TUNNELING EFFECT INDUCED BY A CURVED MAGNETIC EDGE

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ABSTRACT. Experimentally observed magnetic fields with nanoscale variations are theoretically modeled by a piece-wise constant function with jump discontinuity along a smooth curve, the magnetic edge. Assuming the edge is a closed curve with an axis of symmetry and the field is sign changing and with exactly two distinct values, we prove that semi-classical tunneling occurs and calculate the magnitude of this tunneling effect.

This paper is dedicated to Elliott H. Lieb on the occasion of his 90th birthday.

1. Introduction

The purpose of this paper is to study the magnetic Laplacian in $\mathbb{R}^2$,

$$\mathcal{P}_h := (ih\nabla + \mathbf{A})^2 = \sum_{j=1}^{2}(ih\partial_{x_j} + A_j)^2, \tag{1.1}$$

where $\mathbf{A} := (A_1, A_2) \in H^1_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)$, is a magnetic potential generating the magnetic field $B = \operatorname{curl}\mathbf{A} := \partial_{x_1}A_2 - \partial_{x_2}A_1 \in L^2_{\rm loc}(\mathbb{R}^2;\mathbb{R})$. We will also discuss the case of the Neumann or Dirichlet realizations of $\mathcal{P}_h$ in smooth bounded planar domains.

Here h is a positive parameter that tends to 0, which can be interpreted as the semi-classical parameter. By writing $h^{-2}\mathcal{P}_h = (i\nabla + h^{-1}\mathbf{A})^2$, we observe that the semi-classical limit, $h \to 0_+$, is equivalent to the strong magnetic field limit, $h^{-1}|B| \to +\infty$.

The spectrum of the operator $\mathcal{P}_h$ has been the subject of an intense study in the past decades, particularly in the context of superconductivity where the magnetic field B is typically a constant function [6, 11, 17, 21, 23, 32].

There is an interesting analogy between the results for the Neumann realization of $\mathcal{P}_h$ in a bounded smooth domain and those for the Schrödinger operator, $-h^2\Delta+V$, with an electric potential V, in the full plane. The Schrödinger operator was intensively studied by Helffer–Sjöstrand [24, 25] and Simon [36], notably in the context of quantum tunneling. Bound states of $-h^2\Delta + V$ concentrate near the 'well' $\Gamma_V := \{x \in \mathbb{R}^2 \,|\, V(x) = \min_{\mathbb{R}^2} V\}$; if furthermore Γ_V is a regular manifold (i.e. we have a degenerate well), bound states could concentrate near some points of Γ_V, the 'mini-wells'. We have the same picture in the purely magnetic case with a Neumann boundary condition: bound states concentrate near the boundary of the domain, whereby the boundary plays the role of a (degenerate) well, and the set of points of maximum curvature plays the role of mini-wells, where bound states decay away from them. Optimal estimates describing the concentration of bound states are very important, since they lead to accurate asymptotics for the low lying eigenvalues. The proof of the decay away from the mini-wells (points of maximum curvature), is more delicate compared to that of the decay away from the well (boundary).

In this paper, our main focus will be on magnetic fields having a jump-discontinuity. Magnetic fields that vary on very short scales (nanoscales) have been observed experimentally, see e.g. [15]. Their theoretical investigations, in the context of quantum mechanics [34, 35] or graphene [18], involve the operator $\mathcal{P}_h$ but with the magnetic field B being a step function having a discontinuity along a curve, that we will refer to as the *magnetic edge*.

Earlier rigorous results were devoted to the case of a flat edge [28, 29, 30]. More recently, non-flat edges have been considered in the context of spectral asymptotics [1, 3] and in the context of superconductivity [4]. The magnetic edge will play the role of the 'well', while the 'mini-wells' are the points of maximum curvature of the magnetic edge [5], which is interestingly in analogy with the setting of the Neumann realization with a constant magnetic field in a bounded smooth domain.

The case of a single mini-well, where the curvature of the edge has a unique and non-degenerate maximum, was analyzed by Assaad–Helffer–Kachmar [3]. The present paper investigates the situation of a symmetric edge with several mini-wells, the simplest case being when there are two non-degenerate maxima of the boundary curvature. We establish a sharp asymptotics of the splitting between the energies of the ground and first excited state, which measures a tunneling effect induced by the geometry of the edge, see Theorem 3.4 below which is our main result.

Let us recall how the general strategy of Helffer–Sjöstrand [24, 25] has been applied recently to understand the tunneling effect for the Neumann realization in a bounded domain with the breakthrough paper [7] by Bonnaillie-Noël–Hérau–Raymond as the crowning achievement. The first step, already performed in [21] and [16], was the analysis of a model with a flat boundary (de Gennes model), which yields localization of bound states near the boundary of the domain (the well), and consequently, leads to a full asymptotics of the low-lying eigenvalues. The second step is a formal WKB expansion of bound states [8]. The third step consists of optimal decay estimates of bound states recently achieved in [7]. The importance of this step is that it allows one to rigorously approximate the bound states by the formal WKB expansions, which eventually paves the way for the analysis of an interaction matrix whose eigenvalues measure the tunneling effect. The same approach has been successfully applied in the context of thin domains

[31] and the Robin Laplacian [19, 20], where the proof of the tangential estimates was less technical.

We will follow the same approach outlined above in the case of our discontinuous magnetic field. The model problem with a flat edge was analyzed in [5] (see also [4, 28]), while the full asymptotics for the low lying eigenvalues are obtained in [3]. So we still need WKB expansions and optimal tangential estimates of bound states, which we do in the present contribution. Finally, after establishing the WKB approximation, the analysis of the interaction matrix is quite standard.

Let us give an informal statement of our main result (Theorem 3.4 below). Suppose that Γ is a smooth, closed curve in $\mathbb{R}^2$, symmetric with respect to an axis, and with two points of maximum curvature, denoted by s_ℓ and s_r (ℓ refers to "left" and r to "right", see Fig. 1). Consider the magnetic field satisfying $B = 1$ in the interior of Γ, and $B = a \in (-1, 0)$ in the exterior of Γ. Under these assumptions, we prove that, as $h \to 0_+$, the spectral gap of the operator $\mathcal{P}_h$ is of exponential order,

$$\lambda_2(h) - \lambda_1(h) \approx \exp\Big(-\frac{\mathsf{S}^a}{h^{1/4}}\Big), \tag{1.2}$$

where S^a is the Agmon distance between the "wells" s_ℓ and s_r defined by an appropriate potential that depends on the magnetic field (through the parameter a) and the geometry of Γ (through the curvature). The asymptotics in (1.2) (more precisely that in Theorem 3.4) is a consequence of quantum tunneling. It is important to note that it is induced purely by the magnetic field, thereby providing an example of a purely magnetic quantum tunneling—where the case of [7] also required the interaction with the boundary. If we look at earlier results on the tunneling effect, with or without magnetic field, we observe that the tunneling is induced by an external potential [24, 14] or by confinement to a bounded/thin domain [20, 31, 9]. For the Neumann realization of $\mathcal{P}_h$, the presence of the magnetic field adds a challenging difficulty in the estimate of the magnitude of the tunneling that was recently solved in [7]. Our proof of (1.2) is very close to that of [7], but it relies on new elements that follow from a deep investigation of magnetic steps [5, 3].

Let us give some of the heuristics behind the computations leading to (1.2). We can construct two quasi-modes having the following structure

$$\begin{aligned}\Psi_{h,\ell}(s,t) &\approx e^{i\theta_{h,\ell}(s)}\, e^{-\Phi_\ell(s)/h^{1/4}}\, f_{0,\ell}(s)\phi_a(h^{1/2}t),\\ \Psi_{h,r}(s,t) &\approx e^{i\theta_{h,r}(s)} e^{-\Phi_r(s)/h^{1/4}}\, f_{0,r}(s)\phi_a(h^{1/2}t),\end{aligned}$$

where s denotes the arc-length parameter along Γ, t denotes the normal distance to Γ with the convention that $t > 0$ in the interior of Γ and $t < 0$ in the exterior of Γ. The functions Φ_ℓ and Φ_r are non-negative and satisfy $\Phi_\ell(s_\ell) = 0$ and $\Phi_r(s_r) = 0$, so that $\Psi_{h,\ell}$ (resp. $\Psi_{h,r}$) is localized near s_ℓ (resp. near s_r). The phase functions $\theta_{h,\ell}$ and $\theta_{h,r}$ involve the topology of the discontinuity curve and a spectral constant. The function ϕ_a is the ground state eigenfunction of a model operator related

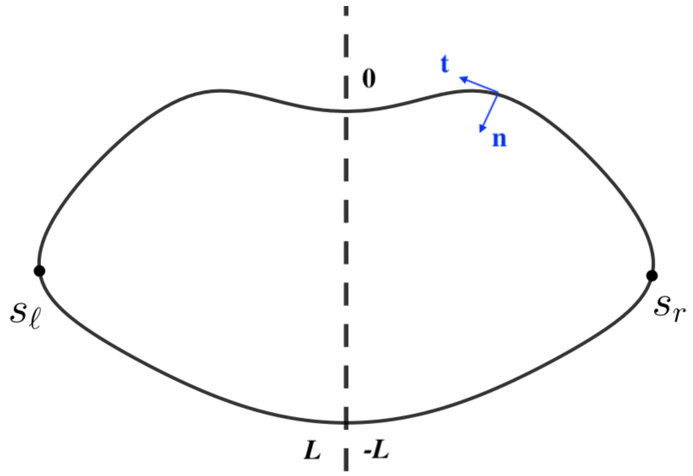

FIGURE 1. A symmetric domain with respect to the y-axis (dashed line). The orientation of the boundary is defined by the direct frame $(\mathbf{t},\mathbf{n})$, where $\mathbf{n}$ is the inward normal vector and $\mathbf{t}$ is the unit tangent. The curvature along the boundary has two non-degenerate maxima at the points a_1 and a_2, with arc-length coordinates $s_\ell \in [0,L)$ and $s_r \in (-L,0]$, connected by upward and downward geodesics oriented counterclockwise and represented by $[s_r,0]\cup(0,s_\ell]$ and $[s_\ell,L]\cup(-L,s_r]$ respectively. These upward and downward geodesics will be denoted by $[s_r,s_\ell]$ and $[s_\ell,s_r]$ respectively.

to the discontinuity of the magnetic field (see Sec. 3). Finally, $f_{0,\ell}$ and $f_{0,r}$ are solutions of appropriate transport equations (see Theorem 5.1).

Up to truncation, the quasi-modes $\Psi_{h,\ell}$ and $\Psi_{h,r}$ are approximations of actual bound states

$$g_{h,\ell}(s,t) \approx \Psi_{h,\ell}(s,t), \quad g_{h,r}(s,t) \approx \Psi_{h,r}(s,t), \tag{1.3}$$

where the bound states $g_{h,\ell}$ and $g_{h,r}$ are defined via the orthogonal projection Π on $V := \oplus_{i=1}^2 \operatorname{Ker}(\mathcal{P}_h - \lambda_i(h))$ as follows

$$g_{h,\ell}(s,t) := \Pi\Psi_{h,\ell}(s,t), \quad g_{h,r}(s,t) := \Pi\Psi_{h,r}(s,t).$$

By the Gram-Schmidt process, we transform $\{g_{h,\ell}, g_{h,r}\}$ to an orthonormal basis $\mathcal{B}$ of V and we denote by M_h the matrix relative to $\mathcal{B}$ of the restriction of $\mathcal{P}_h$ to V. The spectral gap for the operator $\mathcal{P}_h$ is the same as that for the matrix M_h,

$$\lambda_2(h) - \lambda_1(h) = \lambda_2(\mathsf{M}_h) - \lambda_1(\mathsf{M}_h)\,.$$

Using the approximation in (1.3), we get an approximate matrix $\widehat{\mathsf{M}}_h$ of M_h whose spectral gap can be explicitly estimated (compare with (1.2))

$$\lambda_2(\widehat{\mathsf{M}}_h) - \lambda_1(\widehat{\mathsf{M}}_h) \approx \exp\Big(-\frac{\mathsf{S}^a}{h^{1/4}}\Big)\,.$$

We have then to show that the spectral gap for the matrix $\widehat{\mathsf{M}}_h$ is a good approximation of that of $\mathcal{P}_h$ up to an appropriate remainder, more precisely

$$\lambda_2(\mathsf{M}_h) - \lambda_1(\mathsf{M}_h) = \big(\lambda_2(\widehat{\mathsf{M}}_h) - \lambda_1(\widehat{\mathsf{M}}_h)\big)\big(1 + o(1)\big)\,.$$

Such an estimate is closely related to optimal decay estimates of bound states (resp. approximate bound states) of the operator $\mathcal{P}_h$, which yield accurate errors for the approximation in (1.3).

Organisation. In Section 2, we review the recent result of [7] for the Neumann magnetic Laplacian with a constant magnetic field, and introduce the related de Gennes model for a flat boundary. In Section 3, we introduce the magnetic edge along with the related flat edge model, and state our main result, Theorem 3.4, for the operator $\mathcal{P}_h$ with a magnetic step. In Section 4, we express $\mathcal{P}_h$ in a Frénet frame and reduce the spectral analysis to an operator defined near the edge. In Section 5, we introduce operators with a single well (with ground states localized near a single point of maximum curvature), and perform a WKB expansion for an approximate ground state (see Theorem 5.1). In Section 6, we explain how optimal tangential estimates can be derived along the lines of the proof of the similar statement in [7]. Finally, in Section 7, we introduce the interaction matrix and finish the proof of Theorem 3.4, by referring to [7] for the detailed computations, which are essentially the same in our setting.

2. Uniform magnetic fields

In this section, we review some results on the Neumann realization of the operator $\mathcal{P}_h$ with a constant magnetic field. We assume that

$$\left\{\begin{array}{l}\Omega \subset \mathbb{R}^2 \text{ is a simply connected open set,}\\ \Sigma := \partial\Omega \text{ is a } C^\infty \text{ smooth closed curve.}\end{array}\right\} \tag{2.1}$$

and

$$\mathbf{A} = \frac{1}{2}(-x_2, x_1) \quad \text{and} \quad B = \operatorname{curl}\mathbf{A} \equiv 1\,, \tag{2.2}$$

We consider $\mathcal{P}_h$ introduced in (1.1), as a self-adjoint operator in $L^2(\Omega)$, with domain,

$$\mathsf{Dom}(\mathcal{P}_h) = \{u \in H^2(\Omega) \mid \mathbf{n}\cdot(h\nabla - i\mathbf{A})u|_{\partial\Omega} = 0\},$$

where $H^2(\Omega)$ denotes the Sobolev space $W^{2,2}(\Omega)$, and $\mathbf{n}$ the unit normal vector of Σ, pointing inwards with respect to Ω.

2.1. Full asymptotics and decay of bound states.

The conditions in (2.1) ensure that Ω is bounded and that $\mathcal{P}_h$ has compact resolvent. Let $(\lambda_n(h))_{n\geqslant 1}$ be the sequence of eigenvalues of $\mathcal{P}_h$. In generic situations, that will be explained precisely later on, there exist complete expansions of the eigenvalues of $\mathcal{P}_h$, in the form [16],

$$\lambda_n(h) \sim \Theta_0 h - k_{\max} C_1 h^{\frac{3}{2}} + C_1 \Theta_0^{\frac{1}{4}}(2n-1)\sqrt{-\frac{3}{2}k_2}\, h^{\frac{7}{4}} + \sum_{j\geqslant 15} \zeta_{j,n} h^{j/8}\,. \tag{2.3}$$

The coefficients Θ_0 and C_1 appearing in (2.3) are universal positive constants related to the de Gennes model in the half-plane (see Sec. 2.2). The coefficients $k_{\max}$ and k_2 are related to the curvature on the boundary. Let Σ be parameterized by arc-length s and denote by $k(s)$ the curvature of Σ at s (see Sec. 4 for the precise definition of k; in particular the orientation is chosen so that $k \geqslant 0$ if Ω is convex). The asymptotics in (2.3) holds provided the curvature k attains its maximum value non-degenerately and at a unique point, i.e.

$$k_{\max} := \max_{\Sigma} k(s) = k(0) \quad \text{ with } \quad k_2 := k''(0) < 0\,. \tag{2.4}$$

The sequence $(\zeta_{j,n})_{j\geqslant 15}$ is constructed recursively, and it can be shown that $\zeta_{j,1} = 0$ for odd j [8].

The derivation of (2.3) is related to the decay of bound states. Assume that n is fixed and for all $h > 0$ that $u_{h,n}$ is an eigenfunction of $\mathcal{P}_h$, normalized in $L^2(\Omega)$ and with eigenvalue $\lambda_n(h)$. There exist constants $\alpha_1, C_{1,n} > 0$ such that

$$\int_\Omega |u_{h,n}|^2 \exp\Big(\frac{\alpha_1 \operatorname{dist}(x,\Sigma)}{h^{1/2}}\Big) dx \leqslant C_{1,n}\,.$$

This estimate says that the bound state $u_{h,n}$ concentrates near the boundary Σ and is valid even when (2.4) is not satisfied [21]. If moreover (2.4) holds, then $u_{h,n}$ concentrates near the point of maximal curvature as follows: There exist constants $\epsilon_0, \alpha_2, C_{2,n} > 0$ such that [16]

$$\int_{\operatorname{dist}(x,\Sigma)<\epsilon_0} |u_{h,n}|^2 \exp\Big(\frac{\alpha_2 |s(x)|^2}{h^{1/4}}\Big) dx \leqslant C_{2,n}\,, \tag{2.5}$$

where $s(x)$ denotes the arc-length coordinate of the point $p(x) \in \partial\Omega$ defined by $\operatorname{dist}(x, p(x)) = \operatorname{dist}(x,\Sigma)$. The decay estimate (2.5) is a key ingredient in the derivation of the asymptotics in (2.3), but is not sufficient to handle the case of symmetries that we shall discuss below.

Let us examine the case where the curvature attains its maximum at several points $s_1, \cdots, s_N$. For all $j \in \{1, \cdots, N\}$ and $m \in \mathbb{N}$, we introduce

$$\lambda^{\rm app}_{m,j}(h) = \Theta_0 h - k_{\max} C_1 h^{\frac{3}{2}} + C_1 \Theta_0^{\frac{1}{4}}(2m-1)\sqrt{-\frac{3}{2}k''(s_j)}\, h^{\frac{7}{4}}\,.$$

Consider a relabeling $(m_n, j_n)_{n\geqslant 1}$ of $(m,j)_{m\geqslant 1, 1\leqslant j\leqslant N}$ such that

$$\lambda^{\rm app}_{m_1,j_1} \leqslant \lambda^{\rm app}_{m_2,j_2} \leqslant \cdots\,.$$

Then, (2.3) is replaced with

$$\lambda_n(h) = \lambda^{\rm app}_{m_n,j_n} + o(h^{\frac{7}{4}})\,. \tag{2.6}$$

If additionally $k''(s_{j_1}) = k''(s_{j_2})$, then $\lambda_2(h) - \lambda_1(h) = o(h^{\frac{7}{4}})$ and we loose the information on the simplicity of the eigenvalues. Consequently, we need a more detailed analysis in the case of symmetries, which will rely on an optimal tangential decay estimate improving the one given in (2.5). We will discuss these decay estimates later in Sec. 6. Our next step is the review of an important model with a flat boundary.

2.2. **The de Gennes model: flat boundary.** The analysis of the model case where $\Omega = \mathbb{R}\times\mathbb{R}_+$ and $B = \operatorname{curl}\mathbf{A} = 1$ leads us naturally to the family (parametrized by $\xi\in\mathbb{R}$) of harmonic oscillators (de Gennes model)

$$\mathfrak{h}^N[\xi] = -\frac{d^2}{d\tau^2} + (\xi+\tau)^2, \tag{2.7}$$

on the semi-axis $\mathbb{R}_+$ with Neumann boundary condition at $\tau = 0$. Let us denote by $(\mu^N_j(\xi))_{j\geqslant 1}$ the sequence of eigenvalues of $\mathfrak{h}^N[\xi]$. The de Gennes constant is then defined as follows

$$\Theta_0 = \inf_{\xi\in\mathbb{R}} \mu^N_1(\xi)\,. \tag{2.8}$$

There exists a unique minimum $\xi_0 < 0$ such that

$$\Theta_0 = \mu^N_1(\xi_0)\,.$$

Furthermore, $\xi_0 = -\sqrt{\Theta_0}$, $(\mu^N_1)''(\xi) > 0$ and $\frac{1}{2} < \Theta_0 < 1$. Denoting by u_0 the positive and normalized ground state of $\mathfrak{h}^N[\xi_0]$, we can introduce the constant C_1 appearing in (2.3),

$$C_1 = \frac{|u_0(0)|^2}{3}\,. \tag{2.9}$$

2.3. **Symmetric domains and tunneling.** We continue to work under the conditions in (2.1) but we assume furthermore that the domain Ω is symmetric with respect to an axis and the curvature of its boundary Γ has exactly two non-degenerate maxima. More precisely, the hypotheses are (see Fig 1):

Assumption 2.1.

i) *Ω is symmetric with respect to the y-axis.*
ii) *The curvature k on Σ attains its maximum at exactly two symmetric points $a_1 = (a_{1,1}, a_{1,2})$ and $a_2 = (a_{2,1}, a_{2,2})$ with $a_{1,1} < 0$ and $a_{2,1} > 0$.*
iii) *Denoting by s_r and s_ℓ the arc-length coordinates of a_1 and a_2 respectively, we have $k''(s_r) = k''(s_\ell) < 0$.*

This situation induces a tunneling effect where the energy difference between the ground and first excited states is exponentially small. The magnitude of this splitting has been rigorously computed recently in [7].

Let us introduce the following effective quantities:

$$V(s)=\frac{2C_1(k_{\max}-k(s))}{(\mu_1^N)''(\xi_0)}\,, \tag{2.10}$$

and

$$\begin{aligned}\mathsf{A_u}&=\exp\left(-\int_{[s_r,0]}\frac{(V^{\frac12})'(s)+g}{\sqrt{V(s)}}ds\right)\,,\\ \mathsf{A_d}&=\exp\left(-\int_{[s_\ell,L]}\frac{(V^{\frac12})'(s)-g}{\sqrt{V(s)}}ds\right)\,,\\ g&=(V''(s_r)/2)^{\frac12}=(V''(s_\ell)/2)^{\frac12}\,.\end{aligned} \tag{2.11}$$

In the above formulae, 0 and L are the arc-length coordinates of the points of intersection between the y-axis and the curve Σ, with the convention that 0 represents the point on the upper part of Σ (see Fig. 1).

Theorem 2.2 (Bonnaillie-Noël–Hérau–Raymond [7]). *Suppose that* (2.1), (2.2) *and Assumption 2.1 hold. Then the first and second eigenvalues of $\mathcal{P}_h$ satisfy, as $h\to 0_+$,*

$$\lambda_2(h)-\lambda_1(h)=2|w(h)|+o(h^{\frac{13}{8}}e^{-\mathsf{S}/h^{\frac14}})\,,$$

where

$$\begin{aligned}w(h)&=(\mu_1^N)''(\xi_0)h^{\frac{13}{8}}\pi^{-\frac12}g^{\frac12}\\ &\quad\times\left(\mathsf{A_u}\sqrt{V(0)}e^{-\mathsf{S_u}/h^{1/4}}e^{iLf(h)}+\mathsf{A_d}\sqrt{V(L)}e^{-\mathsf{S_d}/h^{1/4}}e^{-iLf(h)}\right)\,,\end{aligned}$$

and

i. *The potential V is introduced in* (2.10)*;*
ii. S *is the Agmon distance between the wells,*

$$\mathsf{S}=\min(\mathsf{S_u},\mathsf{S_d})\,,\ \ \mathsf{S_u}=\int_{[s_r,s_\ell]}\sqrt{V(s)}\,ds,\ \ \mathsf{S_d}=\int_{[s_\ell,s_r]}\sqrt{V(s)}\,ds\,; \tag{2.12}$$

iii. $\mathsf{A_u}$, $\mathsf{A_d}$ *and g are defined in* (2.11)*;*
iv. $f(h)=\gamma_0/h+\xi_0/h^{1/2}-\alpha_0$ *with*

$$\gamma_0=\frac{|\Omega|}{|\Sigma|},$$

where $|\Sigma|$ is the length of Σ, and α_0 is a constant dependent on Ω.

Theorem 2.2 can be extended to the situation of $N\geqslant 3$ wells, which corresponds to a domain having symmetry by rotation of angle $2\pi/N$ and N points of maximum curvature (see Fig. 2).

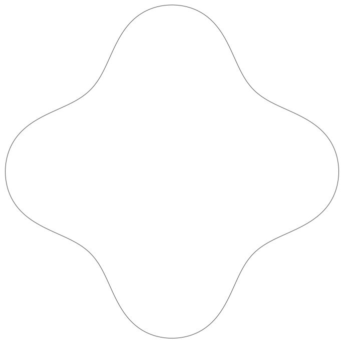

FIGURE 2. A symmetric domain with respect to the origin with $N = 4$ points of maximum curvature.

3. Magnetic steps

The tunneling effect in Theorem 2.2 is a consequence of the magnetic field and imposing the Neumann boundary condition (if a magnetic field were not present, the first eigenvalue would be simple and equal to 0, while the Neumann boundary condition inforces bound states to concentrate near the boundary points of maximum curvature thereby inducing a phenomenon of *multiple wells*).

The present contribution is concerned with the following question:
Can we observe a tunneling effect, similar to the one in Theorem 2.2, but induced purely by the magnetic field ?
That is, we would like to construct an example where the tunneling is not a consequence of imposing a boundary condition, but rather a consequence of the nature of the magnetic field. We will give an affirmative answer by working in the full plane $\mathbb{R}^2$ and considering a magnetic field with a discontinuity along a smooth curve[1] (the magnetic edge). In the case of a flat edge, we get a model in the full plane which plays the role of the de Gennes model for uniform magnetic fields. When the edge is non-flat and has symmetries, we observe an interesting tunneling effect.

3.1. **A new model: flat edge.**

Let us recall the model in $\mathbb{R}^2$ where $B = \operatorname{curl} \mathbf{A} = \mathbf{1}_{\mathbb{R}_+\times\mathbb{R}} + a\mathbf{1}_{\mathbb{R}_-\times\mathbb{R}}$ and $a \in [-1, 0)$ is a fixed constant[2]. We get in this case a family

[1]From a technical perspective, the magnetic discontinuity curve plays the same role in our case as the boundary does in Theorem 2.2.

[2]It is important for us to have $a < 0$, because in the opposite case, $a \in (0, 1)$, $\mu_a(\xi)$ defined in (3.3) becomes a monotone increasing function with $\inf_{\xi\in\mathbb{R}} \mu_a(\xi) = a$. This implies that the magnetic step will no longer attract the ground state, i.e. we do not expect localization near the magnetic step in this case.

of Schrödinger operators [28]

$$\mathfrak{h}_a[\xi] = -\frac{d^2}{d\tau^2} + V_a(\xi,\tau), \tag{3.1}$$

on $L^2(\mathbb{R})$, where $\xi \in \mathbb{R}$ is a parameter and

$$V_a(\xi,\tau) = \big(\xi + b_a(\tau)\tau\big)^2, \quad b_a(\tau) = \mathbf{1}_{\mathbb{R}_+}(\tau) + a\mathbf{1}_{\mathbb{R}_-}(\tau)\,. \tag{3.2}$$

We introduce the ground state energy of $\mathfrak{h}_a[\xi]$,

$$\mu_a(\xi) = \inf_{u\in B^1(\mathbb{R}),u\neq 0} \frac{\|u'\|^2_{L^2(\mathbb{R})} + \|\sqrt{V_a}\,u\|^2_{L^2(\mathbb{R})}}{\|u\|^2_{L^2(\mathbb{R})}}\,, \tag{3.3}$$

along with the following constant

$$\beta_a := \inf_{\xi\in\mathbb{R}} \mu_a(\xi) = \mu_a(\zeta_a)\,, \tag{3.4}$$

where $\zeta_a < 0$, is the unique minimum of $\mu_a(\cdot)$. Let ϕ_a be the *positive* and L^2-normalized ground state of $\mathfrak{h}_a[\zeta_a]$. We have [5]

$$c_2(a) := \frac{1}{2}\mu_a''(\zeta_a) > 0 \tag{3.5}$$

and

$$|a|\Theta_0 < \beta_a < \min(|a|,\Theta_0), \quad \phi_a'(0) < 0 \quad (-1 < a < 0)\,. \tag{3.6}$$

For $a = -1$, we have by a symmetry argument

$$\beta_{-1} = \Theta_0\,, \quad \zeta_{-1} = \xi_0\,, \quad \phi_{-1}(\tau) = u_0(|\tau|)\,, \tag{3.7}$$

thereby returning to the de Gennes model introduced in Sec. 2.2.

Later on, the following negative constant will be of particular interest,

$$M_3(a) = \frac{1}{3}\Big(\frac{1}{a} - 1\Big)\zeta_a\phi_a(0)\phi_a'(0) < 0. \tag{3.8}$$

3.2. **Curved edge and single well.** We return to the operator $\mathcal{P}_h$ in (1.1). Here and in the rest of the paper, we will work under the following assumption[3]

$$\left\{\begin{array}{l} \Omega_1 \subset \mathbb{R}^2 \text{ is a simply connected open set, } \Omega_2 = \mathbb{R}^2\backslash\overline{\Omega}_1, \\ \Gamma := \partial\Omega_1 \text{ is a } C^\infty \text{ smooth closed curve.} \end{array}\right\} \tag{3.9}$$

and that the magnetic field is a step function (see Fig. 3)

$$B = \mathbf{1}_{\Omega_1} + a\mathbf{1}_{\Omega_2} \quad \text{ where } -1 < a < 0\,. \tag{3.10}$$

The operator $\mathcal{P}_h$ is then self-adjoint in $L^2(\mathbb{R}^2)$ with domain[4]

$$\mathsf{Dom}(\mathcal{P}_h) = \{u \in L^2(\mathbb{R}^2)\ :\ (h\nabla - i\mathbf{A})^j u \in L^2(\mathbb{R}^2),\ j = 1,2\}. \tag{3.11}$$

[3]Our results are likely to hold when Γ is C^N smooth for some integer $N \geqslant 1$. We impose the C^∞ hypothesis since we use psudo-differential calculus and sought errors of order $\mathcal{O}(h^\infty)$.

[4]Since $\mathbf{A} \in H^1_{\rm loc}(\mathbb{R}^2)$, there is no jump across Γ of u and $\mathbf{n}\cdot(h\nabla - i\mathbf{A})u$, $\forall u \in \mathrm{Dom}(\mathcal{P}_h)$.

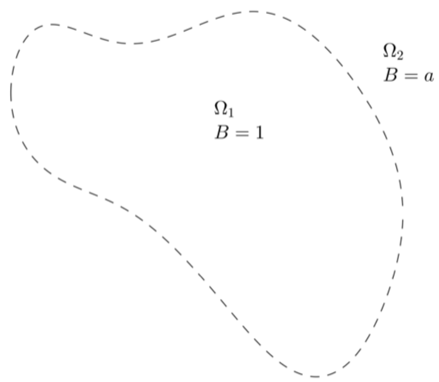

FIGURE 3. The plane $\mathbb{R}^2 = \Omega_1 \cup \Omega_2 \cup \Gamma$ with the non symmetric edge $\Gamma = \partial\Omega_1$ dashed.

By Persson's lemma [33], the essential spectrum of $\mathcal{P}_h$ is determined by the magnetic field at infinity (in our case it is equal to a), so

$$\inf \sigma_{\text{ess}}(\mathcal{P}_h) = |a|h\,.$$

Since $\beta_a < \min(|a|, \Theta_0)$, bound states of $\mathcal{P}_h$ are localized near the edge [5]. More precisely, for every $n \in \mathbb{N}$, there exist constants $\alpha, h_0, C_n > 0$ such that,

$$\int_{\mathbb{R}^2} \left(|u_{h,n}|^2 + h^{-1}|(h\nabla - i\mathbf{A})u_{h,n}|^2\right) \exp\left(\frac{\alpha\ \operatorname{dist}(x,\Gamma)}{h^{1/2}}\right) dx \leqslant C_n, \tag{3.12}$$

for all $h \in (0, h_0]$, where $u_{h,n}$ is a normalized eigenfunction associated to the n'th eigenvalue of $\mathcal{P}_h$.

Remark 3.1 (The case of bounded domains). We can also consider the Dirichlet or Neumann realizations of $\mathcal{P}_h$ in a bounded smooth domain Ω, in which case the spectrum is purely discrete. Related to our setting is [3, Thm. 1.2] dealing with a somehow different geometric condition, where the operator $\mathcal{P}_h$ is considered in $L^2(\Omega)$ with Dirichlet boundary condition, $\Omega_1 \subset \Omega$ and Γ a smooth curve that meets $\partial\Omega$ transversely, see Fig. 5. However, the proofs are not altered by considering the new setting of $\mathcal{P}_h$ above ($\mathcal{P}_h$ in the full plane and closed curve Γ). The main reason is that the property $\beta_a < |a|$ for $-1 < a < 0$ ensures the localization of the bound states near the edge Γ.

So the following result essentially follows from [3, Thm. 1.2]:

Theorem 3.2. *Assume that* (3.9) *and* (3.10) *hold and that the curvature* k *of* Γ *has a unique non-degenerate maximum, i.e.*

$$k_{\max} := \max_{\Gamma} k(s) = k(0) \quad \textit{with} \quad k_2 := k''(0) < 0$$

Then, for all $n \in \mathbb{N}^*$ the n-th eigenvalue $\lambda_n(h)$ of $\mathcal{P}_h$, defined in (1.1)*, satisfies as $h \to 0$,*

$$\lambda_n(h) = \beta_a h + k_{\max} M_3(a) h^{\frac{3}{2}} + (2n-1)\sqrt{\frac{k_2 M_3(a) c_2(a)}{2}}\, h^{\frac{7}{4}} + \mathcal{O}(h^{\frac{15}{8}}),$$

where β_a, $c_2(a)$ and $M_3(a)$ are introduced in (3.4)*,* (3.5) *and* (3.8) *respectively.*

Looking more closely at Theorem 3.2, we observe that the third term in the expansion of $\lambda_n(h)$ is effectively given (up to the factor of $h^{3/2}$) by the n-th eigenvalue of the following 1D operator on $L^2\big(\mathbb{R}/(2L\mathbb{Z})\big)$,

$$\mathfrak{L}_h^{\mathrm{eff}} = \frac{\mu_a''(\zeta_a)}{2}\Big(-h^{\frac{1}{2}}\partial_s^2 + V_a(s)\Big)\,, \quad V_a(s) = \frac{2M_3(a)(k(s)-k_{\max})}{\mu_a''(\zeta_a)}, \tag{3.13}$$

where $L = |\Gamma|/2$ and $|\Gamma|$ denotes the arc-length of Γ. Notice, that $V_a \geqslant 0$, due to the sign of $M_3(a)$ (see (3.8)). This point of view is important in order to discuss the case where Γ has symmetries and the splitting of the eigenvalues is no more of fractional order in h.

In the presence of several points of maximal curvature, a variant of Theorem 3.2 continues to hold but we may loose the information on the simplicity of the eigenvalues, exactly in the same manner observed for the Neumann problem (see (2.6)).

3.3. **Symmetric edge and tunneling.** Suppose that, in addition to (3.9) and (3.10), the following holds (see Fig 1):

Assumption 3.3.

i) *Ω_1 is symmetric with respect to the y-axis.*
ii) *The curvature k on Γ attains its maximum at exactly two symmetric points $a_1 = (a_{1,1}, a_{1,2})$ and $a_2 = (a_{2,1}, a_{2,2})$ with $a_{1,1} < 0$ and $a_{2,1} > 0$.*
iii) *Denoting by s_r and s_ℓ the arc-length coordinates of a_1 and a_2 respectively, we have $k''(s_r) = k''(s_\ell) < 0$.*

This is exactly the same geometric assumption on Ω as Assumption 2.1 for the Neumann realization in $L^2(\Omega)$, with the edge Γ playing the role of Σ, the boundary of Ω.

The presence of a symmetric edge yields a symmetric potential, and consequently two wells, in the effective operator introduced in (3.13), which in turn will induce a tunneling effect whose order of magnitude can be measured by the following quantities (similarly to what we have seen in Theorem 2.2):

$$\begin{aligned}\mathsf{A}_{\mathsf{u}}^a &= \exp\left(-\int_{[s_r,0]} \frac{(V_a^{\frac12})'(s)+g_a}{\sqrt{V_a(s)}}ds\right),\\ \mathsf{A}_{\mathsf{d}}^a &= \exp\left(-\int_{[s_\ell,L]} \frac{(V_a^{\frac12})'(s)-g_a}{\sqrt{V(s)}}ds\right),\\ g_a &= (V_a''(s_r)/2)^{\frac12} = (V_a''(s_\ell)/2)^{\frac12}\,.\end{aligned} \tag{3.14}$$

Up to leading order, the operator in (3.13) continues to be effective under the new assumptions on the edge, modulo additional terms related to the circulation of the magnetic field and the geometry.

Theorem 3.4. *Suppose that Assumption 3.3 holds in addition to* (3.9) *and* (3.10)*. The first and second eigenvalues of $\mathcal{P}_h$ satisfy as $h\to 0_+$,*

$$\lambda_2(h)-\lambda_1(h) = 2|w_a(h)| + o(h^{\frac{13}{8}}e^{-\mathsf{S}^a/h^{\frac14}}),$$

where:

$$\begin{aligned} w_a(h) &= \mu_a''(\zeta_a)h^{\frac{13}{8}}\pi^{-\frac12}g_a^{\frac12}\\ &\quad\times\left(\mathsf{A}_{\mathsf{u}}^a\sqrt{V_a(0)}e^{-\mathsf{S}_{\mathsf{u}}^a/h^{1/4}}e^{iLf_a(h)} + \mathsf{A}_{\mathsf{d}}^a\sqrt{V_a(L)}e^{-\mathsf{S}_{\mathsf{d}}^a/h^{1/4}}e^{-iLf_a(h)}\right),\end{aligned}$$

with μ_a and ζ_a introduced in Section 3.1, and

i. The potential V_a is introduced in (3.13)*;*
ii. S^a is the Agmon distance between the wells,

$$\mathsf{S}^a = \min(\mathsf{S}_{\mathsf{u}}^a,\mathsf{S}_{\mathsf{d}}^a),\ \mathsf{S}_{\mathsf{u}}^a = \int_{[s_r,s_\ell]}\sqrt{V_a(s)}\,ds,\ \mathsf{S}_{\mathsf{d}}^a = \int_{[s_\ell,s_r]}\sqrt{V_a(s)}\,ds\,; \tag{3.15}$$

iii. $\mathsf{A}_{\mathsf{u}}^a$, $\mathsf{A}_{\mathsf{d}}^a$ and g_a are defined in (3.14)*;*
iv. $f_a(h)=\gamma_0/h+\zeta_a/h^{1/2}-\alpha_0(a)$ with

$$\gamma_0 = \frac{|\Omega_1|}{|\Gamma|}, \tag{3.16}$$

and $\alpha_0(a)$ is a constant dependent on a and Ω_1.

Theorem 3.4 is the analogue of Theorem 2.2 but for the situation where tunneling is due to the discontinuity of the magnetic field (without the need for imposing a Neumann boundary condition). As in the proof of Theorem 2.2 in [7], the proof of Theorem 3.4 relies on an optimal tangential decay estimate of ground states.

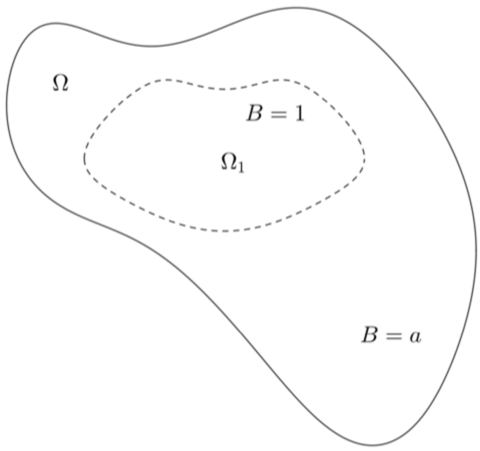

FIGURE 4. The domain Ω is split into two parts with the edge Γ (dashed) is a closed curve.

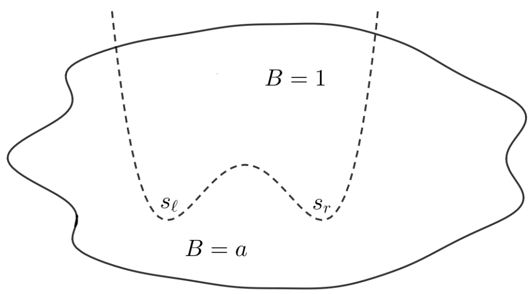

FIGURE 5. The edge Γ (dashed) splits the domain Ω into two parts and intersects the boundary $\partial\Omega$ transversely.

3.4. **Bounded domains.** Theorem 3.4 continues to hold if we consider the Dirichlet or Neumann realization of the operator $\mathcal{P}_h$ in $L^2(\Omega)$, where Ω is a domain with a C^2 boundary such that $\overline{\Omega}_1 \subset \Omega$ (see Fig 4). Thanks to (3.6), bound states of $\mathcal{P}_h$ are localized near $\Gamma = \partial\Omega_1$, and the proof of Theorem 3.4 is not altered. We can also modify the configuration of our domains Ω_1 and Ω_2 in (3.9) and still get the tunneling effect but without oscillatory terms. Let Ω be a domain with a C^1 boundary such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, where Ω_1 and Ω_2 are disjoint simply connected open sets. We consider a magnetic field as in (3.10) and notice that the edge $\Gamma = \Omega \cap \partial\Omega_1 = \Omega \cap \partial\Omega_2$ (see Fig 5). We assume that Γ is a smooth curve and consider the Dirichlet[5] realization of $\mathcal{P}_h$ in $L^2(\Omega)$. This is the situation considered in [3]. Now we assume that the curvature k along Γ has a non-degenerate maximum

[5]The Neumann realization leads to a completely different behavior, reminiscent of domains with corners [1].

attained at two points, with arc-length coordinates $s_\ell < 0$ and $s_r = -s_\ell$, and that it is an even function in a neighborhood of $[s_\ell, s_r]$. In this situation, the splitting between the first eigenvalues is given as follows:

$$\lambda_2(h) - \lambda_1(h) = 2|w_a(h)| + o(h^{\frac{13}{8}} e^{-\mathsf{S}^a/h^{\frac{1}{4}}}),$$

where

$$w_a(h) = 2\mu_a''(\zeta_a) h^{\frac{13}{8}} \pi^{-\frac{1}{2}} g_a^{\frac{1}{2}} \mathsf{A}_a \sqrt{V_a(0)}\, e^{-\mathsf{S}_a/h^{1/4}},$$

and

$$\mathsf{A}_a = 2\exp\left(-\int_{[s_\ell,s_r]} \frac{(V_a^{\frac{1}{2}})'(s) - g_a}{\sqrt{V_a(s)}} ds\right), \quad \mathsf{S}_a = \int_{[s_\ell,s_r]} \sqrt{V(s)} ds.$$

4. Reduction to a neighborhood of the edge

It will be convenient to work in Frénet coordinates, (s,t), along the edge Γ, valid in a neighborhood of Γ of the form

$$\Gamma(\epsilon) = \{x \in \mathbb{R}^2 \ : \ \operatorname{dist}(x,\Gamma) < \epsilon\} \quad (\epsilon > 0)\,. \tag{4.1}$$

Let us briefly recall these coordinates. Consider an arc-length parameterization of Γ , $M : (-L, L] \to \Gamma$, so that (see Assumption 3.3)

$$M(s_\ell) = a_1, \quad M(s_r) = a_2, \quad 0 < s_\ell < L, \quad -L < s_r < 0\,,$$

and

$$\Gamma \cap \{(x,y) \in \mathbb{R}^2 \,|\, \ x = 0\} = \{M(0) =: (0,y_0), M(L) =: (0,y_L)\} \quad \text{ with } y_0 > y_L\,.$$

Let $\mathbf{n}(s)$ be the unit normal to Γ pointing inward to Ω_1 (see Fig. 1), $\mathbf{t}(s) = \dot{\mathbf{n}}(s)$ the unit oriented tangent, so that $\det(\mathbf{t}(s), \mathbf{n}(s)) = 1$. Let us represent the torus $\mathbb{R}/2L\mathbb{Z}$ by the interval $(-L, L]$. We can pick $\epsilon_0 > 0$ such that

$$\Phi : \mathbb{R}/(2L\mathbb{Z}) \times (-\epsilon_0, \epsilon_0) \ni (s,t) \mapsto M(s) + t\mathbf{n}(s) \in \Gamma(\epsilon_0)$$

is a diffeomorphism whose Jacobian is

$$\mathfrak{a}(s,t) = 1 - tk(s)\,,$$

with $k(s)$ the curvature at $M(s)$, defined by $\ddot{\mathbf{n}}(s) = k(s)\mathbf{n}(s)$. The Hilbert space $L^2(\Gamma(\epsilon_0))$ is transformed to the weighted space

$$L^2\big(\mathbb{R}/2L\mathbb{Z} \times (-\epsilon_0, \epsilon_0); \mathfrak{a}\, dsdt\big)$$

and the operator $\mathcal{P}_h$ is transformed into the following operator (after a gauge transformation $(u, \mathbf{A}) \to (v = e^{i\phi/h}u, \mathbf{A}' = \mathbf{A} - \nabla\phi)$ to eliminate the normal component of $\mathbf{A}$, see [17, App. F]):

$$\begin{aligned}\tilde{\mathcal{P}}_h &:= -h^2 \mathfrak{a}^{-1} \partial_t \mathfrak{a} \partial_t \\ &+ \mathfrak{a}^{-1}\left(-ih\partial_s + \gamma_0 - b_a(t)t + \frac{k}{2} b_a(t) t^2\right) \mathfrak{a}^{-1} \left(-ih\partial_s + \gamma_0 - b_a(t)t + \frac{k}{2} b_a(t)t^2\right)\end{aligned}$$

where $b_a(t)$ is introduced in (3.2) and γ_0 is the circulation introduced in (3.16).

Following the presentation of [7] (see also references therein), it is convenient to introduce a truncated version of the operator $\tilde{P}_h$ so that it can be defined on $\mathbb{R}/2L\mathbb{Z} \times \mathbb{R}$ instead of $\mathbb{R}/2L\mathbb{Z} \times (-\epsilon_0, \epsilon_0)$. This will be useful when rescaling the t variable. What is handy in this situation is that the actual bound states of the operator $\mathcal{P}_h$ decay exponentially away from the edge, at the length scale $\hbar := h^{1/2}$, see (3.12). This motivates the change of variables, $t = \hbar\tau$ and $s = \sigma$, that will allow the same spectral reduction as in [7, Prop. 2.7]. We will skip the details which are the same as in [7].

From now on we set

$$\mu = h^{\frac{1}{4}+\eta} \text{ for a fixed } \eta \in (0, \frac{1}{4}) \tag{4.2}$$

and we introduce the function

$$c_\mu(\tau) = c(\mu\tau)\,, \tag{4.3}$$

where $c \in C_c^\infty(\mathbb{R})$ satisfies $c = 1$ on $[-1, 1]$ and $c = 0$ on $\mathbb{R}\backslash(-2, 2)$. Consider the new weight term

$$\tilde{\mathfrak{a}}_h(\sigma, \tau) = 1 - h^{1/2} c_\mu(\tau)\tau k(\sigma)\,,$$

and the self-adjoint operator $\tilde{\mathcal{N}_h}$ on the Hilbert space $L^2(\mathbb{R}/2L\mathbb{Z} \times \mathbb{R}; \tilde{\mathfrak{a}}_h d\sigma d\tau)$,

$$\begin{aligned}\tilde{\mathcal{N}_h} &= -\tilde{\mathfrak{a}}_h^{-1}\partial_\tau \mathfrak{a}_\hbar \partial_\tau \\ &\quad + \tilde{\mathfrak{a}}_h^{-1}\left(-ih^{1/2}\partial_\sigma + h^{-1/2}\gamma_0 - b_a\tau + h^{1/2}c_\mu \frac{k}{2} b_a \tau^2\right) \\ &\quad \times \tilde{\mathfrak{a}}_h^{-1}\left(-ih^{1/2}\partial_\sigma + h^{-1/2}\gamma_0 - b_a\tau + h^{1/2}c_\mu \frac{k}{2} b_a \tau^2\right),\end{aligned} \tag{4.4}$$

with domain

$$\begin{aligned}\mathsf{Dom}(\tilde{\mathcal{N}_h}) = \{u \in L^2(\mathbb{R}/2L\mathbb{Z} \times \mathbb{R}) \mid \partial_\tau^2 u \in L^2(\mathbb{R}/2L\mathbb{Z} \times \mathbb{R}),\\ (-ih^{1/2}\partial_\sigma + h^{-1/2}\gamma_0 - b_a\tau)^2 u \in L^2(\mathbb{R}/2L\mathbb{Z} \times \mathbb{R})\}.\end{aligned}$$

We have now the following spectral reduction[6]:

Proposition 4.1. *Let $a \in (-1, 0)$ and S^a be the Agmon distance introduced in (3.15). There exist $K > \mathsf{S}^a$, $C, h_0 > 0$ such that, for all $h \in (0, h_0)$, we have*

$$\lambda_n(h) - Ce^{-K/h^{\frac{1}{4}}} \leqslant h\lambda_n(\tilde{\mathcal{N}_h}) \leqslant \lambda_n(h) + Ce^{-K/h^{\frac{1}{4}}}\,,$$

where $\lambda_n(h)$ and $\lambda_n(\tilde{\mathcal{N}_h})$ are the n-th (min-max) eigenvalues of the operators $\mathcal{P}_h$ and $\tilde{\mathcal{N}_h}$ respectively.

[6]The eigenvlaues of the operator $\tilde{\mathcal{N}_h}$ depend on η in (4.2). However, the estimates in Proposition 4.1 hold uniformly with respect to $\eta \in (0, \epsilon)$ for any fixed $\epsilon \in (0, \frac{1}{4})$.

Looking at the operator in (4.4), the effective semi-classical parameter is $\hbar = h^{1/2}$ (this is the parameter appearing in front of ∂_σ). So with

$$\hbar = h^{\frac{1}{2}}, \quad \mu = \hbar^{\frac{1}{2}+2\eta} \text{ for a fixed } \eta \in (0, \frac{1}{4}), \tag{4.5}$$

we introduce the new weight term

$$\mathfrak{a}_\hbar(\sigma, \tau) = 1 - \hbar c_\mu(\tau)\tau k(\sigma),$$

and the self-adjoint operator $\mathcal{N}_\hbar$ on the Hilbert space $L^2(\mathbb{R}/2L\mathbb{Z} \times \mathbb{R}; \mathfrak{a}_\hbar d\sigma d\tau)$, which is nothing but the operator in (4.4) but with a change of parameter according to (4.5),

$$\begin{aligned} \mathcal{N}_\hbar = &-\mathfrak{a}_\hbar^{-1}\partial_\tau \mathfrak{a}_\hbar \partial_\tau \\ &+ \mathfrak{a}_\hbar^{-1}\left(-i\hbar\partial_\sigma + \hbar^{-1}\gamma_0 - b_a\tau + \hbar c_\mu \frac{k}{2} b_a \tau^2\right)\mathfrak{a}_\hbar^{-1} \\ &\times \left(-i\hbar\partial_\sigma + \hbar^{-1}\gamma_0 - b_a\tau + \hbar c_\mu \frac{k}{2} b_a \tau^2\right). \end{aligned} \tag{4.6}$$

The domain of the operator $\mathcal{N}_\hbar$ is

$$\begin{aligned} \mathsf{Dom}(\mathcal{N}_\hbar) = \{u \in L^2(\Gamma \times \mathbb{R}) \mid \partial_\tau^2 u \in L^2(\mathbb{R}/2L\mathbb{Z} \times \mathbb{R}), \\ (-i\hbar\partial_\sigma + \hbar^{-1}\gamma_0 - b_a\tau)^2 u \in L^2(\mathbb{R}/2L\mathbb{Z} \times \mathbb{R})\}. \end{aligned}$$

With Proposition 4.1 in hand, it is enough to compute the leading order term of $\nu_2(\hbar) - \nu_1(\hbar)$ to prove Theorem 3.4, where, for $n \geqslant 1$, we denote by $\nu_n(\hbar)$ the n'th min-max eigenvalue of $\mathcal{N}_\hbar$.

5. Single well and WKB construction

We will adjust the edge Γ so that we only have a single point of maximum curvature, s_r or s_ℓ. This procedure will give us two new operators, the "right well" and "left well" operators, $\mathcal{N}_{\hbar,r,\gamma_0}$ and $\mathcal{N}_{\hbar,\ell,\gamma_0}$ respectively. The same procedure appears, for similar problems in the context of geometrically induced tunneling effects [20, 31], but we follow here [7, Sec. 2.4] which is slightly different, but more convenient for dealing with the symbol of the operator later on.

5.1. **Right well operator.**

We present the construction for the right well operator, $\mathcal{N}_{\hbar,r,\gamma_0}$ and deduce the other one by symmetry. Let us fix $\hat{\eta}$ as follows

$$0 < \hat{\eta} < \min\left(\frac{1}{4}, \frac{L}{4}\right) \quad \text{where } L = \frac{|\Gamma|}{2}. \tag{5.1}$$

First, we identify Γ with $(s_\ell - 2L, s_\ell]$ (by periodicity and translation of the s variable), then we extend the curvature k to a function k_r on $\mathbb{R}$ as follows:

$$\begin{aligned} k_r &= k \quad &&\text{on} \quad I_{2\hat{\eta},r} := (s_\ell - 2L + \hat{\eta}, s_\ell - \hat{\eta}), \\ k_r &= 0 \quad &&\text{on} \quad (-\infty, s_\ell - 2L] \cup [s_\ell, +\infty), \end{aligned} \tag{5.2}$$

and k_r has a unique non-degenerate maximum at s_r. Consequently, k_r satisfies (2.4).

We consider now the operator in $L^2(\mathbb{R}^2; \mathfrak{a}_{\hbar,r} d\sigma d\tau)$,

$$\begin{aligned}\mathcal{N}_{\hbar,r,\gamma_0} &= -\mathfrak{a}_{\hbar,r}^{-1}\partial_\tau \mathfrak{a}_{\hbar,r}\partial_\tau \\ &\quad + \mathfrak{a}_{\hbar,r}^{-1}\left(-i\hbar\partial_\sigma + \hbar^{-1}\gamma_0 - b_a\tau + \hbar c_\mu \frac{k_r}{2} b_a \tau^2\right)\mathfrak{a}_{\hbar,r}^{-1} \\ &\quad \times \left(-i\hbar\partial_\sigma + \hbar^{-1}\gamma_0 - b_a\tau + \hbar c_\mu \frac{k_r}{2} b_a \tau^2\right)\end{aligned} \tag{5.3}$$

where

$$\mathfrak{a}_{\hbar,r}(\sigma,\tau) = 1 - \hbar c_\mu(\tau)\tau k_r(\sigma)\,. \tag{5.4}$$

Since, k_r satisfies (2.4), we have, for an arbitrarily fixed $n \in \mathbb{N}$ (with β_a, $c_2(a)$ and $M_3(a)$ from (3.4), (3.5) and (3.8)),

$$\lambda_n(\mathcal{N}_{\hbar,r,\gamma_0}) = \beta_a h + k_{max} M_3(a) h^{\frac{3}{2}} + (2n-1)\sqrt{\frac{k_2 M_3(a) c_2(a)}{2}}\, h^{\frac{7}{4}} + \mathcal{O}(h^{\frac{15}{8}})\,. \tag{5.5}$$

We are now in a simply connected domain, so the operators $\mathcal{N}_{\hbar,r,\gamma_0}$ and $\mathcal{N}_{\hbar,r,0}$ are unitarily equivalent (we can gauge away the flux term $\mathcal{N}_{\hbar,r,\gamma_0}$). Denote by $u_{\hbar,r}$ a normalized ground state of $\mathcal{N}_{\hbar,r,0}$ (the operator without flux term), a corresponding normalized ground state of $\mathcal{N}_{\hbar,r,\gamma_0}$ is given by:

$$\breve{\phi}_{\hbar,r}(\sigma,\tau) = e^{-i\gamma_0\sigma/\hbar^2} u_{\hbar,r}(\sigma,\tau). \tag{5.6}$$

5.2. **Left well operator.** Using the symmetry operator

$$Uf(\sigma,\tau) := \overline{f(-\sigma,\tau)}\,,$$

we can define the left well operator on $L^2(\mathbb{R}^2; \mathfrak{a}_{\hbar,\ell}(\sigma,\tau)$ by :

$$\mathcal{N}_{\hbar,\ell,\gamma_0} = U^{-1}\mathcal{N}_{\hbar,r,\gamma_0} U\,, \tag{5.7}$$

where

$$\mathfrak{a}_{\hbar,\ell}(\sigma,\tau) = \mathfrak{a}_{\hbar,r}(-\sigma,\tau)\,.$$

The left and right operators have the same spectrum, and a normalized ground state of $\mathcal{N}_{\hbar,\ell,\gamma_0}$ is

$$\breve{\phi}_{\hbar,\ell} := U\breve{\phi}_{\hbar,r} = e^{-i\gamma_0\sigma/\hbar^2} u_{\hbar,\ell}(\sigma,\tau) \tag{5.8}$$

where $u_{\hbar,\ell} = Uu_{\hbar,r}$.

5.3. **WKB expansions.** We focus on the right well operator and construct an approximate eigenvalue and an approximate ground state by WKB expansions, involving *formal series* in the sense of [8, Notation 1.13]. The construction can be translated to the left operator by symmetry.

Let us introduce the Agmon distance

$$\Phi_r(\sigma) = \int_{[s_r,\sigma]} \sqrt{V_{a,r}(s)}ds \tag{5.9}$$

related to the "right well" potential[7]

$$V_{a,r}(\sigma) = \frac{2M_3(a)(k_r(s) - k_{\max})}{\mu_a''(\zeta_a)} \tag{5.10}$$

Theorem 5.1. *There exist two sequences* $(b_j)_{j\geqslant 0} \subset \mathit{Dom}(\mathcal{N}_{\hbar,r})$, $(\delta_j)_{j\geqslant 0} \subset \mathbb{R}$, *a family of functions* $(\Psi_{\hbar,r})_{\hbar\in(0,\hbar_0]} \subset L^2(\mathbb{R}^2)$ *and a family of real numbers* $(\delta(\hbar))_{\hbar\in(0,\hbar_0]}$ *such that*

$$e^{\Phi_r(\sigma)/\hbar^{\frac{1}{2}}} e^{-i\sigma\zeta_a/\hbar}\Psi_{\hbar,r}(\sigma,\tau) \underset{\hbar\to 0}{\sim} \hbar^{-\frac{1}{8}} \sum_{j\geqslant 0} b_j(\sigma,\tau)\hbar^{\frac{j}{2}}, \tag{5.11}$$

$$\delta(\hbar) \underset{\hbar\to 0}{\sim} \sum_{j\geqslant 0} \delta_j \hbar^{\frac{j}{2}} \,,$$

and

$$e^{\Phi_r(\sigma)/\hbar^{\frac{1}{2}}} \left(\mathcal{N}_{\hbar,r} - \delta(\hbar)\right) \Psi_{\hbar,r} = \mathcal{O}(\hbar^\infty)\,. \tag{5.12}$$

Furthermore

$$\delta_0 = \beta_a\,, \quad \delta_1 = 0, \quad \delta_2 = M_3(a)k_{\max}, \quad \delta_3 = \sqrt{\frac{k_2 M_3(a) c_2(a)}{2}},$$

$$b_0(\sigma,\tau) = f_0(\sigma)\phi_a(\tau), \tag{5.13}$$

and f_0 *solves the effective transport equation*

$$\frac{\mu_a''(\zeta_a)}{2}(\Phi_r'\partial_\sigma + \partial_\sigma\Phi_r')f_0 + iF(\sigma)f_0 = \sqrt{\frac{k_2 M_3(a)c_2(a)}{2}} f_0\,, \tag{5.14}$$

where F *is a smooth real-valued function, introduced in* (5.15), *such that* $F(s_r) = 0$.

Remark 5.2. Let us explain precisely how the asymptotics in Theorem 5.1 are interpreted. For every $N \geqslant 1$ we introduce the function $\psi^N_{\hbar,r}(\sigma,\tau)$ and the real number $\delta^N(\hbar)$ as follows:

$$\Psi^N_{\hbar,r}(\sigma,\tau) := e^{-\Phi_r(\sigma)/\hbar^{\frac{1}{2}}} e^{i\sigma\zeta_a/\hbar}\,\hbar^{-\frac{1}{8}} \sum_{j=0}^N b_j(\sigma,\tau)\hbar^{\frac{j}{2}}, \quad \delta^N(\hbar) = \sum_{j=0}^N \delta_j \hbar^{\frac{j}{2}}\,.$$

Then (5.12) means

$$e^{\Phi_r(\sigma)/\hbar^{\frac{1}{2}}} \left\|\left(\mathcal{N}_{\hbar,r} - \delta(\hbar)\right) \Psi^N_{\hbar,r}\right\|_{L^2(\mathbb{R}_\tau)} = \mathcal{O}(\hbar^N)$$

[7] Recall from (3.8) that $M_3(a) < 0$, so $V_{a,r} \geqslant 0$.

locally uniformly with respect to σ.

Proof of Theorem 5.1. We work in an arbitrary bounded set of $\mathbb{R}^2$, so, in the below computations, we take $c_\mu = 1$ in (5.3) at the cost of an error $\mathcal{O}(h^\infty)$. That is possible because our constructions will involve functions decaying exponentially with respect to the normal variable τ.

Let us introduce the operator

$$\widehat{\mathcal{N}}_{\hbar,r} := e^{\Phi_r(\sigma)/\hbar^{\frac{1}{2}}} e^{i\sigma\zeta_a/\hbar} \mathcal{N}_{\hbar,r} e^{-i\sigma\zeta_a/\hbar} e^{-\Phi_r(\sigma)/\hbar^{\frac{1}{2}}} .$$

It admits the formal expansion

$$\widehat{\mathcal{N}}_{\hbar,r} = \mathcal{L}_0 + \hbar^{1/2}\mathcal{L}_1 + \hbar\mathcal{L}_2 + \hbar^{3/2}\mathcal{L}_3 + \hbar^2\mathcal{L}_4 + \cdots$$

where

$$\begin{aligned}
\mathcal{L}_0 &= -\partial_\tau^2 + (\zeta_a + b_a\tau)^2 \\
\mathcal{L}_1 &= -2(\zeta_a + b_a\tau) i\Phi_r'(\sigma) \\
\mathcal{L}_2 &= k_r\partial_\tau - 2(\zeta_a + b_a\tau)\Big(- i\partial_\sigma + \frac{k_r}{2} b_a\tau^2\Big) - \Phi_r'(\sigma)^2 + 2k_r\tau(\zeta_a + b_a\tau)^2 \\
\mathcal{L}_3 &= \Big(- i\partial_\sigma + \frac{k_r}{2} b_a\tau^2\Big) i\Phi'(\sigma) + i\Phi_r'(\sigma)\Big(- i\partial_\sigma + \frac{k_r}{2} b_a\tau^2\Big) \\
&\quad - 4\Phi_r'(\sigma)\tau k_r(\zeta_a + b_a\tau) \\
\mathcal{L}_4 &= -\partial_\sigma^2 + 2k_r^2\tau^2(\zeta_a + b_a\tau)^2 \\
&\quad - (\zeta_a + b_a\tau)\Big[\Big(- i\partial_\sigma + \frac{k_r}{2} b_a\tau^2\Big) k_r + k_r\Big(- i\partial_\sigma + \frac{k_r}{2} b_a\tau^2\Big)\Big] \\
&\ \vdots
\end{aligned}$$

Let $b(\sigma,\tau;\hbar) := \sum_{j\geqslant 0} b_j(\sigma,\tau)\hbar^{\frac{j}{2}}$ and let us formally solve the equation

$$\big(\widehat{\mathcal{N}}_{\hbar,r} - \delta(\hbar)\big) b(\sigma,\tau;\hbar) = \mathcal{O}(\hbar^\infty).$$

Expanding the foregoing equation in powers of $\hbar^{1/2}$, the vanishing of the coefficient of each $\hbar^{j/2}$, $j \geqslant 0$, yields the following equations

$$\begin{aligned}
(\mathcal{L}_0 - \delta_0)b_0 &= 0 \\
(\mathcal{L}_0 - \delta_0)b_1 &= (\delta_1 - \mathcal{L}_1)b_0 \\
(\mathcal{L}_0 - \delta_0)b_2 &= (\delta_2 - \mathcal{L}_2)b_0 + (\delta_1 - \mathcal{L}_1)b_1 \\
(\mathcal{L}_0 - \delta_0)b_3 &= (\delta_3 - \mathcal{L}_3)b_0 + (\delta_2 - \mathcal{L}_2)b_1 + (\delta_1 - \mathcal{L}_1)b_2 \\
&\ \vdots
\end{aligned}$$

We will find solutions to these equations one by one. The first equation leads us to choose $\delta_0 = \zeta_a$ and $b_0(\sigma,\tau) = f_0(\sigma)\phi_a(\tau)$, where $f_0(\sigma)$ is to be determined at a later stage. The function f_0 will actually be free untill the first equation involving $\mathcal{L}_3$.

For the equation for b_1, we determine δ_1 by assuming that $(\delta_1 - \mathcal{L}_1)b_0$ is orthogonal to ϕ_a in $L^2(\mathbb{R})$. Then, we take the inner product with ϕ_a in $L^2(\mathbb{R})$, use (9.2) and get

$$\delta_1 = 0, \quad b_1(\sigma,\tau) = 2i\Phi_r'(\sigma)f_0(\sigma)\mathcal{R}_a\big((\zeta_a + b_a\tau)\phi_a\big),$$

where $\mathcal{R}_a$ the regularized resolvent introduced in (9.1). Since we are applying $\mathcal{R}_a$ on functions orthogonal to ϕ_a, we can slightly abuse notation and say that it is equal to $(\mathcal{L}_0 - \delta_0)^{-1}$.

The equation for b_2 will determine δ_2. This equation can be solved if $(\delta_2 - \mathcal{L}_2)b_0 - \mathcal{L}_1 b_1$ is orthogonal to ϕ_a in $L^2(\mathbb{R})$, which we assume henceforth. Taking the inner product with ϕ_a in $L^2(\mathbb{R})$, using (9.3) and Remark 9.1, we get

$$\delta_2 f_0(\sigma) + \frac{\mu_a''(\zeta_a)}{2}\Phi_r'(\sigma)^2 f_0(\sigma) - M_3(a)k_r(\sigma)f_0(\sigma) = 0,$$

since

$$2\int_{\mathbb{R}} \tau(\zeta_a + b_a(\tau)\tau)^2|\phi_a(\tau)|^2\, d\tau - \int_{\mathbb{R}} b_a(\tau)\tau^2(\zeta_a + b_a(\tau)\tau)|\phi_a(\tau)|^2\, d\tau = M_3(a)\,.$$

So, we choose $\delta_2 = k_r(0) = k_{\max}$, and the foregoing equation involving f_0 is valid everywhere in light of (5.9), independently of the choice of f_0. At the same time, we choose b_2 as follows:

$$b_2(\sigma,\tau) = (\mathcal{L}_0 - \delta_0)^{-1}\Big((\delta_2 - \mathcal{L}_2)b_0 - \mathcal{L}_1 b_1\Big).$$

From the equation of b_3, we will determine δ_3 and $f_0(\sigma)$. Taking the inner product with ϕ_a in $L^2(\mathbb{R})$ and using (9.3), we get

$$\Big\langle(\delta_3 - \mathcal{L}_3)b_0 + (\delta_2 - \mathcal{L}_2)b_1 + (\delta_1 - \mathcal{L}_1)b_2, \phi_a\Big\rangle_{L^2(\mathbb{R})} = \\ \Big(\delta_3 - \frac{\mu''(\zeta_a)}{2}\big(\Phi_r'\partial_\sigma + \partial_\sigma\Phi_r'\big)\Big)f_0(\sigma) + iF(\sigma)f_0(\sigma),$$

where $F(\sigma)$ is the real-valued function

$$F(\sigma) = |\Phi_r'(\sigma)|^2\int_{\mathbb{R}} g(\sigma,\tau)\phi_a(\tau)d\tau, \tag{5.15}$$

and

$$\begin{aligned} g(\sigma,\tau) &= (\mathcal{L}_0 - \delta_0)^{-1}\big(g_1(\sigma,\tau) + g_2(\sigma,\tau)\big)\\ g_1(\sigma,\tau) &= -\Big(k_{\max} + k_r(\sigma)\big(\zeta_a + b_a\tau)^2 - b_a(\zeta_a + b_a\tau)\tau^2\big) - |\Phi(\sigma)|^2\Big)\phi_a(\tau)\\ &\quad + k_r(\sigma)\phi_a'(\tau)\\ g_2(\sigma,\tau) &= -4|\Phi_r(\sigma)|^2(\mathcal{L}_0 - \delta_0)^{-1}\big((\zeta_a + b_a\tau)\phi_a(\tau)\big). \end{aligned}$$

Since $\Phi'_r(s_r) = 0$, we observe that $F(s_r) = 0$. We can solve the equation of b_3 if $(\delta_3 - \mathcal{L}_3)b_0 + (\delta_2 - \mathcal{L}_2)b_1 + (\delta_1 - \mathcal{L}_1)b_2$ is orthogonal to ϕ_a in $L^2(\mathbb{R})$, which yields the following equation for f_0:

$$\Big(\delta_3 - \frac{\mu''_a(\zeta_a)}{2}\big(\Phi'_r\partial_\sigma + \partial_\sigma\Phi'_r\big)\Big) f_0(\sigma) + F(\sigma)f_0(\sigma) = 0.$$

Since $F(s_r) = \Phi'_r(s_r) = 0$, the foregoing equation has a solution satisfying $f_0(s_r) \neq 0$ if $\delta_3 = \frac{\mu''_a(\zeta_a)}{2}\Phi''_r(s_r)$, thereby determining δ_3 (from (5.10)) and f_0. The procedure can be continued to any preassigned order. □

Remark 5.3 (Solving (5.14) & normalization of $\Psi_{\hbar,r}$).

In (5.14), we make the ansatz $f_0(\sigma) = e^{i\alpha_0(\sigma)}\tilde{f}_0(\sigma)$, with $\tilde{f}_0$ and α_0 are real-valued functions such that $\tilde{f}_0(0) > 0$. Then we get from (5.14):

$$\frac{\mu''_a(\zeta_a)}{2}\big(\Phi'_r\partial_\sigma + \partial_\sigma\Phi'_r\big)\tilde{f}_0(\sigma) = \sqrt{\frac{k_2M_3(a)c_2(a)}{2}}f_0(\sigma)$$

and

$$\mu''_a(\zeta_a)\alpha'_0(\sigma)\Phi'_r(\sigma) + F(\sigma) = 0\,.$$

This will determine $\tilde{f}_0(\sigma)$ uniquely up to the choice of $\tilde{f}_0(0)$, and also $\alpha_0(\sigma)$ uniquely up to an additive constant (see [7, Eq. (2.14) & Rem. 2.9]). We choose $\tilde{f}(0) = \big(\frac{g_a}{\pi}\big)^{1/4}(\mathsf{A}^a_{\mathsf{u}})^{1/2}$ which yields that the WKB solution $\Psi_{\hbar,r}$ in Theorem 5.1 is almost normalized, $\|\Psi_{\hbar,r}\| \sim 1$. The constant α_a appearing in Theorem 3.4 is

$$\alpha_a = \frac{\alpha_0(0) - \alpha_0(-L)}{L}. \tag{5.16}$$

6. Optimal tangential Agmon estimates

The challenge of obtaining optimal decay estimates of bound states of the Neumann magnetic Laplacian matching with the WKB solutions was recently taken up in [7] by introducing pseudo-differential calculus with operator-valued symbols. Fortunately, the method is quite general and can handle our situation of magnetic steps.

6.1. **A tangential elliptic estimate.** We work with the single 'right well' flux free operator, $\mathcal{N}_{\hbar,r} := \mathcal{N}_{\hbar,r,0}$, introduced in (5.3). For the sake of simplicity, we will omit the reference to 'right well' in the notation and write $\mathcal{N}_\hbar$ and k instead of $\mathcal{N}_{\hbar,r}$ and k_r.

The optimal estimates, on the bound states of $\mathcal{N}_\hbar$, will hold in spaces with an exponential weight, defined via a sub-solution of an effective eikonal equation. More precisely, we consider a family of Lipschitz functions $(\varphi_\hbar)_{\hbar\in(0,1]} \subset C(\mathbb{R};\overline{\mathbb{R}_+})$ satisfying the following hypothesis:

Assumption 6.1. *For all $M > 0$ there exist $\hbar_0, C, R > 0$ such that, for all $\hbar \in (0,\hbar_0)$, the function $\varphi := \varphi_\hbar$ satisfies* *(i) for all* $\sigma \in \mathbb{R}$, $\mathfrak{v}(\sigma) - \frac{\mu_a''(\zeta_a)}{2}\varphi'(\sigma)^2 \geqslant 0$,

(ii) for all σ *such that* $|\sigma - s_r| \geqslant R\hbar^{\frac{1}{2}}$, $\mathfrak{v}(\sigma) - \frac{\mu_1''(\xi_0)}{2}\varphi'(\sigma)^2 \geqslant M\hbar$,

where $\mathfrak{v}(\sigma) = M_3(a)(k(\sigma) - k_{\max})$.

In the sequel, to lighten the notation, we write φ instead of $\varphi_\hbar$. We consider the conjugate operator, with the same domain as $\mathcal{N}_\hbar$, and defined by:

$$\mathcal{N}_\hbar^\varphi = e^{\varphi/\hbar^{\frac{1}{2}}}\mathcal{N}_\hbar e^{-\varphi/\hbar^{\frac{1}{2}}} = -a_\hbar^{-1}\partial_\tau a_\hbar \partial_\tau + a_\hbar^{-1}\mathcal{T}_\hbar^\varphi a_\hbar^{-1}\mathcal{T}_\hbar^\varphi \tag{6.1}$$

where

$$\mathcal{T}_\hbar^\varphi := \left(-i\hbar\partial_\sigma - b_a\tau + i\hbar^{\frac{1}{2}}\varphi' + \hbar c_\mu \frac{\kappa_r}{2} b_a \tau^2\right)$$

Theorem 6.2 (Bonnaillie-Noël–Hérau–Raymond)**.** *Let* $c_0 > 0$ *and* $\chi_0 \in C_c^\infty(\mathbb{R})$ *be* 1 *in a neighborhood of* 0*. Under Assumption 6.1, there exist* $c, \hbar_0 > 0$ *such that for all* $\hbar \in (0, \hbar_0)$, $z \in [\beta_a + M_3(a)k_{\max}\hbar - c_0\hbar^2, \beta_a + M_3(a)k_{\max}\hbar + c_0\hbar^2]$ *and all* $\psi \in \mathsf{Dom}(\mathcal{N}_\hbar^\varphi)$,

$$c\hbar^2\|\psi\| \leqslant \|\langle\tau\rangle^6(\mathcal{N}_\hbar^\varphi - z)\psi\| + \hbar^2\|\chi_0(\hbar^{-\frac{1}{2}}R^{-1}(\sigma - s_r))\psi\|,$$

and

$$c\hbar^2\|\hbar^2\partial_\sigma^2\psi\| \leqslant \|\langle\tau\rangle^6(\mathcal{N}_\hbar^\varphi - z)\psi\| + \hbar^2\|\chi_0(\hbar^{-\frac{1}{2}}R^{-1}(\sigma - s_r))\psi\|,$$

where $\langle\tau\rangle = (1 + \tau^2)^{1/2}$.

Modulo the decomposition of the symbol of the operator $\mathcal{N}_\hbar^\varphi$ and its parametrix, the proof of Theorem 6.2 is the same as that of [7, Thm. 5.1]. In the sequel, we give only the new ingredients.

Let us write

$$\mathcal{N}_\hbar^\varphi u = \operatorname{Op}_\hbar^{\mathrm{W}}(n_\hbar)\, u = \frac{1}{(2\pi\hbar)}\iint_{\mathbb{R}^2} e^{i(\sigma - s)\cdot\xi/\hbar} n_\hbar\left(\frac{\sigma + s}{2}, \xi\right) u(s)\,ds\,d\xi \tag{6.2}$$

where the foregoing quantization formula is formal, unless we consider it on, say, for u in the space $\mathcal{S}\big(\mathbb{R}; \widehat{\mathcal{S}}(\mathbb{R})\big)$ where

$$\widehat{\mathcal{S}}(\mathbb{R}) = \{v \in H^2(\mathbb{R})) \mid v|_{\overline{\mathbb{R}_\pm}} \in \mathcal{S}(\overline{\mathbb{R}_\pm})\}. \tag{6.3}$$

The operator-valued symbol $n_\hbar$ can be decomposed as follows

$$n_\hbar = n_0 + \hbar^{\frac{1}{2}} n_1 + \hbar n_2 + \hbar^{\frac{3}{2}} n_3 + \hbar^2 \tilde{r}_\hbar,$$

where

$$\begin{aligned} n_0(\sigma,\xi) &= -\partial_\tau^2 + (\xi - b_a(\tau)\tau)^2\,,\\ n_1(\sigma,\xi) &= 2i(\xi - b_a(\tau)\tau)\varphi'(\sigma)\,,\\ n_2(\sigma,\xi) &= -\varphi'(\sigma)^2 + \kappa c_\mu(\tau)\partial_\tau + c_\mu\kappa(\sigma)(\xi - b_a(\tau)\tau)b_a(\tau)\tau^2\\ &\quad + 2\kappa(\sigma)\tau c_\mu(\tau)(\xi - b_a\tau)^2 + \kappa(\sigma)\tau c_\mu'(\tau)\,,\\ \mathrm{Re}\, n_3(\sigma,\xi) &= 0\,,\\ \tilde r_\hbar(\sigma,\xi) &= \mathcal{O}(\tau^4, (\xi - b_a(\tau)\tau)^2\tau^2, (\xi - b_a\tau)\tau, \tau^2\partial_\tau). \end{aligned} \tag{6.4}$$

The notation $\mathcal{O}$ is defined in [7, Notation 3.1]:
For differential operators $A, B, C, \cdots$ on $\mathbb{R}_\tau$, writing $A = \mathcal{O}(B, C, \cdots)$ means the following:

$$\exists\, c > 0,\ \forall\, u \in \widehat{\mathcal{S}}(\mathbb{R}), \quad \|Au\|_{L^2(\mathbb{R})} \leqslant c\big(\|Bu\|_{L^2(\mathbb{R})} + \|Cu\|_{L^2(\mathbb{R})} + \cdots\big), \tag{6.5}$$

where $\widehat{\mathcal{S}}(\mathbb{R})$ is the space introduced in (6.3) and the constant c is independent of $A, B, C, \cdots$ (in particular, in (6.4), the estimate is uniform with respect to (σ,ξ)).

Let us introduce a modified symbol by truncating the frequency variable. Recall that $\zeta_a < 0$ is the unique minimum of the model operator with a flat edge (see (3.4)). It will be convenient to introduce

$$\hat\zeta_a := -\zeta_a > 0\,. \tag{6.6}$$

Pick a smooth bounded and increasing function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(\xi) = \xi$ for $\xi \in (-\hat\zeta_a/2, \hat\zeta_a/2)$, and $\eta_+ := \lim_{\xi\to+\infty}\chi(\xi) \in (0, \hat\zeta_a)$. We introduce the function

$$\chi_1(\xi) = \hat\zeta_a + \chi(\xi - \hat\zeta_a), \tag{6.7}$$

and the operator

$$\mathrm{Op}_\hbar^{\mathrm{W}}(p_\hbar) \quad \text{where } p_\hbar(\sigma,\xi) := n_\hbar(\sigma, \chi_1(\xi)). \tag{6.8}$$

Now consider the Grushin problem defined by the matrix operator

$$\mathcal{P}_z(\sigma,\xi) = \begin{pmatrix} p_\hbar - z & \cdot v_\xi \\ \langle\cdot, v_\xi\rangle & 0 \end{pmatrix} \in \mathcal{S}\big(\mathbb{R}^2_{\sigma,\xi}, \mathcal{L}(\ \mathsf{Dom}(p_0) \times \mathbb{C}, L^2(\mathbb{R}) \times \mathbb{C})\big) \tag{6.9}$$

where $\mathcal{S}\big(\mathbb{R}^2_{\sigma,\xi}, \mathcal{L}(\ \mathsf{Dom} p_0 \times \mathbb{C}, L^2(\mathbb{R}) \times \mathbb{C})\big)$ is defined in [7, Notation 3.1],

$$p_0(\sigma,\xi) = -\partial_\tau^2 + (\chi_1(\xi) - b_a\tau)^2 \tag{6.10}$$

is the principal symbol of $p_\hbar$ and v_ξ is the positive normalized ground state of p_0, with corresponding eigenvalue $\mu_1(\chi_1(\xi)) = \mu_a(-\chi_1(\xi))$ (see (3.3)).

From the decomposition of $n_\hbar$, we can decompose $\mathcal{P}_z$ as follows:

$$\mathcal{P}_z = \mathcal{P}_z^{[3]} + \hbar^2\mathcal{R}_\hbar, \quad \mathcal{P}_z^{[3]} = \mathcal{P}_{0,z} + \hbar^{1/2}\mathcal{P}_1 + \hbar\mathcal{P}_2 + \hbar^{3/2}\mathcal{P}_3,$$

where

$$\mathcal{P}_{0,z} = \begin{pmatrix} p_0 - z & \cdot v_\xi \\ \langle\cdot, v_\xi\rangle & 0 \end{pmatrix}, \quad \forall j \geqslant 1,\ \mathcal{P}_j = \begin{pmatrix} p_j & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R}_\hbar = \begin{pmatrix} r_\hbar & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned}
p_1 &= 2i(\chi_1(\xi) - b_a\tau)\varphi',\\
p_2 &= -\varphi'^2 + \kappa c_\mu \partial_\tau + c_\mu \kappa(\chi_1(\xi) - b_a\tau)b_a\tau^2 + 2\kappa\tau c_\mu(\chi_1(\xi) - b_a\tau)^2\\
&\quad + \kappa\tau c'_\mu(\tau)\,,\\
\operatorname{Re} p_3 &= 0,\\
\tilde r_\hbar &= \mathcal{O}(\tau^4, \tau^2\partial_\tau),
\end{aligned}$$

where $\mathcal{O}(\tau^4, \tau^2\partial_\tau)$ is understood in the sense of (6.5).

Then one can construct a parametrix of $\mathrm{Op}_\hbar^{\mathrm{W}}(\mathcal{P}_z)$ (see [7, Thm. 3.5] for details)

$$\mathcal{L}_z^{[3]} = \begin{pmatrix} q_z & q_z^+ \\ q_z^- & q_z^\pm \end{pmatrix}, \qquad \mathrm{Op}_\hbar^{\mathrm{W}}(\mathcal{L}_z^{[3]})\ \mathrm{Op}_\hbar^{\mathrm{W}}(\mathcal{P}_z) = \mathrm{Id} + \hbar^2\mathcal{O}(\langle\tau\rangle^6),$$

where

$$q_z^\pm = q_{0,z}^\pm + \hbar^{1/2}q_{1,z}^\pm + \hbar q_{2,z}^\pm + \hbar^{3/2}q_{3,z}^\pm\,,$$

with

$$\begin{aligned}
q_{0,z}^\pm &= z - \mu_1(\chi_1(\xi)),\\
q_{1,z}^\pm &= -i\varphi'(\sigma)\partial_\xi\mu_1(\chi_1(\xi)),\\
q_{2,z}^\pm &= k(\sigma)C_1(\xi,\mu) + C_2(\xi,z)\varphi'(\sigma)^2,\\
\operatorname{Re} q_{3,z}^\pm &= 0,
\end{aligned}\tag{6.11}$$

and

$$\begin{aligned}
C_1(\xi,\mu) &= -\big\langle\big(c_\mu\partial_\tau + c_\mu(\chi_1(\xi) - b_a\tau)b_a\tau^2 + 2\tau c_\mu(\chi_1(\xi) - b_a\tau)^2\big)v_\xi, v_\xi\big\rangle\\
&\qquad - \langle\tau c'_\mu\partial_\tau\big)v_\xi, v_\xi\rangle,\\
C_2(\xi,\mu) &= 1 - \langle(p_0 - z)^{-1}\Pi^\perp(\chi_1(\xi) - b_a\tau)v_\xi, (\chi_1(\xi) - b_a\tau)v_\xi\rangle.
\end{aligned}\tag{6.12}$$

Here $\Pi = \Pi_\xi$ is the orthogonal projection on v_ξ and $\Pi^\perp = \mathrm{Id} - \Pi$. Note that, by (6.6), Remark 9.1 and (9.3),

$$C_1(\hat\zeta_a, 0) = -M_3(a), \quad C_2(\hat\zeta_a, \beta_a) = \frac{\mu''_a(\zeta_a)}{2}.$$

Now we argue like [7, Prop 4.4]. Recall that $|z - \beta_a - M_3(a)k_{\max}\hbar| \leqslant c_0\hbar^2$ and that $\mu \to 0$ as $h \to 0$. Expanding $C_1(\xi,\mu)$ and $C_2(\xi,z)$ near $\xi = \hat\zeta_a$, we get

$$\begin{aligned}
C_1(\xi,\mu) &= -M_3(a)k_{\max}\hbar + \mathcal{O}\big(\hbar\min(1, |\xi - \hat\zeta_a|)\big),\\
C_2(\xi,z) &= \frac{\mu''_a(\zeta_a)}{2} + \mathcal{O}\big(\hbar\min(1, |\xi - \hat\zeta_a|)\big).
\end{aligned}$$

Furthermore, since $\mu'_a(\zeta_a) = 0$ and $\mu''_a(\zeta_a) > 0$, there exists a constant $c_1 > 0$ such that

$$\mu(\chi_1(\xi)) - z \geqslant c_1\min(1, |\xi - \hat\zeta_a|^2).$$

Now we have the following lower bound

$$-\operatorname{Re} q_z^\pm \geqslant \hbar(\mathfrak{v}(\sigma) - C_2(\hat\zeta_a, \beta_a)\varphi'(\sigma)^2\big) - C\hbar^2,$$

where $\mathfrak{v}(\sigma)$ is introduced in Assumption 6.1. We apply the Fefferman-Phong inequality [12, Thm. 3.2] (see also [37, Thm. 4.3.2]) on the symbol

$$\mathfrak{a}(\hat{\sigma},\hat{\xi};\hbar) := \mathscr{A}(\hbar^{1/2}\hat{\sigma},\hbar^{1/2}\hat{\xi};\hbar),$$

where

$$\mathscr{A}(\sigma,\xi;\hbar) := -\operatorname{Re} q_z^{\pm}(\sigma,\xi) - \hbar\big(\mathfrak{v}(\sigma) - C_2(\hat{\zeta}_a,\beta_a)\varphi'(\sigma)^2\big) - C\hbar^2.$$

In that way, we have

$$-\operatorname{Re}\langle \operatorname{Op}_\hbar^{\mathrm{W}}(q_z^{\pm})\psi,\psi\rangle \geqslant \hbar\int_{\mathbb{R}}\left(\mathfrak{v}(\sigma) - \frac{\mu_a''(\zeta_a)}{2}|\varphi'(\sigma)|^2 - C\hbar\right)|\psi|^2 d\sigma,$$

from which we get the following estimate (see [7, Thm. 4.2])

$$cR^2\hbar^2\|\psi\| \leqslant \|(\operatorname{Op}_\hbar^{\mathrm{W}}(p_\hbar)-z)\psi\| + C_R\hbar^2\|\chi_0(\hbar^{-1/2}R^{-1}(\sigma-s_r))\psi\| + \hbar^2\|\tau^6\psi\|\,,$$

which is almost the inequality in Theorem 6.2, but with the operator $\operatorname{Op}_\hbar^{\mathrm{W}}(p_\hbar)$ instead of the operator $\mathcal{N}_\hbar^{\varphi} = \operatorname{Op}_\hbar^{\mathrm{W}}(n_\hbar)$. The only difference between the two operators is the frequency cut-off in the symbol, which can be removed following the same argument in [7, Thm. 5.1].

6.2. **Applications.** By appropriate choices of the function φ in Theorem 6.2, we get optimal tangential estimates for the bound states of the 'single' and 'double' well operators. For details, see [7, Corol. 5.7, Corol. 6.1 & Prop. 6.2].

Proposition 6.3 (Decay of bound states)**.** *Let $\theta,\varepsilon\in(0,1)$ and $K>0$. There exist $C,\hbar_0>0$ such that for all $\hbar\in(0,\hbar_0)$, the following is true.*

If λ eigenvalue of the operator $\mathscr{N}_\hbar$ in (4.6), $|\lambda-(\beta_a+M_3(a)k_{\max}\hbar)|\leqslant K\hbar^2$ and $u\in \mathsf{Dom}(\mathscr{N}_\hbar)$ is an eigenfunction associated to λ, then

$$\int_{[-L,L)\times\mathbb{R}} e^{2\varphi/\hbar^{\frac12}}|u|^2 ds d\tau \leqslant Ce^{\varepsilon/\hbar^{\frac12}}\|u\|^2_{L^2([-L,L)\times\mathbb{R})}\,.$$

where

$$\varphi = (1-\theta)^{1/2}\min(\tilde{\Phi}_r,\tilde{\Phi}_\ell)\,,$$

and $\tilde{\Phi}_r,\tilde{\Phi}_\ell$ are $2L$-periodic functions satisfying, for η sufficiently small,

$$\tilde{\Phi}_r(\sigma)\,|_{-L\leqslant\sigma\leqslant s_\ell-\eta} = \Phi_r(\sigma) := \sqrt{\frac{-2M_3(a)}{\mu_a''(\zeta_a)}}\int_{[s_r,\sigma]}\sqrt{k_{\max}-k(s)}\,ds,$$

$$\tilde{\Phi}_\ell(\sigma)\,|_{-L\leqslant\sigma\leqslant s_r-\eta} = \Phi_\ell(\sigma) := \sqrt{\frac{-2M_3(a)}{\mu_a''(\zeta_a)}}\int_{[s_\ell,\sigma]}\sqrt{k_{\max}-k(s)}\,ds\,.$$

Remark 6.4. The estimate in Proposition 6.3 continues to hold if λ is an eigenvalue of the right or left well operator, $\mathcal{N}_{\hbar,r}$ or $\mathcal{N}_{\hbar,\ell}$, and u is a corresponding eigenfunction.

Returning to the 'one well' operators, $\mathcal{N}_{\hbar,r,\gamma_0}$ and $\mathcal{N}_{\hbar,\ell,\gamma_0}$ introduced in (5.3) and (5.7) respectively, we get from Proposition 6.3 and the min-max principle, the following rough estimate, important for the analysis of tunneling later on,

$$\mu_1^{\mathrm{sw}}(\hbar) - \tilde{\mathcal{O}}(e^{-\mathsf{S}^a/\sqrt{\hbar}}) \leqslant \nu_{1,a}(\hbar) \leqslant \nu_{2,a}(\hbar) \leqslant \mu_1^{\mathrm{sw}}(\hbar) + \tilde{\mathcal{O}}(e^{-\mathsf{S}^a/\sqrt{\hbar}}), \tag{6.13}$$

where S^a is introduced in (3.15),

$$\mu_1^{\mathrm{sw}}(\hbar) = \inf \sigma(\mathcal{N}_{\hbar,r,\gamma_0}) = \inf \sigma(\mathcal{N}_{\hbar,r,\gamma_0})\,,$$

$(\nu_{j,a}(\hbar))_{j\geqslant 1}$ is the sequence of eigenvalues of the operator $\mathcal{N}_\hbar$, and the notation $\tilde{\mathcal{O}}(e^{-\mathsf{S}^a/\sqrt{\hbar}})$ means

$$\mathcal{O}(e^{(\epsilon-\mathsf{S}^a)/\sqrt{\hbar}}) \quad \text{for any fixed } \varepsilon > 0. \tag{6.14}$$

The analysis of the tunneling requires an explicit approximation of the ground state of the single well operators. Recall the Agmon distance Φ_r and the WKB solution $\Psi_{\hbar,r}$ introduced in (5.9) and Theorem 5.1 respectively. Consider the flux free 'right well' operator $\mathcal{N}_{\hbar,r} := \mathcal{N}_{\hbar,r,0}$. By (5.5), the low lying eigenvalues of this operator are simple; we denote by Π_r the orthogonal projection on its first eigenspace. By [7, Prop. 6.3], it results from Theorem 6.2:

Proposition 6.5 (WKB approximation)**.** *We have*

$$\|\psi_{\hbar,r} - \Pi_r\psi_{\hbar,r}\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(\hbar^\infty)$$

and

$$\langle\tau\rangle\, e^{\Phi_r/\sqrt{\hbar}}(\Psi_{\hbar,r} - u_{\hbar,r}) = \mathcal{O}(\hbar^\infty) \quad \text{in } \mathscr{C}^1(K; L^2(\mathbb{R})), \tag{6.15}$$

where $K \subset I_{2\hat{\eta},r} := (s_\ell - 2L + \hat{\eta}, s_\ell - \hat{\eta})$ is a compact set,

$$\psi_{\hbar,r}(\sigma,\tau) := \chi_{\hat{\eta},r}(\sigma)\Psi_{\hbar,\tau}(\sigma,\tau)\,,$$

and $\chi_{\hat{\eta},r}$ is a cut-off function supported in $I_{\hat{\eta},r}$ such that $\chi_{\hat{\eta},r} = 1$ on $I_{2\hat{\eta},r}$.

7. Interaction matrix and tunneling

We return to the operator $\mathcal{N}_\hbar$ introduced in (4.6). In order to estimate the splitting between the first and second eigenvalues, $\nu_2(\hbar) - \nu_1(\hbar)$, we will write the matrix of this operator in a specific basis of

$$E = \oplus_{i=1}^2 \operatorname{Ker}\big(\mathcal{N}_\hbar - \nu_i(\hbar)\big).$$

Let Π be the orthogonal projection on E. We introduce the two functions

$$f_{\hbar,r} = \chi_{\hat{\eta},r}\phi_{\hbar,r}, \qquad f_{\hbar,\ell} = \chi_{\hat{\eta},\ell}\phi_{\hbar,\ell},$$

where $\chi_{\hat{\eta},r}$ is the cut-off function introduced in Proposition 6.5, $\chi_{\hat{\eta},\ell} = U\chi_{\hat{\eta},r}$ is defined by the symmetry operator (see Sec. 5.2), $\phi_{\hbar,r}$, $\phi_{\hbar,\ell}$ enjoy periodicity properties and are defined by inspiration from the functions in (5.6) and (5.8). In fact, $\phi_{\hbar,r}(\sigma,\tau)$ need to be defined in the support of $\chi_{\hat{\eta},r}$. Starting on $[-L, s_\ell - \frac{\hat{\eta}}{2})\times\mathbb{R}$, we take $\phi_{\hbar,r}(\sigma,\tau)$ the same as the function in (5.6); on $[s_\ell + \frac{\hat{\eta}}{2}, L)\times\mathbb{R}$, we do a change of variable, and modify the function in (5.6) so that its module satisfies a periodic boundary condition on $\pm L$. More precisely, we have

$$\phi_{\hbar,r}(\sigma,\tau)=\begin{cases} e^{-i\gamma_0\sigma/\hbar^2}u_{\hbar,r}(\sigma,\tau), & \text{if } -L\leqslant\sigma\leqslant s_\ell-\frac{\hat\eta}{2},\\ e^{-i\gamma_0(\sigma-2L)/\hbar^2}u_{\hbar,r}(\sigma-2L,\tau), & \text{if } s_\ell+\frac{\hat\eta}{2}<\sigma<L,\end{cases} \tag{7.1}$$

and so $f_{\hbar,r}$ is well defined on $[-L,L)$. In a similar fashion,

$$\phi_{\hbar,\ell}(\sigma,\tau)=\begin{cases} e^{-i\gamma_0(\sigma+2L)/\hbar^2}u_{\hbar,\ell}(\sigma+2L,\tau), & \text{if } -L\leqslant\sigma\leqslant s_r-\frac{\hat\eta}{2},\\ e^{-i\gamma_0\sigma/\hbar^2}u_{\hbar,\ell}(\sigma,\tau), & \text{if } s_r+\frac{\hat\eta}{2}<\sigma<L,\end{cases} \tag{7.2}$$

and $f_{\hbar,\ell}$ is well defined on $[-L,L)$.

Also we introduce the following actual bound states by projecting on the eigenspace E,

$$g_{\hbar,r}=\Pi f_{\hbar,r},\qquad g_{\hbar,\ell}=\Pi f_{\hbar,\ell}\,.$$

We use the notation $\tilde{\mathcal{O}}$ in (6.14). By Proposition 6.3, we have (see [7, Sec. 7.1] and [9, Sec. 3] for details)

$$\|f_{\hbar,r}\|^2=1+\tilde{\mathcal{O}}(e^{-2\mathsf{S}^a/\sqrt{\hbar}}),\quad \|f_{\hbar,\ell}\|^2=1+\tilde{\mathcal{O}}(e^{-2\mathsf{S}^a/\sqrt{\hbar}}),$$
$$\langle f_{\hbar,r},f_{\hbar,\ell}\rangle=\tilde{\mathcal{O}}(e^{-\mathsf{S}^a/\sqrt{\hbar}}),$$

and

$$\|g_{\hbar,\alpha}-f_{\hbar,\alpha}\|+\|\partial_\tau(g_{\hbar,\alpha}-f_{\hbar,\alpha})\|=\tilde{\mathcal{O}}(e^{-\mathsf{S}^a/\sqrt{\hbar}})\qquad \alpha\in\{r,\ell\}.$$

Now, we construct an orthonormal basis $\mathcal{B}_\hbar:=\{\tilde g_{\hbar,r},\tilde g_{\hbar,\ell}\}$ of E from $\{g_{\hbar,r},g_{\hbar,\ell}\}$ by the Gram-Schmidt process. Let M be the matrix of $\mathcal{N}_\hbar$ relative to the basis $\mathcal{B}_\hbar$. Then

$$\nu_2(\hbar)-\nu_1(\hbar)=2|w_{\ell,r}|+\tilde{\mathcal{O}}(e^{-2\mathsf{S}/\sqrt{\hbar}})\,,\quad w_{\ell,r}=\langle r_{\hbar,\ell},f_{\hbar,r}\rangle, \tag{7.3}$$

where

$$r_{\hbar,\ell}=(\mathcal{N}_{\hbar,\ell}-\mu^{\,\mathrm{sw}}(\hbar))f_{\hbar,\ell}.$$

All we have to do now is the computation of $w_{\ell,r}$ by the WKB approximation in Proposition 6.5. By [7, Lem. 7.1](which is essentially an integration by parts formula)

$$\begin{aligned} w_{\ell,r}=i\hbar\int_{\mathbb{R}} a_\hbar^{-1}\left(\phi_{\hbar,\ell}\overline{\mathscr{D}_\hbar\phi_r}+[\mathscr{D}_\hbar\phi_{\hbar,\ell}]\overline{\phi_{\hbar,r}}\right)(0,\tau)d\tau\\ -i\hbar\int_{\mathbb{R}} a_\hbar^{-1}\left(\phi_{\hbar,\ell}\overline{\mathscr{D}_\hbar\phi_{\hbar,r}}+[\mathscr{D}_\hbar\phi_\ell]\overline{\phi_{\hbar,r}}\right)(-L,\tau)d\tau, \end{aligned}$$

where

$$\mathscr{D}_\hbar=\hbar D_\sigma+\hbar^{-1}\gamma_0-b_a(\tau)\tau+\hbar c_\mu\frac{k}{2}b_a(\tau)\tau^2\,.$$

Writing $a_\hbar=1+o(1)$, $\hbar c_\mu\tau^2=o(\hbar^{-2\eta})$, and approximating $\phi_{\hbar,r}$, $\phi_{\hbar,\ell}$ by using (7.1), (7.2) and Proposition 6.5, we get (see [7, Sec. 7.2.2] for details)

$$w_{r,\ell}=i\hbar(w^{\mathsf{u}}_{r,\ell}+w^{\mathsf{d}}_{r,\ell}) \tag{7.4}$$

where

$$w_{\ell,r}^{\mathsf{u}} = \int_{\mathbb{R}} a_\hbar^{-1}\Psi_{\hbar,\ell}\overline{\Big(\hbar D_\sigma - b_a\tau + \hbar c_\mu \frac{\kappa}{2} b_a \tau^2\Big)\Psi_{\hbar,r}}(0,\tau)d\tau$$
$$+ \int_{\mathbb{R}} a_\hbar^{-1}\Big[\Big(\hbar D_\sigma - b_a\tau + \hbar c_\mu \frac{\kappa}{2} b_a \tau^2\Big)\Psi_{\hbar,\ell}\Big]\overline{\Psi_{\hbar,r}}(0,\tau)d\tau + \mathcal{O}(\hbar^\infty)e^{-\mathsf{S}_{\mathsf{u}}^a/\hbar^{1/2}},$$

and

$$w_{\ell,r}^{\mathsf{d}} = \int_{\mathbb{R}} a_\hbar^{-1}\Psi_{\hbar,\ell}\overline{\Big(\hbar D_\sigma - b_a\tau + \hbar c_\mu \frac{\kappa}{2} b_a \tau^2\Big)\Psi_{\hbar,r}}(-L,\tau)d\tau$$
$$+ \int_{\mathbb{R}} a_\hbar^{-1}\Big[\Big(\hbar D_\sigma - b_a\tau + \hbar c_\mu \frac{\kappa}{2} b_a \tau^2\Big)\Psi_{\hbar,\ell}\Big]\overline{\Psi_{\hbar,r}}(-L,\tau)d\tau + \mathcal{O}(\hbar^\infty)e^{-\mathsf{S}_{\mathsf{d}}^a/\hbar^{1/2}}.$$

Eventually, using (3.13) and (5.11), we get (see [7, Eq. (7.11)])

$$\hbar^{\frac14} e^{\mathsf{S}_{\mathsf{u}}^a/\hbar^{1/2}} w_{\ell,r}^{\mathsf{u}} = -i\hbar^{\frac12}\mu_a''(\zeta_a)\pi^{-\frac12} g_a^{\frac12}\sqrt{V_a(0)}\mathsf{A}_{\mathsf{u}}^a e^{-2i\alpha_0(0)} + \mathcal{O}(\hbar)$$

and, in a similar fashion,

$$\hbar^{\frac14} e^{\mathsf{S}_{\mathsf{d}}^a/\hbar^{1/2}} w_{\ell,r}^{\mathsf{d}}$$
$$= -i\hbar^{\frac12}\mu_a''(\zeta_a)\pi^{-\frac12} g_a^{\frac12}\sqrt{V_a(-L)}\mathsf{A}_{\mathsf{d}}^a e^{-2i\alpha_0(-L)} e^{i(-2L\gamma_0/\hbar^2 - 2L\zeta_a/\hbar)} + \mathcal{O}(\hbar)\,,$$

where α_0 is the function introduced in Remark 5.14.

Collecting (7.4) and (7.3) and using that $\hbar = h^{1/2}$, we get

$$\begin{aligned}\nu_2(\hbar) - \nu_1(\hbar) &= 2|e^{iLf(h)}w_{\ell,r}| + \tilde{\mathcal{O}}(e^{-2\mathsf{S}^a/\sqrt{\hbar}})\\ &= h^{-1}|w_a(h)| + \tilde{\mathcal{O}}(h^{-1}e^{-2\mathsf{S}^a/\sqrt{\hbar}}),\end{aligned}$$

where $f(h)$ and $\tilde{w}_a(h)$ are the expressions in Theorem 3.4. In light of Proposition 4.1, this finishes the proof of Theorem 3.4.

8. Conclusion and open problems

Until now, examples of magnetic tunneling effects are rare in the literature. Very few articles have been dealing with the measure of the tunneling effect due to the presence of the magnetic field. In the presence of an electric potential with multiple wells, the article [27] was only considering a case when the magnetic field was a perturbation and the tunneling was mainly created by the electric potential. Other examples include the case of a pure flux [31]. After the recent contributions of Bonnaillie-Hérau-Raymond [9] and Fefferman-Shapiro-Weinstein [14] (see also references therein), we have presented a new magnetic tunneling effect due to the curvature of the magnetic edge.

Both for the Neumann problem occurring in surface superconductivity [22] or for the problem considered here [2], it would be interesting to consider the $(3D)$-case.

Excluded in this paper is the case $a = -1$, where localization near the point(s) of maximum curvature no more occurs ($M_3(a) = 0$ in the asymptotics of Theorem 3.2). In contrast, this case seems to feature an interesting new phenomenon where localization near the whole edge Γ occurs, which also has a nice analogy to what was observed in the multiple wells situation in [26]. We hope to come back to the treatment of this case rather soon.

Finally we mention that the standard purely magnetic double well problem seems at the moment a difficult challenge. Here we consider a purely semi-classical magnetic Laplacian (say in $\mathbb{R}^2$) where the magnetic field has two symmetric non degenerate positive minima.

9. Appendix: Regularized resolvent and moments

Let us return back to the flat edge model in Sec. 3.1 and recall some necessary computational results.

We can invert the operator $\mathfrak{h}_a[\zeta_a]$ on the orthogonal complement of the ground state ϕ_a. Extending by linearity, we get the regularized resolvent $\mathfrak{R}_a$ defined on $L^2(\mathbb{R})$ by

$$\mathfrak{R}_a\, u = \begin{cases} 0 & \text{if } u \parallel \phi_a \\ (\mathfrak{h}_a[\zeta_a] - \beta_a)^{-1} u & \text{if } u \perp \phi_a \end{cases}. \tag{9.1}$$

By [5], $(\zeta_a + b_a(\tau)\tau)\phi_a$ and ϕ_a are orthogonal in $L^2(\mathbb{R})$:

$$\int_{\mathbb{R}} (\zeta_a + b_a(\tau)\tau)|\phi_a(\tau)|^2\, d\tau = 0\,. \tag{9.2}$$

Later on we will encounter the following integral [3, Prop. 2.5]

$$I_2(a) := \int_{\mathbb{R}} \phi_a(\tau)\mathfrak{R}_a[(\zeta_a + b_a(\tau)\tau)\phi_a]\, d\tau = \frac{1}{4} - \frac{\mu''_a(\zeta_a)}{8}\,. \tag{9.3}$$

We recall some identities from [5] involving for $n \in \mathbb{N}$ quantities of the form

$$M_n(a) = \int_{\mathbb{R}} \frac{1}{b_a(\tau)}(\zeta_a + b_a(\tau)\tau)^n|\phi_a(\tau)|^2\, d\tau\,. \tag{9.4}$$

We have

$$M_1(a) = 0\,, \tag{9.5}$$

$$M_2(a) = -\frac{1}{2}\beta_a \int_{\mathbb{R}} \frac{1}{b_a(\tau)}|\phi_a(\tau)|^2\, d\tau + \frac{1}{4}\Big(\frac{1}{a} - 1\Big)\zeta_a\phi_a(0)\phi'_a(0)\,, \tag{9.6}$$

$$M_3(a) = \frac{1}{3}\Big(\frac{1}{a} - 1\Big)\zeta_a\phi_a(0)\phi'_a(0)\,. \tag{9.7}$$

The case $a = -1$ is special because

$$M_3(-1) = 0 \quad \text{and} \quad M_3(a) < 0 \quad \text{for} \quad -1 < a < 0\,.$$

Remark 9.1. The next identities follow in a straightforward manner [3, Rem. 2.3],

$$\int_{\mathbb{R}} \tau(\zeta_a + b_a(\tau)\tau)|\phi_a(\tau)|^2 \, d\tau = M_2(a),$$

$$\int_{\mathbb{R}} \tau(\zeta_a + b_a(\tau)\tau)^2|\phi_a(\tau)|^2 \, d\tau = M_3(a) - \zeta_a M_2(a),$$

$$\int_{\mathbb{R}} b_a(\tau)\tau^2(\zeta_a + b_a(\tau)\tau)|\phi_a(\tau)|^2 \, d\tau = M_3(a) - 2\zeta_a M_2(a),$$

and

$$\int_{\mathbb{R}} \tau|\phi_a(\tau)|^2 \, d\tau = -\zeta_a \int_{\mathbb{R}} \frac{1}{b_a(\tau)}|\phi_a(\tau)|^2 \, d\tau,$$

$$\int_{\mathbb{R}} \tau|\phi_a'(\tau)|^2 \, d\tau = \beta_a\zeta_a \int_{\mathbb{R}} \frac{1}{b_a(\tau)}|\phi_a(\tau)|^2 \, d\tau + 2M_3(a) - 2\zeta_a M_2(a).$$

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