

The time-dependent harmonic oscillator revisited

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Abstract

We re-examine the time-dependent harmonic oscillator $\ddot{q} = -\omega^2 q$ under various regularity assumptions. Where $\omega(t)$ is continuously differentiable we reduce its integration to that of a single first order equation, i.e. the Hamilton equation for the angle variable ψ *alone* (the action variable \mathcal{I} does *not* appear). Reformulating the generic Cauchy problem for $\psi(t)$ as a Volterra-type integral equation and applying the fixed point theorem we obtain a sequence $\{\psi^{(h)}\}_{h \in \mathbb{N}_0}$ converging uniformly to ψ in every compact time interval; if ω varies slowly or little, already $\psi^{(0)}$ approximates ψ well for rather long time lapses. The discontinuities of ω , if any, determine those of ψ, \mathcal{I} . The zeroes of q, \dot{q} are investigated with the help of two Riccati equations. As a demo of the potential of our approach, we briefly argue that it yields good approximations of the solutions (as well as exact bounds on them) and thereby may simplify the study of: the stability of the trivial one; the occurrence of parametric resonance when $\omega(t)$ is periodic; the adiabatic invariance of \mathcal{I} ; the asymptotic expansions in a slow time parameter ε ; etc.

Contents

1	Introduction	2
2	Preliminaries	4
3	Reduction to Riccati equation, and zeroes of q, \dot{q}	5
4	Solving Hamilton equations for action-angle variables	9
4.1	Reduction to the Hamilton equation for the angle variable	9
4.2	Sequence converging to the solution of (27)	10
5	Usefulness of our approach	13
5.1	Adiabatic invariance of \mathcal{I}	13

5.2	Asymptotic expansion in the slow-time parameter ε	14
5.3	Upper and lower bounds on the solutions	15
5.4	Parametric resonance, beats, damping	16
6	Discussion and outlook	18
7	Appendix	21
7.1	Proof of Proposition 1	21
7.2	Proof of Proposition 2	22
7.3	Proof of the bounds (40)	23
7.4	Proof of the adiabatic invariance properties (44), (45)	24

1 Introduction

The equation of the time-dependent harmonic oscillator

$$\ddot{q}(t) = -\omega^2(t)q(t), \quad \omega(t) > 0 \tag{1}$$

(we abbreviate $\dot{f} \equiv df/dt$, etc.) has countless applications in natural sciences. In physics, it arises e.g. in classical and quantum mechanics, optics, electronics, electrodynamics, plasma physics, astronomy, geo- and astro-physics, cosmology (see e.g. [18, 20, 25, 31, 11, 22, 19, 3, 32, 4, 11, 24, 1]), possibly after reduction from more general equations. Moreover, in the equivalent form of Hamilton equations $\dot{q} = p$, $\dot{p} = -\omega^2 q$ associated to the Hamiltonian

$$H(q, p; t) := \frac{1}{2} [p^2 + \omega^2(t)q^2] \tag{2}$$

it is paradigmatic for investigating general phenomena in non-autonomous Hamiltonian systems, such as: i) the long-time behaviour of the solutions and of the adiabatic invariants under slow or small time-dependences; ii) the characterization of the time-dependences making the trivial solution unstable; in particular, iii) the characterization of the periodic ones leading to periodic solutions or to parametric resonance; iv) the behaviour of solutions under fast or large time-dependences; etc. Moreover, known two independent solutions of (1) we can find the general solution of a linear equation of the form

$$\dot{x} = Ax + a, \quad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}, \quad x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \tag{3}$$

with assigned $a(t)$, $\omega^2(t)$ and unknown $x(t)$. In [8] we use a family of equations of this type to describe the evolution of the Jacobian relating the Eulerian to the Lagrangian variables in a family of (1+1)-dimensional models describing the impact of a very short and intense laser pulse into a cold diluted plasma; incidentally, this has given us the initial motivation for the present study. Actually, one can reduce (see section 2) the resolution of (3a) with a more general matrix A to finding two independent solutions (1) with a related ω^2 .

Here we present a rather general and effective method for approximating the solutions (1), as well as a number of useful bounds or qualitative properties of the latter, that we have not been able to find in the very broad literature (see e.g. the already cited references, the more mathematical ones [7, 28, 16, 23, 21, 34, 32, 27, 30, 35, 10, 29, 17, 2], and the references therein) on the subject. Our key observation is that, passing (section 4.1) from the canonical coordinates q, p to the angle-action variables $\psi, \mathcal{I} = H/\omega$, via

$$q = \sqrt{\frac{2\mathcal{I}}{\omega}} \sin \psi, \quad p = \sqrt{2\mathcal{I}\omega} \cos \psi, \quad (4)$$

the Hamilton equations for ψ, \mathcal{I}

$$\dot{\psi} = \omega + \frac{\dot{\omega}}{2\omega} \sin(2\psi), \quad \frac{\dot{\mathcal{I}}}{\mathcal{I}} = -\frac{\dot{\omega}}{\omega} \cos(2\psi) \quad (5)$$

are such that the first is *decoupled* from the second. This apparently overlooked feature of the harmonic oscillator, among the time-dependent Hamiltonian systems admitting angle-action variables, is at the basis of our approach: we reduce (1) to (5a); afterwards (5b) is solved by quadrature. Reformulating the generic Cauchy problem (5a) with $\psi(t_*) = \psi_*$ as an integral equation and applying the fixed point theorem (section 4.2) we obtain a sequence $\{\psi^{(h)}\}_{h \in \mathbb{N}_0}$ such that $|\psi - \psi^{(h)}|$ - and therefore also the corresponding $|\mathcal{I} - \mathcal{I}^{(h)}|$, $|q - q^{(h)}|$, $|p - p^{(h)}|$ - uniformly go to zero in every compact interval containing t_* . If $\omega(t)$ has slow or small variations, namely if

$$\zeta := \dot{\omega}/\omega^2 \quad (6)$$

(ζ is a dimensionless function measuring the relative variation of ω in the characteristic time $1/\omega$) fulfills $|\zeta| \ll 1$, then already the 0-th order (in ζ) approximation

$$\begin{aligned} \mathcal{I}^{(0)}(t) &= \mathcal{I}_* := \mathcal{I}(t_*), & \psi^{(0)}(t) &= \psi_* + \int_{t_*}^t dz \omega(z), \\ q^{(0)} &= \sqrt{\frac{2\mathcal{I}_*}{\omega}} \sin[\psi^{(0)}], & p^{(0)} &= \sqrt{2\mathcal{I}_*\omega} \cos[\psi^{(0)}] = \dot{q}^{(0)}, \end{aligned} \quad (7)$$

is pretty good for quite long time intervals containing t_* , and much better than the one

$$\tilde{q}(t) := \sqrt{\frac{2\mathcal{I}_*}{\omega_*}} \sin[\psi_* + \omega(t)(t - t_*)], \quad \tilde{q}'(t) = [\omega(t) + \dot{\omega}(t)(t - t_*)] \sqrt{\frac{2\mathcal{I}_*}{\omega_*}} \cos[\psi_* + \omega(t)(t - t_*)] \quad (8)$$

obtained from the solution of (1) with constant $\omega \equiv \omega_* := \omega(t_*)$ by the naive substitution $\omega_* \mapsto \omega(t)$; for shorter time lapses (7) gives a reasonable approximation also if ω varies not so slow nor little, see fig.s 2, 3, for examples. Eq. (5) make sense in intervals where ω is continuously differentiable (or is so piecewise, while keeping continuous and with bounded derivative); but all solution (ψ, \mathcal{I}) can be extended across the discontinuities of ω (if any) via related matching conditions.

The plan of the paper, beside section 4, is as follows. In section 2, while fixing the notation, we recall for which A , and how, (3) can be reduced to a homogeneous system

$\dot{u} = Au$ with A of the form (3b), i.e. (setting $q = u^1$) to (1). In section 3 we prove that if $\inf_{t \in \mathbb{R}} \{\omega(t)\} > 0$ then each solution $t \in \mathbb{R} \mapsto q(t) \in \mathbb{R}$ admits a sequence $\{t_h\}_{h \in \mathbb{Z}} \subset \mathbb{R}$ of interlacing zeroes of q, \dot{q} , and we study how these t_h depend on the initial condition fulfilled by $q(t)$; rather than via (5), we do this by reducing (1) patch by patch to a Riccati equation, which is well-defined also if ω is not continuous. In section 5 we sketch how our approach may help for several purposes: proving the adiabatic invariance of \mathcal{I} (section 5.1) faster; determining the asymptotic expansions of ψ, \mathcal{I} in a slow time parameter ε (section 5.2) faster; determining upper/lower bounds on the solutions (section 5.3); studying the stability of the trivial one, or the occurrence of parametric resonance in the case of a periodic $\omega(t)$ (section 5.4). In section 6 we summarize our results, compare them to the literature, list some possible domains of application, point out open problems and directions for further investigations. When possible we have concentrated tedious proofs in the appendix 7.

2 Preliminaries

For generic a, A , the solution of the Cauchy problem (3) with $x(t_*) = x_*$ can be expressed as

$$x(t) = \mathbf{V}(t, t_*) \left[x_* + \int_{t_*}^t dz \mathbf{V}^{-1}(z, t_*) a(z) \right], \quad (9)$$

where $\mathbf{V}(t, t_*)$ is the (non-degenerate) family (parametrized by t_*) of 2×2 matrix solutions of $\dot{V} = AV$ fulfilling $\mathbf{V}(t_*, t_*) = I_2$ (the unit matrix). This in turn can be expressed as $\mathbf{V}(t, t_*) = V(t)V^{-1}(t_*)$, where V is any nondegenerate matrix solution of $\dot{V} = AV$; this means that its two columns are independent solutions of the vector equation $\dot{u} = Au$. If A has zero trace, as in the case (3b), the Wronskian $W := \det V$ is a non-zero constant.

Actually, given a non-diagonal \tilde{A} one can transform $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{a}$ into (3), with A of the form (3b), but ω^2 not necessarily positive; under additional assumptions on A it is $\omega^2 > 0$. In fact, if $\tilde{A}_2^1 \neq 0$ let Λ be a solution of the equation $2\dot{\Lambda} + \tilde{A}_1^1 + \tilde{A}_2^2 + \dot{\tilde{A}}_2^1/\tilde{A}_2^1 = 0$; the Ansatz

$$x := e^\Lambda B \tilde{x}, \quad a := e^\Lambda B \tilde{a}, \quad B := \begin{pmatrix} 1 & 0 \\ b & \tilde{A}_2^1 \end{pmatrix}, \quad b := \dot{\Lambda} + \tilde{A}_1^1 = \frac{1}{2} \left(\tilde{A}_1^1 - \tilde{A}_2^2 - \frac{\dot{\tilde{A}}_2^1}{\tilde{A}_2^1} \right)$$

does the job, with $-\omega^2 := \dot{b} + b^2 + \tilde{A}_2^1 \tilde{A}_1^2$. The additional requirement is that this is negative; it can be fulfilled assuming e.g. that $-\tilde{A}_2^1 \tilde{A}_1^2 > 0$ is sufficiently large¹. If $\tilde{A}_1^2 \neq 0$ one can do the transformation with the indices 1, 2 exchanged. One can reduce \tilde{A} to the form (3b) also by a change of the ‘time’ variable.

Let v_1, v_2 be the solutions of (1) fulfilling $v_1(0) = \dot{v}_2(0) = 1$, $v_2(0) = \dot{v}_1(0) = 0$. The fundamental matrix solution of $\dot{V} = AV$ and its inverse are given by

$$V = \begin{pmatrix} v_1 & v_2 \\ \dot{v}_1 & \dot{v}_2 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} \dot{v}_2 & -v_2 \\ -\dot{v}_1 & v_1 \end{pmatrix}. \quad (10)$$

¹This applies e.g. to the equations of motion $m\dot{x} = \pi$, $\dot{\pi} = -\kappa x - \eta\dot{x} + f$ of a particle with mass m subject to an elastic force $-\kappa x$ and possibly a viscous one $-\eta\dot{x}$, a forcing one f , with time-dependent $m, \kappa > 0$, $\eta \geq 0$, f , provided $\Omega^2 \equiv \kappa/m$ is sufficiently large.

In fact, from $V(0) = I_2$ it follows $W = 1$, namely

$$v_1 \dot{v}_2 - v_2 \dot{v}_1 = 1. \quad (11)$$

The solutions $\mathbf{v}_1, \mathbf{v}_2$ of (1) appearing in the decomposition of the kernel $\mathbf{V}(t, t_*)$ of (9)

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \\ \dot{\mathbf{v}}_1 & \dot{\mathbf{v}}_2 \end{pmatrix}, \quad \mathbf{V}(t_*, t_*) = I_2 \quad (12)$$

are the following combinations of v_1, v_2 :

$$\begin{aligned} \mathbf{v}_2(t, t_*) &= v_2(t)v_1(t_*) - v_1(t)v_2(t_*), \\ \mathbf{v}_1(t, t_*) &= v_1(t)\dot{v}_2(t_*) - v_2(t)\dot{v}_1(t_*). \end{aligned} \quad (13)$$

Therefore they fulfill also $\partial \mathbf{v}_1 / \partial t_*|_{t=t_*} = 0$, $\partial \mathbf{v}_2 / \partial t_*|_{t=t_*} = -1$, and $\partial^2 \mathbf{v}_2 / \partial t_*^2 = -\omega^2(t_*)\mathbf{v}_2$ (whereas $\partial^2 \mathbf{v}_1 / \partial t_*^2$ is more complicated). One can express v_2 in terms of v_1 (and viceversa) solving (11), but the expression becomes more and more complicated as t gets far from 0^2 ; it is more convenient to look for both v_1, v_2 globally in terms of the angle-action variables.

3 Reduction to Riccati equation, and zeroes of q, \dot{q}

The first reduction of (1) to a single first order ordinary differential equation (ODE) is based on the well-known relation between linear homogeneous second order ODE's and Riccati equations, and makes sense also if $\omega(t)$ is not continuous. We use it to study the zeroes of $q, p = \dot{q}$. Given a solution $q(t)$ of (1), we define r and its inverse s by

$$r = \frac{\dot{q}}{q}, \quad s = \frac{q}{\dot{q}} \quad (15)$$

respectively in an interval K where q, \dot{q} does not vanish. There r (resp. s) fulfills the Riccati equation

$$\dot{r} = -\omega^2 - r^2, \quad (\text{ resp. } \dot{s} = 1 + \omega^2 s^2) \quad (16)$$

²Known $v_1(t)$ we can express $v_2(t)$ in $]0, t_2[$ and $[t_2, t_3[$ respectively as the central or right expression in

$$v_2(t) = v_1(t) \int_0^t \frac{dz}{v_1^2(z)} = v_1(t) \left[\int_0^{\bar{t}} \frac{dz}{v_1^2(z)} + \int_{\bar{t}}^t dz \frac{\omega^2(z)}{\dot{v}_1^2(z)} + \frac{1}{\dot{v}_1(\bar{t})v_1(\bar{t})} \right] - \frac{1}{\dot{v}_1(t)} \quad (14)$$

with some $\bar{t} \in]0, t_2[$; $\{t_h\}_{h \in \mathbb{Z}}$, with $t_1 = 0$, is the sequence of zeroes of v_1, \dot{v}_1 studied by Proposition 1. The proof of the first equality is straightforward. The divergence of the integrand (and of the integral) as $t \rightarrow t_2^-$ is compensated by the vanishing of $v_1(t)$, in agreement with $v_2(t_1) = -1/\dot{v}_1(t_1)$, which follows from (11). This is manifest noting that it is $0 < v_1(\bar{t}) < \infty$, and, using (14a) and integrating by parts,

$$\frac{v_2(t)}{v_1(t)} - \int_0^{\bar{t}} \frac{dz}{v_1^2(z)} = \int_{\bar{t}}^t \frac{dz}{v_1^2(z)} = - \int_{\bar{t}}^t \frac{dz}{\dot{v}_1(z)} \frac{d}{dz} \frac{1}{v_1(z)} = \frac{-1}{\dot{v}_1(z)v_1(z)} \Big|_{\bar{t}}^t - \int_{\bar{t}}^t \frac{dz}{v_1(z)} \frac{\ddot{v}_1(z)}{\dot{v}_1^2(z)} = \frac{1}{\dot{v}_1(z)v_1(z)} \Big|_{\bar{t}}^t + \int_{\bar{t}}^t dz \frac{\omega^2(z)}{\dot{v}_1^2(z)}$$

(the last integral is manifestly finite as $t \rightarrow t_2^-$), whence the second equality in (14). The right expression and its derivative are actually well-defined for all $t \in]\bar{t}, t_3[$. Arguing in a similar manner one can express $v_2(t)$ in other intervals $]t_h, t_{h+2}[$ as a sum of a number of terms increasing with $|h|$.

and is strictly decreasing (resp. growing). In each such interval (3) is equivalent to the first order system (15a-16a) in the unknowns r, q [resp. (15b-16b) in the unknowns s, q]. Eq. (16) has the *only* unknown $r(t)$ [resp. $s(t)$]. Once it is solved, solving (15) for $q(t)$ we obtain

$$q(t) = q_* \exp \left\{ \int_{t_*}^t dz r(z) \right\}, \quad \left(\text{resp. } q(t) = q_* \exp \left\{ \int_{t_*}^t \frac{dz}{s(z)} \right\} \right). \quad (17)$$

Here we have parametrized the solution through its values at some point $t_* \in K$: $q_* := q(t_*)$ and $r_* := r(t_*) = q_*/p_*$ or $s_* := s(t_*) = p_*/q_*$ respectively, where $p_* := p(t_*) = \dot{q}(t_*)$. Finally, $\dot{q}(t)$ is obtained replacing these results again in (15). Summing up, locally we can reduce the resolution of (3) to that of a *single* first order equation [(16a) or (16b)] of Riccati type. The associated Cauchy problem is equivalent to the Volterra-type integral equation

$$r(t) = r_* - \int_{t_*}^t dz [r^2 + \omega^2](z), \quad \left(\text{resp. } s(t) = s_* + \int_{t_*}^t dz [1 + \omega^2 s^2](z) \right). \quad (18)$$

The zeroes of q, p interlace. More precisely, in the appendix we prove

Proposition 1 *If $\omega_l := \inf_{t \in \mathbb{R}} \{\omega(t)\} > 0$, then every nontrivial solution $t \in \mathbb{R} \mapsto q(t)$ of (1) admits a strictly increasing sequence $\{t_h\}_{h \in \mathbb{Z}} \subset \mathbb{R}$ such that for all $j \in \mathbb{Z}$:*

1. q vanishes, and p has a positive maximum at $t = t_{4j}$;
2. p vanishes, and q has a positive maximum at $t = t_{4j+1}$;
3. q vanishes, and p has a negative minimum at $t = t_{4j+2}$;
4. p vanishes, and q has a negative minimum at $t = t_{4j+3}$;
5. $(q(t), p(t))$ belongs to the first, second, third, fourth quadrant of the (q, p) phase plane for all t respectively in $]t_{4j}, t_{4j+1}[$, $]t_{4j+1}, t_{4j+2}[$, $]t_{4j+2}, t_{4j+3}[$, $]t_{4j+3}, t_{4j+4}[$.

More generally, if $\omega(t) \geq \bar{\omega}_l$, with some $\bar{\omega}_l > 0$, holds for all t belonging to an interval $J \subseteq \mathbb{R}$, then there is a subset of consecutive integers $\mathbb{Z}_J \subseteq \mathbb{Z}$ such that 1.-5. hold for all t_h , $h \in \mathbb{Z}_J$.

This generalizes the case $\omega = \text{const}$, with $q(t) = \sin(\omega t)$, $p(t) = \omega \cos(\omega t)$, and $t_h = \frac{h\pi}{2\omega}$. The labelling of these special points is defined up to a shift $h \mapsto h + 4k$, with a fixed $k \in \mathbb{Z}$.

Under the assumptions of proposition 1 the function r is defined and strictly decreasing in each interval $]t_{2k}, t_{2k+2}[$, diverges at the extremes and vanishes at the middle point t_{2k+1} ; in each such interval (1) is equivalent to the first order system (15a-16a) in the unknowns r, q . Similarly, s is defined and strictly growing in each interval $]t_{2k-1}, t_{2k+1}[$ diverges at the extremes, and vanishes at the middle point t_{2k} ; in each such interval (1) is equivalent to the first order system (15b-16b) in the unknowns s, q .

If ω_u, ω_d are positive constants such that $\omega_d \leq \omega(t) \leq \omega_u$ for $t \in]t_h, t_{h+1}[$, then we easily obtain the following rough bounds on the length of this interval³:

$$\frac{\pi}{2\omega_u} \leq t_{h+1} - t_h \leq \frac{\pi}{2\omega_d}. \quad (19)$$

These inequalities are most stringent if we adopt

$$\omega_u \equiv \sup_{]t_h, t_{h+1}[} \{\omega(t)\}, \quad \omega_d \equiv \inf_{]t_h, t_{h+1}[} \{\omega(t)\}.$$

In general we are not able to determine such sup, inf, because we don't know the exact locations of t_h, t_{h+1} . However the choice of ω_u, ω_d can be improved recursively⁴. More stringent bounds on the length of the interval will be determined in section 5.3.

Now let $Q(t; t_*, q_*, p_*)$ be the family of solutions of (1) fulfilling the conditions

$$Q(t_*; t_*, q_*, p_*) = q_*, \quad \dot{Q}(t_*; t_*, q_*, p_*) = p_*. \quad (20)$$

We ask how the sequence of special points $\{t_h\}_{h \in \mathbb{Z}}$ for Q (in the sense of Proposition 1) depends on the parameters $(t_*, q_*, p_*) \in \mathbb{R}^3$. Here we partly investigate this question. First we note that, since $Q(t_*; t_*, aq_*, ap_*) = aQ(t_*; t_*, q_*, p_*)$, then $s := Q/\dot{Q}$, $r := \dot{Q}/Q$ and therefore also the sequence is invariant under all rescalings $(q_*, p_*) \mapsto (aq_*, ap_*)$, $a \in \mathbb{R}^+$; the latter map each quadrant of the (q, p) plane into itself. In the appendix we prove

Proposition 2 *For all $(q_i, p_i) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, all the special points t_h associated to the family of solutions $Q(t; t_i, q_i, p_i)$, seen as functions of t_i , are strictly growing.*

This applies in particular to the $\{t_h(t_i)\}_{h \in \mathbb{Z}}$ of the families of solutions $Q = \mathbf{v}_1, \mathbf{v}_2$ defined by (13); we can remove the residual ambiguity $h \mapsto h+4k$ in their definitions setting $t_0(t_i) = t_i$ for $\mathbf{v}_2(t, t_i)$, and $t_1(t_i) = t_i$ for $\mathbf{v}_1(t, t_i)$. If $\omega = \text{const}$ it is $\mathbf{v}_2(t, t_i) = \sin[\omega(t-t_i)]$, $t_h(t_i) = t_i + \frac{h\pi}{2\omega}$ and $\mathbf{v}_1(t, t_i) = \cos[\omega(t-t_i)]$, $t_h(t_i) = t_i + \frac{(h-1)\pi}{2\omega}$ respectively.

Chosen any compact interval $K_k \subset]t_k, t_{k+2}[$, applying the fixed point theorem one can recursively build via

$$r_k^{(h)}(t) := r_k^* - \int_{t_k}^t dz \left[(r^{(h-1)})^2 + \omega^2 \right](z), \quad \left(\text{resp. } s_k^{(h)}(t) := s_k^* + \int_{t_*}^t dz \left[1 + \omega^2 (s^{(h-1)})^2 \right](z) \right) \quad (21)$$

³In fact, by these bounds every solution r of (16) in a neighbourhood of any $t_* \in]t_{2k}, t_{2k+2}[$ fulfills

$$\begin{aligned} -\dot{r} \geq \omega_d^2 + r^2 &\Rightarrow \frac{-\dot{r}/\omega_d}{1 + (r/\omega_d)^2} = \frac{d}{dy} \cot^{-1} \left(\frac{r}{\omega_d} \right) \geq \omega_d \Rightarrow \cot^{-1} \left[\frac{r(t)}{\omega_d} \right] - \cot^{-1} \left[\frac{r(t_*)}{\omega_d} \right] \geq \omega_d(t-t_*) \\ -\dot{r} \leq \omega_u^2 + r^2 &\Rightarrow \frac{-\dot{r}/\omega_u}{1 + (r/\omega_u)^2} = \frac{d}{dy} \cot^{-1} \left(\frac{r}{\omega_u} \right) \leq \omega_u \Rightarrow \cot^{-1} \left[\frac{r(t)}{\omega_u} \right] - \cot^{-1} \left[\frac{r(t_*)}{\omega_u} \right] \leq \omega_u(t-t_*). \end{aligned}$$

Choosing $t = t_{2k+2}$ and $t_* = t_{2k+1}$ leads to (19) for $h = 2k+1$; choosing $t = t_{2k+1}$ and $t_* = t_{2k}^+$ leads to (19) for $h = 2k$. Here we have used $r(t_{2k+1}) = 0$, $r(t_{2k+2}^-) = -\infty$, $r(t_{2k}) = +\infty$.

⁴Assuming for simplicity that t_h is known, a candidate ω_u can be accepted if $\omega_u \geq \sup_{]t_h, t_{h+2\frac{\pi}{\omega_u}}[} \{\omega(t)\}$, must be rejected otherwise; if the inequality is strict a better candidate will be $\omega'_u \equiv \sup_{]t_h, t_{h+2\frac{\pi}{\omega'_u}}[} \{\omega(t)\}$; and so on. Similarly one argues for ω_d .

a sequence $\{r_k^{(h)}\}_{h \in \mathbb{N}_0} \subset C(K)$ if k is even (resp. a sequence $\{s_k^{(h)}\}_{h \in \mathbb{N}_0} \subset C(K)$ if k is odd) that converges uniformly to the unique solution r_k (resp. s_k) of (18) in K_k ⁵. Replacing r (resp. s) by $\{r_k^{(h)}\}$ (resp. $\{s_k^{(h)}\}$) in (17) we obtain the associated sequence $\{q_k^{(h)}\}_{h \in \mathbb{N}_0} \subset C(K)$. Incidentally, we note that such a resolution procedure may be applied also if ω vanishes at some $t \in K$. To obtain a solution of (1) in all of \mathbb{R} in this way one can choose all two consecutive intervals K_k with a non-empty intersection and match the initial values r_k^* (resp. s_k^*) in K_k so that the globally defined q, \dot{q} are continuous. On the contrary, solving the first order ODE (5) yields the global solution $q(t)$ of (1) at once. This is discussed in the next sections.

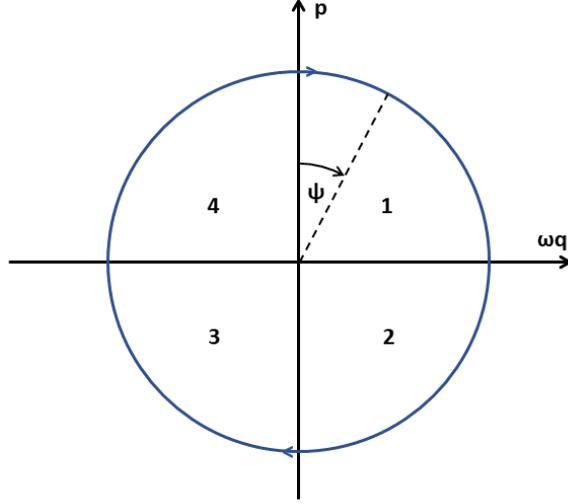


Figure 1: The angle ψ . We order the quadrants clockwise, i.e. in the direction of the motion.

⁵One possibility is to start with $r_k^{(0)} := r_k^*$ (resp. $s_k^{(0)} := s_k^*$); this is especially convenient when ω is 'small'. The corresponding first order approximations are

$$q_k^{(0)}(t) = q_k^* \exp\{r_k^*(t-t_*)\}, \quad r_k^{(1)}(t) = r_k^* - r_k^{*2}(t-t_*) - \int_{t_*}^t dz \omega^2(z),$$

$$q_k^{(1)}(t) = q_k^* \exp\left\{r_k^*(t-t_*) - \frac{r_k^{*2}}{2}(t-t_*)^2 - \int_{t_*}^t dz (t-z) \omega^2(z)\right\}, \quad (22)$$

$$\left(\text{resp. } q_k^{(0)}(t) = q_k^* \exp\left\{\frac{t-t_*}{s_k^*}\right\}, \quad s_k^{(1)}(t) = s_k^* + (t-t_*) + s_k^{*2} \int_{t_*}^t dz \omega^2(z) \quad , \right.$$

$$\left. q_k^{(1)}(t) = q_k^* \exp\left\{\int_{t_*}^t \frac{dz}{s_k^{(1)}(z)}\right\} \right). \quad (23)$$

In particular, we find that if the initial conditions are $q(0) = 0, \dot{q}(0) = 1$ then $q(t) \simeq t$ for small t . If the initial conditions are $q(0) = 1, \dot{q}(0) = 0$ then $q(t) \simeq q^{(1)}(t) = \exp\left\{-\int_0^t dz (t-z) \omega^2(z)\right\}$ for small t .

4 Solving Hamilton equations for action-angle variables

4.1 Reduction to the Hamilton equation for the angle variable

We recall that the transformation $(q, p) \mapsto (\mathcal{I}, \psi)$ to the action, angle⁶ variables defined by

$$\mathcal{I} = \frac{H}{\omega} = \frac{p^2 + \omega^2 q^2}{2\omega}, \quad \psi = \arg(p, \omega q), \quad (24)$$

as well as its inverse $(\mathcal{I}, \psi) \mapsto (q, p)$ defined by (4), remain canonical also if ω - and therefore also H and themselves - depend on time⁷. Therefore the evolution of \mathcal{I}, ψ is still ruled by the Hamilton equations, but with the transformed Hamiltonian $K(\mathcal{I}, \psi, t)$

$$\dot{\psi} = \frac{\partial K}{\partial \mathcal{I}}, \quad \dot{\mathcal{I}} = -\frac{\partial K}{\partial \psi}, \quad K = \left[\omega + \frac{\dot{\omega}}{2\omega} \sin(2\psi) \right] \mathcal{I}$$

that can be obtained itself from a generating function of the transformation⁸. More explicitly these equations read (5). The reader may check also by a direct computation that (1) and (5) are equivalent. From (4) it follows that the sequence of special points $\{t_h\}_{h \in \mathbb{Z}}$ (in the sense of Proposition 1) of every solution $q(t)$ is characterized, up to a multiple of 2π , by

$$\psi(t_h) = h \frac{\pi}{2}, \quad h \in \mathbb{Z}. \quad (26)$$

The Cauchy problem (5a) with $\psi(t_*) = \psi_*$ is equivalent to the Volterra-type integral equation

$$\psi(t) = \varphi(t) + \int_{y_*}^t dz \left[\frac{\dot{\omega}}{2\omega} \sin(2\psi) \right](z), \quad (27)$$

$$\text{where } \varphi(t) := \psi_* + \int_{y_*}^t dz \omega(z) =: \psi^{(0)}(t).$$

Once this is solved, (5b) with initial condition $\mathcal{I}(t_*) = \mathcal{I}_*$ is solved by

$$\mathcal{I}(t) = \mathcal{I}_* \exp \left\{ - \int_{t_*}^t dz \left[\frac{\dot{\omega}}{\omega} \cos(2\psi) \right](z) \right\}; \quad (28)$$

⁶ $\arg(\omega q, p)$ is the azimuthal angle of the point $(\omega q, p)$ in the (q, p) plane, see fig. 1: $\cot \psi = \frac{p}{\omega q} = \frac{r}{\omega}$, $\tan \psi = \frac{\omega q}{p} = s\omega$; these equations make both sense and are equivalent in the intersection of any two adjacent intervals $]t_{h-1}, t_{h+1}[$, $]t_h, t_{h+2}[$ (where $rs = 1$), while one makes sense and the other not at $t = t_h, t_{h+1}$.

⁷The calculation leading to the Poisson bracket $\{\psi, \mathcal{I}\} = 1$ holds regardless of the time dependence of ω .

⁸The generating function $F(q, \psi, t)$ of type 1 of the transformation $(q, p, H) \mapsto (\mathcal{I}, \psi, K)$

$$I = -\frac{\partial F}{\partial \psi}, \quad p = \frac{\partial F}{\partial q}, \quad K := H + \frac{\partial F}{\partial t} \quad (25)$$

must fulfill $p dq - H dt = \mathcal{I} d\psi - K dt + dF$, see e.g. [17]. This equation is solved by $F(q, \psi, t) = \frac{\omega(t)}{2} q^2 \cot \psi$.

the time-dependence of H, q, p is obtained expressing them in terms of $\psi, \mathcal{I}, \omega$. If $|\zeta| \equiv |\dot{\omega}/\omega^2| \ll 1$ (slowly/slightly varying ω) then the integral in (27) can be neglected, the variation of \mathcal{I} over time intervals $[t_*, t]$ containing many cycles⁹ can be approximated replacing $\cos(2\psi)$ by its mean 0 over a cycle, whence [cf. (7)]

$$\psi(t) \simeq \varphi(t), \quad \mathcal{I}(t) \simeq \mathcal{I}(t_*), \quad H(t) \simeq H(t_*) \omega(t)/\omega(t_*). \quad (29)$$

The same applies if $\dot{\omega}/\omega$ oscillates about zero much faster than φ ; such oscillations almost wash out the integrals in (27), (28). Improved estimates of such integrals lead to the better approximations reported below.

Eq. (5) make sense in intervals where ω is continuously differentiable, or is so at least piecewise while keeping continuous and with a bounded derivative. If $\omega(t)$ is continuous in $]t_1, t_2[$ except at some point t_d , then in general so are $\ddot{q}(t), \psi(t), \mathcal{I}(t), H(t)$, whereas $q(t), p(t), r(t), s(t)$ are continuous in the whole $]t_1, t_2[$. Let

$$\omega_{\pm} := \lim_{t \rightarrow t_d^{\pm}} \omega(t), \quad \psi_{\pm} := \lim_{t \rightarrow t_d^{\pm}} \psi(t), \quad \mathcal{I}_{\pm} := \lim_{t \rightarrow t_d^{\pm}} \mathcal{I}(t); \quad (30)$$

note that by their definitions [cf. (24)] both ψ_+, ψ_- must belong to the same quadrant of the (q, p) plane. The continuities of r, q, p at the discontinuity point t_d of $\omega(t)$ imply the matching relations

$$\frac{\tan \psi_+}{\omega_+} = \frac{\tan \psi_-}{\omega_-}, \quad \mathcal{I}_+ = \mathcal{I}_- \frac{\omega_-}{\omega_+} \frac{1 + \left(\frac{\omega_+}{\omega_-} \tan \psi_-\right)^2}{1 + \tan^2 \psi_-}. \quad (31)$$

The second determines the ‘sudden’ change of \mathcal{I} due to the ‘sudden’ change of ω ; this depends not only on ω_{\pm} , but also on the value of ψ_- (or, equivalently, ψ_+). If $t_d = t_h$, $h \in \mathbb{Z}$, then ψ remains continuous because $\tan \psi(t_h) = 0, \pm\infty$, while

$$\mathcal{I}_+ \omega_+ = \mathcal{I}_- \omega_- \quad \text{if } h \text{ even}, \quad \mathcal{I}_+ \omega_- = \mathcal{I}_- \omega_+ \quad \text{if } h \text{ odd}.$$

The maximum, minimum possible ratio $\mathcal{I}_+/\mathcal{I}_-$ are therefore the ratios ω_+/ω_- , ω_-/ω_+ , in either order. In general $\psi(t)$ can be extended from $]t_1, t_d[$ (resp. $]t_d, t_2[$) to the whole $]t_1, t_2[$ solving (5a) with (31a) as initial (resp. final) conditions in the complementary part. $\mathcal{I}(t)$ can be extended from $]t_1, t_d[$ (resp. $]t_d, t_2[$) to the whole $]t_1, t_2[$ applying (28) with (31b) as initial (resp. final) conditions in the complementary part.

4.2 Sequence converging to the solution of (27)

We can reformulate (27) in every compact time interval $K := [a, b] \supset t_*$ as the fixed point equation $\mathcal{T}\psi = \psi$, where \mathcal{T} is the linear map $\mathcal{T} : C(K) \rightarrow C(K)$ defined by

$$[\mathcal{T}u](t) := \varphi(t) + \int_{t_*}^t dz f[z, u(z)], \quad f(t, u) := \frac{\dot{\omega}(t)}{2\omega(t)} \sin(2u). \quad (32)$$

⁹We define $[t_1, t_2]$ as a *cycle* if $\psi(t_2) - \psi(t_1) = 2\pi$.

By means of the fixed point theorem one can build sequences $\{\psi^{(h)}\}_{h \in \mathbb{N}_0} \subset C(K)$ converging to its unique solution ψ . A very convenient one starts with $\psi^{(0)} := \varphi$ and continues with

$$\psi^{(h)}(t) := [\mathcal{T}^h \varphi](t) = \varphi(t) + \int_{t_*}^t dz \left[\frac{\dot{\omega}}{2\omega} \sin(2\psi^{(h-1)}) \right](z). \quad (33)$$

The corresponding sequence $\{\mathcal{I}^{(h)}\}_{h \in \mathbb{N}_0}$ converging to \mathcal{I} arises replacing $\psi \mapsto \psi^{(h-1)}$ in (28)

$$\mathcal{I}^{(h)}(t) := \mathcal{I}(t_*) \exp \left\{ - \int_{t_*}^t dz \left[\frac{\dot{\omega}}{\omega} \cos(2\psi^{(h-1)}) \right](z) \right\}, \quad (34)$$

and the ones of H, q, p are obtained replacing $(\mathcal{I}, \psi) \mapsto (\mathcal{I}^{(h)}, \psi^{(h)})$ in (24), (4). The ‘errors’ of approximation are conveniently bounded via the *total variation* of $\log \omega(t)$ between t_*, t

$$g(t) := \int_{t_*}^t dz \left| \frac{\dot{\omega}(z)}{\omega(z)} \right|. \quad (35)$$

This grows with t , and $g(t_*) = 0$. If $\omega(t)$ is monotone in $[t_*, t]$ then $g(t) = \left| \log \frac{\omega(t)}{\omega(t_*)} \right|$; otherwise g is the sum of a term of this kind for each monotonicity interval contained in $[t_*, t]$.

Proposition 3 *If $\omega \in C^1(K)$, with $\omega > 0$, then for all $h \in \mathbb{N}$*

$$\begin{aligned} 2 |\psi(t) - \psi^{(h)}(t)| &\leq \frac{|g(t)|^{h+1}}{(h+1)!}, \\ |\log \mathcal{I}(t) - \log \mathcal{I}^{(h)}(t)| &\leq \frac{|g(t)|^{h+1}}{(h+1)!}. \end{aligned} \quad (36)$$

Consequently, $\psi^{(h)} \rightarrow \psi$, $\mathcal{I}^{(h)} \rightarrow \mathcal{I}$ in the sup norm of $C(K)$, $\|u\|_\infty := \sup_{t \in K} \{|u(t)|\}$.

Proof We start by deriving the following inequalities: for all $\alpha, \delta \in \mathbb{R}$

$$\begin{aligned} |\sin(\alpha + \delta) - \sin \alpha| &\leq \sqrt{2(1 - \cos \delta)} \leq |\delta|, \\ |\cos(\alpha + \delta) - \cos \alpha| &\leq \sqrt{2(1 - \cos \delta)} \leq |\delta|. \end{aligned} \quad (37)$$

We abbreviate $f(\alpha, \delta) := \sin(\alpha + \delta) - \sin \alpha = \sin \alpha (\cos \delta - 1) + \cos \alpha \sin \delta$. For fixed δ the points α' of maximum, minimum for f fulfill

$$\begin{aligned} 0 = \frac{\partial f}{\partial \alpha}(\alpha', \delta) &= \cos(\alpha' + \delta) - \cos \alpha' = \cos \alpha' (\cos \delta - 1) - \sin \alpha' \sin \delta \quad \Rightarrow \quad \tan \alpha' = \frac{\cos \delta - 1}{\sin \delta} \\ f(\alpha', \delta) &= \cos \alpha' [(\cos \delta - 1) \tan \alpha' + \sin \delta] = \frac{\cos \alpha'}{\sin \delta} [(\cos \delta - 1)^2 + \sin^2 \delta] = 2 \cos \alpha' \frac{1 - \cos \delta}{\sin \delta} \\ \Rightarrow |f(\alpha, \delta)| &\leq |f(\alpha', \delta)| = \frac{2}{\sqrt{1 + \tan^2 \alpha'}} \frac{1 - \cos \delta}{|\sin \delta|} = \frac{2(1 - \cos \delta)}{\sqrt{\sin^2 \delta + (1 - \cos \delta)^2}} = \sqrt{2(1 - \cos \delta)}, \end{aligned}$$

as claimed; the right inequality in (37a) follows from the left one and the one $1 - \cos \delta \leq \frac{\delta^2}{2}$ (obtained integrating $\sin x \leq x$ over $[0, \delta]$). Inequalities (37b) follow via the shift $\alpha \mapsto \alpha + \pi/2$.

By the assumptions it is $\mu := \max_K |\frac{\dot{\omega}}{\omega}| < \infty$ and $M := \max_K \{|g(t)|\} = \max\{|g(a)|, |g(b)|\} < \infty$. Now abbreviate $\sigma^{(h+1)}(t) := \psi(t) - \psi^{(h)}(t)$. Eq. (27) implies

$$2\sigma^{(1)}(t) = \int_{t_*}^t dz \left[\sin(2\psi) \frac{\dot{\omega}}{\omega} \right](z), \quad 2|\sigma^{(1)}(t)| \leq \int_{t_*}^t dz \left| \frac{\dot{\omega}(z)}{\omega(z)} \right| = g(t) \quad \text{if } t \geq t_*, \quad (38)$$

i.e. (36) for $h = 0$. If (36) holds for $h = j-1 \geq 0$ it does also for $h = j$, since by eq. (27), (33)

$$\begin{aligned} 2\sigma^{(j+1)}(t) &= \int_{t_*}^t dz \left\{ \frac{\dot{\omega}}{\omega} [\sin(2\psi) - \sin(2\psi^{(j-1)})] \right\}(z), & (39) \\ 2|\sigma^{(j+1)}(t)| &\leq \int_{t_*}^t dz \left\{ \left| \frac{\dot{\omega}}{\omega} \right| |\sin(2\psi^{(j-1)} + 2\sigma^{(j)}) - \sin(2\psi^{(j-1)})| \right\}(z) \leq \int_{t_*}^t dz \left| \frac{\dot{\omega}}{\omega} 2\sigma^{(j)} \right|(z) \\ &\leq \int_{t_*}^t dz \left[\left| \frac{\dot{\omega}}{\omega} \right| \frac{g^j}{j!} \right](z) = \int_{t_*}^t \frac{dz}{(j+1)!} \frac{d}{dz} g^{j+1}(z) = \frac{[g(t)]^{j+1}}{(j+1)!} \quad \text{if } t \geq t_*. \end{aligned}$$

Similarly, eq. (28), (34) imply $\log[\mathcal{I}(t)/\mathcal{I}^{(j)}(t)] = \int_{t_*}^t dz \left\{ \frac{\dot{\omega}}{\omega} [\cos(2\psi^{(j-1)}) - \cos(2\psi)] \right\}$, whence

$$\begin{aligned} \left| \log \frac{\mathcal{I}(t)}{\mathcal{I}^{(j)}(t)} \right| &\leq \int_{t_*}^t dz \left\{ \left| \frac{\dot{\omega}}{\omega} \right| |\cos(2\psi^{(j-1)}) - \cos(2\psi^{(j-1)} + 2\sigma^{(j)})| \right\}(z) \leq \int_{t_*}^t dz \left| \frac{\dot{\omega}}{\omega} 2\sigma^{(j)} \right|(z) \\ &\leq \int_{t_*}^t dz \left[\left| \frac{\dot{\omega}}{\omega} \right| \frac{g^j}{(j)!} \right](z) = \int_{t_*}^t \frac{dz}{(j+1)!} \frac{d}{dz} g^{j+1}(z) = \frac{[g(t)]^{j+1}}{(j+1)!} \quad \text{if } t \geq t_*, \end{aligned}$$

as claimed. If $t < t_*$ the previous inequalities hold if we exchange the extremes of integration t, t_* , thus proving (36) also in this case. From $|g(t)| \leq M$ we find $\text{rhs}(36) \leq M^{h+1}/(h+1)!$, which goes to zero as $h \rightarrow \infty$; this proves the mentioned convergences $\psi^{(h)} \rightarrow \psi$, $\mathcal{I}^{(h)} \rightarrow \mathcal{I}$. \square

Corollary 1 *If $\omega \in C^1(\mathbb{R})$, with $\omega > 0$, then $\psi^{(h)} \rightarrow \psi$, $\mathcal{I}^{(h)} \rightarrow \mathcal{I}$ pointwise in all of \mathbb{R} .*

Remarks:

1. The rate of the convergence $\psi^{(h)} \rightarrow \psi$ for large $|t-t_*|$ is typically much better than what the bounds (36) suggest. This makes the approximating sequence $\{\psi^{(h)}\}_{h \in \mathbb{N}_0}$ very useful also for practical purposes and numerical computations. In fact, if $|\varphi(t) - \psi_*| \gg \pi$ then the integrand of (38a) changes sign many times in $[t, t_*]$, thus reducing the magnitude of the integral and leading to $2|\sigma^{(1)}(t)| \ll |g(t)|$, unless $\dot{\omega}/\omega$ oscillates in phase with $\sin(2\varphi)$ (resonance). In the latter case, one can show that $\text{lhs} \ll \text{rhs}$ in (36) holds at least for the following corrections (i.e. for $h > 1$).
2. Since $|g(t)| \leq \mu|t-t_*|$, the bounds (36) hold also replacing the rhs by $[\mu|t-t_*|]^h/h!$.

3. The standard application of the fixed point theorem (see the appendix) also leads to $\|\psi - \psi^{(h)}\|_\infty \rightarrow 0$, but via the class of pointwise bounds (parametrized by $\lambda > \mu/2$)

$$|\psi^{(h)}(t) - \psi(t)| < e^{\lambda|t-t_*|} \frac{\nu^h}{1-\nu} \|\varphi - \psi^{(1)}\|_{\lambda,K} \quad (40)$$

where $\nu := \frac{\mu}{2\lambda} \in]0, 1[$, $\|u\|_{\lambda,K} := \max_{t \in K} \{|e^{-\lambda|t-t_*|} u(t)|\}$; the infimum $\sigma_{h,K}$ of the rhs over $\lambda \in]\mu/2, \infty[$ gives the most stringent bound in the class. Nevertheless, (40) is less manageable and stringent than (36a): as $h \rightarrow \infty$ the latter goes faster to zero, also due to the factorial in the denominator.

4. Once φ is computed, through it we can readily use the approximation

$$\hat{q}(t) := \sqrt{\frac{2\mathcal{I}^{(1)}(t)}{\omega(t)}} \sin \psi^{(0)}(t) = \sqrt{\frac{2\mathcal{I}(t_*)}{\omega(t)}} \exp \left\{ -\int_{t_*}^t dz \left[\frac{\dot{\omega}}{2\omega} \cos(2\varphi) \right](z) \right\} \sin \varphi(t), \quad (41)$$

which is an intermediate one between the above defined ones $q^{(0)}$ and $q^{(1)}$.

In the h -th approximation the discontinuity relations (31) become

$$\frac{\tan \psi_+^{(h)}}{\omega_+} = \frac{\tan \psi_-^{(h)}}{\omega_-}, \quad \frac{\mathcal{I}_+^{(h)}}{\omega_+} \sin^2 \psi_+^{(h)} = \frac{\mathcal{I}_-^{(h)}}{\omega_-} \sin^2 \psi_-^{(h)}, \quad \mathcal{I}_+ \omega_+ \cos^2 \psi_+^{(h)} = \mathcal{I}_- \omega_- \cos^2 \psi_-^{(h)}. \quad (42)$$

Consequently, $\psi^{(h)}(t)$ is given by (33) for $t \in]t_*, t_d[$, while for $t \in]t_d, t_f[$ it is given by

$$\psi^{(h)}(t) = \psi_+^{(h)} + \int_{t_d}^t dz \left[\omega + \frac{\dot{\omega}}{2\omega} \sin(2\psi^{h-1}) \right](z). \quad (43)$$

From these formulae one can derive an upper bound for the ‘error’ $\sigma_+^{(h+1)} := \psi_+ - \psi_+^{(h)}$ from one for the ‘error’ $\sigma_-^{(h+1)} := \psi_- - \psi_-^{(h)}$, and conversely.

5 Usefulness of our approach

5.1 Adiabatic invariance of \mathcal{I}

The action variable $\mathcal{I} = H/\omega$ of the harmonic oscillator is the simplest example of an *adiabatic invariant* in a Hamiltonian system¹⁰, in the sense that it remains approximately constant for slowly varying ω s, as heuristic arguments suggest. Fixed a function $\tilde{\omega}(\tau)$ and a ‘slow time’ parameter $\varepsilon > 0$, consider the family of equations (1) with $\omega(t; \varepsilon) = \tilde{\omega}(\varepsilon t)$. At least two precise senses are ascribed to the property of ‘adiabatic invariance’ of \mathcal{I} :

¹⁰It allowed among other things to interpret the Planck’s quantization rule as the first instance of the Bohr-Sommerfeld-Ehrenfest quantization rules in the so-called *Old Quantum Mechanics* [?]. At the Solvay Congress of 1911 on the old quantum theory Lorentz asked how the amplitude of a simple pendulum would vary if its period were slowly changed by shortening its string. Would the number of quanta of its motion change? Einstein answered that the action variable E/ω , where E is its energy and ω its frequency, would remain constant and thus the number of quanta would remain unchanged, if $\dot{\omega}/\omega_0$ were small enough [5].

1. For all $(q_0, p_0) \in \mathbb{R} \times \mathbb{R}^+$ and $T, \delta > 0$ one can choose $\varepsilon > 0$ so small that for all $t \in [0, T/\varepsilon]$ the solution of (1) with initial conditions $(q(0; \varepsilon), \dot{q}(0; \varepsilon)) = (q_0, p_0)$ satisfies

$$|\mathcal{I}(t; \varepsilon) - \mathcal{I}(0; \varepsilon)| < \delta. \quad (44)$$

Namely, if we slow down the rate of variation of ω proportionally to ε and simultaneously dilate the time interval proportionally to $1/\varepsilon$, so that the τ -interval $[0, T]$ and thus the variation of ω in $[0, T/\varepsilon]$ don't change, the corresponding variation of \mathcal{I} vanishes with ε . As known (see e.g. [2]), a sufficient condition for this is $\tilde{\omega} \in C^2(\mathbb{R}^+)$.

2. If $\tilde{\omega} \in C^{k+1}(\mathbb{R})$ ($k \in \mathbb{N}$) and $\frac{d^h \tilde{\omega}}{d\tau^h} \in L^1(\mathbb{R})$ for all $h = 1, \dots, k+1$ (whence $\tilde{\omega}(\tau)$ has well-defined limits and vanishing derivatives as $\tau \rightarrow \pm\infty$), then every solution of (1) satisfies (see e.g. [17])

$$|\mathcal{I}(\infty; \varepsilon) - \mathcal{I}(-\infty; \varepsilon)| = O(\varepsilon^k); \quad (45)$$

if $\frac{d\tilde{\omega}}{d\tau} \in \mathcal{S}(\mathbb{R})$, i.e. is a Schwarz function (in particular, with compact support), then (45) holds for all $k \in \mathbb{N}$, namely the lhs goes to zero with ε faster than any power.

In the appendix we prove both the sufficient condition in 1. and the implications in 2. quite fast via our approach (an example of its usefulness).

In section 6 we briefly mention some applications where these properties are important.

5.2 Asymptotic expansion in the slow-time parameter ε

Similarly, from the sequence $\{\psi^{(h)}\}_{h \in \mathbb{N}_0}$ one can obtain the asymptotic expansion of ψ, \mathcal{I} in the slow time parameter ε introduced in section 5.1. One can iteratively show¹¹ that

$$\sigma^{(k)} = O(\varepsilon^k), \quad \chi^{(h)} := \psi^{(h)} - \psi^{(h-1)} = O(\varepsilon^h), \quad h = 1, \dots, k-1. \quad (46)$$

Therefore the decomposition of $\psi = \psi^{(k-1)} + \sigma^{(k)}$ into a sum of $\psi^{(0)}$ and subsequent corrections

$$\psi = \psi^{(0)} + \chi^{(1)} + \dots + \chi^{(k-1)} + \sigma^{(k)} \quad (47)$$

is automatically a decomposition into terms $O(1), O(\varepsilon), \dots, O(\varepsilon^k)$. Replacing in (28) one obtains the expansion for \mathcal{I} via the Taylor formula for the exponential. Integrating by parts one can iteratively extract from each integral $\chi^{(h)} = \int_{t_*}^t dz \left\{ \frac{\dot{\omega}}{\omega} [\sin(2\psi^{(h-1)}) - \sin(2\psi^{(h-2)})] \right\}(z)$ the leading contribution as a more explicit function times ε^h , putting the rest in the remainder. In particular, for $k = 2$ we obtain (choosing $t_* = 0$ and abbreviating $\psi_0 = \psi(0)$, $\tilde{\zeta} := \frac{1}{\tilde{\omega}^2} \frac{d\tilde{\omega}}{d\tau}$)

$$\psi(t) = \varphi(t) - \frac{\varepsilon}{4} \left\{ \cos[2\varphi(t)] \tilde{\zeta}(\varepsilon t) - \cos(2\psi_0) \tilde{\zeta}(0) \right\} + O(\varepsilon^2), \quad (48)$$

$$\mathcal{I}(t) = \mathcal{I}_0 \left\{ 1 - \frac{\varepsilon}{2} \left[\sin[2\varphi(t)] \tilde{\zeta}(\varepsilon t) - \sin(2\psi_0) \tilde{\zeta}(0) \right] \right\} + O(\varepsilon^2). \quad (49)$$

¹¹Applying the definitions and the sine-of-a-sum rule we decompose the square bracket in (39) into the sum of two terms, one proportional to $\sin(2\sigma^{(j)}) = O(\sigma^{(j)})$ and another to $[\cos(2\sigma^{(j)}) - 1] = O((\sigma^{(j)})^2)$; therefore it is a $O(\varepsilon^j)$ by the induction hypothesis. As this is multiplied by $\dot{\omega}/\omega = \varepsilon d\tilde{\omega}/d\tau$ the integrand is thus $O(\varepsilon^{j+1})$, leading to (46a). The proof of (46b) is analogous.

5.3 Upper and lower bounds on the solutions

We can easily bound $\psi(t), \mathcal{I}(t), H(t), \dots$ in an interval $[t_h, t_{h+1}]$ where $\omega(t)$ is monotone.

If either $\dot{\omega} \geq 0$ and h is even, or $\dot{\omega} \leq 0$ and h is odd, then $\dot{\omega} \sin(2\psi) \geq 0$ and by eq (5) $0 \leq \dot{\psi} - \omega \leq (-1)^h \dot{\omega} / 2\omega$; integrating in $[t_h, t]$ with the initial condition $\varphi(t_h) = \psi(t_h)$ we find

$$\varphi(t) \leq \psi(t) \leq \varphi(t) + (-1)^h \log \sqrt{\frac{\omega(t)}{\omega(t_h)}} =: \varphi^{(1)}(t) \quad (50)$$

for all $t \in [t_h, t_{h+1}]$. In particular, choosing $t = t_{h+1}$ we find

$$0 \leq \frac{\pi}{2} - \int_{t_h}^{t_{h+1}} dz \omega(z) \leq (-1)^h \log \sqrt{\frac{\omega(t_{h+1})}{\omega(t_h)}} \quad (51)$$

which implicitly yield bounds on the length $t_{h+1} - t_h$ of the interval $[t_h, t_{h+1}]$ that are more stringent than (19). Moreover, let $\bar{t}_h \in [t_h, t_{h+1}]$ be defined by the condition $\varphi^{(1)}(\bar{t}_{h+1}) = (h+1)\pi/2$. By (26), $\cos(2w)$ decreases, grows with $w \in [\psi(t_h), \psi(t_{h+1})] = \left[\frac{h\pi}{2}, \frac{(h+1)\pi}{2}\right]$ if h is even, odd respectively. This and (50) imply in either case

$$-\frac{\dot{\omega}}{\omega} \cos(2\varphi) \leq -\frac{\dot{\omega}}{\omega} \cos(2\psi) \leq -\frac{\dot{\omega}}{\omega} \cos(2\varphi^{(1)}), \quad (52)$$

which replaced in (28) give

$$\exp \left\{ -\int_{t_h}^t dz \left[\frac{\dot{\omega}}{\omega} \cos(2\varphi) \right] (z) \right\} \leq \frac{\mathcal{I}(t)}{\mathcal{I}(t_h)} \leq \exp \left\{ -\int_{t_h}^t dz \left[\frac{\dot{\omega}}{\omega} \cos(2\varphi^{(1)}) \right] (z) \right\}; \quad (53)$$

the left bounds in (52-53) hold for all $t \in [t_h, t_{h+1}]$, while the right ones only for $t \in [t_h, \bar{t}_{h+1}]$.

Similarly, if $\dot{\omega} \leq 0$ and h is even, or $\dot{\omega} \geq 0$ and h is odd, then $\dot{\omega} \sin(2\psi) \leq 0$ and by eq. (5) $(-1)^h \dot{\omega} / 2\omega \leq \dot{\psi} - \omega \leq 0$; integrating in $[t_h, t]$ with the initial condition $\varphi(t_h) = \psi(t_h)$ we find

$$\varphi^{(1)}(t) := \varphi(t) + (-1)^h \log \sqrt{\frac{\omega(t)}{\omega(t_h)}} \leq \psi(t) \leq \varphi(t) \quad (54)$$

for all $t \in [t_h, t_{h+1}]$. In particular, choosing $t = t_{h+1}$ we find

$$(-1)^h \log \sqrt{\frac{\omega(t_{h+1})}{\omega(t_h)}} \leq \frac{\pi}{2} - \int_{t_h}^{t_{h+1}} dz \omega(z) \leq 0 \quad (55)$$

which implicitly yield more stringent bounds on $t_{h+1} - t_h$ than (19). Moreover, let $\bar{t}'_{h+1} \in [t_h, t_{h+1}]$ be defined by the condition $\varphi(\bar{t}'_{h+1}) = (h+1)\pi/2$. Eq. (54) implies in either case again (52) and (53); the only difference is that now the right bounds hold for all $t \in [t_h, t_{h+1}]$, while the left ones only for $t \in [t_h, \bar{t}'_{h+1}]$.

From (4), (53) and (50) or (54) one can now easily derive bounds on q, p .

If $\omega(t)$ is monotone in a larger interval $[t_h, t_{h+k}]$ one can obtain there upper and lower bounds for $\psi(t) - \psi(t_h)$ and the ratio $\mathcal{I}(t)/\mathcal{I}(t_h)$ by using (50), (53) recursively in adjacent intervals. From the previous formulae and (24), (4) one obtains corresponding bounds for $H(t), q(t), p(t)$. The derivative $\dot{H} = \dot{\omega}q^2 = 2\dot{\omega}\mathcal{I}\sin^2\psi$ of H along the solutions of (1) has the same sign of $\dot{\omega}$; therefore $H(t)$ grows, decreases where $\omega(t)$ does.

One could obtain upper and lower bounds also applying Liapunov's direct method to the family of Liapunov functions $V(t, \bar{\omega}) := \dot{q}^2(t) + \bar{\omega}^2 q^2(t)$ parametrized by a positive constant $\bar{\omega}$. In each candidate interval $]t_h, t_{h+1}[$ one makes two choices of $\bar{\omega}$, so as to make the derivative $\dot{V} = 2\dot{q}q(\bar{\omega}^2 - \omega^2)$ of V along a solution $q(t)$ once positive and the other negative. Thus one determines upper and lower bounds first for V , then also for q, \dot{q} . However, it turns out that these bounds are rather less stringent and manageable than the ones found above.

5.4 Parametric resonance, beats, damping

Let us apply the previous results to the Hill equation, i.e. (1) with a periodic ω , $\omega(t+T) = \omega(t)$. We recall that, by Floquet theorem, the fundamental matrix solution (10) of $\dot{V} = AV$ fulfills

$$V(t+nT) = V(t)M^n, \quad M := V(T) \quad (56)$$

for all $t \in [0, T]$ and $n \in \mathbb{N}$, so it suffices to determine V in $\in [0, T]$. As $\det M = 1$, the eigenvalues of the monodromy matrix M are $\lambda_{\pm} = \mu \pm \sqrt{\mu^2 - 1}$, where $2\mu := \text{Tr}(M)$, hence fulfill $\lambda_+ \lambda_- = 1$. Consequently the two eigenvalues must do one of the following: i) be complex conjugate and of modulus 1 if $|\mu| < 1$; coincide if $|\mu| = 1$; be both real one larger, and one smaller than 1, if $|\mu| > 1$. The trivial solution of (1) is stable if $|\mu| < 1$, unstable if $|\mu| > 1$ (most initial conditions near the trivial one yield *parametrically resonant* solutions), see e.g. [2, 17]. In terms of the solutions ψ_a of (5a) fulfilling $\psi_1(0) = \frac{\pi}{2}, \psi_2(0) = 0$, we find¹²

$$2\mu = e^{-\Psi_1(T)} \sin[\psi_1(T)] + e^{-\Psi_2(T)} \cos[\psi_2(T)], \quad (57)$$

where $\Psi_a(t) := \int_0^t dz \left[\frac{\dot{\omega}}{2\omega} \cos(2\psi_a) \right](z)$, $a = 1, 2$. If ω depends on a parameter η so that $\dot{\omega}/\omega = O(\eta)$, then at leading order in η (i.e., up to corrections of order 1,2,4) we find¹³

$$\psi_2(t) = \psi_1(t) - \frac{\pi}{2} = \bar{\omega}t,$$

$$\Psi_2(t) = -\Psi_1(t) = \chi(t)$$

$$\mu = \cos(\bar{\omega}T) \cosh[\chi(T)], \quad (58)$$

¹²The ψ_a associated to v_a ($a=1, 2$) of (10) fulfill $\psi_1(0) = \frac{\pi}{2}, \mathcal{I}_1(0) = \frac{\omega_0}{2}, \psi_2(0) = 0, \mathcal{I}_2(0) = \frac{1}{2\omega_0}$. It follows

$$\begin{aligned} v_1 &= \sqrt{\frac{\omega_0}{\omega}} \sin(\psi_1) e^{-\Psi_1}, & \dot{v}_1 &= \sqrt{\omega\omega_0} \cos(\psi_1) e^{-\Psi_1}, \\ v_2 &= \frac{1}{\sqrt{\omega\omega_0}} \sin(\psi_2) e^{-\Psi_2}, & \dot{v}_2 &= \sqrt{\frac{\omega_0}{\omega}} \cos(\psi_2) e^{-\Psi_2}. \end{aligned}$$

where $\omega_0 \equiv \omega(0)$. From $M = \begin{pmatrix} v_1(T) & v_2(T) \\ \dot{v}_1(T) & \dot{v}_2(T) \end{pmatrix}$ and $2\mu = \text{Tr}(M) = v_1(T) + \dot{v}_2(T)$ one obtains (57).

¹³In fact, according to (27b), $\psi_2^{(0)}(t) = \psi_1^{(0)}(t) - \frac{\pi}{2} = \varphi(t) = \int_0^t dz \omega(z) = \bar{\omega}t + O(\eta)$.

where the characteristic (i.e. average) angular frequency $\bar{\omega}$ and $\chi(t)$ are defined by

$$\bar{\omega} := \frac{\varphi(T)}{T}, \quad \chi(t) := \int_0^t dz \left[\frac{\dot{\omega}}{2\omega} \cos(2\bar{\omega}t) \right](z).$$

Clearly $\chi(T) = O(\eta)$. We recover parametric resonance if the period T of variation of ω is one half of the characteristic time $2\pi/\bar{\omega}$, or a multiple thereof, namely if for some $j \in \mathbb{N}$

$$\bar{\omega} = j \frac{\pi}{T}, \quad (59)$$

because then $\cos(\bar{\omega}T) = \pm 1$, and $|\mu| \simeq \cosh[\chi(T)] > 1$, unless $\chi(T) = 0$. Evaluating (57) one can determine the region in parameter space $(\bar{\omega}, \eta)$ in the vicinity of a special value (59) leading to stable or unstable solutions. At lowest order in the small parameters η and $b := \bar{\omega}T - j\pi$ this amounts to evaluating (58). It is $\cos(\bar{\omega}T) = (-1)^j \cos b$, so that $2|\mu| - 2 = [\chi(T)]^2 - b^2 + O(\eta^4, b^4)$; therefore for sufficiently small η, b the trivial solution will be stable if $|\chi(T)| < |b|$, parametrically resonant (hence unstable) if $|\chi(T)| > |b|$.

As an illustration, we determine at leading order in $|\eta| < 1$ the solutions of (5) and μ if

$$\omega(t) = \bar{\omega} \sqrt{1 + \eta \sin(\alpha t)}, \quad (60)$$

with some constant $\alpha, \bar{\omega} > 0$; the corresponding (1) is the well-known Mathieu equation (which rules many different phenomena, e.g. the forced motion of a swing, the stability of ships, the behaviour of parametric amplifiers based on electronic [13] or superconducting devices [22], parametrically excited oscillations in microelectromechanical systems; see e.g. [33] and references therein), and

$$T = \frac{2\pi}{\alpha}, \quad \frac{\dot{\omega}}{\omega} = \frac{\eta \alpha \cos(\alpha t)}{2[1 + \eta \sin(\alpha t)]}$$

Denoting $\psi(0) = \psi_*$, $\mathcal{I}(0) = \mathcal{I}_*$, we find $\psi(t) = \psi^{(0)}(t) + O(\eta) = \psi_* + \bar{\omega}t + O(\eta)$, whence

$$\begin{aligned} \left[\frac{\dot{\omega}}{\omega} \cos(2\psi) \right](t) &= \frac{\eta \alpha}{2} \cos(\alpha t) \cos(2\psi_* + 2\bar{\omega}t) + O(\eta^2) \\ &= \frac{\eta \alpha}{4} \{ \cos[2\psi_* + (2\bar{\omega} - \alpha)t] + \cos[2\psi_* + (2\bar{\omega} + \alpha)t] \} + O(\eta^2) \end{aligned}$$

Replacing in (28) we find up to $O(\eta^2)$ $\log[\mathcal{I}_*/\mathcal{I}(t)] = 2\chi(t)$, with

$$\chi(t) = \begin{cases} \frac{\eta}{4} \bar{\omega} \left\{ \cos(2\psi_*) t + \frac{\sin(2\psi_* + 4\bar{\omega}t) - \sin(2\psi_*)}{4\bar{\omega}} \right\} & \text{if } \alpha = 2\bar{\omega}, \\ \frac{\eta \alpha}{8} \left\{ \frac{\sin[2\psi_* + (2\bar{\omega} - \alpha)t]}{2\bar{\omega} - \alpha} + \frac{\sin[2\psi_* + (2\bar{\omega} + \alpha)t]}{2\bar{\omega} + \alpha} - \frac{4\bar{\omega} \sin(2\psi_*)}{4\bar{\omega}^2 - \alpha^2} \right\} & \text{if } \alpha \neq 2\bar{\omega}. \end{cases}$$

This leads to the following long-time behaviour. If $\alpha = 2\bar{\omega}$ we find $\mathcal{I}(t) \sim \exp[\eta \bar{\omega} \cos(2\psi_*)t/2]$, which exponentially vanishes with t if $\eta \cos(2\psi_*) < 0$ (damping) or diverges with t if

$\eta \cos(2\psi_*) > 0$ (parametric resonance), respectively. Otherwise $\log \mathcal{I}(t)$ is the superposition of two sinusoids. In particular, for $\alpha \simeq 2\bar{\omega}$ the behaviour of $\log \mathcal{I}(t)$ is a typical beat: $\log[\mathcal{I}(t)/\mathcal{I}(0)]$ oscillates approximately between $\pm|\eta|\alpha/4|2\bar{\omega}-\alpha|$ with a period $T = 2\pi/(2\bar{\omega}-\alpha)$. In fig.s 2, 3 we have plotted (in a short interval starting from $t = 0$) the exact fundamental solutions v_a introduced in section 2 and the corresponding action variables \mathcal{I}_a ($a = 1, 2$), as well as their lowest approximations [in the sense of (8), (41)] $\tilde{v}_a, \tilde{\mathcal{I}}_a, \hat{v}_a, \hat{\mathcal{I}}_a$ for a non-resonant and a resonant choice of the parameters η, α . As evident, our approximations $\hat{v}_a, \hat{\mathcal{I}}_a$ fit the exact solutions much better than the ones $\tilde{v}_a, \tilde{\mathcal{I}}_a$.

$\chi(T)$ is obtained setting $\psi_* = 0, t = T$. At leading order in η we find

$$\chi(T) = \begin{cases} \eta \frac{\pi}{4} & \text{if } \alpha = 2\bar{\omega}, \\ \frac{\eta\alpha\bar{\omega}}{2[4\bar{\omega}^2 - \alpha^2]} \sin\left(\frac{4\pi\bar{\omega}}{\alpha}\right) & \text{if } \alpha \neq 2\bar{\omega}. \end{cases}$$

Near the resonance point $T = \pi/\bar{\omega}$, i.e. for small $b = \bar{\omega}T - \pi \equiv \pi(2\bar{\omega}/\alpha - 1)$, we find $\chi(T) = \eta[\pi/4 + O(b)]$ at lowest order in b, η . Therefore near $(\bar{\omega}, \eta) = (\alpha/2, 0)$ the regions fulfilling the inequalities $\pi|\eta| < 4|b|$ and $\pi|\eta| > 4|b|$, or equivalently

$$|\eta| < 4 \left| \frac{2\bar{\omega}}{\alpha} - 1 \right| \quad \text{and} \quad |\eta| > 4 \left| \frac{2\bar{\omega}}{\alpha} - 1 \right|,$$

are respectively inside the stability and instability region, as known (see e.g. [17]). Near the resonance points $T = j \frac{\pi}{\bar{\omega}}$ ($j > 1$) one can determine the (in)stability regions by a more precise evaluation of (57)¹⁴, which is out of the scope of this work.

6 Discussion and outlook

Solving equation (1) with $\omega(t)$ as general as possible, with high degrees of accuracy and/or for long times is paramount both for a deeper understanding of many natural phenomena and for developing sophisticated technological applications. For instance, the boundedness and stability of the solutions, or the knowledge of the tiny evolution of the adiabatic invariants under slowly varying ω 's after millions or even billions of cycles $T = 2\pi/\omega$, are crucial for many phenomena and problems in electrodynamics (in vacuum and plasmas)¹⁵ applied to

¹⁴In fact, for small $b = \bar{\omega}T - j\pi \equiv \pi(2\bar{\omega}/\alpha - j)$ the previous formulae give $\chi(T) = \eta b \frac{j}{j^2-1} + O(b^2)$; for small $|\eta|$ the inequality $|\chi(T)| < |b|$ (whence $|\mu^{(0)}| < 1$) is fulfilled for (small) $|b| \neq 0$, while for $b = 0$ formula (58) gives $|\mu^{(0)}| = 1$.

¹⁵The equation of motion of a particle of charge q in a uniform magnetic field \mathbf{B} having fixed direction and time-dependent magnitude B has projection of the form (67), with gyrofrequency $\omega = qB/m$, in the plane orthogonal to \mathbf{B} , while is free in the direction of B . If the variation of B is slow (with respect to the cyclotron period $2\pi/\omega$) the angular momentum L and the magnetic moment $qL/2m$ of the particle (the so-called first adiabatic invariant, proportional to the sum of the action variables associated to the cartesian coordinates q_1, q_2) is conserved with high accuracy. If \mathbf{B} varies slowly in space or time this remains approximately true locally. Combined with the conservation of energy (and the so-called second adiabatic invariant) this enables also magnetic mirrors and has extremely important consequences in plasma physics (e.g. for confinement of plasmas in nuclear fusion reactors, the formation of the Van Allen belts, etc.).

geo- and astro-physics [24], accelerator physics [3, 32, 4], plasma confinement in nuclear fusion reactors [11]; in particular, rigorous mathematical results [16, 15, 23, 14, 34, 26]¹⁶ on the adiabatic invariance of $\mathcal{I} = H/\omega$ have allowed to dramatically increase the predictive power for these and other phenomena. A number of important classical and quantum control problems (like the stability of atomic clocks [31], the behaviour of parametric amplifiers based on electronic [13] or superconducting devices [22], or of parametrically excited oscillations in microelectromechanical systems [33]) are ruled by linear oscillator equations reducible to (1) (as sketched in section 2), where the interplay between time-dependent driving (parametric and/or external) and/or damping plays a crucial role [35]. In the quantum framework (1) arises e.g. as the evolution equation of the observable q (a Hilbert space operator) in the Heisenberg picture (with important applications e.g. in quantum optics []), but also as the time-independent Schroedinger equation of a particle on \mathbb{R} in a bounded potential $U(t)$ and with energy E , if we interpret t as a space coordinate, $q(t)$ as the wave-function (which is again valued in numbers, but complex) and set $\omega^2 \equiv 2m(E - U)/\hbar^2$; in particular, $E > 0$ and a periodic $U(t)$, leading to periodic $\omega^2(t)$, determine (via Bloch theorem, an application of the Floquet one) the electronic bands structure of a crystal in solid state physics.

Here we have reduced (section 4) the integration of (1), or of more general linear systems (3) (section 2), to that of the first order ODE (5a), or equivalently the integral equation (27), in the globally defined unknown phase ψ and adopted the iterative resolution procedure (33) to obtain sequences $\{\psi^{(h)}\}_{h \in \mathbb{N}_0}$ *converging uniformly* (and quite rapidly) to the solutions $\psi(t)$ in every compact interval where ω is differentiable; $\psi(t)$ can be extended beyond discontinuity points of $\omega(t)$, if any, by suitable matching conditions. As a preliminary step we have studied (section 3) the instants t_h where ψ is a multiple of $\pi/2$ (these are the interlacing zeroes of $q(t), \dot{q}(t)$). The applications sketched in the previous section 5 illustrate hopefully in a convincing way that our approach is economical and effective, and that its interplay with the existing wisdom and alternative approaches is rather promising for improving (both analitically and numerically) our knowledge of the solutions of (1) and of various delicate aspects of theirs. To that end, a comparison in particular with the Ermakov reformulation [7] seems appropriate now.

Looking for q in the form of the product of an amplitude $\rho(t)$ and the sine of a phase $\theta(t)$,

$$q(t) = \rho(t) \sin[\theta(t)], \quad (61)$$

one easily finds that a sufficient condition for this to be a solution of (1) is that ρ, θ fulfill the ODEs

$$\rho \ddot{\theta} + 2\dot{\rho}\dot{\theta} = 0, \quad \ddot{\rho} = -\omega^2 \rho + \rho \dot{\theta}^2, \quad (62)$$

which make the coefficients of $\sin \theta, \cos \theta$ vanish *separately*. By (62a), $\rho^2 \dot{\theta}$ has zero derivative and hence is a constant L . Replacing $\rho^2 \dot{\theta} \equiv L$ in (62b) we arrive at the Ermakov equation [7] in the unknown ρ

$$\ddot{\rho} = -\omega^2 \rho + \frac{L^2}{\rho^3}. \quad (63)$$

¹⁶For a short history and more detailed list of references about the adiabatic invariant of the harmonic oscillator see e.g. the introduction of [30]. For the general theory of the adiabatic invariants in Hamiltonian systems see e.g. [17, 2, 12].

Note that: i) the last term prevents ρ to vanish anywhere; ii) the equation makes sense and yields solutions having continuous $\rho(t), \dot{\rho}(t)$ even if $\omega(t)$ is not. Given $L \neq 0$ and a *particular* solution $\rho(t)$ of (63), the corresponding $\theta(t)$ is found integrating $\dot{\theta} = L/\rho^2$. Actually the fulfillment of (62), or equivalently of (63) and $\theta(t) = \int^t d\tau L/\rho^2(\tau)$, is a sufficient condition for

$$q_{A,\alpha}(t) := A \rho(t) \sin[\theta(t) + \alpha], \quad A, \alpha \in \mathbb{R} \quad (64)$$

to be the *general solution* of (1). In fact, fixed any conditions $(q(t_*), \dot{q}(t_*)) = (q_*, p_*)$, if we set $\alpha = \{\cot^{-1}[(p_*\rho^2/q_* - \dot{\rho}\rho)/L] - \theta\}(t_*)$, $A = q_* / [\rho \sin(\theta + \alpha)](t_*)$ then $q_{A,\alpha}$ satisfies them. Conversely, assume q, ρ solve (1), (63) with some constant $L \neq 0$. Then [7]

$$2I := (q\dot{\rho} - \dot{q}\rho)^2 + L^2 \left(\frac{q}{\rho} \right)^2, \quad (65)$$

is a constant because $\dot{I} = 0$. Of course, this *exact invariant* I (usually dubbed after the names of Ermakov, Pinney, Courant, Snyder, Lewis, Riesenfeld, in various combinations) must not be confused with the adiabatic one \mathcal{I} . Only when $\omega = \text{const}$ eq. (62) admits the constant solution $\rho = \sqrt{L/\omega}$, which, replaced in (65), gives $I = L\mathcal{I}$, namely coinciding I, \mathcal{I} , up to normalization. In general, the value of I can be determined replacing in (65) the initial conditions fulfilled by q, ρ . In principle (65) allows to determine ρ from q , and conversely, but not in closed form. A solution ρ of (63) can be expressed explicitly [28] in terms of *two* independent solutions u, v of (1) as follows:

$$\rho(t) = \sqrt{u^2(t) + \frac{L^2}{w^2} v^2(t)} \equiv \sqrt{q_1^2(t) + q_2^2(t)}; \quad (66)$$

here $w = uv - \dot{u}v = \text{const} \neq 0$ is the Wronskian of u, v , and in the last expression we have renamed $q_1 := u$, $q_2 := vL/w$. A nice way [6] to derive and interpret these results is to note that the vector $\boldsymbol{\rho} \equiv (q_1, q_2)$ satisfies again (1) regarded as a vector equation:

$$\ddot{\boldsymbol{\rho}} = -\omega^2(t)\boldsymbol{\rho}; \quad (67)$$

this is the equation of motion of a particle in a plane under the action of a time-dependent, but *central* elastic force ($\boldsymbol{\rho}$ is its position vector with respect to the center O). Therefore the angular momentum $L := \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} = q_1\dot{q}_2 - q_2\dot{q}_1$ (which coincides with the Wronskian of q_1, q_2) is conserved. Decomposing $\boldsymbol{\rho} = (\rho \cos \theta, \rho \sin \theta)$, it is immediate to check that (62) amount to (67) written in the polar coordinates ρ, θ , and $L = \rho^2\dot{\theta}$. Replacing $q \mapsto q_1 = \rho \cos \theta$ or $q \mapsto q_2 = \rho \sin \theta$ in (65) one immediately finds that $2I = L^2$. If q_1, q_2 are proportional then $L=0$, $\theta = \text{const}$, and the particle oscillates along a straight line passing through O ; otherwise $L \neq 0$, and the particle goes around O . Finally, the invariant (65) is related also to the symmetries of the equation (1) (see [29, 35] and references therein). Probably it is worth underlining that in general $\rho \neq \sqrt{2\mathcal{I}/\omega}$, $\theta \neq \psi$ and, as already noted, $\mathcal{I} \neq I$, albeit (4a), (61) look similar, and both $\sqrt{2\mathcal{I}/\omega}, \rho(t)$ - contrary to $q(t)$ - keep their sign, so that they can play the role of modulating amplitudes (envelopes) for the solutions of (1) resp. in (4), (61).

The exact invariant (65) is theoretically remarkable both in classical and quantum physics [however, in the latter case q, p are *operators*, while ρ remains a *numerical* solution of (63)].

For instance, in accelerator physics I is a powerful constraint used to characterize the motion of a charged particle in alternating-gradient field configurations [3, 4]. In quantum mechanics the eigenvectors of the *operator* I have time-independent eigenvalues, make up an orthonormal basis of the Hilbert space of states and may be normalized so as to be solutions of the Schroedinger equation; they can be built using ladder operators a, a^\dagger as in the $\omega = \text{const}$ case, and $I = a^\dagger a$ [18, 21, 20] (see e.g. also [25, 9]). However, the concrete use of I is based on the knowledge of a solution of (63) for the specific problem at hand, but solving the second order ODE (63) is not easier than solving the one (1), except for special cases, and in general is more difficult than solving the first order ODE (5a). Therefore our iterative resolution method (33) could be used also for constructing solutions ρ of (63) via (66).

Finding the most general conditions for the asymptotic stability of the trivial solution when there are no external forces and only the damping depends on time is also of interest [10].

Using our approach to determine the parametric resonance regions of (1) for specific classes of $\omega(t)$ (see e.g. [27]) is certainly also worth investigation.

7 Appendix

7.1 Proof of Proposition 1

For all $t_* \in \mathbb{R}$ it is $(q_*, p_*) \neq (0, 0)$, because $q(t)$ is a nontrivial solution. If $q_* = 0$ then respectively set $t_0 := t_*$ if $p_* > 0$, $t_2 := t_*$ if $p_* < 0$. If $q_* \neq 0$ then $r_* \in \mathbb{R}$; in the largest interval containing t_* where $q(t)$ keeps its sign we find $-\dot{r} \geq \omega_l^2 + r^2$, whence

$$\frac{-\dot{r}/\omega_l}{1 + (r/\omega_l)^2} = \frac{d}{dt} \cot^{-1}\left(\frac{r}{\omega_l}\right) \geq \omega_l \quad \Rightarrow \quad \cot^{-1}\left[\frac{r(t)}{\omega_l}\right] \geq \beta_* + \omega_l(t - t_*) =: \varphi_l(t). \quad (68)$$

where $\beta_* \equiv \cot^{-1}\left(\frac{r_*}{\omega_l}\right) \in]0, \pi[$. Let \bar{t}_\pm be defined by the conditions $\varphi_l(\bar{t}_-) = 0$, $\varphi_l(\bar{t}_+) = \pi$; by construction $t_* \in]\bar{t}_-, \bar{t}_+[$. The last inequality implies $r(t)/\omega_l \leq \cot[\varphi_l(t)]$; since the rhs goes to $-\infty$ as $t \uparrow \bar{t}_+$, there must exist a $t_+ \in]t_*, \bar{t}_+[$ such that $r(t) \rightarrow -\infty$ as $t \uparrow t_+$. If $p(t) > 0$, $q(t) < 0$ (resp. $p(t) < 0$, $q(t) > 0$) in a left neighbourhood of t_+ then the point $t_0 := t_+$ (resp. $t_2 := t_+$) has indeed the property mentioned in the claim.

In either case, now set $t_* \equiv t_0$ (resp. $t_* \equiv t_2$), $u \equiv \omega_l s$. In the largest interval $]t_*, t'_+[$ where $p(t)$ keeps its sign we find $\dot{s} \geq 1 + u^2$, whence, integrating over $[t_*, t]$,

$$\frac{\dot{u}}{1+u^2} = \frac{d}{dt} \tan^{-1}u \geq \omega_l \quad \Rightarrow \quad \tan^{-1}[u(t)] \geq \omega_l(t - t_*) \quad \Rightarrow \quad \omega_l s(t) \geq \tan[\omega_l(t - t_*)] \quad (69)$$

Since the rhs diverges as $t \uparrow t_* + \pi/2\omega_l$, it must be $t'_+ \in]t_*, t_* + \pi/2\omega_l[$ and $s(t) \rightarrow +\infty$ as $t \uparrow t'_+$; the point $t_1 := t'_+$ (resp. $t_3 := t'_+$) has indeed the property mentioned in the claim.

Setting $t_* = t_1$ (resp. $t_* = t_3$) and using again (68), with $\beta_* = \pi/2$, we prove the existence of t_2 (resp. t_4); setting $t_* = t_2$ (resp. $t_* = t_4$) and using again (69) we prove the existence of t_3 (resp. t_5); and so on. Replacing $t \mapsto -t$ in the equation and using the previous results one iteratively proves the existence of the t_h for $h \in \mathbb{Z}$ going to $-\infty$.

Finally, if $\omega(t) \geq \bar{\omega}_l > 0$ for all $t \in J \subset \mathbb{R}$ the claim follows from the previous case after extending $\omega(t)$ so that $\omega(t) \geq \bar{\omega}_l$ for all $t \in \mathbb{R}$.

7.2 Proof of Proposition 2

Lemma 1 *Let q_1, q_2 be two independent solutions of (1) having initial data at t_* in the same quadrant of the (q, p) plane, and let $\hat{t}_1 > t_*$, $\hat{t}_2 > t_*$ be the points where q_1, q_2 change quadrant, respectively. Then $\hat{t}_1 < \hat{t}_2$ provided the initial data of both q_1, q_2 at t_* fulfill either $\dot{q}_{1*}q_{2*} < \dot{q}_{2*}q_{1*} \leq 0$, or $0 \leq \dot{q}_{1*}q_{2*} < \dot{q}_{2*}q_{1*}$.*

Proof The difference of (16a) for the corresponding ratios $r_1 = \dot{q}_1/q_1, r_2 = \dot{q}_2/q_2$ yields $\dot{r}_1 - \dot{r}_2 = -(r_1 + r_2)(r_1 - r_2)$. If the initial data of q_1, q_2 at t_* fulfill $\dot{q}_{1*}q_{2*} < \dot{q}_{2*}q_{1*} \leq 0$ (so that they both belong either to the first or to the third quadrant), then $r_1(t_*) < r_2(t_*) \leq 0$, $\dot{r}_1 - \dot{r}_2$ is negative at $t = t_*$, keeps negative for all $t \in [t_*, t''[$, and so does $r_1 - r_2$; here $t'' \equiv \min\{\hat{t}_1, \hat{t}_2\}$. Hence r_2 cannot diverge before r_1 , i.e. $t'' = \hat{t}_1 \leq \hat{t}_2$. Dividing by $(r_1 - r_2)$ we find that

$$\frac{d}{dy} \log(r_2 - r_1) = -(r_1 + r_2) \xrightarrow{t \rightarrow \hat{t}_1} +\infty$$

and this excludes that r_2 diverges together with r_1 , i.e. we find $\hat{t}_1 < \hat{t}_2$, as claimed.

Similarly, the difference of (16b) for the ratios $s_1 = q_1/\dot{q}_1, s_2 = q_2/\dot{q}_2$ yields $\dot{s}_1 - \dot{s}_2 = \omega^2(s_1 + s_2)(s_1 - s_2)$. If the initial data of q_1, q_2 at t_* fulfill $0 \leq \dot{q}_{1*}q_{2*} < \dot{q}_{2*}q_{1*}$ (so that they both belong either to the second or to the fourth quadrant) then $s_1(t_*) > s_2(t_*) \geq 0$, whence, arguing as before, $\hat{t}_1 < \hat{t}_2$. \square

To show that $t'_i > t_i$ implies $t_h(t'_i) > t_h(t_i)$ for all $h \in \mathbb{Z}$ we apply the lemma setting $q_1(t) \equiv Q(t; t_i, q_i, p_i)$, $q_2(t) \equiv Q(t; t'_i, q_i, p_i)$.

We first consider the case that $q_i \leq 0, \dot{q}_i > 0$ (fourth quadrant). Let us denote by $t_0(t_i)$ the smallest $t \geq t_i$ such that $q_1(t) = 0$ and by $t_0(t'_i)$ the smallest $t \geq t'_i$ such that $q_2(t) = 0$. In particular, if $q_i = 0$, then $t_0(t_i) = t_i > t'_i = t_0(t'_i)$. If $q_i < 0$ and $t'_i \geq t_0(t_i)$, then a fortiori $t_0(t'_i) > t'_i \geq t_0(t_i)$; if $q_i < 0$ and $t'_i \in]t_i, t_0(t_i)[$, setting $t_* \equiv t_0(t_i)$, we find $0 = \dot{q}_{2*}q_{1*} > \dot{q}_{1*}q_{2*}$ and, applying the previous lemma, $t_0(t_i) = \hat{t}_1 < \hat{t}_2 = t_0(t'_i)$. Namely, in all cases we find $t_0(t'_i) > t_0(t_i)$, as claimed.

If $t_0(t'_i) \geq t_1(t_i)$ then a fortiori $t_1(t'_i) > t_1(t_i)$; if $t_0(t'_i) \in]t_0(t_i), t_1(t_i)[$, then setting $t_* \equiv t_0(t'_i)$ we find $0 = \dot{q}_{1*}q_{2*} < \dot{q}_{2*}q_{1*}$ and, by the previous lemma, $t_1(t_i) = \hat{t}_1 < \hat{t}_2 = t_1(t'_i)$; namely, in both cases we find $t_1(t'_i) > t_1(t_i)$, as claimed.

If $t_1(t'_i) \geq t_2(t_i)$ then a fortiori $t_2(t'_i) > t_2(t_i)$; if $t_1(t'_i) \in]t_1(t_i), t_2(t_i)[$, then setting $t_* \equiv t_1(t'_i)$ we find $\dot{q}_{1*}q_{2*} < \dot{q}_{2*}q_{1*} = 0$ and, by the previous lemma, $t_2(t_i) = \hat{t}_1 < \hat{t}_2 = t_2(t'_i)$; namely, in both cases we find $t_2(t'_i) > t_2(t_i)$, as claimed.

If $t_2(t'_i) \geq t_3(t_i)$, then a fortiori $t_3(t'_i) > t_3(t_i)$; if $t_2(t'_i) \in]t_2(t_i), t_3(t_i)[$, then setting $t_* \equiv t_2(t'_i)$, we find again $\dot{q}_{2*}q_{1*} > \dot{q}_{1*}q_{2*} = 0$ and, by the previous lemma, $t_3(t_i) = \hat{t}_1 < \hat{t}_2 = t_3(t'_i)$;

namely, in both cases we find $t_3(t'_i) > t_3(t_i)$, as claimed. And so on, for all nonnegative h . The claim for negative h follows after replacing $t \mapsto -t$.

In the case that $q_i > 0$, $\dot{q}_i \geq 0$ (first quadrant) we denote by $t_1(t_i)$ the smallest $t \geq t_i$ such that $\dot{q}_1(t) = 0$ and by $t_1(t'_i)$ the smallest $t \geq t'_i$ such that $\dot{q}_2(t) = 0$. In particular, if $\dot{q}_i = 0$, then $t_1(t_i) = t_i > t'_i = t_1(t'_i)$. If $\dot{q}_i > 0$ and $t'_i \geq t_1(t_i)$, then a fortiori $t_1(t'_i) > t'_i \geq t_1(t_i)$; if $\dot{q}_i > 0$ and $t'_i \in]t_i, t_1(t_i)[$, setting $t_* \equiv t_1(t_i)$, we find $0 = \dot{q}_{1*}q_{2*} < \dot{q}_{2*}q_{1*}$ and, applying the previous lemma, $t_1(t_i) = \hat{t}_1 < \hat{t}_2 = t_1(t'_i)$. Namely, in all cases we find $t_1(t'_i) > t_1(t_i)$, as claimed. The rest of the proof goes as above.

In the case that $q_i \geq 0$, $\dot{q}_i < 0$ (second quadrant) we denote by $t_2(t_i)$ the smallest $t \geq t_i$ such that $q_1(t) = 0$ and by $t_2(t'_i)$ the smallest $t \geq t'_i$ such that $q_2(t) = 0$. In particular, if $q_i = 0$, then $t_2(t_i) = t_i > t'_i = t_2(t'_i)$. If $q_i < 0$ and $t'_i \geq t_2(t_i)$, then a fortiori $t_2(t'_i) > t'_i \geq t_2(t_i)$; if $q_i < 0$ and $t'_i \in]t_i, t_2(t_i)[$, setting $t_* \equiv t_2(t_i)$, we find $\dot{q}_{1*}q_{2*} < \dot{q}_{2*}q_{1*} = 0$ and, applying the previous lemma, $t_2(t_i) = \hat{t}_1 < \hat{t}_2 = t_2(t'_i)$. Namely, in all cases we find $t_2(t'_i) > t_2(t_i)$, as claimed. The rest of the proof goes as above.

In the case that $q_i < 0$, $\dot{q}_i \leq 0$ (third quadrant) we denote by $t_3(t_i)$ the smallest $t \geq t_i$ such that $\dot{q}_1(t) = 0$ and by $t_3(t'_i)$ the smallest $t \geq t'_i$ such that $\dot{q}_2(t) = 0$. In particular, if $\dot{q}_i = 0$, then $t_3(t_i) = t_i > t'_i = t_3(t'_i)$. If $\dot{q}_i < 0$ and $t'_i \geq t_3(t_i)$, then a fortiori $t_3(t'_i) > t'_i \geq t_3(t_i)$; if $\dot{q}_i < 0$ and $t'_i \in]t_i, t_3(t_i)[$, setting $t_* \equiv t_3(t_i)$, we find $\dot{q}_{2*}q_{1*} > \dot{q}_{1*}q_{2*} = 0$ and, applying the previous lemma, $t_3(t_i) = \hat{t}_1 < \hat{t}_2 = t_3(t'_i)$. Namely, in all cases we find $t_3(t'_i) > t_3(t_i)$, as claimed. The rest of the proof goes as above.

7.3 Proof of the bounds (40)

First note that $f(t, u)$ is continuous in t and uniformly Lipschitz continuous in u , since

$$|f(t, u_1) - f(t, u_2)| \leq \frac{\mu}{2}|u_1 - u_2|, \quad \mu := \sup_K \left| \frac{\dot{\omega}}{\omega} \right| < \infty. \quad (70)$$

For all $\lambda \in \mathbb{R}^+$ $C(K)$ is a Banach space w.r.t. the norm $\|\cdot\|_{\lambda, K}$, which is equivalent to the one $\|\cdot\|_\infty$. Eq. (70) implies $|f[t, u_1(t)] - f[t, u_2(t)]| e^{-\lambda|t-t_*|} \leq (\mu/2)\|u_1 - u_2\|_{\lambda, K}$, whence

$$\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{\lambda, K} < \frac{\mu}{2\lambda} \|u_1 - u_2\|_{\lambda, K}$$

for all $u_1, u_2 \in C(K)$. This follows from the inequalities

$$\begin{aligned} |[\mathcal{T}u_1 - \mathcal{T}u_2](t)| e^{\lambda(t_*-t)} &\leq \int_{t_*}^t dz |f[z, u_1(z)] - f[z, u_2(z)]| e^{\lambda(t_*-z)} \\ &\leq \frac{\mu}{2} \|u_1 - u_2\|_{\lambda, K} \int_{t_*}^t dz e^{\lambda(t_*-z)} < \frac{\mu}{2\lambda} \|u_1 - u_2\|_{\lambda, K} \end{aligned}$$

if $t > t_*$, and from the ones with t, t_* exchanged if $t < t_*$. Hence \mathcal{T} is a contraction of $C(K)$ provided we choose $2\lambda > \mu$; the fixed point theorem can be applied. Eq. (40) follows from the byproduct of that theorem

$$\|\psi^{(h)} - \psi\|_{\lambda, K} < \frac{\nu^h}{1-\nu} \|\varphi - \psi^{(1)}\|_{\lambda, K}, \quad \text{where } \nu := \frac{\mu}{2\lambda} \in]0, 1[. \quad (71)$$

7.4 Proof of the adiabatic invariance properties (44), (45)

If we adopt $\tau = \varepsilon t$ as the ‘time’ variable eq. (5a) takes the form

$$\frac{d\psi}{d\tau} = \frac{\tilde{\omega}}{\varepsilon} + \frac{1}{2\tilde{\omega}} \frac{d\tilde{\omega}}{d\tau} \sin(2\psi). \quad (72)$$

The function $\varphi(\tau) := \varphi_0 + \int_0^\tau d\tau' \tilde{\omega}(\tau')$ is strictly growing (here $\varphi_0 \equiv \psi(0)$); to shorten the proof we make the further change of independent variable $\tau \mapsto \varphi$. We denote a transformed function putting a hat above its symbol, abbreviate $\hat{u}'(\varphi) \equiv d\hat{u}/d\varphi$ and more generally $\hat{u}^{[h]} \equiv d^h \hat{u}/d\varphi^h$. As φ is dimensionless, not only $\hat{\zeta} = \hat{\omega}'/\hat{\omega}$, but also all derivatives $\hat{\zeta}', \hat{\zeta}'', \dots$ are; moreover, $\hat{\omega}(\varphi)$ fulfills the same conditions as $\tilde{\omega}(\tau)$ ¹⁷, namely

$$\begin{aligned} \sup_{\tau \in [0, T]} \left| \frac{d^j \tilde{\omega}}{d\tau^j} \right| < \infty, \quad c_h := \sup_{\alpha \in [\psi_0, \varphi(T)]} \left| \hat{\zeta}^{[h]}(\alpha) \right| < \infty, \quad j = 0, 1, 2, \quad h = 0, 1, \quad \text{in case 1,} \\ \sup_{\tau \in \mathbb{R}} \left| \frac{d^j \tilde{\omega}}{d\tau^j} \right| < \infty, \quad \sup_{\mathbb{R}} \left| \hat{\zeta}^{[h]} \right| < \infty, \quad \hat{\zeta}^{[h]} \in L^1(\mathbb{R}), \quad j = 0, \dots, k+1, \quad h = 0, \dots, k \quad \text{in case 2.} \end{aligned} \quad (73)$$

Eq. (72), (28) take the form

$$\hat{\psi}' = \frac{f}{\varepsilon}, \quad f := 1 + \varepsilon \frac{\hat{\zeta}}{2} \sin(2\hat{\psi}) \quad (74)$$

$$-\log \frac{\hat{\mathcal{I}}(\varphi)}{\hat{\mathcal{I}}(\varphi_*)} = \int_{\varphi_*}^{\varphi} d\beta \left[\hat{\zeta} \cos(2\hat{\psi}) \right](\beta); \quad (75)$$

If $\tilde{\omega} \in C^{k+1}(\mathbb{R})$ then $\hat{\zeta} \in C^k(\mathbb{R})$, and the integral at the rhs of (75) can be transformed integrating by parts k times. The result can be extracted as the real part of

$$\begin{aligned} \int_{\varphi_*}^{\varphi} d\beta \left[\hat{\zeta} e^{i2\hat{\psi}} \right](\beta) &= \int_{\varphi_*}^{\varphi} d\beta \frac{\varepsilon \hat{\zeta}(\beta)}{i2f(\beta)} \frac{d}{d\beta} e^{i2\hat{\psi}(\beta)} \\ &= \left[e^{i2\hat{\psi}} \frac{\varepsilon \hat{\zeta}}{i2f} \right]_{\beta=\varphi_*}^{\beta=\varphi} + \int_{\varphi_*}^{\varphi} d\beta \left[e^{i2\hat{\psi}} \frac{d}{d\beta} \left(\frac{i\varepsilon \hat{\zeta}}{2f} \right) \right](\beta) = \dots \\ &= \left\{ e^{i2\hat{\psi}} \sum_{h=0}^{k-1} \left(\frac{i\varepsilon}{2f} \frac{d}{d\beta} \right)^h \frac{\varepsilon \hat{\zeta}}{i2f} \right\}_{\beta=\varphi_*}^{\beta=\varphi} + \varepsilon^k \int_{\varphi_*}^{\varphi} d\beta \left[e^{i2\hat{\psi}} \left(\frac{d}{d\beta} \frac{i}{2f} \right)^k v \right](\beta). \end{aligned} \quad (76)$$

In particular, if $\tilde{\omega} \in C^2(\mathbb{R})$ then

$$\begin{aligned} -\log \frac{\hat{\mathcal{I}}(\varphi)}{\hat{\mathcal{I}}(\varphi_0)} &= \varepsilon \left[\sin(2\hat{\psi}) \frac{\hat{\zeta}}{2f} \right]_{\varphi_0}^{\varphi} - \frac{\varepsilon}{2} \int_{\varphi_0}^{\varphi} d\beta \left[\sin(2\hat{\psi}) \frac{d}{d\beta} \left(\frac{\hat{\zeta}}{f} \right) \right](\beta) \\ &= \varepsilon \left[\sin(2\hat{\psi}) \frac{\hat{\zeta}}{2f} \right]_{\varphi_0}^{\varphi} - \frac{\varepsilon}{2} \int_{\varphi_0}^{\varphi} d\beta \left[\sin(2\hat{\psi}) \left(\frac{\hat{\zeta}'}{f^2} - \frac{\hat{\zeta}^2}{f} \cos(2\hat{\psi}) \right) \right](\beta), \end{aligned}$$

¹⁷This follows from expressing $\frac{d}{d\varphi} = \frac{1}{\tilde{\omega}} \frac{d}{d\tau}$ and using the bounds for $d\varphi/d\tau = \tilde{\omega}(\tau)$ and its derivatives.

whence, noting that $f > 1 - \varepsilon c_0/2$,

$$\left| \log \frac{\hat{\mathcal{I}}(\varphi)}{\hat{\mathcal{I}}(\varphi_0)} \right| \leq \varepsilon \frac{\hat{\zeta}(\varphi) + \hat{\zeta}(0)}{2} + \frac{\varepsilon}{2} \int_0^\varphi d\beta \left(\left| \frac{\hat{\zeta}'}{f^2} \right| + \left| \frac{\hat{\zeta}^2}{f} \right| \right) (\beta) \leq \varepsilon \left\{ c_0 + \frac{\varphi - 0}{2(1 - \varepsilon \frac{c_0}{2})} \left[\frac{c_1}{1 - \varepsilon \frac{c_0}{2}} + c_0^2 \right] \right\},$$

or, using again the time as the independent variable,

$$\left| \log \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right| \leq \varepsilon \left\{ c_0 + \frac{T\omega_u}{2 - \varepsilon c_0} \left[\frac{c_1}{1 - \varepsilon \frac{c_0}{2}} + c_0^2 \right] \right\} =: M(\varepsilon), \quad \Rightarrow \quad |\mathcal{I}(t) - \mathcal{I}(0)| < \mathcal{I}(0) [e^{M(\varepsilon)} - 1].$$

Clearly $M(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since $0 < x \leq 1$ implies $e^x - 1 < 2x$, choosing ε so small that $M(\varepsilon) < \min \{1, \delta/2\mathcal{I}(0)\}$ it follows $e^{M(\varepsilon)} - 1 < 2M(\varepsilon) < \delta/\mathcal{I}(0)$, and (44) is fulfilled, as claimed. In case 2, by (73b) the terms in (76) outside the integral go to zero in the limits $\varphi_* \rightarrow -\infty$, $\varphi \rightarrow \infty$ because $\hat{\zeta}$ does, while the integral itself is bounded, so that $\left| \log \frac{\hat{\mathcal{I}}(\infty; \varepsilon)}{\hat{\mathcal{I}}(-\infty; \varepsilon)} \right| < N\varepsilon^k$ for some constant $N > 0$. Arguing as above we conclude the proof of (45).

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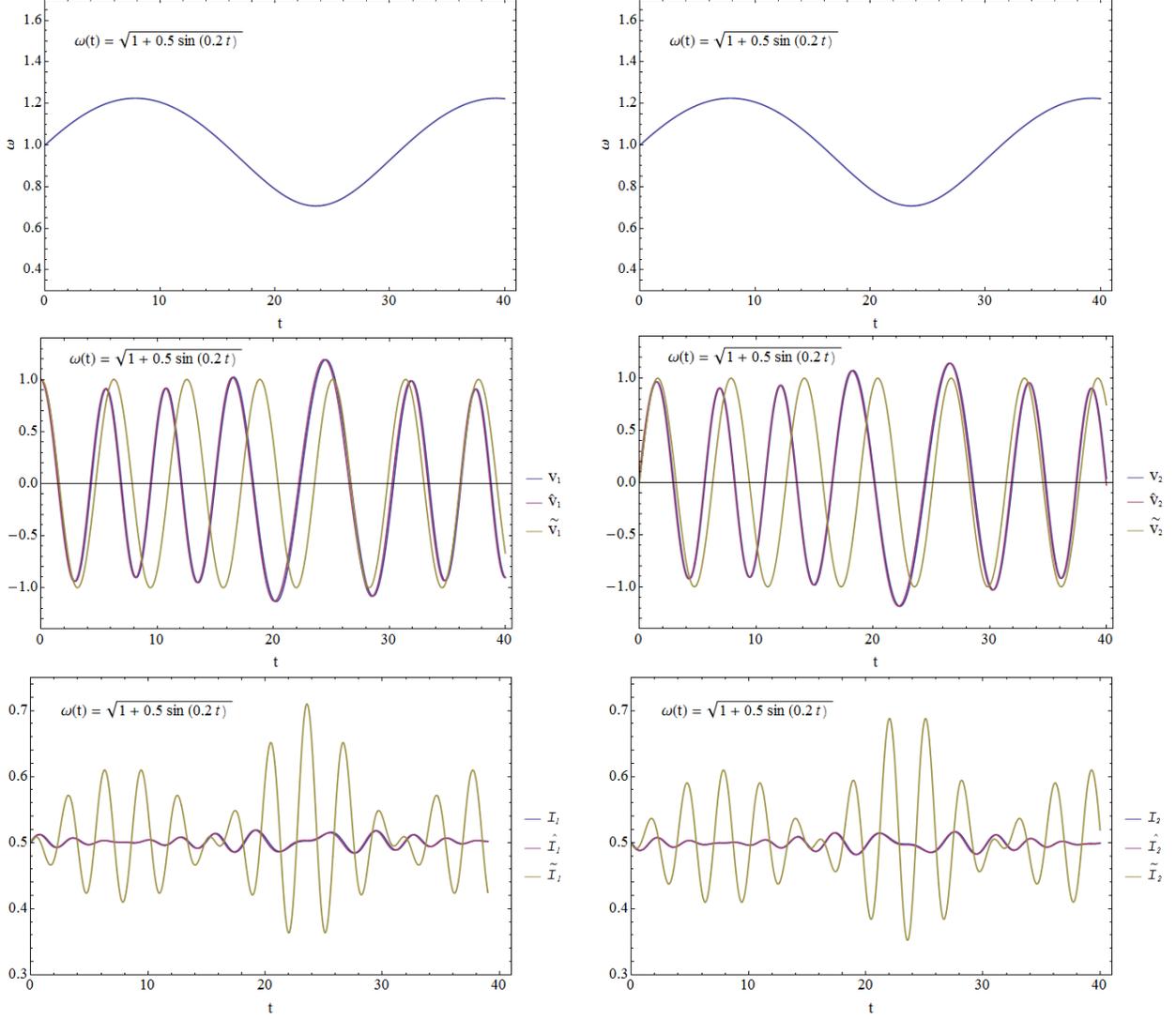


Figure 2: Mathieu equation with $\omega(t) = \bar{\omega} \sqrt{1 + 0.5 \sin(0.2t)}$ (this is a non-resonant case). Up: graph of ω ; center: graphs of the two solutions v_a ($a = 1, 2$) of (1) fulfilling $v_1(0) = \dot{v}_2(0) = 1$, $v_2(0) = \dot{v}_1(0) = 0$ (see section 2) and of their approximations \hat{v}_a, \tilde{v}_a , with $a = 1$ on the left, $a = 2$ on the right; down: graphs of the corresponding action variable \mathcal{I}_a and of their approximations $\hat{\mathcal{I}}_a, \tilde{\mathcal{I}}_a$. As we can see, $\hat{v}_a \simeq v_a$, $\hat{\mathcal{I}}_a \simeq \mathcal{I}_a$, namely the approximation (41) is rather good, and much better than the one (8).

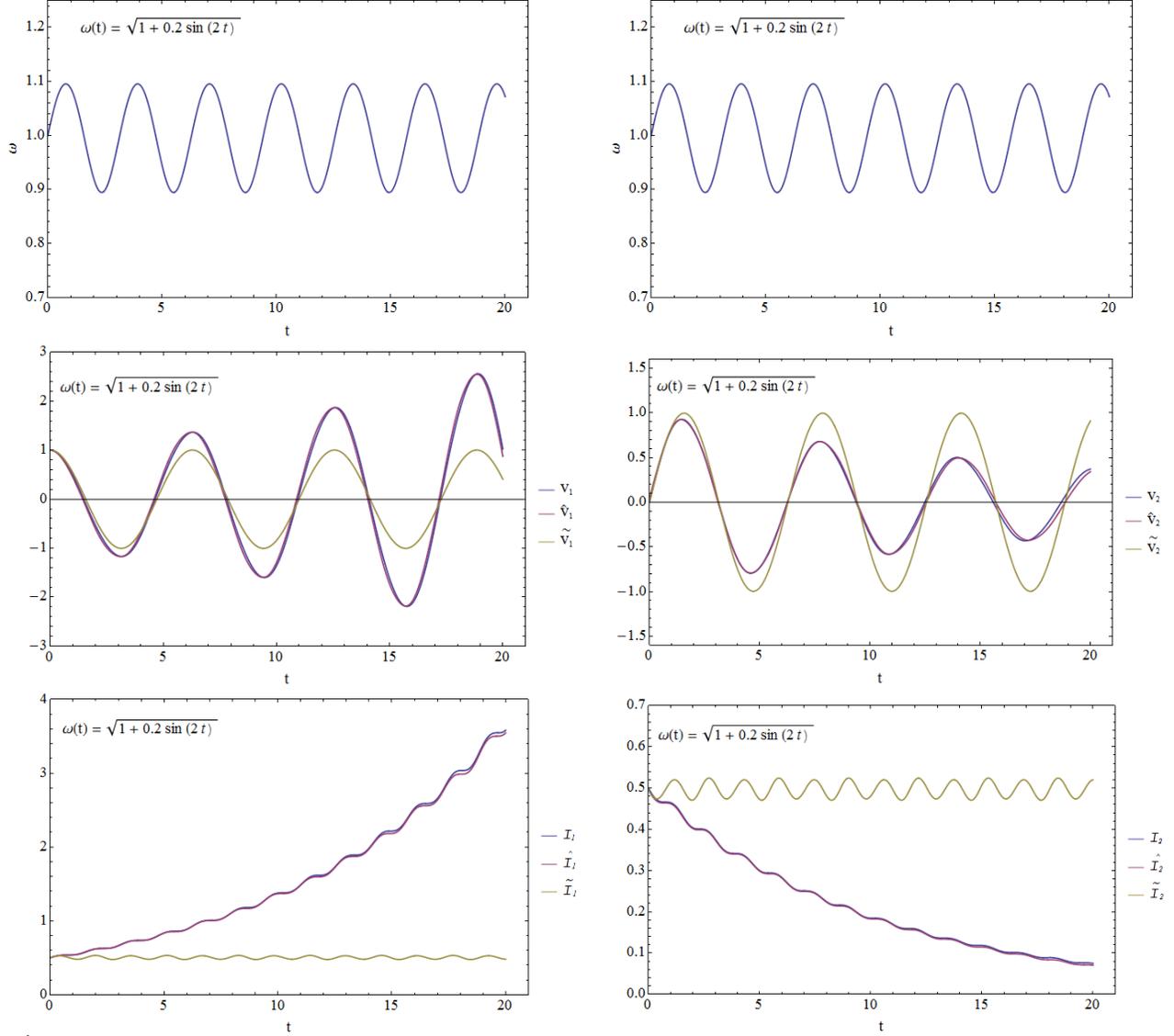


Figure 3: Mathieu equation with $\omega(t) = \bar{\omega}\sqrt{1 + 0.2 \sin(2t)}$ (this is a resonant case). Up: graph of ω ; center: graphs of the two solutions v_a ($a = 1, 2$) of (1) fulfilling $v_1(0) = \dot{v}_2(0) = 1$, $v_2(0) = \dot{v}_1(0) = 0$ (see section 2) and of their approximations \hat{v}_a, \tilde{v}_a , with $a = 1$ on the left, $a = 2$ on the right; down: graphs of the corresponding action variable \mathcal{I}_a and of their approximations $\hat{\mathcal{I}}_a, \tilde{\mathcal{I}}_a$. Again, as we can see, $\hat{v}_a \simeq v_a$, $\hat{\mathcal{I}}_a \simeq \mathcal{I}_a$, namely the approximation (41) is rather good, and much better than the one (8).