

Strongly correlated crystalline higher-order topological phases in two-dimensional systems: A coupled-wire study

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Coupled-wire constructions have been widely applied to quantum Hall systems and symmetry-protected topological (SPT) phases. In this Letter, we use the coupled one-dimensional nonchiral Luttinger liquids with domain-wall structured mass terms as quantum wires to construct the crystalline higher-order topological superconductors (HOTSC) in two-dimensional interacting fermionic systems by two representative examples: D_4 -symmetric class-D HOTSC and C_4 -symmetric class-BDI HOTSC, with Majorana corner modes on the edge. Furthermore, based on the coupled-wire constructions, the quantum phase transition between different phases of 2D HOTSC by tuning the inter-wire coupling are investigated in a straightforward way.

Introduction – Topological phases of quantum matter have become one of the greatest triumph of condensed matter physics since the discovery of fractional quantum Hall effect [1, 2]. Topological order defined by patterns of long-range entanglement provides a systematic way of understanding topological phases of quantum matter [3]. Furthermore, the interplay between symmetry and topology plays a central role in the topological phases of quantum matter. In particular, symmetry-protected topological (SPT) phases has been systematically constructed and classified in short-range entangled systems [4–26]. An elegant example of SPT phases is topological insulator, protected by time-reversal and charge-conservation symmetry [27, 28]. Recently, crystalline SPT phases have been intensively studied [29–60], with great opportunities for experimental realizations [61–64]. In particular, different from internal SPT phases, the boundaries of 2D crystalline SPT phases are almost gapped but with protected 0D *corner* zero modes. This type of topological phases are called *higher-order topological phases* [65–76].

The study of higher-order topological phases mainly focus on free-fermion systems, because interactions and crystalline symmetries restrict the analytical study of lattice model, only numerics on finite size lattice can give some insights. On the other hand, a clear and powerful tool of studying topological phases of quantum matter is coupled-wire construction [77–88]. One decomposes a higher-dimensional system into an assembly of 1D quantum wires, and topological properties then arise from the suitable couplings of them. A unique advantage of coupled-wire construction is that different from higher-dimensional quantum field theory, the powerful bosonization technique of one-dimensional subsystems can be used to challenge the strong interaction effects. Different phases are manifested by patterns of coupled wires and the quantum phase transition of different phases is controlled by tuning inter-wire couplings directly. Therefore, an important open question arises: if the strongly correlated higher-order topological phases can be constructed by coupled-wire perspective?

In this Letter, we systematically construct the crystalline HOTSC in two-dimensional interacting fermionic systems by coupling the circular 1D nonchiral Luttinger liquids with domain-wall structured mass terms as quantum wires, via two typical intriguing interacting examples: D_4 -symmetric class-D HOTSC and C_4 -symmetric class-BDI HOTSC, whose higher-order edge modes are Majorana zero modes (MZMs) [45, 46]. By suitable inter-wire tunneling/interaction, several 1D quantum wires are assembled and fully gapped, living few dangling quantum wires at the edge or near the center of the systems. Near the center, the dangling quantum wires are fully gapped by intra-wire interactions; on the edge, the dangling quantum wires explicitly manifest the higher-order topological edge modes of 2D HOTSC by their domain-wall structure. Different 2D HOTSCs are characterized by different patterns of coupled-wire. Lattice translation symmetry can also be imposed straightforwardly. Furthermore, with concrete coupled-wire construction of 2D HOTSC, we directly investigate the quantum phase transitions by tuning different coupling constants of inter-wire interactions. We stress that our arguments are not sensitive to specific geometry of quantum wires: the calculations are applicable to any geometry respecting the specific crystalline symmetry, we choose circular geometry for calculational convenience.

D_4 -symmetric class-D HOTSC – For 2D D_4 -symmetric systems with spinless fermions, there is an intriguing interacting 2D HOTSC with protected Majorana corner modes ξ_k and ξ'_k ($k = 1, 2, 3, 4$) that can be reformulated to complex fermions $c_k^\dagger = (\xi_k + i\xi'_k)/\sqrt{2}$ (see Fig. 1). In this section we construct this phase by an “almost free” coupled-wires, with necessary interaction only defined near the D_4 -center. These Majorana corner modes are also reformulated in terms of domain walls of 1D nonchiral Luttinger liquids [89]. Consider $2n$ decoupled 1D quantum wires with circular geometry (see Fig. 1), the Lagrangian of j^{th} quantum wire is:

$$\mathcal{L}_0^j = \frac{K_{IJ}^j}{4\pi} \left(\partial_\theta \phi_I^j \right) \left(\partial_t \phi_J^j \right) + \frac{V_{IJ}^j}{8\pi} \left(\partial_\theta \phi_I^j \right) \left(\partial_\theta \phi_J^j \right) \quad (1)$$

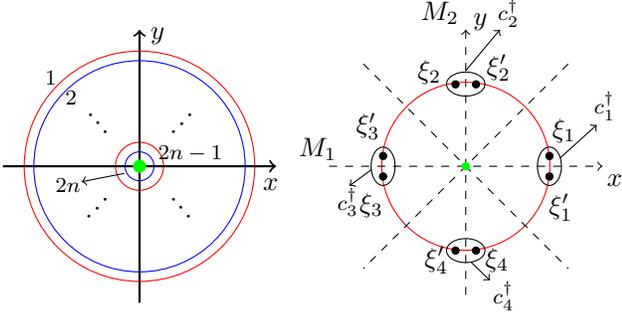


FIG. 1. Coupled-wire construction of 2D fermionic crystalline HOTSC. Right panel: dangling gapless modes of D_4 -symmetric class-D or C_4 -symmetric class-BDI HOTSC. Dashed lines are reflection axes for D_4 symmetry.

where $\phi^j = (\phi_1^j, \phi_2^j)^T$ is the 2-component bosonic field of j^{th} quantum wire and $K^j = \sigma^z$ as the K -matrix of the topological term [90]. The total Lagrangian of decoupled wires is: $\mathcal{L}_0 = \sum_{j=1}^{2n} \mathcal{L}_0^j$. The D_4 symmetry properties of these bosonic fields are ($\mathbf{R} \in C_4/\mathbf{M}_1 \in \mathbb{Z}_2^M$ is rotation/reflection generator of $D_4 = C_4 \times \mathbb{Z}_2^M$ symmetry):

$$\mathbf{R} : \begin{cases} \phi_1^j(\theta) \mapsto -\phi_1^j(\theta + \pi/2) \\ \phi_2^j(\theta) \mapsto -\phi_2^j(\theta + \pi/2) + \pi \end{cases} \quad (2)$$

$$\mathbf{M}_1 : \begin{cases} \phi_1^j(\theta) \mapsto -\phi_2^j(2\pi - \theta) + \pi/2 \\ \phi_2^j(\theta) \mapsto -\phi_1^j(2\pi - \theta) + \pi/2 \end{cases}$$

To figure out the Majorana corner modes of D_4 -symmetric HOTSC, we should further introduce the mass term with domain wall structure of each quantum wire:

$$\mathcal{L}_{\text{wall}}^j = m \sin(2\theta) \cdot \cos[\phi_1^j(\theta) + \phi_2^j(\theta)] \quad (3)$$

where $\mathcal{L}_{\text{wall}}^j$ is symmetric under (2), and $\mathcal{L}_{\text{wall}} = \sum_{j=1}^{2n} \mathcal{L}_{\text{wall}}^j$. For each quantum wire with a domain-wall structured mass term, there are four complex fermion zero modes at poles $c_{1,2,3,4}^\dagger$ of the circle, with $\theta = 0, \pi/2, \pi, 3\pi/2$ because of the vanishing mass term, which are equivalent to eight MZMs. These dangling 0D gapless modes cannot be gapped in a D_4 -symmetric way.

Subsequently we define two types of D_4 -symmetric (2) inter-wire tunneling that couple the $(2j - 2 + k)^{\text{th}}$ and $(2j - 1 + k)^{\text{th}}$ quantum wires ($m_1, m_2 < m, k = 1, 2$):

$$\mathcal{L}_{ck}^j = m_k \sum_{\alpha=1}^2 \cos[\phi_\alpha^{2j-2+k}(\theta) - \phi_\alpha^{2j-1+k}(\theta)] \quad (4)$$

and $\mathcal{L}_{ck} = \sum_{j=1}^n \mathcal{L}_{ck}^j$. There are two extreme cases: $m_1 \neq 0, m_2 = 0/m_1 = 0, m_2 \neq 0$ corresponds to the phase that $\mathcal{L}_{c1}/\mathcal{L}_{c2}$ dominates the inter-wire physics. For \mathcal{L}_{c1} -dominant phase, the $(2j - 1)^{\text{th}}$ and $2j^{\text{th}}$ wires are paired up and gapped, hence the corresponding system is fully gapped on an open circle and topological trivial.

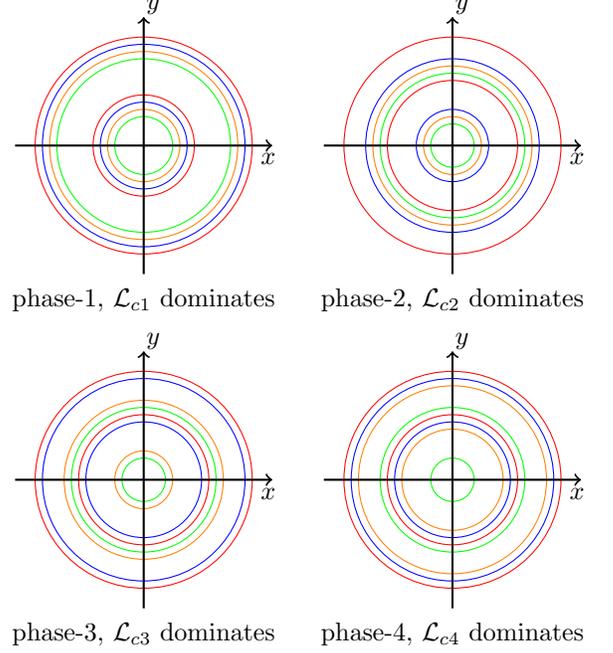


FIG. 2. 4 distinct phases of 2D BDI-class ($C_4 \times \mathbb{Z}_2^T$)-symmetric system. Each four 1D wires with narrower intervals can be gapped by inter-wire interactions \mathcal{L}_{ck} , and the 1D wire near C_4 -center can be gapped by intra-wire interactions \mathcal{L}_{int} .

For \mathcal{L}_{c2} -dominant phase, the $2j^{\text{th}}$ and $(2j + 1)^{\text{th}}$ wires are paired up and gapped, hence all 1D wires except 1^{st} and $2n^{\text{th}}$ are gapped. The 1^{st} wire on the edge of the system presents 4 complex fermions/8 MZMs at poles of circle, which are exactly the second-order topological surface modes of 2D D_4 -symmetric class-D HOTSC. Near the D_4 -center, there are also gapless modes on the $2n^{\text{th}}$ quantum wire. Distinct from quantum wires away from D_4 -center, bosonic field ϕ^{2n} of the $2n^{\text{th}}$ quantum wires with different polar angles can tunnel to/interact with the field at other places. Consider two interacting terms of $2n^{\text{th}}$ quantum wire near the D_4 -center:

$$\mathcal{L}_{\text{int}} = m' \sum_{\beta=1}^2 \cos \left(\sum_{\alpha=1}^2 [\phi_\alpha^{2n}(\theta) - \phi_\alpha^{2n}(\beta\pi - \theta)] \right) \quad (5)$$

i.e., the intra-wire couplings of the $2n^{\text{th}}$ wire lead to a fully gapped bulks. Equivalently, a nontrivial 2D class-D D_4 -symmetric HOTSC are described by 1D coupled wires with Lagrangian $\mathcal{L}_{D_4}^D = \mathcal{L}_0 + \mathcal{L}_{\text{wall}} + \mathcal{L}_{c2} + \mathcal{L}_{\text{int}}$. The intriguing interacting nature of this HOTSC is reflected by \mathcal{L}_{int} near the D_4 -center. On the other hand, the physics away from the D_4 -center is well-understood on the noninteracting level. The classification of 2D class-D HOTSC is \mathbb{Z}_2 , composed by phases dominated by inter-wire coupling \mathcal{L}_{c1} and \mathcal{L}_{c2} (see Fig. 5).

C_4 -symmetric class-BDI HOTSC – For 2D BDI-class systems with C_4 -symmetry, there is another type of in-

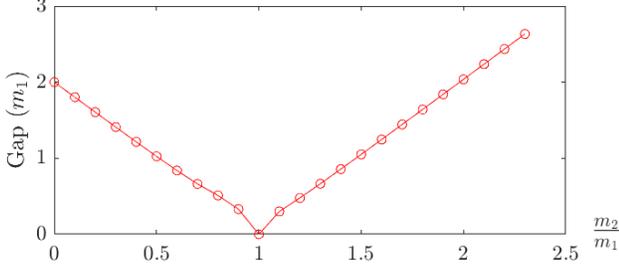


FIG. 3. Bulk gap of 2D D_4 -symmetric Lagrangian $\mathcal{L}_0 + \mathcal{L}_{c1} + \mathcal{L}_{c2} + \mathcal{L}_{\text{int}}$, with respect to the ratio m_2/m_1 .

triguing interacting 2D HOTSCs [45]. We construct these phases by “interacting” coupled-wires in this section. Consider $4n$ 1D circular quantum wires, each wire is described by Lagrangian (1) with $\phi^j = (\phi_1^j, \phi_2^j, \phi_3^j, \phi_4^j)^T$ as 4-component bosonic field of j^{th} quantum wire, $K^j = \sigma^z \oplus \sigma^z$ as the topological K -matrix, and the total Lagrangian of all $4n$ 1D quantum wires is $\mathcal{L}_0 = \sum_{j=1}^{4n} \mathcal{L}_0^j$. The $(C_4 \times \mathbb{Z}_2^T)$ -symmetry properties are defined as [90]:

$$\mathbf{R}: \begin{cases} \phi_1^j \mapsto \phi_1^j + \phi_3^j - \phi_4^j - \pi/2 \\ \phi_2^j \mapsto \phi_2^j - \phi_3^j + \phi_4^j + \pi/2 \\ \phi_3^j \mapsto \phi_1^j + \phi_2^j - \phi_3^j + \pi/2 \\ \phi_4^j \mapsto \phi_1^j + \phi_2^j - \phi_4^j + \pi/2 \end{cases}, \quad \theta \mapsto \theta + \pi/2 \quad (6)$$

$$\mathcal{L}_{ck}^j = m_k \sum_{\alpha=1}^2 \cos \left[\phi_{\alpha}^{4j-4+k} - \phi_{5-\alpha}^{4j-4+k} + \phi_{\alpha}^{4j-3+k} - \phi_{5-\alpha}^{4j-3+k} \right] + \cos \left[\phi_{\alpha}^{4j-4+k} - \phi_{5-\alpha}^{4j-4+k} + \phi_{\alpha}^{4j-2+k} - \phi_{5-\alpha}^{4j-2+k} \right] \\ + \cos \left[\phi_{\alpha}^{4j-2+k} - \phi_{5-\alpha}^{4j-2+k} + \phi_{\alpha}^{4j-1+k} - \phi_{5-\alpha}^{4j-1+k} \right] + \cos \left[\phi_{\alpha}^{4j-3+k} - \phi_{5-\alpha}^{4j-3+k} + \phi_{\alpha}^{4j-1+k} - \phi_{5-\alpha}^{4j-1+k} \right] \quad (9)$$

and the total Lagrangian of inter-wire couplings is $\mathcal{L}_{ck} = \sum_{j=1}^{n-1} \mathcal{L}_{ck}^j$. There are four extreme cases: $m_k \neq 0$ ($k = 1, 2, 3, 4$) as the only nonzero index in $m_{1,2,3,4}$, which corresponds to the phase that \mathcal{L}_{ck} deminate the inter-wire physics. For \mathcal{L}_{c1} -dominant phase, the $(4j - k)^{\text{th}}$ ($k = 0, 1, 2, 3$) quantum wires are assembled and gapped, hence the spectrum is fully gapped on a 2D open circle, and the corresponding phase is topological trivial.

For \mathcal{L}_{c4} -dominant phase with $m_4 \neq 0$ and $m_{1,2,3} = 0$, by applying $\mathcal{L}_{\text{wall}}$ and \mathcal{L}_{c4} , the $(4j + 3 - k)^{\text{th}}$ quantum wires are assembled and gapped, and there are only 4 quantum wires remain gapless: 1st, 2nd, 3rd on the edge, and $4n^{\text{th}}$ near the C_4 -center. On the edge, 1st, 2nd and 3rd quantum wires with dangling gapless modes are treated as the higher-order edge state of 2D C_4 -symmetric class-BDI HOTSC; near the C_4 -center, in order to obtain a HOTSC, we should further add some

$$\mathcal{T}: \begin{cases} \phi_1^j(\theta) \mapsto \phi_2^j(\theta) - \phi_3^j(\theta) + \phi_4^j(\theta) \\ \phi_2^j(\theta) \mapsto \phi_1^j(\theta) + \phi_3^j(\theta) - \phi_4^j(\theta) + \pi \\ \phi_3^j(\theta) \mapsto \phi_1^j(\theta) + \phi_2^j(\theta) - \phi_4^j(\theta) + \pi \\ \phi_4^j(\theta) \mapsto \phi_1^j(\theta) + \phi_2^j(\theta) - \phi_3^j(\theta) \end{cases} \quad (7)$$

Then we repeatedly figure out the Majorana corner modes by backscattering terms with domain-wall structure: for each quantum wire, we introduce a symmetric [cf. Eqs. (6) and (7)] mass term:

$$\mathcal{L}_{\text{wall}}^j = m \sum_{\alpha=1}^2 \cos \left(\theta - \frac{\alpha\pi}{2} \right) \cdot \cos \left[\phi_{\alpha}^j(\theta) - \phi_{5-\alpha}^j(\theta) \right] \quad (8)$$

And $\mathcal{L}_{\text{wall}} = \sum_{j=1}^{4n} \mathcal{L}_{\text{wall}}^j$. For each quantum wire, there are 4 gapless complex fermions c_k^{\dagger} (8 MZMs ξ_k and ξ'_k) at poles of the circle, two of them at north and south poles are from the first term of (8) and other two at east and west poles are from the second term of (8). These dangling gapless modes cannot be gapped in a $(C_4 \times \mathbb{Z}_2^T)$ -symmetric way.

Subsequently we consider $(C_4 \times \mathbb{Z}_2^T)$ -symmetric [cf. Eqs. (6) and (7)] inter-wire interactions including four 1D quantum wires ($k = 1, 2, 3, 4$) [90]:

intra-wire interactions to fully gap the $4n^{\text{th}}$ quantum wire in order to get a fully-gapped bulk state. Consider the 4-body interacting terms of $4n^{\text{th}}$ quantum wire, composed by the backscatterings of bosonic fields $\phi_{1,2,3,4}^{4n}$ with different polar angles [90]:

$$\mathcal{L}_{\text{int}} = m' \sum_{\alpha,\beta=1}^2 \cos \left[\phi_{\alpha}^{4n}(\theta) - \phi_{5-\alpha}^{4n}(\theta) + \phi_{\alpha}^{4n}(\theta + \beta\pi/2) - \phi_{5-\alpha}^{4n}(\theta + \beta\pi/2) \right] \quad (10)$$

i.e., the intra-wire interactions of the $4n^{\text{th}}$ quantum wire lead to a fully gapped bulk, and a nontrivial 2D $(C_4 \times \mathbb{Z}_2^T)$ -symmetric HOTSC with spinless fermions are described by 1D coupled quantum wires with Lagrangian $\mathcal{L}_{C_4}^{\text{BDI}} = \mathcal{L}_0 + \mathcal{L}_{\text{wall}} + \mathcal{L}_{c4} + \mathcal{L}_{\text{int}}$. Similar for \mathcal{L}_{c2} and \mathcal{L}_{c3} dominant phases, and there are 4 topological distinct phases for 2D BDI-class $(C_4 \times \mathbb{Z}_2^T)$ -symmetric system, see

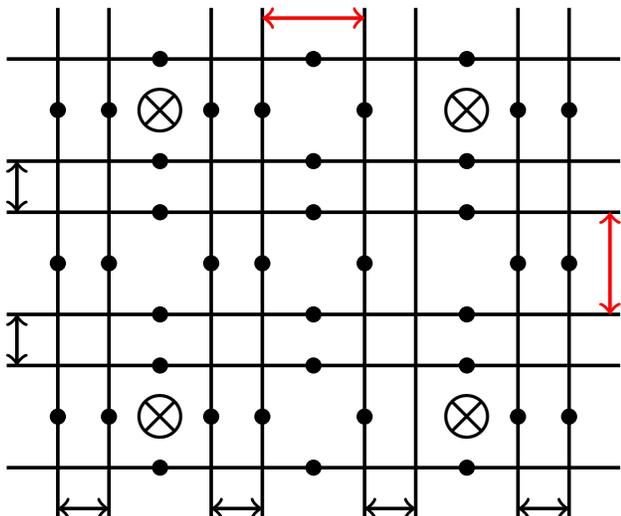


FIG. 4. Coupled wires of 2D $p4mm$ -symmetric class-D HOTSC. Each “ \otimes ” symbol depicts a center of D_4 , each line (horizontal or vertical) depicts a quantum wire, and each dot depicts a domain wall described by $\mathcal{L}_{\text{wall}}^j$. Red and black double arrows depict different inter-wire couplings \mathcal{L}_{c1} and \mathcal{L}_{c2} .

Fig. 2. The interacting nature of these topological phases are reflected by inter-wire interactions \mathcal{L}_{ck} and intra-wire interactions near the C_4 -center, \mathcal{L}_{int} . The classification of 2D class-BDI HOTSC is \mathbb{Z}_4 , composed by phases dominated by inter-wire couplings \mathcal{L}_{ck} ($k = 1, 2, 3, 4$).

The coupled-wire construction is not limited to superconductors, it is also applicable to topological insulators: the only difference is that the Luttinger liquid (1) should respect the $U(1)$ charge conservation.

Imposition of Lattice Translation – With point group symmetric cases, by D_4 -symmetric class-D example, we demonstrate that the imposition of lattice translation symmetry is straightforward: impose the lattice translation to D_4 leads to $p4mm$ wallpaper group. We arrange 8 quantum wires near each D_4 -center (4 vertical and 4 horizontal, see Fig. 4), different topological phases are also controlled by patterns of inter-wire couplings: topological trivial phase is dominated by \mathcal{L}_{c1} (black double arrows in Fig. 4), and nontrivial phase is dominated by \mathcal{L}_{c2} (red double arrows in Fig. 4) and \mathcal{L}_{int} at each D_4 -center in order to the fully gapped bulk.

Quantum phase transition of HOTSC – Coupled-wire picture serves a unique platform for investigating the quantum phase transition (QPT) of 2D HOTSC because of its clear formulations. In this section, we elucidate the QPT of 2D intriguing interacting D_4 -symmetric HOTSC as a representative example. Consider the D_4 -symmetric Lagrangian $\mathcal{L}_0 + \mathcal{L}_{c1} + \mathcal{L}_{c2} + \mathcal{L}_{\text{int}}$, above we have discussed two extreme cases with $m_1 = 0/m_2 = 0$, derive two distinct phases characterized by appearance of Majorana corner modes on the edge (1st quantum wire). Now

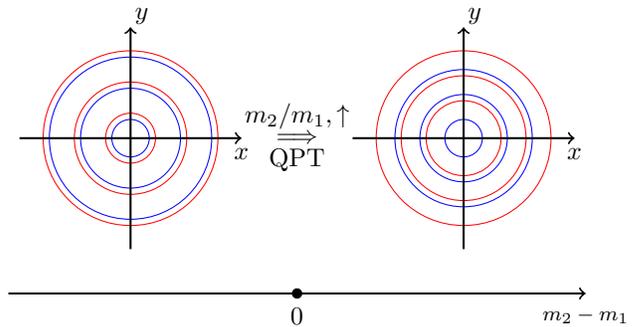


FIG. 5. Quantum phase transition of 2D D_4 -symmetric HOTSC, with respect to $m_2 - m_1$. Each pair of quantum wires with narrower intervals are coupled.

we suppose $m = 10m_1$ and set both m_1 and m_2 finite and study the possible QPT by tuning their ratio m_2/m_1 . As summarized in Fig. 3, turn on m_2 in $m_2 < m_1$ regime, the system remains fully gapped with narrower gap; at $m_1 = m_2$, the gap closes and the system becomes critical; keep increasing m_2 toward $m_2 > m_1$ regime, the system reopens a *bulk* gap but leaving several gapless modes on the edge, which are exactly the Majorana domain walls of 1D quantum wire on the edge. Therefore, we conclude that there is a clear quantum phase transition from trivial state to 2D D_4 -symmetric HOTSC at $m_1 = m_2$ point. Equivalently, this quantum phase transition is characterized by different inter-wire entanglement patterns of 1D quantum wires, as illustrated in Fig. 5.

For 2D C_4 -symmetric class-BDI system, the quantum phase transitions can be described in a similar way with little complications. For this case, there are four distinct phases controlled by four different parameters $m_{1,2,3,4}$ (see Fig. 2). As an example, for the quantum phase transition between phase-2 and phase-3, we set $m_1 = m_4 = 0$ and investigate the bulk gap by tuning the ratio m_2/m_3 . Heuristically, we see that the system will be critical for $m_2 = m_3$, hence there will be a quantum phase transition at this point [90]. As a matter of fact, distinct phases of 2D HOTSC are controlled by different patterns of inter-wire entanglements, and their quantum phase transitions can be manipulated by tuning the intensities of different types of inter-wire couplings. In other words, coupled-wire construction provides a straightforward way of comprehending the quantum phase transitions of 2D HOTSCs, by tuning the inter-wire couplings to control the patterns of inter-wire entanglement. Different phases are manifested by different numbers of Majorana zero modes at each pole of the outer-most quantum wire.

Experimental Implications – In this Letter, the explicit manifestations of second-order modes on the edge of the systems by domain-wall structured quantum wires serves a direct opportunity for observing the higher-order topological phases by tunneling spectroscopic measure-

ments. Recently, the coupled-wire picture is straightforwardly manifested in two-dimensional Moiré superlattices [91, 92]. In particular, in Ref. [92], one-dimensional Luttinger liquids behavior has been explicitly observed in 2D bilayer WTe_2 Moiré superlattice by direct transport measurements. Hence our approach can directly be applied to Moiré superlattice.

Conclusion and Discussion – Coupled-wire construction is a celebrated aspect in topological phases of quantum matter, for both long-range and short-range entangled systems. In this Letter, we establish the coupled-wire construction of 2D intriguing interacting fermionic crystalline HOTSC, with two representative examples: 2D D_4 -symmetric class-D and C_4 -symmetric class-BDI HOTSC phases. An indispensable advantage of coupled-wire construction is that the powerful bosonization technique can be utilized, and the inter-wire couplings can be straightforwardly involved by many-body backscattering terms in the Lagrangian. With this advantage, we use the 1D nonchiral Luttinger liquid with a domain-wall structured mass term as an “almost gapped” 1D quantum wire. Based on these quantum wires, we introduce some suitable inter-wire couplings in order to gap out the bulk by assemblies of quantum wires. The remaining ungapped quantum wires on the edge are treated as the edge theory of 2D HOTSC. Near the center of point group, the ungapped quantum wires are gapped by interactions of bosonic fields at different places. The lattice translation symmetry can be straightforwardly imposed. Distinct HOTSCs are manifested by different patterns of inter-wire entanglement. Furthermore, the concrete coupled-wire constructions serve a straightforward way to comprehend the quantum phase transitions of 2D HOTSCs, by directly tuning the inter-wire couplings to control the inter-wire entanglement patterns. The coupled-wire construction can also be generalized to the systems with arbitrary crystalline symmetry SG and internal symmetry G_0 in arbitrary dimensions, and especially in 2D Moiré superlattice, with more complicated inter-wire entanglement patterns, and their quantum phase transitions should also be controlled by inter-wire entanglement patterns of quantum wires. Furthermore, with explicit corner modes, the 2D HOTSC may directly be justified by tunneling spectroscopic measurements on the edge.

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Supplemental Materials of “Strongly correlated crystalline higher-order topological phases in two-dimensional systems: A coupled-wire study”

K -matrix formalism of fSPT

In the main text, we define the 1D quantum wire based on the nonchiral Luttinger liquid with topological K -matrix. In this section we review the K -matrix formalism of fermionic symmetry-protected topological (fSPT) phases. A $U(1)$ Chern-Simons theory has the form:

$$\mathcal{L} = \frac{K_{IJ}}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu^I \partial_\nu a_\lambda^J + a_\mu^I j_I^\mu + \dots \quad (\text{S1})$$

where K is a symmetric integral matrix, $\{a^I\}$ is a set of one-form gauge fields, and $\{j_I\}$ are the corresponding currents that couple to the gauge fields a^I . The symmetry is defined as: two theories $\mathcal{L}[a^I]$ and $\mathcal{L}[\tilde{a}^I]$ correspond to the same phase if there is an $n \times n$ integral unimodular matrix W satisfying $\tilde{a}^I = W_{IJ} a^J$.

The topological order described by Abelian Chern-Simons theory hosts Abelian anyon excitations. An anyon is labeled by an integer vector $l = (l_1, l_2, \dots, l_n)^T$. The self and mutual statistics of anyons are:

$$\begin{aligned} \theta_l &= \pi l^T K^{-1} l \\ \theta_{l,l'} &= 2\pi l^T K^{-1} l' \end{aligned} \quad (\text{S2})$$

The total number of anyons and the ground-state degeneracy (GSD) on a torus are both given by $|\det K|$. For SPT phase, there is no GSD or anyon, hence we require $|\det K| = 1$ for SPT phases.

The K -matrix Chern-Simons theory has a well-known bulk-boundary correspondence [1, 2]. In a system with open boundary, the edge theory of (S1) has the form:

$$\mathcal{L}_{\text{edge}} = \frac{K_{IJ}}{4\pi} (\partial_x \phi^I) (\partial_t \phi^J) + \frac{V_{IJ}}{8\pi} (\partial_x \phi^I) (\partial_x \phi^J) \quad (\text{S3})$$

where $\phi = (\{\phi^J\})^T$ are chiral bosonic fields on the edge and related to dynamical gauge field a_μ^I in the bulk by $a_\mu^I = \partial_\mu \phi^I$, and an anyon on the edge can be created by the operator $e^{il^T \phi}$.

Assemble of quantum wires of C_4 -symmetric class-BDI HOTSC

In the main text, the inter-wire couplings of 2D C_4 -symmetric class-BDI HOTSC are defined by backscatterings of four 1D quantum wires as an assembly. In this section we demonstrate that the minimal number of quantum wires of an assembly should be four, equivalently, two quantum wires cannot be gapped in a symmetric way.

The 1D quantum wire building block for coupled-wire construction of 2D C_4 -symmetric class-BDI HOTSC is described by 1D nonchiral Luttinger liquid on a circle, with the Lagrangian:

$$\mathcal{L}_0^j = \frac{K_{IJ}^j}{4\pi} (\partial_\theta \phi_j^I) (\partial_t \phi_j^J) + \frac{V_{IJ}^j}{8\pi} (\partial_\theta \phi_j^I) (\partial_\theta \phi_j^J) \quad (\text{S4})$$

where θ is the polar angle of the circle, $\phi^j(\theta) = (\phi_1^j(\theta), \phi_2^j(\theta), \phi_3^j(\theta), \phi_4^j(\theta))^T$ as 4-component bosonic fields of j^{th} quantum wire, and $K^j = \sigma^z \oplus \sigma^z$ as the topological K -matrix. In the main text, the $(C_4 \times \mathbb{Z}_2^T)$ symmetry has been defined on the bosonic fields as Eqs. (6) and (7), which can be reformulated to:

$$\mathbf{R}: \phi^j \mapsto W_j^{\mathbf{R}} \phi^j + \delta \phi_j^{\mathbf{R}}, \quad \mathcal{T}: \phi^j \mapsto W_j^{\mathcal{T}} \phi + \delta \phi_j^{\mathcal{T}} \quad (\text{S5})$$

where Θ is the operator that transforms the polar angle $\theta \mapsto \theta + \pi/2$

$$W_j^{\mathbf{R}} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \Theta, \quad \delta \phi_j^{\mathbf{R}} = \frac{\pi}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (\text{S6})$$

and

$$W_j^{\mathcal{T}} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}, \quad \delta \phi_j^{\mathcal{T}} = \begin{pmatrix} 0 \\ \pi \\ \pi \\ 0 \end{pmatrix} \quad (\text{S7})$$

Now we consider two copies of such quantum wire and investigate whether they can be fully gapped out. The corresponding Lagrangian has the similar form to Eq. (S4), with $\phi(\theta) = (\phi^1(\theta), \phi^2(\theta))^T$ as 8-component bosonic fields, and $K = (\sigma^z)^{\oplus 4}$ as the topological K -matrix. For this case, the $(C_4 \times \mathbb{Z}_2^T)$ symmetry is defined on ϕ as:

$$\mathbf{R}: \phi \mapsto W^{\mathbf{R}} \phi + \delta \phi^{\mathbf{R}}, \quad \mathcal{T}: \phi \mapsto W^{\mathcal{T}} \phi + \delta \phi^{\mathcal{T}} \quad (\text{S8})$$

where

$$W^{\mathbf{R}} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \otimes \mathbb{1}_{2 \times 2} \Theta \quad (\text{S9})$$

$\delta \phi^{\mathbf{R}} = \delta \phi_1^{\mathbf{R}} \oplus \delta \phi_2^{\mathbf{R}}$, $\delta \phi^{\mathcal{T}} = \delta \phi_1^{\mathcal{T}} \oplus \delta \phi_2^{\mathcal{T}}$, and

$$W^{\mathcal{T}} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \otimes \mathbb{1}_{2 \times 2} \quad (\text{S10})$$

We now try to construct interaction terms that gap out the edge without breaking \mathbf{R} and \mathcal{T} symmetries, neither explicitly nor spontaneously. Consider the backscattering term of the form:

$$U = \sum_k U(\Lambda_k) = \sum_k U(\theta) \cos [\Lambda_k^T K \phi - \alpha(\theta)] \quad (\text{S11})$$

The backscattering term (S11) can gap out the edge as long as the vectors $\{\Lambda_k\}$ satisfy the “null-vector” conditions [3] for $\forall i, j$:

$$\Lambda_i^T K \Lambda_j = 0 \quad (\text{S12})$$

For the present case, there are only two linear independent solutions to this problem:

$$\begin{aligned} \Lambda_1^T &= (1, 0, 0, 1, 1, 0, 0, 1)^T \\ \Lambda_2^T &= (0, 1, 1, 0, 0, 1, 1, 0)^T \end{aligned} \quad (\text{S13})$$

Nevertheless, there are eight bosonic fields $\phi^{1,2}$, we should introduce at least 4 independent backscattering terms in order to fully gap them out, hence we cannot fully gap out the two copies of 1D quantum wires for the cases of 2D C_4 -symmetric class-BDI HOTSC.

For the case with 4 copies of 1D quantum wires, we have introduced eight linear independent 4-component backscattering terms in the main text that can fully gap all four quantum wires. As the consequence, in the coupled-wire construction of 2D C_4 -symmetric class-BDI HOTSCs, we should assemble the 1D quantum wires by four, and the corresponding classification from the coupled-wire constructions is \mathbb{Z}_4 .

Symmetry and couplings of quantum wires

In this section, we explain the symmetry defined in the main text in details and demonstrate the inter-wire and intra-wire couplings are symmetric under arbitrary crystalline symmetry operations.

D_4 -symmetric class- D HOTSC

In the main text, we have defined the D_4 -symmetry in Eq. (2). Here we justify the couplings introduced in the main text are D_4 -symmetric.

Firstly, it is easy to verify that the D_4 symmetry operations defined in Eq. (2) satisfy the group structure of 4-fold dihedral symmetry. Firstly, the rotation and reflection generators, \mathbf{R} and \mathbf{M} satisfy the following condition for spinless fermions:

$$\mathbf{R}^4 = \mathbf{M}^2 = \mathbb{1} \quad (\text{S14})$$

and the symmetry operation \mathbf{MRM} will transform the bosonic fields $\phi_{1,2}^j$ as:

$$\begin{aligned} \phi_1^j(\theta) &\mapsto -\phi_1^j(\theta + \frac{3\pi}{2}) \\ \phi_2^j(\theta) &\mapsto -\phi_2^j(\theta + \frac{3\pi}{2}) + \pi \end{aligned} \quad (\text{S15})$$

which is identical to \mathbf{R}^3 . We can straightforwardly verify that the backscattering terms [cf. Eqs. (3)-(5) in the main text] are D_4 -symmetric.

Then we focus on the inter-wire and intra-wire backscattering terms and demonstrate that they can fully gap out the bulk of the D_4 -symmetric class- D HOTSC. Consider the $(2j - 2 + k)^{\text{th}}$ and $(2j - 1 + k)^{\text{th}}$ quantum wires, there are 4 independent bosonic fields $\phi_{1,2}^{2j-2+k}$ and $\phi_{1,2}^{2j-1+k}$. In order to gap them out, we need to introduce at least two linear independent backscattering terms satisfying the Haldane’s “null-vector” condition [cf. Eq. (S12)]. It is straightforwardly to verify that the inter-wire coupling \mathcal{L}_{ck}^j includes two linear independent backscattering terms satisfying the Haldane’s “null-vector” condition, hence the related $(2j - 2 + k)^{\text{th}}$ and $(2j - 1 + k)^{\text{th}}$ quantum wires are fully gapped out by \mathcal{L}_{ck}^j .

For nontrivial HOTSC dominated by \mathcal{L}_{c2} , the innermost quantum wire remains gapless at each pole. The neighborhoods of all four poles includes four segments of bosonic field, $\phi_{1,2}(\theta)$, where $\theta \in (-\epsilon + \beta\pi/2, \epsilon + \beta\pi/2)$, $\beta = 0, 1, 2, 3$. The intra-wire coupling \mathcal{L}_{int} includes only two linear independent backscattering terms globally, nonetheless, if we only focus on the gapless segments at poles of the circular quantum wire, there are four linear independent backscattering terms, two of them take the value $\theta = 0$ and the other two take the value $\theta = \pi/2$. Hence all gapless modes at poles are fully gapped by \mathcal{L}_{int} .

C_4 -symmetric class- B DI HOTSC

In the main text, we have defined the C_4 symmetry in Eq. (6) and time-reversal symmetry in Eq. (7). Now we justify the couplings introduced in the main text that are $(C_4 \times \mathbb{Z}_2^T)$ -symmetric.

Firstly, it is easy to verify that the generators of 4-fold rotation and time-reversal, \mathbf{R} and \mathcal{T} satisfy the condition:

$$\mathbf{R}^4 = \mathcal{T}^2 = \mathbb{1}$$

We note that the interactions we have introduced in the main text as Eqs. (8)-(10) are $(C_4 \times \mathbb{Z}_2^T)$ -symmetric only if we define the rotation and time-reversal symmetries as Eqs. (6) and (7) in the main text.

Then we focus on the inter-wire and intra-wire backscattering terms and demonstrate that they can fully gap out the bulk of the C_4 -symmetric class- B DI HOTSC. Consider the $(4j - k)^{\text{th}}$ ($k = 0, 1, 2, 3$) quantum wires,

there are 16 independent bosonic fields $\phi_{1,2}^{4j-k}$. In order to gap them out, we should introduce at least 8 linear independent backscattering terms satisfying the Haldane’s “null-vector” condition [cf. Eq. (S12)]. It is straightforwardly to verify that the inter-wire coupling \mathcal{L}_{ck}^j includes 8 linear independent backscattering terms satisfying the Haldane’s “null-vector” condition, hence the related four quantum wires are fully gapped by \mathcal{L}_{ck}^j .

For nontrivial HOTSC dominated by \mathcal{L}_{c4} , the innermost quantum wire remains gapless at each pole. The neighborhoods of all four poles includes sixteen segments of bosonic field, $\phi_{1,2}(\theta)$, where $\theta \in (-\epsilon + \beta\pi/2, \epsilon + \beta\pi/2)$, $\beta = 0, 1, 2, 3$. The intra-wire coupling \mathcal{L}_{int} includes only four linear independent backscattering terms globally, nonetheless, if we only focus on the gapless segments at poles of the quantum wire, there are eight linear independent backscattering terms, four of them take the value

$\theta = 0$, and the other four take the value $\theta = \pi/2$. Hence all gapless modes at poles are fully gapped by \mathcal{L}_{int} .

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