Distillation of Indistinguishable Photons

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A reliable source of identical (indistinguishable) photons is a prerequisite for exploiting interference effects, which is a necessary component for linear optical based quantum computing, and applications thereof such as Boson sampling. Generally speaking, the degree of distinguishability will determine the efficacy of the particular approach, for example by limiting the fidelity of constructed resource states, or reducing the complexity of an optical circuits output distribution. It is therefore of great practical relevance to engineer sources of highly indistinguishable photons. Inspired by magic state distillation, we present a protocol using standard linear optics (such as beamsplitters) which can be used to increase the indistinguishability of a photon source, to arbitrary accuracy. In particular, in the asymptotic limit of small error, we show that using 9 (16) photons one can distill a single purer photon, with error decreasing by 1/3 (1/4) per iteration. We demonstrate the scheme is robust to detection errors (such as dark counts) to second order.

Introduction—Linear optical quantum computing (LOQC) is an attractive paradigm for realizing fault-tolerance, since photons in free space have extremely long coherence times, and can be manipulated via high fidelity linear optics which may not require the same level of cooling as other approaches [1]. In LOQC, qubits are constructed out of photons which can exist in two modes, common choices being spatial modes, or using the polarization degrees of freedom. Fault tolerance can in principle be achieved via the KLM protocol (named after the authors Knill, Laflamme and Milburn) with sufficient numbers of qubits and using error correction [2], or using cluster states in a measurement-based approach to quantum computing [1, 3–7].

In order to make use of photons for computational purposes requires a source of highly indistinguishable photons. The Hong-Ou-Mandel (HOM) effect [8] is the prototypical example which shows fundamental differences in which identical versus distinguishable photons interfere (or do not). In this conceptually simple experiment, two photons are incident upon a 50:50 beamsplitter, which results in a bunching of the two photons in the case they are indistinguishable. On the other hand, when the input photons are distinguishable, the signal from an HOM experiment (the HOM ‘dip’) is diminished by an amount related to the infidelity of the two photons [9].

The HOM effect is a crucial ingredient for realizing linear optical quantum computing, for the interference between identical photons can be used to create entanglement over computational degrees of freedom [2]. For example, fusion measurements can be used to create large cluster states out of primitive entangled states, such as Bell states or small GHZ states [10].

Various schemes and proposals exist for the creation of such primitive entangled states [2, 11–13], which rely on multiple identical photons incident on a linear optical circuit, and post-selecting upon the measurement of a desired outcome. However, the presence of distinguishability will generally lead to a dephased version of the ideal state, with less entanglement in the computational degrees of freedom, compared to the ideal state [14].

Similarly, for specific applications of LOQC, such as Boson sampling [15], multi-photon interference is the key ingredient to generate a computationally intractable distribution. Indeed, it has been shown that the classical simulability of a sampling circuit is directly related to the degree of distinguishability of the photons [16].

It is therefore necessary to be able to generate photons with as high an overlap as possible. In this Letter, we present a technique inspired by magic state distillation [17], which is used to ‘distill’ indistinguishable photons from a photon source which outputs photons that are partly distinguishable.

A cartoon example of our general idea is shown in Fig. 1, whereby $n$ copies of a noisy photon state are used to produce single photons, with a lower degree of distinguishability. Input photons to the circuit populate spatial modes (horizontal lines), which we will often refer to as ‘rails’, and can be implemented physically via optical fibres, for example. The black box is at this point unspecified but will be an array of beamsplitters between the rails to enact interference.
The key observation behind our scheme is that identical photons interfere in a fundamentally different manner than partly distinguishable ones, which can be exploited using beamsplitters, and ultimately used to reduce the distinguishability of a noisy photon source.

Related Work—Whilst preparing this manuscript, we became aware of a morally similar scheme proposed by Sparrow and Birchall (SB) in Ref. [14], under the name ‘HOM filtering’. In this scheme, \( n \geq 2 \) photons are incident upon \( n \) spatial rails, which are post-selected upon bunching in a single rail. Photon subtraction is then used to output a single photon of a higher fidelity. This scheme is conceptually elegant, and results in asymptotic scaling of the error \( \epsilon \to \epsilon/n \). However, it is apparent that the scheme becomes prohibitive for even modest \( n \), as the probability to measure the desired outcome falls worse than exponentially in \( n \) [[18]]; we compute in App. A the post-selection success probability to be asymptotically (i.e. at error approaching zero)

\[
P^\text{SB}_{\text{p.s.}} \leq \frac{n}{2^n} \prod_{m=2}^{n} \frac{m}{2^m} = \frac{n^2(n-1)!}{2^{2n}n!},
\]

meaning huge numbers of photons are required to distill a single purer one (for \( n = 2, 3, 4, 5, 6 \), one requires on average 8, 42, 341, 4369, 93206 photons respectively).

The introduced scheme of the present work, as we will show, can distill a single purer photon using 9 noisy photons, with error reduction approaching \( \epsilon \to \epsilon/3 \) in the asymptotic limit of ‘small’ \( \epsilon \) (in App. C we additionally outline a circuit which uses 16 photons with error reduction scaling as \( \epsilon/4 \)). We will compare our main protocol with the SB approach for \( n = 2, 3 \), since we deem larger \( n \) as too resource heavy for any practical cases. Our scheme overcomes two issues identified by SB in their protocol, namely we achieve higher success probabilities, and do not require explicit multiple photon subtraction. Eventually we believe a hybrid scheme can be invoked, as in regimes of high error, the SB scheme can outperform the present approach, whereas at lower errors, our scheme is more efficient. We will discuss this in the Results section.

Theory—An arbitrary single photon state can be written as a sum over modes:

\[
|\psi\rangle = \sum_{s \in \{h, v\}} \int d\omega \; c_s(\omega)|s, \omega\rangle = \sum_{i=0}^{\infty} c_i |\hat{a}^\dagger_i |0\rangle = \sum_{i=0}^{\infty} c_i |\psi_i\rangle.
\]

The term after the first equals sign represents the explicit representation over the polarization (\( s \) being e.g. horizontal \( h \) or vertical \( v \)) and frequency (\( \omega \)) domains, and going to the second equals sign we have picked a countable orthonormal basis in the separable Hilbert space to represent the continuous degrees of freedom (and absorbed the \( s \) index into the new sum). The state \(|0\rangle\) is the vacuum state, and \(\hat{a}^\dagger_i\) creates a photon in the \(i\)'th mode, where for now we use the explicit state representation \(\hat{a}^\dagger_i |0\rangle = |\psi_i\rangle\). By construction, these basis states are orthogonal \(\langle \psi_i |\psi_j\rangle = \delta_{ij}\), and the amplitudes \(c_i \in \mathbb{C}\) square sum to 1: \(\sum_i |c_i|^2 = 1\).

We now describe the model of a noisy photon source which is used in this work. A non-ideal photon source will output photons according to Eq. (2), but with realization dependent coefficients \(c_i\) (that is, they are different for each generated photon). Without loss of generality we can pick the basis so that the desired mode to populate is the 0'th one, i.e. \(|\psi_0\rangle\) is the state which would be generated each time by a perfect photon source. We consider fluctuations around this ideal by assuming the source can generate photons in the 0'th mode with probability 1 - \(\epsilon\), i.e. \(\langle |c_0|^2 \rangle = 1 - \epsilon\), where the angle brackets indicate the realization average. We will similarly define \(p_i := \langle |c_i|^2 \rangle\), where \(\sum_{i=0}^{\infty} p_i = \epsilon\).

We further make a random phase approximation so that \(\langle c_i c_j^\ast \rangle = 0\) for \(i \neq j\), which means the photon source can be equivalently described as a dephased mixture:

\[
\rho(\epsilon) = (1 - \epsilon) |\psi_0\rangle \langle \psi_0| + \sum_{i=0}^{\infty} p_i |\psi_i\rangle \langle \psi_i|.
\]

This approximation amounts to the ‘error amplitudes’ \(c_{j>0} = |c_j| e^{i\phi_j}\) receiving a random phase \(\phi_j\) (independent of the norm) on each realization. With this, we can therefore interpret the photon source as generating a photon in the ideal state \(|\psi_0\rangle\) with probability 1 - \(\epsilon\), or with probability \(\epsilon\) an orthogonal ‘error mode’ is populated (i.e. from one of the \(\hat{a}^\dagger_{i>0}\)). We will similarly call the \(|\psi_{i>0}\rangle\) as an ‘error state’ (orthogonal to \(|\psi_0\rangle\)). We define the indistinguishability as the mean overlap of two pure states sampled from \(\rho\) (HOM visibility), which is simply the purity \(\text{tr}(\rho^2)\), and the aim is to maximise this quantity (by minimising \(\epsilon\)).

To simplify the analysis, we can consider the small error (small \(\epsilon\)) limit. At sufficiently small \(\epsilon\) it is unlikely to observe more than one error state according to the above statistical description; if we draw \(n\) samples from distribution \(\rho\), we either get \(n\) copies of \(|\psi_0\rangle\), or \(n - 1\) copies of \(|\psi_0\rangle\), and one copy of some orthogonal error state \(|\psi_{\perp}\rangle\) (i.e. \(|\psi_{\perp}\rangle\) is one of the \(|\psi_{i>0}\rangle\)). Note, in our subsequent analysis we will still take into account the cases when more than one error mode is populated, but for now we can work in the limit of only single errors, for convenience. We can write the \(n\) photon state, to first order as

\[
\rho^\otimes n = (1 - \epsilon)^n |\psi_0\rangle \langle \psi_0| + \epsilon (1 - \epsilon)^{n-1} \sum_{k=1}^{n} |\Psi_k\rangle \langle \Psi_k| + O(\epsilon^2),
\]

where we have introduced notation \(|\Psi_0\rangle = |\psi_0\rangle^\otimes n\) and \(|\Psi_k\rangle = |\psi_0\rangle^\otimes (k-1)|\psi_{\perp}\rangle |\psi_0\rangle^\otimes (n-k)\). The tensor structure comes from the spatial mode representation, as in Fig. 1. For now we write the error state generically as \(|\psi_{\perp}\rangle\), as we will later see at first order it is unimportant for our analysis which particular error mode \(i > 0\) is populated in state \(|\Psi_k\rangle\).

In order to enact interference between photons of the above form, we will utilise a beamsplitter. In our nota-
tation a beamsplitter is described by 4 parameters, and acts
on (spatial) mode creation operators $\hat{a}^\dagger, \hat{b}^\dagger$ as follows:
\begin{align}
\hat{a}^\dagger &\to e^{i(\phi_0 + \phi_R)} \sin(\theta) \hat{a}^\dagger + e^{i(\phi_0 - \phi_T)} \cos(\theta) \hat{b}^\dagger \\
\hat{b}^\dagger &\to e^{i(\phi_0 - \phi_T)} \cos(\theta) \hat{a}^\dagger - e^{i(\phi_0 - \phi_R)} \sin(\theta) \hat{b}^\dagger.
\end{align}
(5)

We assume the parameters $\{\theta, \phi_0, \phi_R, \phi_T\}$ are agnostic to the
impinging photons internal state $|19\rangle$, and therefore any
single photon incident upon such a beamsplitter will be
‘split’ in the same manner as any other. A 50:50 beamsplitter
refers to the case $\theta = \pi/4$, where there is equal
transmission to the other mode $(T)$, or reflection to the
same mode $(R)$. Throughout we use the the convention
for the phases $\phi_0 = \pi/2, \phi_R = -\pi/2, \phi_T = 0$.

Since we utilise optical components that are state-
agnostic, and any single photon in state $|\psi_i\rangle$ will not interfere
with the ideal state $|\psi_0\rangle$ (by orthogonality), it has no bearing on the output statistics of a circuit of
form Fig. 1 which particular error mode $i > 0$ is actually
populated when state $|\Psi_k\rangle$ is sampled from $\rho^{\otimes n}$. For this
reason we can write the single error state simply as $|\psi^+_k\rangle$, as mentioned above.

Now that we have described the basic components in
our construction, all that remains is to outline the post-
selection over detection events. We will require access
to photon number resolving detectors which we assume
are ideal; it will always detect the exact number of pho-
tons present (though it will in fact be enough to distin-
guish between 0,1,2,3 photons, which will be clear later).
The post-selection on a detection event of $m$ photons
can be described by taking the partial trace of the mea-
sured rail(s) after applying a measurement operator on
the state $|14, 20\rangle$. If before measurement the state is $\rho$, and we place a detector at the $k$’th rail to detect $m$
photons, the post-selected state will be $\text{Tr}_k[\Pi_k^{(m)} / \rho^{\otimes n}] / N$, where $\Pi_k^{(m)}$ sums over all rank 1 projectors onto pure
states which contain $m$ photons in the $k$’th rail. $N$ is for
normalization.

Results—The central question we wish to answer is
whether one can engineer the schematic diagram Fig. 1
with a suitable number $n$ of photons, and linear optical
components in the black box, so that the output state has
less error than Eq. (3), upon a suitable post-selection. If
one can do this, the process can be repeated indefinitely
until arbitrary accuracy (i.e. $\epsilon$ is arbitrarily small).

From our studies, this in fact defines a large class of
optical circuit of varying numbers of photons and linear
optical components. We however will focus our attention
on the ‘best’ performing that we found (where here best
has a precise meaning, in terms of the number of photons
required to distill a photon to some particular accuracy).
Indeed, there is scope for the discovery of improved cir-
cuits. We will assume all components and detectors are
perfect, so that the only source of error is in the photon
generator.

The circuit of present interest is shown in Fig. 2, com-
posed of three rails (each taking one incident photon),
and three beamsplitters, two which are symmetric, and
one which is asymmetric, biased to higher reflectivity (to
stay in same mode). Note permutations of this circuit
also perform identically (keeping the angle of the middle
beamsplitter tangent to the same mode). Note permutations of this circuit
also perform identically (keeping the angle of the middle
beamsplitter tangent to the black circles intersect. The first and third beamsplitters are
50:50 ($\pi/4$ in the diagram), and the middle is asymmetric with $\theta = \tan^{-1} \sqrt{2} \approx 0.955$ (less likely to transmit). In the
asymptotic limit of small $\epsilon$, the error is reduced by a factor
of $1/3$, and post-selection succeeds with probability $1/3$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{Three photon distillation scheme. A successful measurement corresponds to a single photon registered in each of the two measured rails (indicated by the ‘1’ subscript on the detectors). The vertical lines with black circles represent beamsplitters between the rails on which the black circles intersect. The first and third beamsplitters are 50:50 ($\pi/4$ in the diagram), and the middle is asymmetric with $\theta = \tan^{-1} \sqrt{2} \approx 0.955$ (less likely to transmit). In the asymptotic limit of small $\epsilon$, the error is reduced by a factor of $1/3$, and post-selection succeeds with probability $1/3$.}
\end{figure}

\[\frac{i}{\sqrt{3}}|111, 0\rangle + \frac{\sqrt{2}}{3}(|300, 0\rangle + i|030, 0\rangle + |003, 0\rangle),\]
(6)
which has probability of $1/3$ to obtain the correct post-
selected state $|21\rangle$.

If on the other hand a single error state is present, i.e.
one of $|\Psi_k\rangle$ is sampled (each occurring with proba-
bility $\epsilon(1 - \epsilon^2)$), the output in the relevant subspace,
before measurement, is $\frac{1}{2\sqrt{2}} \sum_{k=1}^{3} |\Psi_k\rangle$, up to a phase.
The post-selection therefore succeeds with total probability
$1/9$, and the outputted (unmeasured) photon is
ideal $|\psi_0\rangle$ with post-selected probability $2/3$ $|22\rangle$.

The key observation behind the scheme is that the ideal
input is successfully post-selected upon three times as of-
ten than the case where an error is present (1/3 Vs 1/9),
which allows the errors to be filtered out, approximately
at a rate of 1/3 error reduction per round. The use of the
asymmetric beamsplitter is fundamental to this, as it
allows various other Fock states – which would otherwise
be present in Eq. (6) – to undergo perfect destructive
interference (for example we see there are no rails con-
taining exactly two photons).

One can produce an upper bound on the error reduc-
tion (see App. B), $\epsilon \to \epsilon'$ under the scheme:
\[\epsilon' \leq \frac{\epsilon}{3} - 2\epsilon^2 + 3\epsilon^3 = \frac{\epsilon + 4\epsilon^2}{3} + O(\epsilon^3).
(7)\]
Figure 3. **Error reduction comparison of our scheme (‘Present’), and those of SB for n = 2, 3.** Given a photon source with error $\epsilon$ (Eq. (3)), the post-selected output has error $\epsilon'$. The shaded region indicates the upper and lower bound on the error reduction of our scheme, as discussed in the main text (and App. B). For SB, we use the best case error reduction, Eq. (7.11) in Ref. [14] (also see Eq. (B5)).

Inset: Zoom in on region $\epsilon < 0.1$. The reason this is a bound, instead of equality, is that the error reduction depends on the specifics of the distribution of errors in Eq. (3). In App. B we also produce a lower bound on the error, $\epsilon' \geq \frac{\gamma}{3} + \frac{2\sigma^2}{\epsilon} + O(\epsilon^3)$. The scheme can be used to reduce errors ($\epsilon' < \epsilon$) so long as the initial error $\epsilon$ is below around 43%.

The error reduction capabilities of our scheme is shown in Fig. 3, where we also compare to the SB protocol for $n = 2, 3$. We see our scheme outperforms SB for $n = 2$ for errors less than around 15%, and that our scheme converges with SB $n = 3$ at around 3% error. Note, for the SB scheme we plot the best case error reduction, whereas in reality it may perform worse than this, depending on the distribution of error modes, see Ref. [14] (though for small $\epsilon$ the difference becomes negligible).

In Fig. 4 we plot the probability of obtaining a valid post-selection measurement outcome (i.e. detection of a single photon at each of the two detectors), which succeeds with probability $(1 - 2\epsilon)/3 + O(\epsilon^2)$ (App. B). Since our scheme consumes 3 photons per use, we require around 9 photons to distill a single purer one to 1/3 the error (or 81 photons to error 1/9). In comparison to SB for $n = 2, 3$, around 8 and 42 photons are required respectively to obtain 1/2, 1/3 error respectively. This implies in the asymptotic limit our scheme is the most efficient. We also note for SB $n = 4$, although it results in a reduction of $\epsilon/4$ asymptotically, over 340 photons are required to distill a single photon (since the post-selection success probability is so small). Since we could run our scheme around 113 times at this cost (to distill over 36 purer photons), we could iteratively run our scheme twice to distill around 4 photons to error $\epsilon/9$.

Lastly, we wish to mention we also discovered an $n = 4$ photon circuit, which is essentially a generalization of the presented $n = 3$ circuit (though with only 50:50 beamsplitters), which can reduce errors by $\epsilon/4$, at the expense of a lower success probability – asymptotically $1/4$ – meaning around 16 photons are required on each iteration. Due to the fairly small improvement in the error, but using nearly twice as many photons, we leave this analysis to App. C.

**Discussion** – We have introduced a scheme that can be used to distill perfectly indistinguishable photons, which in the small error limit is the most efficient, compared to another similar scheme outlined by Sparrow and Birchall.

We wish to comment at this point that our scheme has some built in robustness to detection errors since the Hamming distance between states in the ideal output Eq. (6) (the dominant contribution at low error) is at least 2 over the measured rails; e.g. to mistake an outcome $|3, 0, 0\rangle$ or $|0, 3, 0\rangle$ with $|1, 1, 1\rangle$ by detection of the last two rails requires either two dark counts, or a dark count and incorrect photon resolution respectively. If errors of this sort occur at a rate $\epsilon_d$, the overall contribution is $O(\epsilon^2_d)$, i.e. errors from faulty detections are only present as a second order effect. Similarly, receiving an error $|\psi_{i>0}\rangle$ in addition to a dark count could also result in a false measurement pattern, but is still a second order process $O(\epsilon_d)$. As discussed in the Results section, this robustness is ultimately due to the use of the asymmetric beamsplitter which causes destructive interference of lower Hamming distance terms.

In practice loss errors will also occur, which could mean no photon is present at the output, despite a successful detection pattern in the measured rails. In principle this is a first order effect, $O(\epsilon_L)$ where $\epsilon_L$ is the loss rate, since a single lost photon can yield the outcome $|0, 1, 1\rangle$.
meaning the vacuum is outputted despite a valid detection pattern. It is important to note that such errors will generally not build up however, since as soon as more than one photon has leaked, we can not register a valid detection event (unless there are also dark counts, which again is a second order process).

Overall, in realistic scenarios, imperfect detectors as well as photon loss will determine the upper bound on the fidelity of our scheme. Although our scheme is somewhat robust to faulty detections, it is not robust to photon loss in the same way. An interesting follow up would be to find a protocol which can also map these errors to higher order, perhaps in a scheme that can output more than a single photon, thus also increasing the yield.

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[18] Numerically we find Eq. (1) can be approximated by $\exp(-0.29n^{1/6})$, see Fig. 5.
[21] Note this scheme can also be used to prepare the superposition $|3,0\rangle + |0,3\rangle$, upon measuring zero photons in the middle rail of Eq. (6), which occurs with probability $4/9$, which can be considered a 3 photon generalization of the standard HOM state.
[22] The output in the case of a single error mode populated can be evaluated by allowing each spatial mode (rail) to support two ‘physical’ modes $(\hat{a}_0^\dagger,\hat{a}_0^\dagger)^j$, representing the ideal and error mode respectively. Each beamsplitter in the circuit then acts on these physical modes independently (as they are orthogonal). One can show, in the relevant subspace of a single photon per rail, the output state (before measurement) is an equal weight superposition $\sum_{k=1}^{3} |\Psi_k\rangle$, up to a phase. Post-selection therefore succeeds with probability $3/27 = 1/9$, by projecting to one of these three distinct outcomes (and tracing out the measured photons), according to the measurement protocol outlined in the main text.
Appendix A: Derivation of Eq. (1)

We first direct the reader to Ch. 7.2 of Ref. [14] for explicit outline of the SB protocol.

We consider the case where all photons are identical (i.e. case of zero distinguishability), which is the dominant contribution to the post-selection probability as the error $\epsilon \to 0$. There are two independent calculations to arrive at Eq. (1).

The first involves computing the probability for an input of $m$ and 1 photons in separate rails to bunch, that is, $|m, 1\rangle \to |m + 1, 0\rangle$, upon post-selection by measuring 0 photons in the second rail.

To do that, let us denote the initial state using creation operators as

$$|m, 1\rangle := \frac{1}{\sqrt{m!}}(\hat{a}^\dagger)^m \hat{b}^\dagger |0, 0\rangle.$$ 

Under our convention outlined in the main text, a 50:50 beamsplitter acts as (though the choice of phase angles is unimportant here)

$$\hat{a}^\dagger \rightarrow \frac{1}{\sqrt{2}} (\hat{a}^\dagger + i\hat{b}^\dagger)$$

$$\hat{b}^\dagger \rightarrow \frac{1}{\sqrt{2}} (i\hat{a}^\dagger + \hat{b}^\dagger).$$

We wish to compute the amplitude of the coefficient $(\hat{a}^\dagger)^{m+1}$ in the expansion $(\hat{a}^\dagger + i\hat{b}^\dagger)^m(\hat{a}^\dagger + \hat{b}^\dagger)/\sqrt{2^{m+1}}$, which one can see by inspection is simply $i/\sqrt{2^{m+1}}$. The states amplitude is computed via

$$\frac{1}{\sqrt{m!} \sqrt{2^{m+1}}} i \frac{i}{\sqrt{2^{m+1}}} (\hat{a}^\dagger)^{m+1} |0, 0\rangle = i \sqrt{\frac{m + 1}{2^{m+1}}} |m + 1, 0\rangle,$$

and therefore sampled with probability $p_m = \frac{m + 1}{2^{m+1}}$.

For the $n$ photon circuit, the above is applied iteratively, starting with $m = 1$, up to $m = n - 1$, which gives probability proportional to

$$\prod_{m=1}^{n-1} p_m = \prod_{m=2}^n \frac{m}{2^m}.$$

The other independent contribution comes from the photon subtraction after the above post-selection scheme, taking $|n, 0\rangle$ to $|1, n - 1\rangle$ (again by measuring the second rail, this time to find $n - 1$ photons). By similar analysis, this can be shown to give probability $n/2^n$, which combining with the above result, completes the proof of Eq. (1).

Note that the case of partial distinguishability ($\epsilon > 0$) will result in a lower success probability, hence why we write Eq. (1) as an upper bound.

We plot Eq. (1) in Fig. 5, as well as a numerically found approximation shown in the legend, which decreases exponentially in $n^{1.96}$.

Figure 5. Asymptotic SB post-selection probability. Also plotted is a numerical approximation which decreases exponentially in $n^{1.96}$.

Appendix B: Derivation of Eq. (7)

Recall that our model is derived from Eq. (3), which is equivalent to receiving the ideal photon state $|\psi_0\rangle$ with probability $1 - \epsilon$, or an orthogonal error state $|\psi_{1}\rangle$ with probability $\epsilon$.

There are four cases to consider to derive a bound on the output post-selected fidelity from our circuit Fig. 2, which we will denote as

$$f' = \langle \psi_0 | \rho_{p,s} | \psi_0\rangle = 1 - \epsilon', \quad (B1)$$

where $\rho_{p,s}$ is the post-selected density matrix. Since the output fidelity will depend on the error model, namely the distribution $\{p_i\}$ in Eq. (3), we will find lower and upper bounds on the fidelity.

0 errors: With probability $(1 - \epsilon)^3$ no error modes are populated, and the input to the circuit is simply $|\psi_0\rangle \otimes 3$. Using Eq. (6) we see the post-selection succeeds with probability 1/3, of which the outputted photon is the ideal one.

1 error: With probability $3\epsilon(1 - \epsilon)^2$ a single error mode is populated, and the input to the circuit is one of the $\{|\Psi_k\rangle\}_{k=1}^3$ (as defined in the main text). As discussed in footnote [[22]], the post-selection succeeds with probability 1/9 on such an input, but only 2/3 of the outputted photons are in the ideal state $|\psi_0\rangle$.

2 errors: With probability $3\epsilon^2(1 - \epsilon)$ two error modes are populated. Within this there are two sub-cases to consider, i) when the error modes are the same, $|\psi_0, \psi_1, \psi_1\rangle$, or ii) when they are unique $|\psi_0, \psi_1, \psi_2\rangle$ (and permutations thereof). Note we picked the labels 1, 2 for the error modes, however this is completely arbitrary, and we can do this without loss of generality. We find that the former case yields a valid measurement pattern with probability 1/9, whereas the latter is 2/9. In both cases, the outputted photon is the ideal one only 1/3 of the time.
3 errors: With probability $\epsilon^3$ three error modes are populated. There are technically three classes to consider: modes are all distinct, two the same, or three the same. Of course, in all three cases, there is 0 chance of the ideal photon $|\psi_0\rangle$ being outputted. Since we seek a bound on the output fidelity, we wish to find the case which minimizes/maximizes the post-selection probability. When all three error modes are identical the post-selection success probability is maximal for this case, with probability $1/3$ (for the same reason as the 0 error case). The converse case achieves lowest post-selection success with probability $1/9$, occurring when two modes are identical.

The above 4 cases give all possible sampling outcomes, and we can use this to find a lower bound on the output fidelity $f'$, which is simply the ratio of the post-selected probability to obtain the ideal photon state $|\psi_0\rangle$, normalized by the total post-selection success probability. This is achieved taking the $2/9$ post-selection case for two errors, and $1/3$ for three errors, and yields, writing the terms suggestively:

$$f' \geq \frac{(1 - \epsilon)^3}{3} + 3\epsilon(1 - \epsilon)^2 \frac{1}{3} + 3\epsilon^2 (1 - \epsilon) \frac{1}{3} = \frac{1}{3}[(1 - \epsilon)^3 + \epsilon(1 - \epsilon)^2 + 2\epsilon^2 (1 - \epsilon) + \epsilon^3]$$

$$= 1 - \frac{\epsilon^2}{3} + O(\epsilon^3).$$

This can be rearranged to Eq. (7) of the main text.

We can similarly obtain an upper (lower) bound on the fidelity (error), with final result

$$\epsilon' \geq \frac{\epsilon}{3 - 6\epsilon + 6\epsilon^2 - 2\epsilon^3} = \frac{\epsilon}{3} + \frac{2\epsilon^2}{3} + O(\epsilon^3).$$

In the same manner we find a lower bound on the post-selection success probability:

$$P_{p.s.} \geq \frac{1}{3}[(1 - \epsilon)^3 + \epsilon(1 - \epsilon)^2 + \epsilon^2 (1 - \epsilon) + \frac{1}{3} \epsilon^3]$$

$$= \frac{1}{3}[(1 - 2\epsilon^2 - 2\epsilon^3)].$$

Calculation of fidelity for $SB \; n = 2, 3$: Via a similar analysis as above we can find an explicit version of Eq. (7.11) in Ref. [14] for the case $n = 2, 3$ (which is what we plot in Fig. 3).

The result is, writing the error $\epsilon' = 1 - f'$:

$$\epsilon'_{SB, n = 2} \geq \frac{\epsilon}{2 - 2\epsilon + \epsilon^2} = \frac{\epsilon}{2} + \frac{\epsilon^2}{2} + O(\epsilon^3),$$

$$\epsilon'_{SB, n = 3} \geq \frac{2\epsilon - 2\epsilon^2 + \epsilon^3}{6 - 12\epsilon + 9\epsilon^2 - 2\epsilon^3} = \frac{\epsilon}{3} + \frac{\epsilon^2}{3} + O(\epsilon^3).$$

As discussed in Ref. [14], this is the best case performance of their protocol (the bound becomes tight for small $\epsilon$).

Appendix C: 4 Photon Circuit

Here we outline an equivalent circuit as shown in the main text, for $n = 4$, where the error reduction scales as $\epsilon/4$, and post-select success probability $1 - 3\epsilon/4 + O(\epsilon^3)$. The circuit uses only 50:50 beamsplitters (in contrast to Fig. 2) and is shown in Fig. 6. This is due to the symmetry of the circuit (the $n = 3$ circuit requires the asymmetric beamsplitter to effectively symmetrize the output, Eq. (6)).

First, let us consider the circuit output (pre-measurement), given four identical photons input on each rail $|1, 1, 1, 1\rangle$:

$$\frac{1}{2} |1, 1, 1, 1\rangle +$$

$$\frac{1}{4} (|2, 2, 0, 0\rangle - |2, 0, 2, 0\rangle + |2, 0, 0, 2\rangle +$$

$$|0, 2, 2, 0\rangle - |0, 2, 0, 2\rangle + |0, 0, 2, 2\rangle +$$

$$\sqrt{\frac{3}{32}} (|4, 0, 0, 0\rangle + |0, 4, 0, 0\rangle + |0, 0, 4, 0\rangle + |0, 0, 0, 4\rangle).$$

We see the ideal output is now robust to detection errors to third order (e.g. requiring 2 dark counts as well as incorrect photon number resolution). In the absence of such errors, there is a 1/4 probability of detecting the desired output, and outputting a single photon in the unmeasured rail.

Moreover, one can calculate that in the event of a single error being sampled (each with probability $\epsilon(1 - \epsilon)^3$), i.e., one of the $\{|\Psi_k\rangle\}_{k=1}^4$, post-selection succeeds with probability $1/16$, of which $3/4$ of the outputted photons are the ideal $|\psi_0\rangle$. This follows from the observation that before measurement, in the subspace of a single photon per rail, the state is $\frac{1}{2} \sum_{k=1}^4 |\Psi_k\rangle$.

The measurement of a single photon per detected rail projects on to one of these four distinct outcomes, each with probability $1/64$. With total probability $3/64$ the outputted photon is ideal, and $1/64$ is an error state $|\psi_{>0}\rangle$. This is directly analogous to the 3 photon circuit (see footnote [[22]]).

Using this, one can bound the post-selection success.
probability by
\[ P_{p.s.}^{(n=4)} \geq \frac{1}{4}(1 - \epsilon)^3 = \frac{1}{4} - \frac{3}{4}\epsilon + O(\epsilon^2), \]
and the error reduction is
\[ \epsilon' \leq \frac{\epsilon}{4} + O(\epsilon^2). \]
By performing a more rigorous analysis as in App. B (taking into account cases of more than a single error) we could achieve tighter bounds.

Overall this scheme requires, asymptotically, 16 photons to distill a single photon to 1/4 the original error. Since it uses an additional beamsplitter, and requires an additional detection, it is likely the case that in a physical setting the \( n = 3 \) circuit would be the best performing.