

Efimov effect for two particles on a semi-infinite line

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(Dated: September 19, 2022)

Abstract

The Efimov effect (in a broad sense) refers to the onset of a geometric sequence of many-body bound states as a consequence of the breakdown of continuous scale invariance to discrete scale invariance. While originally discovered in three-body problems in three dimensions, the Efimov effect has now been known to appear in a wide spectrum of many-body problems in various dimensions. Here we introduce a simple, exactly solvable toy model of two identical bosons in one dimension that exhibits the Efimov effect. We consider the situation where the bosons reside on a semi-infinite line and interact with each other through a pairwise δ -function potential with a particular position-dependent coupling strength that makes the system scale invariant. We show that, for sufficiently attractive interaction, the bosons are bound together and a new energy scale emerges. This energy scale breaks continuous scale invariance to discrete scale invariance and leads to the onset of a geometric sequence of two-body bound states. We also study the two-body scattering off the boundary and derive the exact reflection amplitude that exhibits a log-periodicity. This article is intended for students and non-specialists interested in discrete scale invariance.

1 Introduction

In his seminal paper in 1970, Efimov considered three identical bosons with short-range pairwise interactions [1]. He pointed out that, when the two-body scattering length diverges, an infinite number of three-body bound states appear, with energy levels $\{E_n\}$ forming a geometric sequence. This phenomenon—generally known as the Efimov effect—has attracted much attention because the ratio $E_{n+1}/E_n \approx 1/(22.7)^2$ is independent of the details of the interactions as well as of the nature of the particles: it is universal. More than thirty-five years after its prediction, this effect was finally observed in cold atom experiments [2–6], which has triggered an explosion of research on the Efimov effect. For more details, see the reviews [7–11]. (See also Refs. [12–14] for a more elementary exposition.)

Aside from its universal eigenvalues ratio, the Efimov effect takes its place among the greatest theoretical discoveries in modern physics because it was the first quantum many-body phenomenon to demonstrate *discrete scale invariance*—an invariance under enlargement or reduction in the system size by a single scale factor [15]. It is now known that the emergence of a geometric sequence in the bound states’ discrete energies is associated with the breakdown of continuous scale invariance to discrete scale invariance [16], and can be found in a wide spectrum of quantum many-body problems in various dimensions [17–23]. The notion of the Efimov effect has therefore now been broadened to include those generalizations, so that its precise meaning varies in the literature. In the present paper, we will use the term “Efimov effect” to simply refer to the onset of a geometric sequence in the energies of many-body bound states as a consequence of the breakdown of continuous scale invariance to discrete scale invariance.

To date, there exist several theoretical approaches to study the Efimov effect. The most common approach is to directly analyze the many-body Schrödinger equation, which normally involves the use of Jacobi coordinates, hyperspherical coordinates, the adiabatic approximation, and the Faddeev equation [7]. Another popular approach is to use second quantization, or quantum field theory [8]. Though the problem itself is conceptually simple, it is hard for students and non-specialists to master these techniques and to work out the physics of the Efimov effect. The essential part of this phenomenon, however, can be understood from undergraduate-level quantum mechanics without using any fancy techniques.

This paper is aimed at introducing a simple toy model for a two-body system that exhibits the Efimov effect. We consider two identical bosons on the half-line $\mathbb{R}_+ = \{x : x \geq 0\}$ with a pairwise δ -function interaction. The Hamiltonian of such a system is given by

$$H = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + g(x_1)\delta(x_1 - x_2), \quad (1)$$

where m is the mass of each particle and $x_j \in \mathbb{R}_+$ ($j = 1, 2$) is the coordinate of the j th particle. Here $g(x)$ is a coupling strength. In this paper, we will focus on the position-dependent coupling strength that satisfies the scaling law $g(e^t x) = e^{-t} g(x)$, where t is an arbitrary real number. Notice that, up to an overall constant factor, this scaling law has a unique solution $g(x) \propto 1/x$. For the following discussion, it is convenient to choose

$$g(x) = \frac{\hbar^2 g_0}{m x}, \quad (2)$$

where g_0 is a dimensionless real number that can either be positive or negative. Physically, Eq. (2) models the situation where the interaction strength becomes stronger as the particles come closer to the boundary $x_1 = x_2 = 0$ (see Fig. 1). This two-body interaction is essentially equivalent to the so-called *scaling trap* introduced in Ref. [19], where the Efimov effect was discussed in the context of two non-identical particles on the whole line \mathbb{R} . As we will see shortly, our two-identical-particle problem on \mathbb{R}_+ enjoys simple solutions and is more tractable than the corresponding two-non-identical-particle problem on \mathbb{R} .

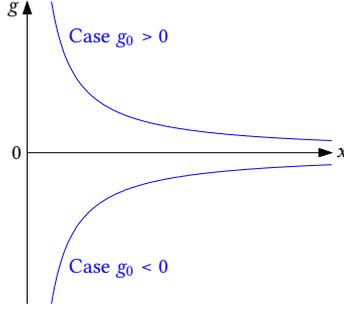


Figure 1: Position dependence of the coupling strength $g(x)$.

The rest of the paper is devoted to the detailed analysis of the eigenvalue problem of H . Before going into details, however, it is worth summarizing the symmetry properties of the model. Of particular importance are the following:

- *Permutation invariance.* Thanks to the relation $g(x_1)\delta(x_1 - x_2) = g(x_2)\delta(x_2 - x_1)$, the Hamiltonian (1) is invariant under the permutation of coordinates, $(x_1, x_2) \mapsto (x_2, x_1)$. Note that this permutation invariance is necessary for Eq. (1) to be a Hamiltonian of indistinguishable particles, where, for bosons, the two-body wavefunction should satisfy $\psi(x_1, x_2) = \psi(x_2, x_1)$. We will see in Sec. 3.1 that this invariance greatly simplifies the analysis.
- *Scale invariance.* Thanks to the relations $g(e^t x_1) = e^{-t} g(x_1)$ and $\delta(e^t x_1 - e^t x_2) = e^{-t} \delta(x_1 - x_2)$, the Hamiltonian (1) transforms as $H \mapsto e^{-2t} H$ under the scale transformation $(x_1, x_2) \mapsto (e^t x_1, e^t x_2)$. This transformation law has significant implications for the spectrum of H . Let $\psi_E(x_1, x_2)$ be a solution to the eigenvalue equation $H\psi_E(x_1, x_2) = E\psi_E(x_1, x_2)$. Then, $\psi_E(e^t x_1, e^t x_2)$ automatically satisfies $H\psi_E(e^t x_1, e^t x_2) = e^{2t} E\psi_E(e^t x_1, e^t x_2)$; that is, $\psi_E(e^t x_1, e^t x_2)$ is proportional to the eigenfunction $\psi_{e^{2t} E}(x_1, x_2)$ corresponding to the eigenvalue $e^{2t} E$. The proportionality coefficient can be determined by requiring that both ψ_E and $\psi_{e^{2t} E}$ be normalized. The result is the following scaling law:

$$\psi_{e^{2t} E}(x_1, x_2) = e^t \psi_E(e^t x_1, e^t x_2). \quad (3)$$

If this indeed holds for any $t \in \mathbb{R}$, $e^{2t} E$ can take any arbitrary (positive) value so that the spectrum of H is continuous. As we will see in Sec. 3.2, however, if g_0 is smaller than a critical value g_* , Eq. (3) holds only for some discrete $t \in t_* \mathbb{Z} = \{0, \pm t_*, \pm 2t_*, \dots\}$; that is, continuous scale invariance is broken to discrete scale invariance, defined by a characteristic scale t_* . As a consequence, there appears a geometric sequence of (negative) energy eigenvalues, $\{E_0, E_0 e^{\pm 2t_*}, E_0 e^{\pm 4t_*}, \dots\}$, where $E_0 (< 0)$ is a newly emergent energy scale. One of the goal of this paper is to show this result using only undergraduate-level calculus.

It should be noted that there is no translation invariance in our model: it is explicitly broken by the boundary at $x = 0$ as well as by the position-dependent coupling strength (2). This non-invariance means that the total momentum—the canonical conjugate of the center-of-mass coordinate—is not a well-defined conserved quantity. In other words, the two-body wavefunction cannot be of the separation-of-variable form $\psi(x_1, x_2) = e^{iPX/\hbar} \phi(x)$, where $X = (x_1 + x_2)/2$ is the center-of-mass coordinate, P the total momentum, $x = x_1 - x_2$ the relative coordinate, and ϕ the wavefunction of relative motion. In the next section, we will first introduce an alternative coordinate system that is more suitable for the two-body problem on the half-line \mathbb{R}_+ , before solving the problem in Sec. 3.

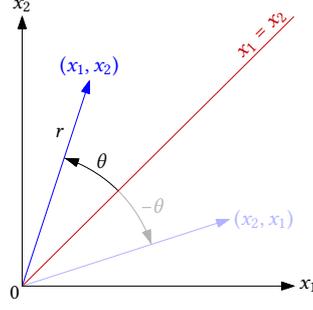


Figure 2: Polar coordinates (r, θ) in the (x_1, x_2) -space. The red line represents the set of two-body coincidence points.

2 Two-body problem without translation invariance

Let us first introduce a new coordinate system in the (x_1, x_2) -space. In what follows, we will work with the polar coordinate system (r, θ) defined as follows (see Fig. 2):

$$x_1 = r \cos(\theta + \frac{\pi}{4}), \quad (4a)$$

$$x_2 = r \sin(\theta + \frac{\pi}{4}), \quad (4b)$$

or, equivalently,

$$r = \sqrt{x_1^2 + x_2^2}, \quad (5a)$$

$$\theta = \frac{1}{2i} \log \left(\frac{x_1 + ix_2}{x_1 - ix_2} \right) - \frac{\pi}{4}, \quad (5b)$$

where $r \in [0, \infty)$ and $\theta \in [-\pi/4, \pi/4]$. Note that $\theta = 0$ and $\theta = \pm\pi/4$ correspond to the two-body coincidence point $x_1 = x_2$ and to the boundaries $x_1 = 0$ and $x_2 = 0$, respectively. Note also that the permutation $(x_1, x_2) \mapsto (x_2, x_1)$ corresponds to the parity transformation $(r, \theta) \mapsto (r, -\theta)$; see Fig. 2.

In the coordinate system (r, θ) , the kinetic energy part of the two-body Hamiltonian (1) takes the following form:

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \\ &= -\frac{\hbar^2}{2m} \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \\ &= \frac{\hbar^2}{2m} r^{-\frac{1}{2}} \left(-\frac{\partial^2}{\partial r^2} + \frac{-\partial_\theta^2 - \frac{1}{4}}{r^2} \right) r^{\frac{1}{2}}, \end{aligned} \quad (6)$$

where $\partial_\theta = \partial/\partial\theta$. Likewise, the potential energy part is rewritten as

$$\begin{aligned} V(x_1, x_2) &= \frac{\hbar^2 g_0}{m x_1} \delta(x_1 - x_2) \\ &= \frac{\hbar^2 g_0}{m r \cos(\theta + \frac{\pi}{4})} \delta(r \cos(\theta + \frac{\pi}{4}) - r \sin(\theta + \frac{\pi}{4})) \\ &= \frac{\hbar^2 g_0}{m \sqrt{2} r^2 \cos(\frac{\pi}{4})} \delta(\theta) \\ &= \frac{\hbar^2 g_0}{m r^2} \delta(\theta) \quad \text{for } \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]. \end{aligned} \quad (7)$$

In the third equality we have used the formula $\delta(f(\theta)) = (1/|f'(\theta_0)|)\delta(\theta - \theta_0)$, where $f(\theta) = r \cos(\theta + \pi/4) - r \sin(\theta + \pi/4)$, $f' = df/d\theta$, and θ_0 is a root of the equation $f(\theta) = 0$ for $\theta \in [-\pi/4, \pi/4]$ and given

by $\theta_0 = 0$. The total Hamiltonian $H = H_0 + V$ can then be cast into the following form:

$$H = \frac{\hbar^2}{2m} r^{-\frac{1}{2}} \left(-\frac{\partial^2}{\partial r^2} + \frac{\Delta_\theta - \frac{1}{4}}{r^2} \right) r^{\frac{1}{2}}, \quad (8)$$

where

$$\Delta_\theta = -\frac{\partial^2}{\partial \theta^2} + 2g_0 \delta(\theta). \quad (9)$$

Now we are ready to analyze the Schrödinger equation by means of the separation of variables. Suppose that the two-body wavefunction is of the form

$$\psi(x_1, x_2) = r^{-\frac{1}{2}} R(r) \Theta(\theta). \quad (10)$$

Let λ be an eigenvalue of the operator Δ_θ . Then, the time-independent Schrödinger equation $H\psi = E\psi$ can be reduced to the following set of differential equations:

$$\left(-\frac{d^2}{d\theta^2} + 2g_0 \delta(\theta) \right) \Theta(\theta) = \lambda \Theta(\theta), \quad (11a)$$

$$\left(-\frac{d^2}{dr^2} + \frac{\lambda - \frac{1}{4}}{r^2} \right) R(r) = \frac{2mE}{\hbar^2} R(r). \quad (11b)$$

The energy eigenvalues are determined by the inverse-square potential, which has been widely studied over the years in the context of quantum anomaly (symmetry breaking by quantization) or renormalization [24–31], and is known to support a geometric sequence of bound states if $\lambda < 0$ [32]. As we will see shortly, if g_0 is smaller than a critical value g_* , the lowest eigenvalue λ_0 in the eigenvalue equation (11a) becomes negative. Hence, in such a λ_0 -channel, continuous scale invariance can be broken down to discrete scale invariance, and the two-body bound-state energies follow a geometric sequence. Let us next see this by solving the differential equations (11a) and (11b) explicitly.

3 Two-body Efimov effect with boundary

3.1 Solution to the angular equation

Let us first solve the angular equation (11a). To this end, we need to specify the connection conditions at $\theta = 0$ and the boundary conditions at $\theta = \pm\pi/4$. We start with the connection conditions equivalent to the δ -function potential.

As is well known, the δ -function potential system (11a) is equivalent to the differential equation $-\Theta'' = \lambda\Theta$ for $\theta \neq 0$ with the following connection conditions at $\theta = 0$ [33]:

$$-\Theta'(0_+) + \Theta'(0_-) + g_0(\Theta(0_+) + \Theta(0_-)) = 0, \quad (12a)$$

$$\Theta(0_+) = \Theta(0_-), \quad (12b)$$

where the prime ($'$) indicates the derivative with respect to θ .

Let us next take into account the symmetry of the two-body wavefunction. Since we are dealing with identical bosons, the wavefunction must be symmetric under the permutation $\psi(x_1, x_2) = \psi(x_2, x_1)$.¹ In the polar coordinate system, this is equivalent to $\Theta(\theta) = \Theta(-\theta)$, whose derivative gives

¹For fermions, we have $\psi(x_1, x_2) = -\psi(x_2, x_1)$, which is equivalent to $\Theta(\theta) = -\Theta(-\theta)$. The connection conditions (12a) and (12b) are then reduced to the Dirichlet boundary conditions $\Theta(0_\pm) = 0$, in which case λ cannot be negative (if we impose $\Theta(\pm\pi/4) = 0$). Hence, for fermions, the Efimov effect cannot be realized with the Hamiltonian (1). In order to realize the Efimov effect for fermions with a pairwise contact interaction, one has to use the Hamiltonian $H = -(\hbar^2/(2m))(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2) + (\hbar^2/m)\varepsilon(x_1 - x_2; x_1/g_0)$, where ε is the so-called ε -function potential defined by $\varepsilon(x; c) = \lim_{a \rightarrow 0^+} (1/(2c) - 1/a)(\delta(x+a) + \delta(x-a))$. For simplicity, in this paper we will not touch upon the fermionic case. For more details of the ε -function potential, see Refs. [34, 35].

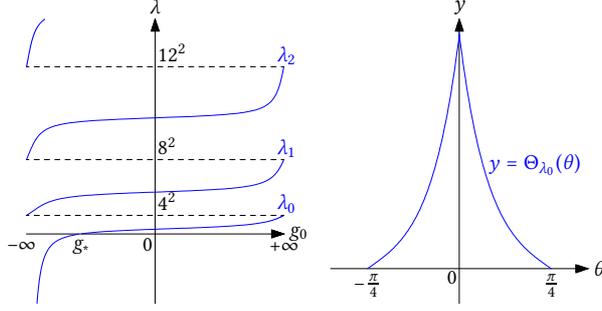


Figure 3: Left: g_0 dependence of $\{\lambda_0, \lambda_1, \lambda_2, \dots\}$. The lowest eigenvalue λ_0 becomes negative if g_0 goes below the critical value $g_* = -4/\pi$. Right: A typical profile of $\Theta_{\lambda_0}(\theta)$ for $g_0 < g_*$.

$\Theta'(\theta) = -\Theta'(-\theta)$. Hence, at the two-body coincidence point $\theta = 0$, there must hold the following additional conditions:

$$\Theta(0_+) = \Theta(0_-) \quad \text{and} \quad \Theta'(0_+) = -\Theta'(0_-). \quad (13)$$

Thus, for identical bosons, Eq. (12a) can be reduced to the following Robin boundary conditions [36]:

$$\mp \Theta'(0_{\pm}) + g_0 \Theta(0_{\pm}) = 0. \quad (14)$$

Let us now specify the boundary conditions at $\theta = \pm\pi/4$. For simplicity, we will impose the following Dirichlet boundary conditions:²

$$\Theta(\pm\frac{\pi}{4}) = 0. \quad (15)$$

Now it is straightforward to solve the angular equation (11a). Thanks to the property $\Theta(\theta) = \Theta(-\theta)$, it is sufficient to solve the differential equation $-\Theta'' = \lambda\Theta$ in the $0 \leq \theta \leq \pi/4$ region under the boundary conditions $-\Theta'(0_+) + g_0\Theta(0_+) = 0$ and $\Theta(\pi/4) = 0$. The resulting solution for $\lambda \neq 0$ is

$$\Theta_{\lambda}(\theta) = A_{\lambda} \sin\left(\sqrt{\lambda}\left(\frac{\pi}{4} - |\theta|\right)\right), \quad (16)$$

where A_{λ} is a normalization constant. Here λ is a root of the transcendental equation

$$g_0 = -\sqrt{\lambda} \cot\left(\frac{\pi}{4}\sqrt{\lambda}\right). \quad (17)$$

For $\lambda < 0$, the square root should be understood as $\sqrt{\lambda} = i\sqrt{|\lambda|}$. In this case, the angular wavefunction $\Theta_{\lambda}(\theta) \propto \sinh(\sqrt{|\lambda|}(\pi/4 - |\theta|))$ sharply localizes to the two-body coincidence point $\theta = 0$; that is, it describes a two-body bound state (see the right panel of Fig. 3).

Though the transcendental equation (17) cannot be solved analytically, the g_0 dependence of its solutions can be seen by plotting the graph of $g_0 = -\sqrt{\lambda} \cot(\pi\sqrt{\lambda}/4)$. As can be observed in the left panel of Fig. 3, the lowest eigenvalue λ_0 becomes negative for $g_0 < g_*$, where g_* is the critical value given by

$$g_* := \lim_{\lambda \rightarrow 0} \left[-\sqrt{\lambda} \cot\left(\frac{\pi}{4}\sqrt{\lambda}\right) \right] = -\frac{4}{\pi}. \quad (18)$$

Hence, in the λ_0 -channel, continuous scale invariance must be broken down to discrete scale invariance for $g_0 < g_*$. Let us next see this by solving the radial equation (11b).

3.2 Boundary-localized two-body Efimov states

From here on, we will consider the situation where $g_0 < g_*$, and focus on the case $E < 0$ in the channel $\lambda = \lambda_0 < 0$. In this case, there exists a square-integrable solution to the differential equation (11b)

²Alternatively, one can impose, e.g., the Neumann boundary conditions $\Theta'(\pm\pi/4) = 0$, in which case the critical value is $g_* = 0$.

whose asymptotic behavior as $r \rightarrow \infty$ is $R_{\kappa\lambda}(r) \rightarrow N_{\kappa\lambda} e^{-\kappa r}$, where $N_{\kappa\lambda}$ is a normalization constant and $\kappa = \sqrt{2m|E|/\hbar^2} > 0$. The full solution is given by³

$$R_{\kappa\lambda}(r) = N_{\kappa\lambda} \sqrt{\frac{2\kappa r}{\pi}} K_{iv}(\kappa r), \quad v = \sqrt{|\lambda|}, \quad (19)$$

where K_{iv} is the modified Bessel function of the second kind whose asymptotic behavior is $K_{iv}(\kappa r) \rightarrow \sqrt{\pi/(2\kappa r)} e^{-\kappa r}$ as $\kappa r \rightarrow \infty$. Note that Eq. (19) together with Eq. (16) describes a two-body wavefunction that localizes to $\theta = 0$ and $r = 0$; that is, it describes the two-body bound state that is localized to the boundary ($x_1 = x_2 = 0$).

It should be noted that at this stage κ is an arbitrary positive real constant. To determine its possible values, we follow the argument in Ref. [32] and require the orthonormality of the radial wavefunctions. Let $R_{\kappa\lambda}$ and $R_{\kappa'\lambda}$ be two distinct solutions to Eq. (11b). Then we have

$$-R_{\kappa\lambda}'' + \frac{\lambda - \frac{1}{4}}{r^2} R_{\kappa\lambda} = -\kappa^2 R_{\kappa\lambda}, \quad (20a)$$

$$-\overline{R_{\kappa'\lambda}''} + \frac{\lambda - \frac{1}{4}}{r^2} \overline{R_{\kappa'\lambda}} = -\kappa'^2 \overline{R_{\kappa'\lambda}}, \quad (20b)$$

where the overline ($\overline{}$) stands for the complex conjugate and the prime here indicates the derivative with respect to r . By multiplying $\overline{R_{\kappa'\lambda}}$ to Eq. (20a) and $R_{\kappa\lambda}$ to Eq. (20b) and then subtracting one from the other, we get

$$\begin{aligned} (-\kappa'^2 + \kappa^2) \overline{R_{\kappa'\lambda}} R_{\kappa\lambda} &= \overline{R_{\kappa'\lambda}} R_{\kappa\lambda}'' - \overline{R_{\kappa'\lambda}'} R_{\kappa\lambda}' \\ &= \frac{d}{dr} \left(\overline{R_{\kappa'\lambda}} R_{\kappa\lambda}' - \overline{R_{\kappa'\lambda}'} R_{\kappa\lambda} \right). \end{aligned} \quad (21)$$

By integrating both sides from $r = 0$ to ∞ , we find

$$\begin{aligned} (-\kappa'^2 + \kappa^2) \int_0^\infty dr \overline{R_{\kappa'\lambda}(r)} R_{\kappa\lambda}(r) &= \int_0^\infty dr \frac{d}{dr} \left(\overline{R_{\kappa'\lambda}(r)} R_{\kappa\lambda}'(r) - \overline{R_{\kappa'\lambda}'(r)} R_{\kappa\lambda}(r) \right) \\ &= -\lim_{r \rightarrow 0} \left(\overline{R_{\kappa'\lambda}(r)} R_{\kappa\lambda}'(r) - \overline{R_{\kappa'\lambda}'(r)} R_{\kappa\lambda}(r) \right) \\ &= \frac{2\sqrt{\kappa\kappa'} \sin(v \log \frac{\kappa}{\kappa'})}{\sinh(v\pi)} \overline{N_{\kappa'\lambda}} N_{\kappa\lambda}, \end{aligned} \quad (22)$$

where the second equality follows from $R_{\kappa\lambda}, R_{\kappa'\lambda} \rightarrow 0$ in the limit $r \rightarrow \infty$, and the last equality follows from the short-distance behavior of the modified Bessel function (see Eq. (A.4) in Appendix A).

Now, Eq. (22) enables us to determine the normalization constant as well as the energy eigenvalues. First, the normalization constant is determined by requiring that $R_{\kappa\lambda}$ have the unit norm:

$$\begin{aligned} 1 &= \int_0^\infty dr |R_{\kappa\lambda}(r)|^2 \\ &= \lim_{\kappa' \rightarrow \kappa} \int_0^\infty dr \overline{R_{\kappa'\lambda}(r)} R_{\kappa\lambda}(r) \\ &= \lim_{\kappa' \rightarrow \kappa} \frac{2\sqrt{\kappa\kappa'} \sin(v \log \frac{\kappa}{\kappa'})}{\sinh(v\pi)(\kappa^2 - \kappa'^2)} \overline{N_{\kappa'\lambda}} N_{\kappa\lambda} \\ &= \frac{v}{\kappa \sinh(v\pi)} |N_{\kappa\lambda}|^2, \end{aligned} \quad (23)$$

³For $E = -\hbar^2 \kappa^2 / (2m)$ and $\lambda = -v^2$, Eq. (11b) is equivalent to the modified Bessel differential equation ($r^2 d^2/d r^2 + r d/d r - \kappa^2 r^2 + v^2$)($r^{-\frac{1}{2}} R$) = 0. The two independent solutions to this equation are the modified Bessel functions of the first and second kind, $I_{iv}(\kappa r)$ and $K_{iv}(\kappa r)$, respectively, where the former is non-square integrable while the latter is square integrable. For more details of the modified Bessel functions, see, e.g., Ref. [37].

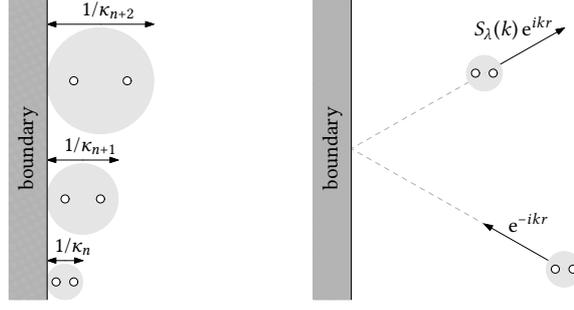


Figure 4: Schematic pictures of the boundary-localized two-body bound states with different binding energies (left) and the scattering of a two-body bound state off the boundary (right). The boundary plays the role of the (infinitely heavy) third particle (see Appendix B for details).

where the last equality follows from $\sin(\nu \log(\kappa/\kappa')) = \sin(\nu \log(1 + (\kappa - \kappa')/\kappa')) = \nu(\kappa - \kappa')/\kappa' + O((\kappa - \kappa')/\kappa')^2$ as $\kappa' \rightarrow \kappa$. Thus we find

$$|N_{\kappa\lambda}| = \sqrt{\frac{\kappa \sinh(\nu\pi)}{\nu}}. \quad (24)$$

Second, the energy eigenvalues are determined by requiring that $R_{\kappa\lambda}$ and $R_{\kappa'\lambda}$ be orthogonal for $\kappa \neq \kappa'$; that is, $\int_0^\infty dr \overline{R_{\kappa'\lambda}(r)} R_{\kappa\lambda}(r) = 0$ for $\kappa \neq \kappa'$, which is attained if and only if $\sin(\nu \log(\kappa/\kappa')) = 0$. Thus, $\nu \log(\kappa/\kappa')$ must be an integer multiple of π :

$$\nu \log \frac{\kappa}{\kappa'} = -n\pi, \quad n \in \mathbb{Z}, \quad (25)$$

where the minus sign on the right hand side is just a convention. The solution to this condition is given by

$$\kappa_n = \kappa_* \exp\left(-\frac{n\pi}{\nu}\right), \quad (26)$$

where $\kappa_*(> 0)$ is an arbitrary reference scale with the dimension of inverse length, which must be introduced on dimensional grounds. Putting these together, we obtain an infinite number of discrete negative energy eigenvalues:

$$E_n = -\frac{\hbar^2 \kappa_n^2}{2m} \exp\left(-\frac{2n\pi}{\nu}\right), \quad n \in \mathbb{Z}. \quad (27)$$

These are the binding energies of the boundary-localized two-body bound states. Note that the spatial extent of these bound states is about $r \approx 1/\kappa_n = \kappa_*^{-1} e^{n\pi/\nu}$, which follows from the asymptotic behavior $R_{\kappa_n\lambda}(r) \rightarrow N_{\kappa_n\lambda} e^{-\kappa_n r}$ as $r \rightarrow \infty$;⁴ see the left panel of Fig. 4. Note also that E_n and $R_{\kappa_n\lambda}(r)$ fulfill the relations $E_{n-1} = E_n e^{2\pi/\nu}$ and $R_{\kappa_{n-1}\lambda}(r) = e^{\pi/(2\nu)} R_{\kappa_n\lambda}(e^{\pi/\nu} r)$, which, through Eq. (10), guarantees the scaling law (3) discussed in the introduction with the scaling factor $e^t = e^{\pi/\nu}$.

3.3 Two-body scattering off the boundary

Let us finally consider the case $E > 0$ in the channel $\lambda = \lambda_0 < 0$. In this case, we are interested in the solution to the radial equation (11b) whose asymptotic behavior as $r \rightarrow \infty$ is the linear combination of plane waves $R_{k\lambda}(r) \rightarrow e^{-ikr} + S_\lambda(k) e^{ikr}$, where $S_\lambda(k)$ is a linear combination coefficient and $k = \sqrt{2mE/\hbar^2} > 0$. The full solution is given by⁵

$$R_{k\lambda}(r) = \sqrt{\frac{2kr}{\pi}} \left(e^{\frac{i\pi}{4}} K_{i\nu}(e^{\frac{i\pi}{2}} kr) + S_\lambda(k) e^{-\frac{i\pi}{4}} K_{i\nu}(e^{-\frac{i\pi}{2}} kr) \right). \quad (28)$$

⁴For more precise estimation, one should compute the expectation value of the distance $|x_1 - x_2| = |r \sin \theta|$.

⁵For $E = \hbar^2 k^2/(2m)$ and $\lambda = -\nu^2$, Eq. (11b) is equivalent to the Bessel differential equation $(r^2 d^2/dr^2 + rd/dr + k^2 r^2 + \nu^2)(r^{-\frac{1}{2}} R) = 0$. The two independent solutions to this equation are the Hankel functions of the first and second kind, $H_{i\nu}^{(1)}(\kappa r)$ and $H_{i\nu}^{(2)}(\kappa r)$, respectively. Note that the Hankel functions and the modified Bessel function of the second kind are related as $H_{i\nu}^{(1)}(x) = (2/(i\pi)) e^{-\frac{i\pi}{2}} K_{i\nu}(e^{-\frac{i\pi}{2}} x)$ and $H_{i\nu}^{(2)}(x) = (2/(i\pi)) e^{\frac{i\pi}{2}} K_{i\nu}(e^{\frac{i\pi}{2}} x)$. For more details of the Hankel functions, see, e.g., Ref. [37].

Note that Eq. (28) together with Eq. (16) describes the superposition of an incoming wave to $r = 0$ and an outgoing wave from $r = 0$, both of which localize to $\theta = 0$; that is, it describes the two-body bound state scattered off the boundary, where $S_\lambda(k)$ plays the role of the reflection amplitude (see the right panel of Fig. 4). Note also that the scattering solution (28) is no longer localized to the boundary $r = 0$.

It should be noted that at this stage $S_\lambda(k)$ is an arbitrary constant. In order to determine $S_\lambda(k)$, we require that the scattering solution (28) be orthogonal to all the bound-state solutions (19). (Note that the energy eigenfunctions should be orthogonal if their eigenvalues are different.) In exactly the same way as for Eq. (22), one obtains the following relation:

$$\begin{aligned} (-\kappa_n^2 - k^2) \int_0^\infty dr \overline{R_{\kappa_n\lambda}(r)} R_{k\lambda}(r) &= - \lim_{r \rightarrow 0} \left(\overline{R_{\kappa_n\lambda}(r)} R'_{k\lambda}(r) - \overline{R'_{\kappa_n\lambda}(r)} R_{k\lambda}(r) \right) \\ &= \frac{2\sqrt{\kappa_n k}}{\pi \sinh(\nu\pi)} \overline{N_{\kappa_n\lambda}} \left[e^{\frac{i\pi}{4}} \sin \left(\nu \log \left(\frac{k}{\kappa_n} \right) + \frac{i\nu\pi}{2} \right) \right. \\ &\quad \left. + S_\lambda(k) e^{-\frac{i\pi}{4}} \sin \left(\nu \log \left(\frac{k}{\kappa_n} \right) - \frac{i\nu\pi}{2} \right) \right]. \end{aligned} \quad (29)$$

Hence, in order to guarantee the orthogonality relation $\int_0^\infty dr \overline{R_{\kappa_n\lambda}(r)} R_{k\lambda}(r) = 0$ for any $k > 0$ and $n \in \mathbb{Z}$, the coefficient $S_\lambda(k)$ must be of the following form:

$$S_\lambda(k) = - e^{\frac{i\pi}{2}} \frac{\sin \left(\nu \log \left(\frac{k}{\kappa_*} \right) + \frac{i\nu\pi}{2} \right)}{\sin \left(\nu \log \left(\frac{k}{\kappa_*} \right) - \frac{i\nu\pi}{2} \right)}. \quad (30)$$

This is the reflection amplitude off the boundary for the two-body bound state. This amplitude, which satisfies the unitarity condition $\overline{S_\lambda(k)} S_\lambda(k) = 1$, is a periodic function of $\log k$ with the period π/ν . This log-periodicity is a manifestation of discrete scale invariance $S_\lambda(e^{n\pi/\nu} k) = S_\lambda(k)$ in the scattering problem.⁶ We also note that Eq. (30) has simple poles at $k = i\kappa_n = \kappa_* e^{i\pi/2 - n\pi/\nu}$ in the complex k -plane. In fact, it behaves as follows:

$$S_\lambda(k) \rightarrow \frac{i|N_{\kappa_n\lambda}|^2}{k - i\kappa_n} + O(1) \quad \text{as } k \rightarrow i\kappa_n. \quad (31)$$

These simple poles are the manifestation of the presence of infinitely many bound states that satisfy the geometric scaling $E_{n+1}/E_n = \kappa_{n+1}^2/\kappa_n^2 = e^{-2\pi/\nu}$.

We note in closing that the reflection amplitude (30) can be regarded as the scattering matrix (S-matrix) element $S_{k\lambda, k'\lambda'} = (\Psi_{k\lambda}^{\text{out}}, \Psi_{k'\lambda'}^{\text{in}})$ for $\lambda = \lambda' = \lambda_0 (< 0)$, where $\Psi_{k\lambda}^{\text{in}} = r^{-1/2} R_{k\lambda} \Theta_\lambda$ is the in-state, $\Psi_{k\lambda}^{\text{out}} = \overline{\Psi_{k\lambda}^{\text{in}}}$ is the out-state given by the complex conjugate (i.e., time reversal) of the in-state, and (\cdot, \cdot) is the inner product defined by $(f, g) = \int_0^\infty \int_0^\pi \overline{f} g \, dx_1 dx_2 = \int_0^\infty \int_{-\pi/4}^{\pi/4} \overline{f} g \, r dr d\theta$. In fact, it follows from the orthonormality $\int_{-\pi/4}^{\pi/4} \Theta_\lambda \Theta_{\lambda'} d\theta = \delta_{\lambda\lambda'}$,⁷ the identity $(k^2 - k'^2) R_{k\lambda} R_{k'\lambda} = (d/dr)(R_{k\lambda} R'_{k'\lambda} - R'_{k\lambda} R_{k'\lambda})$, and the asymptotic behavior $R_{k\lambda} \rightarrow e^{-ikr} + S_\lambda(k) e^{ikr}$ that this S-matrix element takes the form $S_{k\lambda, k'\lambda'} = 2\pi \delta(k - k') \delta_{\lambda\lambda'} S_\lambda(k)$. One nice thing in this formulation is that it is obvious that there is no scattering between different channels λ and λ' .

⁶Once given the relation $S_\lambda(e^{n\pi/\nu} k) = S_\lambda(k)$ for any $n \in \mathbb{Z}$, we can say that the reflection amplitude is of the form $S_\lambda(k) = f_\lambda(\log k)$, where f_λ is a periodic function with the period π/ν . In general, in scattering problems discrete scale invariance manifests itself in a periodic oscillation of the S-matrix as a function of $\log k$. For more details, see Ref. [22].

⁷Without any loss of generality, the angular wavefunction Θ_λ can be chosen to be real for any $\lambda \in \{\lambda_0, \lambda_1, \dots\}$.

4 Conclusion

In this paper, we have introduced a toy scale-invariant model of two identical bosons on the half-line \mathbb{R}_+ , where interparticle interaction is described by the pairwise δ -function potential with the particular position-dependent coupling strength given by Eq. (2). We have seen that, if the two-body interaction is sufficiently attractive, continuous scale invariance is broken down to discrete scale invariance. In the bound-state problem where the bosons are bound together and localized to the boundary, this discrete scale invariance manifests itself in the onset of the geometric sequence of binding energies. In the scattering problem where the two-body bound state is scattered by the boundary, on the other hand, this discrete scale invariance manifests itself in the log-periodic behavior of the reflection amplitude. Hence, by breaking translation invariance of this one-dimensional problem, we can construct a two-body model that exhibits the Efimov effect. In contrast to the ordinary Efimov effect in three-body problems in three dimensions, our model can be solved exactly by just using undergraduate-level calculus.

Finally, it should be mentioned the stability issue of the model and its cure. As is evident from Eq. (27), there is no lower bound in the energy spectrum $\{E_n\}$ for $g_0 < g_*$. This absence of ground state is inevitable if the system is invariant under the full discrete scale invariance that forms the group \mathbb{Z} . (As discussed in the introduction, the full discrete scale invariance leads to the geometric sequence $\{E_0, E_0 e^{\pm 2t}, E_0 e^{\pm 4t}, \dots\}$, which cannot be bounded from below if $E_0 < 0$.) In order to make the spectrum lower-bounded, we therefore have to break this invariance under \mathbb{Z} . The easiest way to do this is to replace the short-distance singularity of the inverse-square potential by, e.g., a square-well potential. Such regularization procedures have been widely studied over the years in the context of renormalization of the inverse-square potential. For more details, we refer to Refs. [24–31].

Appendix A Modified Bessel functions of imaginary order

In this section, we summarize the short- and long-distance behaviors of the modified Bessel functions. For details, we refer to Ref. [37].

First of all, the modified Bessel function of the second kind with imaginary order is defined as follows:

$$K_{i\nu}(z) = \frac{i\pi}{2} \frac{I_{i\nu}(z) - I_{-i\nu}(z)}{\sinh(\nu\pi)}, \quad \nu \in \mathbb{R} \setminus \{0\}, \quad (\text{A.1})$$

where $I_{i\nu}$ is the modified Bessel function of the first kind given by the following series:

$$I_{i\nu}(z) = e^{i\nu \log \frac{z}{2}} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1+n+i\nu)} \left(\frac{z}{2}\right)^{2n}. \quad (\text{A.2})$$

Here Γ is the gamma function. It follows immediately from the definition (A.1) that $K_{-i\nu}(z) = K_{i\nu}(z)$. It also follows from Eqs. (A.1) and (A.2) that $\bar{K}_{i\nu}(z) = K_{i\nu}(z)$ for $z > 0$.

The short-distance behavior of $K_{i\nu}$ is governed by the $n = 0$ term in Eq. (A.2). By using the polar form of the gamma function,

$$\begin{aligned} \Gamma(1+i\nu) &= |\Gamma(1+i\nu)| e^{i \arg \Gamma(1+i\nu)} \\ &= \sqrt{\frac{\nu\pi}{\sinh(\nu\pi)}} e^{i \arg \Gamma(1+i\nu)}, \end{aligned} \quad (\text{A.3})$$

where $\arg \Gamma(1+i\nu)$ stands for the argument of $\Gamma(1+i\nu)$, we see that $K_{i\nu}(z)$ behaves as follows:

$$K_{i\nu}(z) \rightarrow -\sqrt{\frac{\pi}{\nu \sinh(\nu\pi)}} \sin\left(\nu \log \frac{z}{2} - \arg \Gamma(1+i\nu)\right) + O(z^2) \quad \text{as } |z| \rightarrow 0. \quad (\text{A.4})$$

The long-distance behavior, on the other hand, is known to be of the following form:

$$K_{i\nu}(z) \rightarrow \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + O\left(\frac{1}{z}\right)\right] \quad \text{as } |z| \rightarrow \infty. \quad (\text{A.5})$$

Appendix B Boundary as the infinitely heavy third particle

In this section we show that the two-body problem on the half-line \mathbb{R}_+ discussed in the main text is equivalent to a three-body problem on the whole line \mathbb{R} with an infinitely heavy third particle. We note that this section is not necessary for understanding the main text.

To begin with, let us first define some notation. Let $z_1, z_2 \in \mathbb{R}$ be the coordinates of two identical bosons of mass m and $z_3 \in \mathbb{R}$ be the coordinate of a third particle of mass M . The Hamiltonian of these particles is assumed to be of the following form:

$$H_{3\text{-body}} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} \right) - \frac{\hbar^2}{2M} \frac{\partial^2}{\partial z_3^2} + g(z_1 - z_3)\delta(z_1 - z_2) + U(z_1 - z_3) + U(z_2 - z_3), \quad (\text{B.1})$$

where $g(x) = \hbar^2 g_0/(m|x|)$ is the position-dependent coupling strength that makes the two-body interaction between the identical bosons scale invariant. Note that the two bosons interact only when they are in the same position, and that the interaction strength depends on their position relative to the massive third particle. U is a pairwise interaction between one of the identical bosons and the third particle whose explicit form is irrelevant for the moment. Note that Eq. (B.1) is invariant under the translation $(z_1, z_2, z_3) \mapsto (z_1 + a, z_2 + a, z_3 + a)$ for any $a \in \mathbb{R}$. Hence the total momentum is conserved and the center-of-mass motion is that of a free particle. Below, we will show that, by taking the limit $M \rightarrow \infty$, the relative Hamiltonian that describes the relative motion of this three-body system can be reduced to the two-body Hamiltonian on the half-line discussed in the main text.

Let us first separate the center-of-mass Hamiltonian from Eq. (B.1) and identify the relative Hamiltonian. To this end, it is convenient to introduce the Jacobi coordinates (y_1, y_2, y_3) as follows:

$$y_1 = z_1 - z_2, \quad (\text{B.2a})$$

$$y_2 = \frac{z_1 + z_2}{2} - z_3, \quad (\text{B.2b})$$

$$y_3 = \frac{mz_1 + mz_2 + Mz_3}{2m + M}. \quad (\text{B.2c})$$

Physically, y_1 is the relative coordinate for the identical bosons; y_2 is the relative coordinate for the center-of-mass of the identical bosons and the heavy particle; and y_3 is the center-of-mass coordinate for the three particles. A main advantage in this coordinate system is the following equality:

$$\frac{1}{m} \frac{\partial^2}{\partial z_1^2} + \frac{1}{m} \frac{\partial^2}{\partial z_2^2} + \frac{1}{M} \frac{\partial^2}{\partial z_3^2} = \frac{1}{\mu_1} \frac{\partial^2}{\partial y_1^2} + \frac{1}{\mu_2} \frac{\partial^2}{\partial y_2^2} + \frac{1}{\mu_3} \frac{\partial^2}{\partial y_3^2}, \quad (\text{B.3})$$

where

$$\mu_1 = \left(\frac{1}{m} + \frac{1}{m} \right)^{-1} = \frac{m}{2}, \quad (\text{B.4a})$$

$$\mu_2 = \left(\frac{1}{2m} + \frac{1}{M} \right)^{-1} = \frac{2mM}{2m + M}, \quad (\text{B.4b})$$

$$\mu_3 = 2m + M. \quad (\text{B.4c})$$

Physically, μ_1 is the reduced mass for the identical bosons; μ_2 is the reduced mass for the center-of-mass of the identical bosons and the heavy particle; and μ_3 is the total mass of the three particles. It is now straightforward to show that the three-body Hamiltonian (B.1) can be written as $H_{3\text{-body}} = H_{\text{cm}} + H_{\text{rel}}$, where

$$H_{\text{cm}} = -\frac{\hbar^2}{2\mu_3} \frac{\partial^2}{\partial y_3^2}, \quad (\text{B.5a})$$

$$H_{\text{rel}} = -\frac{\hbar^2}{2\mu_1} \frac{\partial^2}{\partial y_1^2} - \frac{\hbar^2}{2\mu_2} \frac{\partial^2}{\partial y_2^2} + g(y_2 + \frac{1}{2}y_1)\delta(y_1) + U(y_2 + \frac{1}{2}y_1) + U(y_2 - \frac{1}{2}y_1). \quad (\text{B.5b})$$

H_{cm} is the center-of-mass Hamiltonian so that $H_{\text{rel}} = H_{3\text{-body}} - H_{\text{cm}}$ describes the relative motion and internal energy of the three-body system. In the following, we focus on H_{rel} .

Now let us consider the situation where the third particle is much heavier than the identical bosons, $m/M \ll 1$. In the extreme case $M \rightarrow \infty$, where $\mu_2 \rightarrow 2m$, Eq. (B.5b) reduces to

$$\begin{aligned} H_{\text{eff}} &= \lim_{M \rightarrow \infty} H_{\text{rel}} \\ &= -\frac{\hbar^2}{m} \frac{\partial^2}{\partial y_1^2} - \frac{\hbar^2}{4m} \frac{\partial^2}{\partial y_2^2} + g(y_2 + \frac{1}{2}y_1)\delta(y_1) + U(y_2 + \frac{1}{2}y_1) + U(y_2 - \frac{1}{2}y_1). \end{aligned} \quad (\text{B.6})$$

To standardize the expression, let us introduce a new coordinate system (x_1, x_2) defined by

$$x_1 = y_2 + \frac{1}{2}y_1 = z_1 - z_3, \quad (\text{B.7a})$$

$$x_2 = y_2 - \frac{1}{2}y_1 = z_2 - z_3, \quad (\text{B.7b})$$

in which Eq. (B.6) takes the following form:

$$H_{\text{eff}} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + g(x_1)\delta(x_1 - x_2) + U(x_1) + U(x_2). \quad (\text{B.8})$$

This Hamiltonian describes the relative motion of the three-body system in the center-of-mass frame and in the $M \rightarrow \infty$ limit. Notice that $y_3 \rightarrow z_3$ as $M \rightarrow \infty$ so that the infinitely heavy third particle remains at the origin in the center-of-mass frame. In this limit, the three-body system is described by the two-body Hamiltonian (B.8).

Finally, let us consider the confinement of the identical bosons on the half-line \mathbb{R}_+ by specifying U . One way to achieve this is to choose $U(x) = 0$ for $x > 0$ and $U(x) = \infty$ for $x < 0$. An alternative way is to use the strong-coupling limit of the following δ -function potential:

$$U(x) = \frac{\hbar^2}{m} \gamma \delta(x), \quad (\text{B.9})$$

where γ is a coupling constant. As discussed in Sec. 3.1, $U(x_1)$ is equivalent to the following connection conditions for the two-body wavefunction $\psi(x_1, x_2)$:

$$-\frac{\partial \psi}{\partial x_1}(0_+, x_2) + \frac{\partial \psi}{\partial x_1}(0_-, x_2) + \gamma (\psi(0_+, x_2) + \psi(0_-, x_2)) = 0, \quad (\text{B.10a})$$

$$\psi(0_+, x_2) = \psi(0_-, x_2), \quad (\text{B.10b})$$

where $x_2 \neq 0$. In the strong-coupling limit $\gamma \rightarrow \infty$, Eq. (B.10a) reduces to $\psi(0_+, x_2) + \psi(0_-, x_2) = 0$, which, together with Eq. (B.10b), leads to the Dirichlet boundary condition $\psi(0_+, x_2) = \psi(0_-, x_2) = 0$ that corresponds to $\Theta(\pi/4) = 0$ in Eq. (15). Similarly, for $U(x_2)$, one can obtain $\psi(x_1, 0_+) = \psi(x_1, 0_-) = 0$ ($x_1 \neq 0$) that corresponds to $\Theta(-\pi/4) = 0$. Notice that, under the Dirichlet boundary conditions, the two regions $x > 0$ and $x < 0$ are physically disconnected because the probability current density vanishes at $x = 0$ and hence there is no probability current flow across the origin. Alternatively, one can say that particles cannot penetrate through the origin because the transmission amplitude for the δ -function potential vanishes in the strong-coupling limit. Hence, if the identical bosons are initially on the positive half-line \mathbb{R}_+ , they remain on this region forever. This is the two-body problem on the half-line discussed in the main text. Note that the effective two-body Hamiltonian (B.8) is no longer invariant under the translation $(x_1, x_2) \mapsto (x_1 + a, x_2 + a)$.

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