

Spectrum of the transfer matrices of the spin chains associated with the $A_3^{(2)}$ Lie algebra

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Abstract

We study the exact solution of quantum integrable system associated with the $A_3^{(2)}$ twist Lie algebra, where the boundary reflection matrices have non-diagonal elements thus the $U(1)$ symmetry is broken. With the help of the fusion technique, we obtain the closed recursive relations of the fused transfer matrices. Based on them, together with the asymptotic behaviors and the values at special points, we obtain the eigenvalues and Bethe ansatz equations of the system. We also show that the method is universal and valid for the periodic boundary condition where the $U(1)$ symmetry is reserved. The results in this paper can be applied to studying the exact solution of the $A_n^{(2)}$ -related integrable models with arbitrary n .

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1 Introduction

Since the pioneer work of Sklyanin [1], the quantum integrable systems with open boundary conditions draw many attentions. The open boundary conditions are characterized by the reflection matrices. The integrability of the system requires that the reflection matrix satisfies the reflection equation. If the reflection matrix is diagonal, the conventional Bethe ansatz methods including the coordinate [2] and algebraic [3–5] ones can be applied to solve it successfully. However, if the reflection matrix has some non-diagonal elements, the $U(1)$ symmetry is broken and these traditional methods do not work because of lacking the vacuum/reference state. Then many interesting methods such as the q -Qnsager algebra [6–9], the separation of variables [10–13], the off-diagonal Bethe ansatz (ODBA) [14, 15], and the modified algebraic Bethe ansatz [16–19] have been proposed.

Recently, the study of quantum integrable systems with high ranks becomes a hot topic due to the many applications in the quantum field theory, AdS/CFT correspondence in string theory and high energy physics. The most typical and simple case is the integrable models associated with A -series Lie algebras. The model with periodic or diagonal open boundary conditions have been studied extensively [20–24]. Then the results of the system with non-diagonal boundary reflections are necessary. The exact solution of q -deformed $su(n+1)$ invariant quantum spin chain, which is connected with the $A_n^{(1)}$ Lie algebra, has been obtained by using the nested ODBA [25]. The next task is to study the quantum integrable models associated with the $A_n^{(2)}$ twist Lie algebra. For the simplest case, the exact solution of Izergin-Korepin model [26], which is connected with the $A_2^{(2)}$ Lie algebra, with generic integrable open boundary condition has been obtained [27]. However, the results with $n \geq 3$ are still missing. We shall note that the generic integrable boundary reflection of quantum integrable models related with other twist Lie algebra such as $D_n^{(2)}$ is also an interesting issue [28–31].

In this paper, we study the exact solution of the $A_3^{(2)}$ model with open boundary condition where the reflection matrices have non-diagonal elements. We use the fusion technique [32–38]. We find that the fusion properties of present system are quite different from the $A_n^{(1)}$ case. In the latter case, only the anti-symmetric fusion is used. For the present case, the R -matrix has two degenerated points. Based on this fact, we obtain two projectors. These two projectors give different fused behaviors. With the help of fused transfer matrices, we

find that the fusion processes can be closed. From the analyzing of polynomials, instead of constructing the eigenstates, we obtain the eigenvalues of the system, where the asymptotic behaviors and special points are used. Then we obtain the energy spectrum of the model Hamiltonian. In order to show the universality of this method, we also give the corresponding results of the system with periodic boundary condition.

The paper is organized as follows. In section 2, we give the description of the model, where the transfer matrix, Hamiltonian, R -matrix and reflection matrices are introduced. In section 3, we study the fusion properties. In section 4, the closed recursive fusion relations among the fused transfer matrices are given. In section 5, by constructing the inhomogeneous $T - Q$ relations, we obtain the eigenvalues and the corresponding Bethe ansatz equations of the system with non-diagonal boundary reflections. In section 6, the results associated with the periodic boundary condition are given. The summary of main results and some concluding remarks are presented in section 7. Some detailed calculations are given in Appendix A.

2 Associated conserved quantities

For the open boundary condition, the one-dimensional quantum integrable systems associated with the $A_3^{(2)}$ twist Lie algebra is generated by the transfer matrix $t(u)$

$$t(u) = \text{tr}_0\{K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\}, \quad (2.1)$$

where u is the spectral parameter, tr_0 means the trace in the four-dimensional auxiliary space V_0 , $K_0^-(u)$ is the reflection matrix at one end and is defined in the auxiliary space V_0 , $K_0^+(u)$ is the dual one at the other end, $T_0(u)$ is the monodromy matrix and the $\hat{T}_0(u)$ is the reflecting one. $T_0(u)$ and $\hat{T}_0(u)$ are constructed by the R -matrices as [1]

$$T_0(u) = R_{01}(u - \theta_1)R_{02}(u - \theta_2) \cdots R_{0N}(u - \theta_N), \quad (2.2)$$

$$\hat{T}_0(u) = R_{N0}(u + \theta_N) \cdots R_{20}(u + \theta_2)R_{10}(u + \theta_1). \quad (2.3)$$

Here $\{\theta_j | j = 1, \dots, N\}$ are the inhomogeneous parameters and N is the number of sites. The subscript j means the four-dimensional quantum space V_j . Thus the physical space is $\otimes_{j=1}^N V_j$. The R -matrix defined in the tensor space $V_1 \otimes V_2$ is the 16×16 matrix [39]

$$R_{12}(u) = a(u) \sum_{\alpha \neq \alpha'} [e_1]_{\alpha}^{\alpha} \otimes [e_2]_{\alpha}^{\alpha} + b(u) \sum_{\alpha \neq \beta, \beta'} [e_1]_{\alpha}^{\alpha} \otimes [e_2]_{\beta}^{\beta}$$

$$+\left\{e(u) \sum_{\alpha < \beta, \alpha \neq \beta'} + \bar{e}(u) \sum_{\alpha > \beta, \alpha \neq \beta'}\right\} [e_1]_{\beta}^{\alpha} \otimes [e_2]_{\alpha}^{\beta} + \sum_{\alpha, \beta} a_{\alpha\beta}(u) [e_1]_{\beta}^{\alpha} \otimes [e_2]_{\beta'}^{\alpha'}, \quad (2.4)$$

where $\alpha, \beta = 1, \dots, 4$, $\alpha' = 5 - \alpha$, $\beta' = 5 - \beta$, $[e_k]_{\beta}^{\alpha}$ is the 4×4 Weyl basis of the space V_k . The matrix elements in Eq.(2.4) are

$$\begin{aligned} a(u) &= 2 \sinh\left(\frac{u}{2} - \eta\right) \cosh\left(\frac{u}{2} - 2\eta\right), \quad b(u) = 2 \sinh\frac{u}{2} \cosh\left(\frac{u}{2} - 2\eta\right), \\ e(u) &= -2e^{-\frac{u}{2}} \sinh \eta \cosh\left(\frac{u}{2} - 2\eta\right), \quad \bar{e}(u) = e^u e(u), \\ a_{\alpha\beta}(u) &= 2 \sinh \eta e^{\mp \frac{u}{2}} \left[\mp e^{(\pm 2 + \bar{\alpha} - \bar{\beta})\eta} \sinh \frac{u}{2} - \delta_{\alpha\beta'} \cosh\left(\frac{u}{2} - 2\eta\right) \right], \quad \text{if } \alpha \leq \beta, \\ a_{\alpha\beta}(u) &= 2 \sinh \frac{u}{2} \cosh\left(\frac{u}{2} - \eta\right), \quad \text{if } \alpha = \beta, \alpha \neq \alpha', \end{aligned} \quad (2.5)$$

where η is the crossing parameter, $\bar{\alpha} = \alpha + \frac{1}{2}$ if $1 \leq \alpha \leq 2$ and $\bar{\alpha} = \alpha - \frac{1}{2}$ if $3 \leq \alpha \leq 4$. The R -matrix (2.4) has the properties

$$\begin{aligned} \text{unitarity : } R_{12}(u)R_{21}(-u) &= \rho_1(u) \times \text{id} = a(u)a(-u) \times \text{id}, \\ \text{crossing unitarity : } R_{12}(u)^{t_1} M_1 R_{21}(-u + 8\eta + 2i\pi)^{t_1} M_1^{-1} \\ &= R_{12}(u)^{t_2} M_2^{-1} R_{21}(-u + 8\eta + 2i\pi)^{t_2} M_2 = \rho_1(u - 4\eta - i\pi), \\ \text{regularity : } R_{12}(0) &= \rho_1(0)^{\frac{1}{2}} \mathcal{P}_{12}, \end{aligned} \quad (2.6)$$

where M_k is the 4×4 diagonal matrix $M_k = \text{diag}(e^{2\eta}, 1, 1, e^{-2\eta})$, \mathcal{P}_{12} is the permutation operator with the matrix elements $[\mathcal{P}_{12}]_{\beta\delta}^{\alpha\gamma} = \delta_{\alpha\delta} \delta_{\beta\gamma}$, t_k denotes the transposition in the k -th space, $R_{21}(u) = \mathcal{P}_{12} R_{12}(u) \mathcal{P}_{12}$. Besides, the R -matrix (2.4) satisfies the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \quad (2.7)$$

The integrability of the system requires that the boundary reflection matrix $K^-(u)$ satisfies the reflection equation

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \quad (2.8)$$

while $K^+(u)$ satisfies the dual one

$$\begin{aligned} &R_{12}(-u+v)K_1^+(u)M_1^{-1}R_{21}(-u-v+8\eta+2i\pi)M_1K_2^+(v) \\ &= K_2^+(v)M_1R_{12}(-u-v+8\eta+2i\pi)M_1^{-1}K_1^+(u)R_{21}(-u+v). \end{aligned} \quad (2.9)$$

The general solution of reflection equations (2.8)-(2.9) for the $A_n^{(2)}$ vertex model has been constructed by Lima-Santos et al [40], Malara et al [41] and Nepomechie et al [23], where the reflection matrices could have the non-diagonal elements. Here, we focus on the non-diagonal boundary reflections. Without losing the generality of our method, we chose

$$K_k^-(u) = \begin{pmatrix} e^{-u} & 0 & 0 & e^\epsilon \sinh u \\ 0 & \frac{-\sinh(u-\eta)}{\sinh \eta} & 0 & 0 \\ 0 & 0 & \frac{-\sinh(u-\eta)}{\sinh \eta} & 0 \\ e^{-\epsilon \frac{\sinh u}{\sinh^2 \eta}} & 0 & 0 & e^u \end{pmatrix}, \quad (2.10)$$

where ϵ is the boundary parameter at one side. The dual reflection matrix $K^+(u)$ is obtained by the mapping

$$K_k^+(u) = M_k K_k^-(-u + 4\eta + i\pi)|_{\epsilon \rightarrow \epsilon'}, \quad (2.11)$$

and ϵ' is the boundary parameter at the other side. It is easy to check that the matrices $K^-(u)$ and $K^+(u)$ cannot be diagonalized simultaneously for generic values of ϵ and ϵ' . Although the $U(1)$ symmetry is broken, the integrability of the system is still held.

From the Yang-Baxter equation (2.7), reflection equation (2.8) and dual one (2.9), one can prove [1] that the transfer matrices with different spectral parameters commute with each other, i.e., $[t(u), t(v)] = 0$. Thus, expanding $t(u)$ with respect to u , all the coefficients are the conserved quantities. The Hamiltonian is constructed by taking the derivative of the logarithm of the transfer matrix

$$\begin{aligned} H &= \left. \frac{\partial \ln t(u)}{\partial u} \right|_{u=0, \{\theta_j\}=0} \\ &= \sum_{j=1}^{N-1} \mathcal{P}_{jj+1} \left. \frac{\partial R_{jj+1}(u)}{\partial u} \right|_{u=0} + \frac{K_N^-(0)'}{2K_N^-(0)} + \frac{\text{tr}_0\{K_0^+(0)H_{10}\}}{\text{tr}_0 K_0^+(0)} + \text{constant}, \end{aligned} \quad (2.12)$$

where $H_{10} = \mathcal{P}_{10} \frac{\partial R_{10}(u)}{\partial u} \Big|_{u=0}$. We shall note that because the R -matrix (2.4) reduces to the permutation operator at the point of $u = 0$, the interaction in the bulk is the nearest neighbor one.

3 Fusion procedure

3.1 Fusion of R -matrices

The next task is to exact diagonal the transfer matrix (2.1). According to the definition, we know that $t(u)$ is an operator-valued polynomial of e^u with degrees $4N + 4$, up to an overall

factor $e^{-2Nu-2u}$. Thus $t(u)$ can be completely determined by $4N + 5$ constraints. In order to obtain these constraints, we adopt the method of fusion.

It is easy to check that the R -matrix (2.4) degenerates into the projectors at some special points. For examples, the R -matrix degenerates into an one-dimensional projector $P_{12}^{(1)}$ if $u = 4\eta + i\pi$, and a six-dimensional projector $P_{12}^{(6)}$ if $u = 2\eta$. These conclusions are achieved by the facts

$$R_{12}(4\eta + i\pi) = P_{12}^{(1)} S_{12}^{(1)}, \quad R_{12}(2\eta) = P_{12}^{(6)} S_{12}^{(6)}, \quad (3.1)$$

where $S_{12}^{(1)}$ and $S_{12}^{(6)}$ are the irrelevant constant matrices omitted here, $P_{12}^{(1)}$ and $P_{12}^{(6)}$ are the projectors

$$P_{12}^{(1)} = |\psi_0\rangle\langle\psi_0|, \quad P_{12}^{(6)} = \sum_{i=1}^6 |\phi_i\rangle\langle\phi_i|. \quad (3.2)$$

The basis vectors of the related projectors are

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{2 \cosh \eta} (e^{-\eta}|14\rangle + |23\rangle + |32\rangle + e^{\eta}|41\rangle), \\ |\phi_1\rangle &= \frac{1}{\sqrt{2} \cosh \eta} (e^{-\frac{\eta}{2}}|12\rangle - e^{\frac{\eta}{2}}|21\rangle), \quad |\phi_2\rangle = \frac{1}{\sqrt{2} \cosh \eta} (e^{-\frac{\eta}{2}}|13\rangle - e^{\frac{\eta}{2}}|31\rangle), \\ |\phi_3\rangle &= \frac{1}{\sqrt{2} \cosh \eta} (\sinh \eta|23\rangle + \sinh \eta|32\rangle + |14\rangle - |41\rangle), \\ |\phi_4\rangle &= \frac{1}{\sqrt{2} \cosh \eta} (e^{-\frac{\eta}{2}}|23\rangle - e^{\frac{\eta}{2}}|32\rangle), \quad |\phi_5\rangle = \frac{1}{\sqrt{2} \cosh \eta} (e^{-\frac{\eta}{2}}|24\rangle - e^{\frac{\eta}{2}}|42\rangle), \\ |\phi_6\rangle &= \frac{1}{\sqrt{2} \cosh \eta} (e^{-\frac{\eta}{2}}|34\rangle - e^{\frac{\eta}{2}}|43\rangle), \end{aligned} \quad (3.3)$$

Exchanging two spaces V_1 and V_2 , we obtain $P_{21}^{(1)}$ and $P_{21}^{(6)}$, where the bases are

$$|\psi_0\rangle_{|kl\rangle \rightarrow |lk\rangle}, \quad |\phi_i\rangle_{\eta \rightarrow -\eta, |kl\rangle \rightarrow |lk\rangle}. \quad (3.4)$$

where $\{|k\rangle, k = 1, \dots, 4\}$ and $\{|l\rangle, l = 1, \dots, 4\}$ are the orthogonal bases of four-dimensional linear space V_1 and V_2 , respectively.

From the Yang-Baxter equation (2.7) and using the properties of projector, we obtain

$$P_{12}^{(1)} R_{23}(u) R_{13}(u + 4\eta + i\pi) P_{12}^{(1)} = a(u) c(u + 4\eta + i\pi) P_{12}^{(1)}, \quad (3.5)$$

$$P_{21}^{(1)} R_{32}(u) R_{31}(u + 4\eta + i\pi) P_{21}^{(1)} = a(u) c(u + 4\eta + i\pi) P_{21}^{(1)}, \quad (3.6)$$

$$P_{12}^{(6)} R_{23}(u) R_{13}(u + 2\eta) P_{12}^{(6)} = \tilde{\rho}_0(u) R_{(12)3}(u + \eta), \quad (3.7)$$

$$P_{21}^{(6)} R_{32}(u) R_{31}(u + 2\eta) P_{21}^{(6)} = \tilde{\rho}_0(u) R_{3\langle 12 \rangle}(u + \eta), \quad (3.8)$$

where $c(u) = 2 \sinh \frac{u}{2} \cosh(\frac{u}{2} - \eta)$, $\tilde{\rho}_0(u) = \sinh \frac{u+\eta}{2} \cosh \frac{u-5\eta}{2}$ and the subscript $\langle 12 \rangle$ denotes the six-dimensional fused space $V_{\langle 12 \rangle} = V_{\bar{1}}$. From Eqs.(3.5)-(3.6), we see that the fusion with one-dimensional projectors gives an one-dimensional vector. From Eqs.(3.7)-(3.8), we know that the fusion with six-dimensional projectors gives a new fused R -matrix $R_{\bar{1}2}(u)$, whose matrix elements are given in Appendix A (see (A.1)-(A.2) below). Moreover, we have checked that $R_{\bar{1}2}(u)$ also satisfies the properties

$$\begin{aligned} \text{unitarity : } & R_{\bar{1}2}(u) R_{2\bar{1}}(-u) \times \text{id} = \rho_2(u) = a_1(u) a_1(-u) \times \text{id}, \\ \text{crossing unitarity : } & R_{\bar{1}2}(u)^{t_1} \bar{M}_{\bar{1}} R_{2\bar{1}}(-u + 8\eta + 2i\pi)^{t_1} \bar{M}_{\bar{1}}^{-1} \\ & = R_{\bar{1}2}(u)^{t_2} M_2^{-1} R_{2\bar{1}}(-u + 8\eta + 2i\pi)^{t_2} M_2 = \rho_2(u - 4\eta - i\pi), \\ \text{periodicity : } & R_{\bar{1}2}(u + i\pi) = -\bar{V}_{\bar{1}} R_{\bar{1}2}(u) \bar{V}_{\bar{1}}^{-1}, \end{aligned} \quad (3.9)$$

where $a_1(u) = 2 \sinh(u - 3\eta)$, $\bar{M}_{\bar{1}}$ is the diagonal matrix $\bar{M}_{\bar{1}} = P_{12}^{(6)} M_1 M_2 P_{12}^{(6)} = \text{diag}(e^{2\eta}, e^{2\eta}, 1, 1, e^{-2\eta}, e^{-2\eta})$ and $\bar{V}_{\bar{1}}$ is a 6×6 matrix with the form of

$$\bar{V}_{\bar{1}} = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix}. \quad (3.10)$$

The above properties are very useful later for us to derive some important polynomial properties of the associated transfer matrices $\bar{t}(u)$ given by (3.25) and $\bar{t}^{(p)}(u)$ given by (6.1).

It is remarked that the fused R -matrix $R_{\bar{1}2}(u)$ becomes a 4×4 matrix at the point of $u = 3\eta$

$$R_{\bar{1}2}(3\eta) = P_{12}^{(4)} S_{12}^{(4)}, \quad P_{12}^{(4)} = \sum_{i=1}^4 |\varphi_i\rangle \langle \varphi_i|, \quad (3.11)$$

where $S_{12}^{(4)}$ is an irrelevant constant matrix omitted here, and $P_{12}^{(4)}$ is a 4-dimensional projector with the basis vectors

$$\begin{aligned} |\varphi_1\rangle &= \frac{1}{\sqrt{2 \cosh \eta + e^{3\eta}}} (\sqrt{\cosh \eta} |13\rangle - \sqrt{\cosh \eta} |22\rangle - e^{\frac{3\eta}{2}} |41\rangle), \\ |\varphi_2\rangle &= \frac{1}{\sqrt{1 + 2 \cosh 2\eta}} (e^{-\frac{\eta}{2}} \sqrt{\cosh \eta} |14\rangle - \cosh \eta |32\rangle - \sinh \eta |42\rangle + e^{\frac{\eta}{2}} \sqrt{\cosh \eta} |51\rangle), \end{aligned}$$

$$\begin{aligned}
|\varphi_3\rangle &= \frac{1}{\sqrt{1+2\cosh 2\eta}}(e^{-\frac{\eta}{2}}\sqrt{\cosh \eta}|24\rangle - \cosh \eta|33\rangle + \sinh \eta|43\rangle + e^{\frac{\eta}{2}}\sqrt{\cosh \eta}|61\rangle), \\
|\varphi_4\rangle &= \frac{1}{\sqrt{2\cosh \eta + e^{-3\eta}}}(\sqrt{\cosh \eta}|62\rangle - \sqrt{\cosh \eta}|53\rangle + e^{-\frac{3\eta}{2}}|44\rangle).
\end{aligned} \tag{3.12}$$

Exchanging the two spaces $V_{\bar{1}}$ and V_2 , we deduce another 4-dimensional projector $P_{2\bar{1}}^{(4)}$ with the bases $|\varphi_i\rangle_{\eta \rightarrow -\eta, |kl\rangle \rightarrow |lk\rangle}$, where $\{|k\rangle, k=1, \dots, 6\}$ and $\{|l\rangle, l=1, \dots, 4\}$ are the orthogonal bases of six-dimensional linear space $V_{\bar{1}}$ and four-dimensional linear space V_2 , respectively. Starting from the Yang-Baxter equation

$$R_{\bar{1}2}(u-v)R_{\bar{1}3}(u)R_{23}(v) = R_{23}(v)R_{\bar{1}3}(u)R_{\bar{1}2}(u-v), \tag{3.13}$$

and using the properties of projector, we have

$$P_{\bar{1}2}^{(4)}R_{23}(u)R_{\bar{1}3}(u+3\eta)P_{\bar{1}2}^{(4)} = \tilde{\rho}_1(u)S_{\langle\bar{1}2\rangle}R_{\langle\bar{1}2\rangle 3}(u+2\eta+i\pi)S_{\langle\bar{1}2\rangle}^{-1}, \tag{3.14}$$

$$P_{2\bar{1}}^{(4)}R_{32}(u)R_{3\bar{1}}(u+3\eta)P_{2\bar{1}}^{(4)} = \tilde{\rho}_1(u)S_{\langle\bar{1}2\rangle}R_{3\langle\bar{1}2\rangle}(u+2\eta+i\pi)S_{\langle\bar{1}2\rangle}^{-1}, \tag{3.15}$$

where $\tilde{\rho}_1(u) = -4\sinh(\frac{u}{2} + \eta)\cosh(\frac{u}{2} - 2\eta)$, the subscript $\langle\bar{1}2\rangle$ denotes the fused four-dimensional space $V_{\langle\bar{1}2\rangle}$, and $S_{\langle\bar{1}2\rangle}$ is a diagonal matrix

$$\begin{aligned}
S_{\langle\bar{1}2\rangle} &= \text{diag} \left(-e^{-\frac{\eta}{2}} \frac{\sinh \eta}{\sinh 3\eta} s(\eta), 1, -1, e^{\frac{\eta}{2}} \frac{\sinh \eta}{\sinh 3\eta} s(-\eta) \right), \\
s(\eta) &= \sqrt{(1+2\cosh 2\eta)(e^{3\eta} + 2\cosh \eta)}.
\end{aligned} \tag{3.16}$$

From Eqs.(3.14) and (3.15), we see that the fused R -matrices $R_{\langle\bar{1}2\rangle 3}(u)$ and $R_{3\langle\bar{1}2\rangle}(u)$ differ from the fundamental ones only by a similar transformation up to a constant. By introducing the one-to-one correspondence, we can map the fused space $V_{\langle\bar{1}2\rangle}$ into V_1 . Then the fused R -matrix $R_{\langle\bar{1}2\rangle 3}(u)$ becomes the fundamental R -matrix $R_{13}(u)$ given by (2.4). Then we conclude that the fusion processes of R -matrices are closed.

3.2 Fusion of monodromy matrices

From the fused R -matrices (3.7)-(3.8), we construct the fused monodromy matrices

$$T_{\bar{0}}(u) = R_{\bar{0}1}(u - \theta_1)R_{\bar{0}2}(u - \theta_2) \cdots R_{\bar{0}N}(u - \theta_N), \tag{3.17}$$

$$\hat{T}_{\bar{0}}(u) = R_{N\bar{0}}(u + \theta_N) \cdots R_{2\bar{0}}(u + \theta_2)R_{1\bar{0}}(u + \theta_1). \tag{3.18}$$

We should note that the fusions are taken in the auxiliary space, thus all the quantum spaces of $T_0(u)$, $\hat{T}_0(u)$, $T_{\bar{0}}(u)$ and $\hat{T}_{\bar{0}}(u)$ are the same.

From the Yang-Baxter equations (2.7) and (3.13), we can prove that the monodromy matrices satisfy the Yang-Baxter relations

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v), \quad (3.19)$$

$$R_{1\bar{2}}(u-v)T_1(u)T_{\bar{2}}(v) = T_{\bar{2}}(v)T_1(u)R_{1\bar{2}}(u-v). \quad (3.20)$$

By using the fusion identities (3.5), (3.7) and (3.14), we obtain

$$\begin{aligned} P_{21}^{(1)}T_1(u)T_2(u+4\eta+i\pi)P_{21}^{(1)} &= P_{21}^{(1)}\prod_{j=1}^N a(u-\theta_j)c(u-\theta_j+4\eta+i\pi) \times \text{id}, \\ P_{12}^{(6)}T_2(u)T_1(u+2\eta)P_{12}^{(6)} &= \prod_{j=1}^N \tilde{\rho}_0(u-\theta_j)T_{\langle 12 \rangle}(u+\eta), \\ P_{\bar{1}2}^{(4)}T_2(u)T_{\bar{1}}(u+3\eta)P_{\bar{1}2}^{(4)} &= \prod_{j=1}^N \tilde{\rho}_1(u-\theta_j)S_{\langle \bar{1}2 \rangle}T_{\langle \bar{1}2 \rangle}(u+2\eta+i\pi)S_{\langle \bar{1}2 \rangle}^{-1}. \end{aligned} \quad (3.21)$$

The reflecting monodromy matrices satisfy the Yang-Baxter relations

$$R_{21}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{21}(u-v), \quad (3.22)$$

$$R_{1\bar{2}}(u-v)\hat{T}_2(v)\hat{T}_1(u) = \hat{T}_1(u)\hat{T}_2(v)R_{1\bar{2}}(u-v). \quad (3.23)$$

From Eqs.(3.6), (3.8) and (3.15), we obtain the fusion identities among the reflecting monodromy matrices

$$\begin{aligned} P_{12}^{(1)}\hat{T}_1(u)\hat{T}_2(u+4\eta+i\pi)P_{12}^{(1)} &= P_{12}^{(1)}\prod_{j=1}^N a(u+\theta_j)c(u+\theta_j+4\eta+i\pi) \times \text{id}, \\ P_{21}^{(6)}\hat{T}_2(u)\hat{T}_1(u+2\eta)P_{21}^{(6)} &= \prod_{j=1}^N \tilde{\rho}_1(u+\theta_j)\hat{T}_{\langle 12 \rangle}(u+\eta), \\ P_{\bar{2}1}^{(4)}\hat{T}_2(u)\hat{T}_{\bar{1}}(u+3\eta)P_{\bar{2}1}^{(4)} &= \prod_{j=1}^N \tilde{\rho}_1(u+\theta_j)S_{\langle \bar{1}2 \rangle}\hat{T}_{\langle \bar{1}2 \rangle}(u+2\eta+i\pi)S_{\langle \bar{1}2 \rangle}^{-1}. \end{aligned} \quad (3.24)$$

3.3 Fusion of reflection matrices

Using the fusion technique [37, 38], now, we need to connect the fusions of monodromy matrices and those of the reflecting ones, which gives the fusion behavior of the reflection matrices. We first define the fused transfer matrix

$$\bar{t}(u) = \text{tr}_{\bar{0}}\{K_{\bar{0}}^+(u)T_{\bar{0}}(u)K_{\bar{0}}^-(u)\hat{T}_{\bar{0}}(u)\}, \quad (3.25)$$

where the trace is taken in the fused auxiliary space $\bar{0}$ and $K_{\bar{0}}^{\pm}(u)$ are the fused reflection matrices. Then, we calculate the quantities

$$t(u)t(u+\Delta) = \rho_1^{-1}(2u+\Delta-4\eta-i\pi)tr_{12}\{K_2^+(u+\Delta)M_2^{-1}R_{12}(-2u+8\eta+2i\pi-\Delta) \\ \times M_2K_1^+(u)T_1(u)T_2(u+\Delta)K_1^-(u)R_{21}(2u+\Delta)K_2^-(u+\Delta)\hat{T}_1(u)\hat{T}_2(u+\Delta)\}, \quad (3.26)$$

$$t(u)\bar{t}(u+\Delta) = \rho_2^{-1}(2u+\Delta-4\eta-i\pi)tr_{1\bar{2}}\{K_2^+(u+\Delta)\bar{M}_2^{-1}R_{1\bar{2}}(-2u+8\eta+2i\pi-\Delta) \\ \times \bar{M}_2K_1^+(u)T_1(u)T_2(u+\Delta)K_1^-(u)R_{21}(2u+\Delta)K_2^-(u+\Delta)\hat{T}_1(u)\hat{T}_2(u+\Delta)\}, \quad (3.27)$$

where Δ is the shift of spectral parameter. From the fusion of monodromy matrices, we know that Δ should be chosen as $4\eta+i\pi$, 2η in Eq.(3.26) and as 3η in (3.27), which gives the fusion relations of reflection matrices.

The $\Delta = 4\eta+i\pi$ in Eq.(3.26) corresponds to the fusion with one-dimensional projectors. According to Eq.(3.26) and using the reflection equations (2.8)-(2.9) and the properties of projector, we obtain

$$P_{21}^{(1)}K_1^-(u)R_{21}(2u+4\eta+i\pi)K_2^-(u+4\eta+i\pi)P_{12}^{(1)} \\ = \frac{1}{\sinh^2\eta} \sinh(u+4\eta) \sinh(2u+2\eta) \sinh(u-\eta)P_{12}^{(1)}, \quad (3.28)$$

$$P_{12}^{(1)}K_2^+(u+4\eta+i\pi)M_1R_{12}(-2u+4\eta+i\pi)M_1^{-1}K_1^+(u)P_{21}^{(1)} \\ = -\frac{1}{\sinh^2\eta} \sinh(u-4\eta) \sinh(2u-2\eta) \sinh(u+\eta)P_{21}^{(1)}. \quad (3.29)$$

We shall remark that the inserted R -matrices with fixed spectral parameters in Eqs.(3.28) and (3.29) is to reserve the integrability of the system. The fused results are the one-dimensional vectors.

The $\Delta = 2\eta$ in Eq.(3.26) corresponds to the fusion with six-dimensional projectors. From Eq.(3.26), we obtain that the fused reflection matrices should be constructed as

$$P_{12}^{(6)}K_2^-(u)R_{12}(2u+2\eta)K_1^-(u+2\eta)P_{21}^{(6)} \\ = \frac{2}{\sinh\eta} \cosh(u-\eta) \sinh(u-\eta) \sinh(u+\eta) \sinh(u+2\eta)K_{\langle 12 \rangle}^-(u+\eta), \quad (3.30)$$

$$P_{21}^{(6)}K_1^+(u+2\eta)\bar{M}_1^{-1}R_{21}(-2u+6\eta)\bar{M}_1K_2^+(u)P_{12}^{(6)} \\ = -\frac{2}{\sinh\eta} \cosh(u-\eta) \sinh(u-\eta) \sinh(u-3\eta) \sinh(u-4\eta)K_{\langle 12 \rangle}^+(u+\eta), \quad (3.31)$$

where $K_{\langle 12 \rangle}^-(u)$ is the 6×6 fused reflection matrix defined in the fused space $V_{\langle 12 \rangle}$ with the

matrix form of

$$K_{\langle 12 \rangle}^-(u) = \begin{pmatrix} 0 & 0 & 0 & 0 & e^\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & e^\epsilon \\ 0 & 0 & -\frac{1}{\sinh \eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sinh \eta} & 0 & 0 \\ \frac{e^{-\epsilon}}{\sinh^2 \eta} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{e^{-\epsilon}}{\sinh^2 \eta} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.32)$$

and $K_{\langle 12 \rangle}^+(u)$ is the dual one

$$K_{\langle 12 \rangle}^+(u) = \bar{M}_{\langle 12 \rangle} K_{\langle 12 \rangle}^-(-u + 4\eta + i\pi)|_{\epsilon \rightarrow \epsilon'}. \quad (3.33)$$

With the definition $V_{\bar{1}} = V_{\langle 12 \rangle}$, the fused reflection matrices satisfy the reflection equations

$$R_{\bar{1}2}(u-v)K_{\bar{1}}^-(u)R_{2\bar{1}}(u+v)K_2^-(v) = K_2^-(v)R_{\bar{1}2}(u+v)K_{\bar{1}}^-(u)R_{2\bar{1}}(u-v), \quad (3.34)$$

$$\begin{aligned} R_{\bar{1}2}(-u+v)K_{\bar{1}}^+(u)\bar{M}_{\bar{1}}^{-1}R_{2\bar{1}}(-u-v+8\eta+2i\pi)\bar{M}_{\bar{1}}K_2^+(v) \\ = K_2^+(v)\bar{M}_{\bar{1}}R_{\bar{1}2}(-u-v+8\eta+2i\pi)\bar{M}_{\bar{1}}^{-1}K_{\bar{1}}^+(u)R_{2\bar{1}}(-u+v). \end{aligned} \quad (3.35)$$

The $\Delta = 3\eta$ in Eq.(3.27) corresponds to the fusion with four-dimensional projectors. According to Eq.(3.27), the fusion of reflection matrices are

$$\begin{aligned} P_{\bar{1}2}^{(4)}K_2^-(u)R_{\bar{1}2}(2u+3\eta)K_{\bar{1}}^-(u+3\eta)P_{2\bar{1}}^{(4)} \\ = \frac{4}{\sinh \eta} \cosh(u) \sinh(u-\eta) S_{\langle \bar{1}2 \rangle} K_{\langle \bar{1}2 \rangle}^-(u+2\eta+i\pi) S_{\langle \bar{1}2 \rangle}^{-1}, \end{aligned} \quad (3.36)$$

$$\begin{aligned} P_{2\bar{1}}^{(4)}K_{\bar{1}}^+(u+3\eta)\bar{M}_{\bar{1}}^{-1}R_{2\bar{1}}(-2u+5\eta)\bar{M}_{\bar{1}}K_2^+(u)P_{\bar{1}2}^{(4)} \\ = -\frac{4}{\sinh \eta} \cosh(u-\eta) \sinh(u-4\eta) S_{\langle \bar{1}2 \rangle} K_{\langle \bar{1}2 \rangle}^+(u+2\eta+i\pi) S_{\langle \bar{1}2 \rangle}^{-1}. \end{aligned} \quad (3.37)$$

With the same one-to-one correspondence as used in the fusion of R -matrices, the fused reflection matrices $K_{\langle \bar{1}2 \rangle}^\pm(u)$ become the original ones given by Eqs.(2.10)-(2.11). Thus the fusion processes of reflection matrices are also closed.

The fusion does not break the integrability. From the fused Yang-Baxter equation (3.13) and the fused reflection equations (3.34)-(3.35), one can prove that the transfer matrices $t(u)$ and $\bar{t}(u)$ commute with each other, i.e.,

$$[t(u), \bar{t}(u)] = 0. \quad (3.38)$$

Thus $t(u)$ and $\bar{t}(u)$ have common eigenstates.

4 Closed operators identities

From the definitions of R -matrices (2.4), (3.9) and reflection matrices (2.10), (3.32), we know that the $t(u)$ (resp. $\bar{t}(u)$) is an operator-valued polynomial of e^u with degree $4N + 4$ up to an overall factors $e^{-2Nu-2u}$ (an operator-valued polynomial of e^{2u} with degree $2N$ up to an overall factor e^{-2Nu} respectively). Denote the eigenvalues of $t(u)$ and $\bar{t}(u)$ acting on a common eigenstate as $\Lambda(u)$ and $\bar{\Lambda}(u)$, respectively. Then the eigenvalue $\Lambda(u)$ (resp. $\bar{\Lambda}(u)$) is a polynomial of e^u with degree $4N + 4$ (is a polynomial of e^{2u} with degree $2N$) up to an overall known factor. Therefore, $\Lambda(u)$ and $\bar{\Lambda}(u)$ can be completely determined by the values of them at $6N + 6$ points. The next task is to find these complete constraints.

From Eqs.(3.26)-(3.27), we see that for arbitrary values of spectral parameter u , the fusion relations among the transfer matrices $t(u)$ and $\bar{t}(u)$ are not closed. However, we find that at the inhomogeneous points $\{\theta_j\}$, the fusions of $t(u)$ and $\bar{t}(u)$ can be closed. The detailed derivation is as follows. From the Yang-Baxter relation (3.19) at the points of $\{u = \theta_j, v = \{\theta_j + 4\eta + i\pi, \theta_j + 2\eta\}\}$, (3.20) at the points of $\{u = \theta_j, v = \theta_j + 3\eta\}$ and using the properties of projectors, we obtain

$$\begin{aligned} T_1(\theta_j)T_2(\theta_j + 4\eta + i\pi) &= P_{21}^{(1)}T_1(\theta_j)T_2(\theta_j + 4\eta + i\pi), \\ T_2(\theta_j)T_1(\theta_j + 2\eta) &= P_{12}^{(6)}T_2(\theta_j)T_1(\theta_j + 2\eta), \\ T_2(\theta_j)T_{\bar{1}}(\theta_j + 3\eta) &= P_{\bar{1}2}^{(4)}T_2(\theta_j)T_{\bar{1}}(\theta_j + 3\eta), \quad j = 1, \dots, N. \end{aligned} \quad (4.1)$$

We see that we can obtain three projectors by the suitable choices of spectral parameters in the monodromy matrices. The role of introducing inhomogeneous parameters $\{\theta_j\}$ is to generate the projectors. The generated projectors allow us to taken the fusion, which is valid for arbitrary u and the only requirement is the shift Δ . Substituting Eq.(4.1) into (3.26)-(3.27) with $u = \theta_j$, we obtain one set of fusion relations between $t(u)$ and $\bar{t}(u)$. The Yang-Baxter relation (3.22) at the points of $\{u = -\theta_j, v = \{-\theta_j + 4\eta + i\pi, -\theta_j + 2\eta\}\}$ and (3.23) at the points of $\{u = -\theta_j, v = -\theta_j + 3\eta\}$ give

$$\begin{aligned} \hat{T}_1(-\theta_j)\hat{T}_2(-\theta_j + 4\eta + i\pi) &= P_{12}^{(1)}\hat{T}_1(-\theta_j)\hat{T}_2(-\theta_j + 4\eta + i\pi), \\ \hat{T}_2(-\theta_j)\hat{T}_1(-\theta_j + 2\eta) &= P_{21}^{(6)}\hat{T}_2(-\theta_j)\hat{T}_1(-\theta_j + 2\eta), \\ \hat{T}_2(-\theta_j)\hat{T}_{\bar{1}}(-\theta_j + 3\eta) &= P_{\bar{1}2}^{(4)}\hat{T}_2(-\theta_j)\hat{T}_{\bar{1}}(-\theta_j + 3\eta), \quad j = 1, \dots, N. \end{aligned} \quad (4.2)$$

We see that three projectors can also be generated in this situation. Substituting Eq.(4.2)

into (3.26)-(3.27) with $u = -\theta_j$, we obtain another set of fusion relations between $t(u)$ and $\bar{t}(u)$.

Now, we are ready to seek the closed fusion relations among the transfer matrices. Substituting Eqs.(3.21), (3.24), (3.28)-(3.31), (3.36)-(3.37), (4.1)-(4.2) into Eq.(3.26) and considering the cases of $\{u = \pm\theta_j, \Delta = 4\eta + i\pi\}$ and $\{u = \pm\theta_j, \Delta = 2\eta\}$, and into Eq.(3.27) with $\{u = \pm\theta_j, \Delta = 3\eta\}$, we arrive at

$$\begin{aligned}
t(\pm\theta_j)t(\pm\theta_j + 4\eta + i\pi) &= \frac{\sinh(\pm 2\theta_j - 2\eta) \sinh(\pm 2\theta_j + 2\eta) \sinh(\pm\theta_j - 4\eta) \sinh(\pm\theta_j + 4\eta)}{4 \sinh^4 \eta \cosh(\pm\theta_j - 2\eta) \cosh(\pm\theta_j + 2\eta)} \\
&\quad \times \prod_{l=1}^N a(\pm\theta_j - \theta_l) c(\pm\theta_j - \theta_l + 4\eta + i\pi) a(\pm\theta_j + \theta_l) c(\pm\theta_j + \theta_l + 4\eta + i\pi) \times \text{id}, \\
t(\pm\theta_j)t(\pm\theta_j + 2\eta) &= \frac{\sinh^2(\pm 2\theta_j - 2\eta) \sinh(\pm\theta_j + 2\eta) \sinh(\pm\theta_j - 4\eta)}{4 \sinh^2 \eta \cosh(\pm\theta_j) \cosh(\pm\theta_j - 2\eta)} \\
&\quad \times \prod_{l=1}^N \tilde{\rho}_0(\pm\theta_j - \theta_l) \tilde{\rho}_0(\pm\theta_j + \theta_l) \bar{t}(\pm\theta_j + \eta), \\
t(\pm\theta_j)\bar{t}(\pm\theta_j + 3\eta) &= \frac{2 \cosh(\pm\theta_j) \sinh(\pm 2\theta_j - 2\eta) \sinh(\pm\theta_j - 4\eta)}{\sinh^2 \eta \sinh(\pm 2\theta_j + 2\eta) \sinh(\pm 2\theta_j - 4\eta)} \\
&\quad \times \prod_{l=1}^N \tilde{\rho}_1(\pm\theta_j - \theta_l) \tilde{\rho}_1(\pm\theta_j + \theta_l) t(\pm\theta_j + 2\eta + i\pi), \quad j = 1, \dots, N.
\end{aligned} \tag{4.3}$$

We shall note that the recursive equations (4.3) are closed, which give the $6N$ constraints of $t(u)$ and $\bar{t}(u)$ at the inhomogeneous points.

Besides, from the direct calculation, we also obtain the values of $t(u)$ and $\bar{t}(u)$ at some special points

$$\begin{aligned}
t(0) &= 4 \cosh^2 \eta \prod_{j=1}^N \rho_1(-\theta_j) \times \text{id}, \quad t(4\eta + i\pi) = 4 \cosh^2 \eta \prod_{j=1}^N \rho_1(-\theta_j) \times \text{id}, \\
t(2\eta) &= \frac{\cosh^2 \eta}{2 \cosh 2\eta (1 + 2 \cosh 2\eta)} \bar{t}(\eta).
\end{aligned} \tag{4.4}$$

In the derivation, we have used the relations

$$\begin{aligned}
\text{tr}[K^+(0)K^-(0)] &= 4 \cosh^2 \eta, \quad \text{tr}[K^+(4\eta + i\pi)K^-(4\eta + i\pi)] = 4 \cosh^2 \eta, \\
\text{tr}_1\{M_2^{-1}R_{12}(6\eta + 2i\pi)M_2K_1^+(0)R_{21}(2\eta)\} &= 4 \cosh 2\eta (1 + 2 \cosh 2\eta) \sinh^2 2\eta \times \text{id}.
\end{aligned}$$

The asymptotic behaviors of $t(u)$ and $\bar{t}(u)$ read

$$t(u)|_{u \rightarrow \pm\infty} = Q_{\pm} e^{\pm(2N+2)u} + \dots, \quad \bar{t}(u)|_{u \rightarrow \pm\infty} = \bar{Q}_{\pm} e^{\pm 2Nu} + \dots, \tag{4.5}$$

where Q_{\pm} and \bar{Q}_{\pm} are the conserved quantities

$$\begin{aligned}
Q_+ &= \frac{1}{4 \sinh^2 \eta} \left\{ e^{\epsilon' - \epsilon} e^{-2\eta} [T_+]_4^4 [\hat{T}_+]_1^1 + e^{\epsilon - \epsilon'} e^{-6\eta} [T_+]_1^1 [\hat{T}_+]_4^4 \right. \\
&\quad \left. + e^{-4\eta} [T_+]_2^2 [\hat{T}_+]_2^2 + e^{-4\eta} [T_+]_3^3 [\hat{T}_+]_3^3 \right\}, \\
Q_- &= \frac{1}{4 \sinh^2 \eta} \left\{ e^{\epsilon' - \epsilon} e^{6\eta} [T_-]_4^4 [\hat{T}_-]_1^1 + e^{\epsilon - \epsilon'} e^{2\eta} [T_-]_1^1 [\hat{T}_-]_4^4 \right. \\
&\quad \left. + e^{4\eta} [T_-]_2^2 [\hat{T}_-]_2^2 + e^{4\eta} [T_-]_3^3 [\hat{T}_-]_3^3 \right\}, \\
\bar{Q}_+ &= \frac{1}{\sinh^2 \eta} \left\{ e^{\epsilon' - \epsilon} e^{2\eta} \left([\bar{T}_+]_5^5 [\hat{\bar{T}}_+]_1^1 + [\bar{T}_+]_6^6 [\hat{\bar{T}}_+]_2^2 \right) \right. \\
&\quad \left. + [\bar{T}_+]_3^3 [\hat{\bar{T}}_+]_3^3 + [\bar{T}_+]_4^4 [\hat{\bar{T}}_+]_4^4 + e^{\epsilon - \epsilon'} e^{-2\eta} \left([\bar{T}_+]_1^1 [\hat{\bar{T}}_+]_5^5 + [\bar{T}_+]_2^2 [\hat{\bar{T}}_+]_6^6 \right) \right\}, \\
\bar{Q}_- &= \frac{1}{\sinh^2 \eta} \left\{ e^{\epsilon' - \epsilon} e^{2\eta} \left([\bar{T}_-]_5^5 [\hat{\bar{T}}_-]_1^1 + [\bar{T}_-]_6^6 [\hat{\bar{T}}_-]_2^2 \right) \right. \\
&\quad \left. + [\bar{T}_-]_3^3 [\hat{\bar{T}}_-]_3^3 + [\bar{T}_-]_4^4 [\hat{\bar{T}}_-]_4^4 + e^{\epsilon - \epsilon'} e^{-2\eta} \left([\bar{T}_-]_1^1 [\hat{\bar{T}}_-]_5^5 + [\bar{T}_-]_2^2 [\hat{\bar{T}}_-]_6^6 \right) \right\}. \tag{4.6}
\end{aligned}$$

Here $[T_{\pm}]_{\beta}^{\alpha}$, $[\hat{T}_{\pm}]_{\beta}^{\alpha}$, $[\bar{T}_{\pm}]_{\beta}^{\alpha}$ and $[\hat{\bar{T}}_{\pm}]_{\beta}^{\alpha}$ are the operators acting on the quantum space $V_1 \otimes V_2 \otimes \dots \otimes V_N$ with the explicit expressions

$$\begin{aligned}
[T_{\pm}]_{\beta}^{\alpha} &= \sum_{\{\delta_i\}=1, \{\gamma_i\}=1}^4 [R_{01}^{(\pm)}]_{\alpha_1 \delta_1}^{\alpha \gamma_1} [R_{02}^{(\pm)}]_{\alpha_2 \delta_2}^{\alpha_1 \gamma_2} \dots [R_{0N}^{(\pm)}]_{\beta}^{\alpha_{N-1} \gamma_N}_{\delta_N}, \\
[\hat{T}_{\pm}]_{\beta}^{\alpha} &= \sum_{\{\delta_i\}=1, \{\gamma_i\}=1}^4 [R_{N0}^{(\pm)}]_{\delta_N \alpha_N}^{\gamma_N \alpha} [R_{N-10}^{(\pm)}]_{\delta_{N-1} \alpha_{N-1}}^{\gamma_{N-1} \alpha_N} \dots [R_{20}^{(\pm)}]_{\delta_1 \beta}^{\gamma_1 \alpha_2}, \\
[\bar{T}_{\pm}]_{\beta}^{\alpha} &= \sum_{\{\delta_i\}=1, \{\gamma_i\}=1}^4 [R_{01}^{(\pm)}]_{\alpha_1 \delta_1}^{\alpha \gamma_1} [R_{02}^{(\pm)}]_{\alpha_2 \delta_2}^{\alpha_1 \gamma_2} \dots [R_{0N}^{(\pm)}]_{\beta}^{\alpha_{N-1} \gamma_N}_{\delta_N}, \\
[\hat{\bar{T}}_{\pm}]_{\beta}^{\alpha} &= \sum_{\{\delta_i\}=1, \{\gamma_i\}=1}^4 [R_{N0}^{(\pm)}]_{\delta_N \alpha_N}^{\gamma_N \alpha} [R_{N-10}^{(\pm)}]_{\delta_{N-1} \alpha_{N-1}}^{\gamma_{N-1} \alpha_N} \dots [R_{10}^{(\pm)}]_{\delta_1 \beta}^{\gamma_1 \alpha_2}, \tag{4.7}
\end{aligned}$$

where the repeated indicators should be summarized, and $R_{0j}^{(\pm)}$ and $R_{0j}^{(\pm)}$ are the leading terms of $e^{\mp u} R_{0j}(u)|_{u \rightarrow \pm \infty}$ and $e^{\mp u} R_{0j}(u)|_{u \rightarrow \pm \infty}$, respectively. The detailed calculation shows that the eigenvalues of conserved quantities Q_{\pm} and \bar{Q}_{\pm} can be characterized by a quantum number m (an integer $|m| \in [0, N]$). Then we obtain the asymptotic behaviors of $\Lambda(u)$ and $\bar{\Lambda}(u)$ as

$$\begin{aligned}
\Lambda(u)|_{u \rightarrow \pm \infty} &= \frac{2}{4^{N+1} \sinh^2 \eta} \left[\cosh(\epsilon - \epsilon' - 2\eta) + \cosh(2m\eta) \right] e^{\pm(2Nu + 2u - 4N\eta - 4\eta)} + \dots, \\
\bar{\Lambda}(u)|_{u \rightarrow \pm \infty} &= \frac{2}{\sinh^2 \eta} \left[2 \cosh(\epsilon - \epsilon' - 2\eta) \cosh(2m\eta) + 1 \right] e^{\pm(2Nu - 4N\eta)} + \dots. \tag{4.8}
\end{aligned}$$

5 Inhomogeneous T-Q relations

Acting the operator identities (4.3) on a common eigenstate, we obtain the functional relations among the eigenvalues $\Lambda(u)$ and $\bar{\Lambda}(u)$ as

$$\begin{aligned}
\Lambda(\pm\theta_j)\Lambda(\pm\theta_j + 4\eta + i\pi) &= \frac{\sinh(\pm 2\theta_j - 2\eta) \sinh(\pm 2\theta_j + 2\eta) \sinh(\pm\theta_j - 4\eta) \sinh(\pm\theta_j + 4\eta)}{4 \sinh^4 \eta \cosh(\pm\theta_j - 2\eta) \cosh(\pm\theta_j + 2\eta)} \\
&\quad \times \prod_{l=1}^N a(\pm\theta_j - \theta_l) c(\pm\theta_j - \theta_l + 4\eta + i\pi) a(\pm\theta_j + \theta_l) c(\pm\theta_j + \theta_l + 4\eta + i\pi), \\
\Lambda(\pm\theta_j)\Lambda(\pm\theta_j + 2\eta) &= \frac{\sinh^2(\pm 2\theta_j - 2\eta) \sinh(\pm\theta_j + 2\eta) \sinh(\pm\theta_j - 4\eta)}{4 \sinh^2 \eta \cosh(\pm\theta_j) \cosh(\pm\theta_j - 2\eta)} \\
&\quad \times \prod_{l=1}^N \tilde{\rho}_0(\pm\theta_j - \theta_l) \tilde{\rho}_0(\pm\theta_j + \theta_l) \bar{\Lambda}(\pm\theta_j + \eta), \\
\Lambda(\pm\theta_j) \bar{\Lambda}(\pm\theta_j + 3\eta) &= \frac{2 \cosh(\pm\theta_j) \cosh(\pm 2\theta_j - 2\eta) \sinh(\pm\theta_j - 4\eta)}{\sinh^2 \eta \sinh(\pm 2\theta_j + 2\eta) \sinh(\pm 2\theta_j - 4\eta)} \\
&\quad \times \prod_{l=1}^N \tilde{\rho}_1(\pm\theta_j - \theta_l) \tilde{\rho}_1(\pm\theta_j + \theta_l) \Lambda(\pm\theta_j + 2\eta + i\pi), \quad j = 1, \dots, N.
\end{aligned} \tag{5.1}$$

According to Eqs.(4.4) and (4.5), we also have other seven constraints of $\Lambda(u)$ and $\bar{\Lambda}(u)$ as

$$\Lambda(0) = 4 \cosh^2 \eta \prod_{j=1}^N \rho_1(-\theta_j), \quad \Lambda(4\eta + i\pi) = 4 \cosh^2 \eta \prod_{j=1}^N \rho_1(-\theta_j), \tag{5.2}$$

$$\Lambda(2\eta) = \frac{\cosh^2 \eta}{2 \cosh 2\eta (1 + 2 \cosh 2\eta)} \bar{\Lambda}(\eta). \tag{5.3}$$

The $6N$ function relations (5.1) and 7 constraints (4.8) and (5.2)-(5.3) allow us sufficient information to determine the values of $\Lambda(u)$ and $\bar{\Lambda}(u)$, which can be expressed in terms of inhomogeneous $T - Q$ relations as

$$\begin{aligned}
\Lambda(u) &= \frac{\sinh(u - 4\eta) \sinh(2u - 2\eta)}{2 \sinh^2 \eta \cosh(u - 2\eta)} \prod_{j=1}^N a(u - \theta_j) a(u + \theta_j) \frac{Q^{(1)}(u + 2\eta)}{Q^{(1)}(u)} \\
&\quad + \frac{\sinh(u) \sinh(u - 4\eta)}{\sinh^2 \eta \sinh(2u - 4\eta)} \prod_{j=1}^N b(u - \theta_j) b(u + \theta_j) \left[\sinh(2u - 2\eta) \right. \\
&\quad \times \frac{Q^{(1)}(u - 2\eta) Q^{(2)}(u + 2\eta)}{Q^{(1)}(u) Q^{(2)}(u)} + \sinh(2u - 6\eta) \frac{Q^{(1)}(u - i\pi) Q^{(2)}(u - 2\eta)}{Q^{(1)}(u - 2\eta - i\pi) Q^{(2)}(u)} \left. \right] \\
&\quad + \frac{\sinh(u) \sinh(2u - 6\eta)}{2 \sinh^2 \eta \cosh(u - 2\eta)} \prod_{j=1}^N c(u - \theta_j) c(u + \theta_j) \frac{Q^{(1)}(u - 4\eta - i\pi)}{Q^{(1)}(u - 2\eta - i\pi)}
\end{aligned}$$

$$+h \prod_{j=1}^N b(u - \theta_j) b(u + \theta_j) \frac{\sinh u \sinh(u - 4\eta)}{\sinh^2 \eta} \frac{Q^{(1)}(u - 2\eta) Q^{(1)}(u - i\pi)}{Q^{(2)}(u)}, \quad (5.4)$$

$$\begin{aligned} \bar{\Lambda}(u) = & \frac{1}{\sinh^2 \eta} \left\{ \prod_{j=1}^N a_1(u - \theta_j) a_1(u + \theta_j) \frac{\sinh(u - 3\eta)}{\sinh(u - \eta)} \right. \\ & \times \left[\frac{\sinh(2u)}{\sinh(2u - 4\eta)} \frac{Q^{(2)}(u + 3\eta)}{Q^{(2)}(u + \eta)} + \frac{Q^{(1)}(u + \eta) Q^{(1)}(u + \eta - i\pi) Q^{(2)}(u - \eta)}{Q^{(1)}(u - \eta) Q^{(1)}(u - \eta - i\pi) Q^{(2)}(u + \eta)} \right] \\ & + \prod_{j=1}^N b_1(u - \theta_j) b_1(u + \theta_j) \frac{\sinh(u - \eta)}{\sinh(u - 3\eta)} \\ & \times \left[\frac{\sinh(2u - 8\eta)}{\sinh(2u - 4\eta)} \frac{Q^{(2)}(u - 3\eta)}{Q^{(2)}(u - \eta)} + \frac{Q^{(1)}(u - 3\eta) Q^{(1)}(u - 3\eta - i\pi) Q^{(2)}(u + \eta)}{Q^{(1)}(u - \eta) Q^{(1)}(u - \eta - i\pi) Q^{(2)}(u - \eta)} \right] \\ & + \prod_{j=1}^N c_1(u - \theta_j) c_1(u + \theta_j) \frac{Q^{(1)}(u + \eta) Q^{(1)}(u - 3\eta - i\pi)}{Q^{(1)}(u - \eta) Q^{(1)}(u - \eta - i\pi)} \\ & + \prod_{j=1}^N c_1(u - \theta_j - i\pi) c_1(u + \theta_j - i\pi) \frac{Q^{(1)}(u + \eta - i\pi) Q^{(1)}(u - 3\eta)}{Q^{(1)}(u - \eta) Q^{(1)}(u - \eta - i\pi)} \\ & + h \frac{\cosh(u - \eta) \sinh(u - 3\eta)}{\sinh(2u - 4\eta)} \prod_{j=1}^N a_1(u - \theta_j) a_1(u + \theta_j) \\ & \times \frac{Q^{(1)}(u + \eta) Q^{(1)}(u + \eta - i\pi)}{Q^{(2)}(u + \eta)} + h \frac{\cosh(u - 3\eta) \sinh(u - \eta)}{\sinh(2u - 4\eta)} \\ & \times \left. \prod_{j=1}^N b_1(u - \theta_j) b_1(u + \theta_j) \frac{Q^{(1)}(u - 3\eta) Q^{(1)}(u - 3\eta - i\pi)}{Q^{(2)}(u - \eta)} \right\}, \quad (5.5) \end{aligned}$$

where h is a parameter to be determined later (see (5.9) below), the Q -functions are

$$\begin{aligned} Q^{(1)}(u) &= \prod_{l=1}^{L_1} \sinh \frac{1}{2}(u - \mu_l^{(1)} - \eta) \sinh \frac{1}{2}(u + \mu_l^{(1)} - \eta), \\ Q^{(2)}(u) &= \prod_{k=1}^{L_2} \sinh(u - \mu_k^{(2)} - 2\eta) \sinh(u + \mu_k^{(2)} - 2\eta), \\ a_1(u) &= 2 \sinh(u - 3\eta), \quad b_1(u) = 2 \sinh(u - \eta), \\ c_1(u) &= 4 \sinh \frac{1}{2}(u - 3\eta) \cosh \frac{1}{2}(u - \eta). \end{aligned} \quad (5.6)$$

L_1 is the number of Bethe roots $\{\mu_l^{(1)}\}$ and L_2 is the number of Bethe roots $\{\mu_k^{(2)}\}$. The regularities of eigenvalues $\Lambda(u)$ and $\bar{\Lambda}(u)$ require that the Bethe roots $\{\mu_l^{(1)}\}$ and $\{\mu_k^{(2)}\}$ satisfy the Bethe ansatz equations (BAEs)

$$\frac{Q^{(1)}(\mu_l^{(1)} + 3\eta) Q^{(2)}(\mu_l^{(1)} + \eta)}{Q^{(1)}(\mu_l^{(1)} - \eta) Q^{(2)}(\mu_l^{(1)} + 3\eta)} = - \frac{\sinh(\mu_l^{(1)} + \eta)}{\sinh(\mu_l^{(1)} - \eta)}$$

$$\times \prod_{j=1}^N \frac{\sinh \frac{1}{2}(\mu_l^{(1)} + \eta - \theta_j) \sinh \frac{1}{2}(\mu_l^{(1)} + \eta + \theta_j)}{\sinh \frac{1}{2}(\mu_l^{(1)} - \eta - \theta_j) \sinh \frac{1}{2}(\mu_l^{(1)} - \eta + \theta_j)}, \quad l = 1, \dots, L_1, \quad (5.7)$$

$$\begin{aligned} & \frac{\sinh(2\mu_k^{(2)} + 2\eta)}{\sinh(2\mu_k^{(2)})} \frac{Q^{(2)}(\mu_k^{(2)} + 4\eta)}{Q^{(1)}(\mu_k^{(2)} + 2\eta)Q^{(1)}(\mu_k^{(2)} + 2\eta - i\pi)} \\ & + \frac{\sinh(2\mu_k^{(2)} - 2\eta)}{\sinh(2\mu_k^{(2)})} \frac{Q^{(2)}(\mu_k^{(2)})}{Q^{(1)}(\mu_k^{(2)})Q^{(1)}(\mu_k^{(2)} - i\pi)} = -h, \quad k = 1, \dots, L_2. \end{aligned} \quad (5.8)$$

From the degree analysis of polynomials $\Lambda(u)$ and $\bar{\Lambda}(u)$, we obtain that the numbers of Bethe roots $\{\mu_l^{(1)}\}$ and $\{\mu_k^{(2)}\}$ should be the same, i.e., $L_1 = L_2$. According to the asymptotic behaviors of $\Lambda(u)$ and $\bar{\Lambda}(u)$, the value of h is determined as

$$h = (-1)^{L_1} 4^{L_1} \left\{ 2 \cosh(\epsilon - \epsilon' - 2\eta) - 2 \cosh[2(L_1 + 1)\eta] \right\}. \quad (5.9)$$

Meanwhile, the quantum number m is related with the number of Bethe roots as $m = L_1 - N^3$.

Here we present the numerical solutions of the $T - Q$ relation (5.4) with BAEs (5.7)-(5.8) for the $N = 2$ case in Table 1. The eigenvalue calculated from (5.4) is the same as that from the exact diagonalization of the transfer matrix $t(u)$ (2.1) with open boundary conditions. Numerical solutions with random choice of u and η for some small size imply that the solution (5.4) indeed gives the complete solutions of the model. We further remark that for $L = 0$, the $T - Q$ relation will give the same eigenvalue as the $L = 4$ in the Table.

³If $m \leq 0$, we have $0 \leq L_1 \leq N$. When $m \geq 0$, we have $N \leq L_1 \leq 2N$.

Table 1: Solutions of BAEs (5.7)-(5.8) for the $T - Q$ relation (5.4), $N = 2$, $u = 0.2$, $\eta = 0.4$, $\{\theta_j\} = 0$, $\epsilon = \epsilon' = 0$. The numbers of the two sets of Bethe roots are the same $L = L_1 = L_2$. The symbol n indicates the number of the spectrum $\Lambda(u)$.

$\mu_1^{(1)}$	$\mu_2^{(1)}$	$\mu_3^{(1)}$	$\mu_4^{(1)}$	$\mu_1^{(2)}$	$\mu_2^{(2)}$
2.8213 - 7.3824 <i>i</i>	2.8566 - 11.7923 <i>i</i>	2.6075 - 9.6328 <i>i</i>	-15.4519 + 9.5555 <i>i</i>	5.7360 - 20.1799 <i>i</i>	-15.9439 + 3.2723 <i>i</i>
-1.8106 - 5.5833 <i>i</i>	-0.6073 - 3.3582 <i>i</i>	-9.6710 - 14.6796 <i>i</i>	-1.7855 - 0.7885 <i>i</i>	8.1575 - 2.9733 <i>i</i>	-10.2358 - 11.4398 <i>i</i>
-1.1119 - 3.1416 <i>i</i>	—	—	—	1.7606 + 0.0000 <i>i</i>	—
-1.1119 + 3.1416 <i>i</i>	-12.7134 + 2.2785 <i>i</i>	-12.7134 - 4.0046 <i>i</i>	—	13.5955 + 4.0046 <i>i</i>	1.7606 - 3.1416 <i>i</i>
-12.8331 + 0.7576 <i>i</i>	-0.0000 - 3.1416 <i>i</i>	—	—	20.7772 - 1.5018 <i>i</i>	20.9923 - 0.1902 <i>i</i>
-0.7650 + 1.2091 <i>i</i>	-0.7650 - 1.2091 <i>i</i>	—	—	10.3808 + 0.9039 <i>i</i>	1.0705 - 1.5708 <i>i</i>
0.0751 - 0.3828 <i>i</i>	—	—	—	-0.7428 - 2.6351 <i>i</i>	—
8.5971 + 2.8481 <i>i</i>	0.0751 - 0.3828 <i>i</i>	-8.5971 + 0.2935 <i>i</i>	—	-9.1416 + 0.6080 <i>i</i>	-0.7428 + 6.7896 <i>i</i>
-0.0751 - 0.3828 <i>i</i>	—	—	—	0.7428 + 0.5064 <i>i</i>	—
-0.0751 + 5.9004 <i>i</i>	8.8382 + 6.7463 <i>i</i>	-8.8382 + 8.9616 <i>i</i>	—	-9.3827 + 2.9929 <i>i</i>	-0.7428 + 12.0599 <i>i</i>
0.0051 + 0.4591 <i>i</i>	-0.0676 + 1.4934 <i>i</i>	—	—	9.3206 - 0.5054 <i>i</i>	-9.0327 + 0.6314 <i>i</i>
0.1816 - 1.3704 <i>i</i>	-0.0175 - 0.4666 <i>i</i>	—	—	-9.1912 - 0.4531 <i>i</i>	-9.3950 + 2.4425 <i>i</i>
0.0000 - 0.0000 <i>i</i>	-14.0480 + 2.8382 <i>i</i>	—	—	22.9513 - 2.7003 <i>i</i>	22.9864 - 1.7707 <i>i</i>
-0.0393 + 0.0000 <i>i</i>	—	—	—	0.4925 - 0.0000 <i>i</i>	—
0.0393 + 0.0000 <i>i</i>	8.6900 - 0.0624 <i>i</i>	8.6901 - 3.2040 <i>i</i>	—	-9.2346 + 2.8895 <i>i</i>	9.2346 - 0.3770 <i>i</i>
-0.0000 - 1.6458 <i>i</i>	-0.0000 + 0.0049 <i>i</i>	—	—	-0.4919 - 0.0000 <i>i</i>	7.3047 - 1.4074 <i>i</i>
$\mu_3^{(2)}$	$\mu_4^{(2)}$	$\Lambda(u)$	L	n	
-5.7344 + 12.8484 <i>i</i>	-5.7362 + 5.5171 <i>i</i>	0.6400 - 0.0000 <i>i</i>	4	1	
9.4903 + 3.1677 <i>i</i>	-8.0876 - 4.9070 <i>i</i>	0.6400 - 0.0000 <i>i</i>	4	2	
—	—	0.6884 - 0.0000 <i>i</i>	1	3	
-12.9204 + 5.4201 <i>i</i>	—	0.6884 - 0.0000 <i>i</i>	3	4	
—	—	0.7049 - 0.0000 <i>i</i>	2	5	
—	—	0.7782 - 0.0000 <i>i</i>	2	6	
—	—	1.7412 - 0.3231 <i>i</i>	1	7	
9.1416 + 3.1626 <i>i</i>	—	1.7412 - 0.3231 <i>i</i>	3	8	
—	—	1.7412 + 0.3231 <i>i</i>	1	9	
-9.3827 + 5.5055 <i>i</i>	—	1.7412 + 0.3231 <i>i</i>	3	10	
—	—	1.7907 - 0.0000 <i>i</i>	2	11	
—	—	1.7907 - 0.0000 <i>i</i>	2	12	
—	—	6.0819 - 0.0000 <i>i</i>	2	13	
—	—	6.2595 - 0.0000 <i>i</i>	1	14	
0.4925 - 3.1416 <i>i</i>	—	6.2595 - 0.0000 <i>i</i>	3	15	
—	—	6.9792 - 0.0000 <i>i</i>	2	16	

We shall give some remarks about the obtained eigenvalues. The BAEs (5.7) are homogeneous while the BAEs (5.8) are inhomogeneous. This is because that the non-diagonal boundary reflections break the $U(1)$ symmetry of the system. Due to the present form of reflection matrix (2.10), which is diagonal in a 2×2 subspace, the Bethe roots in this subspace satisfy the homogeneous BAEs. The existence of one sets of homogeneous BAEs consists with the fact that there is only one good quantum number m . Another things we should mentioned is that during the construction of $T - Q$ relation, the BAEs obtained from the regularities of $\Lambda(u)$ and $\bar{\Lambda}(u)$ should be the same. It is easy to check that $\Lambda(u)$ and $\bar{\Lambda}(u)$ satisfy the functional relations (5.1) and the additional constraints (5.3). Therefore, we conclude that the analytical expressions (5.4) and (5.5) are the eigenvalues of the transfer matrices $t(u)$ and $\bar{t}(u)$, respectively. It is noted that the eigenvalues and associated BAEs have the well-defined homogeneous limit.

Based on the exact solution (5.4) and (5.7)-(5.8) of $\Lambda(u)$, we can obtain the energy spectrum of the Hamiltonian (2.12)

$$E = \frac{\partial \ln \Lambda(u)}{\partial u} \Big|_{u=0, \{\theta_j\}=0}. \quad (5.10)$$

6 Periodic boundary condition case

We shall note that the above method is universal, which is also valid for the quantum integrable systems with $U(1)$ symmetry. For this purpose, we consider the exact solution of the $A_3^{(2)}$ model with the periodic boundary condition.

In the periodic case, the transfer matrix $t^{(p)}(u)$ and the fused one $\bar{t}^{(p)}(u)$ are defined as

$$t^{(p)}(u) = \text{tr}_0 T_0(u), \quad \bar{t}^{(p)}(u) = \text{tr}_0 T_{\bar{0}}(u), \quad (6.1)$$

where the monodromy matrices $T_0(u)$ and $T_{\bar{0}}(u)$ are given by (2.2) and (3.17), respectively. From the Yang-Baxter equations (2.7) and (3.13), we can prove that the transfer matrices $t^{(p)}(u)$ and $\bar{t}^{(p)}(u)$ satisfy the commutation relations

$$[t^{(p)}(u), t^{(p)}(v)] = [t^{(p)}(u), \bar{t}^{(p)}(v)] = 0. \quad (6.2)$$

Taking the partial trace of Eq.(3.21) in the auxiliary spaces and using the relation (4.1), we obtain the operator product identities

$$t^{(p)}(\theta_j) t^{(p)}(\theta_j + 4\eta + i\pi) = \prod_{l=1}^N a(\theta_j - \theta_l) c(\theta_j - \theta_l + 4\eta + i\pi) \times \text{id},$$

$$\begin{aligned}
t^{(p)}(\theta_j) t^{(p)}(\theta_j + 2\eta) &= \prod_{l=1}^N \tilde{\rho}_0(\theta_j - \theta_l) \bar{t}^{(p)}(\theta_j + \eta), \\
t^{(p)}(\theta_j) \bar{t}^{(p)}(\theta_j + 3\eta) &= \prod_{l=1}^N \tilde{\rho}_1(\theta_j - \theta_l) t^{(p)}(\theta_j + 2\eta + i\pi), \quad j = 1, \dots, N.
\end{aligned} \tag{6.3}$$

From the definitions (6.1), we obtain the asymptotic behaviors of fused transfer matrices

$$\begin{aligned}
t^{(p)}(u)|_{u \rightarrow \pm\infty} &= e^{\pm(Nu - \sum_{j=1}^N \theta_j)} \sum_{\alpha=1}^4 [T_{\pm}]_{\alpha}^{\alpha} + \dots, \\
\bar{t}^{(p)}(u)|_{u \rightarrow \pm\infty} &= e^{\pm(Nu - \sum_{j=1}^N \theta_j)} \sum_{\alpha=1}^6 [\bar{T}_{\pm}]_{\alpha}^{\alpha} + \dots,
\end{aligned} \tag{6.4}$$

where $\sum_{\alpha=1}^4 [T_{\pm}]_{\alpha}^{\alpha}$ and $\sum_{\alpha=1}^6 [\bar{T}_{\pm}]_{\alpha}^{\alpha}$ are the conserved quantities. From the direct calculation, we find that the eigenvalues of $\sum_{\alpha=1}^4 [T_{\pm}]_{\alpha}^{\alpha}$ and $\sum_{\alpha=1}^6 [\bar{T}_{\pm}]_{\alpha}^{\alpha}$ can be quantified by two quantum numbers m_1 and m_2 as $2^{1-N} [\cosh(m_1\eta) + \cosh(m_2\eta)] e^{\pm 2N\eta}$ and $2\{1 + \cosh[(m_1 + m_2)\eta] + \cosh[(m_1 - m_2)\eta]\} e^{\mp 2N\eta}$, respectively, where $m_1 \in [0, N]$ and $0 \leq |m_2| \leq N - m_1$. Then the asymptotic behaviors of transfer matrices $t^{(p)}(u)$ and $\bar{t}^{(p)}(u)$ on some subspace which can be parameterized by two integers m_1 and m_2 read

$$\begin{aligned}
t^{(p)}(u)|_{u \rightarrow \pm\infty} &= 2^{1-N} [\cosh(m_1\eta) + \cosh(m_2\eta)] e^{\pm(Nu - \sum_{j=1}^N \theta_j - 2N\eta)} + \dots, \\
\bar{t}^{(p)}(u)|_{u \rightarrow \pm\infty} &= 2\{1 + \cosh[(m_1 + m_2)\eta] + \cosh[(m_1 - m_2)\eta]\} e^{\pm(Nu - \sum_{j=1}^N \theta_j - 2N\eta)} + \dots.
\end{aligned} \tag{6.5}$$

Suppose the eigenvalues of $t^{(p)}(u)$ and $\bar{t}^{(p)}(u)$ as $\Lambda^{(p)}(u)$ and $\bar{\Lambda}^{(p)}(u)$, respectively. From Eqs.(6.3) and (6.5), we obtain that $\Lambda^{(p)}(u)$ and $\bar{\Lambda}^{(p)}(u)$ should satisfy the constraints

$$\begin{aligned}
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j + 4\eta + i\pi) &= \prod_{l=1}^N a(\theta_j - \theta_l) c(\theta_j - \theta_l + 4\eta + i\pi), \\
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j + 2\eta) &= \prod_{l=1}^N \tilde{\rho}_0(\theta_j - \theta_l) \bar{\Lambda}^{(p)}(\theta_j + \eta), \\
\Lambda^{(p)}(\theta_j) \bar{\Lambda}^{(p)}(\theta_j + 3\eta) &= \prod_{l=1}^N \tilde{\rho}_1(\theta_j - \theta_l) \Lambda^{(p)}(\theta_j + 2\eta + i\pi), \quad j = 1, \dots, N, \\
\Lambda^{(p)}(u)|_{u \rightarrow \pm\infty} &= 2^{1-N} [\cosh(m_1\eta) + \cosh(m_2\eta)] e^{\pm(Nu - \sum_{j=1}^N \theta_j - 2N\eta)} + \dots, \\
\bar{\Lambda}^{(p)}(u)|_{u \rightarrow \pm\infty} &= 2\{1 + \cosh[(m_1 + m_2)\eta] + \cosh[(m_1 - m_2)\eta]\} \\
&\quad \times e^{\pm(Nu - \sum_{j=1}^N \theta_j - 2N\eta)} + \dots.
\end{aligned} \tag{6.6}$$

The eigenvalues $\Lambda^{(p)}(u)$ (resp. $\bar{\Lambda}^{(p)}(u)$) is a polynomial of e^u with degree $2N$ (a polynomial of e^{2u} with degree N respectively) up to an overall factor e^{-Nu} . Therefore, $\Lambda^{(p)}(u)$ and $\bar{\Lambda}^{(p)}(u)$

can be completely determined by at least $3N + 2$ constraints. Then we arrive at that $3N$ functional relations together with 4 asymptotic behaviors (6.6) can determine the eigenvalues $\Lambda^{(p)}(u)$ and $\bar{\Lambda}^{(p)}(u)$, which are expressed by the $T - Q$ relations as

$$\begin{aligned} \Lambda^{(p)}(u) = & \prod_{j=1}^N a(u - \theta_j) \frac{Q_p^{(1)}(u + 2\eta)}{Q_p^{(1)}(u)} + \prod_{j=1}^N b(u - \theta_j) \left\{ \frac{Q_p^{(1)}(u - 2\eta) Q_p^{(2)}(u + 2\eta)}{Q_p^{(1)}(u) Q_p^{(2)}(u)} \right. \\ & \left. + \frac{Q_p^{(1)}(u - i\pi) Q_p^{(2)}(u - 2\eta)}{Q_p^{(1)}(u - 2\eta - i\pi) Q_p^{(2)}(u)} \right\} + \prod_{j=1}^N c(u - \theta_j) \frac{Q_p^{(1)}(u - 4\eta - i\pi)}{Q_p^{(1)}(u - 2\eta - i\pi)}, \end{aligned} \quad (6.7)$$

$$\begin{aligned} \bar{\Lambda}^{(p)}(u) = & \prod_{j=1}^N a_1(u - \theta_j) \left[\frac{Q_p^{(2)}(u + 3\eta)}{Q_p^{(2)}(u + \eta)} + \frac{Q_p^{(1)}(u + \eta) Q_p^{(1)}(u + \eta - i\pi) Q_p^{(2)}(u - \eta)}{Q_p^{(1)}(u - \eta) Q_p^{(1)}(u - \eta - i\pi) Q_p^{(2)}(u + \eta)} \right] \\ & + \prod_{j=1}^N b_1(u - \theta_j) \left[\frac{Q_p^{(2)}(u - 3\eta)}{Q_p^{(2)}(u - \eta)} + \frac{Q_p^{(1)}(u - 3\eta) Q_p^{(1)}(u - 3\eta - i\pi) Q_p^{(2)}(u + \eta)}{Q_p^{(1)}(u - \eta) Q_p^{(1)}(u - \eta - i\pi) Q_p^{(2)}(u - \eta)} \right] \\ & + \prod_{j=1}^N c_1(u - \theta_j) \frac{Q_p^{(1)}(u + \eta) Q_p^{(1)}(u - 3\eta - i\pi)}{Q_p^{(1)}(u - \eta) Q_p^{(1)}(u - \eta - i\pi)} \\ & + \prod_{j=1}^N c_1(u - \theta_j - i\pi) \frac{Q_p^{(1)}(u + \eta - i\pi) Q_p^{(1)}(u - 3\eta)}{Q_p^{(1)}(u - \eta) Q_p^{(1)}(u - \eta - i\pi)}, \end{aligned} \quad (6.8)$$

where the definition of the functions $a_1(u)$, $b_1(u)$, and $c_1(u)$ is in (5.6), and

$$Q_p^{(1)}(u) = \prod_{l=1}^{L_1} \sinh \frac{1}{2}(u - \mu_l^{(1)} - \eta), \quad Q_p^{(2)}(u) = \prod_{k=1}^{L_2} \sinh(u - \mu_k^{(2)} - 2\eta). \quad (6.9)$$

The regularity analyses of the $T - Q$ relations (6.7)-(6.8) lead to that the Bethe roots $\{\mu_l^{(1)}\}$ and $\{\mu_k^{(2)}\}$ should satisfy the BAEs

$$\frac{Q_p^{(1)}(\mu_l^{(1)} + 3\eta) Q_p^{(2)}(\mu_l^{(1)} + \eta)}{Q_p^{(1)}(\mu_l^{(1)} - \eta) Q_p^{(2)}(\mu_l^{(1)} + 3\eta)} = - \prod_{j=1}^N \frac{\sinh \frac{1}{2}(\mu_l^{(1)} + \eta - \theta_j)}{\sinh \frac{1}{2}(\mu_l^{(1)} - \eta - \theta_j)}, \quad l = 1, \dots, L_1, \quad (6.10)$$

$$\frac{Q_p^{(1)}(\mu_k^{(2)}) Q_p^{(1)}(\mu_k^{(2)} - i\pi) Q_p^{(2)}(\mu_k^{(2)} + 4\eta)}{Q_p^{(1)}(\mu_k^{(2)} + 2\eta) Q_p^{(1)}(\mu_k^{(2)} + 2\eta - i\pi) Q_p^{(2)}(\mu_k^{(2)})} = -1, \quad k = 1, \dots, L_2, \quad (6.11)$$

where $L_1 \leq N$ and $L_2 \leq L_1$. We shall note that the BAEs (6.10) and (6.11) are homogeneous, because the periodic boundary condition does not break the $U(1)$ symmetry. The quantum numbers m_1 and m_2 characterizing the conserved quantities $\sum_{\alpha=1}^4 [T_{\pm}]_{\alpha}^{\alpha}$ and $\sum_{\alpha=1}^6 [\bar{T}_{\pm}]_{\alpha}^{\alpha}$ are related with the numbers of Bethe roots as

$$m_1 = N - L_1, \quad m_2 = L_1 - 2L_2. \quad (6.12)$$

The eigenvalues (6.7)-(6.8) and associated BAEs (6.10)-(6.11) have the well-defined homogeneous limit. These results with the constraint $\{\theta_j\} = 0$ coincide with the previous ones obtained by using the functional or nested algebraic Bethe ansatz [20, 21].

7 Discussion

In this paper, we have studied the exact solution of quantum integrable model associated with the $A_3^{(2)}$ twisted Lie algebra. We give a detailed analysis of the fusion properties, including the open chain and the periodic one. We obtain the closed recursive fusion relations and additional constraints among the fused transfer matrices. Based on them and with the help of polynomials analysis, we obtain the eigenspectrum and related Bethe ansatz equations of the system. The results provided in this paper can be generalized to the $A_n^{(2)}$ model with arbitrary n and integrable models with the other twisted Lie algebras.

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Appendix A. Expression of the R -matrix $R_{\bar{1}2}(u)$

In this appendix, we give the explicit expression of the R -matrix $R_{\bar{1}2}(u)$ define in (3.7) as

$$R_{\bar{1}2}(u) = \begin{pmatrix} r_1 & r_1 & r_2 & r_2 & & \\ & r_7 & & r_8 & r_9 & \\ & & r_1 & r_2 & r_1 & \\ r_{13} & r_{14} & r_{13} & & r_9 & \\ & & & r_{14} & & r_{12} \\ r_{16} & r_{17} & -r_{16} & & r_{15} & -r_{15} \\ & & & -r_{17} & -r_{15} & r_{15} \\ & & & & r_5 & r_6 \\ & & & & & r_6 \\ & & & & & r_5 \\ & & & & & -r_{11} \\ & & & & & -r_{10} \\ & & & & & r_2 & r_1 & r_2 & r_1 \\ & & & & & & & r_7 & \\ & & & & & & & & r_2 & r_2 & r_1 & r_1 \end{pmatrix}, \quad (\text{A.1})$$

$$\begin{aligned} r_1 &= 2 \sinh(u - 3\eta), \quad r_2 = 2 \sinh(u - \eta), \quad r_3 = 4 \sinh \frac{1}{2}(u - 3\eta) \cosh \frac{1}{2}(u - \eta), \\ r_4 &= 2(\sinh(u - 2\eta) + \sinh 2\eta \sinh \eta), \quad r_5 = 4 \sinh \frac{1}{2}(u - \eta) \cosh \frac{1}{2}(u - 3\eta), \\ r_6 &= 2(\sinh(u - 2\eta) - \sinh 2\eta \sinh \eta), \quad r_7 = -2 \sinh 2\eta, \\ r_8 &= -4e^{-\frac{u}{2}} \sinh \eta \sqrt{\cosh \eta} \sinh \frac{1}{2}(u - 3\eta), \quad r_9 = -4e^{-\frac{u}{2}+\eta} \sinh \eta \sqrt{\cosh \eta} \cosh \frac{1}{2}(u - \eta), \\ r_{10} &= 4e^{-\frac{u}{2}} \sinh \eta \sqrt{\cosh \eta} \cosh \frac{1}{2}(u - 3\eta), \quad r_{11} = 4e^{-\frac{u}{2}+\eta} \sinh \eta \sqrt{\cosh \eta} \sinh \frac{1}{2}(u - \eta), \\ r_{12} &= 2e^{-\eta} \sinh 2\eta, \quad r_{13} = -e^u r_8, \quad r_{14} = e^{u-2\eta} r_9, \quad r_{15} = -2 \sinh \eta \sinh 2\eta, \\ r_{16} &= e^u r_{10}, \quad r_{17} = -e^{u-2\eta} r_{11}, \quad r_{18} = 2e^\eta \sinh 2\eta. \end{aligned} \quad (\text{A.2})$$

The above expression allows us to derive the very properties (3.9) of the resulting fused R -matrix $R_{\bar{1}2}(u)$.

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