Locally finite free space as limiting case of PT-symmetric medium

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Abstract

We explicitly prove that the transfer matrix of a finite layered PT-symmetric system of fix length L consisting of N units of the potential system '+iV' and '-iV' of equal thickness becomes a unit matrix in the limit $N \to \infty$. This result is true for waves of arbitrary wave vector k. This shows that in this limit, the transmission coefficient is always unity while the reflection amplitude is zero for all waves traversing this length L. Therefore, a free space of finite length L can be represented as a PT-symmetric medium.

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1 Introduction

Around two decades ago, it was discovered that certain class of non-Hermitian Hamiltonian can support real energy eigen values provided the Hamiltonian is invariant under a combine parity (P) and time-reversal (T) operation [1]. It was also noted that that a fully consistent quantum theory can be developed for non-Hermitian system in a modified Hilbert space through the modification of inner product which restore the equivalent Hermiticity and the unitary time evolution of the system[2, 3]. Since then a new dimension in quantum mechanics has emerged known as PT-symmetric quantum mechanics [4]. The non-Hermitian Hamiltonian display several new features which are originally absent in Hermitian Hamiltonians. The important features are exceptional points (EPs) [5, 6], spectral singularity (SS) [7]-[10], coherent perfect absorption (CPA) [10]-[15], critical coupling (CC) [16]-[19] and CPA-laser [20]. Others notable features are invisibility [21, 22, 23] and reciprocity [24]. CPA and SS have also been studied in the context of non-Hermitian space fractional quantum mechanics [25]. Phenomena of SS have also been studied in the domain of quaternionic quantum mechanics [26].

A quantum vacuum has fluctuations due to particle and anti-particle creations and annihilations in such a way that creations and annihilations balances each other to keep the net charge neutrality of the vacuum. In a sufficiently small time interval, a snap shot of the vacuum will contain equal number of particle/anti-particles pairs or waves of complex frequencies having positive and negatives components of imaginary part. If the creation and annihilation of the particles are respectively 'gain' and 'loss' component for the vacuum system (and vice-versa for anti-particles), then the snap shot of the vacuum will be a PT-symmetric system. In other words quantum vacuum is stable under PT-symmetry.

Our motivation for the present work arises due to the arguments presented in the above paragraph. In order to check that whether vacuum can be represented as PT-symmetric system, we consider a finite layered PT-symmetric system of fix length L consisting of N units of the potential systems '+iV' and '-iV' of equal thickness 'b' without any intervening gap between the individual potential systems. It is shown that in the limit $N \to \infty$ such that 2Nb = L, the entire PT-symmetric system of finite length L is equivalent to an empty space of length L in all aspect. We prove this by showing that the transfer matrix of our PT-symmetric system of length L is a unit matrix in the above limiting case for particles of any wave vector k incident on this system.

We organize the paper as follows: In section 2 we briefly discuss the transfer matrix for one dimensional scattering. In section 3 we calculate the transfer matrix for our layered PT-symmetric system and evaluate the limiting case $N \to \infty$ of the finite length PT-symmetric medium in section 4. We present results and associated discussion in section 5.

2 Transfer matrix

The Hamiltonian operator in one dimension for a non-relativistic particle is (in the unit $\hbar = 1$ and 2m = 1)

$$H = -\frac{d^2}{dx^2} + V(x),\tag{1}$$

where $V(x) \in C$. $V(x) \to 0$ as $x \to \pm \infty$. If $\int U(x)dx$, where U(x) = (1 + |x|)V(x) is finite over all x, then the Hamiltonian given above admits a scattering solution with the following asymptotic values

$$\psi(k, x \to +\infty) = A_{+}(k)e^{ikx} + B_{+}(k)e^{-ikx},$$
 (2)

$$\psi(k, x \to -\infty) = A_{-}(k)e^{ikx} + B_{-}(k)e^{-ikx}.$$
(3)

The coefficients A_{\pm} , B_{\pm} are connected through a 2×2 matrix M, called as transfer matrix as given below,

$$\begin{pmatrix} A_{+}(k) \\ B_{+}(k) \end{pmatrix} = M(k) \begin{pmatrix} A_{-}(k) \\ B_{-}(k) \end{pmatrix}. \tag{4}$$

Where,

$$M(k) = \begin{pmatrix} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{pmatrix}.$$
 (5)

With the knowledge of the transfer matrix M(k), the transmission and reflection coefficient are obtained as,

$$t_l(k) = \frac{1}{M_{22}(k)} = t_r(k), \quad r_l(k) = -\frac{M_{12}}{M_{22}(k)}, r_r(k) = \frac{M_{21}}{M_{22}(k)}.$$
 (6)

The transfer matrix shows composition property. If the transfer matrix for two non-overlapping finite scattering regions V_1 and V_2 , where V_1 is to the left of V_2 , are M_1 and M_2 respectively, then the net transfer matrix M_{net} of the whole system (V_1 and V_2) is

$$M_{net} = M_2.M_1 (7)$$

The composition result can be generalized for arbitrary numbers of non-overlapping finite scattering regions. Knowing the transfer matrix, one easily compute the scattering coefficients by using Eq. 6. From Eq. 6, it is also seen that if the diagonal elements are unity and off-diagonal elements are zero, then we always have $t_l(k) = 1 = t_r(k)$ and $r_l(k) = 0 = r_r(k)$. This case of transfer matrix represent empty space.

3 Transfer matrix of layered PT-symmetric system

Fig 2 shows the layered PT-symmetric system which is made by periodic repetitions of 'unit cell' PT-symmetric system shown in Fig 1. It can be shown that the transfer matrix

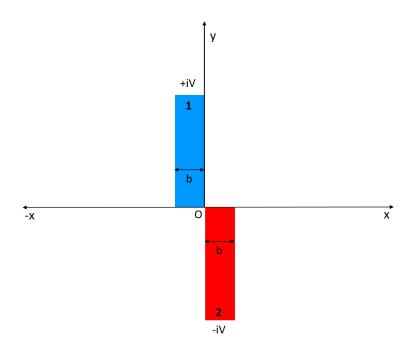


Figure 1: A PT-symmetric 'unit cell' consisting of a pair of complex conjugate barrier. y-axis represent the imaginary height of the potential.

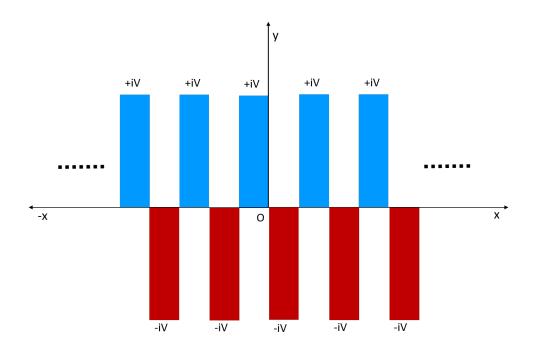


Figure 2: A periodic PT-symmetric potential made by the periodic repetition of the 'unit cell' potential shown in Fig 1. y-axis represent the imaginary height of the potential.

of the 'unit cell' system is

$$M(k) = \begin{pmatrix} (\xi + i\chi)e^{-2ikb} & i(\eta - \tau)e^{-2ikb} \\ i(\eta + \tau)e^{2ikb} & (\xi - i\chi)e^{2ikb} \end{pmatrix}.$$
 (8)

$$\xi = \frac{1}{2}(\cos 2\alpha + \cosh 2\beta) - \cos 2\phi(\cosh^2 \beta \sin^2 \alpha + \cos^2 \alpha \sinh^2 \beta),\tag{9}$$

$$\chi = \frac{1}{2}(U_{+}\cos\phi\sin 2\alpha + U_{-}\sin\phi\sinh 2\beta). \tag{10}$$

$$\eta = \frac{1}{2}(\cosh 2\beta - \cos 2\alpha)\sin 2\phi. \tag{11}$$

$$\tau = \frac{1}{2}(U_{+}\sin\phi\sinh 2\beta + U_{-}\cos\phi\sin 2\alpha). \tag{12}$$

In the above equations, $U_{\pm} = \frac{k}{\rho} \pm \frac{\rho}{k}$, $\alpha = b\rho\cos\phi$, $\beta = b\rho\sin\phi$. ρ and ϕ are the modulus and phase of $k_2 = \sqrt{k^2 + iV} = \rho e^{i\phi}$ respectively such that $\rho = (k^4 + V^2)^{\frac{1}{4}}$ and $\phi = \frac{1}{2}\tan^{-1}\left(\frac{V}{k^2}\right)$. It can be noted that $k_1 = \rho e^{-i\phi}$. From the knowledge of transfer matrix of a 'unit cell' potential, one can find the transfer matrix for the corresponding locally periodic potential consisting N such cells [28]. Using the approach outlined in [28, 29], we obtain the following transfer matrix for the layered PT-symmetric system,

$$\Omega(k) = \begin{pmatrix} [T_N(\xi) + i\chi U_{N-1}(\xi)]e^{-ikL} & i(\eta - \tau)U_{N-1}(\xi)e^{-ikL} \\ i(\eta + \tau)U_{N-1}(\xi)e^{ikL} & [T_N(\xi) - i\chi U_{N-1}(\xi)]e^{ikL} \end{pmatrix}.$$
(13)

 $T_N(\xi)$ and $U_N(\xi)$ are the Chebyshev polynomials of first and second kind respectively. L=2Nb is the net spatial extent of the layered PT-symmetric system.

4 Special case of layered PT-symmetric medium

In this section we show that a finite length L of our layered PT-symmetric system consisting infinitely many cells is analogous to an empty one dimensional space of length L. To show this we take limiting case $N \to \infty$ of each elements of transfer matrix 24 such that $b = \frac{L}{2N}$ where L is fixed (and is finite). Various steps of the calculations are discussed below.

The limiting of case of ξ and χ in the leading order of L can be shown to be,

$$\lim_{N \to \infty} \xi = 1 - \frac{(kL)^2}{2N^2}, \quad \lim_{N \to \infty} \chi = \frac{kL}{N}$$
(14)

In arriving at the above limit, we have used $b = \frac{L}{2N}$. We also observe $\lim_{N\to\infty} \xi < 1$. With the above limiting value of ξ , we also evaluate

$$\lim_{N \to \infty} \cos^{-1} \xi = \frac{kL}{N}.$$
 (15)

It is further known that for $|\varepsilon| < 1$, one can express $T_N(\varepsilon) = \cos(N\cos^{-1}\varepsilon)$. Therefore,

$$\lim_{N \to \infty} T_N(\xi) = \cos(N \cos^{-1} \xi).$$

Using Eq. 15 in the above, we find

$$\lim_{N \to \infty} T_N(\xi) = \cos kL. \tag{16}$$

Next we evaluate the limiting value of Chebyshev polynomial of second kind for the present problem. We use the following identity,

$$U_{N-1}(\xi) = \frac{\sin(N\cos^{-1}\xi)}{\sin(\cos^{-1}\xi)}.$$

Using Eq. 15 in the above, the limiting value is given by,

$$\lim_{N \to \infty} U_{N-1}(\xi) = \frac{\sin kL}{\sin \frac{kL}{N}}.$$
 (17)

From the above result, we arrive at

$$\lim_{N \to \infty} \chi U_{N-1}(\xi) = \sin kL. \tag{18}$$

In the above we have used $\lim_{N\to\infty} \sin\frac{kL}{N} = \frac{kL}{N}$ Combining the limiting values of Eq. 16 and Eq. 18 we have the following results,

$$\lim_{N \to \infty} T_N(\xi) \pm i\chi U_{N-1}(\xi) = e^{\pm ikL}.$$
 (19)

Using Eq. 19 in the diagonal values of transfer matrix (Eq. 24) we obtain

$$\lim_{N \to \infty} \Omega_{11} = 1 = \lim_{N \to \infty} \Omega_{22},\tag{20}$$

i.e. the diagonal elements are unity in the limit $N \to \infty$ provided the support L is finite. Now we evaluate the limiting values of off-diagonal terms of the transfer matrix. The limiting value of $\eta \pm \tau$ in the leading order of b is,

$$\lim_{N \to \infty} (\eta \pm \tau) = Vb^2. \tag{21}$$

Using Eq.17 it can be easily shown that,

$$\lim_{N \to \infty} (\eta \pm \tau) U_{N-1}(\xi) = \frac{Vb}{2k} \sin kL = 0.$$
 (22)

Therefore, the off-diagonal terms of the transfer matrix (Eq. 24) are zero, i.e.,

$$\lim_{N \to \infty} \Omega_{12} = 0 = \lim_{N \to \infty} \Omega_{21},\tag{23}$$

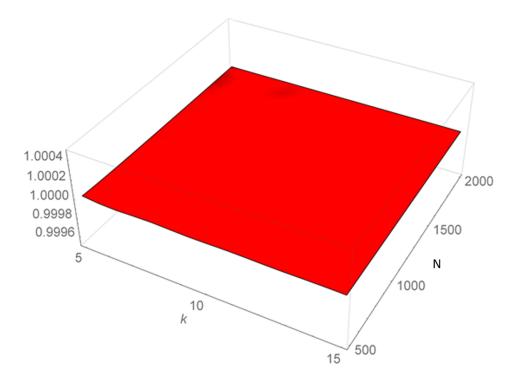


Figure 3: Plot of transmission amplitude T(N,k) as a function of N and k for V=40 and L=1. It is observed that the transmission is unity for large N. For a better clarity, the plot range of T is chosen in the range from 0.9995 to 1.0005.

for finite support L. Thus it is proved that for all k,

$$\lim_{N \to \infty} \begin{pmatrix} [T_N(\xi) + i\chi U_{N-1}(\xi)]e^{-ikL} & i(\eta - \tau)U_{N-1}(\xi)e^{-ikL} \\ i(\eta + \tau)U_{N-1}(\xi)e^{ikL} & [T_N(\xi) - i\chi U_{N-1}(\xi)]e^{ikL} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (24)$$

and therefore transmission coefficient is always unity and reflection coefficient is always zero for all wave number k in the limit $N \to \infty$. Fig 3 shows the plot of transmission amplitude T(N,k) as a function of N and k for V=40 and L=1. It is seen from the figure that transmission amplitude is unity for large N as proven theoretically. The range of N in the figure is taken from 500 to 2000.

This is to be noted that when PT-symmetry is not respected, the vacuum configuration is not obtained. If one take the general case where the potential +iV is replaced by V_1+iV_2 and potential -iV with $V_1-i\varepsilon V_2$, $\{V_1,V_2\}\in R$, the net configuration of length L in the limit $N\to\infty$ corresponds to a barrier potential of height $V_1+i(1-\varepsilon)V_2$ and length L. When PT-symmetry is respected $(\varepsilon=1)$, the limiting case $N\to\infty$ correspond to a real barrier of height V_1 and length L. The case presented in this letter is the special case $V_1=0, \varepsilon=1$. The calculations for the more general case is much lengthy and is planned to be reported elsewhere.

5 Results and Discussions

We have shown that a finite layered PT-symmetric system of fix length L consisting of N units of adjacently arranged 'unit cell' PT-symmetric system represent a free space of length L in the limit $N \to \infty$ at all wave number k. The 'unit cell' PT-symmetric system is made by potential '+iV' and '-iV' ($V \in R^+$), of same thickness and arranged adjacently without an intervening gap. This is proven by showing that the transfer matrix of such a layered PT-symmetric system over fix length L is a unity matrix at all wave number k for large number of unit cells. Therefore for such a system in this limit, the transmission coefficient is always unity while the reflection coefficient is always zero. Thus a free space of finite length L can be represented as PT-symmetric medium. The result is also shown numerically for transmission coefficient.

It is to be noted that in the present case of the layered PT-symmetric system, the effect of gain (+iV) and loss part (-iV) cancel each other in the limit $b \to 0$ (or $N \to \infty$ for the present problem) as the wave traverses through it. This case is different than considering vanishing strength of balanced gain and loss component of the non-Hermitian potential. The present results shows that vacuum can be represented as the special case of PT-symmetric medium. More investigations are needed to understand the significant of this result. This finite PT-symmetric system for large L is also invisible for left and right incidence. If particle production is represented as +iV (the gain part) and particle annihilation is represented as -iV (the lossy part), then the present limiting case of layered PT-symmetric medium represent the static snap shot of vacuum fluctuation. However it will be worth investigating the nature of transfer matrix when the height of each 'unit cell' is oscillatory in nature where frequency of oscillation is different for different cells. This may represent a more realistic picture of vacuum fluctuation when represented in non-Hermitian quantum mechanics.

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