# One-Dimensional Quantum Systems with Ground-State of Jastrow Form Are Integrable

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The exchange operator formalism (EOF) describes many-body integrable systems using phase-space variables involving an exchange operator that acts on any pair of particles. We establish an equivalence between models described by EOF and the complete infinite family of parent Hamiltonians (PHJ) describing quantum many-body models with ground-states of Jastrow form. This makes it possible to identify the invariants of motion for any model in the PHJ family and establish its integrability, even in the presence of an external potential. Using this construction we establish the integrability of the long-range Lieb-Liniger model, describing bosons in a harmonic trap and subject to contact and Coulomb interactions in one dimension. We give a variety of examples exemplifying the integrability of Hamiltonians in this family.

Integrability in both classical and quantum many-body systems is associated with the existence of conserved quantities. At the quantum level, the latter correspond to operators that commute with the system Hamiltonian and govern the nonequilibrium dynamics and thermalization of a system in isolation [1, 2]. Several integrable models have been realized in the laboratory, prompting their use as a test-bed for quantum many-body physics, statistical mechanics, and nonequilibrium phenomena [3, 4].

The integrability of a system may be proven by finding the set of conserved quantities. In one spatial dimension, this is possible in systems that are exactly solved using Bethe ansatz, which posits that the wavefunction of any quantum eigenstate admits an expansion in terms of plane waves with suitable coefficients and quasimomenta. The latter set the integrals of motion, are also known as the Bethe roots or rapidities, and serve as "good" quantum numbers [5, 6]. An alternative framework is the exchange operator formalism (EOF) [7, 8], in which the Hamiltonian of the quantum system admits a decoupled form in terms of generalized momenta, which readily allows for the identification of integrals of motion. This approach can be applied to the study of excited states, as demonstrated in systems with inverse-square interactions [9, 10]. An encompassing notion of quantum integrability relies on scattering without diffraction, encoded in the Yang-Baxter equation [11–13], when collisions between particles can be described exclusively as a sequence of two-body scattering events. The system is then solvable by algebraic Bethe ansatz, i.e., using the quantum inverse scattering method. Integrals of motion can be derived from the transfer matrix [13] or invoking the asymptotic Bethe ansatz [14, 15]. While a definite notion of quantum integrability remains under debate, many of these approaches are closely interrelated [6, 16]. In particular, EOF is related to the Yang-Baxter equation and asymptotic Bethe ansatz [17, 18].

An important class of quantum systems is characterized by a ground-state of (Bijl-Dingle-) Jastrow form, in which the wave function is simply the pairwise product of a pair function [19–21]. This facilitates the computation of correlation functions in these systems [14]. The family of parent Hamiltonians with Jastrow wave functions (PHJ, for short) can be

determined by solving an inverse problem: by acting with the kinetic energy operator in the ground-state wave function, one can recast the resulting terms in the form of a many-body Schrödinger equation, thus identifying the parent hamiltonian. This approach has its roots in the early works by Calogero and Sutherland [22–24]. It has been extended in a number of ways [25, 26] and by now, for identical particles without internal degrees of freedom, the complete family of PHJ is known both in one and higher spatial dimensions, provided that the ground-state wave function includes at most the product of one-particle and two-particle functions [27, 28]. The corresponding Hamiltonians generally contain two-body and three-body interactions. It was shown by Kane et al. [29] that the three-body contribution does not affect the low-energy physics. Further, the conditions for the three-body term to vanish or reduce to a constant have been long-established in the homogeneous case, in the absence of an external potential [14, 30, 31].

Paradigmatic instances of PHJ are integrable. Hard-core bosons in the Tonks-Girardeau regime, realized in the laboratory with ultracold gases [32, 33], have ground state of Jastrow form [34–36] and are integrable, being related to noninteracting fermions via the Bose-Fermi duality [3, 34, 37]. The Calogero-Sutherland model with a Jastrow ground state has a harmonic spectrum, it can be mapped to a set of independent harmonic oscillators [38–40], and satisfies the asymptotic Bethe ansatz [14, 15]. Similarly, the attractive Lieb-Liniger (LL) model of bosons subject to contact interactions, used to describe ultracold gases in tight waveguides [41, 42], has a bright quantum soliton as Jastrow ground-state [43]. This system is solvable by coordinate Bethe ansatz, which yields the Bethe roots as integrals of motion [5, 6, 44, 45].

One may thus wonder the extent to which the ground-state correlations can determine the complete integrability of the system, and what are the required conditions for this to be the case. In this Letter, we show that the complete family of one-dimensional many-body quantum models with ground-state of Jastrow form is integrable. To this end, we first establish the equivalence between this family and models described by EOF. In doing so, we identify explicitly the integrals of motion. Our construction holds in the presence of an exter-

nal potential, which allows us to show the integrability of the long-range Lieb-Liniger model, describing bosons confined in a harmonic trap and subject to both contact and Coulomb interactions in one spatial dimension [27, 46].

Systems described by EOF— Consider the family of onedimensional systems of identical particles without internal degrees of freedom. It will prove useful to consider those models subject to pair-wise interactions that are possibly supplemented with three-body interactions. In this context, EOF is a powerful framework due to Polychronakos that explicitly exhibits the integrability of a many-body quantum system in one spatial dimension [7, 8]. Its application has been particularly fruitful in Calogero-Sutherland-Moser systems involving twobody inverse-square interactions [14, 22, 23, 25, 47, 48], as discussed in [7, 8].

Let  $M_{ij}$  denote the exchange operator, which exchanges the positions of two particles labelled by i and j, respectively. This operator is Hermitian, idempotent  $M_{ij}^2 = \mathbb{I}$  and symmetric with respect to the indices, i.e.,  $M_{ij} = M_{ji}$ . For any 1-body operator  $A_j \equiv A(x_j)$ , it obeys the relations  $M_{ij}A_j = A_iM_{ij}$  and  $M_{ij}A_k = A_kM_{ij}$  for distinct i, j, k [7, 8, 49]. Note that when for spinless identical particles,  $M_{ij}$  can be identified with the permutation of two particles. In terms of the canonical position and momentum coordinates,  $x_i$  and  $p_j = -i\hbar\partial/\partial x_j$ , one can introduce the generalized momenta

$$\pi_i = p_i + i \sum_{i \neq i} V_{ij} M_{ij}, \tag{1}$$

for particles  $j=1,\cdots,N$ . The so-called prepotential function  $V_{ij}=V(x_i-x_j)$  should be antisymmetric (i.e.,  $V_{ij}=-V_{ji}$ ) to guarantee the Hermiticity of the generalized momenta. Using the latter, one can construct a permutation-invariant quantities  $I_n \equiv \sum_i \pi_i^n$ . In particular,  $I_2$  is quadratic in  $p_i$ 's, and resembles the Hamiltonian of many-body systems. To describe states of N particles, consider the tensor product of the single-particle Hilbert space  $\mathcal{H}$ , i.e.,  $\mathcal{H}^{\otimes N}$ . For indistinguishable particles, states are restricted to the bosonic or fermionic subspaces of  $\mathcal{H}^{\otimes N}$ , denoted as  $\mathcal{H}_{\zeta}$  with  $\zeta=+1$  for spinless bosons and  $\zeta=-1$  for spinless fermions. We define the projector  $\mathcal{P}_{\zeta}$  onto  $\mathcal{H}_{\zeta}$  as [50]  $\mathcal{P}_{\zeta}\psi(x_1, x_2, \cdots, x_N) = \frac{1}{N!}\sum_{\sigma}\zeta^{\sigma}\psi(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_N})$ , where  $\sigma$  denotes a permutation of the tuple  $(1, 2, \ldots, N)$ . Projecting  $I_2$  onto the subspace  $\mathcal{H}_{\zeta}$  and using

$$M_{ij}\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}M_{ij} = \zeta\mathcal{P}_{\zeta},$$
 (2)

we obtain  $\mathcal{P}_{\zeta}I_2\mathcal{P}_{\zeta}/(2m) = \mathcal{P}_{\zeta}H_0\mathcal{P}_{\zeta}$ , where  $H_0$  is the translation-invariant quantum many-body Hamiltonian defined as follows

$$H_0 = \sum_{i} \frac{p_i^2}{2m} + \frac{1}{m} \left[ \sum_{i < i} (\zeta \hbar V'_{ij} + V_{ij}^2) - \sum_{i < i < k} V_{ijk} \right], \quad (3)$$

where  $V_{ijk} = V_{ij}V_{jk} + V_{jk}V_{ki} + V_{ki}V_{ij}$  is fully symmetric and a prime denotes the spatial derivative. The form of  $H_0$  will play an important role in proving the integrability of the family of

Hamiltonians generated by EOF and PHJ. Specific choices of the prepotential function V(x) gives rise to well-known models. For  $V(x) = \lambda/x$ ,  $V_{ijk}$  vanishes by permutation symmetry and one recovers the Hamiltonian of identical particles with inverse-square interactions [14, 22]. For  $V(x) = \lambda \cot(ax)$ ,  $V_{ijk}$ is constant and  $H_0$  involves the inverse sine square potentials. The case  $V(x) = c \operatorname{sgn}(x)$ , corresponding  $V_{ijk}$  being a negative constant, gives rise to the celebrated Lieb-Liniger (LL) model [44, 45] describing ultracold gases in tight-waveguides [3, 41]. For all these cases where  $V_{ijk}$  vanishes or is constant,  $I_n$  commute with each other. As the system Hamiltonian coincides with  $I_2$  on the bosonic or fermionic sector, the set of  $I_n$ can be identified as invariants of motion, i.e.,  $[I_n, I_m] = 0$ . We note that all the models that have been shown to be integrable by the EOF in Ref. [7] happen to have a ground-state wave function of Jastrow form, which we discuss next.

Parent Hamiltonians with Jastrow ground-state.— Consider a homogenous one-dimensional many-body quantum system described by a ground state of Jastrow form [19–21],

$$\Phi_0(x_1,\ldots,x_N) = \prod_{i < i} f_{ij},\tag{4}$$

this is, the pairwise product of the pair function  $f_{ij} = f(x_i - x_j)$  [14]. In the "beautiful models" [14] that concern us here, quantum statistics is encoded in the symmetry of f(x) which is an even function for bosons and odd for fermions, i.e., without resorting to the use of permanents or determinants. The case of one dimensional anyons can similarly be taken into account by including a phase factor  $\theta$ , i.e.,  $f(x) = e^{-i\theta}f(-x)$  [51–53]. The complete family of PHJ of ground-state of the Jastrow form (4) has been identified in one spatial dimension [27] and includes paradigmatic models such as the LL gas with contact interactions [44, 45] and the rational Calogero-Sutherland model with inverse-square interactions [22, 23], as well as the recently introduced long-range LL model [46]. For a given choice of f, the parent Hamiltonian  $H_0$  takes the form

$$H_{0} = \sum_{i} \frac{p_{i}^{2}}{2m} + \frac{\hbar^{2}}{m} \left[ \sum_{i < j} \frac{f_{ij}''}{f_{ij}} + \sum_{i < j < k} \left( \frac{f_{ij}' f_{ik}'}{f_{ij} f_{ik}} - \frac{f_{ij}' f_{jk}'}{f_{ij} f_{jk}} + \frac{f_{ik}' f_{jk}'}{f_{ik} f_{jk}} \right) \right]. \tag{5}$$

Here, f' and f'' denote the first and second spatial derivatives of f, respectively. The explicit expressions for this Hamiltonian directly follow from evaluating the Laplacian on the Jastrow wave function (4) and recasting all resulting terms in the form of a Schrödinger equation.

Equivalence of EOF and PHJ for spinless indistinguishable particles.— We now establish the correspondence between EOF and PHJ for spinless identical particles. Comparing the EOF Hamiltonian and the PHJ in Eqs. (3) and (5), the two-body terms are equal if  $\hbar^2 f''(x_{ij})/f(x_{ij}) = \zeta \hbar V'_{ij} + V^2_{ij}$ . Thus, the prepotential reads

$$V_{ij} = \zeta \hbar \frac{d}{dx_{ij}} \log(f_{ij}) = \zeta \hbar \frac{f'_{ij}}{f_{ij}}.$$
 (6)

Independently of whether the pair function is symmetric or antisymmetric, its logarithmic derivative is guaranteed to be odd  $f'_{ij}/f_{ij} = -f'_{ji}/f_{ji}$ . Thus, this property holds for spinless bosons and fermions. The antisymmetry of the prepotential in Eq. (6) guarantees the Hermiticity condition of the associated generalized momenta in the EOF,

$$\pi_i = p_i + i\zeta\hbar \sum_{i \neq i} \frac{f'_{ij}}{f_{ij}} M_{ij}. \tag{7}$$

The prepotential in Eq. (6) further ensures the equivalence of the three-body interaction in the EOF and the PHJ. Thus, any spinless system described by EOF, as in Eq. (3), has a ground-state of Jastrow form with a pair function  $f_{ij} = \exp[\int^{x_{ij}} dy V(y)/(\zeta \hbar)]$ . Conversely, the complete infinite family of PHJ can be recast in the EOF provided (6) is satisfied. This makes it possible to identify the class of PHJ that is integrable as we shall see later.

Embedding in an external potential.— The embedding of a system described by EOF in an external potential is known in the case of a harmonic trap [7]. For the embedding of a homogenous system in an arbitrary trapping potential, we draw inspiration from supersymmetric quantum mechanics [54] and introduce the one-body superpotential  $W_i \equiv W(x_i)$  in terms of which the external trapping potential  $U_i$  will be identified. We define the operators

$$a_i = \frac{\pi_i}{\sqrt{2m}} - iW_i, \quad a_i^{\dagger} = \frac{\pi_i}{\sqrt{2m}} + iW_i, \tag{8}$$

and the permutation-invariant quantities  $\tilde{I}_n \equiv \sum_i h_i^n$ , where  $h_i \equiv a_i^{\dagger} a_i$ . Projecting  $\tilde{I}_1$  onto  $\mathcal{H}_{\zeta}$ , we find  $\mathcal{P}_{\zeta} \tilde{I}_1 \mathcal{P}_{\zeta} = \mathcal{P}_{\zeta} H \mathcal{P}_{\zeta}$ , where H is the Hamiltonian of the system in the presence of the trap, i.e.,

$$H = H_0 + \sum_{i} U_i - \zeta \sqrt{\frac{2}{m}} \sum_{i \le i} V_{ij} (W_i - W_j),$$
 (9)

with the external potential  $U_i$  being determined by the Riccati equation

$$U_i = W_i^2 - \frac{\hbar}{\sqrt{2m}} W_i'. \tag{10}$$

As a familiar example, when  $H_0$  is the homogeneous Calogero model with inverse-square interactions [22] and  $W_i = \sqrt{m/2}\omega x_i$ , Eq. (9) reduces to the rational Calogero-Sutherland model [23, 47] including a harmonic trap.

In PHJ, the ground-state wave functions is not limited to the homogeneous form (4), but also includes more general ground-states

$$\Psi_0 = \prod_{i < i} f_{ij} \prod_i \exp(\nu_i) = \Phi_0 \prod_i \exp(\nu_i), \quad (11)$$

where the one-body function  $v_i = v(x_i)$  accounts for the role of the external potential  $U_i = U(x_i)$  that breaks translational invariance [27]. Specifically, if  $H_0$  is the parent hamiltonian of  $\Phi_0$  in Eq. (4), then  $\Psi_0$  has the parent Hamiltonian

$$H = H_0 + \sum_{i} U_i + \frac{\hbar^2}{m} \sum_{i < j} (v_i' - v_j') \frac{f_{ij}'}{f_{ij}}, \tag{12}$$

with the one-body local external potential  $U_i$  given in terms of the function  $v_i$  by

$$U_i = \frac{\hbar^2}{2m} \left[ (v_i')^2 + v_i'' \right]. \tag{13}$$

As a result, the Hamiltonian H includes the external potential  $U_i$  and an additional pairwise (two-body) potential which is generally of long-range character.

The equivalence between EOF and PHJ require that the one-body and potential and the additional long-range term are equal in both representations. Comparing Eq. (10) and Eq. (13), the superpotential  $W_i$  and the function  $v_i$  entering the one-body function of the Jastrow form are related by

$$W_i = -\frac{\hbar}{\sqrt{2m}} v_i'. \tag{14}$$

Upon substituting Eq. (14) into Eqs. (9, 12), we find that the additional long-range potentials coincide, given the correspondence Eq. (6) is identified. The ground state of the Hamiltonian with the external potential in terms of the prepotential and the superpotential is

$$\Psi_0 = \exp\left(-\frac{\sqrt{2m}}{\hbar} \sum_i \int^{x_i} W(y) dy\right) \prod_{i < j} \exp\left[\frac{\int^{x_{ij}} dy V(y)}{\zeta \hbar}\right]. \tag{15}$$

This establishes the equivalence between EOF and PHJ in the presence of external potential.

Integrability via projection formalism.— For quantum systems with classical analog, as the PHJ, one can define quantum integrability by promoting the Poisson bracket into commutators in the definition of classical integrability. Polychronakos [7] pursued along this line and showed that  $I_n \equiv \sum_i \pi_i^n$  become integrals of motion, i.e.,  $[I_n, I_m] = 0$ ,  $\forall n, m$ , in the restricted case in which  $V_{ijk}$  vanishes or is constant. Having shown that any spinless model described by EOF is a PHJ with a Jastrow ground state, we next establish the integrability of the complete family of PHJ models, i.e., without restrictions on the three-body potential  $V_{ijk}$  or the external potential  $U_i$ .

Note that any physical observable O for spinless indistinguishable particles must be permutation invariant, i.e.,  $O(x_1, x_2, \dots, x_N) = O(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_N}), \forall$  permutation  $\sigma$ . As a consequence,

$$[\mathcal{P}_{\ell}, O] = 0, \tag{16}$$

which can be easily checked by acting on any wave function in  $\mathcal{H}^{\otimes N}$  [49]. Eq. (16) implies the a permutation-invariant observable is block diagonal on  $\mathcal{H}_{\zeta}$  and its orthogonal complement. We define an observable is local if it only involves derivatives with respect to the coordinates up to a finite order. Then permutation invariance and locality implies that if a permutation-invariant and local observable O vanishes on  $\mathcal{H}_{\zeta}$ , then it also vanishes on the full Hilbert space  $\mathcal{H}^{\otimes N}$ . That is [55],

$$\mathcal{P}_{\zeta}O\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}O = O\mathcal{P}_{\zeta} = 0 \iff O = 0,$$
 (17)

for a permutation-invariant and local observable O.

Eqs. (16, 17) lead to the following theorem regarding the commutators of two permutation-invariant observables, which is extremely useful in proving integrability.

**Theorem 1.** For two permutation-invariant and local observable  $O_n$  and  $O_m$ , the following three conditions are equivalent to each other (i)  $[O_n, O_m] = 0$  (ii)  $\mathcal{P}_{\zeta}[O_n, O_m]\mathcal{P}_{\zeta} = 0$  (iii)  $[\mathcal{P}_{\zeta}O_n\mathcal{P}_{\zeta}, \mathcal{P}_{\zeta}O_m\mathcal{P}_{\zeta}] = 0$ .

The equivalence between (i) and (ii) is a consequence of Eq. (17). The equivalence between (ii) and (iii) follows from

$$\mathcal{P}_{\zeta}[O_{n}, O_{m}]\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}O_{n}O_{m}\mathcal{P}_{\zeta}^{2} - \mathcal{P}_{\zeta}O_{m}O_{n}\mathcal{P}_{\zeta}^{2}$$

$$= \mathcal{P}_{\zeta}O_{n}\mathcal{P}_{\zeta}O_{m}\mathcal{P}_{\zeta} - \mathcal{P}_{\zeta}O_{m}\mathcal{P}_{\zeta}O_{n}\mathcal{P}_{\zeta}$$

$$= [\mathcal{P}_{\zeta}O_{n}\mathcal{P}_{\zeta}, \mathcal{P}_{\zeta}O_{m}\mathcal{P}_{\zeta}],$$

where we have used Eq. (16).

**Theorem 2.** Both the quantum mechanical homogenous model (3) and the inhomogeneous model (9) generated in EOF are integrable, with the integral of motion being  $I_n$  for the homogenous model and  $\tilde{I}_n$  for the inhomogeneous model.

To prove Theorem 2, let us first observe a very interesting property due to the projection  $\mathcal{P}_{\zeta}$  and the exchange operator  $M_{ij}$ . Although the generalized momentum  $\pi_i$  involves N degrees of freedom due to the prepotential term, when it is multiplied by  $M_{ij}$  from the left, it still satisfies the exchange rule for 1-body operators [49]. As a consequence,

$$\mathcal{P}_{\zeta}\pi_{i}^{n}\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}M_{ij}^{2}\pi_{i}^{n}\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}M_{ij}\pi_{i}^{n}M_{ij}\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}\pi_{j}^{n}\mathcal{P}_{\zeta}. \quad (18)$$

A similar equation also holds for  $h_i$ . For a more general version of this identity, see [49].

On the other hand, since the integrals of motions  $I_n$  and  $\tilde{I}_n$  are permutation invariant and local, one can reduce their commutativity to condition (iii) in Theorem 1. Using Eq. (18) or the analogous equation for  $h_i$ , it follows that the condition (iii) in Theorem 1 is satisfied, with  $O_n = I_n$  or  $O_n = \tilde{I}_n$ . This concludes the proof of the integrability of the Hamiltonians (3) and (9).

A few comments are in order. First, since we have proved the equivalence between EOF and PHJ, the Hamiltonians (5) and (12) are therefore also integrable. Second, it is possible to build integrals of motions for families of classical models generated by the EOF and PHJ according to the quantum-classical correspondence. One can expand the powers in  $I_n$  and  $\tilde{I}_n$  and compute  $\mathcal{P}_{\zeta}I_{n}\mathcal{P}_{\zeta}$  and  $\mathcal{P}_{\zeta}\tilde{I}_{n}\mathcal{P}_{\zeta}$  explicitly with Eq. (2). Then one is left with the expressions  $\mathcal{P}_{\zeta}K_{n}\mathcal{P}_{\zeta}$  and  $\mathcal{P}_{\zeta}\tilde{K}_{n}\mathcal{P}_{\zeta}$ , where  $K_n$  and  $\tilde{K}_n$  contain only the phase space variables but no exchange operators. In particular, we note that  $K_2 = H_0$  and  $\tilde{K}_1 = H$ ; see [49], where one obtains  $H_0$  by projecting  $I_2$  onto  $\mathcal{H}_{\zeta}$ . According to Theorem 1,  $K_n$ 's and  $\tilde{K}_n$ 's must commute on the whole Hilbert space  $\mathcal{H}^{\otimes N}$ , respectively. Transitioning to the classical model, where the commutator is demoted to Poisson brackets, the Poisson brackets of  $K_n$ 's and  $\tilde{K}_n$ 's must vanish, respectively. Thus, we see that  $K_n$ 's and  $\tilde{K}_n$ 's are also

the integrals of motions for the classical model with Hamiltonians (3, 5) and (9, 12), respectively.

Discussion.— It is worth noting that the Jastrow wave functions  $\Phi_0$ ,  $\Psi_0$  may not be the true ground state of the corresponding PHJ if they cannot be properly normalized. Nevertheless, the family of models generated by EOF and PHJ is always integrable, regardless of the normalization of the Jastrow wave function.

For example, if  $f_{ij} = \exp(g|x_{ij}|)$ ,  $H_0$  becomes the well-known LL model [27]. However, the Jastrow wave function is normalizable only when g < 0, which corresponds to the McGuire bright soliton [43]. Therefore  $\Phi_0$  is no longer the ground state wave function of the repulsive LL model. However, as we have discussed previously, the integrability of the Hamiltonian is not affected, so our result reproduces the integrability of the LL model with the integral of motion being  $I_n$  or  $K_n$ . More interestingly, upon introducing the external harmonic potential, according to Eq. (11),  $\Psi_0$  becomes normalizable even if  $\Phi_0$  is not and Eq. (5) corresponds to the Lieb-Liniger-Coulomb model introduced in Refs. [46], i.e.,

$$H = \sum_{i} \left[ \frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 x_i^2 \right] + g \sum_{i < j} \left[ \frac{2\hbar^2}{m} \delta(x_{ij}) - \frac{m\omega}{\hbar} |x_{ij}| \right], \tag{19}$$

with ground state  $E_0 = \frac{N\hbar\omega}{2} - \frac{g^2\hbar^2}{m} \frac{N(N^2-1)}{6}$ . This system describes harmonically confined bosons subject to contact and Coulomb interactions or gravitational attraction in one spatial dimension. Ref. [46] characterized its EOF representation and ground state properties. Using Theorem 2, we conclude that this system is integrable, with the integrals of motion being  $\tilde{I}_n \equiv \sum_i h_i^n$ .

Further physical examples of integrable PHJ systems are provided in the Supplemental Material [49], which includes Refs. [30, 56]. The proof leading to the integrability of PHJ essentially takes advantage of the permutation invariance and EOF. As a result it can be applied to models defined on the real line as well as those embedded in an external potential. Likewise, it holds for systems with hard-wall confinement or a ring geometry, provided the pair function  $f_{ij}$  and the one body potential  $v_i$  or  $W_i$  fulfill the corresponding boundary conditions.

Conclusion.— We have established the equivalence between the families of one-dimensional many-body quantum systems generated by the exchange operator formalism and parent Hamiltonians with a ground-state wavefunction of Jastrow form, describing indistinguishable particles with no internal degrees of freedom. Making use of the projection operator onto the spinless bosonic or fermionic subspace, we have proved the integrability of all these systems by constructing explicitly the corresponding integrals of motion. Embedding these translation-invariant models in an external potential preserves the integrability, in the presence of long-range interactions, as we have illustrated in the long-range Lieb-Liniger model and related systems.

These findings advance the study of many-body physics by uncovering the implications of ground-state correlations on integrability. They should lead to manifold applications in the study of quantum solitons, quantum quenches and the thermalization of isolated integrable systems (governed by integrals of motion), and strongly-correlated regimes, generalizing the super-Tonks-Girardeau gas [57], among others. Our results bear also implications on numerical methods for strongly-correlated systems such as variational methods and quantum Monte Carlo algorithms, in which the ubiquitous use of Jastrow trial wavefunctions may impose integrability on systems lacking it. It may be possible to extend our results to higher spatial dimensions [28], higher-order correlations [58], the inclusion of spin degrees of freedom [8], mixtures of different species [36], and distinguishable particles [59, 60].

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# **Supplemental Material for**

# "One-Dimensional Quantum Systems with Ground-State of Jastrow Form Are Integrable"

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## I. THE EXCHANGE OPERATOR

In this section, we give a rigorous definition of the exchange operator used in the main text and derive its properties from first principles.

## A. Properties of the two-particle position exchange operator $M_{ij}$

Consider the *N*-fold tensor-product of a single-particle Hilbert space  $\mathcal{H}$ , i.e.,  $\mathcal{H}^{\otimes N}$ , where we do not assume a particular exchange statistics for the particles. This means that the particles may be distinguishable or not. The position exchange operator  $M_{ij}$  acting on the spatial coordinates  $x_i$  and  $x_j$  on a many-particle state  $\psi_s \in \mathcal{H}^{\otimes N}$  is defined as

$$M_{ij}\psi_s(x_1,\dots x_i,\dots,x_j,\dots x_N)=\psi_s(x_1,\dots x_j,\dots,x_i,\dots x_N),\ i\neq j,$$
(S1)

where  $\psi_s(x_1, \dots, x_i, \dots, x_j, \dots, x_N)$  is the wave function on  $\mathcal{H}^{\otimes N}$  and s denotes the internal degrees of freedom, which can account for spin in identical particles. By definition, we find

$$M_{ii} = M_{ii}. (S2)$$

Furthermore, according to this definition, it follows that

$$M_{ij}^2 \psi_s(x_1, \dots x_i, \dots, x_j, \dots x_N) = \psi_s(x_1, \dots x_i, \dots, x_j, \dots x_N), \tag{S3}$$

which implies that  $M_{ij}$  is idempotent, i.e.,

$$M_{ij}^2 = \mathbb{I}. (S4)$$

The inner product on  $\mathcal{H}^{\otimes N}$  is

$$(\phi, \psi) \equiv \sum_{s} \int \prod_{k=1}^{N} dx_k \phi_s^*(x_1, \dots x_i, \dots, x_j, \dots x_N) \psi_s(x_1, \dots x_i, \dots, x_j, \dots x_N).$$
 (S5)

The adjoint of  $M_{ij}$  is defined as  $(M_{ij}^{\dagger}\phi, \psi) \equiv (\phi, M_{ij}\psi)$ . Explicitly expanding the inner product according to Eq (S5), we arrive at

$$\sum_{s} \int \prod_{k=1}^{N} dx_{k} [M_{ij}^{\dagger} \phi_{s}(x_{1}, \dots, x_{i}, \dots x_{j}, \dots x_{N})]^{*} \psi_{s}(x_{1}, \dots x_{i}, \dots, x_{j}, \dots x_{N})$$

$$= \sum_{s} \int \prod_{k=1}^{N} dx_{k} \phi_{s}^{*}(x_{1}, \dots, x_{i}, \dots x_{j}, \dots x_{N}) \psi_{s}(x_{1}, \dots x_{j}, \dots, x_{i}, \dots x_{N})$$

$$= \sum_{s} \int \prod_{k=1}^{N} dx_{k} \phi_{s}^{*}(x_{1}, \dots, x_{j}, \dots x_{i}, \dots x_{N}) M_{ij} \psi_{s}(x_{1}, \dots, x_{j}, \dots, x_{j}, \dots x_{N}), \tag{S6}$$

which indicates that

$$M_{ij}^{\dagger}\phi_s(x_1,\cdots x_i,\cdots,x_j,\cdots x_N)=\phi_s(x_1,\cdots x_j,\cdots,x_i,\cdots x_N). \tag{S7}$$

Comparing above equation with Eq. (S1), we conclude that  $M_{ij}$  is a Hermitian operator,

$$M_{ij} = M_{ij}^{\dagger}. ag{S8}$$

We define operators or observables

$$A_i \equiv A(x_i), \tag{S9}$$

$$A_{ij} \equiv A(x_i, x_j), \tag{S10}$$

$$A_{ijk} \equiv A(x_i, x_j, x_k), \tag{S11}$$

:

where the dependence on the internal degrees of freedom of O is suppressed. Then one can easily find that

$$(\phi, M_{ij}A_k\psi) = (M_{ij}^{\dagger}\phi, A_k\psi) = \sum_s \int \prod_{l=1}^N dx_l \phi_{\sigma}^*(x_1, \dots, x_j, \dots x_i, \dots x_N) O(x_k) \psi_s(x_1, \dots, x_j, \dots x_N)$$
(S12)

For  $k \neq i$ , j,

$$\sum_{s} \int \prod_{l=1}^{N} dx_{l} \phi_{s}^{*}(x_{1}, \dots, x_{j}, \dots x_{i}, \dots x_{N}) O(x_{k}) \psi_{s}(x_{1}, \dots, x_{j}, \dots, x_{j}, \dots x_{N})$$

$$= \sum_{s} \int \prod_{l=1}^{N} dx_{l} \phi_{s}^{*}(x_{1}, \dots, x_{j}, \dots, x_{i}, \dots, x_{N}) O(x_{k}) \psi_{s}(x_{1}, \dots, x_{i}, \dots, x_{j}, \dots, x_{N})$$

$$= \sum_{s} \int \prod_{l=1}^{N} dx_{l} \phi_{s}^{*}(x_{1}, \dots, x_{j}, \dots, x_{i}, \dots, x_{N}) O(x_{k}) M_{ji} \psi_{s}(x_{1}, \dots, x_{j}, \dots, x_{i}, \dots, x_{N})$$

$$= (\phi, O_{k} M_{ji})$$
(S13)

We conclude that

$$M_{ij}A_k = A_k M_{ji} = A_k M_{ij}$$
, for  $i, j, k$ , distinct. (S14)

If k = i, then we find

$$(\phi, M_{ij}A_i\psi) = \sum_{s} \int \prod_{l=1}^{N} dx_l \phi_s^*(x_1, \dots, x_j, \dots x_l, \dots x_N) O(x_i) \psi_s(x_1, \dots, x_j, \dots, x_j, \dots x_N)$$

$$= \sum_{s} \int \prod_{l=1}^{N} dx_l \phi_s^*(x_1, \dots, x_j, \dots x_l, \dots x_N) A(x_i) M_{ji} \psi_s(x_1, \dots, x_j, \dots, x_l, \dots x_N)$$

$$= \sum_{s} \int \prod_{l=1}^{N} dx_l \phi_s^*(x_1, \dots, x_i, \dots x_j, \dots x_N) A(x_j) M_{ij} \psi_s(x_1, \dots, x_j, \dots, x_j, \dots x_N)$$

$$= (\phi, A_i M_{ij} \psi), \tag{S15}$$

where we have used change of the dummy indices  $x_i \to x_j$  and  $x_j \to x_i$  in the second last equation. We find

$$M_{ij}A_i = A_j M_{ij} \tag{S16}$$

and similarly

$$M_{ij}A_i = A_i M_{ij}. (S17)$$

One can extend the above arguments to many-body operators  $A_{ijk\cdots}$  with little effort. For example, for two-body operators  $A_{kl}$ , one can find

$$M_{ij}A_{kl} = A_{kl}$$
, for  $i, j, k, l$ , distinct (S18)

$$M_{ij}A_{jk} = A_{ik}M_{ij}$$
, for  $i, j, k$ , distinct (S19)

$$M_{ij}A_{kj} = A_{ki}M_{ij}$$
, for  $i, j, k$ , distinct (S20)

$$M_{ij}A_{ij} = A_{ii}M_{ij}, (S21)$$

$$M_{ij}A_{ji} = A_{ij}M_{ij}. (S22)$$

## **B.** Properties of the three-particle position exchange operators $M_{ijk}$

The particle exchange operator with three indices is defined as

$$M_{ijk} = M_{ij}M_{jk}$$
, for  $i, j, k$ , distinct. (S23)

With the properties of the two-particle exchange operator, one can easily show that  $M_{ijk}$  is invariant under cyclic permutation, i.e.,

$$M_{ijk} = M_{iki} = M_{kij} \tag{S24}$$

However, it is not fully symmetric in its indices. In particular,  $M_{ijk} \neq M_{jik}$ . Furthemore, for all distinct i, j, k, l,

$$M_{ijk}A_l = A_l M_{ijk} (S25)$$

since  $A_l$  commute with  $M_{ij}$  and  $M_{jk}$ . Furthermore, one can explicit check that

$$M_{ijk}A_i = M_{ij}A_iM_{jk} = A_jM_{ijk}, (S26)$$

$$M_{ijk}A_j = M_{ij}A_kM_{jk} = A_kM_{ijk}, (S27)$$

$$M_{ijk}A_k = M_{ij}A_jM_{jk} = A_iM_{ijk}. (S28)$$

## II. PROJECTION ONTO THE BOSONIC AND FERMIONIC SUBSPACES

In EOF, one may restrict the construction to the bosonic or fermionic subspace to simplify the calculation or motivated on physical grounds. For example, it was shown by Polychronakos [7, 8] that if the prepotential is  $V_{ij} = l/x_{ij}$ , then

$$\frac{1}{2}I_2 = \frac{1}{2m}\sum_i \pi_i^2 = \frac{1}{2m}\sum_i p_i^2 + \frac{\hbar^2}{m}\sum_{i>j} \frac{l(l-M_{ij})}{(x_i-x_j)^2}.$$
 (S29)

This Hamiltonian is still defined on  $\mathcal{H}^{\otimes N}$ . Recall that the Hilbert space  $\mathcal{H}_{\zeta}$  of identical particles is a subspace of  $\mathcal{H}^{\otimes N}$ . We define the projection operator on  $\mathcal{H}^{\otimes N}$  to  $\mathcal{H}_{\zeta}$  as  $\mathcal{P}_{\zeta}$ . It can be readily checked that for any  $\psi(x_1, x_2, \dots, x_N) \in \mathcal{H}^{\otimes N}$ ,

$$M_{ij}\mathcal{P}_{\zeta}\psi(x_{1}, x_{2}, \cdots, x_{N}) = \frac{1}{N!} \sum_{\sigma} (\pm 1)^{\sigma} M_{ij}\psi(x_{\sigma_{1}}, \dots x_{\sigma_{i}}, \dots x_{\sigma_{j}}, \dots, x_{\sigma_{N}})$$

$$= \frac{1}{N!} \sum_{\sigma} (\pm 1)^{\sigma+1} \psi(x_{\sigma_{1}}, \dots x_{\sigma_{i}}, \dots x_{\sigma_{j}}, \dots, x_{\sigma_{N}})$$

$$= \pm \mathcal{P}_{\zeta}\psi(x_{1}, x_{2}, \dots, x_{N}), \tag{S30}$$

$$\mathcal{P}_{\zeta}M_{ij}\psi(x_{1}, x_{2}, \cdots, x_{N}) = \mathcal{P}_{\zeta}\psi(x_{1}, \cdots x_{j}, \cdots x_{i}, \cdots x_{N})$$

$$= (\pm 1)\mathcal{P}_{\zeta}\psi(x_{1}, \cdots x_{i}, \cdots x_{j}, \cdots x_{N})$$

$$= \frac{1}{N!} \sum_{\sigma} (\pm 1)^{\sigma+1} \psi(x_{\sigma_{1}}, \dots x_{\sigma_{i}}, \cdots x_{\sigma_{j}}, \dots, x_{\sigma_{N}})$$

$$= \pm \mathcal{P}_{\zeta}\psi(x_{1}, x_{2}, \dots, x_{N}). \tag{S31}$$

Thus, we conclude

$$[\mathcal{P}_{\zeta}, M_{ij}] = 0, \tag{S32}$$

$$\mathcal{P}_{\ell} M_{ii} = M_{ii} \mathcal{P}_{\ell} = \zeta \mathcal{P}_{\ell}. \tag{S33}$$

Therefore, when projecting onto the bosonic or fermionic subspace, Eq. (\$29) becomes

$$\frac{1}{2}\mathcal{P}_{\zeta}I_{2}\mathcal{P}_{\zeta} = \frac{1}{2}\sum_{i}\pi_{i}^{2} = \mathcal{P}_{\zeta}H_{\text{CS}}\mathcal{P}_{\zeta},\tag{S34}$$

where

$$H_{\rm CS} = \frac{1}{2m} \sum_{i} p_i^2 + \frac{\hbar^2}{m} \sum_{i > i} \frac{l(l \mp \zeta)}{(x_i - x_j)^2}$$
 (S35)

is the rational Calogero model. The procedure in Eq. (S34) is effectively equivalent to replacing  $M_{ij} = \zeta \mathbb{I}$ .

However, we warn the audience that when focusing on spinless bosons or fermions, setting  $M_{ij} = \zeta \mathbb{I}$  requires some caution. For example, when calculating  $[p_i, \sum_{k \neq j} V_{jk} M_{jk}]$  with  $i \neq j$ , had one setting  $M_{jk} = \zeta \mathbb{I}$  before actually calculating the commutator, one would obtain

$$[p_i, \sum_{k \neq i} V_{jk} M_{jk} \mathcal{P}_{\zeta}] \stackrel{?}{=} \zeta i \hbar V'_{ij}. \tag{S36}$$

On the other hand, a rigorous calculation following Eqs. (\$14-\$22) shows that

$$[p_{i}, \sum_{k \neq j} V_{jk} M_{jk}] = [p_{j}, V_{ji} M_{ji}] = [p_{i}, V_{ji}] M_{ji} + V_{ji} [p_{i}, M_{ji}]$$

$$= i\hbar V'_{ij} M_{ij} - V_{ij} (p_{i} - p_{j}) M_{ij}, (i \neq j)$$
(S37)

Upon projecting this relation onto  $\mathcal{H}_{\alpha}$  by setting  $M_{jk} = \mathbb{I}$ , which yields

$$[p_i, \sum_{k \neq j} V_{jk} M_{jk}] \mathcal{P}_{\zeta} = \zeta i \hbar V'_{ij} - \zeta V_{ij} (p_i - p_j).$$
(S38)

This example clearly shows that  $[\mathcal{A}, \mathcal{B}]\mathcal{P}_{\zeta} \neq [\mathcal{A}, \mathcal{B}\mathcal{P}_{\zeta}]$  and in particular

$$[\mathcal{A}, \mathcal{BP}_{\ell}] = [\mathcal{A}, \mathcal{B}]\mathcal{P}_{\ell} + \mathcal{B}[\mathcal{A}, \mathcal{P}_{\ell}]. \tag{S39}$$

The caveat we would like to give to the audience is that whenever one would like to set  $M_{ij} = \pm \mathbb{I}$ , say, on physical grounds, one should bear in mind that at the formal level a projection has been introduced, which has to be taken into account to work out the correct algebra.

Finally, we mention the following interesting lemma for spinless indistinguishable particles, thanks to the EOF and the projection operator:

**Lemma 1.** For any k-body operator  $A_{i_1\cdots i_k}$  acting on states of N-spinless indistinguishable particles,  $\mathcal{P}_{\zeta}A_{i_1\cdots i_k}\mathcal{P}_{\zeta}$  is "supersymmetric", i.e.,

$$\mathcal{P}_{\zeta}A_{i_{1}\cdots i_{k}}\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}A_{j_{1}\cdots j_{k}}\mathcal{P}_{\zeta},\tag{S40}$$

where  $(j_1, j_2, \dots, j_k)$  is arbitrary k-tuple, not necessarily a permutation of  $(i_1, \dots i_k)$ 

*Proof.* The proof is to take advantage of Eq. (S4) and Eq. (2). The intuition for the proof can be easily seen from the case where indices  $(j_1, j_2, \dots, j_k)$  are distinct:

$$\mathcal{P}_{\zeta}A_{i_{1}\cdots i_{k}}\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}M_{i_{1}j_{1}}\cdots M_{i_{k}j_{k}}M_{i_{k}j_{k}}\cdots M_{i_{1}j_{1}}A_{i_{1}\cdots i_{k}}\mathcal{P}_{\zeta}$$

$$= \mathcal{P}_{\zeta}M_{i_{1}j_{1}}\cdots M_{i_{k}j_{k}}A_{j_{1}\cdots j_{k}}M_{i_{k}j_{k}}M_{i_{k}j_{k}}M_{i_{1}j_{1}}\mathcal{P}_{\zeta}$$

$$= \mathcal{P}_{\zeta}\zeta^{k}A_{j_{1}\cdots j_{k}}\zeta^{k}\mathcal{P}_{\zeta}$$

$$= \mathcal{P}_{\zeta}A_{j_{1}\cdots j_{k}}\mathcal{P}_{\zeta}.$$
(S41)

With the above intuition, one can easily prove the general case without any difficulty.

#### III. DERIVATION OF EQ. (3)

Given the generalized momenta

$$\pi_i = p_i + i \sum_{i \neq i} V_{ij} M_{ij},$$

the EOF Hamiltonian is defined as

$$\frac{1}{2m}I_{2} = \frac{1}{2m}\sum_{i}\pi_{i}^{2}$$

$$= \frac{1}{2m}\sum_{i}\left(p_{i} + i\sum_{j\neq i}V_{ij}M_{ij}\right)^{2}$$

$$= \frac{1}{2m}\sum_{i}\left[p_{i}^{2} + i\sum_{j\neq i}(p_{i}V_{ij}M_{ij} + V_{ij}M_{ij}p_{i}) - \sum_{i,k\neq i}V_{ij}M_{ij}V_{ik}M_{ik}\right], \tag{S42}$$

which is the sum of a purely kinetic one-body term, a two-body term, and a three-body term. We next use the fact the two-body

term can be rewritten making use of  $[p_i, V_{ij}] = -i\hbar$ ,  $\partial_i V_{ij} = -i\hbar V'_{ij}$  together with  $V_{ij}M_{ij}p_i = V_{ij}p_jM_{ij}$ . In addition, we note that the third term on r.h.s. of Eq. (S42) with j = k becomes a two-body term  $-V_{ij}M_{ij}V_{ij}M_{ij} = -i\hbar V'_{ij}$  $-V_{ij}V_{ji}M_{ij}^2 = +V_{ij}^2$ , using the fact that  $V_{ij}$  is antisymmetric in its indices. Thus

$$\frac{1}{2m}I_{2} = \frac{1}{2m} \sum_{i} \left[ p_{i}^{2} + \sum_{j \neq i} (\hbar V'_{ij} M_{ij} + iV_{ij} (p_{i} + p_{j}) M_{ij} + V^{2}_{ij}) - \sum_{i \neq j \neq k \neq i} V_{ij} M_{ij} V_{ik} M_{ik} \right] 
= \frac{1}{2m} \sum_{i} \left[ p_{i}^{2} + \sum_{j \neq i} (\hbar V'_{ij} M_{ij} + V^{2}_{ij}) - \sum_{i \neq j \neq k \neq i} V_{ij} V_{jk} M_{ij} M_{ik} \right] 
= \frac{1}{2m} \sum_{i} \left[ p_{i}^{2} + \sum_{j \neq i} (\hbar V'_{ij} M_{ij} + V^{2}_{ij}) - \sum_{i \neq j \neq k \neq i} V_{ij} V_{ki} M_{ijk} \right],$$
(S43)

where in the last step we have swapped the indices i and j and used the fact that  $V_{ij}$  is antisymmetric in its indices, and that  $M_{ijk}$  is the three-particle exchange operator defined in Eq. (S23). Taking advantage of the invariance of  $M_{ijk}$  under the cyclic permutation of i, j, k, we can further symmetrize the last term on the r.h.s. of Eq. (843) as

$$\sum_{i \neq j \neq k \neq i} V_{ij} V_{ki} M_{ijk} = \frac{1}{3} \sum_{j \neq k \neq i} (V_{ij} V_{jk} + V_{jk} V_{ki} + V_{ki} V_{ij}) M_{ijk} = \frac{1}{3} \sum_{j \neq k \neq i} V_{ijk} M_{ijk},$$
 (S44)

where  $V_{ijk} = V_{ij}V_{jk} + V_{jk}V_{ki} + V_{ki}V_{ij}$ . Upon projecting to the bosonic or fermionic subspace, we find,

$$\frac{1}{2m}\mathcal{P}_{\zeta}I_{2}\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta}\left(\frac{1}{2m}\sum_{i}p_{i}^{2} + \frac{1}{2m}\sum_{i\neq j}(\zeta\hbar V_{ij}' + V_{ij}^{2}) - \frac{1}{6m}\sum_{j\neq k\neq i}V_{ijk}\right)\mathcal{P}_{\zeta}$$
(S45)

Since  $V'_{ij}$  and  $V^2_{ij}$  are symmetric in the indices i and j and  $V_{ijk}$  is fully symmetric in permutation of any pair of the indices, the sums in above equation can be further rewritten as

$$\frac{1}{2m} \mathcal{P}_{\zeta} I_2 \mathcal{P}_{\zeta} = \mathcal{P}_{\zeta} \left( \frac{1}{2m} \sum_{i} p_i^2 + \frac{1}{m} \sum_{i < j} (\zeta \hbar V'_{ij} + V_{ij}^2) - \frac{1}{m} \sum_{i < j < k} V_{ijk} \right) \mathcal{P}_{\zeta}. \tag{S46}$$

Therefore, we find Eq. (3).

### IV. RESULTS RELATED TO INTEGRABILITY

In this section, we discuss some results that are used in the proof of integrability discussed in the main text.

#### **A. Proof of Eq. (16)**

For any permutation-invariant observable  $O(x_1, x_2, \dots, x_N)$ ,

$$\mathcal{P}_{\zeta}O(x_{1}, x_{2}, \dots, x_{N})\psi(x_{1}, x_{2}, \dots, x_{N}) = \frac{1}{N!} \sum_{\sigma} O(x_{\sigma_{1}}, x_{\sigma_{2}}, \dots, x_{\sigma_{N}})\psi(x_{\sigma_{1}}, x_{\sigma_{2}}, \dots, x_{\sigma_{N}})$$

$$= O(x_{1}, x_{2}, \dots, x_{N})\mathcal{P}_{\zeta}\psi(x_{1}, x_{2}, \dots, x_{N}), \tag{S47}$$

for any  $\psi(x_1, x_2, \dots, x_N) \in \mathcal{H}^{\otimes N}$ . Therefore, we conclude that

$$[\mathcal{P}_{\zeta}, O] = 0. \tag{S48}$$

## B. Proof of the block-diagonal structure of permutation-invariant observables.

**Proposition.** A permutation-invariant observable is block diagonal on  $\mathcal{H}_{\zeta}$  and  $\mathcal{H}_{\zeta}^{\perp}$ , where  $\mathcal{H}_{\zeta}^{\perp}$  is the orthogonal complement of  $\mathcal{H}_{\zeta}$  on  $\mathcal{H}^{\otimes N}$ .

*Proof.* We denote the projector onto  $\mathcal{H}_{\zeta}^{\perp}$  as  $\mathcal{P}_{\zeta}^{\perp}$ . Then, for a permutation-invariant observable O, according to Eq. (16), it can be easily seen

$$\mathcal{P}_{\zeta}\mathcal{O}\mathcal{P}_{\zeta}^{\perp} = \mathcal{O}\mathcal{P}_{\zeta}\mathcal{P}_{\zeta}^{\perp} = 0, \tag{S49}$$

$$\mathcal{P}_{\zeta}^{\perp} \mathcal{O} \mathcal{P}_{\zeta} = \mathcal{P}_{\zeta} \mathcal{P}_{\zeta}^{\perp} \mathcal{O} = 0, \tag{S50}$$

whence it follows that

$$O = \mathcal{P}_{\zeta} O \mathcal{P}_{\zeta} \oplus \mathcal{P}_{\zeta}^{\perp} O \mathcal{P}_{\zeta}^{\perp}, \tag{S51}$$

which concludes the proof.

## **C. Proof of Eq. (17)**

Eq. (17) is equivalent to the following proposition:

**Proposition.** If a permutation-invariant and local observable O vanishes on  $\mathcal{H}_{\zeta}$ , then it must vanish on the full Hilbert space  $\mathcal{H}^{\otimes N}$ .

*Proof.* Assuming  $\exists \psi(x_1, x_2, \dots, x_N) \in \mathcal{H}^{\otimes N}$ , which is not necessarily symmetric or anti-symmetric. We borrow inspiration from coordinate Bethe ansatz [6, 31] and construct another symmetric or anti-symmetric wave function  $\phi(x_1, x_2, \dots, x_N) \in \mathcal{H}_{\zeta}$ 

$$\phi(x_1, x_2, \dots, x_N) \equiv \sum_{\sigma} \zeta^{\sigma} \psi(x_{\sigma_1}, x_{\sigma_2}, \dots x_{x_{\sigma_N}}) \theta_{H}(x_{\sigma_N} - x_{\sigma_{N-1}}) \dots \theta_{H}(x_{\sigma_3} - x_{\sigma_2}) \theta_{H}(x_{\sigma_2} - x_{\sigma_1})$$
 (S52)

where  $\sigma$  is a permutation of the tuple  $(1, 2, \dots, N)$  and  $\theta_H(x)$  is the Heaviside function. It is worth to note that  $\phi$  is not  $\mathcal{P}_{\zeta}\psi$ . For example, if  $\psi \in \mathcal{H}_{\zeta}^{\perp}$ , then  $\mathcal{P}_{\zeta}\psi = 0$  while  $\phi \neq 0$  as long as  $\psi \neq 0$ . Since O vanishes on  $\mathcal{H}_{\zeta}$  therefore

$$O(x_1, x_2, \dots, x_N)\phi(x_1, x_2, \dots, x_N) = 0$$
 (S53)

which implies

$$O(x_1, x_2, \dots, x_N)\psi(x_{\sigma_1}, x_{\sigma_2}, \dots x_{\sigma_N}) = 0$$
 (S54)

for  $x_{\sigma_1} < x_{\sigma_2} < \cdots < x_{\sigma_N}$ . Eq. (S54) implies that

$$O(x_1, x_2, \dots, x_N)\psi(x_1, x_2, \dots, x_N) = 0$$
 (S55)

for  $x_1 < x_2 < \cdots < x_N$ . However, our goal is to show it holds in all the regions. To reach this goal, let us take advantage of permutation invariance. Note that r.h.s. of Eq. (S54) implies the  $x_i$ 's on l.h.s. are dummy indices. Therefore, Eq. (S54) is equivalence as

$$O(x_{\sigma_1^{-1}}, x_{\sigma_2^{-1}}, \dots, x_{\sigma_N^{-1}})\psi(x_1, x_2, \dots x_N) = O(x_1, x_2, \dots, x_N)\psi(x_1, x_2, \dots x_N) = 0$$
(S56)

where  $\sigma^{-1}$  is the inverse permutation of  $\sigma$ . So far, we have shown that Eq. (S55) holds on *all* the allowed regions of  $(x_1, x_2, \dots x_N)$ . On the boundary where at least two of the coordinates coincide, locality of O implies that the value of  $O\psi$  on the boundary is full determined by the its value on the neighborhood outside the boundary, where  $O\psi$  vanishes. Thus,  $O\psi$  must also vanishes on the boundary.

It can be shown that [7]

$$[\pi_i, \, \pi_j] = \sum_{k \neq i, \, j} V_{ijk} (M_{ijk} - M_{jik}). \tag{S57}$$

Thus  $(M_{ijk} - M_{jik})\mathcal{P}_{\zeta} = 0$  and we obtain

$$[\pi_i, \, \pi_i] \mathcal{P}_{\zeta} = 0. \tag{S58}$$

Furthermore  $[\pi_i, \pi_j]$  is local since they only involve at most second derivative of the coordinate. However, it does not vanish on the whole Hilbert space  $\mathcal{H}^{\otimes N}$  because  $[\pi_i, \pi_j]$  is not permutation-invariant. So here we can see for a local, but not permutation-invariant operator, even it vanishes on  $\mathcal{H}_{\zeta}$ , it may not vanish on  $\mathcal{H}^{\otimes N}$ .

## D. Effective 1-body operator

The operator

$$W_i = i \sum_{k \neq i} V_{ik} M_{ik} \tag{S59}$$

acts an effectively 1-body potential, in the sense that it satisfies the exchange rule for 1-body operators, even if it involves N degrees of freedom! Indeed,

$$M_{ij}W_{j} = i \sum_{k \neq j} M_{ij}V_{jk}M_{jk} = i \sum_{k \neq i,j} M_{ij}V_{jk}M_{jk} + iM_{ij}V_{ji}M_{ji}$$

$$= i \sum_{k \neq i,j} V_{ik}M_{ik}M_{ij} + iV_{ij}M_{ij}M_{ij}$$

$$= i \sum_{k \neq i} V_{ik}M_{ik}M_{ij} = W_{i}M_{ij}.$$
(S60)

Further,

$$M_{ij}W_{k} = i \sum_{l \neq k} M_{ij}V_{kl}M_{kl} = i \sum_{l \neq k, i, j} M_{ij}V_{kl}M_{kl} + iM_{ij}V_{ki}M_{ki} + iM_{ij}V_{ki}M_{ki}$$

$$= i \sum_{l \neq k, i, j} V_{kl}M_{kl}M_{ij} + iV_{ki}M_{ki}M_{ij} + iV_{kj}M_{kj}M_{ij}$$

$$= i \sum_{l \neq k} V_{kl}M_{kl}M_{ij} = W_{k}M_{ij}.$$
(S61)

Therefore,  $\pi_i$ ,  $a_i$ ,  $a_i^{\dagger}$  and  $h_i$  act all as effectively 1-body operators, i.e., they satisfy the exchange rules of a 1-body operator.

### V. GUIDE TO DISCOVER NEW INTEGRABLE SYSTEMS

The systematic investigation of models in the PHJ family [25–28] has been limited to date given that the resulting models were expected to be, at best, quasi exactly solvable, i.e., models in which only part of the spectrum can be determined. Our work establishes the equivalence between the PHJ and EOF family and proves the integrability of the PHJ models by identifying the corresponding integrals of motion. The identification of new integrable models in this family is straightforward. It suffices to choose a prepotential V(x) or pair function f(x), compute their derivatives, and use Eq. (3) or Eq. (5) in the main text to determine the Hamiltonian of the model in the real line. Similarly, Eq. (9) or Eq. (12) in the main text readily gives the Hamiltonian of the model in the presence of an external potential. For the convenience of the reader, we shall give the general integrable Hamiltonian in the EOF and PHJ family explicitly and then provide a user guide to discover new integrable systems. The general integrable Hamiltonian reads

$$H = \sum_{i} \left( \frac{p_i^2}{2m} + U_i \right) + \frac{1}{m} \left[ \sum_{i < j} \left( \zeta \hbar V'_{ij} + V_{ij}^2 - \zeta \sqrt{2m} V_{ij} [W_i - W_j] \right) - \sum_{i < j < k} V_{ijk} \right], \tag{S62}$$

or in term of the Jastrow wave function it reads

$$H = \sum_{i} \left( \frac{p_i^2}{2m} + U_i \right) + \frac{\hbar^2}{m} \left[ \sum_{i \in I} \left( \frac{f_{ij}''}{f_{ij}} + (v_i' - v_j') \frac{f_{ij}'}{f_{ij}} \right) + \sum_{i \in I \in I} \left( \frac{f_{ij}' f_{ik}'}{f_{ij} f_{ik}} - \frac{f_{ij}' f_{jk}'}{f_{ij} f_{jk}} + \frac{f_{ik}' f_{jk}'}{f_{ik} f_{jk}} \right) \right], \tag{S63}$$

where  $V_{ij} = V(x_i - x_j)$  is an odd function,  $U_i = U(x_i)$ ,  $W_i = W(x_i)$ ,  $V_{ijk} = V_{ij}V_{jk} + V_{jk}V_{ki} + V_{ki}V_{ij}$ , and

$$V(x) = \zeta \hbar \frac{f'(x)}{f(x)},\tag{S64}$$

$$W(x) = -\frac{\hbar}{\sqrt{2m}}v'(x),\tag{S65}$$

$$U(x) = W^{2}(x) - \frac{\hbar}{\sqrt{2m}}W'(x),$$
 (S66)

$$=\frac{\hbar^2}{2m}\left[(v_i')^2 + v_i''\right]. \tag{S67}$$

The wave function associated with Eq. (S62) is

$$\Psi_0 = \prod_i \exp(\nu_i) \prod_{i < j} f_{ij} \tag{S68}$$

$$= \exp\left(-\frac{\sqrt{2m}}{\hbar} \sum_{i} \int_{i < j}^{x_i} W(y) dy\right) \prod_{i < j} \exp\left[\hbar \int_{i}^{x_{ij}} dy V(y)\right]. \tag{S69}$$

In some situations, one may prefer to choose the pair function f(x) or the prepotential V(x) such that the three-body potential  $V_{ijk}$  reduces to two-body potential. It has been shown by Calogero [30] that in this case V(x) must take the form

$$V(x) = \alpha \zeta(x; g_2, g_3) + \beta x, \tag{S70}$$

where  $\alpha$ ,  $\beta$ ,  $g_2$ ,  $g_3$  are constants and  $\zeta(x; g_2, g_3)$  is the Weierstrass zeta-function (see e.g., Chap 13 of [61]). In this case, for x + y + z = 0, one can find

$$V(x)V(y) + V(y)V(z) + V(z)V(x) = Q(x) + Q(y) + Q(z),$$
(S71)

where

$$Q(x) = \frac{1}{2} \left[ \alpha \beta - \alpha V'(x) - V^2(x) \right]. \tag{S72}$$

One check that V(x) in Eq. (870) is odd in x and thus Q(x) is even in x. Therefore (862) becomes [24]

$$H = \sum_{i} \frac{p_i^2}{2m} + \sum_{i} U_i + \frac{1}{m} \sum_{i \le i} \left[ \zeta \hbar V'_{ij} + V_{ij}^2 - (N - 2)Q_{ij} - \zeta \sqrt{2m} V_{ij} (W_i - W_j) \right], \tag{S73}$$

where N is the number of particles. The general solution Eq. (S70) immediately implies the following property of V(x):

**Lemma 2.** If V(x) satisfies (S71), so does  $V(x) - \beta x$  [30]. That is,

$$V(x) \to V(x) - \beta x,$$
 (S74)

$$Q(x) \to Q(x) + \beta x V(x) - \frac{1}{2} \beta^2 x^2.$$
 (S75)

Now we are in a position to give the guidelines for constructing an integrable Hamiltonian:

- 1. As we have mentioned in the main text, the Hamiltonian (S62) is always integrable regardless of the normalizability of f(x). When f(x) is not normalizable, the Jastrow wave function (S68) is no longer the ground state wave function of Eq. (S62). However, Eq. (S62) is always legitimate.
- 2. The only requirement on f(x) is that it has well-defined parity; it should be an even or odd function. Both parities lead to an odd prepotential V(x). One can either choose f(x) or V(x) as the starting point for constructing the Hamiltonian.
- 3. When restricting particles in a ring with circumference L, all the potentials must be periodic in L, which dictates that f(x) or V(x) must also have period L. W(x) and v(x) are also required to have the periodicity L.

Following the above guidelines, one should be able to construct an unlimited number of integrable models, as we now illustrate with examples. In what follows, we shall use the notation  $x_{ij} = x_i - x_j$  and  $\bar{x}_{ij} = (x_i + x_j)/2$  so that the many-body Hamiltonian is written in a compact way.

## A. Recovering the integrability of well-known models

Now we show that the PHJ-EOF family includes canonical examples of one-dimensional integrable systems. Thus their integrability is proven immediately with Theorem 2 in the main text.

1. Calogero model with and without a trap

We shall employ Eq. (S63) to generate the model and consider

$$v(x) = -\frac{1}{2} \frac{m\omega}{\hbar} x^2, \tag{S76}$$

$$f(x) = |x|^{\lambda}, \lambda \in [0, \infty). \tag{S77}$$

As we have mentioned in the guidelines, f(x) is unnormalizable since the configuration space is now unbounded. But the integrability of the parent Hamiltonian is not affected. Here, we also would like to emphasize that since the absolute value  $|\cdot|$  is involved, care needs to be taken when taking derivatives. It is straightforward to calculate

$$\frac{d}{dx}|F(x)| = \frac{d}{dx}\left\{F(x)\operatorname{sgn}[F(x)]\right\} = F'(x)\operatorname{sgn}[F(x)] + 2F'(x)F(x)\delta[F(x)]. \tag{S78}$$

We note the following identities

$$F(x)\delta[F(x)] = 0, \ x \neq \xi_k,\tag{S79}$$

$$\int F(x)F'(x)\delta[F(x)]dx = \sum_{k} \int z\delta(z)dz = 0,$$
(S80)

where  $\xi_k$  is the zeros of F(x) and z = F(x). Thus, one can think of  $F(x)\delta[F(x)] = 0$  at all points, as long as it is not multiplied by a factor that is singular at the zeros of F(x). Therefore, we obtain

$$\frac{d}{dx}|F(x)| = F'(x)\operatorname{sgn}[F(x)],\tag{S81}$$

and using Eq. (\$81), we find

$$f'(x) = \lambda |x|^{\lambda - 1} \operatorname{sgn}(x). \tag{S82}$$

As for the second derivative of the pair function, its explicit evaluation yields

$$f''(x) = \lambda |x|^{\lambda} \left[ \frac{(\lambda - 1)}{|x|^2} + \frac{2\delta(x)}{|x|} \right]. \tag{S83}$$

According to Eq. (S64), we find

$$V(x) = \zeta \hbar \frac{\lambda}{|x|} \operatorname{sgn}(x) = \zeta \hbar \frac{\lambda}{x}.$$
 (S84)

Furthermore, we notice that when  $\lambda \neq 1$ 

$$\frac{(\lambda - 1)}{|x|^2} + \frac{2\delta(x)}{|x|} = \frac{1}{|x|^2} [(\lambda - 1) + 2|x|\delta(x)] = \frac{\lambda - 1}{|x|^2},$$
(S85)

where we have used Eq. (S81) to conclude that  $|x|\delta(x)$  is negligible, in comparison with the  $\lambda - 1$  term, whenever the latter is not zero. To summarize,

$$f''(x) = \begin{cases} \frac{\lambda(\lambda - 1)}{|x|^2} & \lambda \in [0, 1) \cup (1, \infty), \\ \frac{2\delta(x)}{|x|} & \lambda = 1. \end{cases}$$
 (S86)

One can further check that the three-body potential vanishes and that the long-range potential induced by the external harmonic potential is also a constant. Thus we find that the parent Hamiltonian (S63) reads

$$H = \sum_{i} \left( \frac{p_i^2}{2m} + \frac{1}{2} m \omega_0^2 x_i^2 \right) - E_0 + \begin{cases} \frac{\hbar^2}{m} \frac{\lambda(\lambda - 1)}{|x_{ij}|^2} & \lambda \in [0, 1) \cup (1, \infty), \\ \frac{\hbar^2}{m} \frac{2\delta(x_{ij})}{|x_{ij}|} & \lambda = 1, \end{cases}$$
(S87)

where

$$E_0 = \frac{N\hbar\omega}{2} + \frac{N(N-1)\lambda\hbar\omega}{2}.$$
 (S88)

The first line of Eq. (S87) in the literature is usually referred to as the (rational) Calogero model [22]. To the best of our knowledge, the careful treatment of the Calogero model in the case  $\lambda = 1$  was first discussed by Mathieu Beau in 2017 [56].

## 2. Tonks-Girardeau gas, describing 1D hard-core bosons in a trap

The Tonks-Girardeau regime describes one-dimensional hardcore bosons [34, 35]. Consider the value  $\lambda = 1$  in Eq. (S77). The interaction between the particles according to Eq. (S87) is set by  $\delta(x_{ij})/|x_{ij}|$ . This interaction is equivalent to a repulsive delta interaction with infinite strength, which describes the hard-core potential in the Tonks-Girardeau gas. In this case, the ground state wave function becomes according to Eq. (S68)

$$\Psi_0 = \exp\left(-\frac{m\omega_0}{2\hbar} \sum_i x_i^2\right) \prod_{i < i} |x_{ij}|,\tag{S89}$$

in agreement with [35]. The corresponding ground state energy is  $E_0 = \frac{N\hbar\omega}{2} + \frac{N(N-1)\hbar\omega}{2}$ .

We shall employ Eq. (S63) to generate the model and start with

$$f(x) = \left| \sin \left( \frac{\pi x}{L} \right) \right|^{\lambda},\tag{S90}$$

$$v(x) = 0. (S91)$$

According to Eq. (S81), we find

$$f'(x) = \frac{\lambda \pi}{L} \left| \sin\left(\frac{\pi x}{L}\right) \right|^{\lambda - 1} \cos\left(\frac{\pi x}{L}\right) \operatorname{sgn}\left[ \sin\left(\frac{\pi x}{L}\right) \right], \tag{S92}$$

and

$$V(x) = \zeta \hbar \frac{f'(x)}{f(x)} = \zeta \hbar \frac{\lambda \pi}{L} \cot\left(\frac{\pi x}{L}\right). \tag{S93}$$

Eq. (S63) becomes the Sutherland model for  $\lambda \in [0, 1)$ ,

$$H = \sum_{i} \frac{p_i^2}{2m} + \frac{\hbar^2}{m} \left(\frac{\pi}{L}\right)^2 \left[\sum_{i < j} \frac{\lambda(\lambda - 1)}{\sin^2\left(\frac{\pi x_{ij}}{L}\right)}\right] - E_0, \tag{S94}$$

with the limiting case  $\lambda \to 1$  reduces to the Tonks-Girardeau gas on a ring, which is similar with Eq. (S87), where

$$E_0 = \left(\frac{\pi}{L}\right)^2 \frac{\lambda^2 \hbar^2 N(N^2 - 1)}{6m}.$$
 (S95)

4. Lieb-Liniger gas

The Lieb-Liniger model reads [44, 45]

$$H = \sum_{i} \frac{p_i^2}{2m} + \frac{\hbar^2}{m} \left[ \sum_{i \le i} 2g\delta(x_{ij}) \right] - E_0,$$
 (S96)

where

$$E_0 = -\frac{\hbar^2 g^2 N(N^2 - 1)}{6m}. ag{S97}$$

We shall employ Eq. (S63) to generate Eq. (S96) and consider [27],

$$f(x) = \exp(g|x|),\tag{S98}$$

$$v(x) = 0, (S99)$$

which leads to

$$V(x) = g \operatorname{sgn}(x). \tag{S100}$$

Since sgn(x)sgn(y) + sgn(y)sgn(z) + sgn(z)sgn(x) = -1 for x + y + z = 0, the three-body potential is a constant. The two-body potential term is the delta potential.

### B. Predict the integrability of existing quasi-solve models or new models

Embedding the Lieb-Liniger gas in a harmonic trap will lead to the long-range Lieb-Liniger model [27, 46]. However, only its ground state property is discussed previously and its integrability is not clear in the previous literature. Theorems proved in the main text show that the long-range Lieb-Liniger model is not only quasi-solvable but also integrable. Below we give more applications of these theorems and show that a variety of related models are actually integrable.

According to Lemma 2, one can shift Eq. (\$100) by a linear function and take

$$V(x) = \zeta \hbar [g \operatorname{sgn}(x) - \beta x]. \tag{S101}$$

Setting W(x) = 0, we find

$$Q(x) = \hbar^2 \left[ -\frac{g^2 N(N-1)}{6} + g\beta |x| - \frac{1}{2}\beta^2 x^2 \right],$$
 (S102)

and Eq. (S62) becomes

$$H = \sum_{i} \frac{p_i^2}{2m} + \frac{\hbar^2}{m} \sum_{i < j} \left[ 2g\delta(x_{ij}) - Ng\beta|x_{ij}| + \frac{N\beta^2}{2} x_{ij}^2 \right] - E_0, \tag{S103}$$

where

$$E_0 = \frac{\hbar^2}{m} \left[ \frac{N(N-1)\beta}{2} - \frac{g^2 N(N^2 - 1)}{6} \right]. \tag{S104}$$

This Hamiltonian was first constructed by Calogero [30] with the Jastrow wave function. Its integrability was not clear previously. Theorem 2 in the main text shows that this Hamiltonian is also integrable.

2. Long-Range Hybrid LL model

Considering the prepotential (\$101) together with

$$W(x) = \sqrt{\frac{m}{2}}\omega x,\tag{S105}$$

one finds Eq. (\$73) becomes

$$H = \sum_{i} \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 \sum_{i} x_i^2 - E_0$$
 (S106)

$$+\frac{\hbar^2}{m}\sum_{i\neq j}\left[2g\delta(x_{ij})-g\left(N\beta+\frac{m\omega}{\hbar}\right)|x_{ij}|+\beta\left(\frac{N\beta}{2}+\frac{m\omega}{\hbar}\right)x_{ij}^2\right],\tag{S107}$$

where

$$E_0 = \frac{N\hbar\omega}{2} - \frac{\hbar^2 g^2 N(N^2 - 1)}{6m}.$$
 (S108)

Eq. (S107) is a further generalization of the Long-Range LL model discussed in the main text and Refs. [27, 46], which we discuss for the first time here. This model is also integrable, according to Theorem 2 in the main text. Interestingly, upon taking

$$\beta = -\frac{m\omega}{N\hbar},\tag{S109}$$

the long-range Coulomb interaction vanishes and Eq. (S107) simplifies to

$$H = \sum_{i} \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 \sum_{i} x_i^2 + \frac{\hbar^2}{m} \sum_{i < i} 2g\delta(x_{ij}) - \frac{m\omega^2}{2N} \sum_{i < i} x_{ij}^2 - E_0.$$
 (S110)

The above Hamiltonian describes the Lieb-Liniger gas in a harmonic trap with quadratic interaction between the particles. Again, it is integrable according to Theorem 2 in the main text.

The same holds true if one takes

$$W(x) = -\sqrt{\frac{m}{2}}\omega x,\tag{S111}$$

which leads to an unnormalizable wave function according to Eq. (S69). However, as we have mentioned the integrability of the Hamiltonian is not affected by the normalization of the wave function. Thus, the sign of  $\omega$  does not matter.

Since we know  $V(x) = \zeta \hbar \lambda a \coth(ax)$  generates the hyperbolic model [7]. According to Lemma 2, we take

$$V(x) = \zeta \hbar [\lambda a \coth(ax) - bx], \tag{S112}$$

and find

$$Q(x) = \hbar^2 \left[ \lambda abx \coth(ax) - \frac{1}{2}b^2x^2 + \text{constant} \right].$$
 (S113)

Following the guidelines, we find in the absence of external potential (i.e.,  $W_i = 0$ ), Eq. (S73) becomes

$$H = \sum_{i} \frac{p_i^2}{2m} + \frac{\hbar^2}{m} \sum_{i < j} \left[ \frac{a^2 \lambda (\lambda - 1)}{\sinh^2(ax_{ij})} - \lambda N a b x_{ij} \coth(ax_{ij}) + \frac{1}{2} N b^2 x_{ij}^2 \right] - E_0, \tag{S114}$$

where

$$E_0 = \frac{\hbar^2}{m} \left[ \frac{bN(N-1)}{2} - \frac{\lambda^2 a^2 N(N^2 - 1)}{6} \right].$$
 (S115)

This model was discussed by Calogero [30] and was only known as quasi-solvable previously, i.e., only the ground state wave function is known. Here, Theorem 2 in the main text indicates that this Hamiltonian is also integrable.

## 4. Sutherland model in a trigonometric trap

To the best of our knowledge, the possibility of embedding a periodic potential for the Sutherland model has not been yet discussed in the literature. We take this step with the PHJ-EOF approach. We shall employ Eq. (S63) and take f(x) to be Eq. (S90) and

$$v(x) = -\frac{m\omega}{2\hbar} \left(\frac{L}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{L}\right),\tag{S116}$$

which yields

$$U(x) = -\frac{\hbar\omega}{2}\cos\left(\frac{2\pi x}{L}\right) - \frac{m\omega^2 L^2}{16\pi^2}\cos\left(\frac{4\pi x}{L}\right) + \frac{m\omega^2 L^2}{16\pi^2}.$$
 (S117)

We have computed before f'(x)/f(x) in Eq. (S93). According to Eq. (S63), we find

$$H = \sum_{i} \frac{p_{i}^{2}}{2m} - \sum_{i} \left[ \frac{\hbar\omega}{2} \cos\left(\frac{2\pi x_{i}}{L}\right) + \frac{m\omega^{2}L^{2}}{16\pi^{2}} \cos\left(\frac{4\pi x_{i}}{L}\right) \right] + \frac{\hbar^{2}}{m} \sum_{i < j} \left[ \left(\frac{\pi}{L}\right)^{2} \frac{\lambda(\lambda - 1)}{\sin^{2}\left(\frac{\pi x_{ij}}{L}\right)} - \frac{\lambda m\omega}{\hbar} \cos\left(\frac{\pi x_{ij}}{L}\right) \cos\left(\frac{2\pi \bar{x}_{ij}}{L}\right) \right] - E_{0},$$
(S118)

where

$$E_0 = \left(\frac{\pi}{L}\right)^2 \frac{\lambda^2 \hbar^2 N(N^2 - 1)}{6m} - \frac{Nm\omega^2 L^2}{16\pi^2}.$$
 (S119)

In the thermodynamic limit L,  $N \to \infty$  with N/L kept fixed, so that one can readily check Eq. (S118) reduces to the Calogero model in a harmonic trap. Thus Eq. (S118) can be viewed as the generalization of the embedded Calogero model to the embedded the Sutherland model. Again, if one flips the sign of the frequency  $\omega$ , the integrability is preserved.

### C. Fixing interactions first: Toda-like interactions in the continuum

The above examples have been found in the PHJ-EOF family by choosing the pair function f(x). As an alternative, one can twist the construction around, by fixing the interactions first. For the sake of illustration, let us consider a two-body potential that decays exponentially with the interparticle distance over a length scale  $\ell$ , e.g.,

$$V_2 = \frac{\hbar^2 g}{m} \sum_{i < j} e^{-|x_{ij}|/\ell}.$$
 (S120)

This potential is a generalization to the continuum of the Toda interactions and can also be considered as a low-density approximation to the hyperbolic potential, e.g., as discussed in [14, 27]. Thus, we expect

$$\frac{f''(x)}{f(x)} = ge^{-|x|/\ell} + \text{singular terms.}$$
 (S121)

Ignoring the singular terms for the moment, this differential equation for x > 0 admits as a specific solution

$$f(x) = I_0 \left( 2\sqrt{g} l e^{-x/\ell} \right), \tag{S122}$$

where  $I_{\alpha}(x)$  is the modified Bessel functions of first kind and order  $\alpha$ . This solution motivates the choice

$$f(x) = I_0 \left( 2\sqrt{g} l e^{-|x|/\ell} \right) \tag{S123}$$

for all x. This pair function decays smoothly as function of x to unit value. To compute f''(x)/f(x), care must be taken for the absolute value. Using Eq. (S81), we can readily calculate

$$\frac{dI_{0}\left(2\sqrt{g}le^{-|x|/\ell}\right)}{dx} = \frac{dI_{0}\left(2\sqrt{g}le^{-|x|/\ell}\right)}{d|x|} \operatorname{sgn}(x) 
= -2e^{-|x|/\ell} \sqrt{g}I_{1}\left(2\sqrt{g}le^{-|x|/\ell}\right) \operatorname{sgn}(x), \tag{S124}$$

$$\frac{d^{2}I_{0}\left(2\sqrt{g}le^{-|x|/\ell}\right)}{dx^{2}} = \frac{d^{2}I_{0}\left(2\sqrt{g}le^{-|x|/\ell}\right)}{d^{2}|x|} + \frac{dI_{0}\left(2\sqrt{g}le^{-|x|/\ell}\right)}{d|x|} \delta(x) 
= \frac{d^{2}I_{0}\left(2\sqrt{g}le^{-|x|/\ell}\right)}{d^{2}|x|} - 2e^{-|x|/\ell} \sqrt{g}I_{1}\left(2\sqrt{g}le^{-|x|/\ell}\right) \delta(x). \tag{S125}$$

Thus we find

$$\frac{f'(x)}{f(x)} = -2\sqrt{g}e^{-|x|/\ell} \frac{I_1(2\sqrt{g}le^{-|x|/\ell})}{I_0(2\sqrt{g}le^{-|x|/\ell})} \operatorname{sgn}(x),$$
 (S126)

$$\frac{f''(x)}{f(x)} = ge^{-|x|/\ell} - c\delta(x),\tag{S127}$$

(S128)

where

$$c = \frac{2\sqrt{g}I_1\left(2\sqrt{g}\ell\right)}{I_0\left(2\sqrt{g}\ell\right)}.$$
 (S129)

The corresponding three-body potential is nonzero, as expected, and takes the form given in Eq. (S63) with f'(x)/f(x) given by Eq. (S126). The expression of f'(x)/f(x) sets the generalized momenta  $\pi_i$  through Eq. (7) in the main text and the corresponding integrals of motion  $I_n$ . The resulting Hamiltonian thus takes the form

$$H = \sum_{i} \frac{p_i^2}{2m} + \frac{\hbar^2}{m} \sum_{i < j} \left[ g e^{-|x_{ij}|/\ell} - c \delta(x_{ij}) \right] + V_3.$$
 (S130)

Given I(0) = 1, i.e., the wave function does not decay when particles are far apart, the Jastrow wave function is not normalizable in the in absence of an external trap. We thus consider the case where system is trapped. The ground state of the trapped system is then

$$\Psi_0 = e^{-\frac{m\omega}{2\hbar} \sum_{i=1}^{N} x_i^2} \prod_{i < j} I_0[2\ell \sqrt{g} \exp(-|x_{ij}|/\ell)].$$
 (S131)

The long-range two-body potential due to the embedding of the external harmonic trap in Eq. (S63) is similarly given in terms of the modified Bessel functions.