

A survey of recent studies concerning the extreme properties of Morris-Thorne wormholes

Peter K.F. Kuhfittig*

*Department of Mathematics, Milwaukee School of Engineering,
Milwaukee, Wisconsin 53202-3109, USA

Abstract

It has been known for a long time that Morris-Thorne wormholes can only be held open by violating the null energy condition, which can be expressed in the form $\tau - \rho c^2 > 0$, where τ is the radial tension. Matter that violates this condition is usually referred to as “exotic.” For any wormhole having a moderately-sized throat, the radial tension is equal to that at the center of a massive neutron star. Attributing this outcome to exotic matter seems reasonable enough, but it ignores the fact that exotic matter was introduced for a completely different reason. Moreover, its problematical nature suggests that the amount of exotic matter be held to a minimum, but this would make the high radial tension harder to explain. If the amount is infinitely small, this explanation breaks down entirely. By invoking $f(R)$ modified gravity, the need for exotic matter at the throat could actually be eliminated, but the negation of the above condition, i.e., $\tau < \rho c^2$, shows that we have not necessarily eliminated the high radial tension. This survey discusses various ways to account for the high radial tension and, in some cases, the possible origin of exotic matter. We conclude with some comments on a possible multiply-connected Universe.

1 Introduction

This paper is a brief survey of recent studies dealing with some of the extreme aspects of Morris-Thorne wormholes, such as the existence and possible origin of exotic matter and the often inexplicably large radial tension at the throat. Most of the discussion is based in Refs. [1, 2, 3, 4].

1.1 Morris-Thorne wormholes

Wormholes are handles or tunnels connecting widely separated regions of our Universe or different universes in a multiverse. Apart from some forerunners, macroscopic traversable

*kuhfitti@msoe.edu

wormholes were first proposed by Morris and Thorne [5] in 1988. They introduced the following static and spherically symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where

$$e^{\lambda(r)} = \frac{1}{1 - \frac{b(r)}{r}}. \quad (2)$$

(We are using units in which $c = G = 1$.) Here $\nu = \nu(r)$ is called the *redshift function*, which must be everywhere finite to prevent the occurrence of an event horizon. The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram [5]. The shape function must satisfy the following conditions: $b(r_0) = r_0$, where $r = r_0$ is the radius of the *throat* of the wormhole, $b'(r_0) < 1$, called the *flare-out condition*, and $b(r) < r$ for $r > r_0$. A final requirement is asymptotic flatness: $\lim_{r \rightarrow \infty} \nu(r) = 0$ and $\lim_{r \rightarrow \infty} b(r)/r = 0$.

The flare-out condition can only be met by violating the null energy condition (NEC), $T_{\alpha\beta} k^\alpha k^\beta \geq 0$, for all null vectors k^α , where $T_{\alpha\beta}$ is the energy-momentum tensor. Matter that violates the NEC is called “exotic” in Ref. [5]. For the outgoing null vector $(1, 1, 0, 0)$, the violation becomes

$$T_{\alpha\beta} k^\alpha k^\beta = \rho + p_r < 0. \quad (3)$$

Here $T^t_t = -\rho$ is the energy density, $T^r_r = p_r$ is the radial pressure, and $T^\theta_\theta = T^\phi_\phi = p_t$ is the lateral (transverse) pressure.

Next, let us list the Einstein field equations, referring to line element (1):

$$8\pi\rho = e^{-\lambda} \left[\frac{\lambda'}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2}, \quad (4)$$

$$8\pi p_r = e^{-\lambda} \left[\frac{1}{r^2} + \frac{\nu'}{r} \right] - \frac{1}{r^2}, \quad (5)$$

and

$$8\pi p_t = \frac{1}{2} e^{-\lambda} \left[\frac{1}{2}(\nu')^2 + \nu'' - \frac{1}{2}\lambda'\nu' + \frac{1}{r}(\nu' - \lambda') \right]. \quad (6)$$

Eq. (4) can also be written

$$8\pi\rho(r) = \frac{b'(r)}{r^2}. \quad (7)$$

1.2 Fundamental problems with Morris-Thorne wormholes

The existence of exotic matter is not a conceptual problem, as we know from the Casimir effect [5]. The question is whether enough exotic matter could be produced to sustain a macroscopic wormhole. According to Ref. [6], however, sufficient fine-tuning might allow such a wormhole to exist. The origin of exotic matter is itself an issue to be discussed in this survey.

A violation of the NEC is a generic feature of any traversable wormhole [7]. For Morris-Thorne wormholes, the violation results from the flare-out condition. Because

of its problematical nature, it is generally understood that the amount of exotic matter should be kept to a minimum. An interesting contrast is provided by $f(R)$ modified gravity: it is possible in principle for the throat to be threaded with ordinary (nonexotic) matter, while the violation of the NEC can be attributed to the higher-order curvature terms [8]. Another possibility is discussed in Ref. [9]. While the NEC is, once again, met at the throat, the unavoidable violation is due to the existence of an extra spatial dimension.

A second problem to be addressed is the enormous radial tension at the throat. We need to recall that the radial tension $\tau(r)$ is the negative of the radial pressure $p_r(r)$. It is noted in Ref. [5] that the Einstein field equations can be rearranged to yield $\tau(r)$: temporarily reintroducing c and G , the tension is given by

$$\tau(r) = \frac{b(r)/r - [r - b(r)]\nu'(r)}{8\pi Gc^{-4}r^2}. \quad (8)$$

It now follows that

$$\tau(r_0) = \frac{1}{8\pi Gc^{-4}r_0^2} \approx 5 \times 10^{41} \frac{\text{dyn}}{\text{cm}^2} \left(\frac{10 \text{ m}}{r_0} \right)^2. \quad (9)$$

In particular, for $r_0 = 3 \text{ km}$, $\tau(r)$ has the same magnitude as the pressure at the center of a massive neutron star [5]. Rewriting Eq. (3) in the form $\tau - \rho c^2 > 0$ helps explain the high radial tension but it also ignores the fact that exotic matter was introduced for a completely different reason, namely ensuring a violation of the NEC. Furthermore, reducing the amount of exotic matter as much as possible makes the high radial tension even harder to explain. If the amount is infinitesimal [10], this explanation breaks down entirely.

One can argue, of course, that the condition $\tau - \rho c^2 > 0$ simply implies that we are not dealing with ordinary matter. We can therefore accept the condition and leave the details to some advanced civilization, as suggested in Ref. [5]. As noted above, however, in $f(R)$ modified gravity, the throat can be lined with ordinary matter, so that the condition $\tau - \rho c^2 > 0$ would not be required. But even if $\tau < \rho c^2$, we could still be dealing with an enormous radial tension.

Problems also occur in certain cosmological settings. For example, it is known that both phantom dark energy and dark matter can support traversable wormholes. For the former, the equation of state is $p = \omega\rho$, $\omega < -1$. Applied to wormholes, $\rho + p_r = \rho + \omega\rho = (1 + \omega)\rho < 0$; so the NEC has been violated. In the case of dark matter, the energy density is extremely low. So it naturally follows from Eq. (7) that $b'(r_0) = 8\pi r_0^2 \rho(r_0) < 1$, so that the flare-out condition is automatically satisfied, and, just as in the case of phantom dark energy, the NEC is violated. However, given the very low energy density, Eq. (9) implies that r_0 would have to be extremely large to yield a realistic value for $\tau(r_0)$. On the other hand, if r_0 is relatively small, then the resulting large radial tension would make little sense and would therefore have to be attributed, once again, to exotic matter. But if exotic matter is needed anyway, then both dark matter and dark energy would become redundant. To emphasize this point, the zero-density case ($\rho = 0$), discussed in Ref. [11], also yields a valid wormhole solution, one that is about as far removed from being a neutron star as could be imagined.

The rest of this survey is devoted to finding various ways to address these issues.

2 Compact stellar objects

As noted in Sec. 1.2, a wormhole with a relatively small throat size could have a radial tension equal to that of a compact stellar object such as a neutron star. It is shown in Ref. [1] that the extreme conditions at the center could indeed result in a topology change, i.e., the formation of wormholes. The argument is based on the fact that quark matter is likely to exist at the center, where neutrons become deconfined to form quark matter due to the extreme conditions.

As a result, the analysis in Ref. [1] is based on a combined model consisting of quark matter and baryonic matter with an isotropic matter distribution:

$$\rho_{\text{effective}} = \rho + \rho_q \quad \text{and} \quad p_{\text{effective}} = p + p_q, \quad (10)$$

where ρ and p correspond to the respective energy density and pressure of the baryonic matter, while ρ_q and p_q correspond to the respective energy density and pressure of the quark matter. (The left-hand sides refer to the effective energy density and pressure, respectively, of the composition.)

In the MIT bag model, the matter equation of state is given by

$$p_q = \frac{1}{3}(\rho_q - 4B), \quad (11)$$

where B is the bag constant. The main conclusion is that the topology change can only take place if the neutron star has a spherical core of quark matter.

Since $r = r_0$ is the throat of the wormhole, the interior $r < r_0$ is necessarily outside the manifold but it still contributes to the gravitational field. This can be compared to a thin-shell wormhole constructed from a Schwarzschild black hole by the standard cut-and-paste technique: although not part of the manifold, the black hole generates the gravitational field. Moreover, the extreme conditions, especially the presence of quark matter, lead to $b'(r_0) < 1$, so that the flare-out condition is satisfied. (See Ref. [1] for details.)

Since the throat of the wormhole is deep inside the neutron star, it cannot be directly observed. According to Ref. [12], however, indirect observations may still be possible: if two neutron stars are connected by a wormhole, they would have similar characteristics and may even exhibit observable variations in their masses.

3 Noncommutative geometry

The last several years have shown that string theory has become ever more influential. An example is the realization that coordinates may become noncommutative operators on a D -brane [13, 14]. The outcome, noncommutative geometry, helps eliminate the divergences that normally occur in general relativity. The reason is that noncommutativity replaces point-like objects by smeared objects: spacetime can thereby be encoded in the commutator $[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix that determines the fundamental cell discretization of spacetime in the same way that Planck's constant \hbar discretizes phase space [15].

A natural way to model the smearing effect is by means of a Gaussian distribution of minimal length $\sqrt{\beta}$ instead of the Dirac delta function [15, 16, 17, 18, 19]. An equally effective way, discussed in Refs. [2, 20, 21], is to assume that the energy density of the static and spherically symmetric and particle-like gravitational source has the form

$$\rho_\beta(r) = \frac{\mu_1\sqrt{\beta}}{\pi^2(r^2 + \beta)^2}, \quad (12)$$

where μ_1 is a constant. Eq. (12) can be interpreted to mean that the gravitational source causes the mass μ_1 of a particle to be diffused throughout a region of linear dimension $\sqrt{\beta}$ due to the uncertainty; so $\sqrt{\beta}$ has units of length. Following Ref. [15], Eq. (12) leads to the mass distribution

$$m(r) = \int_0^r 4\pi(r')^2\rho(r')dr' = \frac{2M}{\pi} \left(\tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r\sqrt{\beta}}{r^2 + \beta} \right), \quad (13)$$

where M is now the total mass of the source. Observe that $m(0) = 0$ but $m(r)$ rapidly rises to M as r increases.

According to Ref. [15], noncommutative geometry is an intrinsic property of space-time and does not depend on any particular features such as curvature. Moreover, the relationship between the radial pressure and energy density is given by

$$p_r = -\rho_\beta. \quad (14)$$

The reason is that the source is a self-gravitating droplet of anisotropic fluid of density ρ_β and the radial pressure is needed to prevent a collapse to a matter point. From Eqs. (12) and (7), we now obtain

$$\begin{aligned} b(r) &= \frac{8M\sqrt{\beta}}{\pi} \int_{r_0}^r \frac{(r')^2 dr'}{[(r')^2 + \beta]^2} + r_0 \\ &= \frac{4M\sqrt{\beta}}{\pi} \left(\frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r}{r^2 + \beta} - \frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r_0}{\sqrt{\beta}} + \frac{r_0}{r_0^2 + \beta} \right) + r_0. \end{aligned} \quad (15)$$

It is shown in Ref. [22] that $b = b(r)$ has the usual properties of a shape function: $b(r_0) = r_0$, $0 < b'(r_0) < 1$, and $\lim_{r \rightarrow \infty} b(r)/r = 0$.

For the purpose of this survey, the most important formula is the density ρ_s of the surface of the throat,

$$\rho_s = \frac{\mu_2\sqrt{\beta}}{\pi^2[(r - r_0)^2 + \beta]^2}, \quad (16)$$

where μ_2 is the mass of the surface. As noted in Sec. 1.2, we are dealing with a moderate throat size. Also, the surface $r = r_0$ is a smeared surface since the individual particles on the surface are smeared. (For further discussion, see Ref. [2].)

Unlike the case of dark matter, a low energy density does not result in a low tension [2]. Recalling that the tension τ is the negative of the pressure p , $p_r + \rho_s < 0$ becomes $\tau - \rho_s > 0$. Eq. (14) implies that we are right on the edge of violating the NEC, i.e., $\tau - \rho_s = 0$. So at $r = r_0$,

$$\rho_s = \frac{\mu_2}{\pi^2} \frac{1}{\beta^{3/2}}, \quad (17)$$

but we still have $\tau - \rho_s = 0$. Since the throat is a smeared surface, however, we only have $r \approx r_0$. So by Eq. (16), ρ_s is reduced in value and we obtain the desired $\tau - \rho_s > 0$. Eq. (17) implies that ρ_s and τ are extremely large at the throat. To check its plausibility, it is argued in Ref. [2] that, according to Eq. (9), for a throat size of 10 m, $\tau \approx 5 \times 10^{41}$ dyn/cm². Suppose μ_2 has the rather minute value 10^{-10} g. Applying Eq. (17), we have

$$\tau = \rho_s c^2 = \frac{\mu_2}{\pi^2} (\sqrt{\beta})^{-3} c^2 = 5 \times 10^{41} \frac{\text{dyn}}{\text{cm}^2}. \quad (18)$$

To satisfy this relationship, the value $\sqrt{\beta} = 10^{-11}$ cm is sufficient. Since $\sqrt{\beta}$ may be much smaller, we can accommodate even larger values of τ .

We conclude that even though we are dealing with a very low energy density, we have a very large tension at the throat thanks to the noncommutative-geometry background.

4 Returning to $f(R)$ modified gravity

Modified gravitational theories, including $f(R)$ modified gravity, have been invoked to explain various phenomena in general relativity. First we need to recall that in $f(R)$ gravity, the Ricci scalar R in the Einstein-Hilbert action $S_{\text{EH}} = \int \sqrt{-g} R dx^4$ is replaced by a nonlinear function $f(R)$. Next, let us state the gravitational field equations in the form used by Lobo and Oliveira [8]:

$$\rho(r) = F(r) \frac{b'(r)}{r^2}, \quad (19)$$

$$p_r(r) = -F(r) \frac{b(r)}{r^3} + F'(r) \frac{rb'(r) - b(r)}{2r^2} - F''(r) \left[1 - \frac{b(r)}{r} \right], \quad (20)$$

and

$$p_t(r) = -\frac{F'(r)}{r} \left[1 - \frac{b(r)}{r} \right] + \frac{F(r)}{2r^3} [b(r) - rb'(r)]; \quad (21)$$

here $F = \frac{df}{dR}$. It is also assumed that $\nu(r) \equiv \text{constant}$, so that $\nu'(r) \equiv 0$. Otherwise, according to Ref. [8], the analysis becomes intractable.

According to Ref. [2], by assuming a noncommutative-geometry background, we obtain in view of Eq. (12),

$$f(R) = \frac{\mu_1 \sqrt{\beta}}{\pi^2} \frac{(\beta R + 2\alpha) \ln(\beta R + 2\alpha) - \beta R}{\beta^2 (\beta R + 2\alpha)} + C, \quad (22)$$

where C and α are constants, thereby providing a motivation for the choice of $f(R)$.

From Eq. (19),

$$b'(r_0) = \frac{r_0^2 \rho(r_0)}{F(r_0)} < 1, \quad (23)$$

even if $F(r_0)$ is quite small, thereby satisfying the flare-out condition. Next, from Eq. (20),

$$\tau(r_0) = -p_r(r_0) = F(r_0) \frac{b(r_0)}{r_0^3} - F'(r_0) \frac{r_0 b'(r_0) - b(r_0)}{2r_0^2}. \quad (24)$$

So given that $b'(r_0) < 1$, $\tau(r_0)$ is large provided that $F'(r_0)$ is large and positive. It is shown in Ref. [2] that this requirement can be met if we retain the connection to noncommutative geometry via $f(R)$ in Eq. (22).

Unlike the situation in classical general relativity, meeting the flare-out condition does not automatically lead to a violation of the NEC, but, as noted in Sec. 1.2, the unavoidable violation can be attributed to the higher-order curvature terms [8]. For this conclusion to hold, noncommutative geometry must be viewed as a special case of $f(R)$ modified gravity. The connection between the two theories had already been suggested in Ref. [23].

5 A small extra spatial dimension

It was brought out in Sec. 1.2 that the throat of a wormhole could be threaded with ordinary (nonexotic) matter if the violation of the NEC can be attributed to either the higher-order curvature terms in $f(R)$ modified gravity [8] or to the existence of a higher spatial dimension [9]. The latter case is based on the line element

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{2\mu(r,l)} dl^2, \quad (25)$$

where l is the extra spatial dimension. To study the radial tension, Ref. [2] starts with the Einstein field equations in the orthonormal frame:

$$G_{\hat{\alpha}\hat{\beta}} = R_{\hat{\alpha}\hat{\beta}} - \frac{1}{2}Rg_{\hat{\alpha}\hat{\beta}} = \kappa T_{\hat{\alpha}\hat{\beta}}, \quad (26)$$

where κ is the coupling constant. (In the five-dimensional case, $\kappa = 3\pi^2$.) Also,

$$g_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (27)$$

and $\tau(r) = -p_r(r) = -T_{11}$. The components of the Ricci scalar are given in Ref. [2]. From $G_{11} = \kappa T_{11}$, we obtain

$$\kappa p_r(r) = R_{11} - \frac{1}{2}Rg_{11}, \quad (28)$$

while

$$R = R^i_i = -R_{00} + R_{11} + R_{22} + R_{33} + R_{44}. \quad (29)$$

It follows that

$$2\kappa p_r(r) = R_{00} + R_{11} - R_{22} - R_{33} - R_{44}. \quad (30)$$

The fifth dimension comes into play because (Ref. [2])

$$2\kappa p_r(r) = -\frac{1}{2r^2} \frac{\partial \mu(r,l)}{\partial r} [rb'(r) - b(r)] - \frac{2b(r)}{r^3}, \quad (31)$$

which is negative if $\partial\mu(r, l)/\partial r < 0$. Eq. (31) is a tension whose magnitude depends on $\partial\mu(r, l)/\partial r$. In principle, then, $\tau(r)$ can be extremely large.

Finally, according to Ref. [2], the extra dimension can be small or even curled up, as in string theory. In fact, the models discussed in Sections 3, 4, and 5 help confirm that whenever we are dealing with an extreme regime, the effects of string theory cannot be neglected [24].

6 Solutions based on the theory of embedding

Using embedding theorems to account for the high radial tension has the advantage of offering a possible explanation for the origin of exotic matter. Embedding theorems have a long history in the general theory of relativity, in large part due to Campbell's theorem [25]. According to Ref. [26], the field equations in terms of the Ricci scalar are $R_{AB} = 0$, $A, B = 0, 1, 2, 3, 4$. The resulting five-dimensional theory explains the origin of matter in the following sense: the vacuum field equations in five dimensions yield the usual Einstein field equations *with matter*, called the induced-matter theory [27, 28]. What we perceive as matter is the impingement of the fifth dimension onto our spacetime – and this would include exotic matter. It even suggests that the amount of exotic matter may be irrelevant. Given its problematical nature, however, we would prefer that the amount be kept to a minimum. Fortunately, the embedding theory fulfills this requirement.

According to Campbell's theorem [25], a Riemannian space can be embedded in a higher-dimensional flat space: an n -dimensional Riemannian space is said to be of embedding class m if $m + n$ is the lowest dimension d of the flat space in which the given space can be embedded; here $d = \frac{1}{2}n(n - 1)$. So a four-dimensional Riemannian space is of class two and can therefore be embedded in a six-dimensional flat space, i.e., $d = 6$. Moreover, a line element of class two can be reduced to a line element of class one by a suitable coordinate transformation [29, 30, 31, 32].

An interesting aspect of Einstein's theory is that the extra dimension can be spacelike or timelike. These cases will be taken up separately.

6.1 An extra timelike dimension

Due to the extra timelike dimension, the embedding space has the form

$$ds^2 = -(dz^1)^2 - (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2. \quad (32)$$

According to Ref. [3], the coordinate transformation is $z^1 = \sqrt{K} e^{\nu/2} \sin \frac{t}{\sqrt{K}}$, $z^2 = \sqrt{K} e^{\nu/2} \cos \frac{t}{\sqrt{K}}$, $z^3 = r \sin \theta \cos \phi$, $z^4 = r \sin \theta \sin \phi$, and $z^5 = r \cos \theta$. Substituting in Eq. (32) yields the line element

$$ds^2 = -e^\nu dt^2 + \left[1 - \frac{1}{4} K e^\nu (\nu')^2\right] dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (33)$$

This metric is equivalent to metric (1) if

$$e^\lambda = 1 - \frac{1}{4} K e^\nu (\nu')^2, \quad (34)$$

where $K > 0$ is a free parameter.

Given Eq. (34), Eq. (5) immediately yields

$$p_r(r) = \frac{1}{8\pi} \left[\frac{1}{1 - \frac{1}{4}K e^\nu (\nu')^2} \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} \right]. \quad (35)$$

Since K is a free parameter, K can be chosen to make $\tau(r) = -p_r(r)$ as large as required without relying on exotic matter. It turns out, however, that to meet the other conditions, some fine-tuning cannot be avoided. For example, Eq. (34) yields

$$b(r) = r \left[1 - \frac{1}{1 - \frac{1}{4}K e^{\nu(r)} [\nu'(r)]^2} \right] + \frac{r_0}{1 - \frac{1}{4}K e^{\nu(r_0)} [\nu'(r_0)]^2}, \quad (36)$$

so that $b(r_0) = r_0$, but the flare-out condition can only be met if $(\nu')^2 + 2\nu''$ is sufficiently small near the throat. (See Ref. [3] for details.) Fortunately, this condition can be easily met “by hand.” The same condition ensures that $\rho + p_r$ can be made as small as desired. So the amount of exotic matter [33],

$$\Omega = \int_0^{2\pi} \int_0^\pi \int_{r_0}^\infty (\rho + p_r) \sqrt{-g} \, dr d\theta d\phi, \quad (37)$$

can also be kept small.

Finally, the resulting wormhole spacetime is asymptotically flat.

6.2 An extra spacelike dimension

If the extra dimension is spacelike, then the embedding space has the form

$$ds^2 = -(dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2. \quad (38)$$

This case is discussed in Ref. [4]. Here the coordinate transformation is [29, 30] $z^1 = \sqrt{K} e^{\nu/2} \sinh \frac{t}{\sqrt{K}}$, $z^2 = \sqrt{K} e^{\nu/2} \cosh \frac{t}{\sqrt{K}}$, $z^3 = r \sin \theta \cos \phi$, $z^4 = r \sin \theta \sin \phi$, and $z^5 = r \cos \theta$. This time we obtain

$$ds^2 = -e^\nu dt^2 + \left[1 + \frac{1}{4}K e^\nu (\nu')^2 \right] dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (39)$$

and

$$e^\lambda = 1 + \frac{1}{4}K e^\nu (\nu')^2. \quad (40)$$

The result is a metric of embedding class one. Eq. (40) can also be obtained from the Karmarkar condition

$$R_{1414} = \frac{R_{1212}R_{3434} + R_{1224}R_{1334}}{R_{2323}}, \quad R_{2323} \neq 0.$$

In fact, Eq. (40) is a solution of the differential equation

$$\frac{\nu' \lambda'}{1 - e^\lambda} = \nu' \lambda' - 2\nu'' - (\nu')^2,$$

making K an arbitrary constant of integration.

To continue, Ref. [4] makes use of the following redshift function first proposed by Lake [34]:

$$\nu(r) = n \ln(1 + Ar^2), \quad n \geq 1, \quad (41)$$

where A is a constant. According to Ref. [34], this class of monotone increasing functions generates all regular static spherically symmetric perfect-fluid solutions of the Einstein field equations. Eq. (41) can be written $e^\nu = (1 + Ar^2)^n$. A slightly more general form, $e^\nu = B(1 + Ar^2)^n$, is also acceptable since the resulting ν is still monotone increasing. For convenience, Ref. [4] assumes that $n = 1$. The resulting form is

$$e^\nu = B(1 + Ar^2), \quad A, B > 0. \quad (42)$$

While A is still a free parameter, B can be determined from the junction conditions discussed below.

This time the shape function is

$$b(r) = r \left[1 - \frac{1}{1 + \frac{1}{4}K e^{\nu(r)} [\nu'(r)]^2} \right] + \frac{r_0}{1 + \frac{1}{4}K e^{\nu(r_0)} [\nu'(r_0)]^2}. \quad (43)$$

It is shown in Ref. [4] that for sufficiently large K ,

$$b'(r_0) = 1 + \frac{1 - Ar_0^2}{1 + Ar_0^2 + KBA^2r_0^2}. \quad (44)$$

It follows that

$$b'(r_0) < 1 \quad \text{for} \quad r_0 > \frac{1}{\sqrt{A}}. \quad (45)$$

So the flare-out condition is met whenever $r_0 > 1/\sqrt{A}$.

While $\lim_{r \rightarrow \infty} b(r)/r = 0$, it is not true that $\lim_{r \rightarrow \infty} \nu(r) = 0$. So the wormhole spacetime is not asymptotically flat and must therefore be cut off at some $r = a$ and joined to an exterior Schwarzschild spacetime. In other words, $e^{\nu(a)} = B(1 + Aa^2) = 1 - 2M/a$; so

$$B = \frac{1 - \frac{2M}{a}}{1 + Aa^2}. \quad (46)$$

For the radial pressure, we get

$$p_r(r) = \frac{A(2 - KBA)}{8\pi(1 + Ar^2 + KBA^2r^2)}, \quad (47)$$

whence

$$\lim_{r_0 \rightarrow 0} \tau(r_0) = \frac{A}{8\pi}(-2 + KBA). \quad (48)$$

So $\tau(r_0)$ can be made as large as required due to the free parameter K , as long as r_0 is relatively small.

Finally, according to Ref. [4], $\lim_{K \rightarrow \infty} 8\pi(\rho + p_r) = 0$, showing that the amount of exotic matter can be kept to a minimum.

7 Could the Universe be multiply connected?

A commonly accepted view is that the Universe is a 3-sphere and hence simply connected. The idea of a multiply-connected Universe has recently been revived [35]. We saw in Sec. 1.2 that this possibility is a direct consequence of Einstein's theory in the following sense: if we assume a dark-matter or dark-energy background, the conditions for the existence of wormholes are satisfied, but such wormholes could only exist on very large scales. An example of such a wormhole would be the huge doughnut shape proposed in Ref. [35].

8 Summary

Wormholes are just as good a prediction of Einstein's theory as black holes, in part because Einstein's theory is able to tolerate strange-sounding features such as space and time warps. Quantum field theory, on the other hand, makes its own demands that cannot be readily dismissed. For example, the need to violate the null energy condition (NEC) is a generic feature of a traversable wormhole, calling for the existence of exotic matter. Not only that, exotic matter must exist in sufficient quantities, which, on a macroscopic scale, is itself problematical. Another serious issue is the enormous radial tension at the throat, which could exceed the tension at the center of a massive neutron star unless the throat radius is extremely large. Attributing this characteristic of exotic matter ignores the fact that exotic matter was introduced for a completely different reason. Reducing the amount thereof, by itself highly desirable, makes matters worse. If the amount is infinitesimal, then exotic matter could not have the slightest effect on the radial tension. In $f(R)$ modified gravity, the throat can be lined with ordinary (nonexotic) matter, but as noted in Sec. 1.2, this does not necessarily avoid a high radial tension.

These issues are addressed in four recent publications [1, 2, 3, 4], covering six different aspects. Sec. 2, which is based on Ref. [1], deals with neutron star interiors. According to Ref. [1], the large radial tension is sufficient for a topology change, i.e., the formation of wormholes, provided that there is a core of quark matter at the center.

Sections 3, 4, and 5 are based on Ref. [2]. Sec. 3 briefly discusses wormholes in a noncommutative-geometry setting. The low energy density causes the flare-out condition to be met, thereby resulting in a violation of the NEC at or near the throat. While this is also true of wormholes supported by dark matter and dark energy, it is only the noncommutative-geometry case that results in a large radial tension. Since the NEC has been duely violated, the large radial tension can indeed be attributed to exotic matter. Sec. 4 combines the noncommutative-geometry background with $f(R)$ modified gravity by deriving the form of $f(R)$, Eq. (22). In this way the large radial tension is retained. The throat can now be threaded with ordinary (nonexotic) matter, however, since the violation of the NEC, the generic feature mentioned earlier, can be attributed to the higher-order curvature terms in the modified theory. Conceptually, this is a considerable advantage since the large radial tension is not directly connected to exotic matter,

Sec. 5 assumes an extra spatial dimension, possibly small or even curled up. It is noted that, once again, the throat could be lined with ordinary matter because the violation of the NEC can be attributed to the extra spatial dimension. It is shown in Ref. [2] that

the extra dimension can also account for the large radial tension.

Sec. 6 summarizes the theory of embedding, including the induced-matter theory: what we perceive as matter is the impingement of the higher-dimensional space onto ours. This may include exotic matter, suggesting that the amount of exotic matter is irrelevant. Given its problematical nature, however, it is generally assumed that the amount should be kept to a minimum. The embedding theory can fulfill both of these requirements.

A Riemannian space can be embedded in a higher-dimensional flat space: an n -dimensional Riemannian space is said to be of embedding class m if $m + n$ is the lowest dimension of the flat space in which the given space can be embedded. In particular, a four-dimensional Riemannian space is of class two but can be reduced to class one by a suitable transformation of coordinates. Furthermore, Einstein's theory allows the extra dimension to be either timelike or spacelike. The timelike case is discussed in Sec. 6.1, based on Ref. [3], and the spacelike case is discussed in Sec. 6.2, based on Ref. [4]. In both cases, the free parameter K in the coordinate transformation can be chosen to make the radial tension as large as required without relying on exotic matter.

Finally, Sec. 7 discusses the possibility of a multiply-connected Universe.

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