A Construction of 2-D Z-Complementary Array Code Sets with Flexible Even Row Lengths and Applications in Massive MIMO

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***Abstract*—The need for two-dimensional (2-D) arrays with good 2-D correlation properties and flexible parameters has been of great interest due to their application in the field of wireless communications such as massive multiple input multiple output (MIMO), phased array antenna, multi-carrier code division multiple access (MC-CDMA), 2D-MC-CDMA, etc. In this paper, we propose a direct construction of a 2-D Z-complementary array code set (ZCACS) with flexible parameters. For this purpose, we first propose a construction of inter-group complementary (IGC) code sets using multivariable function and by using this construction 2-D Z-complementary array code (ZCAC) and 2-D ZCAC set (ZCACS) are provided. In some special case, the proposed 2-D ZCAC reduces to a 2-D Z-complementary array pair (ZCAP), which is not reported till date. The peak-to-mean envelope power ratio (PMEPR) of row and column sequences of 2-D ZCAC is shown to be better than the existing ones for use in MC-CDMA. 2-D Golay complementary array set (GCAS) and Golay complementary set (GCS) are derived from the proposed construction, which can be applied in omnidirectional precoding (OP) based transmission through massive MIMO. The proposed construction can support a more flexible number of antennas for a uniform rectangular array (URA) to transmit space-time block coded (STBC) data, than the existing constructions. The bit-error-rate (BER) simulation result also shows the performance benefits of derived 2-D GCAS and GCS compared to the existing ones.**

***Index Terms*—Golay complementary pair (GCP), inter-group complementary (IGC) code set, multivariable function, 2-D Z-complementary array code set (ZCACS), peak-to-mean envelope power ratio (PMEPR), multi-carrier code division multiple access (MC-CDMA), multiple input multiple output (MIMO), uniform rectangular array (URA), space-time block code (STBC).**

I. Introduction

In the field of sequence design, the introduction of Golay complementary pair (GCP) by Marcel J. E. Golay in 1951 marks a milestone [1]. It is a pair of sequences having zero aperiodic auto-correlation function (AACF) sum for all non-zero time shifts. Due to its ideal AACF property, it has been widely used in many applications such as reducing peak-to-mean envelope power ratio (PMEPR) of orthogonal frequency division multiplexing (OFDM) [2]–[4], channel estimation [5], radar [6] etc. Binary GCPs exist only in lengths of the form $2^\alpha 10^\beta 26^\gamma$, where $\alpha, \beta, \gamma \geq 1$. To overcome the shortage of GCPs on lengths, the concept of Z-complementary pair (ZCP) and Z-complementary sets (ZCS) with a wider range of lengths was introduced by Fan *et al.* in [7], where AACF sum is zero for all non-zero time shifts in a zone around the zero time shift, called the zero-correlation zone (ZCZ). In 1972, Tseng and Liu [8] extended the concept of GCP to the Golay complementary set (GCS), which refers to a set of sequences having the same AACF sum property as GCP. The first direct construction of GCS was given by Paterson [9] by using a generalized Boolean function (GBF). Later, GCS was extended to mutually orthogonal Golay complementary set (MOGCS), which is a collection of multiple GCSs with the property that the aperiodic cross-correlation function (ACCF) sum of two different GCSs is zero for all time shifts. Complete complementary code (CCC) is the special case of MOGCS when the set size achieves its upper bound [10]. In the literature, there are many constructions of CCCs [11]–[17]. The main drawback of CCC is the set size, which is bounded by the number of sequences in each constituting GCS, called flock size and it is a barrier for the increment in the number of supported users for the application in multi-carrier code division multiple access (MC-CDMA) [18]. To overcome this drawback, ZCS was extended to the Z-complementary code set (ZCCS) in [7]. A ZCCS consists of codes that exhibit zero ACCF sum within the ZCZ region and provide zero AACF sum within the same ZCZ, except for the zero time shift. Several direct constructions of ZCCS can be found in the literature using the GBF method [19]–[21], as well as indirect construction using PU matrices [22]. A special case of ZCCS is an inter-group complementary (IGC) code set, where the codes are divided into some groups, and the ACCF sum of codes from two different groups is zero for all time shifts. Several constructions of IGC code sets can be found in the literature [23]–[27].

The discussions above are about one-dimensional (1-D) sequences. However, these 1-D sequences can be generalized to two-dimensional (2-D) arrays. The GCP can be extended to 2-D, which is called a 2-D Golay complementary array pair (GCAP). Some indirect constructions of 2-D GCAP are provided in [28], [29], as well as direct construction in [30] with sequence length being power-of-two only. Similarly, 2-D MOGCS is a generalization of 1-D MOGCS, which was first introduced by Lüke in 1985 [31] having row and column length of the form $2^\alpha 10^\beta 26^\gamma$ (α, β and γ are positive integers). It is important to note that the building blocks of 2-D MOGCS are called 2-D Golay complementary array sets (GCASs),

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which are collections of 2-D arrays exhibiting ideal 2-D AACF sum. In 2003, Zhang *et al.* proposed an application of 2-D GCAS in ultra-wideband multi-carrier code division multiple access (UWB MC-CDMA) system and constructed 2-D GCAS recursively with size $M \times N^n$ ($M = 2^{n_1}, N = 2^{n_2}$) [32]. When the set size of a 2-D MOGCS is equal to the number of constituent arrays, then it is called 2-D CCC [33]. An important application of 2-D CCCs and 2-D GCASs, as well as CSs, lies in omnidirectional precoding (OP) based transmission in massive multiple input multiple output (MIMO) with uniform rectangular array (URA) [34]. Till date, many indirect constructions of 2-D MOGCS and 2-D CCC have appeared in the literature [31], [33], [35], [36]. In [33], [35], indirect construction of 2-D CCC from existing 1-D CCC has been developed. The size of 2-D CCC in [33] is $P^2 \times Q^2$ (where P, Q are orders of 1-D CCCs), whereas in [35] the size is of the form $L \times L$, where L is the length of seed 1-D CCC. This L may be varied according to the existing construction of 1-D CCC, such as $L = N, 2^m N, N^2$, etc. for some positive integer m, N; N being dependent on the choice of some unitary-like matrix. In [36], indirect construction of 2-D CCC of size $p^m \times p^n$ (m, n are positive integers and p is prime) has been proposed by extracting extended Boolean functions (EBFs) from seed paraunitary (PU) matrices.

As the set size is limited in 2-D CCC, the 2-D generalization of the ZCCS is introduced by Zeng *et al.* in 2004, called 2-D Z-complementary array code set (ZCACS) [37]. Each set of 2-D arrays of a 2-D ZCACS which have zero 2-D AACF sum in a rectangular ZCZ, except the $(0,0)$ time shift, is called a 2-D Z-complementary array code (ZCAC). In [37], [38], the theoretical bound for set size has been established for a binary, ternary, or polyphase 2-D ZCACS. When the flock size of 2-D ZCAC is 2, it is called 2-D Z-complementary array pair (ZCAP) [39], [40]. In the MC-CDMA system, 2-D ZCAC can be applied with the aim of reducing PMEPR in one dimension and having high bandwidth efficient spreading in another dimension [41]. In 1-D settings, column sequence PMEPR of a ZCS is of particular interest in MC-CDMA system [21], which needs to be reduced. Hence, for the application of 2-D ZCAC in MC-CDMA we need at least one of the PMEPRs, either row or column, to be bounded by a sufficiently low value. Besides, 2-D ZCACS can support more users than 2-D CCC in a multi-access system, say, 2D-MC-CDMA [33], [42], [43]. Hence, 2-D ZCACS with flexible parameters is an important requirement for use in spreading schemes having flexible parameters.

There are some constructions of 2-D ZCACS in the literature, which also include the construction of 2-D ZCAC, as they are the building blocks of a 2-D ZCACS. It should be noted that 2-D GCAS is a special case of 2-D ZCAC, where $L_1 = Z_1$ and $L_2 = Z_2$. Some indirect constructions of 2-D ZCACSs are [38], [41] and direct construction of 2-D GCASs [32], [44], [45]. In [38], ternary 2-D ZCACS has been constructed indirectly from existing binary 2-D ZCACS by inserting zeroes in the specific places of binary arrays of 2-D ZCACS and the lower bound for set size has been derived. The array size in this construction is $L_1 \times (L_2 + r + 1)$ and ZCZ size is $Z_1 \times Z_2$ ($rZ_2 \leq Z_2'$), where $L_1 \times L_2$ and $Z_1' \times Z_2'$ are the array size and ZCZ size of the initial 2-D arrays, respectively. In 2020, Das *et al.* [41] proposed an indirect construction of polyphase 2-D ZCACS by using 2-D Z-paraunitary (ZPU) matrices. In this construction, the size of each array of 2-D ZCACS is of the form $K \times K$ and the ZCZ size is of the form $M \times K$ for some positive integers K, M. Here P satisfies $K = MP$, which depends on the choice of *Butson-type Hadamard* (BH) matrices, thus making the parameters of array size restricted to the initial choice. Recently, in [46], a direct construction of binary 2-D ZCACS is provided, and in [47], a direct construction of general q-ary 2-D ZCACS is provided using 2-D multivariable function. In [46], the proposed 2-D ZCACS has size of the form $2^m \times (14 \cdot 2^n)$, where $m \geq 3, n \geq 1$. The 2-D ZCACS in [47] has array size of the form $R_1 N_1 \times R_2 N_2$, where $R_1 \geq 1, R_2 \geq 2, N_1 = p_1^{m_1} \cdots p_{k_1}^{m_{k_1}}, N_2 = q_1^{n_1} \cdots q_{k_2}^{n_{k_2}}$, p_i, q_i being prime, $m_i, n_i \geq 1$. In [44], a direct construction of 2-D GCASs has been proposed by GBFs, where the array sizes are of the form $2^n \times 2^m$ ($m, n \geq 2$ are positive integers) restricting the range of array sizes. In [45], the authors have constructed 2-D GCAS of size $p_1^{m_1} \times p_2^{m_2}$ directly, where p_1, p_2 are primes by using multivariable functions. To the best of the authors' knowledge, there is no direct construction of 2-D ZCAC and 2-D ZCACS with flexible row and column lengths in the literature. In the recent wireless communication technology, URA-based massive MIMO has played a very important role where data is encoded with space-time block codes (STBCs), and 2-D arrays are used for precoding the data to provide omnidirectional transmission [34], [48]–[54]. In [55], [56] 2-D CCC, in [45], [57], [58] 2-D GCAS and in [59] 2-D GCAP has been used as precoding matrices. But they are limited by array sizes and flock sizes. The flock size of the 2-D code plays an important role as it determines the number of antennas in the STBC used in the transmission and the array size determines the size of the URA used in the system. One interesting problem that has recently drawn the interest of researchers in communication theory, can be stated as: *"Can 2-D or 1-D codes be constructed for all types of antenna configurations in an OP-based massive MIMO system with good performance benefits?"* The attempts made in [45], [55], [57]–[59] provide a large class of such codes. Still, they do not cover the entirety of possible parameter values of the codes.

Motivated by all the above reasons, in this paper, we develop a direct construction method of 2-D ZCACS for flexible even row length by using multivariable functions. We first propose a construction of IGC code set of length of the form $p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$, where p_α's are prime numbers and $m_\alpha \geq 2$ for all α, by using multivariable functions. Although, the IGC code set is constructed primarily to produce 2-D ZCAC and 2-D ZCACS, a direct construction with the length of this form appears for the first time in this paper. Hence, we have compared our proposed IGC code set with existing literature in TABLE II. Then we propose the construction of 2-D ZCAC and 2-D ZCACS by using a set of multivariable functions, which is obtained from the construction of the IGC code set. For some special cases, the proposed 2-D ZCAC is shown to reduce into 2-D ZCAP of size $2^m p \times 2^{m_1+1}$ ($m \geq 1, m_1 \geq 2$,

p prime). The array size of the proposed 2-D ZCAC and 2-D ZCACS is of the form $L_1 \times L_2$, where $L_1 = 2^m p$ ($p \geq 2$ is a prime number) and $L_2 = 2p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$, making the row length a flexible even number. We have investigated the optimality condition of the proposed 2-D ZCACS. The proposed 2-D ZCACS is not optimal according to the existing upper bound [38], but a similar argument for ZCCS in [7] may be used to obtain a tighter upper bound for 2-D ZCACS and construct optimal code set. We have also investigated the row and column sequence PMEPRs of the arrays of the constructed 2-D ZCAC and shown its advantage against the existing literature. We have derived 2-D GCAS and GCS from the proposed 2-D ZCAC by using truncation and shown its application in OP-based transmission in massive MIMO. The 2-D GCAS and GCS are suitable for increasing the flexibility and efficiency of the STBC-encoded MIMO-URA system and the simulation result shows that it has better bit-error-rate (BER) performance than the existing direct constructions [45], [55], [57]–[59], for some special conditions. Finally, we compared our proposed constructions of 2-D GCAS for OP-based MIMO-URA system, and general q-ary 2-D ZCACS with the existing constructions based on their parameters in TABLE III and IV, respectively.

The rest of the paper is organized as follows. In Section II, some definitions and lemmas related to the proposed construction and its applications are discussed. The construction of the IGC code set and comparison with existing literature are described in Section III. Section IV contains the construction of 2-D ZCAC and 2-D ZCACS. The bounds of row and column sequence PMEPRs have been investigated and simulation results for BER performance have been shown for the derived 2-D GCAS and GCS. The existing constructions of 2-D GCASs have been compared with the proposed 2-D GCAS/GCS from an application point of view in massive MIMO. The parameters of existing 2-D ZCACS also have been compared with the proposed work. Finally, conclusions are drawn in Section V.

II. Preliminaries

A. One dimensional sequences

Definition 1 (ACCF): For two complex-valued sequences $\mathbf{a} = (a_0, a_1, \dots, a_{L-1})$ and $\mathbf{b} = (b_0, b_1, \dots, b_{L-1})$ of length L, the ACCF at the time shift τ is defined by $C(\mathbf{a}, \mathbf{b})(\tau) = \sum_{i=0}^{L-1} a_i b_{i+\tau}^*$, where $(\cdot)^*$ denotes the complex conjugate and $a_i, b_i = 0$, when $i \in \mathbb{Z} \setminus \{0, 1, \dots, L-1\}$. If $\mathbf{a} = \mathbf{b}$, then the function is called an AACF and is denoted by $A(\mathbf{a})(\tau)$.

Definition 2 (MOGCS([12])): A collection $\mathcal{S} = \{\mathcal{C}_0, \mathcal{C}_1, \cdots, \mathcal{C}_{K-1}\}$ of codes, where $\mathcal{C}_\nu = \{\mathbf{a}_0^\nu, \mathbf{a}_1^\nu, \dots, \mathbf{a}_{M-1}^\nu\}$ is called an MOGCS if it satisfies

$$\sum_{\lambda=0}^{M-1} C(\mathbf{a}_\lambda^{\nu_1}, \mathbf{a}_\lambda^{\nu_2})(\tau) = \begin{cases} ML, & \tau = 0, \quad \nu_1 = \nu_2; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

When $K = M$, then $\mathcal{S}$ is called a CCC. Each code of a MOGCS is called a GCS [8]. When $M = 2$, then a GCS is called a GCP [1]. Two GCPs $(\mathbf{a}_1, \mathbf{b}_1)$ and $(\mathbf{a}_2, \mathbf{b}_2)$ are called complementary mates if they satisfy $C(\mathbf{a}_1, \mathbf{a}_2)(\tau) + C(\mathbf{b}_1, \mathbf{b}_2)(\tau) = 0, \quad \forall \tau$.

Definition 3 (IGC code set ([23])): Let $\mathcal{S}(K, M, L, Z) = \{\mathcal{C}_i : i = 0, 1, \dots, K-1\}$ be a code set with K number of codes, M sequences in each code, sequence length L and ZCZ width Z. Suppose, the parameters satisfy $K = M \lfloor L/Z \rfloor$. Let, the K codes be divided into M number of code groups $I^g (g = 0, 1, \dots, M-1)$. Let the code set $I(K, M, L, Z)$ satisfy the following properties:

$$\sum_{\lambda=0}^{M-1} C(\mathbf{a}_\lambda^i, \mathbf{a}_\lambda^j)(\tau) = \begin{cases} ML, & i = j, \quad \tau = 0, \\ 0, & i = j, \quad 0 < |\tau| < Z, \\ 0, & i \neq j, \quad \mathcal{C}_i, \mathcal{C}_j \in I^g, \ |\tau| < Z, \\ 0, & \mathcal{C}_i \in I^{g_1}, \quad \mathcal{C}_j \in I^{g_2}, \forall \tau. \end{cases} \tag{2}$$

Then, the code set $\mathcal{S}(K, M, L, Z)$ is called an IGC code set.

B. Two Dimensional Array

A 2-D array is a two dimensional matrix $\mathbf{X}$, whose elements are denoted by $X_{i,j}$, where $0 \leq i < L_1$, $0 \leq j < L_2$ and the size of the array is $L_1 \times L_2$.

Definition 4 (2-D ACCF): The 2-D ACCF of two arrays $\mathbf{X}$ and $\mathbf{Y}$ both of size (L_1, L_2) at time shift (τ_1, τ_2) is defined by $C(\mathbf{X}, \mathbf{Y})(\tau_1, \tau_2) = \sum_{i=0}^{L_1-1} \sum_{j=0}^{L_2-1} X_{i,j} Y_{i+\tau_1, j+\tau_2}^*$. Here, $X_{i,j}, Y_{i,j} = 0$, when $i \in \mathbb{Z} \setminus \{0, 1, \dots, L_1 - 1\}$ or $j \in \mathbb{Z} \setminus \{0, 1, \dots, L_2 - 1\}$. When $\mathbf{X} = \mathbf{Y}$, then it is called 2-D AACF and is denoted by $A(\mathbf{X})(\tau_1, \tau_2)$. It can be noted that, for $\tau_1, \tau_2 \geq 0$, we have $C(\mathbf{X}, \mathbf{Y})(\tau_1, \tau_2) = \sum_{i=0}^{L_1-\tau_1-1} \sum_{j=0}^{L_2-\tau_2-1} X_{i,j} Y_{i+\tau_1, j+\tau_2}^*$.

Definition 5 (2-D ZCAC ([38])): A set $\mathcal{X} = \{\mathbf{X}_0, \mathbf{X}_1, \dots, \mathbf{X}_{M-1}\}$ of 2-D arrays with equal size $L_1 \times L_2$ is called a 2-D ZCAC if they satisfy

$$\sum_{i=0}^{M-1} A(\mathbf{X}_i)(\tau_1, \tau_2) = \begin{cases} ML_1L_2, & (\tau_1, \tau_2) = (0, 0); \\ 0, & 0 \leq |\tau_1| < Z_1, \\ & 0 \leq |\tau_2| < Z_2, \\ & (\tau_1, \tau_2) \neq (0, 0). \end{cases} \tag{3}$$

where $Z_1 \leq L_1$, $Z_2 \leq L_2$. $Z_1 \times Z_2$ is the rectangular ZCZ size. When $M = 2$, then it is called a 2-D Z-complementary array pair (ZCAP). If $Z_1 = L_1$ and $Z_2 = L_2$, then it is called 2-D GCAS. A 2-D GCAS is called a 2-D GCAP if $M = 2$.

Definition 6 (2-D ZCACS ([38])): A collection of 2-D ZCAC $\{\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_{K-1}\}$ where $\mathcal{X}_\nu = \{\mathbf{X}_0^\nu, \mathbf{X}_1^\nu, \dots, \mathbf{X}_{M-1}^\nu\}$ and each $\mathbf{X}_i^\nu$ has equal size of $L_1 \times L_2$, is called a 2-D ZCACS if they satisfy

$$\begin{aligned} &\sum_{i=0}^{M-1} C(\mathbf{X}_i^{\nu_1}, \mathbf{X}_i^{\nu_2})(\tau_1, \tau_2) \\ &= \begin{cases} ML_1L_2, & \nu_1 = \nu_2, (\tau_1, \tau_2) = (0, 0); \\ 0, & \nu_1 = \nu_2, 0 \leq |\tau_1| < Z_1, 0 \leq |\tau_2| < Z_2, \\ & (\tau_1, \tau_2) \neq (0, 0); \\ 0, & \nu_1 \neq \nu_2, |\tau_1| < Z_1, |\tau_2| < Z_2, \end{cases} \end{aligned} \tag{4}$$

where $Z_1 \leq L_1$, $Z_2 \leq L_2$. We denote it as a 2-D $(K, Z_1 \times Z_2) - \text{ZCACS}_M^{L_1 \times L_2}$. When $L_1 = Z_1$ and $L_2 = Z_2$, then it is called 2-D MOGCS. When $L_1 = 1, Z_1 = 1, L_2 = L$ and $Z_2 = Z$, then it is called ZCCS and denoted by $(K, Z) - \text{ZCCS}_M^L$.

Lemma 1 ([38]): For an aperiodic 2-D $(K, Z_1 \times Z_2) - \text{ZCACS}_M^{L_1 \times L_2}$, the following equation holds true:

$$K \leq \frac{M(L_1 + Z_1 - 1)(L_2 + Z_2 - 1)}{Z_1 Z_2}. \tag{5}$$

A 2-D ZCACS is called optimal if K achieves its upper bound.

Lemma 2 ([60]): For an aperiodic $(K, Z) - \text{ZCCS}_M^L$, the parameters satisfy $K \leq M\left(\frac{L+Z-1}{Z}\right)$. An aperiodic ZCCS is usually said to be optimal [7] if $K = M\left\lfloor \frac{L}{Z} \right\rfloor$.

C. Multivariable Function

Let, for any prime p, $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ be the set of integers modulo p. Then a multivariable function [17], [61] can be defined as $f : \mathbb{Z}_{p_1}^{m_1} \times \mathbb{Z}_{p_2}^{m_2} \times \cdots \times \mathbb{Z}_{p_k}^{m_k} \to \mathbb{Q}_q$, where $p_1, p_2, \ldots, p_k$ are prime numbers, q is a positive integer and $\mathbb{Q}_q = \{\frac{a}{b} : a, b \in \mathbb{Z}_q, b \neq 0\}$. It is worth noting that $\mathbb{Z}_q \subset \mathbb{Q}_q \subset \mathbb{Q}$, because, from a simple set-theoretic point of view, i.e., without considering them as sets of residue classes, any $a \in \mathbb{Z}_q$ may be considered $a \equiv \frac{a}{1}$. So, such $a \in \mathbb{Q}_q$. For ease of understanding, we define the following notations:

- $v_{p_\alpha,1}, v_{p_\alpha,2}, \ldots, v_{p_\alpha,m_\alpha}$ are the m_α variables which take values from $\mathbb{Z}_{p_\alpha}$ for $\alpha = 1, 2, \ldots, k$.
- $\mathbf{u}_{p_\alpha,i_\alpha}$ is the vector representation of i_α with base p_α, i.e., $\mathbf{u}_{p_\alpha,i_\alpha} = (i_{\alpha,1}, i_{\alpha,2}, \ldots, i_{\alpha,m_\alpha})$ where $i_\alpha = \sum_{\gamma=1}^{m_\alpha} p_\alpha^{\gamma-1} i_{\alpha,\gamma}$.
- The set M^r given by
$$M^r = \left\{ \prod_{\alpha=1}^{k} \prod_{\beta=1}^{m_\alpha} v_{p_\alpha,\beta}^{j_\beta^\alpha} : 0 \leq \sum_{\alpha=1}^{k} \sum_{\beta=1}^{m_\alpha} j_\beta^\alpha \leq r \right\}$$
is the set of monomials of degree at most r.

A multivariable function f of order r is a linear combination of the monomials from the sets M^r with coefficients from $\mathbb{Q}_q$, such that at least one monomial of degree r has a non-zero coefficient. The function rule of a multivariable function can be defined in a variety of ways, that follow the fundamental property of a *single-valued function*, i.e., the image of an element from the domain should be unique in the range of the function. Here we follow a particular function rule that is convenient for the proposed constructions. To be specific, we use the concept of modulus operation. But, first note that, the modulus operation is valid only on integers, not on the elements of the rational number field $\mathbb{Q}$. Hence, we extend this concept on $\mathbb{Q}$ and for a multivariable function f, which is a $\mathbb{Q}_q$-linear combination of the monomials, here we define the function rule as

$$(\overrightarrow{i_{p_1}}, \ldots, \overrightarrow{i_{p_k}}) \longmapsto f(\overrightarrow{i_{p_1}}, \ldots, \overrightarrow{i_{p_k}}) \bmod q \coloneqq \frac{a_f \bmod q}{b_f \bmod q}, \tag{6}$$

where $f(\overrightarrow{i_{p_1}}, \overrightarrow{i_{p_2}}, \ldots, \overrightarrow{i_{p_k}}) \in \mathbb{Q}$ is of the form $\frac{a_f}{b_f}$; $a_f, b_f \in \mathbb{Z}$, $b_f \mod q \neq 0$, and $\gcd(a_f, b_f) = 1$. Whenever $b_f \mod q = 0$, then we define the function as

$$(\overrightarrow{i_{p_1}}, \ldots, \overrightarrow{i_{p_k}}) \longmapsto f(\overrightarrow{i_{p_2}}, \ldots, \overrightarrow{i_{p_k}}) \mod q \coloneqq q_0, \tag{7}$$

for some fixed $q_0 \in \mathbb{Z}_q$. Throughout this paper, the term $(f(\overrightarrow{i_{p_1}}, \overrightarrow{i_{p_2}}, \ldots, \overrightarrow{i_{p_k}}) \bmod q)$ and f is used invariably, unless otherwise stated. We illustrate the above-mentioned function rule for multivariable functions in the following example.

Example 1: Let us have a multivariable function $f : \mathbb{Z}_3^2 \times \mathbb{Z}_5 \to \mathbb{Q}_4$ defined as $f(V_3, V_5) = \frac{2}{3} v_{3,1} v_{5,1} + v_{3,2}$, where $V_3 = (v_{3,1}, v_{3,2})$ and $V_5 = v_{5,1}$. We take $\overrightarrow{i_3} = (2, 2)$ and $\overrightarrow{i_5} = 4$. Then we have

$$f(\overrightarrow{i_3}, \overrightarrow{i_5}) = \left(\frac{2}{3} \times 2 \times 4 + 2\right) \bmod 4 = \frac{22 \bmod 4}{3 \bmod 4} = \frac{2}{3}. \tag{8}$$

A multivariable function $f : \mathbb{Z}_{p_1}^{m_1} \times \mathbb{Z}_{p_2}^{m_2} \times \cdots \times \mathbb{Z}_{p_k}^{m_k} \to \mathbb{Q}_q$ can be associated with a $\mathbb{Q}_q$-valued sequence of length $p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$. The $\mathbb{Q}_q$-valued sequence corresponding to the multivariable function f is given by

$$\mathrm{f} = \left(f(\mathbf{u}_{p_1,i_1}, \mathbf{u}_{p_2,i_2}, \ldots, \mathbf{u}_{p_k,i_k}) : 0 \leq i_\alpha < p_\alpha^{m_\alpha} \right).$$

One can also associate a complex-valued sequence corresponding to a multivariable function f, which is given by

$$\psi(f) = \left(\omega_q^{f(\mathbf{u}_{p_1,i_1}, \mathbf{u}_{p_2,i_2}, \ldots, \mathbf{u}_{p_k,i_k})} : 0 \leq i_\alpha < p_\alpha^{m_\alpha} \right), \tag{9}$$

where $\omega_q = \exp(\frac{2\pi\sqrt{-1}}{q})$. We further introduce some notations to produce 2-D matrices by using multivariable functions. Let $F_{\mathbf{d}} : \mathbb{Z}_{p_1}^{m_1} \times \mathbb{Z}_{p_2}^{m_2} \times \cdots \times \mathbb{Z}_{p_k}^{m_k} \to \mathbb{Q}_q$ be a set of multivariable functions for $\mathbf{d} = (d_{p'_1}, d_{p'_2}, \ldots, d_{p'_w}) \in \prod_{\alpha=1}^{w} \mathbb{Z}_{p'_\alpha}$, where p'_α are some prime numbers. Let $K = \{F_{\mathbf{d}} : \mathbf{d} \in \prod_{\alpha=1}^{w} \mathbb{Z}_{p'_\alpha}\}$ be an ordered set, which can be written as a $\mathbb{Q}_q$-valued matrix or array of size $L_1 \times L_2$, where $L_1 = p'_1 p'_2 \ldots p'_w$ and $L_2 = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$. The i-th row of K is determined by the multivariable function $F_{\vec{i}}$, where $\vec{i}$ is the vector representation of i in $\mathbb{Z}_{p'_1} \times \mathbb{Z}_{p'_2} \times \cdots \times \mathbb{Z}_{p'_w}$. So, the (i, j)-th component of the matrix K is defined by $K_{i,j} = F_{\vec{i}}(\vec{j})$, where $\vec{j}$ is the vector representation of j in $\prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}^{m_\alpha}$. We can also associate a complex-valued matrix $\Psi(K)$ with K, where $(\Psi(K))_{i,j} = \omega_q^{K_{i,j}}$ for all i, j.

TABLE I: 2-D array corresponding to the functions $F_{\mathbf{d}}$

$K=$

$\mathbf{d} = (d_2, d_3)$ \ $(v_{2,1}, v_{3,1})$	(0,0)	(1,0)	(0,1)	(1,1)	(0,2)	(1,2)
(0,0)	0	0	0	0	0	0
(1,0)	0	2	0	2	0	2
(0,1)	0	0	3	3	2	2
(1,1)	0	2	3	1	2	0
(0,2)	0	0	2	2	0	0
(1,2)	0	2	2	0	0	2

Example 2: Let $F_{\mathbf{d}} : \mathbb{Z}_2 \times \mathbb{Z}_3 \to \mathbb{Q}_4$ be defined by $F_{\mathbf{d}}(v_{2,1}, v_{3,1}) = 2v_{2,1}d_2 + 3v_{3,1}d_3$, where $\mathbf{d} = (d_2, d_3)$. Then the 6×6 2-D array K corresponding to $F_{\mathbf{d}}$ is generated in the manner as shown in TABLE I.

D. Peak-to-Mean Envelope Power Ratio (PMEPR)

MC-CDMA takes advantage of both CDMA and OFDM technology by using the concept of multiplexing and multi-access technology, simultaneously. In an asynchronous multi-carrier system the column sequence PMEPR is accounted for when a 1-D code is used in that system [18]. For a 2-D code, the 2-D arrays of that code can be used similarly taking into account only one of the dimensions for the AACF sum. Given a $\mathbb{Q}_q$-valued sequence $\mathbf{a} = \{a_0, a_1, \ldots, a_{L-1}\}$, the

MC-CDMA signal is the real part of the complex envelope $S_{\mathbf{a}}(t) = \sum_{i=0}^{L-1} \omega_q^{a_i+qf_it}$, where $f_i = f + i\Delta f$, f is a constant, Δf is an integer multiple of OFDM symbol rate and $0 \leq \Delta ft \leq 1$. The term $\frac{|S_{\mathbf{a}}(t)|^2}{L}$ is called the instantaneous-to-average power ratio (IAPR). The PMEPR of the sequence $\mathbf{a}$ is given by PMEPR$(\mathbf{a}) = \sup_{0\leq \Delta ft\leq 1} \frac{|S_{\mathbf{a}}(t)|^2}{L}$.

E. Omnidirectional Precoding in Massive MIMO

In this subsection, we introduce the concept of omnidirectional precoding in massive MIMO with URA. As described in [34], [56], we consider a downlink transmission from a base station (BS) equipped with URA to single-antenna user equipment (UE). We consider a $P \times Q$ URA with PQ antennas. Let, $A(\phi, \theta)$ be the steering matrix corresponding to the URA at direction (ϕ, θ) where $\phi \in [0, \pi/2]$ and $\theta \in [0, 2\pi]$, then $A(\phi, \theta)$ is given by

$$[A(\phi, \theta)]_{i,j} = e^{-\mathtt{i}\frac{2\pi}{\lambda} id_y \sin\phi \sin\theta - \mathtt{i}\frac{2\pi}{\lambda} jd_x \sin\phi\cos\theta}, \tag{10}$$

where $\mathtt{i} = \sqrt{-1}$, $i = 0, 1, \ldots, P-1$; $j = 0, 1, \ldots, Q-1$; λ is the wavelength of the carrier, d_x and d_y are the inter-element distance of URA along the vertical and horizontal directions, respectively. At first, the data bits are modulated into a $K\times N$ space-time block code (STBC) $\mathbf{S}$, the (n,t)-th entry given by $s_n(t)$, where $n = 0, 1, \ldots, K-1$ and $t = 0, 1, \ldots, N-1$ [62]. This is then beamformed by the precoding matrices $\mathbf{W}_n$ of size $P \times Q$. The received signal in a line-of-sight (LOS) channel without multi-paths is given by

$$y(t) = \sum_{n=0}^{K-1} \left[vec(A(\phi, \theta))^T \cdot vec(\mathbf{W}_n)\right] s_n(t) + w(t), \tag{11}$$

for $t = 0, 1, \ldots, N-1$; where $vec(\cdot)$ denotes the vector produced by vertically stacking the columns of the corresponding matrix and $w(t)$ is the additive white Gaussian noise (AWGN).

Lemma 3 ([56]): Let the precoding matrices $\{\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_K\}$ of size $P \times Q$ constitute a 2-D GCAS. Then the received power $\sum_{n=0}^{K-1} \left|\left[vec(A(\phi, \theta))^T \cdot vec(\mathbf{W}_n)\right]\right|^2$ is independent of the direction (ϕ, θ).

Lemma 4 ([34]): Let $\{c_0, c_1, \ldots, c_{P-1}\}$ be a GCS of length Q. Then the received power $\sum_{n=0}^{K-1} \left|\left[vec(A(\phi, \theta))^T \cdot vec(\mathbf{W}_n)\right]\right|^2$ is independent of the direction (ϕ, θ) if $K = P$ and $vec(\mathbf{W}_n)$ is chosen to be

$$vec(\mathbf{W}_n) = \begin{bmatrix} \underbrace{\mathbf{0}\cdots\mathbf{0}}_{n-\text{times}} & c_n & \underbrace{\mathbf{0}\cdots\mathbf{0}}_{(P-n-1)-\text{times}} \end{bmatrix}^T, \tag{12}$$

where $\mathbf{0}$ denotes the zero matrix of size $1\times Q$, and $(\cdot)^T$ denotes the transpose of the matrix $(\cdot)$.
In other words, *Lemma 3* and *Lemma 4* say that 2-D GCAS and GCS can be used to construct precoding matrices in omnidirectional massive MIMO system with URA.

III. Construction of IGC Code Set

In this section, a construction of IGC code set for lengths of the form $p_1^{m_1}p_2^{m_2}\ldots p_k^{m_k}$, ($m_\alpha \geq 2$, for all α, k is a positive integer) is presented by using multivariable function. Later, the construction of the IGC code set is used to construct 2-D ZCAC and 2-D ZCACS. We take the multivariable functions $f_\alpha : \mathbb{Z}_{p_\alpha}^{m_\alpha-1} \to \mathbb{Q}_q$, for $\alpha = 1, 2, \ldots, k$. Each f_α is defined by

$$f_\alpha = \frac{q}{p_\alpha} \sum_{\beta=1}^{m_\alpha-2} v_{p_\alpha,\pi_\alpha(\beta)} v_{p_\alpha,\pi_\alpha(\beta+1)} + \sum_{\beta=1}^{m_\alpha-1} c_{\alpha,\beta} v_{p_\alpha,\beta} + c_\alpha, \tag{13}$$

where $c_{\alpha,\beta}, c_\alpha \in \mathbb{Q}_q$ for $\beta = 1, 2, \ldots, m_\alpha-1$; $m_\alpha \geq 2$ for all α, π_α is a permutation of the set $\{1, 2, \ldots, m_\alpha - 1\}$ and q is taken in such a way that $p_\alpha \mid q$ for all $\alpha = 1, 2, \ldots, k$. Let the function $f : \mathbb{Z}_{p_1}^{m_1-1} \times \mathbb{Z}_{p_2}^{m_2-1} \times \cdots \times \mathbb{Z}_{p_k}^{m_k-1} \to \mathbb{Q}_q$ be defined by $f = f_1 + f_2 + \cdots + f_k$. Now, for $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, $\mathbf{s} = (s_1, s_2, \ldots, s_k)$, $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$, we define the set of functions $a_{\mathbf{s},\mathbf{t}}^{\boldsymbol{\gamma}} : \mathbb{Z}_{p_1}^{m_1-1} \times \cdots \times \mathbb{Z}_{p_k}^{m_k-1} \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k} \to \mathbb{Q}_q$, given by

$$a_{\mathbf{s},\mathbf{t}}^{\boldsymbol{\gamma}} = R_{\mathbf{t}}^{\boldsymbol{\gamma}} + T_{\mathbf{s}}, \tag{14}$$

where $R_{\mathbf{t}}^{\boldsymbol{\gamma}} : \mathbb{Z}_{p_1}^{m_1-1} \times \cdots \times \mathbb{Z}_{p_k}^{m_k-1} \to \mathbb{Q}_q$ is defined by $R_{\mathbf{t}}^{\boldsymbol{\gamma}} = f + \sum_{\alpha=1}^{k} \frac{q}{p_\alpha} v_{p_\alpha,\pi_\alpha(1)}\gamma_\alpha + \sum_{\alpha=1}^{k} \frac{q}{p_\alpha} v_{p_\alpha,\pi_\alpha(m_\alpha-1)} t_\alpha$, $T_{\mathbf{s}} : \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k} \to \mathbb{Q}_q$ is defined by $T_{\mathbf{s}} = \sum_{\alpha=1}^{k} \frac{q}{p_\alpha} v'_{p_\alpha} s_\alpha$, and v'_{p_α} is the variable corresponding to $\mathbb{Z}_{p_\alpha}$, for $\alpha = 1, 2, \ldots, k$. Next, we define the code $\mathcal{C}_{\mathbf{s},\mathbf{t}}$ which is given by $\mathcal{C}_{\mathbf{s},\mathbf{t}} = [\psi(a_{\mathbf{s},\mathbf{t}}^{\boldsymbol{\gamma}})]_{\boldsymbol{\gamma}\in\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\times\mathbb{Z}_{p_k}}$, which is a $M \times L$ matrix, where $M = p_1p_2\ldots p_k$ and $L = p_1^{m_1}p_2^{m_2}\ldots p_k^{m_k}$. The i-th row is given by $\psi(a_{\mathbf{s},\mathbf{t}}^{(i)_{\boldsymbol{\gamma}}})$, where $(i)_{\boldsymbol{\gamma}}$ is the vector representation of i in $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$. Now, we have our construction of the IGC code set of length $p_1^{m_1}p_2^{m_2}\ldots p_k^{m_k}$.

Theorem 1: The code set $I = \{\mathcal{C}_{\mathbf{s},\mathbf{t}} : \mathbf{s}, \mathbf{t} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}\}$ forms a (K, M, L, Z)-IGC code set with $K = (p_1p_2\ldots p_k)^2$, $M = p_1p_2\ldots p_k$, $L = p_1^{m_1}p_2^{m_2}\ldots p_k^{m_k}$ and $Z = p_1^{m_1-1}p_2^{m_2-1}\ldots p_k^{m_k-1}$. The $\mathbf{t}$-th IGC code group is expressed as $I^{\mathbf{t}} = \{\mathcal{C}_{\mathbf{s},\mathbf{t}} : \mathbf{s} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}\}$.

Proof: See Appendix A. ■

We illustrate *Theorem 1* by using an example.

Example 3: Let $p_1 = 2$, $p_2 = 3$ and $q = 6$ so that $p_1 \mid q$, $p_2 \mid q$ and $\pi_1(1) = 1$, $\pi_2(1) = 1$. Let the functions $f_1 : \mathbb{Z}_2 \to \mathbb{Q}_6$ and $f_2 : \mathbb{Z}_3 \to \mathbb{Q}_6$ be defined by $f_1 = 3v_{2,1}$ and $f_2 = 4v_{3,1}$. Then the function $a_{\mathbf{s},\mathbf{t}}^{\boldsymbol{\gamma}} : \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \to \mathbb{Q}_6$ is defined by $a_{\mathbf{s},\mathbf{t}}^{\boldsymbol{\gamma}} = f_1 + f_2 + (3v_{2,1}\gamma_1 + 2v_{3,1}\gamma_2) + (3v_{2,1}t_1 + 2v_{3,1}t_2) + (3v'_2s_1 + 2v'_3s_2)$ where $\mathbf{s} = (s_1, s_2)$, $\mathbf{t} = (t_1, t_2)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in \mathbb{Z}_2 \times \mathbb{Z}_3$. Then the code set corresponding to functions $a_{\mathbf{s},\mathbf{t}}^{\boldsymbol{\gamma}}$ is an IGC code set of length 36.

Note 1: IGC code sets have promising applications in CDMA and they can be used both in synchronous and asynchronous environments [23]. It is interesting to find that the IGC code set with such flexible parameters has been reported for the first time in this paper. Hence, to distinguish the proposed IGC code set from the existing ones, we draw a brief comparison in TABLE II. It can be seen that all the constructions of IGC code sets are either indirect [23]–[26]

TABLE II: Comparison table for IGC code set

Construction	Length	ZCZ width	Constraints	Based on
[23]	L	Z	$PZ=L$, $P\times P$ = size of orthogonal matrix, Z = length of perfect complementary (PC) code	Kronecker product
[24]	PL	Z	$0<Z\leq L$, $P\times P$ is size of orthogonal matrix, L is length of periodic complementary sequence (PCS), $Z=\left\lfloor\frac{L+1}{P}\right\rfloor\cdot P-1$	Interleaving
[25]	$(N+Z-1)L$	Z	$0<Z\leq N$, Z is ZCZ width of ZCCS, N is length of ZCCS, L is length of perfect periodic cross-correlation (PPCC) set	Kronecker product
[26]	$2L$	Z	$0<Z<L$, L is length of PC code	Interleaving, Matrix operation
[27]	2^m	2^{m-p}	$m\geq 2, 1\leq p<m$	GBF
Theorem 1	$p_1^{m_1}p_2^{m_2}\dots p_k^{m_k}$	$p_1^{m_1-1}p_2^{m_2-1}\dots p_k^{m_k-1}$	p_i is prime and $m_i\geq 2$ for all $i=1,2,\dots,k$	Multivariable function

or direct [27] with limited lengths, whereas the proposed construction is direct and has flexible lengths.

IV. Construction of 2-D ZCAC and 2-D ZCACS

In this section, we give the construction of 2-D ZCAC and 2-D ZCACS of flexible row lengths using multivariable functions obtained from the construction of the IGC code set. But before giving the construction, we state two lemmas.

Lemma 5 ([9]): Let q be an even positive integer and $f:\mathbb{Z}_2^m\rightarrow\mathbb{Z}_q$ be a function given by

$$f=\frac{q}{2}\sum_{\beta=1}^{m-1}x_{\pi(\beta)}x_{\pi(\beta+1)}+\sum_{\beta=1}^{m}g_\beta x_\beta+e, \tag{15}$$

where π is a permutation of the set $\{1,2,\dots,m\}$ and $g_\beta\in\mathbb{Z}_q$. Then the complex-valued sequences $\mathbf{a}_1$ and $\mathbf{b}_1$, corresponding to the functions $a_1=f$, and $b_1=f+\frac{q}{2}x_{\pi(1)}$, respectively, forms a GCP of length 2^m for any $e\in\mathbb{Z}_q$.

Lemma 6 ([12]): Let f, a_1 and b_1 be defined as in *Lemma 5*. Then the complex-valued sequences $\mathbf{a}_2$ and $\mathbf{b}_2$, corresponding to the functions $a_2=\bar{f}+\frac{q}{2}x_{\pi(1)}$, and $b_2=\bar{f}$, respectively, forms a GCP of length 2^m, where $\bar{f}=\frac{q}{2}\sum_{\beta=1}^{m-1}\bar{x}_{\pi(\beta)}\bar{x}_{\pi(\beta+1)}+\sum_{\beta=1}^{m}g_\beta\bar{x}_\beta+e$, and $\bar{x}_i=1-x_i$, $\forall i$. Moreover, $(\mathbf{a}_1,\mathbf{b}_1)$ and $(\mathbf{a}_2,\mathbf{b}_2)$ are complementary mate.

Let, $2\mid q$, $a^{\boldsymbol{\gamma}}_{\mathbf{s},\mathbf{t}}$ be the same function used in *Theorem 1* and $p\geq 2$ be a prime number such that $p\mid q$. For $\tilde{\mathbf{d}}=(d_1,d_2,\dots,d_m,\tilde{d}_p)\in\mathbb{Z}_2^m\times\mathbb{Z}_p$, $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\dots,\gamma_k)\in\prod_{\alpha=1}^k\mathbb{Z}_{p_\alpha}$ and $\mathbf{t}_1\neq\mathbf{t}_2$, and $\boldsymbol{\zeta}=(\mathbf{s}_1,\mathbf{s}_2,\mathbf{t}_1,\mathbf{t}_2)$, we define a set of functions

$$F^{\boldsymbol{\gamma},\zeta}_{\tilde{\mathbf{d}}}:\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}^{m_\alpha-1}\times\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}\times\mathbb{Z}_2\rightarrow\mathbb{Q}_q, \tag{16}$$

given by

$$\begin{aligned}F^{\boldsymbol{\gamma},\zeta}_{\tilde{\mathbf{d}}}=&\bar{v''}\left\{a^{\boldsymbol{\gamma}}_{\mathbf{s}_1,\mathbf{t}_1}+a_1(\mathbf{d})W_1(\tilde{d}_p)+a_2(\mathbf{d})W_2(\tilde{d}_p)\right\}\\&+v''\left\{a^{\boldsymbol{\gamma}}_{\mathbf{s}_2,\mathbf{t}_2}+b_1(\mathbf{d})W_1(\tilde{d}_p)+b_2(\mathbf{d})W_2(\tilde{d}_p)\right\},\end{aligned} \tag{17}$$

where

$$W_i(\tilde{d}_p)=\sum_{j=0}^{p-1}\frac{\prod_{i=0,i\neq j}^{p-1}(i-\tilde{d}_p)}{\prod_{i=0,i\neq j}^{p-1}(i-j)}\delta_j^i, \tag{18}$$

$$\delta_j^1=\begin{cases}1, & 2\mid j;\\0, & 2\nmid j;\end{cases}\delta_j^2=\begin{cases}0, & 2\mid j;\\1, & 2\nmid j;\end{cases} \tag{19}$$

v'' is the variable corresponding to $\mathbb{Z}_2$ in (16), $\bar{v''}=1-v''$; a_1 and b_1 are the functions from *Lemma 5*, a_2 and b_2 are the functions from *Lemma 6* and $\mathbf{d}=(d_1,d_2,\dots,d_m)$. We define the set $\Omega=\{(\mathbf{s}_1,\mathbf{s}_2,\mathbf{t}_1,\mathbf{t}_2):\mathbf{t}_1\neq\mathbf{t}_2\}$. Let $\Lambda\subset\Omega$ such that for any two $\boldsymbol{\zeta}_1=(\mathbf{s}_1^1,\mathbf{s}_2^1,\mathbf{t}_1^1,\mathbf{t}_2^1),\boldsymbol{\zeta}_2=(\mathbf{s}_1^2,\mathbf{s}_2^2,\mathbf{t}_1^2,\mathbf{t}_2^2)\in\Lambda$, $\mathbf{s}_i^j,\mathbf{t}_i^j$ follow the condition: Either $\mathbf{t}_i^1\neq\mathbf{t}_j^2$ or $\mathbf{s}_i^1\neq\mathbf{s}_j^2$ for $1\leq i,j\leq 2$. Now, we consider only those function $\mathcal{F}^{\boldsymbol{\gamma},\boldsymbol{\zeta}}_{\tilde{\mathbf{d}}}:\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}^{m_\alpha-1}\times\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}\times\mathbb{Z}_2\rightarrow\mathbb{Q}_q$, for which $\boldsymbol{\zeta}\in\Lambda$. Let the $\mathbb{Q}_q$-valued 2-D arrays be given by $\mathcal{K}^{\boldsymbol{\gamma},\boldsymbol{\zeta}}=\{F^{\boldsymbol{\gamma},\boldsymbol{\zeta}}_{\tilde{\mathbf{d}}}:\tilde{\mathbf{d}}\in\mathbb{Z}_2^m\times\mathbb{Z}_p\}$ for $\boldsymbol{\gamma}\in\prod_{\alpha=1}^k\mathbb{Z}_{p_\alpha}$ and we define $\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}}=\Psi(\mathcal{K}^{\boldsymbol{\gamma},\boldsymbol{\zeta}})$.

Theorem 2: Each set $\mathcal{X}^{\boldsymbol{\zeta}}=\{\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}}:\boldsymbol{\gamma}\in\prod_{\alpha=1}^k\mathbb{Z}_{p_\alpha}\}$ is a 2-D ZCAC of size $L_1\times L_2$, where $L_1=2^mp$ and $L_2=2p_1^{m_1}p_2^{m_2}\dots p_k^{m_k}$, ZCZ size is $Z_1\times Z_2$, where $Z_1=2^{m+1}$ and $Z_2=p_1^{m_1-1}p_2^{m_2-1}\dots p_k^{m_k-1}$ and flock size $M=p_1p_2\dots p_k$. The set $\{\mathcal{X}^{\boldsymbol{\zeta}}:\boldsymbol{\zeta}\in\Lambda\}$ is a 2-D ZCACS with set size $K=2p_1^2p_2^2\dots p_k^2$.

Proof: See Appendix B. ■

The following example illustrates *Theorem 2*.

Example 4: Let $p_1=2$, $p_2=3$, $q=6$, $\pi_1(1)=1$, $\pi_2(1)=1$, $a^{\boldsymbol{\gamma}}_{\mathbf{s}_i^k,\mathbf{t}_j^k}$, f_1, and f_2 be the same functions in *Example 3*. Let the function $f:\mathbb{Z}_2^2\rightarrow\mathbb{Z}_6$ be defined by $f=3x_1x_2+3$, where $\pi(1)=1,\pi(2)=2$. Then, let the functions $a_1,b_1,a_2,b_2:\mathbb{Z}_2^2\rightarrow\mathbb{Z}_6$ be given by $a_1=3x_1x_2+3$, $b_1=3x_1x_2+3+3x_1$, and a_2,b_2 are generated from *Lemma 6*. We also take $\boldsymbol{\zeta}_1=(\mathbf{s}_1^1,\mathbf{s}_2^1,\mathbf{t}_1^1,\mathbf{t}_2^1)$, where $\mathbf{s}_1^1=\mathbf{s}_2^1=(1,1)$ and $\mathbf{t}_1^1=(0,1)$, $\mathbf{t}_2^1=(1,2)$ and similarly we take $\boldsymbol{\zeta}_2=(\mathbf{s}_1^2,\mathbf{s}_2^2,\mathbf{t}_1^2,\mathbf{t}_2^2)$, where $\mathbf{s}_1^2=\mathbf{s}_2^2=(1,1)$ and $\mathbf{t}_1^2=(1,1)$, $\mathbf{t}_2^2=(1,0)$, so that $\mathbf{t}_1^1,\mathbf{t}_2^1,\mathbf{t}_1^2$ and $\mathbf{t}_2^2$ are all distinct. We also take $\tilde{\mathbf{d}}=(d_1,d_2,\tilde{d}_p)\in\mathbb{Z}_2^2\times\mathbb{Z}_3$, W_1,W_2 are generated from (18). Then, we get a 2-D ZCACS of size 12×72 and the set $\{\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_i}:\boldsymbol{\gamma}\in\mathbb{Z}_2\times\mathbb{Z}_3\}$ contains 6 numbers of 2-D arrays. The 2-D AACF sum of the set of the 2-D arrays $\{\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1}:\boldsymbol{\gamma}\in\mathbb{Z}_2\times\mathbb{Z}_3\}$ is given in the Fig. 1 from which it can be seen that the set is a 2-D ZCAC. The size of the rectangular ZCZ is 8×6. The plot of the 2-D ACCF sum of the set $\{\mathcal{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1},\mathcal{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}\}$ is given in Fig. 2, where it can be noted that the peaks are zero for all values of the 2-D time shifts (τ_1,τ_2).

Remark 1: It is to be noted that, fixing $k=1$, and $p_1=2$, the set $\mathcal{X}=\{\mathbf{X}^\gamma:\gamma\in\mathbb{Z}_2\}$ becomes a 2-D ZCAP $\{\mathbf{X}^0,\mathbf{X}^1\}$ of size $2^mp\times 2^{m_1+1}$ and rectangular ZCZ width is $2^{m+1}\times 2^{m_1-1}$. 2-D ZCAP of this size is reported for the first time in this paper.

Remark 2: In our proposed construction, $L_1=2^mp$, $L_2=2p_1^{m_1}p_2^{m_2}\dots p_k^{m_k}$, $Z_1=2^{m+1}$, $Z_2=p_1^{m_1-1}p_2^{m_2-1}\dots p_k^{m_k-1}$ and $M=p_1p_2\dots p_k$. The value of K depends on the choice of $\zeta=(\mathbf{s}_1,\mathbf{s}_2,\mathbf{t}_1,\mathbf{t}_2)\in\Lambda$ and can be calculated to be $K=2p_1^2p_2^2\dots p_k^2$. So, the proposed 2-D ZCACS is not optimal with respect to the upper bound given in [38]. But, a similar rationale as in [7] could be used to arrive at the conclusion

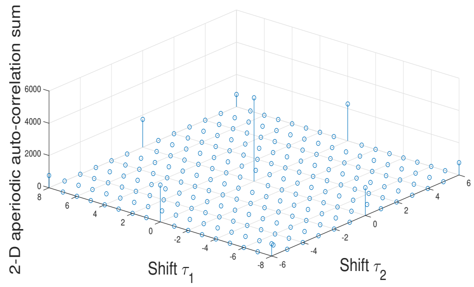

Fig. 1: The 2-D AACF sum of the set of 2-D arrays $\{\mathbf{X}^{\gamma,\zeta_1} : \gamma \in \mathbb{Z}_2 \times \mathbb{Z}_3\}$

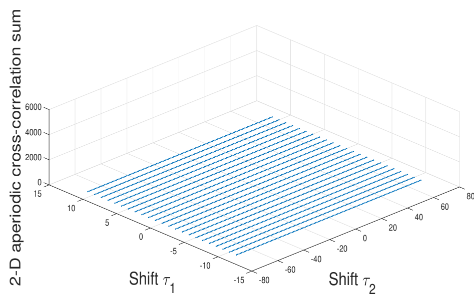

Fig. 2: The 2-D ACCF sum of the set $\{\mathcal{X}^{\gamma,\zeta_1}, \mathcal{X}^{\gamma,\zeta_2}\}$

that a 2-D ZCACS can be said to be optimal in general if $K = M \left\lfloor \frac{L_1}{Z_1} \right\rfloor \left\lfloor \frac{L_2}{Z_2} \right\rfloor$. Considering this optimality condition, the proposed 2-D ZCACS becomes an optimal one when $p = 2$.

Corollary 1: Let $\mathcal{X}^{\zeta}$ be any 2-D ZCAC with $p = 2$. Let $\bar{\mathcal{X}}^{\zeta}$ be the set of 2-D arrays derived from $\mathcal{X}^{\zeta}$ where each 2-D array is produced by deleting the columns from $(\prod_{i=1}^{k} p_i^{m_i-1} + 1)$-th to $(\prod_{i=1}^{k} p_i^{m_i})$-th, and $(\prod_{i=1}^{k} p_i^{m_i-1}(\prod_{i=1}^{k} p_i + 1) + 1)$-th to $(2\prod_{i=1}^{k} p_i^{m_i})$-th position. Then $\bar{\mathcal{X}}^{\zeta}$ is a 2-D GCAS of size $(2^{m+1}) \times (2p_1^{m_1-1} p_2^{m_2-1} \dots p_k^{m_k-1})$ and flock size $p_1 p_2 \dots p_k$.

Proof: The proof follows from *Lemma 7*, *Theorem 1*, *Theorem 2* and the fact that in the generating function $F_{\tilde{\mathbf{d}}}^{\gamma,\mathbf{s}_1,\mathbf{s}_2,\mathbf{t}_1,\mathbf{t}_2}$ of $\bar{\mathcal{X}}^{\zeta}$ the value of $T_{\mathbf{s}} = 0$. ■

Corollary 2: Let $\mathbf{X}$ be any set of a 2-D ZCAC $\mathcal{X}^{\zeta}$ with $p = 2$. Then the column vectors $\{c_0, c_1, \dots, c_{L_1-1}\}$ of $\mathbf{X}$ becomes a GCS of length 2^{m+1} and flock size $2p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$.

Proof: The proof follows from *Theorem 2* directly. ■

Remark 3: The 2-D GCAS and GCS thus derived from *Corollary 1* and *2* can be used to construct precoding matrices for OP-based massive MIMO system with URA. The flock size determines the number of antennas to transmit STBC-encoded data. For the derived 2-D GCAS, flock size is of the form $p_1 p_2 \dots p_k$. Hence, it can be applied for any arbitrary number of antennas to transmit STBC-encoded data in a massive MIMO system. Also, the URA size can be taken flexibly in the proposed cases. In Fig. 3, the supported URA sizes $P \times Q$ are plotted for different existing constructions [45], [55], [57]–[59] and we compare them with the proposed 2-D GCAS and GCS. We restrict the values of P, Q in the range $[0, 40]$. It can be easily seen that the proposed construction supports those URA sizes that cannot be supported by the existing constructions. For example, the transmission of STBC-encoded data through 18 antennas in an OP-based massive MIMO system with URA size 4×18 can be achieved only by the GCS derived from *Corollary 2*, but not by [45], [55]. Similarly, the transmission of STBC-encoded data through 3 antennas with URA size 4×6 can be achieved only by the 2-D GCAS derived in *Corollary 1*, but not by [45], [55]. Similar examples can be stated for other constructions too [57]–[59].

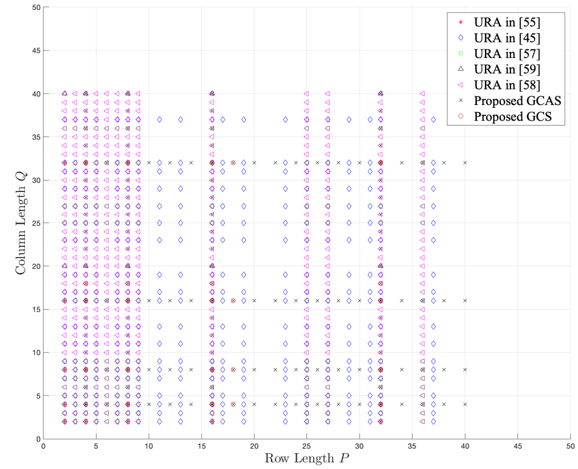

Fig. 3: Plot of supported URA size in different precoding schemes

A. Bound for PMEPR

It is easy to observe that for the m-th row sequence X_m and the n-th column sequence X_n^T we have

$$\begin{aligned} \mid S_{X_n^T}(t) \mid^2 &= L_1 + \sum_{\tau_1 \neq 0} A(X_n^T)(\tau_1) \omega_q^{-q\tau_1 \Delta f t}, \\ \mid S_{X_m}(t) \mid^2 &= L_2 + \sum_{\tau_2 \neq 0} A(X_m)(\tau_2) \omega_q^{-q\tau_2 \Delta f t}. \end{aligned} \tag{20}$$

For a fixed γ we take the $\mathbb{Q}_q$-valued 2-D array $K^{\gamma} = \{F_{\tilde{\mathbf{d}}}^{\gamma,\zeta} : \tilde{\mathbf{d}} \in \mathbb{Z}_2^m \times \mathbb{Z}_p\}$. Now, from *Theorem 1* and *2*, we have

$$\begin{aligned} &\sum_{\gamma \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} \mid S_{K_m^{\gamma}}(t) \mid^2 \\ &\leq L_2 \prod_{\alpha=1}^{k} p_\alpha + 2 \sum_{\beta=1}^{\prod_{\alpha=1}^{k} p_\alpha - 1} \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha} \cdot 2 \left(\prod_{\alpha=1}^{k} p_\alpha - \beta \right) \\ &= L_2 \prod_{\alpha=1}^{k} p_\alpha + 2 \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha+1} \left(\prod_{\alpha}^{k} p_\alpha - 1 \right), \end{aligned} \tag{21}$$

as $\sum_{\gamma \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} A(K_m^{\gamma})(\tau_2) = 0$, for $\beta \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1} < |\tau_2| < (\beta+1) \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$, where $\beta = 0, 1, \dots, 2\prod_{\alpha=1}^{k} p_\alpha - 1$ and $\mid \tau_2 \mid = \delta \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$ where $\delta = \prod_{\alpha=1}^{k} p_\alpha, \prod_{\alpha=1}^{k} p_\alpha +$

$1, \ldots, 2\prod_{\alpha=1}^{k} p_\alpha - 1$. But $\mid S_{K_m^{\gamma}}(t) \mid^2 \geq 0$ for all $\gamma \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}$ and $L_2 = 2\prod_{\alpha=1}^{k} p_\alpha^{m_\alpha}$. So, we have

$$\frac{\mid S_{K_m^{\gamma}}(t) \mid^2}{L_2} \leq \prod_{\alpha=1}^{k} p_\alpha + \prod_{\alpha}^{k} p_\alpha \left(\prod_{\alpha}^{k} p_\alpha - 1 \right). \tag{22}$$

Hence, PMEPR(K_m^{γ}) of the row sequence is upper bounded by $\prod_{\alpha=1}^{k} p_\alpha + \prod_{\alpha}^{k} p_\alpha \left(\prod_{\alpha}^{k} p_\alpha - 1 \right)$. Similarly, we can compute the bound of the PMEPR of the column sequence. For this, we note that either $K{\gamma}_n^T = \mathcal{I}_a$ or $K{\gamma}_n^T = \mathcal{I}_b$, where $\mathcal{I}_a = a^{\gamma}_{\mathbf{s_1},\mathbf{t_1}} + a_1(\mathbf{d})W_1(\tilde{d}_p) + a_2(\mathbf{d})W_2(\tilde{d}_p)$, $\mathcal{I}_b = a^{\gamma}_{\mathbf{s_2},\mathbf{t_2}} + b_1(\mathbf{d})W_1(\tilde{d}_p) + b_2(\mathbf{d})W_2(\tilde{d}_p)$ for $(\mathbf{d}, \tilde{d}_p) \in \mathbb{Z}_2^m \times \mathbb{Z}_p$ and a_1, a_2, b_1, b_2, W_1 and W_2 are the functions used in *Theorem 2*. Note that for $p \neq 2$, we have

$$\begin{aligned} &A(\mathcal{I}_a)(\tau_1) + A(\mathcal{I}_b)(\tau_1) \\ &= \begin{cases} (p-\delta)2^{m+1}, & \mid \tau_1 \mid = \delta 2^m, \delta = 0, 2, \ldots, p-1, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{23}$$

as $a_1(\mathbf{d})W_1(\tilde{d}_p) + a_2(\mathbf{d})W_2(\tilde{d}_p) = [a_1 \otimes a_2 \otimes \cdots \otimes a_1]$ (p times), $b_1(\mathbf{d})W_1(\tilde{d}_p) + b_2(\mathbf{d})W_2(\tilde{d}_p) = [b_1 \otimes b_2 \otimes \cdots \otimes b_1]$ (p times), and (a_2, b_2) is a complementary mate of (a_1, b_1). For $p = 2$, we have $a_1(\mathbf{d})W_1(\tilde{d}_p) + a_2(\mathbf{d})W_2(\tilde{d}_p) = [a_1 \otimes a_2]$ and $b_1(\mathbf{d})W_1(\tilde{d}_p) + b_2(\mathbf{d})W_2(\tilde{d}_p) = [b_1 \otimes b_2]$. In that case, $A(\mathcal{I}_a)(\tau_1) + A(\mathcal{I}_b)(\tau_1) = 0, \forall \tau \neq 0$. The PMEPR$(K{\gamma}_n^T)$ of the column sequence can be computed to be upper bounded by 2, when $p = 2$ and it is upper bounded by $p + \frac{1}{p}$, when $p \neq 2$.

Remark 4: For $p = 2$, $k = 1$ and $p_1 = 2$, *Theorem 2* provides 2-D ZCAC of array size of the form $2^n \times 2^m$ and flock size $M = 2$, which is comparable with the array size of 2-D GCAPs having flock size 2 [44, Th. 12] and 2-D GCASs having flock size greater than 2 [44, Th. 16]. However, the row and column sequence PMEPRs of 2-D GCAPs and 2-D GCASs in [44] are upper bounded by 2^v for some positive integer $v \geq 1$. But, in our construction, PMEPR of the row sequence is upper bounded by 4 and column sequence by 2, considering $p = 2$. It is evident that $4, 2 \leq 2^v$, when $v > 1$. Thus, although the proposed 2-D ZCAC has a flock size of only 2, it emerges as a good candidate to reduce PMEPR as compared to the 2-D GCAPs and 2-D GCASs in [44] for certain cases. The actual PMEPR comparison between the proposed ZCAC and the one in [44] is shown in Fig. 4.

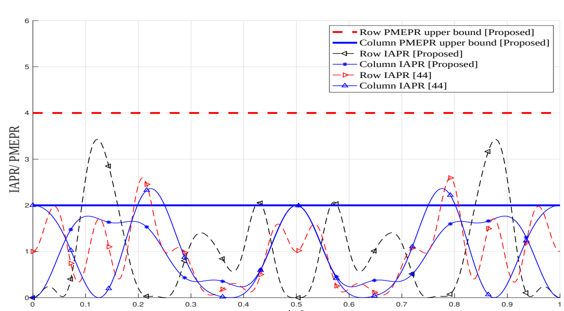

Fig. 4: Row and column sequence IAPR/PMEPR

Example 5: In this example, we take a 2-D ZCAC from the proposed construction and evaluate the row and column sequence PMEPRs. We have taken the function from (17), where $a^{\gamma}_{\mathbf{s_i},\mathbf{t_j}} = v_1v_2 + \gamma v_1 + t_jv_2 + s_iv_3$, $a_1 = x_1x_2 + 1$, a_2, b_1 and b_2 are derived using *Lemma 5* and *Lemma 6*. Then we get a 2-D ZCAC of size 8×16 and flock size 2. The row and column sequence PMEPRs are computed as 3.4271 and 2, respectively. In Fig. 4, we have shown the corresponding plot of row and column sequence IAPR and PMEPR bound. We also have taken a binary 8×16-GCAP from [44] by using the function $f = (z_1z_2 + z_2z_4 + z_4z_5 + z_5z_3 + z_3z_6 + z_6z_7) + 1$ and plotted the corresponding PMEPR/IAPR curve in Fig. 4. It is evident from the figure that column sequence PMEPR of [44] is greater than 2, for this particular example, while the proposed one is always bounded by 2.

B. Performance Analysis and Simulation Results

In this subsection, we provide a performance analysis of the OP-based massive MIMO system for different STBC-URA configurations. We have generated the steering matrix based on (10), where $\lambda = 3 \times 10^{-1}$, $d_x = \lambda/2$, $d_y = \lambda/2$. In this simulation, the URA sizes i.e. $P \times Q$ are varied, as well as the sizes of STBC matrices S required for the particular URA. The received signal is obtained by using (11), where the elements $s_n(t)$ of full-rate STBCs are generated based on the construction method given in [62]. We apply the derived 2-D GCAS and GCS from *Corollary 1* and *Corollary 2*, respectively, to construct precoding matrices for massive MIMO-URA system and compare these performances with that of the 2-D CCC based MIMO-URA system of [55] and 2-D GCAS based precoding system of [45], [57], [58]. The 2-D GCAPs from [59] have not been considered in this simulation, as 8×16 and 4×6 URA-based precoding matrices cannot be produced from the construction of [59]. BER performance is obtained through simulation and plotted against signal-to-noise ratio (SNR) in decibels (dB). We have used randomly generated binary phase shift keying (BPSK) data bits in the simulation and 4×10^5 data bits are considered for each performance curve. The complex AWGN $w(t)$ used in the simulation is obtained using the formula

$$w(t) = \sigma \left[\mathcal{N}(0,1) + \mathtt{i} \times \mathcal{N}(0,1) \right], \tag{24}$$

where $\mathtt{i} = \sqrt{-1}$, $\mathcal{N}(0,1)$ denotes the standard normal distribution with mean 0, variance 1, $\sigma^2 = N_0/2$ and $N_0 = \left(\frac{\text{Signal Power}}{\text{SNR}} \right)$. The set size K of a 2-D GCAS determines the number of antennas in an STBC [56]. The same argument applies to the number of columns $K = P$ in a GCS-based precoding matrix [34]. Apart from the existing 2-D GCAS/CCC-based constructions, we have also shown the performance benefit with respect to Zadoff-Chu (ZC) [63], [64] sequence-based precoder and random matrix (RM)-based precoder. For the 8×16-URA system, the following codes have been used:

1) RM-based precoding is used with random binary matrices $\mathbf{W}_i$'s of size 8×16, with 4×4-STBC.
2) ZC-based precoding matrix is used by generating ZC-sequences of length $8 \cdot 16 = 128$ and 4×4-STBC.

3) 4×4-STBC is used for precoders in [45], [55]. The generating functions of $\mathbf{W}_i$'s used for [45] is given by $f = (x_1x_2 + x_2x_3) + (y_1y_2 + y_2y_3 + y_3y_4) + 1$. For the case of [55] in this simulation, the same $\mathbf{W}_i$'s are used which are explicitly written in [55].
4) 8×8-STBC is used for the case of [57] with the generating function of $\mathbf{W}_i$' given by $f = (x_1x_2+x_2x_3)+(y_1y_2 + y_3y_4) + 1$.
5) 2×2-Alamouti code [65] is used as STBC for the derived 2-D GCAS-based precoding matrices. The 2-D GCAS is constructed from *Corollary 1*, using the function given in (17), where $a^{\gamma}_{\mathbf{s_i},\mathbf{t_j}} = v_1v_2 + v_2v_3 + \gamma v_1 + t_jv_3 + s_iv_4$, $a_1 = x_1x_2 + 1$, a_2, b_1 and b_2 are derived using *Lemma 5* and *Lemma 6*.
6) The derived GCS-based precoding matrix is generated from *Corollary 2* with the function given by (17), where $a^{\gamma}_{\mathbf{s_i},\mathbf{t_j}} = v_1 + \gamma v_1 + t_jv_1 + s_iv_2$, $a_1 = x_1x_2 + x_2x_3 + 1$. It uses 8×8-STBC model given in [62].

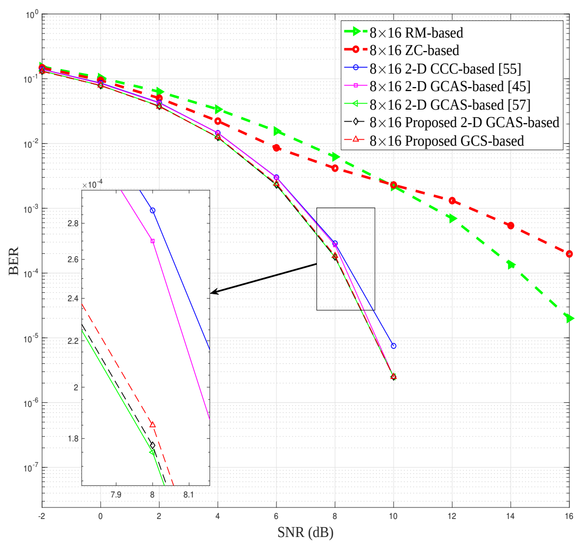

Fig. 5: BER comparison for 8×16-URA configurations

For 4×6 URA configuration, we use the following codes:

1) RM-based precoding is used with random binary matrices $\mathbf{W}_i$'s of size 4×6 and 3×4-STBC.
2) ZC-based precoding matrix is used by generating ZC-sequences of length 24 and same 3×4-STBC model.
3) 4×4-STBC is used for precoder in [58]. The generating functions of $\mathbf{W}_i$'s used for [58] is given by $f = 2(z_1z_2 + z_2z_3 + z_3z_4) + 1$.
4) 3×4-STBC is used for the derived 2-D GCAS-based precoding matrices. The function for 2-D GCAS is given by (17), where $a^{\gamma}_{\mathbf{s_i},\mathbf{t_j}} = 2v_1 + 2\gamma v_1 + t_jv_1 + 2s_iv_2$, $a_1 = 3x_1 + 1$.

In Fig. 5 and Fig. 6, the BER performances of the above-mentioned OP-based massive MIMO URA systems are shown. It can be seen from Fig. 5 that the proposed 2-D GCAS- and GCS-based precoding provide better performances than 2-D CCC and 2-D GCAS-based precoding in [55] and [45], respectively. The performance curve for derived 2-D GCAS and GCS is almost the same as that for [57], although the proposed construction has other benefits in regards to the flexibility of the parameters as seen in *Remark 3*. Also, from the point of view of transmit-diversity, the proposed 2-D GCASs are more efficient than [57], because the precoder for [57] uses a large 8×8-STBC which increases the transmission overhead. Fig. 6 clearly shows the performance benefit of the proposed 2-D GCAS with respect to the existing ones for a 4×6-URA setting.

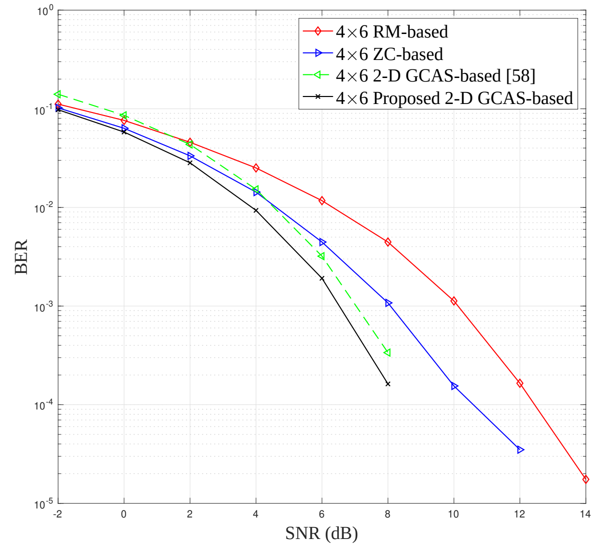

Fig. 6: BER comparison for 4×6-URA configurations

C. Comparison of constructions with existing work

In TABLE III, we compare the direct constructions of 2-D GCASs in [44], [45], [57]–[59] with our derived 2-D GCAS from the proposed 2-D ZCAC, based on their parameters which are involved in the OP-based transmission of massive MIMO-URA system using STBC-encoded data. Also, in TABLE IV, we compare 2-D MOGCS and 2-D CCCs in [31], [33], [35], [36] and 2-D ZCACSs in [38], [41], [46], [47] with our proposed construction of 2-D ZCACS. It can be seen from the comparison TABLE III and IV that our proposed constructions provide a flexible set of parameters that are unreported and also can be formulated in explicit mathematical forms. Also, our proposed construction is direct, which implies the rapid generation of the desired arrays from the generating functions.

V. Conclusion

In this paper, we first constructed an IGC code set, and by using it 2-D ZCAC and 2-D ZCACS with flexible array sizes have been constructed. The 2-D ZCACs and 2-D ZCACS have array size $L_1 \times L_2$, where L_1, L_2 are of the form $L_1 = 2^m p$ and $L_2 = 2p_1^{m_1}p_2^{m_2}\dots p_k^{m_k}$. We have shown that, for special cases, our proposed 2-D ZCAC reduces into 2-D ZCAP, which has not been reported before. The bounds of row and column

TABLE III: Comparison table of 2-D GCAS/GCS for OP-based MIMO-URA Transmission

Construction	**Array/URA size**	**Flock size**	**STBC required (full-rate)**	**Constraints**
[45]	$p^{m_1} \times p^{m_2}$	p^k	$p^k \times N$	p is a prime, $m_1, m_2, k \geq 1$, $m_1 + m_2 \geq k$, $p^k \leq \mu(N)$
[55]	$2^m \times 2^n$	2^k	$2^k \times 2^k$	$m, n, k \geq 1, m, n \geq k$
[57]	$b_1^{m_1} \times b_2^{m_2}$	$N_1^{k_1} N_2^{k_2}$	$N_1^{k_1} N_2^{k_2} \times N$	$N_1 \geq b_1$, $N_1 \geq b_1$, $m_1, m_2 \geq 1$, $b_1, b_2 \geq 2$ $k_1 \leq m_1$, $k_2 \leq m_2$, $N_1^{k_1} N_2^{k_2} \leq \mu(N)$
[58]	$2^n \times N$	2^{k+1}	$2^{k+1} \times 2^{k+1}$	$N \in \mathbb{Z}^+, n \geq 2, k \in \mathbb{Z}^+ \cup \{0\}$
[59]	$2^{m_1} \times 10 \cdot 2^{m_2-4}$, $2^{m_3} \times 26 \cdot 2^{m_4-5}$	2	2×2	$m_1, m_3 \geq 1, m_2 \geq 5, m_4 \geq 6$
Proposed, *Corollary 1*	$2^{m+1} \times 2p_1^{m_1-1} \cdots p_k^{m_k-1}$	$p_1 p_2 \cdots p_k$	$p_1 p_2 \cdots p_k \times N$	$m \geq 1, m_i \geq 2, \forall i, k \in \mathbb{N}, p_1 p_2 \cdots p_k \leq \mu(N)$
Proposed, *Corollary 2*	$2^{m+1} \times 2p_1^{m_1} \cdots p_k^{m_k}$	$2p_1^{m_1} \cdots p_k^{m_k}$	$2p_1^{m_1} \cdots p_k^{m_k} \times N$	$m \geq 1, m_i \geq 2, \forall i, k \in \mathbb{N}, 2p_1^{m_1} \cdots p_k^{m_k} \leq \mu(N)$

Note: $\mu(n) = 8c + 2^d$, where $n = 2^a b$, $2 \nmid b$, $a = 4c + d$, $0 \leq d < 4$ as provided in [62].

TABLE IV: Comparison table for 2-D ZCACS

Construction	**Array type**	**Phase**	**Array size**	**Rectangular ZCZ size**	**Set size**	**Flock size**	**Constraints**	**Based on**
[31]	2-D MOGCS	2	$P \times Q$	$P \times Q$	K	M	$P, Q \in 2^\alpha 10^\beta 26^\gamma, \alpha, \beta, \gamma \geq 0$; M depends on the number of orthogonal element of Welti codes	Welti codes
[33]	2-D CCC	q	$P^2 \times Q^2$	$P^2 \times Q^2$	N	N	P, Q are orders of 1-D CCCs, q is phase of 1-D CCCs, N is the set size of 1-D CCC	Matrix operation
[35]	2-D CCC	q	$L \times L$	$L \times L$	K^2	M^2	$L = N, 2^m N, N^2$; N depends on unitary-like matrix, q is phase of 1-D CCC; K, M are set size and flock size of 1-D CCC	Matrix operation
[36]	2-D CCC	q	$p^m \times p^n$	$p^m \times p^n$	N	N	$m, n \geq 1$, N depends on the choice of seed PU matrix, p is any prime	PU matrix
[38]	2-D ZCACS	Ternary	$L_1 \times (L_2 + r + 1)$	$Z_1 \times Z_2$	$K = K'r$	2	$L_1, L_2 \geq 1$; $Z_1, Z_2 \geq 1$; $rZ_2 \leq Z_2'$; $K \geq 1$; $L_1 \times L_2$ and $Z_1' \times Z_2'$ are the size and ZCZ size of initial 2-D ZCACS with set size K'	Matrix operation
[41]	2-D ZCACS	q	$L_1 \times L_2$	$Z_1 \times Z_2$	K	M	$q \geq 2$, $q = lcm(q_{\mathbf{V}_K}, q_{\mathbf{U}_K}, q_{\mathbf{U}_M})$; $L_1 = L_2 = K$; $Z_1 = M, Z_2 = K, K = MP$	BH matrices
[46]	2-D ZCACS	2	$2^m \times (14 \cdot 2^n)$	$2^m \times (12 \cdot 2^n)$	2^k	2^k	$m \geq 3, n \geq 1, k \geq 1$	Boolean function
[47]	2-D ZCACS	q	$R_1 N_1 \times R_2 N_2$	$N_1 \times N_2$	$R_1 R_2 N_1 N_2$	$N_1 N_2$	$R_1 \geq 1, R_2 \geq 2, N_1 = p_1^{m_1} \cdots p_{k_1}^{m_{k_1}}$, $N_2 = q_1^{n_1} \cdots q_{k_2}^{n_{k_2}}$, p_i, q_i being prime, $m_i, n_i \geq 1$, $p_i \mid q, q_j \mid q, \forall i, j$	Multivariable function
Proposed, *Theorem 2*	2-D ZCACS	q	$L_1 \times L_2$	$Z_1 \times Z_2$	$2p_1^2 p_2^2 \dots p_k^2$	$p_1 p_2 \dots p_k$	$L_1 = 2^m p, L_2 = 2\prod_{\alpha=1}^k p_\alpha^{m_\alpha}$; $Z_1 = 2^{m+1}$, $Z_2 = \prod_{\alpha=1}^k p_\alpha^{m_\alpha - 1}$; p_α is prime, $\forall \alpha = 1, 2, \dots, k$; p is prime; $m \geq 1$, $m_\alpha \geq 2$, $\forall \alpha = 1, 2, \dots, k$; $2 \mid q, p \mid q, p_\alpha \mid q$, $\forall \alpha = 1, 2, \dots, k$	Multivariable function

sequence PMEPRs have been computed and shown to have advantages over existing construction for certain cases. 2-D GCAS and GCS are derived from the proposed 2-D ZCAC and applied as precoding matrices in an OP-based massive MIMO-URA system. Simulation shows that the proposed constructions have better BER performance than the existing constructions. Also, they have increased flexibility in terms of URA size and the number of antennas to transmit STBC-encoded data. The proposed constructions of the IGC code set, 2-D ZCAC, and 2-D ZCACS are compared with the existing ones based on their parameters. The proposed 2-D ZCACS becomes optimal with respect to the analogous upper bound for 1-D ZCCS, when $p = 2$. It is still an open problem to show whether this bound can be tightened for some special cases or not and also, whether optimal 2-D ZCACSs with respect to the tightened upper bound is possible or not. Also, the improvement of PMEPR bound for row and column sequences can be considered an open problem.

Appendix A
Proof of Theorem 1

Note that $\psi(a_{\mathbf{s},\mathbf{t}}^{\boldsymbol{\gamma}})$ can be written as $\psi(a_{\mathbf{s},\mathbf{t}}^{\boldsymbol{\gamma}}) = \psi(T_{\mathbf{s}}) \otimes \psi(R_{\mathbf{t}}^{\boldsymbol{\gamma}})$, where $\otimes$ denotes the Kronecker product. Also note that for two pairs of sequences $(\mathbf{a}_1, \mathbf{a}_2)$ and $(\mathbf{b}_1, \mathbf{b}_2)$, we have

$$
\begin{aligned}
&C(\mathbf{a}_1 \otimes \mathbf{b}_1, \mathbf{a}_2 \otimes \mathbf{b}_2)(\tau) \\
&= C(\mathbf{a}_1, \mathbf{a}_2)\left(\left\lfloor \frac{\tau}{N} \right\rfloor\right) C(\mathbf{b}_1, \mathbf{b}_2)(\tau \bmod N) \\
&+ \Delta_N C(\mathbf{a}_1, \mathbf{a}_2)\left(\left\lfloor \frac{\tau}{N} \right\rfloor + 1\right) C(\mathbf{b}_1, \mathbf{b}_2)(\tau \bmod N - N),
\end{aligned}
\tag{25}
$$

where $\tau \geq 0$, each $\mathbf{a}_i$ has length M and each $\mathbf{b}_i$ has length N for $i = 1, 2$; and $\Delta_N = 0$ for $\tau \mod N = 0$, and 1 otherwise. We give two lemmas before the proof of *Theorem 1*.

Lemma 7 ([17]): For $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$, we have

$$
\begin{aligned}
&\sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^k \mathbb{Z}_{p_\alpha}} C\left(\psi(R_{\mathbf{t}_1}^{\boldsymbol{\gamma}}), \psi(R_{\mathbf{t}_2}^{\boldsymbol{\gamma}})\right)(\tau) \\
&= \begin{cases} \prod_{\alpha=1}^k p_\alpha^{m_\alpha}, & \mathbf{t}_1 = \mathbf{t}_2, \tau = 0; \\ 0, & \mathbf{t}_1 = \mathbf{t}_2, 0 <\mid \tau \mid < \prod_{\alpha=1}^k p_\alpha^{m_\alpha - 1}; \\ 0, & \mathbf{t}_1 \neq \mathbf{t}_2, \mid \tau \mid < \prod_{\alpha=1}^k p_\alpha^{m_\alpha - 1}. \end{cases}
\end{aligned}
\tag{26}
$$

Lemma 8: For $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$, we have

$$
C\left(\psi(T_{\mathbf{s}_1}), \psi(T_{\mathbf{s}_2})\right)(0) = \begin{cases} \prod_{\alpha=1}^k p_\alpha, & \mathbf{s}_1 = \mathbf{s}_2; \\ 0, & \mathbf{s}_1 \neq \mathbf{s}_2. \end{cases}
\tag{27}
$$

Proof: We let $\mathbf{s}_i = (s_1^i, s_2^i, \ldots, s_k^i)$ for $i = 1, 2$ and $V = (v'_{p_1}, v'_{p_2}, \ldots, v'_{p_k})$. As $T_{\mathbf{s}} = \sum_{\alpha=1}^{k} \frac{q}{p_\alpha} v'_{p_\alpha} s_\alpha$, we have

$$\begin{aligned} & C\left(\psi(T_{\mathbf{s}_1}), \psi(T_{\mathbf{s}_2})\right)(0) \\ &= \sum_{V \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}} \omega_q^{T_{\mathbf{s}_1} - T_{\mathbf{s}_2}} \\ &= \sum_{v'_{p_1}=0}^{p_1-1} \sum_{v'_{p_2}=0}^{p_2-1} \cdots \sum_{v'_{p_k}=0}^{p_k-1} \omega_q^{\sum_{\alpha=1}^{k} \frac{q}{p_\alpha} v'_{p_\alpha}(s_\alpha^1 - s_\alpha^2)} \\ &= \sum_{v'_{p_1}=0}^{p_1-1} \omega_{p_1}^{v'_{p_1}(s_1^1 - s_1^2)} \times \cdots \times \sum_{v'_{p_k}=0}^{p_k-1} \omega_{p_k}^{v'_{p_k}(s_k^1 - s_k^2)}. \end{aligned} \tag{28}$$

When $\mathbf{s}_1 = \mathbf{s}_2$, then $s_\alpha^1 = s_\alpha^2$ for all $\alpha = 1, 2, \ldots, k$. So, we have

$$\sum_{v'_{p_\alpha}=0}^{p_\alpha-1} \omega_{p_\alpha}^{v'_{p_\alpha}(s_\alpha^1 - s_\alpha^2)} = p_\alpha, \quad \text{for all} \quad \alpha = 1, 2, \ldots, k. \tag{29}$$

From (28) and (29), we have $C\left(\psi(T_{\mathbf{s}_1}), \psi(T_{\mathbf{s}_2})\right)(0) = \prod_{\alpha=1}^{k} p_\alpha$, when $\mathbf{s}_1 = \mathbf{s}_2$. On the other hand, when $\mathbf{s}_1 \neq \mathbf{s}_2$, then $s_\delta^1 \neq s_\delta^2$ for some $\delta \in \{1, 2, \ldots, k\}$. So, we have

$$\sum_{v'_{p_\delta}=0}^{p_\delta-1} \omega_{p_\delta}^{v'_{p_\delta}(s_\delta^1 - s_\delta^2)} = 0, \tag{30}$$

as $\{\omega_{p_\delta}^{v'_{p_\delta}(s_\delta^1 - s_\delta^2)} : v'_{p_\delta} = 0, 1, \ldots, p_\delta - 1\}$ are all the roots of the polynomial $z^{p_\delta} - 1$. From (28) and (30), we have $C\left(\psi(T_{\mathbf{s}_1}), \psi(T_{\mathbf{s}_2})\right)(0) = 0$, when $\mathbf{s}_1 \neq \mathbf{s}_2$. ∎

$\{\mathcal{C}_{\mathbf{s},\mathbf{t}} : \mathbf{s}, \mathbf{t} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}\}$ will be an IGC code set if it satisfies

$$\begin{aligned} & C(\mathcal{C}_{\mathbf{s}_1,\mathbf{t}_1}, \mathcal{C}_{\mathbf{s}_2,\mathbf{t}_2})(\tau) \\ &= \begin{cases} \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha+1}, & \mathbf{s}_1 = \mathbf{s}_2, \mathbf{t}_1 = \mathbf{t}_2, \tau = 0; \\ 0, & \mathbf{s}_1 = \mathbf{s}_2, \mathbf{t}_1 = \mathbf{t}_2, \\ & 0 < \tau < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}; \\ 0, & \mathbf{s}_1 \neq \mathbf{s}_2, \mathbf{t}_1 = \mathbf{t}_2, \tau < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}; \\ 0, & \mathbf{t}_1 \neq \mathbf{t}_2, \tau < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha}, \end{cases} \end{aligned} \tag{31}$$

keeping in mind that $C(\mathbf{a}, \mathbf{b})(-\tau) = C^*(\mathbf{b}, \mathbf{a})(\tau)$, for a pair of sequences $(\mathbf{a}, \mathbf{b})$ and $\tau \geq 0$. Now,

$$\begin{aligned} & C(\mathcal{C}_{\mathbf{s}_1,\mathbf{t}_1}, \mathcal{C}_{\mathbf{s}_2,\mathbf{t}_2})(\tau) \\ &= \sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\psi(a^{\boldsymbol{\gamma}}_{\mathbf{s}_1,\mathbf{t}_1}), \psi(a^{\boldsymbol{\gamma}}_{\mathbf{s}_2,\mathbf{t}_2})\right)(\tau) \\ &= \Bigg[C\left(\psi(T_{\mathbf{s}_1}), \psi(T_{\mathbf{s}_2})\right)\left(\left\lfloor \frac{\tau}{N} \right\rfloor\right) \\ & \quad \times \sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_1}), \psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_2})\right)(\tau \bmod N) \Bigg] \\ & + \Bigg[\Delta_N C\left(\psi(T_{\mathbf{s}_1}), \psi(T_{\mathbf{s}_2})\right)\left(\left\lfloor \frac{\tau}{N} \right\rfloor + 1\right) \\ & \times \sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_1}), \psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_2})\right)(\tau \bmod N - N) \Bigg], \end{aligned} \tag{32}$$

where $N = \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$. The rest of the proof is split into three cases.

Case I: Let $\tau = 0$. Then $\left\lfloor \frac{\tau}{N} \right\rfloor = 0$, $\tau \bmod N = 0$ and $\Delta_N = 0$. Using *Lemma 7* and *Lemma 8*, we have

$$C(\mathcal{C}_{\mathbf{s}_1,\mathbf{t}_1}, \mathcal{C}_{\mathbf{s}_2,\mathbf{t}_2})(0) = \begin{cases} \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha+1}, & \mathbf{s}_1 = \mathbf{s}_2, \mathbf{t}_1 = \mathbf{t}_2; \\ 0, & \mathbf{s}_1 \neq \mathbf{s}_2, \mathbf{t}_1 = \mathbf{t}_2; \\ 0, & \mathbf{t}_1 \neq \mathbf{t}_2. \end{cases} \tag{33}$$

Case II: Let $0 < \tau < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$. In this case $\left\lfloor \frac{\tau}{N} \right\rfloor = 0$, $0 < \tau \bmod N < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$, $-\prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1} < \tau \bmod N - N < 0$ and $\Delta_N = 1$. Using *Lemma 7*, we have

$$\begin{aligned} \sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_1}), \psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_2})\right)(\tau \bmod N) &= 0, \\ \sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_1}), \psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_2})\right)(\tau \bmod N - N) &= 0, \end{aligned} \tag{34}$$

for all values of $\mathbf{t}_1$ and $\mathbf{t}_2$.

Case III: Let $\prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1} \leq \tau < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha}$ and $\mathbf{t}_1 \neq \mathbf{t}_2$. In this case, we have two sub-cases.

Sub-case 1: When $\tau = \beta \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$ for some $\beta \in \{1, 2, \ldots, \prod_{\alpha=1}^{k} p_\alpha - 1\}$, then $\left\lfloor \frac{\tau}{N} \right\rfloor = \beta$, $\tau \bmod N = 0$ and $\Delta_N = 0$. Then, form *Lemma 7*, we have

$$\sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_1}), \psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_2})\right)(\tau \bmod N) = 0. \tag{35}$$

Sub-case 2: Let $\beta \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1} < \tau < (\beta + 1) \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$ for some $\beta \in \{1, 2, \ldots, \prod_{\alpha=1}^{k} p_\alpha - 1\}$. Then $\left\lfloor \frac{\tau}{N} \right\rfloor = \beta$, $0 < \tau \bmod N < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$, $-\prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1} < \tau \bmod N - N < 0$ and $\Delta_N = 1$. Then using *Lemma 7*, we have

$$\begin{aligned} \sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_1}), \psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_2})\right)(\tau \bmod N) &= 0, \\ \sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_1}), \psi(R^{\boldsymbol{\gamma}}_{\mathbf{t}_2})\right)(\tau \bmod N - N) &= 0. \end{aligned} \tag{36}$$

Hence, the theorem is proved.

Appendix B
Proof of Theorem 2

It is easy to verify that the number of $\zeta \in \Lambda$ is $K = 2p_1^2 p_2^2 \ldots p_k^2$. We need to show that

$$\begin{aligned} & \sum_{\boldsymbol{\gamma} \in \prod_{\alpha=1}^{k} \mathbb{Z}_{p_\alpha}} C\left(\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1}, \mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}\right)(\tau_1, \tau_2) \\ &= \begin{cases} L_1 L_2 \prod_{\alpha=1}^{k} p_\alpha, & \boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_2, (\tau_1, \tau_2) = (0, 0); \\ 0, & \boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_2, \mid \tau_1 \mid < 2^{m+1}, \\ & \mid \tau_2 \mid < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}, (\tau_1, \tau_2) \neq (0, 0); \\ 0, & \boldsymbol{\zeta}_1 \neq \boldsymbol{\zeta}_2, \mid \tau_1 \mid < 2^{m+1}, \\ & \mid \tau_2 \mid < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}, \end{cases} \end{aligned} \tag{37}$$

where $L_1 = 2^m p$ $L_2 = 2\prod_{\alpha=1}^{k} p_\alpha^{m_\alpha}$. It is straightforward to show that $\sum_{\boldsymbol{\gamma}\in\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}} C\left(\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1},\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}\right)(\tau_1,\tau_2) = 2^{m+1}p\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha+1}$, when $\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_2, (\tau_1,\tau_2) = (0,0)$. We shall prove the rest only for $\tau_1 \geq 0$ as for any two 2-D array $\mathbf{X}_1$, $\mathbf{X}_2$, we have $C(\mathbf{X}_1,\mathbf{X}_2)(-\tau_1,\tau_2) = C(\mathbf{X}_2,\mathbf{X}_1)^*(\tau_1,-\tau_2)$. We split the proof into two cases.

Case I: Let $0 \leq \tau_1 < 2^{m+1}$ and $0 \leq \tau_2 < \prod_{\alpha=1}^{k} p_\alpha^{m_\alpha-1}$. For a fixed γ, we have

$$\begin{aligned} &C\left(\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1},\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}\right)(\tau_1,\tau_2)\\ &=\sum_{i=0}^{L_1-\tau_1-1}\sum_{j=0}^{L_2-\tau_2-1}\mathbf{X}_{i,j}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1}{\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}}^*_{i+\tau_1,j+\tau_2}\\ &=\sum_{i=0}^{L_1-\tau_1-1}\sum_{j=0}^{L_2-\tau_2-1}\omega_q^{F_{\vec{i}}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1}(\vec{j})}\omega_q^{-F_{\vec{i'}}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}(\vec{j'})}, \end{aligned} \tag{38}$$

where $L_1 = 2^m p$ and $L_2 = 2p_1^{m_1}p_2^{m_2}\dots p_k^{m_k}$. We let $\vec{j} = (\vec{j}_{res}, j_L)$ and $\vec{j'} = (\vec{j'}_{res}, j'_L)$, where $\vec{j}_{res},\vec{j'}_{res}$ are the vectors in $\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}^{m_\alpha-1}\times\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}$ which are truncated from $\vec{j}$ and $\vec{j'}$, respectively and j_L, j'_L are the last elements of $\vec{j},\vec{j'}$ corresponding to $\mathbb{Z}_2$. We define j_{res}, j'_{res} to be the decimal numbers corresponding to the vectors $\vec{j}_{res},\vec{j'}_{res}$, respectively. Similarly, we define $\vec{i} = (\vec{i}_{res}, i_p)$ and $\vec{i'} = (\vec{i'}_{res}, i'_p)$, where $\vec{i}_{res},\vec{i'}_{res}$ are the vectors corresponding to $\mathbb{Z}_2^m$ which are truncated from $\vec{i},\vec{i'}$ respectively. i_p, i'_p are the last elements of $\vec{i},\vec{i'}$ corresponding to $\mathbb{Z}_p$. Similarly, we define i_{res}, i'_{res} to be the decimal numbers corresponding to the vectors $\vec{i}_{res},\vec{i'}_{res}$, respectively. Now, we have

$$\begin{aligned} &\omega_q^{F_{\vec{i}}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1}(\vec{j})}\omega_q^{-F_{\vec{i'}}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}(\vec{j'})}\\ &=\omega_q^{(1-j_L)\{a^{\boldsymbol{\gamma}}_{\mathbf{s}_1^1,\mathbf{t}_1^1}(\vec{j}_{res})+a_1(\vec{i}_{res})W_1(i_p)+a_2(\vec{i}_{res})W_2(i_p)\}}\\ &\times\omega_q^{j_L\{a^{\boldsymbol{\gamma}}_{\mathbf{s}_2^1,\mathbf{t}_2^1}(\vec{j}_{res})+b_1(\vec{i}_{res})W_1(i_p)+b_2(\vec{i}_{res})W_2(i_p)\}}\\ &\times\omega_q^{-(1-j'_L)\{a^{\boldsymbol{\gamma}}_{\mathbf{s}_1^2,\mathbf{t}_1^2}(\vec{j'}_{res})+a_1(\vec{i'}_{res})W_1(i'_p)+a_2(\vec{i'}_{res})W_2(i'_p)\}}\\ &\times\omega_q^{-j'_L\{a^{\boldsymbol{\gamma}}_{\mathbf{s}_2^2,\mathbf{t}_2^2}(\vec{j'}_{res})+b_1(\vec{i'}_{res})W_1(i'_p)+b_2(\vec{i'}_{res})W_2(i'_p)\}}, \end{aligned} \tag{39}$$

and

$$\begin{aligned} &C\left(\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1},\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}\right)(\tau_1,\tau_2)\\ &=\sum_{i=0}^{L_1-\tau_1-1}\Bigg[\sum_{j=0}^{\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2-1}\Phi_{i,i'}^{j,j'}+\sum_{j=\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2}^{\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-1}\Phi_{i,i'}^{j,j'}\\ &\quad+\sum_{j=\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}}^{2\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2-1}\Phi_{i,i'}^{j,j'}\Bigg], \end{aligned} \tag{40}$$

where $\Phi_{i,i'}^{j,j'} = \omega_q^{F_{\vec{i}}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1}(\vec{j})}\omega_q^{-F_{\vec{i'}}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}(\vec{j'})}$. When $0 \leq j \leq \prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2-1$, then $\tau_2 \leq j' \leq \prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-1$. So, $j_L = 0$, $j'_L = 0$ and $0 \leq j_{res} \leq \prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2-1$, $\tau_2 \leq j'_{res} \leq \prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-1$. From (39), we have

$$\begin{aligned} &\sum_{j=0}^{\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2-1}\Phi_{i,i'}^{j,j'}\\ &=\omega_q^{a_1(\vec{i}_{res})W_1(i_p)+a_2(\vec{i}_{res})W_2(i_p)}\\ &\times\omega_q^{-\left(a_1(\vec{i'}_{res})W_1(i'_p)+a_2(\vec{i'}_{res})W_2(i'_p)\right)}\\ &\times\sum_{j_{res}=0}^{\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2-1}\omega_q^{a^{\boldsymbol{\gamma}}_{\mathbf{s}_1^1,\mathbf{t}_1^1}(\vec{j}_{res})-a^{\boldsymbol{\gamma}}_{\mathbf{s}_1^2,\mathbf{t}_1^2}(\vec{j'}_{res})}. \end{aligned} \tag{41}$$

Sub-case 1: When $\boldsymbol{\zeta}_1 \neq \boldsymbol{\zeta}_2$, from *Theorem 1*, we have

$$\sum_{\boldsymbol{\gamma}\in\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}}\sum_{j_{res}=0}^{\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2-1}\omega_q^{a^{\boldsymbol{\gamma}}_{\mathbf{s}_1^1,\mathbf{t}_1^1}(\vec{j}_{res})-a^{\boldsymbol{\gamma}}_{\mathbf{s}_1^2,\mathbf{t}_1^2}(\vec{j'}_{res})}=0. \tag{42}$$

Arguing in a similar manner, from (40) we can get $\sum_{\boldsymbol{\gamma}\in\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}} C\left(\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1},\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}\right)(\tau_1,\tau_2) = 0$.

Sub-case 2: Let $\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_2$, $(\tau_1,\tau_2) \neq (0,0)$. If $0 < \tau_2 < \prod_{\alpha=1}^{k}p_\alpha^{m_\alpha-1}$, then from *Theorem 1*, we have

$$\sum_{\boldsymbol{\gamma}\in\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}}\sum_{j_{res}=0}^{\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-\tau_2-1}\omega_q^{a^{\boldsymbol{\gamma}}_{\mathbf{s}_1^1,\mathbf{t}_1^1}(\vec{j}_{res})-a^{\boldsymbol{\gamma}}_{\mathbf{s}_1^1,\mathbf{t}_1^1}(\vec{j'}_{res})}=0, \tag{43}$$

and arguing similarly, we have $\sum_{\boldsymbol{\gamma}\in\prod_{\alpha=1}^{k}\mathbb{Z}_{p_\alpha}} C\left(\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1},\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}\right)(\tau_1,\tau_2) = 0$. If $\tau_2 = 0$, then $0 < \tau_1 < 2^{m+1}$. Then, we have

$$\begin{aligned} &C\left(\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_1},\mathbf{X}^{\boldsymbol{\gamma},\boldsymbol{\zeta}_2}\right)(\tau_1,\tau_2)\\ &=\sum_{i=0}^{L_1-\tau_1-1}\Bigg[\sum_{j=0}^{\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-1}\Phi_{i,i'}^{j,j'}+\sum_{j=\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}}^{2\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-1}\Phi_{i,i'}^{j,j'}\Bigg]. \end{aligned} \tag{44}$$

Now, from similar arguments as (41), we have

$$\begin{aligned} &\sum_{j=0}^{\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-1}\Phi_{i,i'}^{j,j'}+\sum_{j=\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}}^{2\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}-1}\Phi_{i,i'}^{j,j'}\\ &=\prod_{\alpha=1}^{k}p_\alpha^{m_\alpha}\Bigg[\omega_q^{a_1(\vec{i}_{res})W_1(i_p)+a_2(\vec{i}_{res})W_2(i_p)}\\ &\qquad\times\omega_q^{-\left(a_1(\vec{i'}_{res})W_1(i'_p)+a_2(\vec{i'}_{res})W_2(i'_p)\right)}\\ &\qquad+\omega_q^{b_1(\vec{i}_{res})W_1(i_p)+b_2(\vec{i}_{res})W_2(i_p)}\\ &\qquad\times\omega_q^{-\left(b_1(\vec{i'}_{res})W_1(i'_p)+b_2(\vec{i'}_{res})W_2(i'_p)\right)}\Bigg]. \end{aligned} \tag{45}$$

For $0 < \tau_1 < 2^m$, when $0 \leq i_{res} \leq 2^m-\tau_1-1$, then $\tau_1 \leq i'_{res} \leq 2^m-1$. When $2^m-\tau_1 \leq i_{res} \leq 2^m-1$, then $0 \leq i'_{res} \leq \tau_1 - 1$. From the definition of W_1, W_2, *Lemma 5* and *Lemma 6*, we have

$$\sum_{i_{res}=0}^{2^m-1} \Bigg[\omega_q^{a_1(\vec{i}_{res})W_1(i_p)+a_2(\vec{i}_{res})W_2(i_p)} \times \omega_q^{-a_1(\vec{i'}_{res})W_1(i'_p)-a_2(\vec{i'}_{res})W_2(i'_p)} + \omega_q^{b_1(\vec{i}_{res})W_1(i_p)+b_2(\vec{i}_{res})W_2(i_p)} \times \omega_q^{-b_1(\vec{i'}_{res})W_1(i'_p)-b_2(\vec{i'}_{res})W_2(i'_p)} \Bigg] = 0. \tag{46}$$

Similar argument follows for $2^m \leq \tau_1 < 2^{m+1}$.

Case II: Let $0 \leq \tau_1 < 2^{m+1}$ and $-\prod_{\alpha=1}^{k} p_\alpha^{m_\alpha - 1} < \tau_2 < 0$. A similar argument as *Case I* follows for this case.

This proves the theorem.

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