $=$}} \uparrow\text{”} \tag{5.15}$$

which leads to a system of three unknowns $(X, Y, \mathfrak{R})$.

We now turn our attention to $:u^2:$ in (5.14). Let $Q_{X,Y}$ to denote a good part of $:u^2:$ defined by

$$Q_{X,Y} = (X + Y)^2 + 2(X + Y)\Upsilon + 2X \mathbin{\textcircled{\scriptsize $<$}} \uparrow + 2X \mathbin{\textcircled{\scriptsize $>$}} \uparrow + 2Y\uparrow. \tag{5.16}$$

In view of $X \mathbin{\textcircled{\scriptsize $<$}} \uparrow$ and $Y\uparrow$, $Q_{X,Y}$ has (expected) regularity $-\frac{1}{2}-$ From (5.11), (5.15), and (5.16), we can write $:u^2:$ as

$$:u^2: = Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon_{\!\uparrow} + \vee, \tag{5.17}$$

where $\curlyvee^{\!\uparrow}$ denotes the product of $\curlyvee$ and $\uparrow$ given by

$$\curlyvee^{\!\uparrow} = \curlyvee \olessthan \uparrow + \curlyvee^{\!\uparrow}_{\circ} + \curlyvee \ogreaterthan \uparrow.$$

By substituting the Duhamel formulation of the X-equation (5.12) and (5.17) into (5.15), we obtain

$$\mathfrak{R} = 2\mathcal{I}\Big((X + Y + \curlyvee) \olessthan \uparrow\Big) \circleddash \uparrow - \mathcal{I}\Big(M\big(Q_{X,Y} + 2\mathfrak{R} + \curlyvee^2 + 2\curlyvee^{\!\uparrow} + \vee\big)\uparrow\Big) \circleddash \uparrow. \tag{5.18}$$

As we see below, both resonant products on the right-hand side are not well defined at this point.

Let us consider the first term on the right-hand side of (5.18):

$$\mathcal{I}\Big((X + Y + \curlyvee) \olessthan \uparrow\Big) \circleddash \uparrow. \tag{5.19}$$

Due to the paraproduct structure (with the high frequency part given by $\uparrow$) under the Duhamel integral operator $\mathcal{I}$, we see that the resonant product in (5.19) is not well defined at this point since a term $\mathcal{I}(w \olessthan \uparrow)$ has (at best) regularity $\frac{1}{2}-$. In order to give a precise meaning to the right-hand side of (5.18), we now recall the paracontrolled operators introduced in [35].[22] We point out that in the parabolic setting, it is at this step where one would introduce commutators and exploit their smoothing properties. For our dispersive problem, however, one of the commutators does not provide any smoothing and thus such an argument does not seem to work. See [35, Remark 1.17].

Given a function w on $\mathbb{T}^3 \times \mathbb{R}_+$, define

$$\begin{aligned} \mathfrak{I}_{\olessthan}(w)(t) &:= \mathcal{I}(w \olessthan \uparrow)(t) \\ &= \sum_{n\in\mathbb{Z}^3} e_n \sum_{\substack{n = n_1 + n_2 \\ |n_1| \ll |n_2|}} \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \widehat{w}(n_1, t') \,\widehat{\uparrow}(n_2, t') dt', \end{aligned} \tag{5.20}$$

where $[\![n]\!]$ is as in (5.2). Here, $|n_1| \ll |n_2|$ signifies the paraproduct $\olessthan$ in the definition of $\mathfrak{I}_{\olessthan}$.[23] As mentioned above, the regularity of $\mathfrak{I}_{\olessthan}(w)$ is (at best) $\frac{1}{2}-$ and thus the resonant product $\mathfrak{I}_{\olessthan}(w) \circleddash \uparrow$ does not make sense in terms of deterministic analysis. Proceeding as in [35], we divide the paracontrolled operator $\mathfrak{I}_{\olessthan}$ into two parts. Fix small $\theta > 0$. Denoting by n_1 and n_2 the spatial frequencies of w and $\uparrow$ as in (5.20), we define $\mathfrak{I}^{(1)}_{\olessthan}$ and $\mathfrak{I}^{(2)}_{\olessthan}$ as the restrictions of $\mathfrak{I}_{\olessthan}$ onto $\{|n_1| \gtrsim |n_2|^\theta\}$ and $\{|n_1| \ll |n_2|^\theta\}$. More concretely, we set

$$\mathfrak{I}^{(1)}_{\olessthan}(w)(t) := \sum_{n\in\mathbb{Z}^3} e_n \sum_{\substack{n = n_1 + n_2 \\ |n_2|^\theta \lesssim |n_1| \ll |n_2|}} \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \widehat{w}(n_1, t') \,\widehat{\uparrow}(n_2, t') dt' \tag{5.21}$$

and

$$\mathfrak{I}^{(2)}_{\olessthan}(w) := \mathfrak{I}_{\olessthan}(w) - \mathfrak{I}^{(1)}_{\olessthan}(w). \tag{5.22}$$

[22]Strictly speaking, the paracontrolled operators introduced in [35] are for the undamped wave equation. Since the local-in-time mapping property remains unchanged, we ignore this minor point.

[23]For simplicity of the presentation, we use the less precise definitions of paracontrolled operators. For example, see (5.41) for the precise definition of the paracontrolled operator $\mathfrak{I}^{(1)}_{\olessthan}$.

As for the first paracontrolled operator $\mathfrak{I}^{(1)}_{\varolessthan}$, the lower bound $|n_1| \gtrsim |n_2|^\theta$ and the positive regularity of w allow us to prove a smoothing property such that the resonant product $\mathfrak{I}^{(1)}_{\varolessthan}(w) \circledequal \uparrow$ is well defined. See Lemma 5.8 below.

As noted in [35], the second paracontrolled operator $\mathfrak{I}^{(2)}_{\varolessthan}$ does not seem to possess a (deterministic) smoothing property. One of the main novelties in [35] was then to directly study the random operator $\mathfrak{I}_{\varolessthan, \circledequal}$ defined by

$$\begin{aligned} \mathfrak{I}_{\varolessthan,\circledequal}(w)(t) &:= \mathfrak{I}^{(2)}_{\varolessthan}(w) \circledequal \uparrow(t) \\ &= \sum_{n \in \mathbb{Z}^3} e_n \int_0^t \sum_{n_1 \in \mathbb{Z}^3} \widehat{w}(n_1, t') \mathcal{A}_{n,n_1}(t,t') dt', \end{aligned} \tag{5.23}$$

where $\mathcal{A}_{n,n_1}(t,t')$ is given by

$$\mathcal{A}_{n,n_1}(t,t') = \mathbf{1}_{[0,t]}(t') \sum_{\substack{n - n_1 = n_2 + n_3 \\ |n_1| \ll |n_2|^\theta \\ |n_1 + n_2| \sim |n_3|}} e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n_1+n_2]\!])}{[\![n_1+n_2]\!]} \widehat{\uparrow}(n_2, t') \widehat{\uparrow}(n_3, t). \tag{5.24}$$

Here, the condition $|n_1 + n_2| \sim |n_3|$ is used to denote the spectral multiplier corresponding to the resonant product $\circledequal$ in (5.23). See (5.43) and (5.44) for the precise definitions. The almost sure bounded property of the random operator $\mathfrak{I}_{\varolessthan,\circledequal}$ was studied in [35, 53]. See Lemma 5.9 below.

Next, we consider the second term on the right-hand side of (5.18):

$$\mathcal{I}\Big(M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\dot{\Upsilon} + \vee\big)\uparrow\Big) \circledequal \uparrow. \tag{5.25}$$

Once again, the resonant product is not well defined since the sum of regularities is negative. The term (5.25) appeared in our previous work [53] on the focusing Hartree Φ^4_3-model, where we introduced the following stochastic term:

$$\mathbb{A}(x,t,t') = \sum_{n \in \mathbb{Z}^3} e_n(x) \sum_{\substack{n = n_1 + n_2 \\ |n_1| \sim |n_2|}} e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n_1]\!])}{[\![n_1]\!]} \widehat{\uparrow}(n_1, t') \widehat{\uparrow}(n_2, t) \tag{5.26}$$

for $t \geq t' \geq 0$, where $|n_1| \sim |n_2|$ signifies the resonant product. Then, we have

$$\Big(\mathcal{I}\big(M(w)\uparrow\big) \circledequal \uparrow\Big)(t) = \int_0^t M(w)(t') \mathbb{A}(t,t') dt'. \tag{5.27}$$

We point out that the Fourier transform $\widehat{\mathbb{A}}(n,t,t')$ corresponds to $\mathcal{A}_{n,0}(t,t')$ defined in (5.24) and thus the analysis for $\mathbb{A}$ is closely related to that for the paracontrolled operator $\mathfrak{I}_{\varolessthan,\circledequal}$ in (5.23). See Lemma 5.10 below for the almost sure regularity of $\mathbb{A}$.

Finally, we are ready to present the full system for the three unknowns $(X, Y, \mathfrak{R})$. Putting together (5.12), (5.13), (5.16), (5.18), (5.21), (5.23), and (5.27), we arrive at the following system:

$$
\begin{aligned}
(\partial_t^2+\partial_t+1-\Delta)X &= 2(X+Y+\<3>)\olessthan \<1> \\
&\quad - M\big(Q_{X,Y}+2\mathfrak{R}+\<3>^2+2\<31p>+\<2>\big)\<1>, \\
(\partial_t^2+\partial_t+1-\Delta)Y &= (X+Y+\<3>)^2+2(\mathfrak{R}+Y\oequal\<1>+\<31p>)+2(X+Y+\<3>)\olessthan\<1> \\
&\quad - M\big(Q_{X,Y}+2\mathfrak{R}+\<3>^2+2\<31p>+\<2>\big)(X+Y+\<3>), \\
\mathfrak{R} &= 2\mathfrak{I}^{(1)}_{\olessthan}\big(X+Y+\<3>\big)\oequal\<1>+2\mathfrak{I}_{\olessthan,\oequal}\big(X+Y+\<3>\big) \\
&\quad -\int_0^t M\big(Q_{X,Y}+2\mathfrak{R}+\<3>^2+2\<31p>+\<2>\big)\mathbb{A}(t,t')dt', \\
(X,\partial_t X,Y,\partial_t Y)|_{t=0} &= (X_0,X_1,Y_0,Y_1).
\end{aligned}
\tag{5.28}
$$

By viewing the following random distributions and operator in the system above:

$$
\<1>,\quad \<2>,\quad \<3>,\quad \<31p>,\quad \mathbb{A},\quad \text{and}\quad \mathfrak{I}_{\olessthan,\oequal}, \tag{5.29}
$$

as predefined deterministic data with certain regularity / mapping properties, we prove the following local well-posedness of the system (5.28).

Theorem 5.1. *Let $\frac14 < s_1 < \frac12 < s_2 \le s_1+\frac14$ and $s_2-1\le s_3<0$. Then, there exist $\theta=\theta(s_3)>0$ and $\varepsilon=\varepsilon(s_1,s_2,s_3)>0$ such that if*

- *$\<1>$ is a distribution-valued function belonging to $C([0,1];W^{-\frac12-\varepsilon,\infty}(\mathbb{T}^3))\cap C^1([0,1];W^{-\frac32-\varepsilon,\infty}(\mathbb{T}^3))$,*
- *$\<2>$ is a distribution-valued function belonging to $C([0,1];W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$,*
- *$\<3>$ is a distribution-valued function belonging to $C([0,1];W^{\frac12-\varepsilon,\infty}(\mathbb{T}^3))\cap C^1([0,1];W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$,*
- *$\<31p>$ is a distribution-valued function belonging to $C([0,1];H^{-\varepsilon}(\mathbb{T}^3))$,*
- *$\mathbb{A}(t,t')$ is a distribution-valued function belonging to $L^\infty_{t'}L^3_t(\Delta_2(1);H^{-\varepsilon}(\mathbb{T}^3))$, where $\Delta_2(T)\subset[0,T]^2$ is defined by*

$$
\Delta_2(T)=\{(t,t')\in\mathbb{R}^2_+ : 0\le t'\le t\le T\}, \tag{5.30}
$$

- *the operator $\mathfrak{I}_{\olessthan,\oequal}$ belongs to the class $\mathcal{L}_2\big(\frac32,1\big)$, where $\mathcal{L}_2(q,T))$ is defined by*

$$
\mathcal{L}_2(q,T):=\mathcal{L}(L^q([0,T];L^2(\mathbb{T}^3)))\,;\,L^\infty([0,T];H^{s_3}(\mathbb{T}^3))), \tag{5.31}
$$

then the system (5.28) is locally well-posed in $\mathcal{H}^{s_1}(\mathbb{T}^3)\times\mathcal{H}^{s_2}(\mathbb{T}^3)$. More precisely, given any $(X_0,X_1,Y_0,Y_1)\in\mathcal{H}^{s_1}(\mathbb{T}^3)\times\mathcal{H}^{s_2}(\mathbb{T}^3)$, there exist $T>0$ and a unique solution $(X,Y,\mathfrak{R})$ to the hyperbolic Φ^3_3-system (5.28) on $[0,T]$ in the class:

$$
Z^{s_1,s_2,s_3}(T)=X^{s_1}(T)\times Y^{s_2}(T)\times L^3([0,T];H^{s_3}(\mathbb{T}^3)). \tag{5.32}
$$

Here, $X^{s_1}(T)$ and $Y^{s_2}(T)$ are the energy spaces at the regularities s_1 and s_2 intersected with appropriate Strichartz spaces defined in (5.47) below. Furthermore, the solution $(X,Y,\mathfrak{R})$ depends Lipschitz-continuously on the enhanced data set:

$$
\big(X_0,X_1,Y_0,Y_1,\<1>,\<2>,\<3>,\<31p>,\mathbb{A},\mathfrak{I}_{\olessthan,\oequal}\big) \tag{5.33}
$$

in the class:

$$\begin{aligned}\mathcal{X}_T^{s_1,s_2,\varepsilon} &= \mathcal{H}^{s_1}(\mathbb{T}^3)\times\mathcal{H}^{s_2}(\mathbb{T}^3)\\ &\quad\times\big(C([0,T];W^{-\frac12-\varepsilon,\infty}(\mathbb{T}^3))\cap C^1([0,T];W^{-\frac32-\varepsilon,\infty}(\mathbb{T}^3))\big)\\ &\quad\times C([0,T];W^{-1-\varepsilon,\infty}(\mathbb{T}^3))\\ &\quad\times\big(C([0,T];W^{\frac12-\varepsilon,\infty}(\mathbb{T}^3))\cap C^1([0,T];W^{-1-\varepsilon,\infty}(\mathbb{T}^3))\big)\\ &\quad\times C([0,T];H^{-\varepsilon}(\mathbb{T}^3))\times L^\infty_{t'}L^3_t(\Delta_2(T);H^{-\varepsilon}(\mathbb{T}^3))\times\mathcal{L}_2\big(\tfrac32,T\big).\end{aligned}$$

Given the a priori regularities of the enhanced data, Theorem 5.1 follows from the standard energy and Strichartz estimates for the wave equation. While the proof is a slight modification of those in [35, 53], we present the proof of Theorem 5.1 in Subsection 5.4 for readers' convenience. The local well-posedness of the hyperbolic Φ^3_3-model (Theorem 1.14) follows from Theorem 5.1 and the almost sure convergence of the truncated stochastic objects:

$$\uparrow_N,\quad \vee_N,\quad \curlyvee_N,\quad \curlyvee^{\circ}_N,\quad \mathbb{A}_N,\quad\text{and}\quad \mathfrak{I}^N_{⊗,⊖} \tag{5.34}$$

to the elements in the enhanced data set in (5.29); see Lemmas 5.4, 5.6, 5.7, 5.8, 5.9, and 5.10 in Subsection 5.3. See Remark 5.2 below.

Remark 5.2. (i) For the sake of the well-posedness of the system (5.28), we considered general initial data $(X_0,X_1,Y_0,Y_1)\in\mathcal{H}^{s_1}(\mathbb{T}^3)\times\mathcal{H}^{s_2}(\mathbb{T}^3)$ in Theorem 5.1. However, in order to go back from the system (5.28) to the hyperbolic Φ^3_3-model (5.1) with the identification (5.15) (in the limiting sense), we need to set $(X_0,X_1)=(0,0)$ since the resonant product of the linear solution $S(t)(X_0,X_1)$ and $\uparrow$ is not well defined in general. As we see in Section 6, we simply use the zero initial data for the system (5.28) in constructing global-in-time invariant Gibbs dynamics for the hyperbolic Φ^3_3-model (5.1).

(ii) Our choice of the norms for $\curlyvee^{\circ}$ is crucial in the globalization argument. See Proposition 6.5 and Remark 6.6.

(iii) In proving the local well-posedness result of the system (5.28) stated in Theorem 5.1, we do not need to use the C^1_T-norms for $\uparrow$ and $\curlyvee$. However, we will need these C^1_T-norms for $\uparrow$ and $\curlyvee$ in the globalization argument presented in Section 6 and thus have included them in the hypothesis and the definition of Theorem 5.1 of the space $\mathcal{X}_T^{s_1,s_2,\varepsilon}$. See also (5.49) and Remark 5.11.

Furthermore, with this definition of the space $\mathcal{X}_T^{s_1,s_2,\varepsilon}$, the map from an enhanced data set in (5.33) (with $(X_0,X_1,Y_0,Y_1)=(0,0,u_0,u_1)$) to $(u,\partial_t u)$, where $u=\uparrow+\curlyvee+X+Y$ as in (5.11) becomes a continuous map from $\mathcal{X}_T^{s_1,s_2,\varepsilon}$ to $C([0,T];\mathcal{H}^{-\frac12-\varepsilon}(\mathbb{T}^3))$.

5.2. **Strichartz estimates.** Given $0\le s\le 1$, we say that a pair (q,r) is s-admissible (a pair $(\widetilde{q},\widetilde{r})$ is dual s-admissible,[24] respectively) if $1\le\widetilde{q}<2<q\le\infty$, $1<\widetilde{r}\le 2\le r<\infty$,

$$\frac1q+\frac3r=\frac32-s=\frac1{\widetilde{q}}+\frac3{\widetilde{r}}-2,\qquad \frac1q+\frac1r\le\frac12,\qquad\text{and}\qquad \frac1{\widetilde{q}}+\frac1{\widetilde{r}}\ge\frac32.$$

[24]Here, we define the notion of dual s-admissibility for the convenience of the presentation. Note that $(\widetilde{q},\widetilde{r})$ is dual s-admissible if and only if $(\widetilde{q}',\widetilde{r}')$ is $(1-s)$-admissible.

We say that u is a solution to the following nonhomogeneous linear damped wave equation:

$$\begin{cases} (\partial_t^2 + \partial_t + 1 - \Delta)u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \tag{5.35}$$

on a time interval containing $t = 0$, if u satisfies the following Duhamel formulation:

$$u = S(t)(u_0, u_1) + \int_0^t \mathcal{D}(t - t')F(t')dt',$$

where $S(t)$ and $\mathcal{D}(t)$ are as in (5.5) and (5.3), respectively. We now recall the Strichartz estimates for solutions to the nonhomogeneous linear damped wave equation (5.35).

Lemma 5.3. *Given $0 \le s \le 1$, let (q, r) and $(\widetilde{q}, \widetilde{r})$ be s-admissible and dual s-admissible pairs, respectively. Then, a solution u to the nonhomogeneous linear damped wave equation* (5.35) *satisfies*

$$\|(u, \partial_t u)\|_{L_T^\infty \mathcal{H}_x^s} + \|u\|_{L_T^q L_x^r} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \|F\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}} \tag{5.36}$$

for all $0 < T \le 1$. The following estimate also holds:

$$\|(u, \partial_t u)\|_{L_T^\infty \mathcal{H}_x^s} + \|u\|_{L_T^q L_x^r} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \|F\|_{L_T^1 H_x^{s-1}} \tag{5.37}$$

for all $0 < T \le 1$. The same estimates also holds for any finite $T > 1$ but with the implicit constants depending on T.

The Strichartz estimates on $\mathbb{R}^d$ are well known; see [28, 45, 40] in the context of the undamped wave equation (with the linear part $\partial_t^2 - \Delta$). For the undamped Klein-Gordon equation (with the linear part $\partial_t^2 + 1 - \Delta$), see [41]. Thanks to the finite speed of propagation, these estimates on $\mathbb{T}^3$ follow from the corresponding estimates on $\mathbb{R}^3$.

As for the current damped case, by setting $v(t) = e^{\frac{t}{2}}u(t)$, the damped wave equation (5.35) becomes

$$\begin{cases} (\partial_t^2 + \frac{3}{4} - \Delta)v = e^{\frac{t}{2}}F \\ (v, \partial_t v)|_{t=0} = (u_0, u_1), \end{cases}$$

to which the Strichartz estimates for the Klein-Gordon equation apply. By undoing the transformation, we then obtain the Strichartz estimates for the damped equation (5.35) on finite time intervals $[0, T]$, where the implicit constants depend on T.

In proving Theorem 5.1, we use the fact that $\left(8, \frac{8}{3}\right)$ and $(4, 4)$ are $\frac{1}{4}$-admissible and $\frac{1}{2}$-admissible, respectively. We also use a dual $\frac{1}{2}$-admissible pair $\left(\frac{4}{3}, \frac{4}{3}\right)$.

5.3. **Stochastic terms and paracontrolled operators.** In this subsection, we collect regularity properties of stochastic terms and the paracontrolled operators. See [35, 53] for the proofs. Note that the stochastic objects are constructed from the stochastic convolution $\uparrow = \Psi$ in (5.4). In particular, in the following, probabilities of various events are measured with respect to the Gaussian initial data and the space-time white noise.[25]

First, we state the regularity properties of $\uparrow$ and $\vee$. See Lemma 3.1 in [35] and Lemma 4.1 in [53].

[25]With the notation in Section 6 (see (6.4)), this is equivalent to saying that we measure various events with respect to $\vec{\mu} \otimes \mathbb{P}_2$.

Lemma 5.4. *Let $T > 0$.*

(i) *For any $\varepsilon > 0$, $\<1>_N$ in (5.8) converges to $\<1>$ in $C([0,T]; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-\frac{3}{2}-\varepsilon,\infty}(\mathbb{T}^3))$ almost surely. In particular, we have*

$$\<1> \in C([0,T]; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-\frac{3}{2}-\varepsilon,\infty}(\mathbb{T}^3))$$

almost surely. Moreover, we have the following tail estimate:

$$\mathbb{P}\Big(\|\<1>_N\|_{C_T W_x^{-\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-\frac{3}{2}-\varepsilon,\infty}} > \lambda \Big) \le C(1+T) \exp\big(-c\lambda^2\big) \tag{5.38}$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N \in \mathbb{N} \cup \{\infty\}$ with the understanding that $\<1>_\infty = \<1>$.

(ii) *For any $\varepsilon > 0$, $\<2>_N$ in (5.9) converges to $\<2>$ in $C([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$ almost surely. In particular, we have*

$$\<2> \in C([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$$

almost surely. Moreover, we have the following tail estimate:

$$\mathbb{P}\Big(\|\<2>_N\|_{C_T W_x^{-1-\varepsilon,\infty}} > \lambda \Big) \le C(1+T) \exp\big(-c\lambda\big)$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N \in \mathbb{N} \cup \{\infty\}$ with the understanding that $\<2>_\infty = \<2>$.

Remark 5.5. A slight modification of the proof of the exponential tail estimate (5.38) shows that there exists small $\delta > 0$ such that

$$\mathbb{P}\Big(N_2^\delta \|\<1>_{N_1} - \<1>_{N_2}\|_{C_T W_x^{-\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-\frac{3}{2}-\varepsilon,\infty}} > \lambda \Big) \le C(1+T) \exp\big(-c\lambda^2\big)$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N_1 \ge N_2 \ge 1$. A similar comment applies to the other elements $\<2>_N$, $\<20>_N$, $\<21p>_N$, $\mathbb{A}_N$, and $\mathfrak{I}^N_{\circledcirc,\circleddash}$ in the truncated enhanced data set in (5.34).

The next two lemmas treat $\<20>$ and the resonant product $\<21p>$, exhibiting an extra $\frac{1}{2}$-smoothing. See Propositions 1.6 and 1.8 in [35]. While the exponential tail estimates (5.39) and (5.40) were not proven in [35], they follow from the second moment bounds on the Fourier coefficients of $\<20>_N$ and $\<21p>_N$ obtained in [35] and arguing as in the proof of Lemma 2.3 in [36], using a version of the Garsia-Rodemich-Rumsey inequality (see Lemma 2.2 in [36]) with the fact that $\<20>_N \in \mathcal{H}_2$ and $\<21p>_N \in \mathcal{H}_{\le 3}$. Since the required argument is verbatim from [36], we omit details.

Lemma 5.6. *Let $T > 0$. Then, $\<20>_N$ converges to $\<20>$ in $C([0,T]; W^{\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$ almost surely for any $\varepsilon > 0$. In particular, we have*

$$\<20> \in C([0,T]; W^{\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$$

almost surely for any $\varepsilon > 0$. Moreover, we have the following tail estimate:

$$\mathbb{P}\Big(\|\<20>_N\|_{C_T W_x^{\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-1-\varepsilon,\infty}} > \lambda \Big) \le C(1+T) \exp\big(-c\lambda\big) \tag{5.39}$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N \in \mathbb{N} \cup \{\infty\}$ with the understanding that $\<20>_\infty = \<20>$.

Lemma 5.7. *Let $T > 0$. Then, $\<21p>_N := \<20>_N \circleddash \<1>_N$ converges to $\<21p>$ in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^3))$ almost surely for any $\varepsilon > 0$. In particular, we have*

$$\<21p> \in C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^3))$$

almost surely for any $\varepsilon > 0$. *Moreover, we have the following tail estimate:*

$$\mathbb{P}\Big(\|\curlyvee_N\|_{C_T W_x^{-\varepsilon,\infty}} > \lambda\Big) \le C(1+T)\exp\big(-c\lambda^{\frac{2}{3}}\big) \tag{5.40}$$

for any $T > 0$ *and* $\lambda > 0$, *uniformly in* $N \in \mathbb{N}\cup\{\infty\}$ *with the understanding that* $\curlyvee_\infty = \curlyvee$.

Next, we state the almost sure mapping properties of the paracontrolled operators. We first consider the paracontrolled operator $\mathfrak{I}^{(1)}_{\olessthan}$ defined in (5.21). By writing out the frequency relation $|n_2|^\theta \lesssim |n_1| \ll |n_2|$ in a more precise manner, we have

$$\begin{aligned}\mathfrak{I}^{(1)}_{\olessthan}(w)(t) = \sum_{n\in\mathbb{Z}^3} e_n \sum_{n=n_1+n_2} \sum_{\theta k + c_0 \le j < k-2} \varphi_j(n_1)\varphi_k(n_2)\\ \times \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \widehat{w}(n_1,t')\,\widehat{\upharpoonleft}(n_2,t')dt',\end{aligned} \tag{5.41}$$

where φ_j is as in (2.1) and $c_0 \in \mathbb{R}$ is some fixed constant. Given a pathwise regularity of $\upharpoonleft$, the mapping property of $\mathfrak{I}^{(1)}_{\olessthan}$ can be established in a deterministic manner. See Lemma 7.1 in [53]. See also Corollary 5.2 in [35].

Lemma 5.8. *Let* $s > 0$ *and* $T > 0$. *Then, given small* $\theta > 0$, *there exists small* $\varepsilon = \varepsilon(s,\theta) > 0$ *such that the following deterministic estimate holds the paracontrolled operator* $\mathfrak{I}^{(1)}_{\olessthan}$ *defined in* (5.21):

$$\|\mathfrak{I}^{(1)}_{\olessthan}(w)\|_{L^\infty_T H_x^{\frac{1}{2}+3\varepsilon}} \lesssim \|w\|_{L^2_T H^s_x} \|\upharpoonleft\|_{L^2_T W_x^{-\frac{1}{2}-\varepsilon,\infty}}. \tag{5.42}$$

In particular, $\mathfrak{I}^{(1)}_{\olessthan}$ *belongs almost surely to the class*

$$\mathcal{L}_1(T) = \mathcal{L}(L^2([0,T];H^s(\mathbb{T}^3))\,;\,C([0,T];H^{\frac{1}{2}+3\varepsilon}(\mathbb{T}^3))).$$

Moreover, by letting $\mathfrak{I}^{(1),N}_{\olessthan}$, $N\in\mathbb{N}$, *denote the paracontrolled operator in* (5.21) *with* $\upharpoonleft$ *replaced by the truncated stochastic convolution* $\upharpoonleft_N$ *in* (5.8), *the truncated paracontrolled operator* $\mathfrak{I}^{(1),N}_{\olessthan}$ *converges almost surely to* $\mathfrak{I}^{(1)}_{\olessthan}$ *in* $\mathcal{L}_1(T)$.

Next, we consider the random operator $\mathfrak{I}_{\olessthan,\circledequal}$ defined in (5.23). By writing out the frequency relations more carefully as in (5.41), we have

$$\mathfrak{I}_{\olessthan,\circledequal}(w)(t) = \sum_{n\in\mathbb{Z}^3} e_n \int_0^t \sum_{j=0}^\infty \sum_{n_1\in\mathbb{Z}^3} \varphi_j(n_1)\widehat{w}(n_1,t')\mathcal{A}_{n,n_1}(t,t')dt', \tag{5.43}$$

where $\mathcal{A}_{n,n_1}(t,t')$ is given by

$$\begin{aligned}\mathcal{A}_{n,n_1}(t,t') = \mathbf{1}_{[0,t]}(t') \sum_{\substack{k=0\\ 0\le j<\theta k+c_0}}^\infty \sum_{\substack{\ell,m=0\\ |\ell-m|\le 2}}^\infty \sum_{n-n_1=n_2+n_3} \varphi_k(n_2)\varphi_\ell(n_1+n_2)\varphi_m(n_3)\\ \times e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n_1+n_2]\!])}{[\![n_1+n_2]\!]} \widehat{\upharpoonleft}(n_2,t')\,\widehat{\upharpoonleft}(n_3,t).\end{aligned} \tag{5.44}$$

Then, we have the following almost sure mapping property of the random operator $\mathfrak{I}_{\olessthan,\circledequal}$. See Proposition 2.5 in [53]. See also Proposition 1.11 in [35].

Lemma 5.9. *Let $s_3 < 0$ and $T > 0$. Then, there exists small $\theta = \theta(s_3) > 0$ such that, for any finite $q > 1$, the paracontrolled operator $\mathfrak{I}_{\circledcirc,\ominus}$ defined by (5.23) and (5.24) belongs to $\mathcal{L}_2(q,T)$ defined in (5.31), almost surely. Furthermore the following tail estimate holds for some $C, c > 0$:*

$$\mathbb{P}\Big(\|\mathfrak{I}_{\circledcirc,\ominus}\|_{\mathcal{L}_2(q,T)} > \lambda \Big) \le C(1+T)\exp(-\lambda) \tag{5.45}$$

for any $\lambda \gg 1$.

If we define the truncated paracontrolled operator $\mathfrak{I}^N_{\circledcirc,\ominus}$, $N \in \mathbb{N}$, by replacing $\uparrow$ in (5.23) and (5.24) with the truncated stochastic convolution $\uparrow_N$ in (5.8), then the truncated paracontrolled operators $\mathfrak{I}^N_{\circledcirc,\ominus}$ converge almost surely to $\mathfrak{I}_{\circledcirc,\ominus}$ in $\mathcal{L}_2(q,T)$. Furthermore, the tail estimate (5.45) holds for the truncated paracontrolled operators $\mathfrak{I}^N_{\circledcirc,\ominus}$, uniformly in $N \in \mathbb{N}$.

Finally, we state the regularity property of $\mathbb{A}$ defined in (5.26). See Lemma 7.2 in [53]. Given $N \in \mathbb{N}$, we define the truncated version $\mathbb{A}_N$:

$$\mathbb{A}_N(x,t,t') = \sum_{n \in \mathbb{Z}^3} e_n(x) \sum_{\substack{n = n_1 + n_2 \\ |n_1| \sim |n_2|}} e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n_1]\!])}{[\![n_1]\!]} \widehat{\uparrow}_N(n_1,t')\widehat{\uparrow}_N(n_2,t) \tag{5.46}$$

by replacing $\uparrow$ by $\uparrow_N$ in (5.26).

Lemma 5.10. *Fix finite $q \ge 2$. Then, given any $T, \varepsilon > 0$ and finite $p \ge 1$, $\{\mathbb{A}_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; L^\infty_{t'} L^q_t(\Delta_2(T); H^{-\varepsilon}(\mathbb{T}^3)))$, converging to some limit $\mathbb{A}$ (formally defined by (5.26)) in $L^p(\Omega; L^\infty_{t'} L^q_t(\Delta_2(T); H^{-\varepsilon}(\mathbb{T}^3)))$, where $\Delta_2(T)$ is as in (5.30). Moreover, $\mathbb{A}_N$ converges almost surely to the same limit in $L^\infty_{t'} L^q_t(\Delta_2(T); H^{-\varepsilon}(\mathbb{T}^3))$. Furthermore, we have the following uniform tail estimate*:

$$\mathbb{P}\Big(\|\mathbb{A}_N\|_{L^\infty_{t'} L^q_t(\Delta_2(T); H^{-\varepsilon}_x)} > \lambda \Big) \le C(1+T)\exp(-\lambda)$$

for any $\lambda \gg 1$, and $N \in \mathbb{N} \cup \{\infty\}$, where $\mathbb{A}_\infty = \mathbb{A}$.

5.4. **Proof of local well-posedness.** In this subsection, we present the proof of Theorem 5.1. In the following, we assume that $s_3 < 0 < s_1 < s_2 < 1$. Recall that $\big(8, \frac{8}{3}\big)$ and $(4,4)$ are $\frac{1}{4}$-admissible and $\frac{1}{2}$-admissible, respectively. Given $0 < T \le 1$, we define $X^{s_1}(T)$ (and $Y^{s_2}(T)$) as the intersection of the energy spaces of regularity s_1 (and s_2, respectively) and the Strichartz space:

$$\begin{aligned} X^{s_1}(T) &= C([0,T]; H^{s_1}(\mathbb{T}^3)) \cap C^1([0,T]; H^{s_1-1}(\mathbb{T}^3)) \cap L^8([0,T]; W^{s_1-\frac{1}{4},\frac{8}{3}}(\mathbb{T}^3)), \\ Y^{s_2}(T) &= C([0,T]; H^{s_2}(\mathbb{T}^3)) \cap C^1([0,T]; H^{s_2-1}(\mathbb{T}^3)) \cap L^4([0,T]; W^{s_2-\frac{1}{2},4}(\mathbb{T}^3)), \end{aligned} \tag{5.47}$$

and set

$$Z^{s_1,s_2,s_3}(T) = X^{s_1}(T) \times Y^{s_2}(T) \times L^3([0,T]; H^{s_3}(\mathbb{T}^3)).$$

By writing (5.28) in the Duhamel formulation, we have

$$
\begin{aligned}
X &= \Phi_1(X, Y, \mathfrak{R}) \\
&:= S(t)(X_0, X_1) + 2\mathcal{I}\Big((X + Y + \Upsilon) \olessthan \upharpoonleft\Big) \\
&\quad - \mathcal{I}\Big(M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\curlyvee + \vee\big)\upharpoonleft\Big), \\
Y &= \Phi_2(X, Y, \mathfrak{R}) \\
&:= S(t)(Y_0, Y_1) + \mathcal{I}\Big((X + Y + \Upsilon)^2\Big) \\
&\quad + 2\mathcal{I}\big(\mathfrak{R} + Y \circeq \upharpoonleft + \curlyvee_{\circ}\big) + 2\mathcal{I}\Big((X + Y + \Upsilon) \ogreaterthan \upharpoonleft\Big) \\
&\quad - \mathcal{I}\Big(M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\curlyvee + \vee\big)(X + Y + \Upsilon)\Big), \\
\mathfrak{R} &= \Phi_3(X, Y, \mathfrak{R}) \\
&:= 2\mathfrak{I}^{(1)}_{\olessthan}\big(X + Y + \Upsilon\big) \circeq \upharpoonleft + 2\mathfrak{I}_{\olessthan, \circeq}\big(X + Y + \Upsilon\big) \\
&\quad - \int_0^t M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\curlyvee + \vee\big)\mathbb{A}(t, t')dt'.
\end{aligned}
\tag{5.48}
$$

In the following, we use $\varepsilon = \varepsilon(s_1, s_2, s_3) > 0$ to denote a small positive number. Given an enhanced data set as in (5.33), we set

$$
\Xi = \big(\upharpoonleft, \vee, \Upsilon, \curlyvee_{\circ}, \mathbb{A}, \mathfrak{I}_{\olessthan, \circeq}\big)
$$

and

$$
\begin{aligned}
\|\Xi\|_{\mathcal{X}^\varepsilon_T} &= \|\upharpoonleft\|_{C_T W_x^{-\frac{1}{2}-\varepsilon,\infty} \cap C^1_T W_x^{-\frac{3}{2}-\varepsilon,\infty}} + \|\vee\|_{C_T W_x^{-1-\varepsilon,\infty}} \\
&\quad + \|\Upsilon\|_{C_T W_x^{\frac{1}{2}-\varepsilon,\infty} \cap C^1_T W_x^{-1-\varepsilon,\infty}} + \|\curlyvee_{\circ}\|_{C_T H_x^{-\varepsilon}} \\
&\quad + \|\mathbb{A}\|_{L^\infty_{t'} L^3_t(\Delta_2; H_x^{-\varepsilon})} + \|\mathfrak{I}_{\olessthan, \circeq}\|_{\mathcal{L}_2(\frac{3}{2}, T)}
\end{aligned}
\tag{5.49}
$$

for some small $\varepsilon = \varepsilon(s_1, s_2, s_3) > 0$. Moreover, we assume that

$$
\|(X_0, X_1)\|_{\mathcal{H}^{s_1}} + \|(Y_0, Y_1)\|_{\mathcal{H}^{s_2}} + \|\Xi\|_{\mathcal{X}^\varepsilon_1} \le K \tag{5.50}
$$

for some $K \ge 1$. Here, we assume the bound on Ξ for the time interval $[0, 1]$.

Remark 5.11. As for proving local well-posedness stated in Theorem 5.1, we do not need to use the $C^1_T W_x^{-\frac{3}{2}-\varepsilon,\infty}$-norm for $\upharpoonleft$ and the $C^1_T W_x^{-1-\varepsilon,\infty}$-norm for Υ. However, in constructing global-in-time dynamics, we need to make use of these norms and thus we have included them in the definition of the $\mathcal{X}^\varepsilon_T$-norm in (5.49).

We first establish preliminary estimates. By Sobolev's inequality, we have

$$
\|f^2\|_{H^{-a}} \lesssim \|f^2\|_{L^{\frac{6}{3+2a}}} = \|f\|^2_{L^{\frac{12}{3+2a}}} \lesssim \|f\|^2_{H^{\frac{3-2a}{4}}} \tag{5.51}
$$

for any $0 \le a < \frac{3}{2}$. By (5.16), (5.51), Lemma 2.2, Lemma 2.3 (ii), and Hölder's inequality with (5.50), we have

$$
\begin{aligned}
\|Q_{X,Y}\|_{L^\infty_T H^{-100}_x} &\lesssim \|(X+Y)^2\|_{L^\infty_T H^{-100}_x} + \|X\Upsilon\|_{L^\infty_T H^{-100}_x} + \|Y\Upsilon\|_{L^\infty_T H^{-100}_x} \\
&\quad + \|X \olessthan \uparrow\|_{L^\infty_T H^{-100}_x} + \|X \ogreaterthan \uparrow\|_{L^\infty_T H^{-100}_x} + \|Y \uparrow\|_{L^\infty_T H^{-100}_x} \\
&\lesssim \|X\|^2_{L^\infty_T H^\varepsilon_x} + \|Y\|^2_{L^\infty_T H^\varepsilon_x} \\
&\quad + \Big(\|X\|_{L^\infty_T L^2_x} + \|Y\|_{L^\infty_T L^2_x} \Big) \|\Upsilon\|_{L^\infty_T L^\infty_x} \\
&\quad + \|X\|_{L^\infty_T L^2_x} \|\uparrow\|_{L^\infty_T W^{-\frac{1}{2}-\varepsilon,\infty}_x} + \|Y\|_{L^\infty_T H^{\frac{1}{2}+\varepsilon}_x} \|\uparrow\|_{L^\infty_T W^{-\frac{1}{2}-\varepsilon,\infty}_x} \\
&\lesssim \|(X,Y,\mathfrak{R})\|^2_{Z^{s_1,s_2,s_3}(T)} + K^2,
\end{aligned}
\tag{5.52}
$$

provided that $s_1 \ge \varepsilon$ and $s_2 \ge \frac{1}{2} + \varepsilon$.

We now estimate $\Phi_1(X,Y,\mathfrak{R})$ in (5.48). By (5.47), Lemmas 5.3 and 2.2, (1.34), and (5.52) with (5.50), we have

$$
\begin{aligned}
\|\Phi_1(X,Y,\mathfrak{R})\|_{X^{s_1}(T)} &\lesssim \|(X_0,X_1)\|_{\mathcal{H}^{s_1}} + \big\|(X+Y+\Upsilon) \olessthan \uparrow\big\|_{L^1_T H^{s_1-1}_x} \\
&\quad + \big\|M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\tilde{\Upsilon} + \vee\big)\uparrow\big\|_{L^1_T H^{s_1-1}_x} \\
&\lesssim \|(X_0,X_1)\|_{\mathcal{H}^{s_1}} + T\|X+Y+\Upsilon\|_{L^\infty_T L^2_x} \|\uparrow\|_{L^\infty_T W^{-\frac{1}{2}-\varepsilon,\infty}_x} \\
&\quad + T^{\frac{1}{3}} \|Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\tilde{\Upsilon} + \vee\|^2_{L^3_T H^{-100}_x} \|\uparrow\|_{L^\infty_T H^{s_1-1}_x} \\
&\lesssim \|(X_0,X_1)\|_{\mathcal{H}^{s_1}} + T^{\frac{1}{3}} K \Big(\|(X,Y,\mathfrak{R})\|^4_{Z^{s_1,s_2,s_3}(T)} + K^4 \Big),
\end{aligned}
\tag{5.53}
$$

provided that $\varepsilon \le s_1 < \frac{1}{2} - \varepsilon$, $s_2 \ge \frac{1}{2} + \varepsilon$, and $s_3 \ge -100$.

Next, we estimate $\Phi_2(X,Y,\mathfrak{R})$ in (5.48). By (5.47) and Lemma 5.3 with the fractional Leibniz rule (Lemma 2.3 (i)), we have

$$
\begin{aligned}
\big\|\mathcal{I}\big((X+Y+\Upsilon)^2\big)\big\|_{Y^{s_2}(T)} &\lesssim \|\langle\nabla\rangle^{s_2-\frac{1}{2}}(X+Y+\Upsilon)^2\|_{L^{\frac{4}{3}}_{T,x}} \\
&\lesssim T^{\frac{1}{4}} \Big(\|\langle\nabla\rangle^{s_2-\frac{1}{2}} X\|^2_{L^8_T L^{\frac{8}{3}}_x} + \|\langle\nabla\rangle^{s_2-\frac{1}{2}} Y\|^2_{L^4_{T,x}} + \|\langle\nabla\rangle^{s_2-\frac{1}{2}} \Upsilon\|^2_{L^\infty_{T,x}} \Big) \\
&\lesssim T^{\frac{1}{4}} \Big(\|(X,Y,\mathfrak{R})\|^2_{Z^{s_1,s_2,s_3}(T)} + K^2 \Big),
\end{aligned}
\tag{5.54}
$$

provided that $\frac{1}{2} \le s_2 \le \min(1-\varepsilon, s_1 + \frac{1}{4})$. By Lemmas 5.3 and 2.2, (5.54), and (5.52) with (5.50), we have

$$
\begin{aligned}
&\|\Phi_2(X,Y,\mathfrak{R})\|_{Y^{s_2}(T)} \\
&\quad \lesssim \|(Y_0,Y_1)\|_{\mathcal{H}^{s_2}} + \big\|\mathcal{I}\big((X+Y+\Upsilon)^2\big)\big\|_{Y^{s_2}(T)} + \|\mathfrak{R}\|_{L^1_T H^{s_2-1}_x} \\
&\qquad + \|Y \circeq \uparrow\|_{L^1_T H^{s_2-1}_x} + \|\Upsilon_{\circeq}\|_{L^1_T H^{s_2-1}_x} + \|(X+Y+\Upsilon) \ogreaterthan \uparrow\|_{L^1_T H^{s_2-1}_x} \\
&\qquad + \big\|M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\tilde{\Upsilon} + \vee\big)(X+Y+\Upsilon)\big\|_{L^1_T H^{s_2-1}_x} \\
&\quad \lesssim \|(Y_0,Y_1)\|_{\mathcal{H}^{s_2}} + T^{\frac{1}{4}} \Big(\|(X,Y,\mathfrak{R})\|^2_{Z^{s_1,s_2,s_3}(T)} + K^2 \Big) + T^{\frac{2}{3}} \|\mathfrak{R}\|_{L^3_T H^{s_3}_x} \\
&\qquad + T\|\Upsilon_{\circeq}\|_{L^\infty_T H^{-\varepsilon}_x} + T\Big(\|X\|_{L^\infty_T H^{s_1}_x} + \|Y\|_{L^\infty_T H^{s_2}_x} + \|\Upsilon\|_{L^\infty_T W^{\frac{1}{2}-\varepsilon}_x} \Big) \|\uparrow\|_{L^\infty_T W^{-\frac{1}{2}-\varepsilon}_x}
\end{aligned}
\tag{5.55}
$$

$$
\begin{aligned}
&+ T^{\frac{1}{3}} \| Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \curlyvee \|^2_{L^3_T H^{-100}_x} \\
&\qquad \times \Big(\|X\|_{L^\infty_T H^{s_1}_x} + \|Y\|_{L^\infty_T H^{s_2}_x} + \|\Upsilon\|_{L^\infty_T W^{\frac{1}{2}-\varepsilon,\infty}_x} \Big) \\
&\lesssim \|(Y_0, Y_1)\|_{\mathcal{H}^{s_2}} + T^{\frac{1}{4}} \Big(\|(X,Y,\mathfrak{R})\|^5_{Z^{s_1,s_2,s_3}(T)} + K^5 \Big),
\end{aligned}
$$

provided that $s_1 \geq \varepsilon$, $\frac{1}{2} + \varepsilon < s_2 \leq \min(1 - 3\varepsilon, s_1 + \frac{1}{4}, s_3 + 1)$, and $s_3 \geq -100$.

Finally, we estimate $\Phi_3(X,Y,\mathfrak{R})$ in (5.48). By Lemma 2.2, Lemma 5.8 (in particular (5.42)), and (5.52) with (5.50), we have

$$
\begin{aligned}
&\|\Phi_3(X,Y,\mathfrak{R})\|_{L^3_T H^{s_3}_x} \\
&\quad \lesssim \big\| \mathfrak{I}^{(1)}_{\text{\textcircled{<}}} (X + Y + \Upsilon) \mathbin{\text{\textcircled{=}}} \shortmid \big\|_{L^3_T H^{s_3}_x} + \big\| \mathfrak{I}_{\text{\textcircled{<}},\text{\textcircled{=}}} (X + Y + \Upsilon) \big\|_{L^3_T H^{s_3}_x} \\
&\qquad + \bigg\| \int_0^t M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \curlyvee\big) \mathbb{A}(t,t') dt' \bigg\|_{L^3_T H^{s_3}_x} \\
&\quad \lesssim T^{\frac{1}{3}} \big\| \mathfrak{I}^{(1)}_{\text{\textcircled{<}}} (X + Y + \Upsilon) \big\|_{L^\infty_T H^{\frac{1}{2}+3\varepsilon}_x} \| \shortmid \|_{L^\infty_T W^{-\frac{1}{2}-\varepsilon,\infty}_x} + T^{\frac{1}{3}} K \|X + Y + \Upsilon\|_{L^\infty_T L^2_x} \\
&\qquad + \int_0^T |M(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \curlyvee)(t')| \cdot \|\mathbb{A}(t,t')\|_{L^3_t([t',T]; H^{s_3}_x)} dt' \\
&\quad \lesssim T^{\frac{1}{3}} K^2 \Big(\|X\|_{L^\infty_T H^{s_1}_x} + \|Y\|_{L^\infty_T H^{s_2}_x} + \|\Upsilon\|_{L^\infty_T W^{\frac{1}{2}-\varepsilon,\infty}_x} \Big) \\
&\qquad + T^{\frac{1}{3}} K \|Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \curlyvee\|^2_{L^3_T H^{-100}_x} \\
&\quad \lesssim T^{\frac{1}{3}} K \Big(\|(X,Y,\mathfrak{R})\|^4_{Z^{s_1,s_2,s_3}(T)} + K^4 \Big)
\end{aligned}
\tag{5.56}
$$

provided that $s_1 > 0$ with sufficiently small $\varepsilon = \varepsilon(s_1) > 0$ (in view of Lemma 5.8), $s_2 \geq \frac{1}{2} + \varepsilon$, and $-100 \leq s_3 \leq -\varepsilon$.

Note that $|x|x$ is differentiable with a locally bounded derivative. In view of (1.34), this allows us to estimate the difference $M(w_1) - M(w_2)$. By repeating a similar computation, we also obtain the difference estimate:

$$
\begin{aligned}
&\|\vec{\Phi}(X,Y,\mathfrak{R}) - \vec{\Phi}(\widetilde{X}, \widetilde{Y}, \widetilde{\mathfrak{R}})\|_{Z^{s_1,s_2,s_3}(T)} \\
&\quad \lesssim T^{\frac{1}{4}} \Big(\|(X,Y,\mathfrak{R})\|^4_{Z^{s_1,s_2,s_3}(T)} + K^4 \Big) \|(X,Y,\mathfrak{R}) - (\widetilde{X}, \widetilde{Y}, \widetilde{\mathfrak{R}})\|_{Z^{s_1,s_2,s_3}(T)},
\end{aligned}
\tag{5.57}
$$

where

$$
\vec{\Phi} := (\Phi_1, \Phi_2, \Phi_3).
$$

Therefore, by choosing $T = T(K) > 0$ sufficiently small, we conclude from (5.53), (5.55), (5.56), and (5.57) that $\vec{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ is a contraction on the closed ball $B_R \subset Z^{s_1,s_2,s_3}(T)$ of radius $R \sim 1 + \|(X_0, X_1)\|_{\mathcal{H}^{s_1}} + \|(Y_0, Y_1)\|_{\mathcal{H}^{s_2}}$ centered at the origin. A similar computation yields Lipschitz continuous dependence of the solution $(X, Y, \mathfrak{R})$ on the enhanced data set $(X_0, X_1, Y_0, Y_1, \Xi)$ measured in the $\mathcal{X}^{s_1,s_2,\varepsilon}_T$-norm by possibly making $T > 0$ smaller. This concludes the proof of Theorem 5.1.

6. Invariant Gibbs dynamics

In this section, we present the proof of Theorem 1.15. In the remaining part of this section, we work in the weakly nonlinear regime. Namely, we fix $\sigma \neq 0$ such that $|\sigma| \leq \sigma_0$, where σ_0 is as in Theorem 1.8 (i). We also fix sufficiently large $A \gg 1$ as in Theorem 1.8 (i) such that the Φ^3_3-measure ρ is constructed as the limit of the truncated Φ^3_3-measures ρ_N in (1.25). With these parameters, consider the truncated Gibbs measure $\vec{\rho}_N$:

$$\vec{\rho}_N = \rho_N \otimes \mu_0 \tag{6.1}$$

for $N \in \mathbb{N}$, where μ_0 is the white noise measure; see (1.15) with $s = 0$. A standard argument [36, 58, 53] shows that the truncated Gibbs measure $\vec{\rho}_N$ is invariant under the truncated hyperbolic Φ^3_3-model (1.38):

$$\begin{aligned} &\partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N \\ &\qquad - \sigma \pi_N\big(:\!(\pi_N u_N)^2\!: \big) + M(:\!(\pi_N u_N)^2\!:) \pi_N u_N = \sqrt{2}\xi, \end{aligned} \tag{6.2}$$

where $:\!(\pi_N u_N)^2\!: = (\pi_N u_N)^2 - \sigma_N$ and π_N and σ_N are as in (1.19) and (1.22), respectively. See Lemma 6.4 below. Moreover, as a corollary to Theorem 1.8 (i), the truncated Gibbs measure $\vec{\rho}_N$ in (6.1) converges weakly to the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ in (1.32).

Our main goal is to construct global-in-time dynamics for the limiting hyperbolic Φ^3_3-model (1.33) almost surely with respect to the Gibbs measure $\vec{\rho}$, and prove invariance of the Gibbs measure $\vec{\rho}$ under the limiting hyperbolic Φ^3_3-dynamics. A naive approach would be to apply Bourgain's invariant measure argument [8, 10], by exploiting the invariance of the truncated Gibbs measure $\vec{\rho}_N$ under the truncated hyperbolic Φ^3_3-dynamics, and to try to construct global-in-time limiting dynamics for the limiting process $u = \lim_{N\to\infty} u_N$. There are, however, two issues in the current situation: (i) the truncated Gibbs measure $\vec{\rho}_N$ converges to the limiting Gibbs measure $\vec{\rho}$ *only weakly* and (ii) the Gibbs measure $\vec{\rho}$ and the base Gaussian measure $\vec{\mu} = \mu \otimes \mu_0$ in (1.16) are *mutually singular*. Moreover, our local theory relies on the paracontrolled approach, which gives additional difficulty. As a result, Bourgain's invariant measure argument [8, 10] is not directly applicable to our problem. In [13], Bringmann encountered a similar problem in the context of the defocusing Hartree NLW on $\mathbb{T}^3$, where he overcame this issue by introducing a new globalization argument, by using the fact that the (truncated) Gibbs measure is absolutely continuous with respect to a shifted measure (as in Appendix A below) [53, 12] in a uniform manner and establishing a (rather involved) large time stability theory, where sets of large probabilities are characterized via the shifted measures.

In the following, we introduce a new alternative globalization argument. This new argument has the advantage of being conceptually simple and straightforward. Our approach consists of several steps:

1. In the first step, we establish a uniform (in N) exponential integrability of the truncated enhanced data set Ξ_N (see (6.10) below) with respect to the truncated measure $\vec{\rho}_N \otimes \mathbb{P}_2$ (Proposition 6.5). Here, $\mathbb{P}_2$ is the measure for the stochastic forcing defined in (6.4) below. By combining the variational approach with space-time estimates, we prove this uniform exponential integrability *without* any reference to (the truncated version of) the shifted measure $\mathrm{Law}(Y(1) + \sigma \mathfrak{Z}(1) + \mathcal{W}(1))$ constructed in Appendix A. As a corollary, we construct the limiting enhanced data set Ξ associated with the Gibbs measure $\vec{\rho}$ (see (6.11) below) by establishing convergence of the truncated enhanced data set Ξ_N almost surely with respect to the limiting measure $\vec{\rho} \otimes \mathbb{P}_2$.

2. In the second step, we establish a stability result (Proposition 6.8). We prove this stability result by a simple contraction argument, where we use a norm with an exponentially decaying weight in time. As a result, the proof follows from a small modification of that of the local well-posedness (Theorem 5.1). As compared to [13], our stability argument is very simple (both in terms of the statements and the proofs).

3. In the third step, we establish a uniform (in N) control on the solution $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated system (see (6.58) below) with respect to the truncated measure $\vec{\rho}_N \otimes \mathbb{P}_2$ (Proposition 6.9). The proof is based on the invariance of the truncated Gibbs measure $\vec{\rho}_N$ and a discrete Gronwall argument.

4. In the fourth step, we study the pushforward measures $(\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ and $(\Xi)_\#(\vec{\rho} \otimes \mathbb{P}_2)$. In particular, by using ideas from theory of optimal transport (the Kantorovich duality) and the Boué-Dupuis variational formula, we prove that the pushforward measure $(\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ converges to $(\Xi)_\#(\vec{\rho} \otimes \mathbb{P}_2)$ in the Wasserstein-1 distance, as $N \to \infty$; see Proposition 6.10 below.

Once we establish Steps 1 - 4, the proof of Theorem 1.15 follows in a straightforward manner. In Subsection 6.1, we first study the truncated dynamics (6.2) and briefly go over almost sure global well-posedness of (6.2) and invariance of the truncated Gibbs measure $\vec{\rho}_N$ (Lemma 6.4). We then discuss the details of Step 1 above. In Subsection 6.2, we first go over the details of Steps 2, 3, and 4 and then present the proof of Theorem 1.15.

Notations: By assumption, the Gaussian field $\vec{\mu} = \mu \otimes \mu_0$ in (1.16) and hence the (truncated) Gibbs measure are independent of (the distribution of) the space-time white noise ξ in (1.33) and (6.2). Hence, we can write the probability space Ω as

$$\Omega = \Omega_1 \times \Omega_2 \tag{6.3}$$

such that the random Fourier series in (1.18) depend only on $\omega_1 \in \Omega_1$, while the cylindrical Wiener process W in (3.1) depends only on $\omega_2 \in \Omega_2$. In view of (6.3), we also write the underlying probability measure $\mathbb{P}$ on Ω as

$$\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2, \tag{6.4}$$

where $\mathbb{P}_j$ is the marginal probability measure on Ω_j, $j = 1, 2$.

With the decomposition (6.3) in mind, we set

$$\upharpoonleft(t; \vec{u}_0, \omega_2) = S(t)\vec{u}_0 + \sqrt{2}\int_0^t \mathcal{D}(t - t')dW(t', \omega_2) \tag{6.5}$$

for $\vec{u}_0 = (u_0, u_1) \in \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)$ and $\omega_2 \in \Omega_2$, where $S(t)$ and $\mathcal{D}(t)$ are as in (5.5) and (5.3), respectively. When it is clear from the context, we may suppress the dependence on $\vec{u}_0$ and/or ω_2. Given $N \in \mathbb{N}$, we set

$$\upharpoonleft_N(\vec{u}_0, \omega_2) = \pi_N \upharpoonleft(\vec{u}_0, \omega_2), \tag{6.6}$$

where π_N is as in (1.19). We also set

$$
\begin{aligned}
\vee_N(\vec{u}_0,\omega_2) &= \uparrow^2_N(\vec{u}_0,\omega_2) - \sigma_N,\\
\Upsilon_N(\vec{u}_0,\omega_2) &= \pi_N \mathcal{I}(\vee_N(\vec{u}_0,\omega_2)),\\
\Upsilon\!\!\uparrow_N(\vec{u}_0,\omega_2) &= \Upsilon_N(\vec{u}_0,\omega_2) \ominus \uparrow_N(\vec{u}_0,\omega_2),
\end{aligned}
\tag{6.7}
$$

and define $\mathbb{A}_N(\vec{u}_0,\omega_2)$ as in (5.46) by replacing $\uparrow_N$ with $\uparrow_N(\vec{u}_0,\omega_2)$. We define the paracontrolled operator $\widetilde{\mathfrak{I}}^N_{\olessthan,\ominus} = \widetilde{\mathfrak{I}}^N_{\olessthan,\ominus}(\vec{u}_0,\omega_2)$ in a manner analogous to $\mathfrak{I}^N_{\olessthan,\ominus}$ in Lemma 5.9, but with an extra frequency cutoff π_N. Namely, instead of (5.20), we first define $\widetilde{\mathfrak{I}}^N_{\olessthan}$ by

$$
\widetilde{\mathfrak{I}}^N_{\olessthan}(w)(t) = \mathcal{I}(\pi_N(w \olessthan \uparrow_N))(t), \tag{6.8}
$$

where $\uparrow_N = \uparrow_N(\vec{u}_0,\omega_2)$ is as in (6.6). We then define $\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}$ and $\widetilde{\mathfrak{I}}^{(2),N}_{\olessthan}$ as in (5.21) and (5.22) with an extra frequency cutoff $\chi_N(n)$, depending on $|n_1| \gtrsim |n_2|^\theta$ or $|n_1| \ll |n_2|^\theta$. Note that the conclusion of Lemma 5.8 (in particular the estimate (5.42)) holds for $\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}$, uniformly in $N \in \mathbb{N}$. Finally, we define $\widetilde{\mathfrak{I}}^N_{\olessthan,\ominus}$ by

$$
\widetilde{\mathfrak{I}}^N_{\olessthan,\ominus}(w)(t) = \widetilde{\mathfrak{I}}^{(2),N}_{\olessthan}(w) \ominus \uparrow_N(t), \tag{6.9}
$$

namely, by inserting a frequency cutoff $\chi_N(n_1+n_2)$ and replacing $\uparrow$ by $\uparrow_N = \uparrow_N(\vec{u}_0,\omega_2)$ in (5.24). We then define the truncated enhanced data set $\Xi_N(\vec{u}_0,\omega_2)$ by

$$
\Xi_N(\vec{u}_0,\omega_2) = \big(\uparrow_N,\, \vee_N,\, \Upsilon_N,\, \Upsilon\!\!\uparrow_N,\, \mathbb{A}_N,\, \widetilde{\mathfrak{I}}^N_{\olessthan,\ominus}\big), \tag{6.10}
$$

where, on the right-hand side, we suppressed the dependence on $(\vec{u}_0,\omega_2)$ for notational simplicity. Note that, given $\vec{u}_0 \in \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)$, the enhanced data set $\Xi_N(\vec{u}_0,\omega_2)$ does not converge in general. Nonetheless, for the notational purpose, let us *formally* define the (untruncated) enhanced data set $\Xi(\vec{u}_0,\omega_2)$ by setting

$$
\Xi(\vec{u}_0,\omega_2) = \big(\uparrow,\, \vee,\, \Upsilon,\, \Upsilon\!\!\uparrow,\, \mathbb{A}, \mathfrak{I}_{\olessthan,\ominus}\big), \tag{6.11}
$$

where each term on the right-hand side is a limit of the corresponding term in (6.10) (if it exists). In Corollary 6.7, we will construct the enhanced data set $\Xi(\vec{u}_0,\omega_2)$ in (6.11) as a limit of the truncated enhanced data set $\Xi_N(\vec{u}_0,\omega_2)$ in (6.10) almost surely with respect to $\vec{\rho}\otimes\mathbb{P}_2$.

In the remaining part of this section, we fix $s_1, s_2, s_3 \in \mathbb{R}$ satisfying

$$
\frac{1}{4} < s_1 < \frac{1}{2} < s_2 < s_1 + \frac{1}{4} \qquad \text{and} \qquad s_2 - 1 < s_3 < 0. \tag{6.12}
$$

Furthermore, we take both s_1 and s_2 to be sufficiently close to $\frac{1}{2}$ (such that the conditions in (6.82) are satisfied, say with $r_1 = r_2 = 3$).

Remark 6.1. (i) In view of (6.6) with (1.19), we have $\uparrow_N(\vec{u}_0,\omega_2) = \uparrow_N(\pi_N\vec{u}_0,\omega_2)$ and thus

$$
\Xi_N(\vec{u}_0,\omega_2) = \Xi_N(\pi_N\vec{u}_0,\omega_2).
$$

Namely, the truncated enhanced data set $\Xi_N(\vec{u}_0,\omega_2)$ in (6.10) depends only on the low frequency part $\pi_N\vec{u}_0$ of the initial data.

(ii) Note that the terms Υ_N, $\Upsilon\!\!\uparrow_N$, and $\widetilde{\mathfrak{I}}^N_{\olessthan,\ominus}$ in (6.10) come with an extra frequency cutoff as compared to the corresponding terms studied in Section 5. When $\mathrm{Law}(\vec{u}_0) = \vec{\mu}$, the results in Lemmas 5.6, 5.7, and 5.9, and Remark 5.5 from Subsection 5.3 also apply to $\Upsilon_N(\vec{u}_0,\omega_2)$, $\Upsilon\!\!\uparrow_N(\vec{u}_0,\omega_2)$, and $\widetilde{\mathfrak{I}}^N_{\olessthan,\ominus}(\vec{u}_0,\omega_2)$.

(iii) Note that the $\mathcal{X}_T^\varepsilon$-norm for enhanced data sets defined in (5.49) also measures the time derivatives of $\uparrow_N$ and Υ_N in appropriate space-time norms. In view of (6.7) and (5.6), the time derivative of $\Upsilon_N(\vec{u}_0, \omega_2)$ is given by

$$\partial_t \Upsilon_N(t; \vec{u}_0, \omega_2) = \pi_N \int_0^t \partial_t \mathcal{D}(t-t') \vee_N(t'; \vec{u}_0, \omega_2) dt'.$$

As for the stochastic convolution, recall that, unlike the heat or Schrödinger case, the stochastic convolution for the damped wave equation is differentiable in time and the time derivative of $\uparrow_N(\vec{u}_0, \omega_2)$ is given by

$$\partial_t \uparrow_N(t; \vec{u}_0, \omega_2) = \pi_N \partial_t S(t)\vec{u}_0 + \sqrt{2}\pi_N \int_0^t \partial_t \mathcal{D}(t-t') dW(t', \omega_2). \tag{6.13}$$

The formula (6.13) easily follows from viewing the stochastic integral in (6.5) (with an extra frequency cutoff π_N) as a Paley-Wiener-Zygmund integral and taking a time derivative.

6.1. **On the truncated dynamics.** In this subsection, we study the truncated hyperbolic Φ^3_3-model (6.2). We first go over local well-posedness of the truncated equation (6.2) and then almost sure global well-posedness and invariance of the truncated Gibbs measure $\vec{\rho}_N$; see Lemmas 6.2 and 6.4. Then, by combining the Boué-Dupuis variational formula (Lemma 3.1) and space-time estimates, we prove uniform (in N) exponential integrability of the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ with respect to $\vec{\rho}_N \otimes \mathbb{P}_2$ on $(\vec{u}_0, \omega_2)$; see Proposition 6.5. As a corollary, we prove that the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.10) converges to the limiting enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.11) almost surely with respect to the limiting measure $\vec{\rho} \otimes \mathbb{P}_2$ (Corollary 6.7).

Given $N \in \mathbb{N}$, let $\vec{u}_0 = (u_0, u_1)$ be a pair of random distributions such that $\mathrm{Law}((u_0, u_1)) = \vec{\rho}_N = \rho_N \otimes \mu_0$. Let u_N be a solution to the truncated equation (6.2) with $(u_N, \partial_t u_N)|_{t=0} = \vec{u}_0$. With $:\!(\pi_N u_N)^2\!: = (\pi_N u_N)^2 - \sigma_N$, we write (6.2) as

$$\begin{cases} \partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N \\ \qquad - \sigma \pi_N\big((\pi_N u_N)^2 - \sigma_N\big) + M\big((\pi_N u_N)^2 - \sigma_N\big)\pi_N u_N = \sqrt{2}\xi \\ (u_N, \partial_t u_N)|_{t=0} = \vec{u}_0, \end{cases} \tag{6.14}$$

where M is as in (1.34). Note that, due to the presence of the frequency projector π_N, the dynamics (6.14) on high frequencies $\{|n| \gtrsim N\}$ and low frequencies $\{|n| \lesssim N\}$ are decoupled. The high frequency part of the dynamics (6.14) is given by

$$\begin{cases} \partial_t^2 \pi_N^\perp u_N + \partial_t \pi_N^\perp u_N + (1-\Delta)\pi_N^\perp u_N = \sqrt{2}\pi_N^\perp \xi \\ (\pi_N^\perp u_N, \partial_t \pi_N^\perp u_N)|_{t=0} = \pi_N^\perp \vec{u}_0. \end{cases} \tag{6.15}$$

The solution $\pi_N^\perp u_N$ to (6.15) is given by

$$\pi_N^\perp u_N = \pi_N^\perp \uparrow(\vec{u}_0), \tag{6.16}$$

where $\uparrow(\vec{u}_0)$ is as in (6.5) with the ω_2-dependence suppressed. With $v_N = \pi_N u_N$, the low frequency part of the dynamics (6.14) is given by

$$\begin{cases} \partial_t^2 v_N + \partial_t v_N + (1-\Delta) v_N \\ \qquad - \sigma \pi_N\big((\pi_N v_N)^2 - \sigma_N\big) + M\big((\pi_N v_N)^2 - \sigma_N\big)\pi_N v_N = \sqrt{2}\pi_N\xi \\ (v_N, \partial_t v_N)|_{t=0} = \pi_N \vec{u}_0, \end{cases} \tag{6.17}$$

where we kept π_N in several places to emphasize that (6.17) depends only on finite many frequencies $\{n \in NQ\}$ with Q as in (1.21). By writing (6.17) in the Duhamel formulation, we have

$$v_N(t) = \pi_N S(t)\vec{u}_0 + \int_0^t \mathcal{D}(t-t')\mathcal{N}_N(v_N)(t')dt' + \upharpoonleft_N(t;0), \tag{6.18}$$

where the truncated nonlinearity $\mathcal{N}_N(v_N)$ is given by

$$\mathcal{N}_N(v_N) = \sigma\pi_N\big((\pi_N v_N)^2 - \sigma_N\big) - M\big((\pi_N v_N)^2 - \sigma_N\big)\pi_N v_N. \tag{6.19}$$

and $\upharpoonleft_N(t;0)$ is as in (6.6) with $\vec{u}_0 = 0$:

$$\upharpoonleft_N(t;0,\omega_2) = \sqrt{2}\int_0^t \mathcal{D}(t-t')\pi_N dW(t',\omega_2).$$

For each fixed $N \in \mathbb{N}$, we have $\upharpoonleft_N(t;0) = \pi_N \upharpoonleft(t;0) \in C^1(\mathbb{R}_+; C^\infty(\mathbb{T}^3))$; see Remark 6.1. By viewing $\upharpoonleft_N(t;0)$ in (6.18) as a perturbation, it suffices to study the following damped NLW with a deterministic perturbation:

$$v_N(t) = \pi_N S(t)(v_0, v_1) + \int_0^t \mathcal{D}(t-t')\mathcal{N}_N(v_N)(t')dt' + F, \tag{6.20}$$

where $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^3)$, σ_N is as in (1.22), and $F \in C^1(\mathbb{R}_+; C^\infty(\mathbb{T}^3))$ is a given deterministic function.

A standard contraction argument with the one degree of smoothing from the Duhamel integral operator $\mathcal{I}$ in (5.6) and Sobolev's inequality yields the following local well-posedness of (6.20). Since the argument is standard, we omit details. See, for example, the proof of Lemma 9.1 in [53].

Lemma 6.2. *Let $N \in \mathbb{N}$. Given any $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^3)$ and $F \in C^1([0,1]; H^1(\mathbb{T}^3))$ with*

$$\|(v_0, v_1)\|_{\mathcal{H}^1} \le R \qquad \text{and} \qquad \|F\|_{C^1([0,1];H^1)} \le K$$

for some $R, K \ge 1$, there exist $\tau = \tau(R, K, N) > 0$ and a unique solution v_N to (6.20) *on $[0, \tau]$, satisfying the bound*:

$$\|v_N\|_{\widetilde{X}^1(\tau)} \lesssim R + K,$$

where

$$\widetilde{X}^1(\tau) = C([0,\tau]; H^1(\mathbb{T}^3)) \cap C^1([0,\tau]; L^2(\mathbb{T}^3)).$$

Moreover, the solution v_N is unique in $\widetilde{X}^1(\tau)$.

Remark 6.3. (i) A standard contraction argument gives $\tau = \tau(R, K, N) \sim (R + K + N)^{-\theta}$ for some $\theta > 0$, in particular the local existence depends on $N \in \mathbb{N}$.

(ii) We also point out that the uniqueness statement for v_N in Lemma 6.2 is unconditional, namely, the uniqueness of the solution v_N holds in the entire class $\widetilde{X}^1(\tau)$. Then, from (6.16) and the unconditional uniqueness of the solution $v_N = v_N(\pi_N\vec{u}_0)$ to (6.17), we obtain the *unique* representation of u_N:

$$u_N = \pi_N^\perp \upharpoonleft(\vec{u}_0) + v_N(\pi_N\vec{u}_0).$$

See for example (6.129) below, where we use a different representation of u_N.

Before proceeding further, let us introduce some notations. Given the cylindrical Wiener process W in (3.1), by possibly enlarging the probability space Ω_2, there exists a family of translations $\tau_{t_0} : \Omega_2 \to \Omega_2$ such that

$$W(t, \tau_{t_0}(\omega_2)) = W(t + t_0, \omega_2) - W(t_0, \omega_2)$$

for $t, t_0 \geq 0$ and $\omega_2 \in \Omega_2$. Denote by $\Phi^N(t)$ the stochastic flow map to the truncated hyperbolic Φ^3_3-model (6.2) constructed in Lemma 6.2 (which is not necessarily global at this point). Namely,

$$\begin{aligned} \vec{u}_N(t) = (u_N(t), \partial_t u_N(t)) &= \Phi^N(t)(\vec{u}_0, \omega_2) \\ &= \big(\Phi^N_1(t)(\vec{u}_0, \omega_2), \Phi^N_2(t)(\vec{u}_0, \omega_2)\big) \end{aligned} \tag{6.21}$$

is the solution to (6.2) with $\vec{u}_N|_{t=0} = \vec{u}_0$, satisfying $\text{Law}(\vec{u}_0) = \vec{\rho}_N$, and the noise $\xi(\omega_2)$. We now extend $\Phi^N(t)$ as

$$\widehat{\Phi}^N(t)(\vec{u}_0, \omega_2) = \big(\Phi^N(t)(\vec{u}_0, \omega_2), \tau_t(\omega_2)\big). \tag{6.22}$$

Note that by the uniqueness of the solution to (6.2), we have

$$\Phi^N(t_1 + t_2)(\vec{u}_0, \omega_2) = \Phi^N(t_2)\big(\Phi^N(t_1)(\vec{u}_0, \omega_2), \tau_{t_1}(\omega_2)\big) = \Phi^N(t_2)\big(\widehat{\Phi}^N(t_1)(\vec{u}_0, \omega_2)\big)$$

for $t_1, t_2 \geq 0$ as long as the flow is well defined.

By writing the truncated dynamics (6.2) as a superposition of the deterministic NLW:

$$\partial_t^2 u_N + (1 - \Delta) u_N - \mathcal{N}_N(u_N) = 0, \tag{6.23}$$

where $\mathcal{N}_N(u_N)$ is as in (6.19), and the Ornstein-Uhlenbeck process (for $\partial_t u_N$):

$$\partial_t(\partial_t u_N) = -\partial_t u_N + \sqrt{2}\xi, \tag{6.24}$$

we see that the truncated Gibbs measure $\vec{\rho}_N$ in (6.1) is formally[26] invariant under the dynamics of (6.2), since $\vec{\rho}_N$ is invariant under the NLW dynamics (6.23), while the white noise measure μ_0 on $\partial_t u_N$ (and hence $\vec{\rho}_N = \rho_N \otimes \mu_0$ on $(u_N, \partial_t u_N)$) is invariant under the Ornstein-Uhlenbeck dynamics (6.24). Then, by exploiting the formal invariance of the truncated Gibbs measure $\vec{\rho}_N$, Bourgain's invariant measure argument [8] yields the following result on almost sure global well-posedness of the truncated hyperbolic Φ^3_3-model (6.2) and invariance of the truncated Gibbs measure $\vec{\rho}_N$. Since the argument is standard (for fixed $N \in \mathbb{N}$), we omit details. See the proof of Lemma 9.3 in [53] for details.

Lemma 6.4. *Let $N \in \mathbb{N}$. Then, the truncated hyperbolic Φ^3_3-model* (6.2) *is almost surely globally well-posed with respect to the random initial data distributed by the truncated Gibbs measure $\vec{\rho}_N$ in* (6.1). *Furthermore, $\vec{\rho}_N$ is invariant under the resulting dynamics and, as a consequence, the measure $\vec{\rho}_N \otimes \mathbb{P}_2$ is invariant under the extended stochastic flow map $\widehat{\Phi}^N(t)$ defined in* (6.22). *More precisely, there exists $\Sigma_N \subset \Omega = \Omega_1 \times \Omega_2$ with $\vec{\rho}_N \otimes \mathbb{P}_2(\Sigma_N) = 1$ such that the solution $u_N = u_N(\vec{u}_0, \omega_2)$ to* (6.2) *exists globally in time and* $\text{Law}(u_N(t), \partial_t u_N(t)) = \vec{\rho}_N$ *for any $t \in \mathbb{R}_+$.*

Next, we establish uniform exponential integrability of the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.10) with respect to the truncated measure $\vec{\rho}_N \otimes \mathbb{P}_2$. We also establish uniform exponential integrability for the difference of the truncated enhanced data sets.

[26]Namely, as long as the dynamics is well defined.

Proposition 6.5. *Let $T > 0$. Then, we have*

$$\int \mathbb{E}_{\mathbb{P}_2}\Big[\exp\Big(\|\Xi_N(\vec{u}_0,\omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T}\Big)\Big]d\vec{\rho}_N(\vec{u}_0) \le C(T,\varepsilon,\alpha) < \infty \tag{6.25}$$

for $0 < \alpha < \frac{1}{3}$, uniformly in $N \in \mathbb{N}$, where the $\mathcal{X}^\varepsilon_T$-norm and the truncated enhanced data set $\Xi_N(\vec{u}_0,\omega_2)$ are as in (5.49) and (6.10), respectively. Here, $\mathbb{E}_{\mathbb{P}_2}$ denotes an expectation with respect to the probability measure $\mathbb{P}_2$ on $\omega_2 \in \Omega_2$ defined in (6.4).

Moreover, there exists small $\beta > 0$ such that

$$\int \mathbb{E}_{\mathbb{P}_2}\Big[\exp\Big(N_2^\beta\|\Xi_{N_1}(\vec{u}_0,\omega_2) - \Xi_{N_2}(\vec{u}_0,\omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T}\Big)\Big]d\vec{\rho}_N(\vec{u}_0) \le C(T,\varepsilon,\alpha) < \infty \tag{6.26}$$

for $0 < \alpha < \frac{1}{3}$, uniformly in $N, N_1, N_2 \in \mathbb{N}$ with $N \ge N_1 \ge N_2$.

Proof. For simplicity, we only prove (6.25) and (6.26) for the random operator $\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}$ defined in (6.9). The other terms in $\Xi_N(\vec{u}_0,\omega_2)$ can be estimated in an analogous manner. See Remark 6.6.

• **Part 1:** We first prove the following uniform exponential integrability:

$$\int \mathbb{E}_{\mathbb{P}_2}\Big[\exp\Big(\|\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}\|^\alpha_{\mathcal{L}_2(q,T)}\Big)\Big]d\vec{\rho}_N(\vec{u}_0) \le C(T,\varepsilon,\alpha) < \infty \tag{6.27}$$

for any $T > 0$, any finite $q > 1$, and $0 < \alpha < \frac{1}{2}$, uniformly in $N \in \mathbb{N}$. Note that the range $0 < \alpha < \frac{1}{2}$ of the exponent in (6.27) comes from the presence of $\|\mathfrak{Z}_N\|^2_{W^{1-\varepsilon,\infty}}$ in (6.41) and (6.45), since $\mathfrak{Z}_N$ defined in one line below (3.11) belongs to $\mathcal{H}_{\le 2}$. Similarly, the overall restriction $0 < \alpha < \frac{1}{3}$ in this proposition comes from the terms involving ψ_1 in (6.51), where ψ_1 is defined in (6.36) with (6.34). Namely, the worst contribution in (6.51) behaves like $\|\mathfrak{Z}_N\|^{3\alpha}_{W^{1-\varepsilon,\infty}}$ which is exponentially integrable only for $\alpha < \frac{1}{3}$; see (6.52).

From (6.8) and (6.9), we see that $\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}$ depends on two entries of $\mathord{\uparrow}_N = \pi_N \mathord{\uparrow}(\vec{u}_0,\omega_2)$. We now generalize the definition of $\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}$ to allow general entries. Given $\psi_j \in C(\mathbb{R}_+;\mathcal{D}'(\mathbb{T}^3))$, $j = 1,2$, we first define $\widetilde{\mathfrak{I}}^N_{\otimes}[\psi_1]$ by

$$\widetilde{\mathfrak{I}}^N_{\otimes}[\psi_1](w) = \mathcal{I}\big(\pi_N(w \otimes (\pi_N\psi_1))\big). \tag{6.28}$$

As in (5.21) and (5.22), define $\widetilde{\mathfrak{I}}^{(2),N}_{\otimes}[\psi_1]$ to be the restriction of $\widetilde{\mathfrak{I}}^N_{\otimes}[\psi_1]$ onto $\{|n_1| \ll |n_2|^\theta\}$:

$$\widetilde{\mathfrak{I}}^{(2),N}_{\otimes}[\psi_1](w) = \mathcal{I}\big(\pi_N(\mathcal{K}^\theta(w,\pi_N\psi_1))\big), \tag{6.29}$$

where $\mathcal{K}^\theta$ is the bilinear Fourier multiplier operator with the multiplier $\mathbf{1}_{\{|n_1|\ll|n_2|^\theta\}}$. More precisely, we have

$$\begin{aligned}\widetilde{\mathfrak{I}}^{(2),N}_{\otimes}[\psi_1](w)(t) = \sum_{n\in\mathbb{Z}^3}\chi_N(n)e_n \sum_{n=n_1+n_2}\sum_{0\le j<\theta k+c_0}\varphi_j(n_1)\varphi_k(n_2)\chi_N(n_2)\\ \times\int_0^t e^{-\frac{t-t'}{2}}\frac{\sin((t-t')[\![n]\!])}{[\![n]\!]}\widehat{w}(n_1,t')\,\widehat{\psi}_1(n_2,t')dt',\end{aligned} \tag{6.30}$$

where χ_N is as in (1.20) and $c_0 \in \mathbb{R}$ is as in (5.41). Then, we define $\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\psi_1,\psi_2]$ by

$$\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\psi_1,\psi_2](w) = \widetilde{\mathfrak{I}}^{(2),N}_{\otimes}[\psi_1](w) \ominus (\pi_N\psi_2). \tag{6.31}$$

Note that $\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi_1,\psi_2]$ is bilinear in ψ_1 and ψ_2. We also set

$$\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi] = \widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi,\psi] \tag{6.32}$$

for simplicity. With this notation, we can write $\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}$ in (6.27) as $\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\upharpoonleft(\vec{u}_0,\omega_2)]$, where $\vec{u}_0=(u_0,u_1)$. Note that we have $\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\pi_N\psi] = \widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi]$. Before proceeding further, we record the following boundedness of $\mathcal{K}^\theta$ defined in (6.29) and (6.30); a slight modification of the proof of (2.7) in Lemma 2.2 yields

$$\|\mathcal{K}^\theta(f,g)\|_{B^{s_2}_{p,q}} \lesssim \|f\|_{L^{p_1}}\|g\|_{B^{s_2}_{p_2,q}} \tag{6.33}$$

for any $s_2\in\mathbb{R}$ and $1\le p,p_1,p_2,q\le\infty$ such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$.

By the Boué-Dupuis variational formula (Lemma 3.1) with the change of variables (3.12), we have

$$\begin{aligned}
&-\log\int \exp\Big(\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\upharpoonleft(\vec{u}_0,\omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,T)}\Big)d\rho_N(u_0)\\
&=\inf_{\dot\Upsilon^N\in\mathbb{H}^1_a}\mathbb{E}\bigg[-\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,T)}\\
&\qquad\qquad + \widehat{R}^\diamond_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N)+\frac{1}{2}\int_0^1\|\dot\Upsilon^N(t)\|^2_{H^1_x}dt\bigg]+\log Z_N,
\end{aligned}$$

where $\widehat{R}^\diamond_N$ is as in (3.33) and

$$\Theta = \Upsilon^N+\sigma\mathfrak{Z}_N. \tag{6.34}$$

Recall the notation $Y_N=\pi_N Y$ and $\Upsilon_N=\pi_N\Upsilon^N$. Then, from Lemmas 3.5 and 3.6 with Lemma 3.2 and (3.25), there exists $\varepsilon_0, C_0>0$ such that

$$\begin{aligned}
&-\log\int \exp\Big(\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\upharpoonleft(\vec{u}_0,\omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,T)}\Big)d\rho_N(u_0)\\
&\ge\inf_{\dot\Upsilon^N\in\mathbb{H}^1_a}\mathbb{E}\Big[-\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,T)}+\varepsilon_0\big(\|\Upsilon^N\|^2_{H^1}+\|\Upsilon_N\|^6_{L^2}\big)\Big]-C_0,
\end{aligned}\tag{6.35}$$

uniformly in u_1 and ω_2.

In view of (6.5), we write $\upharpoonleft(Y+\Theta,u_1,\omega_2)$ as

$$\upharpoonleft(Y+\Theta,u_1,\omega_2)=\upharpoonleft(Y,u_1,\omega_2)+S(t)(\Theta,0)=:\psi_0+\psi_1, \tag{6.36}$$

where $S(t)$ is as in (5.5). By (6.32), we have

$$\begin{aligned}
\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|_{\mathcal{L}_2(q,T)} &\le \big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi_0,\psi_0]\big\|_{\mathcal{L}_2(q,T)}+\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi_0,\psi_1]\big\|_{\mathcal{L}_2(q,T)}\\
&\quad+\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi_1,\psi_0]\big\|_{\mathcal{L}_2(q,T)}+\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi_1,\psi_1]\big\|_{\mathcal{L}_2(q,T)}.
\end{aligned}\tag{6.37}$$

Under the truncated Gibbs measure $\vec{\rho}_N$, we have $\mathrm{Law}(u_1)=\mu_0$ and thus we have $\mathrm{Law}(Y,u_1)=\vec{\mu}=\mu\otimes\mu_0$. Then, from the uniform exponential tail estimates in Lemmas 5.4 and 5.9 (see also Remark 6.1) with (3.11), there exists $K(Y,u_1,\omega_2)$ such that

$$\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi_0]\big\|_{\mathcal{L}_2(q,T)}+\|\psi_0\|^2_{L^\infty_T W^{-\frac{1}{2}-\varepsilon,\infty}_x}+\|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\le K(Y,u_1,\omega_2) \tag{6.38}$$

and

$$\mathbb{E}_{\vec{\mu}\otimes\mathbb{P}_2}\big[\exp(\delta K(Y,u_1,\omega_2))\big]<\infty \tag{6.39}$$

for sufficiently small $\delta > 0$.

We now estimate the last three terms on the right-hand side of (6.37). Let $s_3 < 0$. By Sobolev's inequality, (6.31), Hölder's inequality,[27] (6.29), Sobolev's inequality, Lemma 5.3, and (6.33) with (6.36), we have

$$\begin{aligned}\big\|\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\psi_0,\psi_1](w)\big\|_{L^\infty_T H^{s_3}_x} &\lesssim \big\|\widetilde{\mathfrak{I}}^{(2),N}_{\otimes}[\psi_0](w)\ominus(\pi_N\psi_1)\big\|_{L^\infty_T L_x^{\frac{6}{3-2s_3}}}\\ &\lesssim \big\|\widetilde{\mathfrak{I}}^{(2),N}_{\otimes}[\psi_0](w)\big\|_{L^\infty_T L_x^{\frac{3}{1-s_3-\varepsilon}}}\|\pi_N\psi_1\|_{L^\infty_T L_x^{\frac{6}{1+2\varepsilon}}}\\ &\lesssim \|\mathcal{I}(\mathcal{K}^\theta(w,\pi_N\psi_0))\|_{L^\infty_T H_x^{s_3+\frac12+\varepsilon}}\|\psi_1\|_{L^\infty_T H^{1-\varepsilon}_x}\\ &\lesssim \|\mathcal{K}^\theta(w,\pi_N\psi_0)\|_{L^1_T H_x^{s_3-\frac12+\varepsilon}}\|\Theta\|_{H^{1-\varepsilon}}\\ &\lesssim \|w\|_{L^1_T L^2_x}\|\psi_0\|_{L^\infty_T W_x^{-\frac12-2\varepsilon,\infty}}\|\Theta\|_{H^{1-\varepsilon}},\end{aligned} \tag{6.40}$$

for $\varepsilon > 0$ sufficiently small such that $4\varepsilon \le -s_3$. Hence, by the definition (5.31) of the $\mathcal{L}(q,T)$-norm, Cauchy's inequality, and (6.34), we obtain

$$\begin{aligned}\big\|\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\psi_0,\psi_1]\big\|_{\mathcal{L}_2(q,T)} &\lesssim T^{\frac{q-1}{q}}\|\psi_0\|_{L^\infty_T W_x^{-\frac12-2\varepsilon,\infty}}\|\Theta\|_{H^{1-\varepsilon}}\\ &\lesssim T^{\frac{q-1}{q}}\Big(\|\psi_0\|^2_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} + \|\Upsilon^N\|^2_{H^1} + \|\mathfrak{Z}_N\|^2_{W^{1-\varepsilon,\infty}}\Big).\end{aligned} \tag{6.41}$$

Proceeding as in (6.40) and applying Sobolev's embedding theorem with (6.34) and (6.36), we have

$$\begin{aligned}\big\|\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\psi_1,\psi_1]\big\|_{\mathcal{L}_2(q,T)} &\lesssim T^{\frac{q-1}{q}}\|\psi_1\|_{L^\infty_T W_x^{-\frac12-2\varepsilon,\infty}}\|\Theta\|_{H^{1-\varepsilon}} \lesssim T^{\frac{q-1}{q}}\|\Theta\|^2_{H^{1-\varepsilon}}\\ &\lesssim T^{\frac{q-1}{q}}\Big(\|\Upsilon^N\|^2_{H^1} + \|\mathfrak{Z}_N\|^2_{W^{1-\varepsilon,\infty}}\Big).\end{aligned} \tag{6.42}$$

Finally, from Lemma 2.2, Lemma 5.3, Sobolev's inequality, and (6.33), we have

$$\begin{aligned}\big\|\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\psi_1,\psi_0](w)\big\|_{L^\infty_T H^{s_3}_x} &\le \big\|\widetilde{\mathfrak{I}}^{(2),N}_{\otimes}[\psi_1](w)\ominus(\pi_N\psi_0)\big\|_{L^\infty_T L^2_x}\\ &\lesssim \|\mathcal{I}(\mathcal{K}^\theta(w,\pi_N\psi_1))\|_{L^\infty_T H_x^{\frac12+2\varepsilon}}\|\psi_0\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}}\\ &\lesssim \|\mathcal{K}^\theta(w,\pi_N\psi_1)\|_{L^1_T H_x^{-\frac12+2\varepsilon}}\|\psi_0\|_{L^\infty_1 W_x^{-\frac12-\varepsilon,\infty}}\\ &\lesssim \|\mathcal{K}^\theta(w,\pi_N\psi_1)\|_{L^1_T L_x^{\frac{3}{2-2\varepsilon}}}\|\psi_0\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}}\\ &\lesssim \|w\|_{L^q_T L^2_x}\|\psi_1\|_{L^{q'}_T B^0_{\frac{6}{1-4\varepsilon},2}}\|\psi_0\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}}.\end{aligned} \tag{6.43}$$

[27]To be more precise, this is the Coifman-Meyer theorem on $\mathbb{T}^3$ to estimate a resonant product. The Coifman-Meyer theorem on $\mathbb{T}^3$ follows from the Coifman-Meyer theorem for functions on $\mathbb{R}^d$ [30, Theorem 7.5.3] and the transference principle [25, Theorem 3]. We may equally proceed with (2.9) in Lemma 2.2 with a slight loss of derivative which does not affect the estimate.

Note that $(\frac{1}{3\varepsilon}, \frac{6}{1-4\varepsilon})$ is $(1-\varepsilon)$-admissible. Since $q > 1$, we can choose $\varepsilon > 0$ sufficiently small such that $q' \le \frac{1}{3\varepsilon}$. Then, by Minkowski's integral inequality, (6.36), and Lemma 5.3, we have

$$\|\psi_1\|_{L^{q'}_T B^0_{\frac{6}{1-4\varepsilon},2}} \le \bigg(\sum_{j=0}^\infty \|S(t)(\mathbf{P}_j\Theta, 0)\|^2_{L^{q'}_T L_x^{\frac{6}{1-4\varepsilon}}}\bigg)^{\frac12} \lesssim \|\Theta\|_{H^{1-\varepsilon}}, \tag{6.44}$$

where $\mathbf{P}_j$ is the Littlewood-Paley projector onto the frequencies $\{|n| \sim 2^j\}$. Hence, from (5.31), (6.43), (6.44), and Cauchy's inequality with (6.34), we obtain

$$\begin{aligned} \big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\psi_1, \psi_0]\big\|_{\mathcal{L}_2(q,T)} &\le C(T)\|\psi_0\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} \|\Theta\|_{H^{1-\varepsilon}} \\ &\le C(T)\Big(\|\psi_0\|^2_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} + \|\Upsilon^N\|^2_{H^1} + \|\mathfrak{Z}_N\|^2_{W^{1-\varepsilon,\infty}}\Big). \end{aligned} \tag{6.45}$$

By (6.37), (6.38), (6.41), (6.42), (6.45), and Young's inequality (with $\alpha < 1$) we have

$$\begin{aligned} &\inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\Big[-\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta, u_1, \omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,T)} + \varepsilon_0\big(\|\Upsilon^N\|^2_{H^1} + \|\Upsilon_N\|^6_{L^2}\big)\Big] \\ &\qquad \ge -c\mathbb{E}\Big[K(Y, u_1, \omega_2)^{2\alpha}\Big] + \inf_{\dot\Upsilon^N \in \mathbb{H}^1_a}\Big(-c\|\Upsilon^N\|^{2\alpha}_{H^1} + \varepsilon_0\|\Upsilon^N\|^2_{H^1}\Big) - C_1 \\ &\qquad \gtrsim -\mathbb{E}\Big[K(Y, u_1, \omega_2)^{2\alpha}\Big] - C_2. \end{aligned} \tag{6.46}$$

Therefore, from (6.35), (6.46), Young's inequality, and Jensen's inequality, we obtain

$$\begin{aligned} \int \exp\Big(\big\|\mathfrak{I}^N_{\circledcirc,\ominus}[\upharpoonleft(\vec u_0, \omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,1)}\Big) d\rho_N(u_0) &\lesssim \exp\Big(C\mathbb{E}\big[K(Y, u_1, \omega_2)^{2\alpha}\big]\Big) \\ &\le \exp\Big(\delta\mathbb{E}\big[K(Y, u_1, \omega_2)\big]\Big) \\ &\le \int \exp\big(\delta K(Y, u_1, \omega_2)\big) d\mu(Y) \end{aligned}$$

for $0 < \alpha < \frac12$. Finally, by integrating in (u_1, ω_2) with respect to $\mu_2 \otimes \mathbb{P}_2$, we obtain the desired bound (6.27) from (6.39).

• **Part 2:** Next, we briefly discuss how to prove (6.26) for the random operator $\widetilde{\mathfrak{I}}^N_{\circledcirc,\ominus}$. For $N \ge N_1 \ge N_2 \ge 1$, proceeding as in Part 1, we arrive at

$$\begin{aligned} &-\log \int \exp\Big(N_2^\beta\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\upharpoonleft(\vec u_0, \omega_2)] - \widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\upharpoonleft(\vec u_0, \omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,T)}\Big) d\rho_N(u_0) \\ &\quad \ge \inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\Big[-N_2^\beta\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta, u_1, \omega_2)] - \widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta, u_1, \omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,T)} \\ &\qquad\qquad + \varepsilon_0\big(\|\Upsilon^N\|^2_{H^1} + \|\Upsilon_N\|^6_{L^2}\big)\Big] - C_0, \end{aligned}$$

uniformly in u_1 and ω_2. See (6.35). With ψ_0 and ψ_1 as in (6.36), we write

$$\begin{aligned}
N_2^{\frac{\beta}{\alpha}} &\big\|\widetilde{\mathfrak{I}}^{N_1}_{\oslash,\ominus}[\uparrow(Y+\Theta,u_1,\omega_2)] - \widetilde{\mathfrak{I}}^{N_2}_{\oslash,\ominus}[\uparrow(Y+\Theta,u_1,\omega_2)]\big\|_{\mathcal{L}_2(q,T)} \\
&\le N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\oslash,\ominus}[\psi_0,\psi_0] - \widetilde{\mathfrak{I}}^{N_2}_{\oslash,\ominus}[\psi_0,\psi_0]\big\|_{\mathcal{L}_2(q,T)} \\
&\quad + N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\oslash,\ominus}[\psi_0,\psi_1] - \widetilde{\mathfrak{I}}^{N_2}_{\oslash,\ominus}[\psi_0,\psi_1]\big\|_{\mathcal{L}_2(q,T)} \\
&\quad + N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\oslash,\ominus}[\psi_1,\psi_0] - \widetilde{\mathfrak{I}}^{N_2}_{\oslash,\ominus}[\psi_1,\psi_0]\big\|_{\mathcal{L}_2(q,T)} \\
&\quad + N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\oslash,\ominus}[\psi_0,\psi_1] - \widetilde{\mathfrak{I}}^{N_2}_{\oslash,\ominus}[\psi_1,\psi_1]\big\|_{\mathcal{L}_2(q,T)}.
\end{aligned} \tag{6.47}$$

In view of Remark 6.1 (see also Lemma 5.9 and Remark 5.5), we see that there exists $K(Y,u_1,\omega_2)$ such that

$$\begin{aligned}
N_2^{\frac{\beta}{\alpha}} &\big\|\widetilde{\mathfrak{I}}^{N_1}_{\oslash,\ominus}[\psi_0,\psi_0] - \widetilde{\mathfrak{I}}^{N_2}_{\oslash,\ominus}[\psi_0,\psi_0]\big\|_{\mathcal{L}_2(q,T)} \\
&+ \|\psi_0\|^2_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}} \le \widetilde{K}(Y,u_1,\omega_2)
\end{aligned} \tag{6.48}$$

and

$$\mathbb{E}_{\vec{\mu}\otimes\mathbb{P}_2}\big[\exp(\delta\widetilde{K}(Y,u_1,\omega_2))\big] < \infty \tag{6.49}$$

for sufficiently small $\delta > 0$, provided that $\beta > 0$ is sufficiently small. The last three terms on the right-hand side of (6.47) can be handled as in (6.41), (6.42), and (6.45). By noting that one of the factors comes with $\pi_{N_1} - \pi_{N_2}$, we gain a small negative power of N_2 by losing small regularity in (6.41), (6.42), and (6.45), while keeping the resulting regularities on the right-hand sides unchanged. This allows us to hide $N_2^{\frac{\beta}{\alpha}}$ in (6.47). The rest of the argument follows precisely as in Part 1. $\square$

Remark 6.6. In the proof of Proposition 6.5, we only treated $\widetilde{\mathfrak{I}}^{N}_{\oslash,\ominus}$ from the truncated enhanced data set $\Xi_N(\vec{u}_0,\omega_2)$ in (6.10). Let us briefly discuss how to treat the other terms in $\Xi_N(\vec{u}_0,\omega_2)$ to get the exponential integrability bound (6.25). The second bound (6.26) follows in a similar manner. The terms $\uparrow_N$, $\vee_N$, Υ_N, and $\mathbb{A}_N$ can be estimated in a similar manner since they are (at most) quadratic in $\uparrow(Y+\Theta,u_1,\omega_2)$ and the product $\psi_0\psi_1$ is well defined, where ψ_j, $j=0,1$, is as in (6.36).

As for $\mathring{\Upsilon}_N$, with the notation above and (6.36), we have

$$\begin{aligned}
\mathring{\Upsilon}_N[\uparrow(Y+\Theta,u_1,\omega_2)] &= \mathring{\Upsilon}_N[\psi_0+\psi_1] \\
&= \Upsilon_N[\psi_0+\psi_1] \ominus (\pi_N\psi_0) + \Upsilon_N[\psi_0+\psi_1] \ominus (\pi_N\psi_1).
\end{aligned} \tag{6.50}$$

Let $0<\alpha<\frac13$. Then, by Lemma 2.2 and Young's inequality, we can estimate the second term on the right-hand side as

$$\begin{aligned}
\|\Upsilon_N[\psi_0+\psi_1] \ominus (\pi_N\psi_1)\|^\alpha_{C_T H_x^{-\varepsilon}} &\lesssim \|\Upsilon_N[\psi_0+\psi_1]\|^\alpha_{C_T W_x^{\frac12-\varepsilon,\infty}} \|\psi_1\|^\alpha_{C_T H_x^{1-\varepsilon}} \\
&\lesssim \|\Upsilon_N[\psi_0+\psi_1]\|^{\frac32\alpha}_{C_T W_x^{\frac12-\varepsilon,\infty}} + \|\psi_1\|^{3\alpha}_{C_T H_x^{1-\varepsilon}}.
\end{aligned} \tag{6.51}$$

Noting that $\frac32\alpha < \frac12$ and $3\alpha < 1$, we can control the first term on the right-hand side of (6.51) by the exponential integrability bound for Υ_N under $\vec{\rho}_N \otimes \mathbb{P}_2$, while by Young's inequality with (6.36) and (6.34), we can bound the second term by

$$\delta\Big(\|\Upsilon^N\|_{H^1} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\Big) + C_\delta. \tag{6.52}$$

for any small $\delta > 0$.

Let us consider the first term on the right-hand side of (6.50). In view of (6.7), by writing

$$\begin{aligned}\curlyvee_N[\psi_0 + \psi_1] \circledequal (\pi_N\psi_0) &= \curlyvee_N[\psi_0] \circledequal (\pi_N\psi_0) + 2\Big(\pi_N\mathcal{I}\big((\pi_N\psi_0)(\pi_N\psi_1)\big)\Big) \circledequal (\pi_N\psi_0)\\ &\quad + \Big(\pi_N\mathcal{I}\big((\pi_N\psi_1)^2\big)\Big) \circledequal (\pi_N\psi_0).\end{aligned} \tag{6.53}$$

Note that we have $\curlyvee_N[\psi_0] \circledequal (\pi_N\psi_0) = \curlyvee^{\circ}_N((Y,u_1),\omega_2)$, where the latter term is as in (6.7). While there is an extra frequency cutoff as compared to $\curlyvee^{\circ}_N$ in Lemma 5.7, the conclusion of Lemma 5.7 also holds for $\curlyvee_N[\psi_0] \circledequal (\pi_N\psi_0) = \curlyvee^{\circ}_N((Y,u_1),\omega_2)$. Hence, we can control the first term on the right-hand side of (6.53) by the exponential tail estimate in Lemma 5.7 with $0 < \alpha < \frac{1}{3}$. The third term on the right-hand side of (6.53) causes no issue since the resonant product of $\pi_N\mathcal{I}\big((\pi_N\psi_1)^2\big)$ and $\pi_N\psi_0$ is well defined.

Lastly, let us consider the second term on the right-hand side of (6.53). In view of (6.28), (6.29), and (6.31), we have

$$\begin{aligned}\Big(\pi_N\mathcal{I}\big((\pi_N\psi_0)(\pi_N\psi_1)\big)\Big) \circledequal (\pi_N\psi_0) &= \Big(\pi_N\mathcal{I}\big((\pi_N\psi_1) \ogreaterthan (\pi_N\psi_0)\big)\Big) \circledequal (\pi_N\psi_0)\\ &\quad + \widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}[\psi_0](\pi_N\psi_1) \circledequal (\pi_N\psi_0) + \widetilde{\mathfrak{I}}^{N}_{\olessthan,\circledequal}[\psi_0](\pi_N\psi_1),\end{aligned} \tag{6.54}$$

where $\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}[\psi_0]$ is defined by

$$\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}[\psi_0] := \widetilde{\mathfrak{I}}^{N}_{\olessthan}[\psi_0] - \widetilde{\mathfrak{I}}^{(2),N}_{\olessthan}[\psi_0]. \tag{6.55}$$

From Lemma 2.2 and the one degree of smoothing from the Duhamel integral operator $\mathcal{I}$, we see that $\mathcal{I}\big((\pi_N\psi_1) \ogreaterthan (\pi_N\psi_0)\big) \in C([0,T]; H^{\frac{3}{2}-3\varepsilon}(\mathbb{T}^3))$, which allows us to handle the first term on the right-hand side of (6.54).

Next, we estimate the second term on the right-hand side of (6.54). Recall from (6.36) that $\psi_0 = \upharpoonleft(Y,u_1,\omega_2)$ with $\mathrm{Law}(Y,u_1) = \vec{\mu}$. Namely, $\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}[\psi_0]$ defined in (6.55) is nothing but $\mathfrak{I}^{(1),N}_{\olessthan}$ in Lemma 5.8 with an extra frequency cutoff $\chi_N(n)$. Hence, the conclusion of Lemma 5.8 (in particular (5.42)) holds true for $\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}[\psi_0]$. Then, from Lemma 2.2 and Lemma 5.8, we have

$$\begin{aligned}\big\|\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}[\psi_0](\pi_N\psi_1) \circledequal (\pi_N\psi_0)\big\|^{\alpha}_{C_TH^{-\varepsilon}_x} &\lesssim \big\|\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}[\psi_0](\pi_N\psi_1)\big\|^{\alpha}_{C_TH^{\frac{1}{2}+3\varepsilon}_x}\|\psi_0\|^{\alpha}_{C_TW^{-\frac{1}{2}-\varepsilon,\infty}_x}\\ &\le C(T)\|\psi_1\|^{\alpha}_{C_TH^{1-\varepsilon}_x}\|\psi_0\|^{2\alpha}_{C_TW^{-\frac{1}{2}-\varepsilon,\infty}_x}.\end{aligned}$$

Then, Young's inequality allows us to handle this term.

Finally, we treat the third term on the right-hand side of (6.54). From (5.31) and Young's inequality, we have

$$\begin{aligned}\big\|\widetilde{\mathfrak{I}}^{N}_{\olessthan,\circledequal}[\psi_0](\pi_N\psi_1)\big\|^{\alpha}_{C_TH^{-\varepsilon}_x} &\le \big\|\widetilde{\mathfrak{I}}^{N}_{\olessthan,\circledequal}[\psi_0]\big\|^{\alpha}_{\mathcal{L}(\frac{3}{2},T)}\|\psi_1\|^{\alpha}_{L^{\frac{3}{2}}_TL^2_x}\\ &\lesssim C(T)\Big(\big\|\widetilde{\mathfrak{I}}^{N}_{\olessthan,\circledequal}[\psi_0]\big\|^{\frac{3}{2}\alpha}_{\mathcal{L}(\frac{3}{2},T)} + \|\psi_1\|^{3\alpha}_{L^{\frac{3}{2}}_TL^2_x}\Big),\end{aligned}$$

which can be controlled by (6.27) and (6.52).

Therefore, Proposition 6.5 holds for all the elements in the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.10).

We conclude this subsection by constructing the full enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.11) under $\vec{\rho} \otimes \mathbb{P}_2$ as a limit of the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.10).

Corollary 6.7. *Let $T > 0$. Then, the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.10) converges to the enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.11), with respect to the $\mathcal{X}^\varepsilon_T$-norm defined in (5.49), almost surely and in measure with respect to the limiting measure $\vec{\rho} \otimes \mathbb{P}_2$.*

Proof. Let $0 < \alpha < \frac{1}{3}$ and $\beta > 0$ be as in Proposition 6.5. Then, by Fatou's lemma, the weak convergence of $\vec{\rho}_N \otimes \mathbb{P}_2$ to $\vec{\rho} \otimes \mathbb{P}_2$, and Proposition 6.5, we have

$$\begin{aligned}
&\int \exp\Big(N_2^\beta \|\Xi_{N_1}(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T}\Big) d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2)\\
&\quad \le \liminf_{L\to\infty} \int \exp\Big(\min\big(N_2^\beta \|\Xi_{N_1}(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T}, L\big)\Big) d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2)\\
&\quad = \liminf_{L\to\infty} \lim_{N\to\infty} \int \exp\Big(\min\big(N_2^\beta \|\Xi_{N_1}(\vec{u}_0, \omega_2)\\
&\qquad\qquad - \Xi_{N_2}(\vec{u}_0, \omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T}, L\big)\Big) d(\vec{\rho}_N \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2)\\
&\quad \le \lim_{N\to\infty} \int \exp\Big(N_2^\beta \|\Xi_{N_1}(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T}\Big) d(\vec{\rho}_N \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2)\\
&\quad \lesssim 1,
\end{aligned} \tag{6.56}$$

uniformly in $N_1 \ge N_2 \ge 1$. Then, by Chebyshev's inequality, we have

$$\vec{\rho} \otimes \mathbb{P}_2\Big(\|\Xi_{N_1}(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T} > \lambda\Big) \le Ce^{-cN_2^\beta \lambda^\alpha}$$

for any $\lambda > 0$ and $N_1 \ge N_2 \ge 1$. This shows that $\{\Xi_N(\vec{u}_0, \omega_2)\}_{N\in\mathbb{N}}$ is Cauchy in measure with respect to $\vec{\rho} \otimes \mathbb{P}_2$ and thus converges in measure to the full enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.11). By Fatou's lemma and (6.56), we also have

$$\int \exp\Big(N_2^\beta \|\Xi(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T}\Big) d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2) \lesssim 1,$$

uniformly in $N_1 \ge N_2 \ge 1$, which in turn implies

$$\vec{\rho} \otimes \mathbb{P}_2\Big(\|\Xi(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|^\alpha_{\mathcal{X}^\varepsilon_T} > \lambda\Big) \le Ce^{-cN_2^\beta \lambda^\alpha}$$

for any $\lambda > 0$ and $N_2 \in \mathbb{N}$. By summing in $N_2 \in \mathbb{N}$ and invoking the Borel-Cantelli lemma, we also conclude almost sure convergence $\Xi_N(\vec{u}_0, \omega_2)$ to $\Xi(\vec{u}_0, \omega_2)$ with respect to $\vec{\rho} \otimes \mathbb{P}_2$. □

6.2. **Proof of Theorem 1.15.** In this subsection, we present the proof of Theorem 1.15. The main task is to prove convergence of the solution $(u_N, \partial_t u_N)$ to the truncated hyperbolic Φ^3_3-model (6.2). We first carry out Steps 2, 3, and 4 described at the beginning of this section. Namely, we first establish a stability result (Proposition 6.8) as a slight modification of the local well-posedness argument (Theorem 5.1). Next, we establish a uniform (in N) control on the solution $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated system (see (6.57) below) with respect to the truncated measure $\rho_N \times \mathbb{P}_2$ (Proposition 6.9). Then, by using ideas from theory of optimal transport, we study the convergence property of the pushforward measure $(\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ to $(\Xi)_\#(\vec{\rho} \otimes \mathbb{P}_2)$ with respect to the Wasserstein-1 distance (Proposition 6.10).

Let $\Phi_1^N(t)(\vec{u}_0,\omega_2)$ be the first component of $\Phi^N(t)(\vec{u}_0,\omega_2)$ in (6.21). Then, by decomposing $\Phi_1^N(t)(\vec{u}_0,\omega_2)$ as in (5.11):

$$\Phi_1^N(t)(\vec{u}_0,\omega_2) = \uparrow(t;\vec{u}_0,\omega_2) + \sigma \curlyvee_N(t;\vec{u}_0,\omega_2) + X_N(t) + Y_N(t), \tag{6.57}$$

we see that X_N, Y_N, and $\mathfrak{R}_N := X_N \circleddash \uparrow_N(\vec{u}_0,\omega_2)$ satisfy the following system:

$$\begin{aligned}
&(\partial_t^2+\partial_t+1-\Delta)X_N \\
&\quad = 2\sigma\pi_N\Big(\big(X_N+Y_N+\sigma\curlyvee_N\big) \olessthan \uparrow_N\Big) \\
&\quad\quad - M(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\curlyvee_N^2 + 2\sigma\curlyvee\!\!\!\!\cdot_N + \vee_N)\uparrow_N, \\
&(\partial_t^2+\partial_t+1-\Delta)Y_N \\
&\quad = \sigma\pi_N\Big(\big(X_N+Y_N+\sigma\curlyvee_N\big)^2 + 2\big(\mathfrak{R}_N + Y_N \circleddash \uparrow_N + \sigma \curlyvee_{\!\circ N}\big) \\
&\quad\quad + 2\big(X_N+Y_N+\sigma\curlyvee_N\big) \ogreaterthan \uparrow_N\Big) \\
&\quad\quad - M(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\curlyvee_N^2 + 2\sigma\curlyvee\!\!\!\!\cdot_N + \vee_N)(X_N+Y_N+\sigma\curlyvee_N), \\
&\mathfrak{R}_N = 2\sigma\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}\big(X_N+Y_N+\sigma\curlyvee_N\big) \circleddash \uparrow_N \\
&\quad\quad + 2\sigma\widetilde{\mathfrak{I}}^{N}_{\olessthan,\circleddash}\big(X_N+Y_N+\sigma\curlyvee_N\big) \\
&\quad\quad - \int_0^t M(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\curlyvee_N^2 + 2\sigma\curlyvee\!\!\!\!\cdot_N + \vee_N)(t')\mathbb{A}_N(t,t')dt', \\
&(X_N,\partial_t X_N, Y_N, \partial_t Y_N)|_{t=0} = (0,0,0,0),
\end{aligned} \tag{6.58}$$

where M is as in (1.34), Q_{X_N,Y_N} is as in (5.16) with $\uparrow$ replaced by $\uparrow_N = \uparrow_N(\vec{u}_0,\omega_2)$ as in (6.6), and the enhanced data set is given by $\Xi_N(\vec{u}_0,\omega_2)$ in (6.10).

We first establish the following stability result. The main idea is that by introducing a norm with an exponential decaying weight in time (see (6.63)), the proof essentially follows from a straightforward modification of the local well-posedness argument (Theorem 5.1). A simple, but key observation is (6.65) below.

Proposition 6.8. *Let $T \gg 1$, $K \gg 1$, and $C_0 \gg 1$. Then, there exist $N_0(T,K,C_0) \in \mathbb{N}$ and small $\kappa_0 = \kappa_0(T,K,C_0) > 0$ such that the following statements hold. Suppose that for some $N \geq N_0$, we have*

$$\|\Xi_N(\vec{u}'_0,\omega'_2)\|_{\mathcal{X}^\varepsilon_T} \leq K \tag{6.59}$$

and

$$\|(X_N,Y_N,\mathfrak{R}_N)\|_{Z^{s_1,s_2,s_3}(T)} \leq C_0 \tag{6.60}$$

for the solution to $(X_N,Y_N,\mathfrak{R}_N)$ to the truncated system (6.58) on $[0,T]$ with the truncated enhanced data set $\Xi_N(\vec{u}'_0,\omega'_2)$. Furthermore, suppose that we have

$$\|\Xi(\vec{u}_0,\omega_2) - \Xi_N(\vec{u}'_0,\omega'_2)\|_{\mathcal{X}^\varepsilon_T} \leq \kappa \tag{6.61}$$

for some $0 < \kappa \leq \kappa_0$ and some $(\vec{u}_0,\omega_2)$, where $\Xi(\vec{u}_0,\omega_2)$ denotes the enhanced data set in (6.11). Then, there exists a solution $(X,Y,\mathfrak{R})$ to the full system (5.28) on $[0,T]$ with the zero initial data and the enhanced data set $\Xi(\vec{u}_0,\omega_2)$, satisfying the bound

$$\|(X,Y,\mathfrak{R})\|_{Z^{s_1,s_2,s_3}(T)} \leq C_0 + 1.$$

Conversely, suppose that

$$\|\Xi(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T} \le K$$

and that the full system (5.28) *with the zero initial data and the enhanced data set* $\Xi(\vec{u}_0, \omega_2)$ *has a solution* $(X, Y, \mathfrak{R})$ *on* $[0, T]$*, satisfying*

$$\|(X, Y, \mathfrak{R})\|_{Z^{s_1,s_2,s_3}(T)} \le C_0.$$

Then, if (6.61) *holds for some* $N \ge N_0$*,* $0 < \kappa \le \kappa_0$*, and* $(\vec{u}_0', \omega_2')$*, then there exists a solution* $(X_N, Y_N, \mathfrak{R}_N)$ *to the truncated system* (6.58) *on* $[0, T]$ *with the enhanced data set* $\Xi_N(\vec{u}_0', \omega_2')$*, satisfying*

$$\|(X_N, Y_N, \mathfrak{R}_N) - (X, Y, \mathfrak{R})\|_{Z^{s_1,s_2,s_3}(T)} \le A(T, K, C_0)(\kappa + N^{-\delta}) \tag{6.62}$$

for some $A(T, K, C_0) > 0$ *and some small* $\delta > 0$*.*

Proof. Fix $T \gg 1$. Given $\lambda \ge 1$ (to be determined later), we define $\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)$ by

$$\|(X, Y, \mathfrak{R})\|_{\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)} = \|(e^{-\lambda t}X, e^{-\lambda t}Y, e^{-\lambda t}\mathfrak{R})\|_{Z^{s_1,s_2,s_3}(T)}. \tag{6.63}$$

For notational simplicity, we set $Z = (X, Y, Z)$, $Z_N = (X_N, Y_N, \mathfrak{R}_N)$, $\Xi = \Xi(\vec{u}_0, \omega_2)$, and $\Xi_N = \Xi_N(\vec{u}_0', \omega_2')$.

In the following, given $N \in \mathbb{N}$, we assume that (6.59), (6.60), and (6.61) hold. Without loss of generality, assume that $\kappa \le 1$. Then, from (6.59) and (6.61), we have

$$\|\Xi(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T} \le K + \kappa \le K + 1 =: K_0. \tag{6.64}$$

In the following, we study the difference of the Duhamel formulation[28] (5.48) of the system (5.28) with the zero initial data (i.e. $(X_0, X_1, Y_0, Y_1) = (0, 0, 0, 0)$) and the Duhamel formulation of the truncated system (6.58) with respect to the $\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)$-norm by choosing appropriate $\lambda = \lambda(T, K_0, R) \gg 1$. See (6.72) below.

The main observation is the following bound:

$$e^{-\lambda t}\|e^{\lambda t'}\|_{L^q_{t'}([0,t])} \lesssim \lambda^{-\frac{1}{q}}. \tag{6.65}$$

Let $\mathcal{I}$ be the Duhamel integral operator defined in (5.6). Then, using (6.65), we have

$$\begin{aligned} \|e^{-\lambda t}\mathcal{I}(F)\|_{C_T H^s_x} &\le \bigg\| e^{-\lambda t}\int_0^t e^{\lambda t'}\|e^{-\lambda t'}F(t')\|_{H^{s-1}_x} dt' \bigg\|_{L^\infty_T} \\ &\lesssim \lambda^{-\frac{1}{q}}\|e^{-\lambda t'}F(t')\|_{L^{q'}_T H^{s-1}_x} \end{aligned} \tag{6.66}$$

for any $1 \le q \le \infty$. Let (q_1, r_1) be an s_1-admissible pair with $0 < s_1 < 1$. Then, there exists an s_2-admissible pair (q_2, r_2) with $0 < s_1 < s_2 < 1$ such that

$$\frac{1}{q_1} = \frac{\theta}{\infty} + \frac{1-\theta}{q_2}, \quad \frac{1}{r_1} = \frac{\theta}{2} + \frac{1-\theta}{r_2}, \quad \text{and} \quad s_1 = \theta \cdot 0 + (1-\theta)s_2$$

[28]Recall that we set $\sigma = 1$ in Section 5 for simplicity and thus need to insert σ in appropriate locations of (5.48).

for some $0 < \theta < 1$. By the homogeneous Strichartz estimate ((5.36) with $F = 0$), we have

$$
\begin{aligned}
\|e^{-\lambda t}\mathcal{I}(F)\|_{L^{q_2}_T L^{r_2}_x} &\le \bigg\| \int_0^t e^{-\lambda(t-t')}\mathcal{D}(t-t')(e^{-\lambda t'}F(t'))dt' \bigg\|_{L^{q_2}_T L^{r_2}_x} \\
&\le \int_0^T \|\mathcal{D}(t-t')(e^{-\lambda t'}F(t'))\|_{L^{q_2}_t([0,T];L^{r_2}_x)}dt' \\
&\lesssim \|e^{-\lambda t'}F(t')\|_{L^1_T H^{s_2-1}_x}.
\end{aligned}
\tag{6.67}
$$

Thus, given any $\delta > 0$, it follows from interpolating (6.66) with large $q \gg 1$ and (6.67) that there exists small $\theta = \theta(\delta) > 0$ such that

$$
\|e^{-\lambda t}\mathcal{I}(F)\|_{L^{q_1}_T L^{r_1}_x} \le C(T)\lambda^{-\theta}\|e^{-\lambda t'}F(t')\|_{L^{1+\delta}_T H^{s_1-1}_x}. \tag{6.68}
$$

Recalling that $(4,4)$ is $\frac{1}{2}$-admissible, it follows from (6.66), (6.68), and Sobolev's inequality that

$$
\begin{aligned}
\|e^{-\lambda t}\mathcal{I}(F)\|_{C_T\mathcal{H}^{\frac{1}{2}}_x \cap L^4_T L^4_x} &\le C(T)\lambda^{-\theta}\|e^{-\lambda t'}F(t')\|_{L^{1+\delta}_T H^{-\frac{1}{2}}_x} \\
&\le C(T)\lambda^{-\theta}\|e^{-\lambda t'}F(t')\|_{L^{1+\delta}_T L^{\frac{3}{2}}_x}.
\end{aligned}
\tag{6.69}
$$

By writing (6.58) in the Duhamel formulation, we have

$$
\begin{aligned}
X_N &= \Phi_{1,N}(X_N, Y_N, \mathfrak{R}_N) \\
&:= 2\sigma\pi_N\mathcal{I}\Big(\big(X_N + Y_N + \sigma\mathsf{Y}_N\big) \varolessthan \uparrow_N\Big) \\
&\quad - \mathcal{I}\Big(M(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\mathsf{Y}^2_N + 2\sigma\Psi_N + \vee_N)\uparrow_N\Big), \\
Y_N &= \Phi_{2,N}(X_N, Y_N, \mathfrak{R}_N) \\
&:= \sigma\pi_N\mathcal{I}\Big(\big(X_N + Y_N + \sigma\mathsf{Y}_N\big)^2\Big) + 2\sigma\pi_N\mathcal{I}\big(\mathfrak{R}_N + Y_N \ominus \uparrow_N + \sigma\Psi^{\ominus}_N\big) \\
&\quad + 2\sigma\pi_N\mathcal{I}\Big(\big(X_N + Y_N + \sigma\mathsf{Y}_N\big) \varogreaterthan \uparrow_N\Big) \\
&\quad - \mathcal{I}\Big(M(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\mathsf{Y}^2_N + 2\sigma\Psi_N + \vee_N)(X_N + Y_N + \sigma\mathsf{Y}_N)\Big), \\
\mathfrak{R}_N &= \Phi_{3,N}(X_N, Y_N, \mathfrak{R}_N), \\
&:= 2\sigma\widetilde{\mathfrak{I}}^{(1),N}_{\varolessthan}\big(X_N + Y_N + \sigma\mathsf{Y}_N\big) \ominus \uparrow_N \\
&\quad + 2\sigma\widetilde{\mathfrak{I}}^N_{\varolessthan,\ominus}\big(X_N + Y_N + \sigma\mathsf{Y}_N\big) \\
&\quad - \int_0^t M(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\mathsf{Y}^2_N + 2\sigma\Psi_N + \vee_N)(t')\mathbb{A}_N(t,t')dt'.
\end{aligned}
\tag{6.70}
$$

Then, $Z - Z_N = (X - X_N, Y - Y_N, \mathfrak{R} - \mathfrak{R}_N)$ satisfies the system

$$
\begin{aligned}
X - X_N &= \Phi_1(X, Y, \mathfrak{R}) - \Phi_{1,N}(X_N, Y_N, \mathfrak{R}_N), \\
Y - Y_N &= \Phi_2(X, Y, \mathfrak{R}) - \Phi_{2,N}(X_N, Y_N, \mathfrak{R}_N), \\
\mathfrak{R} - \mathfrak{R}_N &= \Phi_3(X, Y, \mathfrak{R}) - \Phi_{3,N}(X_N, Y_N, \mathfrak{R}_N).
\end{aligned}
\tag{6.71}
$$

By setting

$$
\delta X_N = X - X_N, \quad \delta Y_N = Y - Y_N, \quad \text{and} \quad \delta\mathfrak{R}_N = \mathfrak{R} - \mathfrak{R}_N,
$$

we have

$$X = \delta X_N + X_N, \quad Y = \delta Y_N + Y_N, \quad \text{and} \quad \mathfrak{R} = \delta\mathfrak{R}_N + \mathfrak{R}_N.$$

Then, we can view the system (6.71) for the system for the unknown

$$\delta Z_N = (\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)$$

with given source terms $Z_N = (X_N, Y_N, Z_N)$, Ξ_N, and Ξ. We thus rewrite (6.71) as

$$\begin{aligned} \delta X_N &= \Psi_1(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N), \\ \delta Y_N &= \Psi_2(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N), \\ \delta\mathfrak{R}_N &= \Psi_3(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N), \end{aligned} \tag{6.72}$$

where Ψ_j, $j = 1, 2, 3$, is given by

$$\begin{aligned} &\Psi_j(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N) \\ &\quad = \Phi_j(\delta X_N + X_N, \delta Y_N + Y_N, \delta\mathfrak{R}_N + \mathfrak{R}_N) - \Phi_{j,N}(X_N, Y_N, \mathfrak{R}_N). \end{aligned} \tag{6.73}$$

We now study the system (6.72). We basically repeat the computations in Subsection 5.4 by first multiplying the Duhamel formulation by $e^{-\lambda t}$ and using (6.66), (6.68), and (6.69) as a replacement of the Strichartz estimates (Lemma 5.3). This allows us to place $e^{-\lambda t'}$ on one of the factors of $\delta X_N(t')$, $\delta Y_N(t')$, or $\delta\mathfrak{R}_N(t')$ appearing on the right-hand side of (6.72) under some integral operator (with integration in the variable t'). Our main goal is to prove that

$$\vec{\Psi} = (\Psi_1, \Psi_2, \Psi_3) \tag{6.74}$$

is a contraction on a small ball in $\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)$. In the following, however, we first establish bounds on Ψ_j in (6.73) for $\delta Z_N \in B_1$, where $B_1 \subset Z^{s_1,s_2,s_3}(T)$ denotes the closed ball of radius 1 (with respect to the $Z^{s_1,s_2,s_3}(T)$-norm) centered at the origin. For $\delta Z_N \in B_1$, it follows from (6.60) that

$$\begin{aligned} \|Z\|_{Z^{s_1,s_2,s_3}(T)} &\le \|\delta Z_N\|_{Z^{s_1,s_2,s_3}(T)} + \|Z_N\|_{Z^{s_1,s_2,s_3}(T)} \\ &\le 1 + C_0 =: R. \end{aligned} \tag{6.75}$$

We first study the first equation in (6.72). From (6.73) with (5.48), (6.70), and (6.73), we have

$$e^{-\lambda t}\Psi_1(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)(t) = e^{-\lambda t}\mathrm{I}_1(t) + e^{-\lambda t}\mathrm{I}_2(t) + e^{-\lambda t}\mathrm{I}_3(t), \tag{6.76}$$

where (i) I_1 contains the difference of one of the elements in the enhanced data sets Ξ and Ξ_N, (ii) I_2 contains the terms with the high frequency projection $\pi_N^\perp = \mathrm{Id} - \pi_N$ onto the frequencies $\{|n| \gtrsim N\}$, and (iii) I_3 consists of the rest, which contains at least one of the differences δX_N, δY_N, or $\delta\mathfrak{R}_N$ (other than those in $Z = \delta Z_N + Z_N$).

In view of (6.61), the contribution from I_1 gives a small number κ, while the contribution from I_2 with $\pi_N^\perp$ gives a small negative power of N by losing a small amount of regularity.[29] Proceeding as in (5.53) with (6.59), (6.60), (6.61), (6.64), and (6.75), we have

$$\begin{aligned} \|e^{-\lambda t}\mathrm{I}_1 + e^{-\lambda t}\mathrm{I}_2\|_{X^{s_1}(T)} &\le C(T)(\kappa + N^{-\delta}K_0)(R^4 + K_0^4) \\ &\le C(T)(\kappa + N^{-\delta})K_0(R^4 + K_0^4) \end{aligned} \tag{6.77}$$

[29] We have sharp inequalities in (6.12) as compared to the regularity condition in Theorem 5.1. This allows us to gain a small negative power of N, by losing a small amount of regularity and using $\pi_N^\perp$.

for any $\delta Z_N \in B_1$ and some small $\delta > 0$. As for the last term on the right-hand side of (6.76), we use (6.66) and (6.68) in place of Lemma 5.3. Then, a slight modification of (5.53) yields

$$\|e^{-\lambda t}\mathrm{I}_3\|_{X^{s_1}(T)} \le C(T)\lambda^{-\theta}K_0\Big(R^3\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} + K_0^4\Big) \tag{6.78}$$

for any $\delta Z_N \in B_1$.

Next, we study the second equation in (6.72). As in (6.76), we can write

$$e^{-\lambda t}\Psi_2(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)(t) = e^{-\lambda t}\mathrm{II}_1(t) + e^{-\lambda t}\mathrm{II}_2(t) + e^{-\lambda t}\mathrm{II}_3(t), \tag{6.79}$$

where (i) II_1 contains the difference of one of the elements in the enhanced data sets Ξ and Ξ_N, (ii) II_2 contains the terms with the high frequency projection $\pi_N^\perp = \mathrm{Id} - \pi_N$ onto the frequencies $\{|n| \gtrsim N\}$, and (iii) II_3 consists of the rest, which contains at least one of the differences δX_N, δY_N, or $\delta\mathfrak{R}_N$ (other than those in $Z = \delta Z_N + Z_N$). As for the first two terms on the right-hand side of (6.79), we can proceed as in (5.55) with (6.59), (6.60), (6.61), (6.64), and (6.75), and obtain

$$\|e^{-\lambda t}\mathrm{II}_1 + e^{-\lambda t}\mathrm{II}_2\|_{Y^{s_2}(T)} \le C(T)(\kappa + N^{-\delta})(R^5 + K_0^5) \tag{6.80}$$

for any $\delta Z_N \in B_1$ and some small $\delta > 0$. Before we proceed to study the last term $e^{-\lambda t}\mathrm{II}_3(t)$, let us make a preliminary computation. By the fractional Leibniz rule (Lemma 2.3 (i)) and Sobolev's inequality, we have

$$\begin{aligned} \|\langle\nabla\rangle^{s_2-\frac12}(fg)\|_{L^{\frac32}} &\lesssim \|\langle\nabla\rangle^{s_2-\frac12}f\|_{L^{r_1}}\|g\|_{L^{r_2}} + \|f\|_{L^{r_2}}\|\langle\nabla\rangle^{s_2-\frac12}g\|_{L^{r_1}} \\ &\lesssim \|\langle\nabla\rangle^{s_1-\frac14}f\|_{L^{\frac83}}\|\langle\nabla\rangle^{s_1-\frac14}g\|_{L^{\frac83}}, \end{aligned} \tag{6.81}$$

provided that $\frac{1}{r_1} + \frac{1}{r_2} = \frac23$ with $1 < r_1, r_2 < \infty$,

$$\frac{s_1 - s_2 + \frac14}{3} \ge \frac38 - \frac{1}{r_1} \qquad \text{and} \qquad \frac{s_1 - \frac14}{3} \ge \frac38 - \frac{1}{r_2}. \tag{6.82}$$

This condition is easily satisfied by taking $s_1 < \frac12 < s_2$ both sufficiently close to $\frac12$ and $r_1 = r_2 = 3$. By (6.69), (6.81), and Lemma 2.3 (i), we have

$$\begin{aligned} &\big\|e^{-\lambda t}\mathcal{I}\big((X_1 + Y_1 + \Xi_0)(X_2 + Y_2 + \Xi_0)\big)\big\|_{Y^{s_2}(T)} \\ &\quad \le C(T)\lambda^{-\theta}\big\|e^{-\lambda t}\langle\nabla\rangle^{s_2-\frac12}\big((X_1 + Y_1 + \Xi_0)(X_2 + Y_2 + \Xi_0)\big)\big\|_{L_T^{1+\delta}L_x^{\frac32}} \\ &\quad \le C(T)\lambda^{-\theta}\Big(\|\langle\nabla\rangle^{s_1-\frac14}X_1\|_{L_T^8L_x^{\frac83}} + \|\langle\nabla\rangle^{s_2-\frac12}Y_1\|_{L^4_{T,x}} + \|\langle\nabla\rangle^{s_2-\frac12}\Xi_0\|_{L^\infty_{T,x}}\Big) \\ &\quad\quad \times \Big(\|e^{-\lambda t}\langle\nabla\rangle^{s_1-\frac14}X_2\|_{L_T^8L_x^{\frac83}} + \|e^{-\lambda t}\langle\nabla\rangle^{s_2-\frac12}Y_2\|_{L^4_{T,x}} + \|\langle\nabla\rangle^{s_2-\frac12}\Xi_0\|_{L^\infty_{T,x}}\Big), \end{aligned} \tag{6.83}$$

provided that $s_1 < \frac12 < s_2$ are both sufficiently close to $\frac12$. Compare this with (5.54). Then, from (6.66), (6.68), and (6.83) with (6.59), (6.60), (6.64), and (6.75), a slight modification of (5.55) yields

$$\|e^{-\lambda t}\mathrm{II}_3\|_{Y^{s_2}(T)} \le C(T)\lambda^{-\theta}\Big(R^4\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} + K_0^5\Big) \tag{6.84}$$

for any $\delta Z_N \in B_1$.

Finally, we study the third equation in (6.72). As in (6.76) and (6.79), we can write

$$e^{-\lambda t}\Psi_3(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)(t) = e^{-\lambda t}\mathrm{III}_1(t) + e^{-\lambda t}\mathrm{III}_2(t) + e^{-\lambda t}\mathrm{III}_3(t), \tag{6.85}$$

where (i) III_1 contains the difference of one of the elements in the enhanced data sets Ξ and Ξ_N, (ii) III_2 contains the terms with the high frequency projection $\pi_N^\perp = \mathrm{Id} - \pi_N$ onto the frequencies $\{|n| \gtrsim N\}$, and (iii) III_3 consists of the rest, which contains at least one of the differences δX_N, δY_N, or $\delta \mathfrak{R}_N$ (other than those in $Z = \delta Z_N + Z_N$). Proceeding as in (5.56) with (6.59), (6.60), (6.61), (6.64), and (6.75), we have

$$\|e^{-\lambda t}\text{III}_1 + e^{-\lambda t}\text{III}_2\|_{L^3_T H^{s_3}_x} \le C(T)(\kappa + N^{-\delta})K_0(R^4 + K_0^4) \tag{6.86}$$

for any $\delta Z_N \in B_1$ and some small $\delta > 0$. As for the last term on the right-hand side of (6.85), let us fist consider the terms with the random operator $\mathfrak{I}_{\otimes,\ominus}$. By (6.64) and (6.65), we have

$$\begin{aligned}
&\big\|e^{-\lambda t}\,\mathfrak{I}_{\otimes,\ominus}\big(X_1 + Y_1 + \Xi_0\big)(t) - e^{-\lambda t}\,\mathfrak{I}_{\otimes,\ominus}\big(X_2 + Y_2 + \Xi_0\big)(t)\big\|_{L^3_T H^{s_3}_x} \\
&\le K_0 \Big\| e^{-\lambda t} \|e^{\lambda t'}(e^{-\lambda t'}(X_1 + Y_1 - X_2 - Y_2))\|_{L^{\frac32}_{t'}([0,t];L^2_x)} \Big\|_{L^3_T} \\
&\le C(T)\lambda^{-\theta} K_0 \Big(\|e^{-\lambda t}(X_1 - X_2)\|_{L^\infty_T H^{s_1}_x} + \|e^{-\lambda t}(Y_1 - Y_2)\|_{L^\infty_T H^{s_2}_x} \Big)
\end{aligned}$$

for some $\theta > 0$. The other terms can be estimated in a similar manner and thus we obtain

$$\|e^{-\lambda t}\text{III}_3\|_{L^3_T H^{s_3}_x} \le C(T)\lambda^{-\theta} K_0 \Big(R^3 \|\delta Z_N\|_{\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)} + K_0^4 \Big) \tag{6.87}$$

for any $\delta Z_N \in B_1$.

Hence, putting (6.77), (6.78), (6.80), (6.84), (6.86), and (6.87) together, we obtain

$$\begin{aligned}
\|\vec{\Psi}(\delta Z_N)\|_{\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)} &\le C(T, K_0, R)\lambda^{-\theta} \|\delta Z_N\|_{\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)} \\
&\quad + C(T, K_0, R)(\kappa + N^{-\delta})
\end{aligned} \tag{6.88}$$

for any $\delta Z_N \in B_1$, where $\vec{\Psi}$ is as in (6.74). By a similar computation, we also obtain the difference estimate:

$$\|\vec{\Psi}(\delta Z_N^{(1)}) - \vec{\Psi}(\delta Z_N^{(2)})\|_{\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)} \le C(T, K_0, R)\lambda^{-\theta} \|\delta Z_N^{(1)} - \delta Z_N^{(2)}\|_{\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)} \tag{6.89}$$

for any $\delta Z_N^{(1)}, \delta Z_N^{(2)} \in B_1$. We now introduce small $r = r(T, \lambda) > 0$ such that, in view of (6.63), we have

$$\|\delta Z_N\|_{Z^{s_1,s_2,s_3}(T)} \le e^{\lambda T} \|\delta Z_N\|_{\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)} \le e^{\lambda T} r \le 1 \tag{6.90}$$

for any $\delta Z_N \in B_r^\lambda$, where $B_r^\lambda \subset \mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)$ is the closed ball of radius r (with respect to the $\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)$-norm) centered at the origin. From (6.90), we see that both (6.88) and (6.89) hold on B_r^λ. Therefore, by choosing large $\lambda = \lambda(T, K_0, R) \gg 1$, small $\kappa = \kappa(T, K_0, R) > 0$, and large $N_0 = N_0(T, K_0, R) \in \mathbb{N}$, we conclude that $\vec{\Psi}$ is a contraction on B_r^λ for any $N \ge N_0$. Hence, there exists a unique solution $\delta Z_N \in B_r^\lambda$ to the fixed point problem $\delta Z_N = \vec{\Psi}(\delta Z_N)$. We need to check that by setting $Z = \delta Z_N + Z_N$, Z satisfies the Duhamel formulation (5.48) of the full system (5.28) with the zero initial data and the enhanced data set $\Xi = \Xi(\vec{u}_0, \omega_2)$. From (6.72) and (6.70), we have

$$\begin{aligned}
Z = \delta Z_N + Z_N &= \vec{\Psi}(\delta Z_N) + \vec{\Phi}_N(Z_N) \\
&= \vec{\Phi}(\delta Z_N + Z_N) = \vec{\Phi}(Z),
\end{aligned}$$

where $\vec{\Phi}_N = (\Phi_{1,N}, \Phi_{2,N}, \Phi_{3,N})$. This shows that Z indeed satisfies the Duhamel formulation (5.48) with the zero initial data and the enhanced data set $\Xi = \Xi(\vec{u}_0, \omega_2)$. Lastly, we point out that from (6.64) and (6.75), we have $K_0 = K + 1$ and $R = C_0 + 1$ and thus the parameters λ, κ, and N_0 depend on T, K, and C_0.

As for the second claim in this proposition, we write $Z_N = Z - (Z - Z_N)$ and study the system for $\delta Z_N = Z - Z_N$:

$$\delta Z_N = \vec{\Psi}^N(\delta Z_N)$$

where $\vec{\Psi}^N = (\Psi_1^N, \Psi_2^N, \Psi_3^N)$ and Ψ_j^N, $j = 1, 2, 3$, is given by

$$\begin{aligned}
&\Psi_j^N(\delta X_N, \delta Y_N, \delta \mathfrak{R}_N) \\
&\quad = \Phi_j(X, Y, \mathfrak{R}) - \Phi_{j,N}(X - \delta X_N, Y - \delta Y_N, \mathfrak{R} - \delta \mathfrak{R}_N).
\end{aligned}$$

Here, we view $Z = (X, Y, Z)$, Ξ_N, and Ξ as given source terms. By a slight modification of the computation presented above, we obtain

$$\begin{aligned}
\|\vec{\Psi}^N(\delta Z_N)\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} &\le C(T, K_0, R)\lambda^{-\theta}\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} \\
&\quad + C(T, K_0, R)(\kappa + N^{-\delta})
\end{aligned} \tag{6.91}$$

and

$$\|\vec{\Psi}^N(\delta Z_N^{(1)}) - \vec{\Psi}^N(\delta Z_N^{(2)})\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} \le C(T, K_0, R)\lambda^{-\theta}\|\delta Z_N^{(1)} - \delta Z_N^{(2)}\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)}$$

for any $\delta Z_N, \delta Z_N^{(1)}, \delta Z_N^{(2)} \in B_1$. This shows that there exists a solution

$$Z_N = Z - \delta Z_N = \Phi(Z) - \vec{\Psi}^N(\delta Z_N) = \vec{\Phi}_N(Z_N)$$

to the truncated system (6.58) on $[0, T]$. Furthermore, from (6.91) with $\lambda = \lambda(T, K_0, R) \gg 1$, we have

$$\begin{aligned}
\|Z - Z_N\|_{Z^{s_1,s_2,s_3}(T)} &\le e^{\lambda T}\|\vec{\Psi}^N(\delta Z_N)\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} \\
&\le C(T, K_0, R)e^{\lambda T}(\kappa + N^{-\delta}) \longrightarrow 0,
\end{aligned}$$

as $N \to \infty$ and $\kappa \to 0$. This proves (6.62). This concludes the proof of Proposition 6.8. $\square$

Next, we prove that the solution $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated system (6.58) has a uniform bound with a large probability. The proof is based on the invariance of the truncated Gibbs measure $\vec{\rho}_N$ under the truncated hyperbolic Φ_3^3-model (6.2) (Lemma 6.4) and a discrete Gronwall argument.

Proposition 6.9. *Let $T > 0$. Then, given any $\delta > 0$, there exists $C_0 = C_0(T, \delta) \gg 1$ such that*

$$\vec{\rho}_N \otimes \mathbb{P}_2\Big(\|(X_N, Y_N, \mathfrak{R}_N)\|_{Z^{s_1,s_2,s_3}(T)} > C_0\Big) < \delta, \tag{6.92}$$

uniformly in $N \in \mathbb{N}$, where $(X_N, Y_N, \mathfrak{R}_N)$ is the solution to the truncated system (6.58) *on* $[0, T]$ *with the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in* (6.10).

Proof. Let $(u_N, \partial_t u) = \Phi^N(t)(\vec{u}_0, \omega_2)$ be a global solution to (6.2) constructed in Lemma 6.4, where $\Phi^N(t)(\vec{u}_0, \omega_2)$ is as in (6.21). Then, by the invariance of the truncated Gibbs measure $\vec{\rho}_N$ (Lemma 6.4), we have

$$\int F(\Phi^N(t)(\vec{u}_0, \omega_2))d(\vec{\rho}_N \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2) = \int F(\vec{u}_0)d\rho_N(\vec{u}_0) \tag{6.93}$$

for any bounded continuous function $F : \mathcal{C}^{-100}(\mathbb{T}^3) \times \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$ and $t \in \mathbb{R}_+$. By Minkowski's integral inequality, (6.93), (1.34), and Proposition 6.5, we have, for any finite $p \geq 1$,

$$\begin{aligned} \bigg\| \int_0^T |M(:(\pi_N u_N)^2:)(t)| dt \bigg\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N \otimes \mathbb{P}_2)} &\leq \int_0^T \|M(:(\pi_N u_0)^2:)\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N \otimes \mathbb{P}_2)} dt \\ &\leq C(T,p) < \infty, \end{aligned} \tag{6.94}$$

for any $0 \leq t \leq T$ and $p \geq 1$, uniformly in $N \in \mathbb{N}$. By defining

$$v_N := u_N - \mathord{\upharpoonleft},$$

we see that v_N satisfies the equation

$$(\partial_t^2 + \partial_t + 1 - \Delta) v_N = \sigma \pi_N\big(:(\pi_N u_N)^2: \big) - M(:(\pi_N u_N)^2:) \pi_N u_N$$

with the zero initial data, or equivalently

$$v_N(t) = \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![\nabla]\!])}{[\![\nabla]\!]} \Big(\sigma \pi_N(:(\pi_N u_N)^2:) - M(:(\pi_N u_N)^2:) \pi_N u_N \Big)(t') dt'.$$

Thus, we have

$$\begin{aligned} \|v_N(t)\|_{W_x^{-\varepsilon,\infty}} \leq \int_0^t \bigg(&\bigg\| \frac{\sin((t-t')[\![\nabla]\!])}{[\![\nabla]\!]} \sigma \pi_N(:(\pi_N u_N)^2:)(t') \bigg\|_{W_x^{-\varepsilon,\infty}} \\ &+ \bigg\| M(:(\pi_N u_N)^2:)(t') \frac{\sin((t-t')[\![\nabla]\!])}{[\![\nabla]\!]} \pi_N u_N(t') \bigg\|_{W_x^{-\varepsilon,\infty}} \bigg) dt' \end{aligned}$$

for any $t > 0$. Then, by using Minkowski's integral inequality, (6.93), and Proposition 6.5 once again, we have

$$\begin{aligned} &\Big\| \|v_N(t)\|_{W_x^{-\varepsilon,\infty}} \Big\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N \otimes \mathbb{P}_2)} \\ &\lesssim \int_0^t \bigg(\bigg\| \frac{\sin(\tau[\![\nabla]\!])}{[\![\nabla]\!]} \pi_N(:(\pi_N u_0)^2:) \bigg\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N \otimes \mathbb{P}_2; W_x^{-\varepsilon,\infty})} \\ &\qquad + \bigg\| M(:(\pi_N u_0)^2:) \frac{\sin(\tau[\![\nabla]\!])}{[\![\nabla]\!]} \pi_N u_0 \bigg\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N \otimes \mathbb{P}_2; W_x^{-\varepsilon,\infty})} \bigg) d\tau \\ &\leq C(T,p) < \infty \end{aligned} \tag{6.95}$$

for any $0 \leq t \leq T$, $p \geq 1$, and $\varepsilon > 0$, uniformly in $N \in \mathbb{N}$.

We rewrite the system (6.58) as

$$
\begin{aligned}
&(\partial_t^2 + \partial_t + 1 - \Delta) X_N = 2\sigma \pi_N (v_N \varolessthan \upharpoonleft_N) - M(:(\pi_N u_N)^2:) \upharpoonleft_N, \\
&(\partial_t^2 + \partial_t + 1 - \Delta) Y_N \\
&\quad = \sigma \pi_N \Big(v_N \big(X_N + Y_N + \sigma \Yup_N \big) + 2 \big(\mathfrak{R}_N + Y_N \circledeq \upharpoonleft_N + \sigma \Yup\!\!\!\!\circ_N \big) \\
&\qquad + 2 (X_N + Y_N + \sigma \Yup_N) \varogreaterthan \upharpoonleft_N \Big) - M(:(\pi_N u_N)^2:)(X_N + Y_N + \sigma \Yup_N), \\
&\mathfrak{R}_N = 2\sigma \widetilde{\mathfrak{I}}_{\varolessthan}^{(1),N} \big(X_N + Y_N + \sigma \Yup_N \big) \circledeq \upharpoonleft_N + 2\sigma \widetilde{\mathfrak{I}}_{\varolessthan, \circledeq}^{N} \big(X_N + Y_N + \sigma \Yup_N \big) \\
&\qquad - \int_0^t M(:(\pi_N u_N)^2:)(t') \mathbb{A}_N(t, t') dt',
\end{aligned}
\tag{6.96}
$$

where we used (5.17) (with the frequency truncations and extra σ's in appropriate places) and $v_N = \sigma \Yup + X_N + Y_N$ so that the right-hand side is linear in $(X_N, Y_N, \mathfrak{R}_N)$.

Let $\delta > 0$. In view of Proposition 6.5, we choose $K = K(T, \delta) \gg 1$ such that

$$
\vec{\rho}_N \otimes \mathbb{P}_2 \Big(\| \Xi_N(\vec{u}_0, \omega_2) \|_{\mathcal{X}_T^\varepsilon} > K \Big) < \frac{\delta}{3}, \tag{6.97}
$$

uniformly in $N \in \mathbb{N}$. We also define $L(t)$ by

$$
L(t) = 1 + \| v_N(t) \|_{W_x^{-\varepsilon, \infty}} + | M(:(\pi_N u_N)^2:)(t) |. \tag{6.98}
$$

In view of (6.94) and (6.95), we choose $L_1 = L_1(T, \delta) \gg 1$ such that

$$
\vec{\rho}_N \otimes \mathbb{P}_2 \Big(\| L \|_{L_T^3} > L_1 \Big) < \frac{\delta}{3}. \tag{6.99}
$$

In the following, we work on the set

$$
\| \Xi_N(\vec{u}_0, \omega_2) \|_{\mathcal{X}_T^\varepsilon} \le K \qquad \text{and} \qquad \| L \|_{L_T^3} \le L_1. \tag{6.100}
$$

By applying Lemma 5.3 with (5.47) and Lemma 2.2 to (6.96) and using (5.49), (6.98), and (6.100), we have

$$
\begin{aligned}
\| X_N \|_{X^{s_1}(T)} &\lesssim \int_0^T \Big(\| v_N \varolessthan \upharpoonleft_N(t) \|_{H_x^{s_1 - 1}} + | M(:(\pi_N u_N)^2:)(t) | \cdot \| \upharpoonleft_N(t) \|_{H_x^{s_1 - 1}} \Big) dt \\
&\lesssim K \int_0^T L(t) dt.
\end{aligned}
\tag{6.101}
$$

Since $s_2 < 1$, we can choose sufficiently small $\varepsilon > 0$ such that Lemma 2.3 (ii) yields

$$
\begin{aligned}
\| v_N (X_N + Y_N + \Yup_N) \|_{H_x^{s_2 - 1}} &\lesssim \| v_N \|_{W_x^{-\varepsilon, \infty}} \| X_N + Y_N + \Yup_N \|_{H_x^\varepsilon} \\
&\lesssim \| v_N \|_{W_x^{-\varepsilon, \infty}} \Big(\| X_N \|_{H_x^{s_1}} + \| Y_N \|_{H_x^{s_2}} + \| \Xi_N(\vec{u}_0, \omega_2) \|_{\mathcal{X}_T^\varepsilon} \Big).
\end{aligned}
$$

Hence, by (6.96), Lemma 5.3 with (5.47), Lemma 2.2 (see also (5.55)), (6.98), and (6.100), we have

$$\begin{aligned}
\|Y_N\|_{Y^{s_2}(T)} &\lesssim \int_0^T \Big(\|v_N(t)(X_N(t) + Y_N(t) + \Upsilon_N(t))\|_{H_x^{s_2-1}} \\
&\qquad + \|\mathfrak{R}_N(t) + Y_N(t) \circeq \uparrow_N(t) + \sigma \Upsilon^{\circ}_N(t)\|_{H_x^{s_2-1}} \\
&\qquad + \|(X_N(t) + Y_N(t) + \sigma \Upsilon_N(t)) \olessthan \uparrow_N(t)\|_{H_x^{s_2-1}} \\
&\qquad + |M(:\!(\pi_N u_N)^2\!:)(t)| \cdot \|X_N(t) + Y_N(t) + \sigma \Upsilon_N(t)\|_{H_x^{s_2-1}} \Big) dt \\
&\le C(T)K^2 + K \int_0^T L(t) \Big(1 + \|X_N\|_{X^{s_1}(T)} + \|Y_N\|_{Y^{s_2}(t)}\Big) dt \\
&\quad + \int_0^T \|\mathfrak{R}_N(t)\|_{H_x^{s_3}} dt.
\end{aligned} \tag{6.102}$$

Fix $0 < \tau < 1$ and set

$$L^q_{I_k} = L^q(I_k), \quad \text{where} \quad I_k = [k\tau, (k+1)\tau].$$

By a computation analogous to that in (5.56), we obtain

$$\begin{aligned}
\|\mathfrak{R}_N\|_{L^3_{I_k} H_x^{s_3}} &\lesssim \|\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}\big(X_N + Y_N + \sigma \Upsilon_N\big) \circeq \uparrow_N\|_{L^3_{I_k} H_x^{s_3}} \\
&\quad + \|\widetilde{\mathfrak{I}}^{N}_{\olessthan,\circeq}\big(X_N + Y_N + \sigma \Upsilon_N\big)\|_{L^3_{I_k} H_x^{s_3}} \\
&\quad + \int_0^T |M(:\!(\pi_N u_N)^2\!:)(t')| \cdot \|\mathbb{A}_N(t,t')\|_{L^3_t([t',T]; H_x^{s_3})} dt' \\
&\le C(T)K^2 \Big(K + \|X_N\|_{X^{s_1}(T)} + \|Y_N\|_{Y^{s_2}((k+1)\tau)}\Big) + K \int_0^T L(t) dt.
\end{aligned} \tag{6.103}$$

Given $0 < t \le T$, let $k_*(t)$ be the largest integer such that $k_*(t)\tau \le t$. Then, from (6.102) and (6.103), we have

$$\begin{aligned}
&\|Y_N\|_{Y^{s_2}(t)} \le \|Y_N\|_{Y^{s_2}((k_*(t)+1)\tau)} \\
&\quad \le C(T)K^2 + C_1(T)K^3 \sum_{k=0}^{k_*(t)} \tau^{\frac{2}{3}} \Big(1 + \|L(t)\|_{L^3_{I_k}}\Big)\Big(1 + \|X_N(t)\|_{X^{s_1}(T)}\Big) \\
&\qquad + C_2 K T \sum_{k=0}^{k_*(t)} \tau^{\frac{1}{3}} \|L(t)\|_{L^3_{I_k}} + C_3 K^2 \sum_{k=0}^{k_*(t)} \tau^{\frac{2}{3}} \Big(1 + \|L(t)\|_{L^3_{I_k}}\Big) \|Y_N\|_{Y^{s_2}((k+1)\tau)}.
\end{aligned} \tag{6.104}$$

Now, choose $\tau = \tau(K, L_1) = \tau(T, \delta) > 0$ sufficiently small such that

$$C_3 K^2 \tau^{\frac{2}{3}} L_1 \ll 1. \tag{6.105}$$

In view of (6.94) and (6.95), and define $L_2 = L_2(T, \delta) \gg 1$ such that

$$\vec{\rho}_N \otimes \mathbb{P}_2\bigg(\sum_{k=0}^{k_*(T)} \tau^{\frac{1}{3}} \Big(1 + \|L(t)\|_{L^3_{I_k}}\Big) > L_2 \bigg) < \frac{\delta}{3}. \tag{6.106}$$

In the following, we work on the set

$$\sum_{k=0}^{k_*(T)} \tau^{\frac{1}{3}}\Big(1 + \|L(t)\|_{L^3_{I_k}}\Big) \le L_2. \tag{6.107}$$

It follows from (6.104) with (6.100), (6.101), (6.105), and (6.107) that

$$\|Y_N\|_{Y^{s_2}((k_*(t)+1)\tau)} \le C(T)K^4 L_1 L_2 + C_4 K^2 \sum_{k=0}^{k_*(t)-1} \tau^{\frac{2}{3}} \|L(t)\|_{L^3_{I_k}} \|Y_N\|_{Y^{s_2}((k+1)\tau)}.$$

By applying the discrete Gronwall inequality with (6.107), we then obtain

$$\begin{aligned}\|Y_N\|_{Y^{s_2}(t)} &\le \|Y_N\|_{Y^{s_2}((k_*(t)+1)\tau)} \\ &\le C(T)K^4 L_1 L_2 \exp\bigg(C_4 K^2 \sum_{k=0}^{k_*(t)-1} \tau^{\frac{2}{3}} \|L(t)\|_{L^3_{I_k}}\bigg) \\ &\le C(T)K^4 L_1 L_2 \exp\big(C_4 K^2 L_2\big).\end{aligned} \tag{6.108}$$

Therefore, from (6.101) and (6.108), we have

$$\|X_N\|_{X^{s_1}(T)} + \|Y_N\|_{Y^{s_2}(T)} \le C(T)KL_1 + C(T)K^4 L_1 L_2 \exp\big(C_4 K^2 L_2\big).$$

Together with (6.103), we then obtain

$$\|(X_N, Y_N, \mathfrak{R}_N)\|_{Z^{s_1,s_2,s_3}(T)} \le C_5(T, K, L_1, L_2)$$

under the conditions (6.100) and (6.107). Hence, by choosing $C_0 = C_0(T, \delta) > 0$ in (6.92) such that $C_0 > C_5(T, K, L_1, L_2)$, we have

$$\begin{aligned}\vec{\rho}_N \otimes \mathbb{P}_2\bigg(&\big\{\|(X_N, Y_N, \mathfrak{R}_N)\|_{Z^{s_1,s_2,s_3}(T)} > C_0\big\} \cap \big\{\|\Xi_N(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T} \le K\big\} \\ &\cap \big\{\|L\|_{L^3_T} \le L_1\big\} \cap \Big\{\sum_{k=0}^{k_*(T)} \tau^{\frac{1}{3}} \|L(t)\|_{L^3_{I_k}} \le L_2\Big\}\bigg) = 0.\end{aligned} \tag{6.109}$$

Finally, the bound (6.92) follows from (6.97), (6.99) (6.106), and (6.109). □

Given a map S from a measure space (X, μ) to a space Y, we use $S_\# \mu$ to denote the image measure (the pushforward) of μ under S. Fix $T > 0$ and we set

$$\nu_N = (\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2) \qquad \text{and} \qquad \nu = \Xi_\#(\vec{\rho} \otimes \mathbb{P}_2), \tag{6.110}$$

where we view $\Xi_N = \Xi_N(\vec{u}_0, \omega_2)$ in (6.10) and $\Xi = \Xi(\vec{u}_0, \omega_2)$ in (6.11) as maps from $\mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) \times \Omega_2$ to $\mathcal{X}^\varepsilon_T$ defined in (5.49). In view of the weak convergence of $\vec{\rho}_N \otimes \mathbb{P}_2$ to $\vec{\rho} \otimes \mathbb{P}_2$ (Theorem 1.8 (i)) and the $\vec{\rho} \otimes \mathbb{P}_2$-almost sure convergence of $\Xi_N(\vec{u}_0, \omega_2)$ to $\Xi(\vec{u}_0, \omega_2)$ (Corollary 6.7), we see that ν_N converges weakly to ν. Indeed, given a bounded continuous function $F : \mathcal{X}_T^\varepsilon \to \mathbb{R}$, by the dominated convergence theorem, we have

$$\begin{aligned}
&\bigg| \int F(\Xi) d\nu_N - \int F(\Xi) d\nu \bigg| \\
&\quad = \bigg| \int F(\Xi_N(\vec{u}_0, \omega_2)) d(\vec{\rho}_N \otimes \mathbb{P}_2) - \int F(\Xi(\vec{u}_0, \omega_2)) d(\vec{\rho} \otimes \mathbb{P}_2) \bigg| \\
&\quad \le \|F\|_{L^\infty} \bigg| \int 1 \, d\big((\vec{\rho}_N \otimes \mathbb{P}_2) - (\vec{\rho} \otimes \mathbb{P}_2)\big) \bigg| \\
&\qquad + \bigg| \int \big(F(\Xi_N(\vec{u}_0, \omega_2)) - F(\Xi(\vec{u}_0, \omega_2)) \big) d(\vec{\rho} \otimes \mathbb{P}_2) \bigg| \\
&\quad \longrightarrow 0,
\end{aligned}$$

as $N \to \infty$.

Next, we prove that $\nu_N = (\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ converges to $\nu = \Xi_\#(\vec{\rho} \otimes \mathbb{P}_2)$ in the Wasserstein-1 metric. We view this problem as of Kantorovich's mass optimal transport problem and study the dual problem under the Kantorovich duality, using the Boué-Dupuis variational formula. This proposition plays a crucial role in the proof of almost sure global well-posedness and invariance of the Gibbs measure $\vec{\rho}$ presented at the end of this section.

Proposition 6.10. *Fix $T > 0$. Then, there exists a sequence $\{\mathfrak{p}_N\}_{N \in \mathbb{N}}$ of probability measures on $\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon$ with the first and second marginals ν and ν_N on $\mathcal{X}_T^\varepsilon$, respectively, namely,*

$$\int_{\Xi^2 \in \mathcal{X}_T^\varepsilon} d\mathfrak{p}_N(\Xi^1, \Xi^2) = d\nu(\Xi^1) \qquad \text{and} \qquad \int_{\Xi^1 \in \mathcal{X}_T^\varepsilon} d\mathfrak{p}_N(\Xi^1, \Xi^2) = d\nu_N(\Xi^2), \tag{6.111}$$

such that

$$\int_{\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon} \min(\|\Xi^1 - \Xi^2\|_{\mathcal{X}_T^\varepsilon}, 1) d\mathfrak{p}_N(\Xi^1, \Xi^2) \longrightarrow 0,$$

as $N \to \infty$. Namely, the total transportation cost associated to $\mathfrak{p}_N$ tends to 0 as $N \to \infty$.

Remark 6.11. In view of the weak convergence of the truncated Gibbs measure $\vec{\rho}_N$ to $\vec{\rho}$ (Theorem 1.8) and the almost sure convergence of the truncated enhanced data set Ξ_N to Ξ with respect to $\vec{\rho} \otimes \mathbb{P}_2$ (Corollary 6.7), it suffices to define $\mathfrak{p}_N = (\Xi, \Xi_N)_\#(\vec{\rho} \otimes \mathbb{P}_2)$. In the following, however, we present the full proof of Proposition 6.10, using the Kantorovich duality and the variational approach since we believe that such an argument is of general interest.

Proof of Proposition 6.10. Define a cost function $c(\Xi^1, \Xi^2)$ on $\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon$ by setting

$$c(\Xi^1, \Xi^2) = \min(\|\Xi^1 - \Xi^2\|_{\mathcal{X}_T^\varepsilon}, 1).$$

Then, define the Lipschitz norm for a function $F : \mathcal{X}_T^\varepsilon \to \mathbb{R}$ by

$$\|F\|_{\mathrm{Lip}} = \sup_{\substack{\Xi^1, \Xi^2 \in \mathcal{X}_T^\varepsilon \\ \Xi^1 \ne \Xi^2}} \frac{|F(\Xi^1) - F(\Xi^2)|}{c(\Xi^1, \Xi^2)}.$$

Note that $\|F\|_{\mathrm{Lip}} \le 1$ implies that F is bounded and Lipschitz continuous. From the Kantorovich duality (the Kantorovich-Rubinstein theorem [77, Theorem 1.14]), we have

$$\begin{aligned}&\inf_{\mathfrak{p}\in\Gamma(\nu,\nu_N)} \int_{\mathcal{X}_T^\varepsilon\times\mathcal{X}_T^\varepsilon} c(\Xi^1,\Xi^2)d\mathfrak{p}(\Xi^1,\Xi^2)\\ &\quad = \sup_{\|F\|_{\mathrm{Lip}}\le 1} \bigg(\int F(\Xi)d\nu_N(\Xi) - \int F(\Xi)d\nu(\Xi)\bigg),\end{aligned} \tag{6.112}$$

where $\Gamma(\nu,\nu_N)$ is the set of probability measures on $\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon$ with the first and second marginals ν and ν_N on $\mathcal{X}_T^\varepsilon$, respectively.

For a function F with $\|F\|_{\mathrm{Lip}} \le 1$, let

$$G := F - \inf F + 1.$$

Then, we have

$$\int F(\Xi)d\nu_N(\Xi) - \int F(\Xi)d\nu(\Xi) = \int G(\Xi)d\nu_N(\Xi) - \int G(\Xi)d\nu(\Xi). \tag{6.113}$$

Note that $\|G\|_{\mathrm{Lip}} = \|F\|_{\mathrm{Lip}} \le 1$ and $1 \le G \le 2$. Moreover, the mean value theorem yields that

$$\frac{1}{e} \le \frac{\log x - \log y}{x-y} \le 1 \tag{6.114}$$

for any $x, y \in [1,e]$ with $x \ne y$. Set $\{a\}_+ = \max(a,0)$ for any $a \in \mathbb{R}$. By (6.113) and (6.114), we obtain

$$\begin{aligned}&\int F(\Xi)d\nu_N(\Xi) - \int F(\Xi)d\nu(\Xi)\\ &\quad \lesssim \bigg\{ -\log\bigg(\int G(\Xi)d\nu(\Xi)\bigg) + \log\bigg(\int G(\Xi)d\nu_N(\Xi)\bigg)\bigg\}_+\end{aligned} \tag{6.115}$$

for any $N \in \mathbb{N}$.

Finally, define $H = \log G$. Then, from (6.114) and $1 \le G \le 2$, we have $\|H\|_{\mathrm{Lip}} \lesssim 1$. Hence, it follows from (6.112), (6.113), and (6.115) that

$$\begin{aligned}&\inf_{\mathfrak{p}\in\Gamma(\nu,\nu_N)} \int_{\mathcal{X}_T^\varepsilon\times\mathcal{X}_T^\varepsilon} c(\Xi^1,\Xi^2)d\mathfrak{p}(\Xi^1,\Xi^2)\\ &\quad \lesssim \sup_{\substack{0\le H\le 1\\ \|H\|_{\mathrm{Lip}}\lesssim 1}} \bigg\{ -\log\bigg(\int \exp(H(\Xi))d\nu(\Xi)\bigg) + \log\bigg(\int \exp(H(\Xi))d\nu_N(\Xi)\bigg)\bigg\}_+.\end{aligned}$$

Our goal is to show that the right-hand side tends 0 as $N \to \infty$. Since $\|H\|_{\mathrm{Lip}} \lesssim 1$, H is bounded and Lipschitz continuous. Then, by the weak convergence of $\{\nu_N\}_{N\in\mathbb{N}}$ to ν, it suffices to show that

$$\begin{aligned}\limsup_{N\to\infty} \sup_{\substack{0\le H\le 1\\ \|H\|_{\mathrm{Lip}}\lesssim 1}} \sup_{M\ge N} \bigg\{ &-\log\bigg(\int \exp(H(\Xi))d\nu_M(\Xi)\bigg)\\ &+ \log\bigg(\int \exp(H(\Xi))d\nu_N(\Xi)\bigg)\bigg\}_+ \le 0.\end{aligned} \tag{6.116}$$

From (6.110), (6.1), and (6.114) with $0 \le H \le 1$, we have

$$
\begin{aligned}
&\bigg\{ -\log\Big(\int \exp(H(\Xi))d\nu_M(\Xi)\Big) + \log\Big(\int \exp(H(\Xi))d\nu_N(\Xi)\Big)\bigg\}_+ \\
&\quad = \bigg\{ -\log\Big(\iiint \exp(H(\Xi_M(\vec{u}_0,\omega_2)))d\rho_M(u_0)d\mu_0(u_1)d\mathbb{P}_2(\omega_2)\Big) \\
&\qquad\quad + \log\Big(\iiint \exp(H(\Xi_N(\vec{u}_0,\omega_2)))d\rho_N(u_0)d\mu_0(u_1)d\mathbb{P}_2(\omega_2)\Big)\bigg\}_+ \\
&\quad \lesssim \bigg\{ -\iiint \exp(H(\Xi_M(\vec{u}_0,\omega_2)))d\rho_M(u_0)d\mu_0(u_1)d\mathbb{P}_2(\omega_2) \\
&\qquad\quad + \iiint \exp(H(\Xi_N(\vec{u}_0,\omega_2)))d\rho_N(u_0)d\mu_0(u_1)d\mathbb{P}_2(\omega_2)\bigg\}_+ \\
&\quad \lesssim \iint \bigg[\bigg\{ -\int \exp(H(\Xi_M(\vec{u}_0,\omega_2)))d\rho_M(u_0) \\
&\qquad\qquad + \int \exp(H(\Xi_N(\vec{u}_0,\omega_2)))d\rho_N(u_0)\bigg\}_+\bigg] d\mu_0(u_1)d\mathbb{P}_2(\omega_2) \\
&\quad \lesssim \iint \bigg[\bigg\{ -\log\Big(\int \exp(H(\Xi_M(\vec{u}_0,\omega_2)))d\rho_M(u_0)\Big) \\
&\qquad\qquad + \log\Big(\int \exp(H(\Xi_N(\vec{u}_0,\omega_2)))d\rho_N(u_0)\Big)\bigg\}_+\bigg] d\mu_0(u_1)d\mathbb{P}_2(\omega_2).
\end{aligned}
\tag{6.117}
$$

In the following, we study the integrand of the (u_1,ω_2)-integral. Thus, we fix u_1 and ω_2 and write $\Xi_N(\vec{u}_0,\omega_2) = \Xi_N(u_0,u_1,\omega_2)$ as $\Xi_N(u_0)$ for simplicity of notation. By the Boué-Dupuis variational formula (Lemma 3.1) with the change of variables (3.12), we have

$$
\begin{aligned}
&-\log\bigg(\int \exp(H(\Xi_M(u_0)))d\rho_M(u_0)\bigg) + \log\bigg(\int \exp(H(\Xi_N(u_0)))d\rho_N(u_0)\bigg) \\
&\quad = \inf_{\dot\Upsilon^M\in\mathbb{H}^1_a} \mathbb{E}\bigg[-H(\Xi_M(Y+\Upsilon^M+\sigma\mathfrak{Z}_M)) + \widehat{R}^\diamond_M(Y+\Upsilon^M+\sigma\mathfrak{Z}_M) \\
&\qquad\qquad + \frac12\int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x}dt\bigg] \\
&\qquad - \inf_{\dot\Upsilon^N\in\mathbb{H}^1_a} \mathbb{E}\bigg[-H(\Xi_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N)) + \widehat{R}^\diamond_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N) \\
&\qquad\qquad + \frac12\int_0^1 \|\dot\Upsilon^N(t)\|^2_{H^1_x}dt\bigg] \\
&\qquad + \log Z_M - \log Z_N,
\end{aligned}
\tag{6.118}
$$

where $\widehat{R}^\diamond_N$ is as in (3.33). Given $\delta>0$, let $\underline{\Upsilon}^N$ be an almost optimizer, namely,

$$
\begin{aligned}
&\inf_{\dot\Upsilon^N\in\mathbb{H}^1_a} \mathbb{E}\bigg[-H(\Xi_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N)) + \widehat{R}^\diamond_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N) + \frac12\int_0^1 \|\dot\Upsilon^N(t)\|^2_{H^1_x}dt\bigg] \\
&\quad \ge \mathbb{E}\bigg[-H(\Xi_N(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_N)) + \widehat{R}^\diamond_N(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_N) + \frac12\int_0^1 \|\dot{\underline{\Upsilon}}^N(t)\|^2_{H^1_x}dt\bigg] - \delta.
\end{aligned}
$$

Then, by choosing $\Upsilon^M = \underline{\Upsilon}^N$ and the Lipschitz continuity of H, we have

$$\begin{aligned}
&\inf_{\dot\Upsilon^M \in \mathbb{H}_a^1} \mathbb{E}\bigg[- H(\Xi_M(Y + \Upsilon^M + \sigma \mathfrak{Z}_M)) + \widehat{R}^\diamond_M(Y + \Upsilon^M + \sigma\mathfrak{Z}_M) + \frac12 \int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x} dt\bigg] \\
&\quad - \inf_{\dot\Upsilon^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[- H(\Xi_N(Y + \Upsilon^N + \sigma \mathfrak{Z}_N)) + \widehat{R}^\diamond_N(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) + \frac12 \int_0^1 \|\dot\Upsilon^N(t)\|^2_{H^1_x} dt\bigg] \\
&\quad \le \delta + \mathbb{E}\Big[H(\Xi_N(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N)) - H(\Xi_M(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_M)) \\
&\qquad\qquad + \widehat{R}^\diamond_M(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_M) - \widehat{R}^\diamond_N(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N)\Big] \\
&\quad \le \delta + \|H\|_{\mathrm{Lip}} \cdot \mathbb{E}\Big[\|\Xi_M(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N) - \Xi_N(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_M)\|_{\mathcal{X}^\varepsilon_T}\Big] \\
&\qquad\qquad + \mathbb{E}\Big[\widehat{R}^\diamond_M(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_M) - \widehat{R}^\diamond_N(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N)\Big].
\end{aligned} \tag{6.119}$$

Proceeding as in Subsection 3.3 with $0 \le H \le 1$, we have (3.76). Then, using the computations from (3.67) to (3.78) we obtain

$$\mathbb{E}\Big[\widehat{R}^\diamond_M(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_M) - \widehat{R}^\diamond_N(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N)\Big] \longrightarrow 0, \tag{6.120}$$

as $M \ge N \to \infty$. We also note that as a consequence of (3.76) with (3.24) and Lemma 3.2, we have

$$\mathbb{E}\Big[\|\underline{\Upsilon}^N\|^2_{H^1}\Big] \lesssim 1, \tag{6.121}$$

uniformly in $N \in \mathbb{N}$.

Moreover, by slightly modifying (part of) the proof of Proposition 6.5, we can show that

$$\mathbb{E}\Big[\big\|\Xi_M(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N) - \Xi_N(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_M)\big\|_{\mathcal{X}^\varepsilon_T}\Big] \longrightarrow 0, \tag{6.122}$$

as $M \ge N \to \infty$. Here, we only consider the contribution from $\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}$. The other terms in the truncated enhanced data sets can be handled in a similar manner. With the notations (6.31) and (6.32) (recall that we suppress the dependence on u_1 and ω_2), we have

$$\begin{aligned}
&\widetilde{\mathfrak{I}}^M_{\otimes,\ominus}[\uparrow(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_M)] - \widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\uparrow(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N)] \\
&\quad = \widetilde{\mathfrak{I}}^M_{\otimes,\ominus}[\uparrow(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_M), \uparrow(\sigma(\mathfrak{Z}_M - \mathfrak{Z}_N))] \\
&\qquad + \widetilde{\mathfrak{I}}^M_{\otimes,\ominus}[\uparrow(\sigma(\mathfrak{Z}_M - \mathfrak{Z}_N)), \uparrow(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N)] \\
&\qquad + \Big(\widetilde{\mathfrak{I}}^M_{\otimes,\ominus}[\uparrow(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N)] - \widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\uparrow(Y + \underline{\Upsilon}^N + \sigma\mathfrak{Z}_N)]\Big) \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{6.123}$$

It follows from (6.41), (6.42), and (6.45) together with Remark 5.5 that there exists small $\delta_0 > 0$ such that

$$\begin{aligned}
&\|\mathrm{I}\|_{\mathcal{L}_2(q,T)} + \|\mathrm{II}\|_{\mathcal{L}_2(q,T)} \\
&\quad \le C(T)\Big(\|Y\|_{L^\infty_T W^{-\frac12-\varepsilon,\infty}_x} + \|\underline{\Upsilon}^N\|_{H^1} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\Big)\|\mathfrak{Z}_N - \mathfrak{Z}_M\|_{W^{1-\varepsilon,\infty}} \\
&\quad \le C(T) N^{-\delta_0}\Big(\|Y\|_{L^\infty_T W^{-\frac12-\varepsilon,\infty}_x} + \|\underline{\Upsilon}^N\|_{H^1} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\Big)^2 \\
&\qquad + N^{\delta_0}\|\mathfrak{Z}_N - \mathfrak{Z}_M\|^2_{W^{1-\varepsilon,\infty}}
\end{aligned} \tag{6.124}$$

and

$$\mathbb{E}\Big[N^{\delta_0}\|\mathfrak{Z}_N - \mathfrak{Z}_M\|^2_{W^{1-\varepsilon,\infty}}\Big] \longrightarrow 0, \tag{6.125}$$

as $M \geq N \to \infty$. From (6.29) and (6.31), we have

$$\widetilde{\mathfrak{I}}^N_{\circledcirc,\circleddash}[\psi_1,\psi_2](w) = \mathcal{I}\big(\pi_N(\mathcal{K}^\theta(w,\pi_N\psi_1))\big) \circleddash (\pi_N\psi_2).$$

Hence, when we consider the difference in III, we see that one of the factors comes with $\pi_M - \pi_N$, from which we can gain a small negative power of N. Hence, by repeating the calculation above with this observation, we obtain

$$\begin{aligned}&\big\|\mathrm{III} - (\widetilde{\mathfrak{I}}^N_{\circledcirc,\circleddash}[\uparrow(Y)] - \widetilde{\mathfrak{I}}^M_{\circledcirc,\circleddash}[\uparrow(Y)])\big\|_{\mathcal{L}_2(q,T)}\\ &\lesssim N^{-\delta_0}\Big(\|Y\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} + \|\underline{\Upsilon}^N\|_{H^1} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\Big)^2\end{aligned} \tag{6.126}$$

for any $M \geq N \geq 1$. Lastly, from (6.48) and (6.36), there exists $\delta > 0$ such that

$$\big\|\,\widetilde{\mathfrak{I}}^N_{\circledcirc,\circleddash}[\uparrow(Y)] - \widetilde{\mathfrak{I}}^M_{\circledcirc,\circleddash}[\uparrow(Y)]\big\|_{\mathcal{L}_2(q,T)} \leq N^{-\delta_0}\widetilde{K}(Y,u_1,\omega_2) \tag{6.127}$$

for any $M \geq N \geq 1$, where, in view of (6.49), $\mathbb{E}[\widetilde{K}(Y,u_1,\omega_2)] \leq C(u_1,\omega_2) < \infty$ for almost every u_1 and ω_2. Therefore, from (6.123), (6.124), (6.125), (6.126), and (6.127) with the bound (6.121), we obtain

$$\mathbb{E}\Big[\big\|\widetilde{\mathfrak{I}}^M_{\circledcirc,\circleddash}[Y + \Upsilon^N_\delta + \sigma\mathfrak{Z}_M] - \widetilde{\mathfrak{I}}^N_{\circledcirc,\circleddash}[Y + \Upsilon^N_\delta + \sigma\mathfrak{Z}_N]\big\|_{\mathcal{L}_2(q,T)}\Big] \longrightarrow 0,$$

as $M \geq N \to \infty$.

Note that $\{Z_N\}_{N\in\mathbb{N}}$ is a convergent sequence and $\delta > 0$ was arbitrary. Hence, it follows from (6.118), (6.119), (6.120), and (6.122) that

$$\begin{aligned}\limsup_{N\to\infty} \sup_{\substack{0\leq H\leq 1\\ \|H\|_{\mathrm{Lip}}\lesssim 1}} \sup_{M\geq N}\bigg\{&-\log\bigg(\int \exp(H(\Xi_M(u_0,u_1,\omega_2)))d\rho_M(u_0)\bigg)\\ &+\log\bigg(\int \exp(H(\Xi_N(u_0,u_1,\omega_2)))d\rho_N(u_0)\bigg)\bigg\}_+ \leq 0,\end{aligned} \tag{6.128}$$

for almost every u_1 and ω_2, where the supremum in H was trivially dropped in the last step of (6.119). Therefore, (6.116) follows from (6.117) and (6.128) with Fatou's lemma. This concludes the proof of Proposition 6.10. $\square$

Finally, we present the proof of Theorem 1.15.

Proof of Theorem 1.15. • **Part 1:** We first prove almost sure global well-posedness of the hyperbolic Φ^3_3-model. As in [8, 18, 5], it suffices to prove "almost" almost sure global well-posedness. More precisely, it suffices to prove that given any $T > 0$ and small $\delta > 0$, there exists $\Sigma_{T,\delta} \subset \mathcal{H}^{-\frac12-\varepsilon}(\mathbb{T}^3)\times\Omega_2$ with $\vec{\rho}\otimes\mathbb{P}_2(\Sigma^c_{T,\delta}) < \delta$ such that for each $(\vec{u}_0,\omega_2) \in \Sigma_{T,\delta}$, the solution $(X, Y, \mathfrak{R})$ to (5.28), with the zero initial data and the enhanced data $\Xi(\vec{u}_0,\omega)$ in (6.11), exists on the time interval $[0, T]$.

We assume this "almost" almost sure global well-posedness claim for the moment. Denote by $(X_N, Y_N, \mathfrak{R}_N)$ the solution to the truncated system (6.58) with the truncated enhanced data $\Xi_N(\vec{u}_0,\omega)$ in (6.10) and set

$$u_N(\vec{u}_0,\omega_2) = \uparrow(\vec{u}_0,\omega_2) + \sigma\curlyvee_N(\vec{u}_0,\omega_2) + X_N + Y_N, \tag{6.129}$$

which is the solution to the truncated hyperbolic Φ^3_3-model (6.2) with the initial data $(u_N, \partial_t u_N)|_{t=0} = \vec{u}_0 = (u_0, u_1)$ and the noise $\xi = \xi(\omega_2)$. Here, we used the uniqueness of the solution u_N to (6.2); see Remark 6.3. Then, we conclude from Corollary 6.7 (on the almost sure convergence of $\Xi_N(\vec{u}_0, \omega)$ to $\Xi(\vec{u}_0, \omega)$) and the second part of Proposition 6.8 that $(u_N, \partial_t u_N)(\vec{u}_0, \omega_2)$ in (6.129) converges to $(u, \partial_t u)(\vec{u}_0, \omega_2)$ in $C([0,T]; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ for each $(\vec{u}_0, \omega_2) \in \Sigma_{T,\delta}$, where $u(\vec{u}_0, \omega_2)$ is defined by

$$u(\vec{u}_0, \omega_2) = \overset{\bullet}{\mid}(\vec{u}_0, \omega_2) + \sigma \overset{\bullet}{\curlyvee}(\vec{u}_0, \omega_2) + X + Y. \tag{6.130}$$

Now, we define

$$\Sigma = \bigcup_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \Sigma_{2^j, 2^{-j}k^{-1}}.$$

Then, we have $\vec{\rho} \otimes \mathbb{P}_2(\Sigma) = 1$ and, for each $(\vec{u}_0, \omega_2) \in \Sigma$, the solution $(u_N, \partial_t u_N)(\vec{u}_0, \omega_2)$ to the truncated hyperbolic Φ^3_3-model (6.2) converges to $(u, \partial_t u)(\vec{u}_0, \omega_2)$ in (6.130) in $C(\mathbb{R}_+; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ (endowed with the compact-open topology in time). This proves the almost sure global well-posedness claim in Theorem 1.15, assuming "almost" almost sure global well-posedness.

We now prove "almost" almost sure global well-posedness. Fix $T > 0$ and small $\delta > 0$. Given $\Xi = (\Xi_1, \dots, \Xi_6) \in \mathcal{X}^\varepsilon_T$, let $Z(\Xi) = (X, Y, \mathfrak{R})(\Xi)$ be the solution to (5.28) with the zero initial data and the enhanced data set given by Ξ, namely, Ξ_j replacing the jth element in (5.29). Note that Ξ here denotes a general element in $\mathcal{X}^\varepsilon_T$ and is not associated with any specific $(\vec{u}_0, \omega_2) \in \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) \times \Omega_2$. Similarly, given $N \in \mathbb{N}$ and $\Xi \in \mathcal{X}^\varepsilon_T$, let $Z_N(\Xi) = (X_N, Y_N, \mathfrak{R}_N)(\Xi)$ be the solution to (6.58) with the enhanced data set Ξ, namely, Ξ_j replacing the jth element of $\Xi_N(\vec{u}_0, \omega_2)$ in (6.10).

Given $C_0 > 0$, define the set $\Sigma_{C_0} \subset \mathcal{X}^\varepsilon_T$ such that, for each $\Xi \in \Sigma_{C_0}$, the solution $Z(\Xi)$ to (5.28), with the zero initial data and the enhanced data Ξ, exists on the time interval $[0, T]$, satisfying the bound

$$\|Z(\Xi)\|_{Z^{s_1,s_2,s_3}(T)} \le C_0 + 1. \tag{6.131}$$

Let $N \in \mathbb{N}$. Given $K, C_0 > 0$, we set

$$A_{N,K,C_0} = \big\{ \Xi' \in \mathcal{X}^\varepsilon_T : \|\Xi'\|_{\mathcal{X}^\varepsilon_T} \le K, \ \|Z_N(\Xi')\|_{Z^{s_1,s_2,s_3}(T)} \le C_0 \big\} \tag{6.132}$$

and

$$B_{N,K,C_0} = \big\{ (\Xi, \Xi') \in \mathcal{X}^\varepsilon_T \times \mathcal{X}^\varepsilon_T : \|\Xi - \Xi'\|_{\mathcal{X}^\varepsilon_T} \le \kappa, \ \Xi' \in A_{N,K,C_0} \big\}, \tag{6.133}$$

where $\kappa > 0$ is a small number to be chosen later. Then, from the stability result (the first claim in Proposition 6.8) with (6.131), (6.132), and (6.133), there exists small $\kappa(T, K, C_0) \in (0,1)$ and $N_0 = N_0(T, K, C_0) \in \mathbb{N}$ such that

$$\Sigma_{C_0} \times \mathcal{X}^\varepsilon_T \supset B_{N,K,C_0} \tag{6.134}$$

for any $N \ge N_0$.

Let $C_0 = C_0(T,\delta) \gg 1$ be as in Proposition 6.9 and let $\mathfrak{p}_N$, $N \in \mathbb{N}$, be as in Proposition 6.10. Then, from (6.110), (6.111), and (6.134), we have

$$\begin{aligned}
\vec{\rho} \otimes \mathbb{P}_2\big(\Xi(\vec{u}_0,\omega_2) \in \Sigma_{C_0}\big) &= \int \mathbf{1}_{\Xi \in \Sigma_{C_0}}(\Xi,\Xi')d\mathfrak{p}_N(\Xi,\Xi') \\
&\geq \int \mathbf{1}_{B_{N,K,C_0}}(\Xi,\Xi')d\mathfrak{p}_N(\Xi,\Xi') \\
&\geq 1 - \int \mathbf{1}_{\{\|\Xi-\Xi'\|_{\mathcal{X}^\varepsilon_T} > \kappa\}} d\mathfrak{p}_N(\Xi,\Xi') - \int \mathbf{1}_{A^c_{N,K,C_0}}(\Xi')d\mathfrak{p}_N(\Xi,\Xi') \qquad (6.135) \\
&\geq 1 - \frac{1}{\kappa}\int \min(\|\Xi-\Xi'\|_{\mathcal{X}^\varepsilon_T},1)d\mathfrak{p}_N(\Xi,\Xi') - \vec{\rho}_N \otimes \mathbb{P}_2(\{\Xi_N(\vec{u}'_0,\omega'_2) \in A^c_{N,K,C_0}\}) \\
&> 1 - \frac{1}{\kappa}\int \min(\|\Xi-\Xi'\|_{\mathcal{X}^\varepsilon_T},1)d\mathfrak{p}_N(\Xi,\Xi') - 2\delta,
\end{aligned}$$

where the last step follows from Proposition 6.5 by choosing $K = K(\delta) \gg 1$, together with Proposition 6.9. By Proposition 6.10, we have

$$\frac{1}{\kappa}\int \min(\|\Xi-\Xi'_N\|_{\mathcal{X}^\varepsilon_T},1)d\mathfrak{p}_N(\Xi,\Xi'_N) \longrightarrow 0, \qquad (6.136)$$

as $N \to \infty$. Therefore, we conclude from (6.135) and (6.136) that

$$\vec{\rho} \otimes \mathbb{P}_2\big(\Xi(\vec{u}_0,\omega_2) \in \Sigma_{C_0}\big) > 1 - 2\delta.$$

This proves "almost" almost sure global well-posedness with

$$\Sigma_{T,\delta} = \{(\vec{u}_0,\omega_2) \in \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) \times \Omega_2 : \Xi(\vec{u}_0,\omega_2) \in \Sigma_{C_0}\},$$

and hence almost sure global well-posedness of the hyperbolic Φ^3_3-model, namely, the unique limit $u = u(\vec{u}_0,\omega_2)$ in (6.130) exists globally in time almost surely with respect to $\vec{\rho} \otimes \mathbb{P}_2$.

• **Part 2:** Next, we prove invariance of the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ under the limiting hyperbolic Φ^3_3-dynamics. In the following, we prove

$$\int F(\Phi(t)(\vec{u}_0,\omega_2))d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0,\omega_2) = \int F(\vec{u}_0)d\vec{\rho}(\vec{u}_0) \qquad (6.137)$$

for any bounded Lipschitz functional $F : \mathcal{C}^{-100}(\mathbb{T}^3) \times \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$ and $t \in \mathbb{R}_+$, where $\Phi(\vec{u}_0,\omega_2)$ is the limit of the solution $(u_N, \partial_t u_N) = \Phi^N(\vec{u}_0,\omega_2)$ to the truncated hyperbolic Φ^3_3-model defined in (6.21).

As in Part 1, we use the notation $(X,Y,\mathfrak{R}) = (X,Y,\mathfrak{R})(\Xi)$, etc. Also, let $\mathfrak{p}_N$, $N \in \mathbb{N}$, be as in Proposition 6.10. Then, by the decomposition (6.57) (also for $N = \infty$), (6.110), (6.111), and the invariance of $\vec{\rho}_N$ under the truncated hyperbolic Φ^3_3-model (6.2) (Lemma 6.4), we have

$$\begin{aligned}
&\int F(\Phi(t)(\vec{u}_0,\omega_2))d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0,\omega_2) \\
&\quad = \int F(\Phi(t)(\Xi))d\mathfrak{p}_N(\Xi,\Xi') \\
&\quad = \int F(\Phi^N(t)(\Xi'))d\mathfrak{p}_N(\Xi,\Xi') + \int \big[F(\Phi(t)(\Xi)) - F(\Phi^N(t)(\Xi'))\big]d\mathfrak{p}_N(\Xi,\Xi') \\
&\quad = \int F(\vec{u}_0)d\vec{\rho}_N(\vec{u}_0) + \int \big[F(\Phi(t)(\Xi)) - F(\Phi^N(t)(\Xi'))\big]d\mathfrak{p}_N(\Xi,\Xi').
\end{aligned}$$

By the weak convergence of $\vec{\rho}_N$ to $\vec{\rho}$, we have

$$\lim_{N\to\infty} \int F(\vec{u}_0) d\vec{\rho}_N(\vec{u}_0) = \int F(\vec{u}_0) d\vec{\rho}(\vec{u}_0).$$

Hence, since F is bounded and Lipschitz, (6.137) is reduced to showing that

$$\int \min\big(\|\Phi(t)(\Xi) - \Phi^N(t)(\Xi')\|_{\mathcal{C}^{-100}\times\mathcal{C}^{-100}}, 1\big) d\mathfrak{p}_N(\Xi, \Xi') \longrightarrow 0, \tag{6.138}$$

as $N \to \infty$.

As in (6.21), we write

$$\Phi(t)(\Xi) = \big(\Phi_1(t)(\Xi), \Phi_2(t)(\Xi)\big) \qquad \text{and} \qquad \Phi^N(t)(\Xi') = \big(\Phi_1^N(t)(\Xi'), \Phi_2^N(t)(\Xi')\big),$$

where $\Xi = (\Xi_1, \dots, \Xi_6)$ and $\Xi' = (\Xi_1', \dots, \Xi_6')$ (see also (6.10) and (6.11)). With the decomposition as in (6.57), we have

$$\begin{aligned} \Phi_1(t)(\Xi) &= \Xi_1 + \sigma\Xi_3 + X(\Xi) + Y(\Xi), \\ \Phi_1^N(t)(\Xi') &= \Xi_1' + \sigma\Xi_3' + X_N(\Xi') + Y_N(\Xi'), \end{aligned} \tag{6.139}$$

and $\Phi_2(t)(\Xi) = \partial_t\Phi_1(t)(\Xi)$ and $\Phi_2^N(t)(\Xi') = \partial_t\Phi_1^N(t)(\Xi')$ are given by term-by-term differentiation of the terms on the right-hand sides of (6.139). From the definition (5.49) of the $\mathcal{X}_T^\varepsilon$-norm, we clearly have

$$\begin{aligned} &\|(\Xi_1 + \sigma\Xi_3)(t) - (\Xi_1' + \sigma\Xi_3')(t)\|_{\mathcal{C}^{-100}} \\ &\quad + \|(\partial_t\Xi_1 + \sigma\partial_t\Xi_3)(t) - (\partial_t\Xi_1' + \sigma\partial_t\Xi_3')(t)\|_{\mathcal{C}^{-100}} \lesssim \|\Xi - \Xi'\|_{\mathcal{X}_T^\varepsilon}. \end{aligned}$$

Hence, in view of (5.32) with (5.47), (6.138) is reduced to showing that

$$\int \min\big(\|Z(\Xi) - Z_N(\Xi')\|_{Z^{s_1,s_2,s_3}(T)}, 1\big) d\mathfrak{p}_N(\Xi, \Xi') \longrightarrow 0, \tag{6.140}$$

as $N \to \infty$, where $Z(\Xi) = (X, Y, \mathfrak{R})(\Xi)$ and $Z_N(\Xi') = (X_N, Y_N, \mathfrak{R}_N)(\Xi')$ as in Part 1.

It follows from the second part of Proposition 6.8 (with $\kappa = \|\Xi - \Xi'\|_{\mathcal{X}_T^\varepsilon}$) and Proposition 6.10 that

$$\begin{aligned} &\int \min\big(\|Z(\Xi) - Z_N(\Xi')\|_{Z^{s_1,s_2,s_3}(T)}, 1\big) d\mathfrak{p}_N(\Xi, \Xi') \\ &\quad \le A\big(T, \|\Xi\|_{\mathcal{X}_T^\varepsilon}, \|Z(\Xi)\|_{Z^{s_1,s_2,s_3}(T)}\big) \\ &\qquad \times \int \min\big(\|\Xi - \Xi'\|_{\mathcal{X}_T^\varepsilon} + N^{-\delta}, 1\big) d\mathfrak{p}_N(\Xi, \Xi') \longrightarrow 0, \end{aligned}$$

as $N \to \infty$. This proves (6.140) and therefore, we conclude (6.137), which proves invariance of the Gibbs measure $\vec{\rho}$ under the limiting hyperbolic Φ_3^3-model. □

Appendix A. Absolute continuity with respect to the shifted measure

A.1. **Preliminary lemmas.** In this appendix, we prove that the Φ_3^3-measure ρ in the weakly nonlinear regime ($|\sigma| \ll 1$), constructed in Theorem 1.8 (i), is absolutely continuous with respect to the shifted measure $\text{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$, where Y is as in (3.2), $\mathfrak{Z}$ is defined as the limit of the antiderivative of $\dot{\mathfrak{Z}}^N$ in (3.11) as $N \to \infty$, and the auxiliary process $\mathcal{W}$ is defined by

$$\mathcal{W}(t) = (1 - \Delta)^{-1} \int_0^t \langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\big(\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon} Y(t')\big)^5 dt' \tag{A.1}$$

for some small $\varepsilon > 0$. For the proof, we construct a drift as in the discussion in Section 3 of [4]. See also Appendix C in [53]. The coercive term $\mathcal{W}$ is introduced to guarantee global existence of a drift on the time interval $[0,1]$. See Lemma A.2 below. We closely follow the presentation in Appendix C of our previous work [53].

First, we recall the following general lemma, giving a criterion for absolute continuity. See Lemma C.1 in [53] for the proof.

Lemma A.1. *Let μ_n and ρ_n be probability measures on a Polish space X. Suppose that μ_n and ρ_n converge weakly to μ and ρ, respectively. Furthermore, suppose that for every $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\delta(\varepsilon), \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for every continuous function $F : X \to \mathbb{R}$ with $0 < \inf F \le F \le 1$ satisfying*

$$\mu_n(\{F \le \varepsilon\}) \ge 1 - \delta(\varepsilon)$$

for any $n \ge n_0(F)$, we have

$$\limsup_{n\to\infty} \int F(u) d\rho_n(u) \le \eta(\varepsilon).$$

Then, ρ is absolutely continuous with respect to μ.

By regarding $\dot{\mathfrak{Z}}^N$ in (3.11) and $\mathcal{W}$ in (A.1) as functions of Y, we write them as

$$\begin{aligned} \dot{\mathfrak{Z}}^N(Y)(t) &= (1-\Delta)^{-1} :\! Y_N^2(t)\! :, \\ \mathcal{W}(Y)(t) &= (1-\Delta)^{-1} \int_0^t \langle \nabla \rangle^{-\frac12-\varepsilon} \big(\langle \nabla \rangle^{-\frac12-\varepsilon} Y(t') \big)^5 dt' \end{aligned} \tag{A.2}$$

and we set $\dot{\mathfrak{Z}}_N(Y) = \pi_N \dot{\mathfrak{Z}}^N(Y)$. Then, from (A.2), we have

$$\dot{\mathfrak{Z}}_N(Y + \Theta) - \dot{\mathfrak{Z}}_N(Y) = (1-\Delta)^{-1} \pi_N (2\Theta_N Y_N + \Theta_N^2), \tag{A.3}$$

where $\Theta_N = \pi_N \Theta$. We also define $\mathcal{W}_N(Y)(t)$ by

$$\mathcal{W}_N(Y)(t) = (1-\Delta)^{-1} \pi_N \int_0^t \langle \nabla \rangle^{-\frac12-\varepsilon} \big(\langle \nabla \rangle^{-\frac12-\varepsilon} Y_N(t') \big)^5 dt'. \tag{A.4}$$

Next, we state a lemma on the construction of a drift Θ.

Lemma A.2. *Let $\sigma \in \mathbb{R}$ and $\dot{\Upsilon} \in L^2([0,1]; H^1(\mathbb{T}^3))$. Then, given any $N \in \mathbb{N}$, the Cauchy problem for Θ:*

$$\begin{cases} \dot{\Theta} + \sigma (1-\Delta)^{-1} \pi_N (2\Theta_N Y_N + \Theta_N^2) + \dot{\mathcal{W}}_N(Y+\Theta) - \dot{\Upsilon} = 0 \\ \Theta(0) = 0 \end{cases} \tag{A.5}$$

is almost surely globally well-posed on the time interval $[0,1]$ such that a solution Θ belongs to $C([0,1]; H^1(\mathbb{T}^3))$. Moreover, if $\|\dot{\Upsilon}\|^2_{L^2([0,\tau]; H^1_x)} \le M$ for some $M > 0$ and for some stopping time $\tau \in [0,1]$, then, for any $1 \le p < \infty$, there exists $C = C(M,p) > 0$ such that

$$\mathbb{E}\Big[\|\dot{\Theta}\|^p_{L^2([0,\tau]; H^1_x)} \Big] \le C(M,p), \tag{A.6}$$

where $C(M,p)$ is independent of $N \in \mathbb{N}$.

A.2. **Absolute continuity.** In this subsection, we prove the absolute continuity of the Φ^3_3-measure ρ with respect to $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$ by assuming Lemma A.2. We present the proof of Lemma A.2 at the end of this appendix. For simplicity, we use the same short-hand notations as in Sections 3 and 4, for instance, $Y = Y(1)$, $\mathfrak{Z} = \mathfrak{Z}(1)$, $\mathcal{W} = \mathcal{W}(1)$, and $\mathcal{W}_N = \mathcal{W}_N(1)$.

Given $L \gg 1$, let $\delta(L)$ and $R(L)$ satisfy $\delta(L) \to 0$ and $R(L) \to \infty$ as $L \to \infty$, which will be specified later. In view of Lemma A.1, it suffices to show that if $G : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$ is a bounded continuous function with $G > 0$ and

$$\mathbb{P}\big(\{G(Y + \sigma\mathfrak{Z}_N + \mathcal{W}_N) \ge L\}\big) \ge 1 - \delta(L), \tag{A.7}$$

then we have

$$\limsup_{N\to\infty} \int \exp(-G(u)) d\rho_N(u) \le \exp(-R(L)), \tag{A.8}$$

where ρ_N denotes the truncated Φ^3_3-measure defined in (1.25). Here, think of $\mathrm{Law}(Y + \sigma\mathfrak{Z}_N + \mathcal{W}_N)$ as the measure μ_N, weakly converging to $\mu = \mathrm{Law}(Y + \sigma\mathfrak{Z} + \mathcal{W})$.

By the Boué-Dupuis variational formula (Lemma 3.1) and the change of variables (3.12), we have

$$\begin{aligned} &-\log\bigg(\int \exp(-G(u) - R^\diamond_N(u)) d\mu(u)\bigg)\\ &\quad = \inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\bigg[G(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) + \widehat{R}^\diamond_N(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot\Upsilon^N(t)\|^2_{H^1_x} dt\bigg], \end{aligned}$$

where $\widehat{R}^\diamond_N$ is as in (3.33). We proceed as in Subsection 3.2, using Lemmas 3.5 and 3.6 with Lemma 3.2, (3.25), and the smallness of $|\sigma|$. See (3.17), (3.24), and (3.27). Thus, we have

$$\begin{aligned} &-\log\bigg(\int \exp(-G(u) - R^\diamond_N(u)) d\mu(u)\bigg)\\ &\qquad \ge \inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\bigg[G(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) + \frac{1}{20}\int_0^1 \|\dot\Upsilon^N(t)\|^2_{H^1_x} dt\bigg] - C_1 \end{aligned} \tag{A.9}$$

for some constant $C_1 > 0$. For $\dot\Upsilon^N \in \mathbb{H}^1_a$, let Θ^N be the solution to (A.5) with $\dot\Upsilon$ replaced by $\dot\Upsilon^N$. For any $M > 0$, define the stopping time τ_M as

$$\begin{aligned} \tau_M = \min\bigg(1, &\min\bigg\{\tau : \int_0^\tau \|\dot\Upsilon^N(t)\|^2_{H^1_x} dt = M\bigg\},\\ &\min\bigg\{\tau : \int_0^\tau \|\dot\Theta^N(t)\|^2_{H^1_x} dt = 2C(M,2)\bigg\}\bigg), \end{aligned} \tag{A.10}$$

where $C(M,2)$ is the constant appearing in (A.6) with $p = 2$. Let

$$\Theta^N_M(t) := \Theta^N(\min(t, \tau_M)). \tag{A.11}$$

From (3.2), we have $Y(0) = 0$, while $\mathfrak{Z}^N(0) = 0$ by definition. Then, from the change of variables (3.12) with $\Theta(0) = 0$, we see that $\Upsilon^N(0) = 0$. We also have $\mathcal{W}_N(0) = 0$ from (A.4). Then, substituting (A.3) into (A.5) and integrating from $t = 0$ to 1 gives

$$Y + \Upsilon^N + \sigma\mathfrak{Z}_N = Y + \Theta^N_M + \sigma\mathfrak{Z}_N(Y + \Theta^N_M) + \mathcal{W}_N(Y + \Theta^N_M) \tag{A.12}$$

on the set $\{\tau_M = 1\}$.

From the definition (A.11) with (A.10), we have

$$\|\dot{\Theta}^N_M\|^2_{L^2_t([0,1];H^1_x)} \le 2C(M,2) \tag{A.13}$$

and thus the Novikov condition is satisfied. Then, Girsanov's theorem [20, Theorem 10.14] yields that $\mathrm{Law}(Y+\Theta^N_M)$ is absolutely continuous with respect to $\mathrm{Law}(Y)$; see (A.16) below. Let $\mathbb{Q}=\mathbb{Q}^{\dot{\Theta}^N_M}$ the probability measure whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is given by the following stochastic exponential:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^1 \langle \dot{\Theta}^N_M(t), dY(t)\rangle_{H^1_x} - \frac{1}{2}\int_0^1 \|\dot{\Theta}^N_M(t)\|^2_{H^1_x}dt} \tag{A.14}$$

such that, under this new measure $\mathbb{Q}$, the process

$$W^{\dot{\Theta}^N_M}(t) = W(t) + \langle\nabla\rangle \dot{\Theta}^N_M(t) = \langle\nabla\rangle(Y+\dot{\Theta}^N_M)(t)$$

is a cylindrical Wiener process on $L^2(\mathbb{T}^3)$. By setting $Y^{\dot{\Theta}^N_M}(t) = \langle\nabla\rangle^{-1}W^{\dot{\Theta}^N_M}(t)$, we have

$$Y^{\dot{\Theta}^N_M}(t) = Y(t) + \Theta^N_M(t). \tag{A.15}$$

Moreover, from Cauchy-Schwarz inequality with (A.14) and the bound (A.13), and then (A.15), we have

$$\begin{aligned}\mathbb{P}\big(\{Y+\Theta^N_M\in E\}\big) &= \int \mathbf{1}_{\{Y+\Theta^N_M\in E\}}\frac{d\mathbb{P}}{d\mathbb{Q}}d\mathbb{Q} \le C_M\Big(\mathbb{Q}\big(\{Y^{\dot{\Theta}^N_M}\in E\}\big)\Big)^{\frac{1}{2}}\\ &= C_M\Big(\mathbb{P}\big(\{Y\in E\}\big)\Big)^{\frac{1}{2}}\end{aligned} \tag{A.16}$$

for any measurable set E.

From (A.9), (A.12), and the non-negativity of G, we have

$$\begin{aligned}(\mathrm{A.9}) \ge \inf_{\dot{\Upsilon}^N\in\mathbb{H}^1_a}\mathbb{E}\bigg[&\Big(G\big(Y+\Theta^N_M+\sigma\mathfrak{Z}_N(Y+\Theta^N_M)+\mathcal{W}_N(Y+\Theta^N_M)\big)\\ &+\frac{1}{20}\int_0^1\|\dot{\Upsilon}^N(t)\|^2_{H^1_x}dt\Big)\mathbf{1}_{\{\tau_M=1\}}\\ &+\Big(G(Y+\Upsilon^N+\sigma\mathfrak{Z}_N)+\frac{1}{20}\int_0^1\|\dot{\Upsilon}^N(t)\|^2_{H^1_x}dt\Big)\mathbf{1}_{\{\tau_M<1\}}\bigg]-C_1\\ \ge \inf_{\dot{\Upsilon}^N\in\mathbb{H}^1_a}\mathbb{E}\bigg[&G\big(Y+\Theta^N_M+\sigma\mathfrak{Z}_N(Y+\Theta^N_M)+\mathcal{W}_N(Y+\Theta^N_M)\big)\cdot\mathbf{1}_{\{\tau_M=1\}}\\ &+\frac{1}{20}\int_0^1\|\dot{\Upsilon}^N(t)\|^2_{H^1_x}dt\cdot\mathbf{1}_{\{\tau_M<1\}}\bigg]-C_1.\end{aligned}$$

Then, using the definition (A.10) of the stopping time τ_M and applying (A.16) and (A.7), we have

$$\begin{aligned}
\text{(A.9)} &\geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[L \cdot \mathbf{1}_{\{\tau_M = 1\} \cap \{G(Y+\Theta_M^N + \sigma \mathfrak{Z}_N(Y+\Theta_M^N)) + \mathcal{W}_N(Y+\Theta_M^N)) \geq L\}} \\
&\qquad\qquad + \frac{M}{20} \cdot \mathbf{1}_{\{\tau_M < 1\} \cap \{\int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt < 2C(M,2)\}} \bigg] - C_1 \\
&\geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \bigg\{ L\Big(\mathbb{P}(\{\tau_M = 1\}) - C_M \delta(L)^{\frac{1}{2}}\Big) \\
&\qquad\qquad + \frac{M}{20}\mathbb{P}\bigg(\{\tau_M < 1\} \cap \bigg\{ \int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt < 2C(M,2)\bigg\}\bigg)\bigg\} - C_1.
\end{aligned} \tag{A.17}$$

In view of (A.6) with (A.10) and (A.11), Markov's inequality gives

$$\mathbb{P}\bigg(\int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt = \int_0^{\tau_M} \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt \geq 2C(M,2)\bigg) \leq \frac{1}{2},$$

which yields

$$\mathbb{P}\bigg(\{\tau_M < 1\} \cap \bigg\{ \int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt < 2C(M,2)\bigg\}\bigg) \geq \mathbb{P}(\{\tau_M < 1\}) - \frac{1}{2}. \tag{A.18}$$

Now, we set $M = 20L$. Note from (A.10) that $\mathbb{P}(\{\tau_M = 1\}) + \mathbb{P}(\{\tau_M < 1\}) = 1$. Then, from (A.17) and (A.18), we obtain

$$\begin{aligned}
&- \log\bigg(\int \exp(-G(u) - R_N^\diamond(u)) d\mu(u)\bigg) \\
&\quad \geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \bigg\{ L\Big(\mathbb{P}(\{\tau_M = 1\}) - C_L' \delta(L)^{\frac{1}{2}}\Big) + L\Big(\mathbb{P}(\{\tau_M < 1\}) - \frac{1}{2}\Big)\bigg\} - C_1 \\
&\quad = L\Big(\frac{1}{2} - C_L'\delta(L)^{\frac{1}{2}}\Big) - C_1.
\end{aligned}$$

Therefore, by choosing $\delta(L) > 0$ such that $C_L'\delta(L)^{\frac{1}{2}} \to 0$ as $L \to \infty$, this shows (A.8) with

$$R(L) = L\Big(\frac{1}{2} - C_L'\delta(L)^{\frac{1}{2}}\Big) - C_1 + \log Z,$$

where $Z = \lim_{N\to\infty} Z_N$ denotes the limit of the partition functions for the truncated Φ_3^3-measures ρ_N.

A.3. **Proof of Lemma A.2.** We conclude this appendix by presenting the proof of Lemma A.2.

Proof of Lemma A.2. By Lemma 2.3 (ii) and Sobolev's inequality, we have

$$\begin{aligned}
\|(1-\Delta)^{-1}(2\Theta_N Y_N + \Theta_N^2)(t)\|_{H_x^1} &\lesssim \|(2\Theta_N Y_N + \Theta_N^2)(t)\|_{H_x^{-1}} \\
&\lesssim \|\Theta_N(t)\|_{H_x^{\frac{1}{2}+\varepsilon}} \|Y_N(t)\|_{W_x^{-\frac{1}{2}-\varepsilon,\infty}} + \|\Theta_N^2(t)\|_{L_x^{\frac{6}{5}}} \\
&\lesssim \|\Theta_N(t)\|_{H_x^1} \|Y_N(t)\|_{W_x^{-\frac{1}{2}-\varepsilon,\infty}} + \|\Theta_N(t)\|_{H_x^1}^2
\end{aligned} \tag{A.19}$$

for small $\varepsilon > 0$. Moreover, from (A.1), we have

$$\begin{aligned}\|\dot{\mathcal{W}}_N(Y(t)+\Theta(t))\|_{H^1_x} &\lesssim \|\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}Y_N(t)\|^5_{L^\infty_x} + \|\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t)\|^5_{L^\infty_x}\\ &\lesssim \|Y_N(t)\|^5_{W_x^{-\frac{1}{2}-\varepsilon,\infty}} + \|\Theta_N(t)\|^5_{H^1_x}.\end{aligned} \tag{A.20}$$

Therefore, by studying the integral formulation of (A.5), a contraction argument in $L^\infty([0,T];H^1(\mathbb{T}^3))$ for small $T>0$ with (A.19) and (A.20) yields local well-posedness. Here, the local existence time T depends on $\|\Theta(0)\|_{H^1}$, $\|\dot{\Upsilon}\|_{L^2_T H^1_x}$, and $\|Y_N\|_{L^6_T W_x^{-\frac{1}{2}-\varepsilon,\infty}}$, where the last term is almost surely bounded in view of Lemma 3.2 and (2.4).

Next, we prove global existence on $[0,1]$ by establishing an a priori bound on the H^1-norm of a solution. From (A.5) with (A.4), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta(t)\|^2_{H^1} = &-\sigma\int_{\mathbb{T}^3}(2\Theta_N(t)Y_N(t)+\Theta_N^2(t))\Theta_N(t)dx\\ &-\int_{\mathbb{T}^3}\big(\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}(Y_N(t)+\Theta_N(t))\big)^5\cdot\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t)dx\\ &+\int_{\mathbb{T}^3}\langle\nabla\rangle\Theta(t)\cdot\langle\nabla\rangle\dot{\Upsilon}(t)dx.\end{aligned} \tag{A.21}$$

The second term on the right-hand side of (A.21), coming from $\mathcal{W}$ is a coercive term, allowing us to hide part of the first term on the right-hand side.

From Lemma 2.1 and Young's inequality, we have

$$\begin{aligned}&\bigg|\int_{\mathbb{T}^3}(2\Theta_N(t)Y_N(t)+\Theta_N^2(t))\Theta_N(t)dx\bigg|\\ &\qquad\lesssim \|\Theta_N(t)\|^2_{H^1} + \|\Theta_N(t)\|^3_{L^3} + \|Y_N(t)\|^c_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}\end{aligned} \tag{A.22}$$

for small $\varepsilon>0$ and some $c>0$. We now estimate the second term on the right-hand side of (A.22). By (2.3), we have

$$\begin{aligned}\|\Theta_N(t)\|^3_{L^3} &\lesssim \|\Theta_N(t)\|^{\frac{3+6\varepsilon}{3+2\varepsilon}}_{H^1}\|\Theta_N(t)\|^{\frac{6}{3+2\varepsilon}}_{W^{-\frac{1}{2}-\varepsilon,6}}\\ &\le \|\Theta_N(t)\|^2_{H^1} + \varepsilon_0\|\Theta_N(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}} + C_{\varepsilon_0}\end{aligned} \tag{A.23}$$

for small $\varepsilon, \varepsilon_0 > 0$. As for the coercive term, from (3.40) and Young's inequality, we have

$$\begin{aligned}&\int_{\mathbb{T}^3}\big(\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}(Y_N(t)+\Theta_N(t))\big)^5\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t)dx\\ &\quad\ge \frac{1}{2}\int_{\mathbb{T}^3}(\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t))^6dx - c\int_{\mathbb{T}^3}\big|(\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}Y_N(t))^5\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t)\big|dx\\ &\quad\ge \frac{1}{2}\|\Theta_N(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}} - c\|Y_N(t)\|^5_{W^{-\frac{1}{2}-\varepsilon,6}}\|\Theta_N(t)\|_{W^{-\frac{1}{2}-\varepsilon,6}}\\ &\quad\ge \frac{1}{4}\|\Theta_N(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}} - c\|Y_N(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}}.\end{aligned} \tag{A.24}$$

Therefore, putting (A.21), (A.22), (A.23), and (A.24) together we obtain

$$\frac{d}{dt}\|\Theta(t)\|^2_{H^1} \lesssim \|\Theta(t)\|^2_{H^1} + \|\dot{\Upsilon}(t)\|^2_{H^1} + \|Y(t)\|^c_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|Y(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}} + 1.$$

By Gronwall's inequality, we then obtain

$$\|\Theta(t)\|_{H^1}^2 \lesssim \|\dot{\Upsilon}\|_{L^2([0,t];H_x^1)}^2 + \|Y_N\|_{L^c([0,1];\mathcal{C}_x^{-\frac{1}{2}-\varepsilon})}^c + \|Y\|_{L^6([0,1];W_x^{-\frac{1}{2}-\varepsilon,6})}^6 + 1, \tag{A.25}$$

uniformly in $0 \le t \le 1$. The a priori bound (A.25) together with Lemma 3.2 allows us to iterate the local well-posedness argument, guaranteeing existence of the solution Θ on $[0,1]$.

Lastly, we prove the bound (A.6). From (A.19), (A.20), and (A.25), we have

$$\begin{aligned} &\|\sigma(1-\Delta)^{-1}(2\Theta_N Y_N + \Theta_N^2) + \dot{\mathcal{W}}_N(Y+\Theta)\|_{L^2([0,\tau];H_x^1)} \\ &\quad \lesssim \|\dot{\Upsilon}\|_{L^2([0,\tau];H_x^1)}^5 + \|Y_N\|_{L^q([0,1];\mathcal{C}_x^{-\frac{1}{2}-\frac{1}{2}\varepsilon})}^{c_0} + 1 \end{aligned} \tag{A.26}$$

for some finite $q, c_0 \ge 1$ and for any $0 \le \tau \le 1$. Then, using the equation (A.5), the bound (A.6) follows from (A.26), the bound on $\dot{\Upsilon}$, and the following corollary to Lemma 3.2:

$$\mathbb{E}\Big[\|Y_N\|_{L^q([0,1];\mathcal{C}_x^{-\frac{1}{2}-\frac{1}{2}\varepsilon})}^p\Big] < \infty$$

for any finite $p, q \ge 1$, uniformly in $N \in \mathbb{N}$. $\square$

Remark A.3. A slight modification of the argument presented above shows that the tamed Φ_3^3-measure ν_δ constructed in Proposition 4.1 is absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$. In this setting, we can use the analysis in Subsection 4.2 (Step 1 of the proof of Proposition 4.1) to arrive at (A.9). The rest of the argument remains unchanged. As a consequence, the σ-finite version $\overline{\rho}_\delta$ of the Φ_3^3-measure defined in (4.9) is also absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$ for any $\delta > 0$.

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