Transference for loose Hamilton cycles in random 3-uniform hypergraphs

Kalina Petrova[*,†] Miloš Trujić[*,‡]

Abstract

A loose Hamilton cycle in a hypergraph is a cyclic sequence of edges covering all vertices in which only every two consecutive edges intersect and do so in exactly one vertex. With Dirac's theorem in mind, it is natural to ask what minimum d-degree condition guarantees the existence of a loose Hamilton cycle in a k-uniform hypergraph. For $k = 3$ and each $d \in \{1, 2\}$, the necessary and sufficient such condition is known precisely. We show that these results adhere to a 'transference principle' to their sparse random analogues. The proof combines several ideas from the graph setting and relies on the absorbing method. In particular, we employ a novel approach of Kwan and Ferber for finding absorbers in subgraphs of sparse hypergraphs via a contraction procedure. In the case of $d = 2$, our findings are asymptotically optimal.

1 Introduction

The question of deciding when a given graph is Hamiltonian is in general notoriously difficult and was included in Karp's original list of 21 NP-complete problems [34]. Being a fundamental problem in graph theory (and computer science), Hamiltonicity has inspired a long line of work exploring sufficient conditions for it. Perhaps the best known among those is the classical theorem of Dirac [14]: every graph on $n \geqslant 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. Another regime in which the problem is understood better than in the general case is that of random graphs (also, more broadly, quasi-random graphs and expanders). In that regard, Pósa [59] and independently Korshunov [43] proved that if $p \geqslant C \log n/n$, for some constant $C > 0$, then the Erdős-Rényi binomial random graph $G(n, p)$[1] is *with high probability*[2] Hamiltonian. The more precise value $p = p(n)$ for which the former holds was later determined by Komlós and Szemerédi [42], and an even stronger, so-called *hitting-time* result, was shown by Ajtai, Komlós, and Szemerédi [1] and independently by Bollobás [7].

Inquiring further into properties of random graphs, Sudakov and Vu [63] asked how *resilient* $G(n, p)$ is with respect to having a Hamilton cycle. A graph G is α-resilient with respect to a property $\mathcal{P}$ if after the removal of at most an α-fraction of edges incident to every vertex of G,

[*]Institute of Theoretical Computer Science, ETH Zürich, 8092 Zürich, Switzerland
Email: {kpetrova|mtrujic}@inf.ethz.ch

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[1] $G(n, p)$ stands for a graph on n vertices in which each edge exists with probability $p = p(n) \in (0, 1)$ independently.

[2] With high probability (or w.h.p. for brevity) means with probability going to 1 as n tends to infinity.

the resulting graph (still) contains $\mathcal{P}$. Observe that Dirac's theorem states exactly that: the complete graph on n vertices K_n is $1/2$-resilient with respect to Hamiltonicity (with $1/2$ being optimal as witnessed by two disjoint cliques of size $n/2$). The work of Sudakov and Vu initiated a systematic study of minimum degree requirements in the flavour of Dirac's theorem for random graphs as they showed that for $p \gg \log^4 n/n$, w.h.p. $G(n,p)$ is $(1/2 - o(1))$-resilient with respect to Hamiltonicity. A full analogue of Dirac's theorem for random graphs was later established by Lee and Sudakov [51]: if $p \gg \log n/n$ then w.h.p. $G(n,p)$ is $(1/2 - o(1))$-resilient with respect to Hamiltonicity. Even more, the absolutely best-possible hitting-time results were recently obtained by Montgomery [52] and independently Nenadov, Steger, and the second author [56].

There are, however, certain deficiencies of the basic notion of resilience in the usual binomial random graph—for example, it is unable to capture the behaviour of $G(n,p)$ with respect to containment of large structures riddled with triangles. Namely, as soon as $p = o(1)$, one can easily prevent as many as $\Theta(p^{-2})$ vertices from being in a triangle while removing only $o(np)$ edges incident to each vertex (see [3, 31]). This makes the study of resilience for, e.g., a triangle-factor or any power of a Hamilton cycle futile. Recently, Fischer, Steger, Škorić, and the second author extended the notion of resilience to H-resilience in order to study robustness of $G(n,p)$ with respect to the containment of the square of a Hamilton cycle [21]. Roughly speaking, they determined the smallest $\alpha \in [0,1]$ such that, for $p \gg \log^3 n/\sqrt{n}$, w.h.p. every $G \subseteq G(n,p)$ in which every vertex belongs to at least $(\alpha + o(1))p^3\binom{n}{2}$ triangles contains the square of a Hamilton cycle.

Such a concept naturally corresponds to resilience in hypergraphs, with the added advantage that edges in a random hypergraph are independent, unlike copies of H in $G(n,p)$. Thus, the hypergraph counterpart to the questions above can be a solid test case for developing new ideas and techniques. Of course, studying resilience of Hamiltonicity in hypergraphs is also of independent interest as a natural 'high-dimensional' generalisation of one of the most important questions in graph theory to a radically more challenging setting.

In this paper, we are concerned with Dirac-type conditions, hence resilience, for Hamiltonicity in random k-uniform[3] hypergraphs. The notions of cycles and degrees do not generalise unambiguously to the hypergraph setting, so in what follows we make them more specific. For some $1 \leqslant \ell < k$, an ℓ-cycle in a k-graph is a cyclic sequence of vertices such that every vertex belongs to some edge, every edge consists of k consecutive vertices, and each two consecutive edges overlap in exactly ℓ vertices. The most studied special cases are $\ell = 1$ and $\ell = k-1$, which are referred to as a *loose* and a *tight* cycle, respectively. Note that the number of vertices in any ℓ-cycle is necessarily divisible[4] by $k - \ell$. In a k-graph $\mathcal{H}$, for some $1 \leqslant d < k$, the d-degree $\deg_{\mathcal{H}}(S)$ of a d-element set of vertices $S \subseteq V(\mathcal{H})$ is the number of edges $A \in E(\mathcal{H})$ such that $S \subseteq A$. The 1-degree $\deg_{\mathcal{H}}(v)$ of a vertex v is referred to simply as its degree, and the $(k-1)$-degree of a $(k-1)$-set as its codegree. The minimum d-degree $\delta_d(\mathcal{H})$ of $\mathcal{H}$ is the minimum value among the d-degrees over all d-sets $S \subseteq V(\mathcal{H})$.

As is the case with graphs, there is a large body of research studying various Dirac-type conditions in hypergraphs, that is smallest d-degree which implies existence of Hamilton ℓ-cycles (e.g. [60, 61, 46], surveys [49, 64] and the references within). Most closely related to the present work, for $d = k-1$, Keevash, Kühn, Mycroft, and Osthus [36] and independently

[3]A hypergraph is k-uniform, also called *k-graph* for short, if every edge consists of exactly k vertices.

[4]From now on, we always assume that the hypergraph we are dealing with has its number of vertices divisible by the appropriate integer needed for a loose Hamilton cycle to exist.

Hán and Schacht [28] showed that every (sufficiently large) n-vertex k-graph $\mathcal{H}$ with $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-1)} + o(1))n$ contains a loose Hamilton cycle, which is asymptotically optimal (up to the $o(n)$ term). In case of 3-graphs, it was previously known that $\delta_d(\mathcal{H}) \geqslant (\delta_d + o(1))\binom{n-d}{3-d}$ implies (loose) Hamiltonicity, with $\delta_1 = 7/16$ for $d = 1$ or $\delta_2 = 1/4$ for $d = 2$ (see [10, 47] and their strengthening [29, 13]). Neither of the values $\delta_1 = 7/16, \delta_2 = 1/4$ can be improved upon.

When it comes to (sparse) random structures, Dudek, Frieze, Loh, and Speiss [15] showed that $\mathcal{H}^k(n,p)$[5] w.h.p. contains a loose Hamilton cycle, provided that $p \gg \log n/n^{k-1}$ (this generalises a previous result of Frieze [22]; see also [17] for a short proof and [58] for a much more general framework from which this follows directly). This result is asymptotically optimal, since for p of lower order of magnitude, $\mathcal{H}^k(n,p)$ w.h.p. has isolated vertices.

Our main contribution is a resilience variant of this result for 3-graphs, thus transferring the formerly mentioned Dirac-type statements to the sparse random setting.

Theorem 1.1. *Let $d \in \{1,2\}$. For every $\gamma > 0$ there is a $C > 0$ such that the following holds. Suppose $p \geqslant C \log n \cdot \max\{n^{-3/2}, n^{-3+d}\}$ and n is even. Then $\mathcal{H}^3(n,p)$ w.h.p. has the property that every spanning subgraph $G \subseteq \mathcal{H}^3(n,p)$ with $\delta_d(G) \geqslant (\delta_d + \gamma)p\binom{n-d}{3-d}$ contains a loose Hamilton cycle.*

The constants $\delta_1 = 7/16$ and $\delta_2 = 1/4$ are optimal—in other words, w.h.p. there exists a subgraph G of $\mathcal{H}^3(n,p)$ with minimum d-degree $(\delta_d - o(1))\binom{n-d}{3-d}$ which does not have a loose Hamilton cycle. One can see this by, e.g., considering analogues of dense extremal constructions from [10] (in case $d = 1$) and [47] (in case $d = 2$).

The more interesting part are the limitations for the density p. Observe that one can more concisely summarise the density requirements as $p \geqslant C \log n/n^{2-d/2}$, however we chose to present them separately to underline their distinct origins. In case $d = 2$, if $p = c \log n/n$ for some small $c > 0$, then w.h.p. $\mathcal{H}^3(n,p)$ contains pairs of vertices with 2-degree equal to zero, and thus Theorem 1.1 is optimal (up to the multiplicative constant). However, the bound $p \geqslant Cn^{-3/2} \log n$ for $d = 1$ is 'artificial' but crucial for our proof technique. It is highly likely, but probably quite challenging to prove, that the result should remain true all the way down to $p = \Omega(\log n/n^2)$—the threshold for appearance of a loose Hamilton cycle in $\mathcal{H}^3(n,p)$. It seems that one would need to explore completely different techniques in order to establish this.

Note that this provides a resilience result for $\mathcal{H}^3(n,p)$ with respect to containment of a loose Hamilton cycle—it shows that $\mathcal{H}^3(n,p)$ is $(9/16 - o(1))$-resilient for this property (similarly one can think of 'codegree resilience' in terms of δ_2). This partially answers a question raised by Frieze [23, Problem 56] in his survey on Hamilton cycles in random graphs. Being only the beginning of the story, a natural next step would be an extension to hypergraphs of larger uniformities. One obstacle on the way is that the corresponding dense Dirac condition with $d = 1$ is not known for k-graphs with $k \geqslant 4$ (a 'transference' result could be obtained even without knowing this value, see [20]). In addition, our methods do not seem to straightforwardly generalise to $k \geqslant 4$ regardless of the type of degree considered.

Usually, establishing properties of hypergraphs is substantially more difficult than in the case of their graph counterparts, hence it is not surprising that not much is known about resilience of random hypergraphs. Prior to this, Clemens, Ehrenmüller, and Person [11] studied resilience

[5] $\mathcal{H}^k(n,p)$ stands for the (random) graph on n vertices in which each set of k vertices is chosen as an edge independently with probability p.

of $\mathcal{H}^k(n,p)$ with respect to the containment of a Hamilton Berge cycle, and, very recently, Allen, Parczyk, and Pfenninger [2] proved that $\mathcal{H}^k(n,p)$ is resilient for containment of tight Hamilton cycles. On a related note, Ferber and Kwan [20] proved a very general 'transference principle' concerning perfect matchings in random k-graphs (see also [18] for a specific result when $d = k-1$).

On the whole, our proof follows a standard strategy for embedding relying on the *absorbing method*. In a nutshell, this method allows one to reduce the problem of finding a *spanning* structure to the usually significantly easier one of finding an *almost-spanning* one. For the latter we combine several ideas originating from their graph counterparts such as the DFS technique for finding long paths (Section 4), path connection techniques (Section 5), and the sparse regularity method (Section 3). As always, the most difficult, involved, and creative part comes in designing and finding the absorber (see Definition 6.1 below). The 'finding' part is partially done through a contraction procedure of Ferber and Kwan which helps with finding absorbers inside a regular partition of $\mathcal{H}^3(n,p)$. We discuss this, as well as the absorbing method in general, in much greater detail in Section 6. All these ingredients are mixed together following a usual recipe to give a proof of Theorem 1.1 (Section 7).

2 Preliminaries

Our graph theoretic notation mostly follows standard textbooks in the area, e.g. [8]. More specifically, for a (hyper)graph $G = (V,E)$ we let $v(G)$ and $e(G)$ denote the number of its vertices and edges, respectively. Given a set of vertices $X \subseteq V(G)$, $G - X$ stands for the induced graph $G[V(G) \setminus X]$. For sets $X, Y, Z \subseteq V(G)$ we let $e_G(X,Y,Z)$ denote the number of triples $(x,y,z) \in X \times Y \times Z$ for which $xyz \in E(G)$. Instead of $e_G(\{x\},Y,Z)$, we write $e_G(x,Y,Z)$ for brevity. Similarly, if $\mathcal{P} \subseteq \binom{V(G)}{2}$ is a set of *pairs* of vertices, then $e_G(\mathcal{P},Z)$ counts the number of (ordered) pairs $(\{x,y\},z)$ for which $xyz \in E(G)$ with $\{x,y\} \in \mathcal{P}$ and $z \in Z$. For a set $W \subseteq V(G)$, we use $\deg_G(x,W)$, respectively $\deg_G(\{x,y\},W)$, for the number of distinct pairs $\{y,z\}$ with $y,z \in W$, respectively the number of vertices $z \in W$, for which $xyz \in E(G)$. The neighbourhood $N_G(v,W)$ of a vertex v in a set $W \subseteq V(G)$ refers to the set of vertices $w \in W$ such that $uvw \in G$ for some $u \in W$, and we denote $N_G(v,V(G))$ as $N_G(v)$ for brevity. Similarly, the neighbourhood $N_G(\{v,u\},W)$ of a pair of vertices v,u in a set $W \subseteq V(G)$ denotes the set of vertices $w \in W$ with $uvw \in G$, and we abbreviate $N_G(\{v,u\},V(G))$ to $N_G(v,u)$.

A *loose path* of length $\ell \in \mathbb{N}$ is an ordered sequence of distinct vertices $v_1,\dots,v_{2\ell+1}$ and ℓ edges: $v_{2i-1}v_{2i}v_{2i+1} \in E(G)$ for all $i \in [\ell]$. Note that a loose path always consists of an odd number of vertices. Throughout, whenever we say path we mean loose path and we write xy-path for a path whose start- and end-points are x and y. A (loose) cycle is defined similarly.

All logarithms are in base e. For $n \in \mathbb{N}$, $[n]$ stands for the set of first n integers, that is $[n] := \{1,\dots,n\}$. We use standard asymptotic notation o, O, ω, Ω, and Θ. For a set W and an integer d, $\binom{W}{d}$ denotes the collection of all d-element subsets, d-sets for brevity, of W, and W^d denotes the collection of all distinct d-tuples $(w_1,\dots,w_d)$ with $w_i \in W$ and $w_i \neq w_j$ for each $1 \leqslant i < j \leqslant d$. When using set-theoretic notation, we treat tuples as corresponding sets, e.g. $x \in (w_1,\dots,w_d)$ stands for $x \in \{w_1,\dots,w_d\}$ and two tuples are disjoint if they do not share an element.

2.1 Distribution of edges

In this subsection we list some lemmas that give upper bounds on the number of edges between various vertex sets in $\mathcal{H}^3(n,p)$.

Lemma 2.1. *For every $\gamma > 0$ and $C > 0$ there exists $\lambda > 0$ such that w.h.p. $G \sim \mathcal{H}^3(n,p)$ satisfies the following, provided that $n^2p \gg \log n$. There are no two disjoint sets $A, B \subseteq V(G)$ of sizes a and b such that $1 \leqslant a \leqslant \lambda n$, $b \leqslant Ca$, and $e_G(A,B,V(G)) \geqslant \gamma ap\binom{n-1}{2}$.*

Proof. Set $m := \gamma ap\binom{n-1}{2}$. Fix $a \leqslant \lambda n$, $b \leqslant Ca$, and two disjoint sets A and B of sizes a and b, respectively. The probability that $e_G(A,B,V(G)) \geqslant m$ is at most

$$\binom{ab(n-2)}{m}p^m \leqslant \Big(\frac{2eab(n-2)}{\gamma ap(n-1)(n-2)}\Big)^m p^m \leqslant \Big(\frac{2eb}{\gamma(n-1)}\Big)^m \leqslant \Big(\frac{4e\lambda C}{\gamma}\Big)^m.$$

This is at most $e^{-\omega(a\log n)}$ provided λ is chosen small enough with respect to γ and C. Then the union bound over at most $e^{(a+b)\log n} \leqslant e^{2Ca\log n}$ choices for the sets A and B completes the proof. $\square$

Lemma 2.2. *For every $\varepsilon \in (0, 1/300)$, w.h.p. $G \sim \mathcal{H}^3(n,p)$ satisfies the following. Let $X, Y \subseteq V(G)$ be sets of size x and y.*

(i) If $y \leqslant x \leqslant \varepsilon^{-3}\log n/(np)$ then $e_G(X,Y,V(G)) \leqslant \varepsilon^{-4}x\log n$.

(ii) If $x, y \geqslant \varepsilon^{-3}\log n/(np)$ then $e_G(X,Y,V(G)) \leqslant (1+\varepsilon)xynp$.

Proof. Let $t := \varepsilon^{-3}\log n/(np)$. The probability that there exist X, Y of sizes $y \leqslant x \leqslant t$ for which $e_G(X,Y,V(G)) > \varepsilon^{-4}x\log n =: m$ is at most

$$\sum_{x=1}^{t}\binom{n}{x}\sum_{y=1}^{x}\binom{n}{y}\binom{xyn}{m}p^m \leqslant \sum_{x=1}^{t}\sum_{y=1}^{x} n^{2x}\Big(\frac{ex^2np}{m}\Big)^m \leqslant \sum_{x=1}^{t}\sum_{y=1}^{x} n^{2x}\Big(\frac{extnp}{\varepsilon^{-4}x\log n}\Big)^m$$
$$\leqslant \sum_{x=1}^{t}\sum_{y=1}^{x} n^{2x}e^{-10x\log n} = o(1).$$

If $x, y \geqslant t$, from Chernoff's inequality (see, e.g. [32, Corollary 2.3])

$$\Pr[e_G(X,Y,V(G)) > (1+\varepsilon)xynp] \leqslant e^{-\frac{\varepsilon^2}{3}xynp}.$$

Then by the union bound, the probability that the property of the lemma fails is at most

$$\sum_{x=t}^{n}\binom{n}{x}\sum_{y=t}^{n}\binom{n}{y}\cdot e^{-\frac{\varepsilon^2}{3}xynp} \leqslant \sum_{x=t}^{n}\sum_{y=t}^{n} e^{x\log n}e^{y\log n}\cdot e^{-\frac{\varepsilon^2}{3}xynp}$$
$$\leqslant \sum_{x=t}^{n}\sum_{y=t}^{n} e^{2\max\{x,y\}\log n}\cdot e^{-\frac{1}{3\varepsilon}\max\{x,y\}\log n} = o(1),$$

once again. $\square$

Proving the next two lemmas follows similar steps as for the one above, using a straightforward application of Chernoff's inequality and the union bound. Thus, we omit the proofs.

Lemma 2.3. *For every $\varepsilon \in (0, 1/300)$, w.h.p. $G \sim \mathcal{H}^3(n,p)$ satisfies the following. Let $W \subseteq V(G)$ and $\mathcal{P} \subseteq \binom{V(G)\smallsetminus W}{2}$.*

(i) *If* $|W| \leqslant |\mathcal{P}| \leqslant \varepsilon^{-3}\log n/p$ *then* $e_G(\mathcal{P}, W) \leqslant \varepsilon^{-4}|\mathcal{P}|\log n$.

(ii) *If* $|W|, |\mathcal{P}| \geqslant \varepsilon^{-3}\log n/p$ *then* $e_G(\mathcal{P}, W) \leqslant (1+\varepsilon)|\mathcal{P}||W|p$.

Lemma 2.4. *For every $\varepsilon > 0$, w.h.p. $G \sim \mathcal{H}^3(n,p)$ satisfies the following. Let $X, Y, Z \subseteq V(G)$ be (not necessarily disjoint) sets of size x, y, z, such that $xyzp \geqslant 200\varepsilon^{-2}n$. Then*

$$e_G(X, Y, Z) \leqslant (1+\varepsilon)xyzp.$$

In particular, if $Y = Z$, then $e_G(X, \binom{Y}{2}) \leqslant (1+\varepsilon)x\binom{y}{2}p$.

2.2 Expansion

The following lemma captures the fact that expansion of vertices behaves as expected in a not too sparse subgraph G of the random graph $\mathcal{H}^3(n,p)$. Namely, if a vertex v has the property that for some W the minimum degree of $G[\{v\} \cup W]$ is at least $\Omega(n^2p)$, then there are at least $\sqrt{n}$ vertices $w_1 \in W$ and $\Theta(n)$ vertices $w_2 \in W$ for which there is a vw_i-path of length i in $G[\{v\} \cup W]$. It plays an important role both for proving the Connecting Lemma and finding absorbers.

Lemma 2.5. *For every $\gamma > 0$ there exist $\xi, C > 0$ such that w.h.p. $\mathcal{H}^3(n,p)$ satisfies the following, provided that $p \geqslant Cn^{-3/2}\log n$. Let $G \subseteq \mathcal{H}^3(n,p)$ and $W \subseteq V(G)$ with $\delta_1(G[W]) \geqslant \gamma p\binom{n-1}{2}$. For every $x \in V(G)$ for which $\deg_G(x, W) \geqslant \gamma p\binom{n-1}{2}$, there exist $\mathcal{F}_x \subseteq (W \smallsetminus \{x\})^4$ and $\mathcal{P}_x \subseteq \binom{W \smallsetminus \{x\}}{2}$, all pairwise disjoint, of size $|\mathcal{F}_x| \geqslant \xi n$, $|\mathcal{P}_x| = \sqrt{n}$, and such that*

- **(A1)** *$xuv \in E(G)$, for every $\{u, v\} \in \mathcal{P}_x$, and*
- **(A2)** *for every $(w_1, w_2, w_3, w_4) \in \mathcal{F}_x$ there is some $\{u, v\} \in \mathcal{P}_x$ for which $uw_1w_2, vw_3w_4 \in E(G)$.*

Proof. Suppose $\mathcal{H}^3(n,p)$ has the properties of Lemma 2.2 and Lemma 2.4 for, say, $\varepsilon = 10^{-3}$. Let $\mathcal{P}_x \subseteq \binom{W \smallsetminus \{x\}}{2}$ be a maximal set of disjoint pairs which close an edge together with x, and suppose $|\mathcal{P}_x| < \sqrt{n}$. Let X be a superset of all the vertices that belong to these pairs of size precisely $2\sqrt{n}$. Then, in particular, there are no edges $xuv \in E(G)$ with $u, v \in W \smallsetminus X$. Note that $2\sqrt{n} \geqslant \varepsilon^{-3}\log n/(np)$ by our choice of p for $C > 0$ large enough. From the minimum degree assumption on the one hand, and the property of Lemma 2.2 *(ii)* (considering a superset of $\{x\}$ of size $\varepsilon^{-3}\log n/(np)$ if necessary) on the other, we have

$$\gamma p\binom{n-1}{2} \leqslant e_G(x, X, V(G)) \leqslant (1+\varepsilon)\cdot\max\{\varepsilon^{-3}\log n/(np), 1\}\cdot 2\sqrt{n}\cdot np.$$

This is a contradiction for C large enough.

Similarly, after fixing $\mathcal{P}_x$ and X, let $\mathcal{F}_x$ be a maximal set of disjoint 4-tuples which satisfy the second property of the lemma, and suppose $|\mathcal{F}_x| < \xi n$. Let Y be the vertices that belong to these 4-tuples. We first show there is a set $\mathcal{P}'_x \subseteq \mathcal{P}_x$ of size $|\mathcal{P}'_x| > |\mathcal{P}_x|/2$ such that for every $\{u, v\} \in \mathcal{P}'_x$ there exist distinct (also from other such vertices) $w_1, w_2 \in W \smallsetminus (X \cup Y \cup \{x\})$ which close an edge in G with u. Let $\mathcal{P}'_x$ be the largest such set and let Q denote the union of $X \cup Y \cup \{x\}$ and all such w_1, w_2 (the ones that $u \in \mathcal{P}'_x$ closes an edge with). Then, with X' denoting the vertices in $\mathcal{P}'_x$, we have

$$e_G(X \smallsetminus X', Q, W) \geqslant |X \smallsetminus X'|\gamma n^2p/3.$$

On the other hand, by the property of Lemma 2.2 *(ii)*, assuming $|X \smallsetminus X'| \geqslant \sqrt{n} \geqslant \varepsilon^{-3}\log n/(np)$ and as $|Q| = 6\xi n$ (taking a superset if necessary), we have

$$e_G(X \smallsetminus X', Q, W) \leqslant e_G(X \smallsetminus X', Q, V(G)) \leqslant (1+\varepsilon)|X \smallsetminus X'|6\xi n^2p,$$

leading to a contradiction for ξ small enough. We just need to show that there is $\{u, v\} \in \mathcal{P}'_x$ and $w_3, w_4 \in W \smallsetminus Q$ which comprise an edge in G with v. Indeed, exactly the same computation as above establishes this, which completes the proof. $\square$

2.3 Degree inheritance properties

We first state a slightly strengthened version of [20, Lemma 5.5]. The strengthening comes in the bound on p, which is stated to match the one from Theorem 1.1 (and in fact, most of the statements in this paper). For the case $d = 2$ we require $p = \Theta(\log n/n)$, in contrast to $\Omega(n^{-2}\log^3 n)$, and here the desired property is in fact easily obtained through Chernoff's inequality and the union bound due to independence. In case $d = 1$ the former requirement of $p \geqslant n^{-2}\log^3 n$ is actually necessary for this particular proof.

Lemma 2.6. *For every $\gamma, \mu, \xi > 0$, there is a C such that w.h.p. $\mathcal{H}^3(n, p)$ satisfies the following, provided that $p \geqslant C\log n \cdot \max\{n^{-3/2}, n^{-3+d}\}$. Let $G \subseteq \mathcal{H}^3(n, p)$ and let $W \subseteq V(G)$ be a uniformly random set of size at least ξn. Then w.h.p. every d-set $S \subseteq V(G)$ that satisfies $\deg_G(S) \geqslant (\mu + \gamma)p\binom{|V(G)|-d}{3-d}$ also satisfies $\deg_G(S, W) \geqslant (\mu + \gamma/2)p\binom{|W|-d}{3-d}$.*

Proof. For $d = 2$, the codegree of a pair of vertices into W follows a hypergeometric distribution with mean $\Omega(np)$, for which Chernoff's inequality applies (see, e.g. [32, Theorem 2.10]). The statement thus follows directly from it and the union bound over all pairs of vertices. In case $d = 1$ the assertion holds by [20, Lemma 5.5], since $p \geqslant Cn^{-3/2}\log n \geqslant n^{-2}\log^3 n$. $\square$

The next lemma allows us to start with a graph in which almost all d-sets have at least some degree and pick a random subgraph of it such that in it, with positive probability, *all* d-sets have at least some (slightly smaller) degree.

Lemma 2.7 ([20, Lemma 3.4])**.** *There is a $c > 0$ such that the following holds. Let $d \in \{1, 2\}$ and $\gamma, \delta, \mu > 0$. Let G be an n-vertex 3-graph in which all but $\delta\binom{n}{d}$ of the d-sets have degree at least $(\mu + \gamma)\binom{n-d}{3-d}$. Let S be a uniformly random subset of $s \geqslant 2d$ vertices of G. Then with probability at least $1 - \binom{s}{d}(\delta + e^{-c\gamma^2 s})$, the random induced subgraph $G[S]$ has minimum d-degree at least $(\mu + \gamma/2)\binom{s-d}{3-d}$.*

The last two lemmas establish that in a subgraph of the random hypergraph $\mathcal{H}^3(n, p)$ with sufficiently large minimum degree, after an adversary removes λn vertices, for some tiny $\lambda > 0$, almost all d-sets still keep a significant portion of their original degree in the resulting graph.

Lemma 2.8. *For every $\gamma, \mu > 0$ there exist $\lambda, C > 0$ such that w.h.p. $\mathcal{H}^3(n, p)$ satisfies the following, provided that $p \geqslant Cn^{-3/2}\log n$. Let $G \subseteq \mathcal{H}^3(n, p)$ with $\delta_1(G) \geqslant (\mu + \gamma)p\binom{n-1}{2}$ and let $S \subseteq V(G)$ be of size $|S| \leqslant \lambda n$. Then there exists a set T of size at most $\sqrt{n}$ such that for $U := V(G) \smallsetminus (S \cup T)$, the graph $G[U]$ is of minimum degree at least $(\mu + \gamma/4)p\binom{|U|-1}{2}$. Furthermore, if for some $x, y \in V(G) \smallsetminus S$ we have $\deg_G(x, V(G) \smallsetminus S), \deg_G(y, V(G) \smallsetminus S) \geqslant (\mu + \gamma/2)p\binom{n-1}{2}$, then T can be chosen to avoid x, y.*

Proof. Initially, let $T := \varnothing$. As long as there exists $v \in V(G) \smallsetminus (S \cup T)$ and $v \neq x, y$ with

$$\deg_G(v, V(G) \smallsetminus (S \cup T)) \leqslant (\mu + \gamma/2)p\binom{n-1}{2},$$

add such a vertex to T. Stop this process at the first point in time when $|T| = \varepsilon^{-3}\log n/(np)$, for some small enough $\varepsilon > 0$. Note that by the assumption on p, it also holds that $|T| < \sqrt{n}$. We then have

$$e_G(T, S \cup T, V(G)) \geqslant |T| \cdot (\gamma/2)p\binom{n-1}{2} \geqslant (\gamma/8)|T|n^2p.$$

On the other hand, as $\mathcal{H}^3(n,p)$ w.h.p. satisfies the conclusion of Lemma 2.2 *(ii)*,

$$e_G(T, S \cup T, V(G)) \leqslant (1+\varepsilon)|T||S \cup T|np \leqslant (1+\varepsilon)2\lambda|T|n^2p,$$

which is a contradiction with the former, for $\lambda > 0$ small enough.

Consider now x and its degree into $V(G) \smallsetminus (S \cup T)$. If it does not satisfy the bound promised by the lemma, this means

$$\deg_G(x, T, V(G) \smallsetminus S) \geqslant (\gamma/4)p\binom{n-1}{2} \geqslant (\gamma/16)n^2p.$$

On the other hand, by the property of Lemma 2.2 *(i)*,

$$e_G(x, T, V(G) \smallsetminus S) \leqslant e_G(x, T, V(G)) \leqslant \varepsilon^{-4}|T|\log n \leqslant \varepsilon^{-7}\log^2 n/(np).$$

This is a contradiction with the former for $C > 0$ large enough as $n^3p^2 \geqslant C^2\log^2 n$ by the assumption on p. □

Lemma 2.9. *For every $\gamma, \mu > 0$, there exist $\lambda, C > 0$ such that for every $p \geqslant C\log n/n$ w.h.p. $\mathcal{H}^3(n,p)$ satisfies the following. Let $G \subseteq \mathcal{H}^3(n,p)$ with $\delta_2(G) \geqslant (\mu+\gamma)p(n-2)$. Then, for every $S \subseteq V(G)$ with $|S| \leqslant \lambda n$, all but at most $10^9\log n/p$ pairs of vertices $u, v \in V(G) \smallsetminus S$ have $\deg_{G-S}(u,v) \geqslant (\mu+\gamma/2)p(n-|S|-2)$.*

Proof. Let

$$\mathcal{P} := \big\{\{u,v\} : u, v \in V(G) \smallsetminus S, \deg_G(\{u,v\}, S) \geqslant \gamma np/4\big\}.$$

As $\mathcal{H}^3(n,p)$ w.h.p. has the property of Lemma 2.3 *(ii)* for $\varepsilon = 10^{-3}$, assuming $|\mathcal{P}| > 10^9\log n/p$ we have

$$|\mathcal{P}| \cdot \gamma np/4 \leqslant e_G(\mathcal{P}, S) \leqslant (1+\varepsilon)|\mathcal{P}|\max\{|S|p, 10^9\log n\}.$$

This leads to a contradiction for $\lambda > 0$ sufficiently small and C large enough, as $|S| \leqslant \lambda n$ and $np \gg \log n$. □

2.4 (Hyper)graph theory

The following Dirac-type conditions for the existence of a loose Hamilton cycle were mentioned in the introduction but we state them here explicitly in the form in which we use them later. Recall, $\delta_1 = 7/16$ and $\delta_2 = 1/4$.

Theorem 2.10 ([10, 47])**.** *Let $d \in \{1,2\}$ and $\gamma > 0$. Every sufficiently large 3-uniform hypergraph $\mathcal{H}$ on an even number of vertices n with $\delta_d(\mathcal{H}) \geqslant (\delta_d+\gamma)\binom{n-d}{3-d}$ contains a loose Hamilton cycle.*

The next lemma gives a minimum degree condition for a 3-graph that ensures that any pair of vertices is contained in some loose cycle of length three. We make use of it later for finding absorbers.

Lemma 2.11. *Let $\mathcal{H}$ be a sufficiently large n-vertex 3-uniform hypergraph which satisfies $\delta_1(\mathcal{H}) \geqslant \frac{7}{16}\binom{n}{2}$. Then for every pair of vertices $u, v \in V(\mathcal{H})$, there exist distinct vertices $a_1, a_2, a_3, a_4 \in V(\mathcal{H}) \setminus \{u, v\}$ such that $a_1a_2a_3, a_2a_4u, a_3a_4v \in E(\mathcal{H})$, that is $a_3a_1a_2ua_4va_3$ is a loose cycle.*

Proof. Suppose $\mathcal{H}'$ is a sufficiently large n-vertex 3-graph with $\delta_1(\mathcal{H}') \geqslant (5/8+\gamma)^2\binom{n}{2}$, for some $\gamma > 0$. A simple counting argument (see, e.g., [10, Claim 9]) shows that for every two $u, v \in V(\mathcal{H}')$, there is a set $W = W(u, v) \subseteq V(\mathcal{H}')$ of size γn, such that either:

- $\deg_{\mathcal{H}'}(u, w) \geqslant \gamma n$ and $\deg_{\mathcal{H}'}(v, w) \geqslant 3n/8$ for all $w \in W$, or
- $\deg_{\mathcal{H}'}(v, w) \geqslant \gamma n$ and $\deg_{\mathcal{H}'}(u, w) \geqslant 3n/8$ for all $w \in W$.

Since $7/16 = (5/8+\gamma)^2$ for $\gamma = \sqrt{7}/4 - 5/8 > 0$, by the discussion above there is a $W \subseteq V(\mathcal{H})$ such that, without loss of generality, $\deg_{\mathcal{H}}(u, w) \geqslant \gamma n$ and $\deg_{\mathcal{H}}(v, w) \geqslant 3n/8$ for all $w \in W$. Pick an arbitrary $a_4 \in W$ and an arbitrary $a_2 \in V(\mathcal{H}) \setminus \{v\}$ such that $ua_4a_2 \in E(\mathcal{H})$. Note that $\deg_{\mathcal{H}}(v, a_4) \geqslant 3n/8$ and $|N_{\mathcal{H}}(a_2)| \geqslant \sqrt{7}n/4$ due to the minimum degree condition for a_2. Therefore, since $\sqrt{7}/4 + 3/8 > 1$, we have $|N_{\mathcal{H}}(v, a_4) \cap N_{\mathcal{H}}(a_2)| > 2$ (with plenty of room to spare). Pick $a_3 \in N_{\mathcal{H}}(v, a_4) \cap N_{\mathcal{H}}(a_2) \setminus \{u\}$ and pick $a_1 \in N_{\mathcal{H}}(a_2, a_3) \setminus \{a_4, u, v\}$. ☐

We lastly need a Hall-type matching condition for 'bipartite' hypergraphs due to Haxell, which has been used frequently for embedding problems in random graph theory.

Theorem 2.12 (Haxell's condition [30])**.** *Let $\mathcal{H}$ be an ℓ-graph whose vertex set can be partitioned into sets A and B such that $|e \cap A| = 1$ and $|e \cap B| = \ell - 1$, for every edge $e \in E(\mathcal{H})$. Suppose that for every choice of subsets $A' \subseteq A$ and $B' \subseteq B$ such that $|B'| \leqslant (2\ell - 3)(|A'| - 1)$, there exists an edge $e \in E(\mathcal{H})$ intersecting A' but not B'. Then $\mathcal{H}$ has an A-saturating matching (i.e. a collection of disjoint edges whose union contains A).*

3 The sparse regularity method for hypergraphs

Following [20, 19] in a natural generalisation of the analogous concept for graphs, we say that, given $\varepsilon > 0$ and $p \in (0, 1)$, a 3-partite 3-graph G on sets V_1, V_2, V_3 is (ε, p)-regular if for every $X_i \subseteq V_i$, $|X_i| \geqslant \varepsilon|V_i|$, we have

$$\left|d_G(V_1, V_2, V_3) - d_G(X_1, X_2, X_3)\right| \leqslant \varepsilon p,$$

where $d_G(A, B, C) = e_G(A, B, C)/(|A||B||C|)$ stands for the density of edges of a given triple.

A partition $(V_i)_{i\in[t]}$ of the vertex set of a 3-graph G is said to be (ε, p)-*regular* if it is an equipartition and for all but at most $\varepsilon\binom{t}{3}$ triples V_i, V_j, V_k, the graph induced by them is (ε, p)-regular. For the (hyper)graph regularity lemma to be of any use, it usually needs to prevent too many edges lying within some partition class V_i. A common way to restrict this is the notion of *upper-uniformity*. We say that a 3-graph G is (η, b, p)-upper-uniform, for some $\eta \in (0, 1)$ and $b \geqslant 1$, if $d_G(V_1, V_2, V_3) \leqslant bp$ for all disjoint sets V_1, V_2, V_3 with $|V_i| \geqslant \eta|V(G)|$.

With all these concepts at hand, we state a so-called weak[6] hypergraph regularity lemma, which acts as a natural generalisation from the graph setting, can be proven in the same way (see,

[6]'Weak' comes from the fact that in this variant, the corresponding counting and embedding lemmas are not necessarily true in general—one would require a stronger concept of regularity.

e.g. [38, 41, 25] for the sparse regularity lemma and [40] for the regularity lemma in dense hypergraphs), and appears in the same form in [20, 19].

Theorem 3.1. *For every $\varepsilon > 0$ and $t_0, b \geqslant 1$ there exist $\eta > 0$ and $T \geqslant t_0$ such that for every $p \in (0, 1]$, every (η, b, p)-upper-uniform 3-graph G with at least t_0 vertices admits an (ε, p)-regular partition $(V_i)_{i\in[t]}$, where $t_0 \leqslant t \leqslant T$.*

The upper-uniformity property can be seen as a 'true property of random graphs' and indeed is exhibited by $\mathcal{H}^3(n, p)$ with high probability (e.g. established by a straightforward application of Chernoff's inequality and the union bound).

Lemma 3.2. *For every $\eta \in (0, 1)$ and $b > 1$ the random 3-graph $\mathcal{H}^3(n, p)$ is (η, b, p)-upper-uniform with probability at least $1 - e^{-\omega(n \log n)}$, provided that $p = \omega(n^{-2} \log n)$.*

Given an equipartition $(V_i)_{i\in[t]}$ of the vertex set of a 3-graph G, we define the reduced graph $\mathcal{R} = \mathcal{R}((V_i)_{i\in[t]}, \varepsilon, p, \alpha)$ on vertex set $\{1, \dots, t\}$ corresponding to the sets V_i, whose edges are all 3-element sets of indices $\{i, j, k\}$ such that $d_G(V_i, V_j, V_k) \geqslant \alpha p$ and $G[V_i, V_j, V_k]$ is (ε, p)-regular.

In fact, in the regularity lemma, one can even roughly define where the clusters V_i lie in the graph G. Namely, given a partition $V(G) = P_1 \cup \dots \cup P_h$, the (ε, p)-regular partition resulting from Theorem 3.1 can be made such that all but at most εht clusters V_i each completely belong to one P_j (may be distinct for different V_i). This comes in handy when it comes to finding absorbers.

The property that we use most frequently is that the reduced graph in a way inherits degree properties from its underlying graph G. This is nothing fancy and should come as no surprise to anyone familiar with the (graph) regularity method. Again, very conveniently, one can make it such that degrees are 'controlled' within certain predetermined sets, and not only globally in the whole graph $\mathcal{R}$. The following statement is almost a one-to-one copy of [20, Lemma 4.7] (slightly paraphrased for convenience).

Lemma 3.3 ([20, Lemma 4.7])**.** *For all $h, t_0 \in \mathbb{N}$ and $\varepsilon, \delta, \lambda > 0$, there exist $\eta, b, T > 0$ such that the following holds. Let $p \in [0, 1]$ and let G be a sufficiently large n-vertex (η, b, p)-upper-uniform 3-graph. Let $n_1, \dots, n_h \geqslant \lambda n$ and $P_1, \dots, P_h$ be a partition of $V(G)$ with $|P_i| = n_i$. Then there exists an (ε, p)-regular partition $(V_i)_{i\in[t]}$ of $V(G)$, for some $t \in [t_0, T]$ and a corresponding reduced t-vertex 3-graph $\mathcal{R} = \mathcal{R}((V_i)_{i\in[t]}, \varepsilon, p, 2\varepsilon)$ with the following property. Let $\mathcal{P}_i$ be the set of clusters V_j contained entirely in P_i and let $t_i = |\mathcal{P}_i|$. Then:*

(i) $t_i \geqslant (\frac{n_i}{n} - \varepsilon h)t$ *for every* $i \in [h]$.

(ii) *Let $d \in \{1, 2\}$. Suppose that for some $i, j \in [h]$ all but at most $o(n^d)$ of the d-sets $S \subseteq P_i$ satisfy*

$$\deg_G(S, P_j) \geqslant \delta \binom{n_j - d}{3 - d} p.$$

Then all but at most $\sqrt{\varepsilon}\binom{t}{d}$ of the d-sets $\mathcal{S} \subseteq \mathcal{P}_i$ satisfy

$$\deg_{\mathcal{R}}(\mathcal{S}, \mathcal{P}_j) \geqslant (\delta - (h + 2)\varepsilon - \sqrt{\varepsilon} - 3/t_0)\binom{t_j - d}{3 - d}.$$

We remark that, if one wants to only inherit minimum degree (that is, $d = 1$), then standard double counting methods (see e.g. [57]) show that this can be done without having the $\sqrt{\varepsilon}t$ error term. Namely, actually *all vertices* satisfy the corresponding degree assumption. For $d \geqslant 2$, the 'almost all' is necessary.

Lastly, we use a hypergraph version of the infamous KŁR conjecture[7] whose proof for *linear hypergraphs* was explicitly spelled out in [19] but already observed to hold in the work of Conlon, Gowers, Samotij, and Schacht [12]. For a 3-graph H on vertex set $\{1, \dots, t\}$ we denote by $\mathcal{G}(H, n, m, p, \varepsilon)$ the class of graphs G obtained in the following way. The vertex set of G is a disjoint union $V_1 \cup \cdots \cup V_t$ of sets of size n. For each edge $ijk \in E(H)$ we add to G an (ε, p)-regular 3-graph with m edges between the triple (V_i, V_j, V_k) (and these are the only edges of G). A *canonical copy* of H in G is a t-tuple $(v_1, \dots, v_t)$ with $v_i \in V_i$ for every $i \in V(H)$ and $v_i v_j v_k \in E(G)$ for every $ijk \in E(H)$. We write $G(H)$ for the number of canonical copies of H in G. Lastly, we need the notion of 3-density $m_3(H)$ of a 3-graph H, which is defined as

$$m_3(H) = \max\left\{\frac{e(H') - 1}{v(H') - 3} : H' \subseteq H \text{ with } v(H') \geqslant 4\right\}.$$

Theorem 3.4. *For every linear* 3*-graph* H *and every* $\alpha > 0$*, there exist* $\varepsilon, \xi > 0$ *with the following property. For every* $\eta > 0$*, there is a* $C > 0$ *such that if* $p \geqslant CN^{-1/m_3(H)}$*, then with probability* $1 - e^{-\Omega(N^3 p)}$ *the following holds in* $\mathcal{H}^3(N, p)$*. For every* $n \geqslant \eta N$*,* $m \geqslant \alpha p n^3$*, and every subgraph* G *of* $\mathcal{H}^3(N, p)$ *in* $\mathcal{G}(H, n, m, p, \varepsilon)$*, we have* $G(H) \geqslant \xi n^{v(H)} p^{e(H)}$*.*

Strictly speaking, the subgraph G that we later apply Theorem 3.4 to is not a member of $\mathcal{G}(H, n, m, p, \varepsilon)$, in particular not all (ε, p)-regular 3-graphs have exactly m edges—the number of edges across these are within a constant factor of each other. However, subsampling (see, e.g. [25, Lemma 4.3]) circumvents this. For clarity of presentation, we prefer to not explicitly spell out this argument.

4 Covering random hypergraphs by loose paths

In this section we show a vital part of every strategy relying on the absorbing method which in our specific problem reads as: the majority of vertices of $G \subseteq \mathcal{H}^3(n, p)$ can be covered by a few (loose) paths. For the purposes of this lemma, we consider single vertices to be loose *paths of length zero.*

Lemma 4.1. *Let* $d \in \{1, 2\}$*. For every* $\gamma, \varrho > 0$*, there exists a* $C > 0$ *such that w.h.p.* $\mathcal{H}^3(n, p)$ *has the following property, provided that* $p \geqslant C \log n \cdot \max\{n^{-3/2}, n^{-3+d}\}$*. Let* $G \subseteq \mathcal{H}^3(n, p)$ *with* $v(G) \geqslant n/2$ *in which all but* $o(n^d)$ d*-sets* $S \subseteq V(G)$ *satisfy* $\deg_G(S) \geqslant (\delta_d + \gamma)p\binom{n-d}{3-d}$*. Then there exist at most* ϱn *vertex-disjoint loose paths that cover the vertex set of* G*.*

To a reader familiar with the topic, there is no magic that happens here. We apply the sparse regularity lemma to G, find a desirable structure in the obtained reduced graph, and then use it as a guide to construct loose paths in G itself. Perhaps the most innovative thing comes in the part where, in an (ε, p)-regular triple (V_1, V_2, V_3), we find a long loose path covering all but $o(|V_i|)$ vertices in each V_i. This itself relies on a widely-used *Depth-First Search* (DFS) technique employed explicitly in the context of graphs in [5, 6]. (For an extremely neat application of the method and more in-depth discussion see [45].)

Lemma 4.2. *Let* $\mathcal{H} = (V_1, V_2, V_3; E)$ *be a* 3*-partite* 3*-graph with* $|V_1| = |V_3| = t$ *and* $|V_2| = 2t$ *such that for every choice* $X_i \subseteq V_i$ *of size* $|X_i| = k$*, there exists an edge in* $\mathcal{H}[X_1 \cup X_2 \cup X_3]$*.*

[7]still known by this name, but has since its introduction [39] been proven [4, 62]; for a very recent, short, and self-contained proof see [55].

Then there is a loose uv-path of length $2t-4k$ in $\mathcal{H}$ with $u, v \in V_1 \cup V_3$, and whose every other degree-one vertex lies in V_2.

Proof. We explore the graph $\mathcal{H}$ by using a variant of the *Depth-First Search* (DFS) procedure as follows. To start with, set $T = \varnothing$, $S = \varnothing$, and $U = V(\mathcal{H})$. For as long as $U \cap (V_1 \cup V_3) \neq \varnothing$, do:

- if $S = \varnothing$, pick an arbitrary vertex from $U \cap (V_1 \cup V_3)$ and add it to S;
- if $S \neq \varnothing$ and there is an edge $uvw \in \mathcal{H}$ such that $u \in V_i$ is the last added vertex to S, and $v \in U \cap V_2$ and $w \in U \cap V_{4-i}$, move v, w from U to S in that order;
- otherwise, for $uvw \in \mathcal{H}$ being the last three vertices added to S, move v, w from S to T (if there is no such edge just move the only vertex u in S to T).

Observe that the vertices in S at all times span a loose path as wanted in the lemma, but maybe not of the required length, and, crucially, there is never a time when some edge $uvw \in \mathcal{H}$ is such that $u \in T \cap V_i$, $v \in U \cap V_2$, and $w \in U \cap V_{4-i}$, for $i \in \{1, 3\}$. We aim to show that at some point we have $|S \cap V_2| \geqslant 2t - 4k + 1$ which is sufficient for the lemma to hold.

In every step of the procedure either one (if $S = \varnothing$, say) or two vertices get moved from U to S or from S to T. Consider the first moment in time when $|T \cap V_i| = k$, for some $i \in \{1, 3\}$; we may safely assume this happens for V_1. So, $|T \cap V_3| < k$, and moreover by the description of the procedure above, at this point we necessarily have $|T \cap V_2| < 2k$.

As $|T \cap V_1| = k$, it cannot be that $|U \cap V_2|, |U \cap V_3| \geqslant k$, by the property of the lemma that such three sets must contain an edge. Assuming $|U \cap V_2| < k$, we have $|S \cap V_2| \geqslant 2t - 3k + 2$ as desired.

Otherwise, suppose $|U \cap V_2| \geqslant k$ and $|U \cap V_3| < k$. From the fact that $|S \cap V_1| = |S \cap V_3| \pm 1$ we conclude that the intersections of V_1 and V_3 with $T \cup U$ are the same (up to ± 1). This further implies $|U \cap V_1| < k$ (recall, $|T \cap V_3| < |T \cap V_1| = k$). Putting things together, we have $|S \cap (V_1 \cup V_3)| \geqslant 2t - 4k + 3$, and hence again $|S \cap V_2| \geqslant 2t - 4k + 2$ as desired. $\square$

As a reminder, the exact values of δ_d are known for 3-uniform hypergraphs to be $\delta_1 = 7/16$ and $\delta_2 = 1/4$.

Proof of Lemma 4.1. Take $s \in \mathbb{N}$ sufficiently large and ε sufficiently small, in particular so that $\varepsilon < \min\{\gamma^2/64, \varrho/16\}$ and $\Lambda := \binom{s}{d}(\sqrt{\varepsilon} + e^{-\Omega(\gamma^2 s)})$ is small enough, namely $\Lambda \leqslant \varrho/100$. Pick t_0 large enough so that $t_0 \geqslant s/\Lambda$. Let $\eta = \eta_{3.3}(\varepsilon, \delta_d + \gamma, t_0)$, $b = b_{3.3}(\varepsilon, \delta_d + \gamma, t_0)$, and $T = T_{3.3}(\varepsilon, \delta_d + \gamma, t_0)$.

Since $p = \omega(n^{-2} \log n)$, by Lemma 3.2 $\mathcal{H}^3(n, p)$ is w.h.p. $(\eta/2, b, p)$-upper-uniform, and so G is (η, b, p)-upper-uniform then. Therefore, we can apply Lemma 3.3 to G with $\delta_d + \gamma$ (as δ), to obtain an (ε, p)-regular partition $(V_i)_{i \in [t]}$ for some $t_0 \leqslant t \leqslant T$ and a corresponding reduced graph $\mathcal{R} = \mathcal{R}((V_i)_{i \in [t]}, \varepsilon, p, 2\varepsilon)$. In particular, we get that for all but at most $\sqrt{\varepsilon}\binom{t}{d}$ d-sets $\mathcal{S} \subseteq V(\mathcal{R})$, it holds that $\deg_{\mathcal{R}}(\mathcal{S}) \geqslant (\delta_d + \gamma/2)\binom{t-d}{3-d}$. Let us remove from each V_i at most two vertices to get that they are all of even size $\lfloor n/t \rfloor - 1 \leqslant m \leqslant \lceil n/t \rceil$ (with slight abuse of notation we still refer to them as V_i).

Let $R_1, \dots, R_{t/s}$ be a partition of $V(\mathcal{R})$ into disjoint subsets of size s, chosen uniformly at random. Recall that $\Lambda := \binom{s}{d}(\sqrt{\varepsilon} + e^{-\Omega(\gamma^2 s)})$. By Lemma 2.7, with positive probability all but a Λ-fraction of the subgraphs $\mathcal{R}_i := \mathcal{R}[R_i]$ have minimum d-degree at least $(\delta_d + \gamma/4)\binom{s-d}{3-d}$. As we have picked s to be even and large, each $\mathcal{R}_i$ with this minimum degree has a loose Hamilton cycle by

Theorem 2.10. The vertices not covered by loose cycles in $\mathcal{R}$ are then at most $\Lambda t + s$. We next show how to cover each loose cycle in $\mathcal{R}$ with not too many loose paths in G.

Consider a (loose) Hamilton cycle in some $\mathcal{R}_i$ and suppose without loss of generality that the clusters corresponding to the vertices of $\mathcal{R}_i$ are $V_1, \ldots, V_s$ in the order in which they appear on the cycle. Then we know that (V_i, V_{i+1}, V_{i+2}) is (ε, p)-regular with density at least $2\varepsilon p$, for every $i \in [s-1]$, i odd ($s+1$ is identified with 1). Split every V_i, for i odd, arbitrarily into $V_i = V_i^1 \cup V_i^2$, each of size $m/2$. By the definition of a regular triple, for all $X_i \subseteq V_i^1$, $X_{i+1} \subseteq V_{i+1}$, and $X_{i+2} \subseteq V_{i+2}^2$, each of size εm, we have

$$d_G(X_i, X_{i+1}, X_{i+2}) \geqslant d_G(V_i, V_{i+1}, V_{i+2}) - \varepsilon p \geqslant \varepsilon p > 0,$$

and in particular $e_G(X_i, X_{i+1}, X_{i+2}) > 0$. Lemma 4.2 implies there is a loose uv-path in $G[V_i^1, V_{i+1}, V_{i+2}^2]$ with $u, v \in V_i^1 \cup V_{i+2}^2$, which is of length $2 \cdot m/2 - 4\varepsilon m = (1-4\varepsilon)m$ and thus uses all but at most $8\varepsilon m$ vertices in $V_i^1 \cup V_{i+1} \cup V_{i+2}^2$.

Repeating this for every odd $i \in [s]$, we get $s/2$ loose paths covering all but at most $8\varepsilon m \cdot s/2$ vertices in $V_1, \ldots, V_s$. The whole thing can independently be repeated for every $\mathcal{R}_i$ as well. In total, and counting vertices as paths of length zero, we found at most

$$t/s \cdot s/2 \cdot (1 + 8\varepsilon m) + (\Lambda t + s) \cdot m + 2t \leqslant t \cdot 8\varepsilon n/t + 2\Lambda n + o(n/t) \leqslant \varrho n$$

loose paths that cover the vertex set of G. □

5 Connecting Lemma

In this section we prove a vital ingredient both for constructing absorbers and independently as a part of the proof of Theorem 1.1. Roughly speaking, we show that one can connect prescribed pairs of vertices with short paths through a set of vertices under some degree assumptions. Given a set $W \subseteq V(G)$ in a graph G, an integer ℓ, and a set of pairs $\{x_i, y_i\}_i$ from $V(G) \setminus W$, a $(\{x_i, y_i\}_i, W, \ell)$-*matching* is a collection of internally vertex-disjoint paths P_i, where each P_i is of length at most ℓ, has x_i, y_i as endpoints, and its remaining vertices belong to W.

Lemma 5.1 (Connecting Lemma). *Let $d \in \{1, 2\}$. For every $\gamma, \xi > 0$ there exist $\varepsilon, C > 0$ such that w.h.p. $\mathcal{H}^3(n,p)$ satisfies the following, provided $p \geqslant C \log n \cdot \max\{n^{-3/2}, n^{-3+d}\}$. Let $G \subseteq \mathcal{H}^3(n,p)$ and $U, W \subseteq V(G)$ be disjoint subsets such that*

- *$|W| \geqslant \xi n$,*
- *all d-sets $S \subseteq U$, $S \subseteq W$ have $\deg_G(S, W) \geqslant (\delta_d + \gamma)p\binom{|W|-d}{3-d}$.*

Then, for every family of distinct pairs $\{x_i, y_i\}_{i \in [t]}$ in U such that $t \leqslant \varepsilon |W|$ and every $u \in U$ appears in at most two pairs, there exists a $(\{x_i, y_i\}_i, W, 4)$-matching in G.

Proof. Given ξ and γ, let $\tilde{\gamma}, \varepsilon > 0$ be sufficiently small and $C > 0$ sufficiently large for the arguments below to go through. Condition on $\mathcal{H}^3(n,p)$ having the properties of Lemma 2.2, Lemma 2.3, Lemma 2.4, Lemma 2.5 for $\tilde{\gamma}$ (as γ), and Lemma 2.6, which happens with high probability.

Consider an auxiliary hypergraph $\mathcal{H}$ (an 8-graph in case $d = 1$ and a 2-graph in case $d = 2$) with vertex set $[t] \cup W$ in which for $Z \subseteq W$, with $|Z| = 7$ if $d = 1$ and $|Z| = 1$ if $d = 2$, an edge $\{i\} \cup Z$ exists if and only if there is a $x_i y_i$-path in G all whose internal vertices belong to

Z. Hence, a $[t]$-saturating matching in $\mathcal{H}$ corresponds to a $(\{x_i, y_i\}_i, W, 4)$-matching in G. Our plan is to use Haxell's matching theorem (Theorem 2.12) to show this graph contains such a matching. It is sufficient to, for every $\mathcal{I} \subseteq [t]$ and $Q \subseteq W$ of size $|Q| \leqslant 16|\mathcal{I}|$, find an $i \in \mathcal{I}$ and an $x_i y_i$-path in G whose internal vertices all belong to $W \smallsetminus Q$. Indeed, this implies there is an edge in $\mathcal{H}$ that intersects $\mathcal{I}$ but does not intersect Q, and so the condition in Theorem 2.12 is satisfied. We treat the cases $d = 1$ and $d = 2$ separately, starting with $d = 2$.

Codegree, $d = 2$. Recall, $\delta_2 = 1/4$. Consider any $\mathcal{I} \subseteq [t]$ and $Q \subseteq W$ of size $|Q| \leqslant 16|\mathcal{I}|$. Let $\mathcal{P}$ be a maximal set of pairs $\{x_i, y_i\}$ with $i \in \mathcal{I}$ and all vertices distinct and assume towards a contradiction there is no $x_i y_i$-path of length at most four in G with all internal vertices belonging to $W \smallsetminus Q$. By the assumption of the lemma we have $|\mathcal{P}| \geqslant |\mathcal{I}|/3$. If $|\mathcal{P}| \leqslant \varepsilon^{-3} \log n/p$, using on the one hand the minimum codegree assumption and on the other the property of Lemma 2.3 *(i)*, we obtain

$$|\mathcal{P}| \cdot (1/4 + \gamma)p(|W| - 2) \leqslant e_G(\mathcal{P}, Q) \leqslant 48\varepsilon^{-4}|\mathcal{P}| \log n,$$

which is a contradiction for C large enough, as $|W|p/8 \geqslant (\xi/8) \cdot C \log n \geqslant 48\varepsilon^{-4} \log n$. Otherwise, if $|\mathcal{P}| \geqslant \varepsilon^{-3} \log n/p$, then by the property of Lemma 2.3 *(ii)*,

$$|\mathcal{P}| \cdot (1/4 + \gamma)p(|W| - 2) \leqslant e_G(\mathcal{P}, Q) \leqslant (1 + \varepsilon)48|\mathcal{P}|^2 p,$$

by taking a superset of Q of size $16|\mathcal{I}| \leqslant 48|\mathcal{P}|$, leading to a contradiction as $|W|/8 \geqslant t/(8\varepsilon) > 96|\mathcal{P}|$.

Degree, $d = 1$. Recall, $\delta_1 = 7/16$. Let $W = W_1 \cup W_2 \cup W_3$ be an equipartition of W in which each $v \in U \cup W$, for $\tilde{s} = |W|/3$, satisfies

$$\deg_G(v, W_i) \geqslant (7/16 + \gamma/2)p\binom{\tilde{s} - 1}{2} \qquad (1)$$

for all i. From the assumptions of the lemma, it is straightforward to show that such a partition exists by making use of the property of Lemma 2.6 and the union bound.

Consider any $\mathcal{I} \subseteq [t]$ and $Q \subseteq W$ of size $|Q| \leqslant 16|\mathcal{I}|$. Assume that $\mathcal{I}$ contains indices i so that the collection $\{x_i, y_i\}$ consists of distinct vertices (a maximal subset of such i comprises at least a third of the original $\mathcal{I}$ which has no influence on the proof). The $x_i y_i$-path we are trying to find is going to be such that it intersects each W_1, W_3 in exactly two vertices, and W_2 in exactly three vertices.

Let $\tilde{W}_i := W_i \smallsetminus Q$. It is sufficient to show that there is an index $i \in \mathcal{I}$ for which:

- **(B1)** There exist $S_1 \subseteq N_G(x_i, \tilde{W}_1)$ and $S_3 \subseteq N_G(y_i, \tilde{W}_3)$, both of size $\sqrt{n}$;
- **(B2)** Let G_1 and G_3 be auxiliary 2-graphs on the same vertex set $\tilde{W}_2$ and $uv \in E(G_i)$ if $\{u, v\}$ form an edge in G with a vertex from S_i, $i \in \{1, 3\}$. Then for each $i \in \{1, 3\}$

$$\big|\{v \in \tilde{W}_2 : \deg_{G_i}(v) \geqslant 2\}\big| \geqslant \frac{\sqrt{7}}{4}\tilde{s}.$$

Before proving these statements, let us show how to complete the proof. If we were to find three vertices $u, v, w \in \tilde{W}_2$ so that $e_G(S_1, u, v) > 0$ and $e_G(S_3, w, v) > 0$, this would close a $x_i y_i$-path as desired. A path of length two in $G_1 \cup G_3$ with one edge in each of G_1, G_3 corresponds exactly to a triple $u, v, w \in \tilde{W}_2$ as above. By (B2) and the pigeonhole principle, there must be a $v \in \tilde{W}_2$

with $\deg_{G_1}(v), \deg_{G_3}(v) \geqslant 2$. This implies we can choose u as one of v's neighbours in G_1 and w as one of v's neighbours in G_3 such that $u \neq w$.

We first show there is an $i \in \mathcal{I}$ for which (B1) holds. For this, it is sufficient to show that more than half of the indices $i \in \mathcal{I}$ are such that $\deg_G(x_i, \tilde{W}_1) \geqslant \tilde{\gamma} p\binom{n-1}{2}$. The existence of $i \in \mathcal{I}$ and the corresponding sets S_1 and S_3 as in (B1) follows from the pigeonhole principle and the property of Lemma 2.5.

Let X be a set of vertices x_i, $i \in \mathcal{I}$, with $\deg_G(x_i, \tilde{W}_1) < \tilde{\gamma} p\binom{n-1}{2}$ and suppose $|X| \geqslant |\mathcal{I}|/2$. Owing to the assumption on the minimum degree (1) this means

$$e_G(X, Q, V(G)) \geqslant |X|\Big((7/16 + \gamma/2)p\binom{\tilde{s}-1}{2} - \tilde{\gamma} p\binom{n-1}{2}\Big) \geqslant 0.001\xi^2 |\mathcal{I}| n^2 p. \tag{2}$$

If $|Q|, |X| \leqslant \varepsilon^{-3} \log n/(np)$, then the property of Lemma 2.2 *(i)* implies

$$e_G(X, Q, V(G)) \leqslant \varepsilon^{-4} \max\{|X|, |Q|\} \log n = o(|\mathcal{I}| n^2 p).$$

Otherwise, if $\max\{|Q|, |X|\} \geqslant \varepsilon^{-3} \log n/(np)$ then by the property of Lemma 2.2 *(ii)*

$$e_G(X, Q, V(G)) \leqslant 2 \max\{|X||Q|np, 48\varepsilon^{-3} |\mathcal{I}| \log n\},$$

where we take a superset of X or Q of size exactly $\varepsilon^{-3} \log n/(np)$ if necessary. This again leads to a contradiction with (2) since $|X||Q| \leqslant 48\varepsilon|\mathcal{I}|n$ and by our choice of ε (with room to spare).

For (B2), let

$$\mathcal{P} := E(G_1), \quad S_2 := \{u \in \tilde{W}_2 : \deg_{G_1}(u) \geqslant 2\}, \quad \text{and} \quad \mathcal{P}_1 := \big\{\{u, v\} \in \mathcal{P} : u \notin S_2\big\}.$$

Assume for contradiction $|S_2| < \sqrt{7}\tilde{s}/4$. Note that, by the property of Lemma 2.4 and taking a superset of Q of size $48\varepsilon n$ if necessary,

$$e_G(S_1, Q, V(G)) \leqslant 48(1+\varepsilon)|S_1|\varepsilon n^2 p. \tag{3}$$

By (1) and for ε sufficiently small, we thus get

$$e_G(S_1, \mathcal{P}) \overset{(1)}{\geqslant} |S_1|(7/16 + \gamma/2)p\binom{\tilde{s}-1}{2} - e_G(S_1, Q, V(G)) \overset{(3)}{\geqslant} |S_1|(7/16 + \gamma/4)p\binom{\tilde{s}}{2}. \tag{4}$$

On the other hand, noting that $|\mathcal{P}_1| \leqslant n$, by the property of Lemma 2.3 and taking supersets of $\mathcal{P}_1$ and S_1 if necessary,

$$e_G(S_1, \mathcal{P}_1) \leqslant \max\{\varepsilon^{-4}|\mathcal{P}_1| \log n, (1+\varepsilon)n^2 p\} = o(|S_1| p \tilde{s}^2)$$

and so by the property of Lemma 2.4 again

$$e_G(S_1, \mathcal{P}) = e_G\Big(S_1, \binom{S_2}{2}\Big) + e_G(S_1, \mathcal{P}_1) \leqslant (1+\varepsilon)|S_1|p\binom{\sqrt{7}\tilde{s}/4}{2} + o(|S_1|p\tilde{s}^2),$$

which is smaller than $|S_1|(7/16 + \gamma/4)p\binom{\tilde{s}}{2}$ by our choice of ε, thereby contradicting (4). $\square$

6 The absorbing method in sparse hypergraphs

The absorbing method was initially explicitly introduced by Rödl, Ruciński, and Szemerédi [61] even though the implicit idea has its roots in the works of Krivelevich [44] and Erdős, Gyárfás, and Pyber [16]. Recently it has seen a surge of interest and has been used in a variety of settings: combinatorial designs [27, 35], decompositions [26, 48], Steiner systems [50, 19], Ramsey theory [9, 37], colouring (hyper)graphs [54, 33], embeddings [53, 24], and many, many more.

In principle, the idea behind it is simple. It relies on reducing the problem of finding a spanning subgraph $\mathcal{S}$ in some graph G to the one of finding an almost spanning subgraph $\mathcal{S}' \subseteq \mathcal{S}$, say, one of size $(1-o(1))|V(G)|$. Often times, the latter is significantly easier to solve, if nothing else, just for the fact that we have quite a bit of room for error. In practice, the implementation of this idea typically depends on first embedding a highly structured graph A into G, which is capable of extending any partial embedding to a complete one. The task of designing the graph A with this magical property is where the whole *art of absorption* lies in. Usually, it is specific to the embedding problem at hand and is where the main difficulties arise—it can be, first, quite surprising that such a graph should even exist and, second, challenging to find it in the host graph G in a convenient way (or in any way for that matter).

When dealing with paths, the 'design' of the graph A is not too complex. We make this more rigorous.

Definition 6.1 ((a,b,R)-absorber)**.** An (a,b,R)-absorber is a graph A with $R \subseteq V(A)$, $a, b \in V(A) \smallsetminus R$, with the property that for every $R' \subseteq R$ with $|R'| < |R|/2$ such that $V(A) \smallsetminus R'$ has odd cardinality, there is a loose ab-path in A with vertex set $V(A) \smallsetminus R'$.

The next lemma handles the second step of the method: actually finding the graph A in the subgraph G of the random graph $\mathcal{H}^3(n,p)$.

Lemma 6.2 (Absorbing Lemma)**.** *Let $d \in \{1,2\}$. For every $\gamma, \xi > 0$ there exist $\alpha, C > 0$ such that w.h.p. $\mathcal{H}^3(n,p)$ has the following property, provided that $p \geqslant C \log n \cdot \max\{n^{-3/2}, n^{-3+d}\}$. Let $G \subseteq \mathcal{H}^3(n,p)$ with $\delta_d(G) \geqslant (\delta_d + \gamma)p\binom{n-d}{3-d}$. Then, for a uniform random set $R \subseteq V(G)$ of size αn, w.h.p. there exists an (a,b,R)-absorber in G of order at most ξn.*

It may seem awkward that there are *two probabilistic statements* in the lemma. Indeed, the *first w.h.p.* establishes some typical properties a random 3-uniform hypergraph has (such as, e.g., distribution of the edges), whereas the *second w.h.p.* is over the choice of R. Namely, once we condition on $\mathcal{H}^3(n,p)$ having these 'nice' properties, then for a randomly chosen set there is an absorber with high probability. In fact, the lemma could be written in a quantitative form, saying how w.h.p. in $\mathcal{H}^3(n,p)$ there are at least, say, a $(1-n^{-5})$-fraction of choices for R which yield an absorber. We found this a bit more cumbersome to deal with and decided to go with the former.

As per usual, the 'construction' of such a graph A consists of carefully patching up many small structures.

Definition 6.3 (xy-absorber)**.** An *xy-absorber* is a graph that consists of:

- a path P, which we refer to as the *covering path*, and
- a path Q, which we refer to as the *non-covering path*, with $V(Q) = V(P) \smallsetminus \{x, y\}$ and whose endpoints are identical with those of P.

We now define the xy-absorber we use, depending on d (in doing this we draw inspiration from [10]). Both of these, however, have a much more natural visual representation depicted in Figure 2 and Figure 1.

Definition 6.4. The xy-absorber A^d_{xy} is defined as:

(d = 2) consists of nine vertices $x, y, v_1, v_2, v_3, v_4, v_5, v_6, v_7$, and the edges $v_1v_2v_3$, $v_3v_4v_5$, $v_5v_6v_7$, v_2xv_4, v_4yv_6.

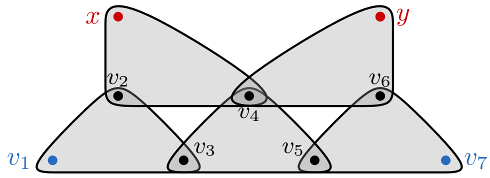
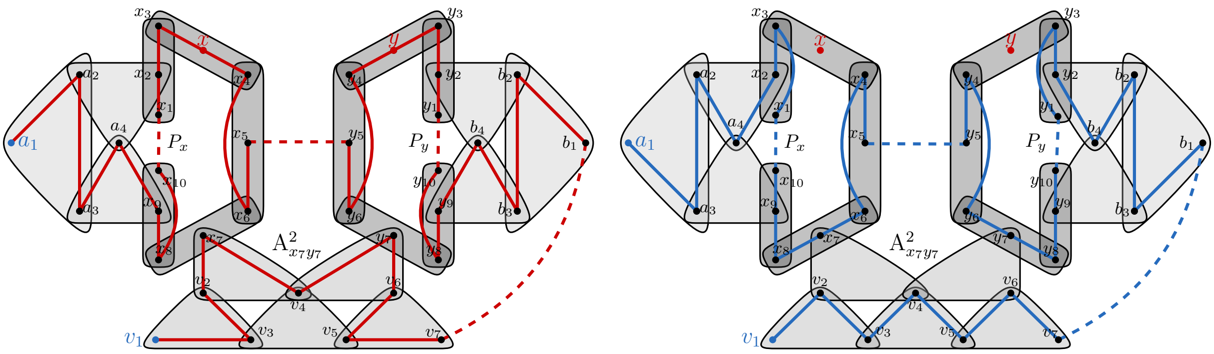


Figure 1: The xy-absorber A^2_{xy}.

(d = 1) consists of:

- a cycle $x_1, x_2, x_3, x, x_4, \dots, x_{10}, P_x, x_1$, where P_x is an x_1x_{10}-path of length four;
- a cycle $y_1, y_2, y_3, y, y_4, \dots, y_{10}, P_y, y_1$, where P_y is a y_1y_{10}-path of length four;
- a copy of $A^2_{x_7y_7}$;
- edges $a_1a_2a_3$, $a_2a_4x_2$, $a_3a_4x_9$ and $b_1b_2b_3$, $b_2b_4y_2$, $b_3b_4y_9$;
- an x_5y_5-path and a v_7b_1-path, both of length four.



(a) The covering a_1v_1-path *(b) The non-covering a_1v_1-path*

Figure 2: The xy-absorber A^1_{xy}. The dashed lines represent paths of length four.

For future reference, the graph obtained by removing P_x, P_y, the x_5y_5-path, and the v_7b_1-path is called a *backbone* of the xy-absorber A^1_{xy} (see Figure 3a below). We refer to all vertices of A^d_{xy} other than x, y as its *internal vertices.*

To build an (a, b, R)-absorber, we plan to string together a number of copies of A^d_{xy}. Clearly, we cannot just build disjoint xy-absorbers for every pair of vertices $x, y \in R$ for the simple reason of there not being enough space for that as $\binom{|R|}{2} \gg n$. A way of dealing with this originated in the work of Montgomery [53], who used an idea of looking at an auxiliary *bounded degree* graph, which serves as a template for which pairs of vertices of R to use. Roughly speaking, there is a graph $T = (R, E)$ on the vertex set R and with $\Delta(T) = O(1)$, so that if for every $xy \in E(T)$ we find disjoint xy-absorbers, then the obtained graph is an (a, b, R)-absorber, for some $a, b \notin R$.

The following lemma is very similar to [20, Lemma 7.3], but it has the additional property that the template graph it describes has bounded maximum degree. The proof, being almost identical to that of the mentioned lemma, is for completeness spelled out in the appendix.

Lemma 6.5. *There is an $L > 0$ such that the following holds. For any sufficiently large m, there exists a graph T with $2m \leqslant v(T) \leqslant Lm$, maximum degree at most L, and a set Z of m vertices, such that for every $Z' \subseteq Z$ with $|Z'| < m/2$ and $|V(T) \smallsetminus Z'|$ even, the graph $T - Z'$ has a perfect matching.*

At this point, we essentially reduced our goal to finding a single xy-absorber for a prescribed pair of vertices $x, y \in R$. This is the key lemma of this section.

Lemma 6.6. *Let $d \in \{1, 2\}$. For any $\gamma > 0$ there exist $\lambda, C > 0$, such that w.h.p. $\mathcal{H}^3(n,p)$ satisfies the following, provided that $p \geqslant C \log n \cdot \max\{n^{-3/2}, n^{-3+d}\}$. Let $G \subseteq \mathcal{H}^3(n,p)$ with $\delta_d(G) \geqslant (\delta_d + \gamma)p\binom{n-d}{3-d}$ and let $S \subseteq V(G)$ with $|S| \leqslant \lambda n$. For any $x, y \in V(G) \smallsetminus S$, that if $d = 1$ additionally satisfy $\deg_{G-S}(x), \deg_{G-S}(y) \geqslant (\delta_d + \gamma/2)p\binom{n-1}{2}$, there exists a copy of A^d_{xy} in $G - S$.*

We remark that the degree assumption for x, y is not needed in the $d = 2$ case. The proof of this lemma is the most intricate part of the argument so we defer it to Section 6.1. With all these ingredients we can now prove the absorbing lemma.

Proof of Lemma 6.2. Let D denote the number of vertices in A^d_{xy} and further set

$$L = L_{6.5}, \quad \lambda = \min\{\lambda_{2.1}(\gamma/64, 8DL), \lambda_{6.6}(\gamma/2)\}, \quad \text{and} \quad \varepsilon = \varepsilon_{5.1}(\gamma/2).$$

Having chosen these, let α be sufficiently small with respect to all constants (in particular, small with respect to the given $\xi > 0$) and C sufficiently large. Write $m := \alpha n$, let T be the auxiliary graph given by Lemma 6.5 for m, and let Z be the set of special vertices in T as in the lemma. Assume $\mathcal{H}^3(n,p)$ is such that it satisfies the conclusion of Lemma 2.1 with $\gamma/64$ (as γ) and $8DL$ (as C), Lemma 2.6, Lemma 5.1, and Lemma 6.6 with $\gamma/2$ (as γ).

Choose a uniform random set $R \subseteq V(G)$ of size αn and let $R_0 \cup V \cup W$ be a partition of $V(G) \smallsetminus R$ such that:

- **(C1)** $|R| = m$, $|R_0| = v(T) - m$, and $|W| = n/100$;
- **(C2)** $\deg_G(S, X) \geqslant (\delta_d + \gamma/2)p\binom{|X|-d}{3-d}$, for every d-set $S \subseteq V(G)$ and $X \in \{R \cup R_0, V, W\}$.

Note that, in particular, $|V| = (1 - 1/100 - o(1))n \geqslant n/2$. By the property of Lemma 2.6 almost any choice of such a partition will do.

Let $f\colon V(T) \to R_0 \cup R$ be a bijection which maps the vertices belonging to the set Z in T to R. For each edge $xy \in E(T)$, we plan to first find an $f(x)f(y)$-absorber $A^d_{f(x)f(y)}$ whose internal vertices belong to $G[V]$ in such a way that all these absorbers are pairwise internally vertex-disjoint. Subsequently, we string all the $f(x)f(y)$-absorbers together by an application of the Connecting Lemma (Lemma 5.1) over the set W.

The first part of this is to be completed using Haxell's condition and Lemma 6.6. Let $\mathcal{H}$ be a $(D-1)$-uniform hypergraph with vertex set $U \cup V$, where $U = \big\{\{f(x), f(y)\} : xy \in E(T)\big\}$. For $\{u_x, u_y\} \in U$ and $S \subseteq V$ with $|S| = D - 2$, we add the edge $\{u_x, u_y\} \cup S$ to $\mathcal{H}$ if and only if there is a copy of $A^d_{u_x u_y}$ in $G[\{u_x, u_y\} \cup S]$. Then what we are looking for is precisely a U-saturating matching in $\mathcal{H}$. Comparing to Haxell's condition (Lemma 2.12), it is sufficient to show that for every $U' \subseteq U$ and $V' \subseteq V$ with $|V'| \leqslant 2D|U'|$, there is some $\{u_x, u_y\} \in U'$ and a copy of $A^d_{u_x u_y}$ in G whose internal vertices are fully contained in $V \setminus V'$.

As $\Delta(T) \leqslant L$, we can greedily find a set of pairwise vertex-disjoint edges $xy \in E(T)$ for which $\{f(x), f(y)\} \in U'$ and which is of size at least $|U'|/(2L)$. For simplicity, we assume U' already consists only of such disjoint edges—this changes nothing in the proof. In case $d = 2$, straightforwardly applying Lemma 6.6 with any $f(x), f(y)$ for which $\{f(x), f(y)\} \in U'$ (as x, y), $G[V \cup \{f(x), f(y)\}]$ (as G), and V' (as S), we are done. Note that we can indeed do so since

$$|V'| \leqslant 4DL|U'| \leqslant 4DL \cdot e(T) \leqslant 4DL^3 m = 4DL^3 \alpha n \leqslant \lambda n.$$

In the other case, $d = 1$, the only thing remaining is to check that there are $f(x), f(y)$ with $\{f(x), f(y)\} \in U'$ that satisfy the degree requirement $\deg_G(f(x), V \setminus V'), \deg_G(f(y), V \setminus V') \geqslant (\delta_1 + \gamma/4)p\binom{|V|-1}{2}$. Towards contradiction, suppose there is no such $f(x), f(y)$. Let $Q \subseteq V(G)$ be the union of all $f(x)$ that violate the prior requirement and such that for some $f(y)$, $\{f(x), f(y)\} \in U'$, and assume $|Q| \geqslant |U'|/2$. On the one hand, this means

$$e_G(Q, V', V) \overset{\text{(C2)}}{\geqslant} |Q| \cdot (\gamma/4)p\binom{|V|-1}{2} \overset{\text{(C1)}}{\geqslant} (\gamma/64)|Q|n^2 p,$$

while on the other, by the property of Lemma 2.1 since $|Q| \leqslant 2|U'| \leqslant \lambda n$ and $|V'| \leqslant 4DL|U'| \leqslant 8DL|Q|$,

$$e_G(Q, V', V) \leqslant e_G(Q, V', V(G)) < (\gamma/64)|Q|p\binom{n-1}{2} \leqslant (\gamma/64)|Q|n^2 p,$$

which is a contradiction. So, $|Q| < |U'|/2$ and by the pigeonhole principle, there has to exist a pair $\{f(x), f(y)\} \in U'$ with the desired degree value into $V \setminus V'$.

Finally, we use Lemma 5.1 to connect all $f(x)f(y)$-absorbers into one loose path. Denote the absorbers we have found as $A_1, \ldots, A_t$, for $t = e(T)$, each A_i a copy of A^d_{xy}. We want to, for every $i \in [t-1]$, connect $v_1 \in V(A_i)$ and $a_1 \in V(A_{i+1})$ if $d = 1$ (see Figure 2) and $v_7 \in V(A_i)$ and $v_1 \in V(A_{i+1})$ if $d = 2$ (see Figure 1). Let $Y \subseteq V$ be the set of (the images of) all these $2(t-1)$ vertices we would like to connect. We use the property of Lemma 5.1 with W, Y (as U) which we can do by (C2) and since $|W| \geqslant n/100$ and the total number of pairs we want to connect with paths is

$$e(T) - 1 \leqslant L^2 m = L^2 \alpha n \leqslant \varepsilon n/100 \leqslant \varepsilon|W|.$$

Lastly, observe that the total number of vertices used by this whole procedure is

$$e(T) \cdot D + (e(T) - 1) \cdot 7 \leqslant L^2 \alpha n \cdot (D + 7) \leqslant \xi n,$$

as desired.

It remains to establish that the graph A obtained by connecting the absorbers $A_1, \ldots, A_t$ as described comprises an (a, b, R)-absorber, where a and b are (the images of) $a_1 \in V(A_1)$ and $v_1 \in V(A_t)$ if $d = 1$ and (the images of) $v_1 \in V(A_1)$ and $v_7 \in V(A_t)$ if $d = 2$. Namely, let $R' \subseteq R$ of size $|R'| < |R|/2$ be such that $V(A) \setminus R'$ has odd cardinality. Note that $V(A) \setminus (R_0 \cup R)$ must also be of odd cardinality as it can be covered by a loose path (by taking the non-covering paths of all individual $f(x)f(y)$-absorbers) and hence, crucially, $R_0 \cup (R \setminus R')$ is of even cardinality. In particular, this means there is a perfect matching for $f^{-1}(R_0 \cup (R \setminus R'))$ in T. Consider the set of $f(x)f(y)$-absorbers corresponding to the edges in this matching. Then, as a witness for the absorbing property of A, we can use the covering path for all these absorbers, the non-covering path for all other $f(x)f(y)$-absorbers, and the short paths connecting the absorbers to get the desired loose path between a and b. $\square$

6.1 Finding an xy-absorber robustly

In this section we prove Lemma 6.6. The case $d = 2$ is much simpler, so we deal with it first.

Proof of Lemma 6.6, case $d = 2$. Recall, $\delta_2 = 1/4$. Let $\lambda = \lambda_{2.9}(\gamma, 1/4)$ and suppose $\mathcal{H}^3(n,p)$ satisfies the conclusion of Lemma 2.4 and Lemma 2.9.

We use the property of Lemma 2.9 to get a set $T \subseteq \binom{V(G)\smallsetminus S}{2}$ of size $|T| \leqslant 10^9 \log n/p \leqslant 10^{-6}n$ (by choosing C sufficiently large) such that all pairs $\{u,v\} \in \binom{V(G)\smallsetminus S}{2} \smallsetminus T$ satisfy $\deg_F(u,v) \geqslant (1/4 + \gamma/2)p(n - |S| - 2) = \Omega(\log n)$, for $F := G - S$. Recall, our goal is to find a copy of A^2_{xy} in F (see Figure 1).

Since T is relatively small, there must be some $v_4 \in V(F) \smallsetminus \{x,y\}$ such that neither $\{x, v_4\}$ nor $\{y, v_4\}$ belong to T. Having chosen v_4, take distinct $v_2, v_6 \in V(F) \smallsetminus \{x, y, v_4\}$ such that v_2xv_4, v_4yv_6 are edges in F. Let

$$V_3 := \{v \in V(F) \smallsetminus \{x, y, v_2, v_4, v_6\} : \{v, v_2\}, \{v, v_4\} \notin T\}$$

and

$$\overline{V_5} := \{x, y, v_2, v_4, v_6\} \cup \{v \in V(F) : \{v, v_6\} \in T\},$$

and note that $|V_3| \geqslant (1 - 10^{-2})n$ and $|\overline{V_5}| \leqslant 10^{-5}n$. It is enough to show that there is an edge $v_3v_4v_5$ in F such that $v_3 \in V_3$ and $v_5 \in V(F) \smallsetminus \overline{V_5}$, because $\{v_2, v_3\}, \{v_5, v_6\} \notin T$ imply that we can choose v_1 and v_7 as desired for A^2_{xy}.

Suppose for contradiction such an edge does not exist in F. Then, from the codegree assumption and the property of Lemma 2.4, we have

$$|V_3|/2 \cdot (1/4 + \gamma/2)p(n - |S| - 2) \leqslant e_F(v_4, V_3, \overline{V_5}) \leqslant e_G(v_4, V_3, \overline{V_5}) \leqslant 2 \cdot 10^{-5}|V_3|np,$$

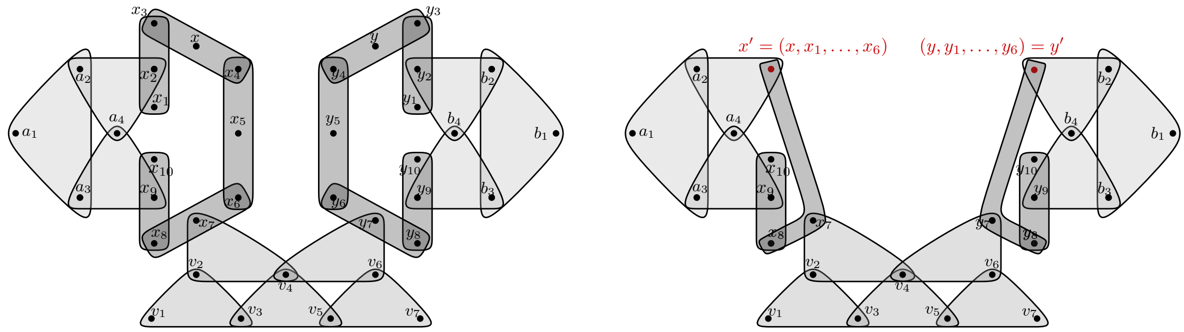
which is a contradiction. Thus, an edge $v_3v_4v_5 \in F$ exists as desired. $\square$

In what follows we provide a proof of Lemma 6.6 for $d = 1$. This is the most involved part of the whole proof and we, for convenience of reading, first give a brief outline. The main idea relies on an intricate combination of the sparse regularity method and the connecting lemma. It consists of two almost independent steps: (1) find a backbone of an A^1_{xy} absorber; (2) use the Connecting Lemma (Lemma 5.1) to find the remaining loose paths which comprise an absorber A^1_{xy} (see Figure 2). Most of the difficulty lies in the first part. To do this, we use the 'contraction technique' of Ferber and Kwan [20] and the regularity method. In order for the next steps to make more sense, we first introduce a definition.

Definition 6.7. A *contracted backbone of an absorber* is a graph that consists of:

- edges $x'x_7x_8$, $x_8x_9x_{10}$ and $a_1a_2a_3$, a_2a_4x', $a_3a_4x_9$;
- edges $y'y_7y_8$, $y_8y_9y_{10}$ and $b_1b_2b_3$, b_2b_4y', $b_3b_4y_9$;
- a copy of $A^2_{x_7y_7}$.

A contracted backbone of an absorber can be thought of as starting with a backbone of A^1_{xy} and *contracting* the edges $x_1x_2x_3, x_3xx_4, x_4x_5x_6$ into a single vertex x', and keeping only the edges 'to the outside' (that is, ones containing x_2 or x_6). The same is done to obtain y'. Note that, in this context, the vertices x' and y' play a special role and we often explicitly mention them when talking about the contracted backbone of an absorber. Strictly speaking, a name for this structure that is more descriptive of the aforementioned contraction operation might be a ‘contracted backbone of an xy-absorber’. However, as the graph itself does not contain the vertices x, y, we omit them from the name to avoid confusion. For a more natural visual representation we depict this contraction operation below.



(a) The backbone of an absorber A^1_{xy}. *(b) The contracted backbone of an absorber A^1_{xy}.*

Figure 3: The backbone and contracted backbone of an absorber A^1_{xy}.

The contraction operation on $G \subseteq \mathcal{H}^3(n,p)$, almost analogous to the one of Ferber and Kwan, is defined as follows. For a 3-graph G, a collection $\mathcal{U}$ of disjoint sets $U_1, U_2 \subseteq V(G)$, and a family of disjoint 4-tuples $\mathcal{F} \subseteq V(G)$ we let $G(\mathcal{U}, \mathcal{F})$ be a 3-graph on vertex set $U_1 \cup U_2 \cup \mathcal{F}$ and whose edge set is given as follows: add first all edges from $G[U_1 \cup U_2]$ and next, for every $\mathbf{w} = (w_1, w_2, w_3, w_4) \in \mathcal{F}$, we add an edge $\mathbf{w}uv$ to $G(\mathcal{U}, \mathcal{F})$ if and only if $u, v \in U_1$ and $w_2uv \in E(G)$ or $u, v \in U_2$ and $w_4uv \in E(G)$.

We can now continue with the outline. Namely, imagine for a moment that we can find a collection of $\Omega(n)$ distinct 4-tuples (x_1, x_2, x_5, x_6) such that, $x_1x_2x_3, x_3xx_4, x_4x_5x_6 \in E(G)$, for some x_3, x_4. Let us, for every such tuple, contract these vertices (and edges) into a single vertex and keep only edges to the outside that contain x_2 or x_6 (as in the contracted graph $G(\cdot,\cdot)$ above). Denote the set of these new vertices as X'. Now, do the same for y while keeping all these disjoint to obtain Y'. If we were to find a copy of the contracted backbone of an absorber with x', y' mapped into some vertices of X' and Y' respectively, we would be done by just undoing the contraction operation.

To do this last part, we rely on the sparse regularity lemma (Lemma 3.3) and Theorem 3.4. If all the previous steps have been done carefully, what remains of the graph still satisfies all the necessary conditions (in particular, the minimum degree will be sufficiently large) to do this. First, we show that in the reduced graph obtained from the application of Lemma 3.3 we can find a copy of the contracted backbone of an absorber with the vertices x' and y' being mapped to their corresponding ‘clusters’ belonging to X' and Y'. Subsequently we use Theorem 3.4 to transfer it into a canonical copy of the same graph in G[8]. As we prove in the appendix (not to interrupt the flow of the main argument), the contracted backbone of an absorber is just sparse enough to exist in the regular partition.

Claim 6.8. *The m_3 density of a contracted backbone of an absorber is $2/3$.*

There is one crucial difference compared to the method used in [20], reflected in the fact that we work with $p = \Theta(n^{-3/2} \log n)$, in comparison to $p = \Omega(n^{-1})$. Namely, the neighbourhood of a

[8]Strictly speaking, this copy is found in a contracted version of G and then unfolded into a copy of A^1_{xy} in G itself.

vertex in our setting is roughly of size $n^2p = \Theta(\sqrt{n}\log n)$, which is much below the point at which we can rely on regularity properties. Hence, we first need to show using ad-hoc density techniques that x and y expand to $\Theta(n)$ vertices in two hops, and from then on start implementing the strategy outlined above. This all also affects the design of our xy-absorbers.

As a final preparation step, we need a statement for dense hypergraphs that enables us to find a copy of the contracted backbone of an absorber in the reduced graph.

Lemma 6.9 (Proposition 8 in [10])**.** *For every $\gamma \in (0, 3/8)$ the following holds for every sufficiently large n. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on n vertices which satisfies $\delta_1(\mathcal{H}) \geqslant (5/8+\gamma)^2\binom{n}{2}$. Then for every pair of vertices $x, y \in V(\mathcal{H})$ the number of 7-tuples that form a copy of A^2_{xy} is at least $(\gamma n)^7/8$.*

Proof of Lemma 6.6, case $d = 1$. Recall, $\delta_1 = 7/16$. Given γ, let $t_0 \in \mathbb{N}$ be a large constant, in particular such that $3/t_0 \ll \gamma$. Next, let $\xi = \xi_{2.5}(\gamma/256)$, $\lambda = \lambda_{2.8}(\gamma)$, choose ε sufficiently small, and let η and b be as given by Lemma 3.3 for $7/16 + \gamma/8$ (as δ) and other respective parameters. Pick μ sufficiently small with respect to ε, t_0, and ξ. Let $G' := G - S$. We condition on $\mathcal{H}^3(n,p)$ satisfying the conclusion of Lemma 2.5, Lemma 2.6, Lemma 2.8, and Lemma 5.1. Furthermore,

- **(D1)** for all disjoint $U_1, U_2 \subseteq V(G)$ and $\mathcal{F} \subseteq V(G)^4$, with $|U_i|, |\mathcal{F}| \geqslant \xi n$, the conclusion of Theorem 3.4 holds for $G(\mathcal{U}, \mathcal{F})$ with $\mathcal{U} = (U_1, U_2)$: for every H with $m_3(H) \leqslant 2/3$, every subgraph G_0 of $G(\mathcal{U}, \mathcal{F})$ which belongs to $\mathcal{G}(H, \mu n, 2\varepsilon p \mu^3 n^3, p, \varepsilon)$ contains a canonical copy of H;
- **(D2)** every $G(\mathcal{U}, \mathcal{F})$ as in (D1) is (η, b, p)-upper-uniform;

Let us establish (D1) and (D2). Observe that $G(\mathcal{U}, \mathcal{F})$ has $\Omega(n)$ vertices and that it can be coupled with a random graph on its vertex set with edge probability p. Namely, there is a bijection φ from edges of $G(\mathcal{U}, \mathcal{F})$ to 3-sets in $V(G)$ such that the existence of e as an edge in $G(\mathcal{U}, \mathcal{F})$ is determined by $\varphi(e)$ being an edge in G or not.

So, for a fixed choice of U_1, U_2, and $\mathcal{F}$, the graph $G(\mathcal{U}, \mathcal{F})$ satisfies both (D1) and (D2) with probability at least $1 - e^{-\Omega(n^3p)} - e^{-\omega(n\log n)}$. As there are at most $2^{2n} \cdot n^{4n} \leqslant e^{10n\log n}$ choices for these sets, recalling that $p \geqslant Cn^{-3/2}\log n$, a simple union bound shows that w.h.p. both (D1) and (D2) hold in $\mathcal{H}^3(n,p)$. Thus, from now on we also condition on (D1)–(D2).

By the property of Lemma 2.8 there is a set $T \subseteq V(G) \smallsetminus \{x, y\}$ with $|T| \leqslant \sqrt{n}$ such that the graph $G'' := G - (S \cup T)$ has minimum degree at least $(7/16 + \gamma/4)p\binom{|V(G'')|-1}{2}$. Hence, for simplicity of notation, we assume that G' already satisfies this.

Let $V_x \cup V_y \cup U_1 \cup U_2 \cup W$ be an equipartition of $V(G') \smallsetminus \{x, y\}$ such that $\deg_{G'}(v, Z) \geqslant (7/16 + \gamma/8)p\binom{|Z|-1}{2}$ for every $v \in V(G')$ and $Z \in \{V_x, V_y, U_1, U_2, W\}$. Observe that $|Z| \geqslant n/6$. A vast number of partitions is such by the property of Lemma 2.6, so we fix one of them. In order to complete the proof it is sufficient to show that

- (i) there exists a backbone of A^1_{xy} in $G'[V_x \cup V_y \cup U_1 \cup U_2 \cup \{x, y\}]$;
- (ii) for a copy of a backbone of A^1_{xy} as above, there is an x_1x_{10}-path, a y_1y_{10}-path, a x_5y_5-path, and a v_7b_1-path, each of length four, and whose internal vertices all belong to W.

Throughout the proof and for ease of reference, it might help to have Figure 2 and especially Figure 3 in mind. Assuming we have found the backbone of A^1_{xy}, step (ii) follows by applying the Connecting Lemma with W and (the images of) $\{\{x_1, x_{10}\}, \{y_1, y_{10}\}, \{x_5, y_5\}, \{v_7, b_1\}\}$ as the family of pairs to be connected by paths. For the rest of the proof, we focus on showing (i).

Note that, for $* \in \{x,y\}$,

$$\deg_{G'}(*, V_*) \geqslant (7/16+\gamma/8)p\binom{n/6-1}{2} \geqslant \frac{\gamma}{256}p\binom{n-1}{2}.$$

By the property of Lemma 2.5 for V_* (as W), for every $* \in \{x,y\}$ there exist a family of 4-tuples $\mathcal{F}_* \subseteq V_*^4$ and a set of pairs $\mathcal{P}_* \subseteq \binom{V_*}{2}$, all pairwise disjoint, of size $|\mathcal{F}_*| = \xi n$, $|\mathcal{P}_*| = \sqrt{n}$, and such that

- $*uv \in E(G')$, for every $\{u,v\} \in \mathcal{P}_*$, and
- for every $(w_1,w_2,w_3,w_4) \in \mathcal{F}_*$ there is some $\{u,v\} \in \mathcal{P}_*$ with $uw_1w_2, vw_3w_4 \in E(G')$.

Let now $J := G'(\{U_1,U_2\}, \mathcal{F}_x \cup \mathcal{F}_y)$. Recall, J is then obtained from G' by 'contracting' each 4-tuple in $\mathcal{F}_x \cup \mathcal{F}_y$ into a single vertex and keeping only specifically selected edges. Note that $\mathcal{F}_*$ become sets of vertices in the contracted graph J. To reiterate, J contains all edges in $G'[U_1 \cup U_2]$ and for every $\mathbf{w} = (w_1,w_2,w_3,w_4) \in \mathcal{F}_x \cup \mathcal{F}_y$ and $u,v \in U_i$, an edge $\mathbf{w}uv$ exists in J if and only if $i=1$ and $w_2uv \in E(G')$ or $i=2$ and $w_4uv \in E(G')$. As every vertex $v \in V(G')$ satisfies $\deg_{G'}(v,U_i) \geqslant (7/16+\gamma/8)p\binom{|U_i|-1}{2}$, it follows that every $v \in V(J)$ has its degree into U_i determined by the same quantity.

If we were to find a contracted backbone of an absorber (see Figure 3b) in J with $\mathbf{w}_x \in \mathcal{F}_x$ (as image of x') and $\mathbf{w}_y \in \mathcal{F}_y$ (as image of y'), we would be done. Indeed, let $\mathbf{w}_x = (x_1,x_2,x_5,x_6)$. By the choice of $\mathcal{F}_x$ and $\mathcal{P}_x$, there is a pair $\{x_3,x_4\} \in \mathcal{P}_x$ for which $x_1x_2x_3, x_4x_5x_6, xx_3x_4 \in E(G')$. Analogously, there are $y_1,\dots,y_6$ for which $y_1y_2y_3, y_4y_5y_6, yy_3y_4 \in E(G')$ and $\mathbf{w}_y = (y_1,y_2,y_5,y_6)$. This, using the fact that edges of (the backbone in) J are actually also edges in G', constructs a copy of the backbone of A^1_{xy} in G' (once again, see Figure 3a).

With this in mind, it remains to show that J contains a copy of the contracted backbone of an absorber for some $\mathbf{w}_x \in \mathcal{F}_x$ and $\mathbf{w}_y \in \mathcal{F}_y$ as x' and y', respectively. Recall, by (D2), J is (η,b,p)-upper-uniform. Thus, we can apply Lemma 3.3 (the sparse regularity lemma) to it with $h=4$, ξ (as λ), $7/16+\gamma/8$ (as δ), and $\mathcal{F}_x, \mathcal{F}_y, U_1, U_2$, to get an (ε,p)-regular partition $(V_i)_{i\in[t]}$ of $V(J)$ and a corresponding reduced graph $\mathcal{R} := \mathcal{R}((V_i)_{i\in[t]}, \varepsilon, p, 2\varepsilon)$.

Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{Z}_1, \mathcal{Z}_2$ be the sets of vertices in $\mathcal{R}$ whose corresponding clusters fully lie in $U_1, U_2, \mathcal{F}_x, \mathcal{F}_y$, respectively. Note then that, for small enough ε, each $\mathcal{U}_i, \mathcal{Z}_i$ contains at least $2\xi t/3$ vertices (clusters). Additionally, all but at most $\sqrt{\varepsilon}t$ vertices $\mathcal{S} \in \mathcal{U}_i \cup \mathcal{Z}_i$ satisfy $\deg_{\mathcal{R}}(\mathcal{S},\mathcal{U}_j) \geqslant (7/16+\gamma/16)\binom{|\mathcal{U}_j|-1}{2}$ (here we used that ε and $3/t_0$ are both extremely small with respect to γ). In fact, as $|\mathcal{U}_i| \geqslant 2\xi t/3$ and by choosing ε much smaller than ξ, by removing these at most $\sqrt{\varepsilon}t$ vertices from each $\mathcal{U}_i \cup \mathcal{Z}_i$ of $\mathcal{R}$, we obtain a subgraph $\mathcal{R}'$ which satisfies the above minimum degree condition with a negligible reduction. All in all, we have a graph $\mathcal{R}' \subseteq \mathcal{R}$ with the following properties:

- **(E1)** there are at least $\xi t/2$ vertices in each $\mathcal{U}_i, \mathcal{Z}_i$, and
- **(E2)** every $\mathcal{S} \in \mathcal{U}_i \cup \mathcal{Z}_i$ satisfies $\deg_{\mathcal{R}'}(\mathcal{S},\mathcal{U}_j) \geqslant (7/16+\gamma/32)\binom{|\mathcal{U}_j|}{2}$, for $i,j \in \{1,2\}$.

We now constructively find a contracted backbone of an absorber in the cluster graph $\mathcal{R}'$, after which we use (D1) to finally complete the proof by finding a corresponding canonical copy of it in J. Let $X' \in \mathcal{Z}_1$ and $Y' \in \mathcal{Z}_2$.

Claim 6.10. *There exists a contracted backbone of an absorber in $\mathcal{R}'$ with x' mapped to X' and y' to Y'.*

Proof. Choose distinct arbitrary vertices $X_7, X_8, X_9, X_{10}, Y_7, Y_8, Y_9, Y_{10} \in \mathcal{U}_2$ such that

$$\{X', X_7, X_8\}, \{X_8, X_9, X_{10}\}, \{Y', Y_7, Y_8\}, \{Y_8, Y_9, Y_{10}\} \in E(\mathcal{R}')$$

which must exist due to (E2).

We can now apply Lemma 6.9 with $\mathcal{R}_2 := \mathcal{R}'[\mathcal{U}_2 \smallsetminus \{X_8, X_9, X_{10}, Y_8, Y_9, Y_{10}\}]$ (as $\mathcal{H}$) and X_7, Y_7 (as x, y), since

$$\delta_1(\mathcal{R}_2) \geqslant \left(\frac{7}{16} + \frac{\gamma}{32}\right)\binom{|\mathcal{U}_2|}{2} - 6|\mathcal{U}_2| - 36 \geqslant \left(\frac{5}{8} + \gamma'\right)^2 \binom{|V(\mathcal{R}_2)|}{2}$$

for some small $\gamma' > 0$. (Note that here we used (E1) and that t can be assumed to be sufficiently large.) We conclude that there is a copy of $A^2_{X_7 Y_7}$ in $\mathcal{R}_2$.

Next, we apply Lemma 2.11 with $\mathcal{R}[\mathcal{U}_1 \cup \{X', X_9\}]$ (as $\mathcal{H}$) and X', X_9 (as u, v) to find vertices $A_1, A_2, A_3, A_4 \in \mathcal{U}_1$ so that

$$\{A_1, A_2, A_3\}, \{A_2, A_4, X'\}, \{A_3, A_4, X_9\} \in E(\mathcal{R}').$$

Almost analogously, there are $B_1, B_2, B_3, B_4 \in \mathcal{U}_1$ so that $\{B_1, B_2, B_3\}, \{B_2, B_4, Y'\}, \{B_3, B_4, Y_9\} \in E(\mathcal{R}')$. This forms the contracted backbone of an absorber in $\mathcal{R}' \subseteq \mathcal{R}$ (see Figure 3b). $\square$

As the 3-density of the contracted backbone of an absorber is at most $2/3$ by Claim 6.8, we can use (D1) in the subgraph of J induced by the clusters corresponding to the found contracted backbone of an absorber in $\mathcal{R}'$. This gives us a canonical copy of the contracted backbone of an absorber in J with vertices x' and y' mapped into X' and Y' as desired, and this finally completes the proof. $\square$

7 Putting everything together: Proof of Theorem 1.1

With all the preparatory lemmas at hand, the proof of our main theorem follows the usual steps: (i) find an appropriate set R and a not-too-large (a, b, R)-absorber in G; (ii) cover almost everything else by $o(n)$ loose paths; (iii) use the Connecting Lemma to patch those paths together over the set R; (iv) absorb the unused vertices of R into a loose Hamilton cycle.

Proof of Theorem 1.1. Choose $\lambda > 0$ sufficiently small with respect to γ so that the argument below works out; in particular, $\lambda < \min\{\lambda_{2.8}(\gamma), \lambda_{2.9}(\gamma)\}$. Next, $\alpha = \alpha_{6.2}(\gamma, \lambda)$, $\varepsilon = \varepsilon_{5.1}(\gamma/2, \alpha)$, and ϱ small enough with respect to ε and α. Condition on $\mathcal{H}^3(n, p)$ having the properties of Lemma 2.6, Lemma 2.8, Lemma 2.9. Lemma 4.1, Lemma 5.1, and Lemma 6.2.

Pick a random set of vertices $R \subseteq V(G)$ of size $|R| = \alpha n$. By the Absorbing Lemma (Lemma 6.2) we get, w.h.p. over the choice of R, an (a, b, R)-absorber A, for some $a, b \in V(G) \smallsetminus R$, of size at most λn. Furthermore, by Lemma 2.6, w.h.p. we have that every d-set $S \subseteq V(G)$ satisfies $\deg_G(S, R) \geqslant (\delta_d + \gamma/2)p\binom{|R|-d}{3-d}$. In particular, as $\mathcal{H}^3(n, p)$ has the property of the Connecting Lemma (Lemma 5.1), the set R can be used as the 'reservoir' (set W in the lemma) to find paths through it. We fix such a choice of R and A for the remainder of the proof.

For step (ii) of the strategy, we aim to cover almost all of $G - V(A)$ with a few loose paths, using Lemma 4.1. If $d = 1$ then by Lemma 2.8 for $V(A)$ (as S) there exists a set T of size $\sqrt{n}$ such that $F := G - (V(A) \cup T)$ satisfies $\delta_1(F) \geqslant (7/16 + \gamma/4)p\binom{|V(F)|-1}{2}$. Otherwise, if $d = 2$, set $T := \varnothing$, and by Lemma 2.9 again for $V(A)$ (as S), in $F := G - V(A)$, all but at most $O(\log n/p) = o(n^2)$ pairs of vertices $u, v \in V(F)$ have $\deg_F(u, v) \geqslant (1/4 + \gamma/2)p(n - |V(A)| - 2)$. Thus, in both cases the graph F satisfies the requirements of Lemma 4.1 and, therefore, there are $k \leqslant \varrho n$ disjoint loose paths that cover its vertices. Denote the i-th such path by P_i and let x_i, y_i be its endpoints (note, for some i we may have $x_i = y_i$).

Step (iii) is to use the Connecting Lemma (Lemma 5.1) to connect everything into a large loose cycle. Let $T = \{v_1, \ldots, v_t\}$ (note, $t = 0$ if $d = 2$). We want to apply it with pairs:

$$\{y_i, x_{i+1}\}_{i \in [k-1]} \cup \{y_k, v_1\} \cup \{v_i, v_{i+1}\}_{i \in [t-1]} \cup \{v_t, a\} \cup \{b, x_1\}.$$

Recall, set R is chosen so that $|R| = \alpha n$ and each d-set $Q \subseteq V(G)$ has $\deg_G(Q, R) \geqslant (\delta_d + \gamma/2)p\binom{|R|-d}{3-d}$. Furthermore, the total number of pairs to connect is at most $k+t+1 \leqslant \sqrt{n}+\varrho n+1 \leqslant \varepsilon|R|$. Therefore, by Lemma 5.1 all these pairs can be connected via disjoint loose paths, each of length at most 4, whose internal vertices belong to R. These paths use a subset $R' \subseteq R$ with $|R'| \leqslant 100\varrho n < \alpha n/2 = |R|/2$. The union of these paths with $P_1, \ldots, P_k$ makes up a loose ab-path P^* with vertex set $(V(G) \smallsetminus V(A)) \cup \{a, b\} \cup R'$. Notice that this implies $n - |V(A)| + |R'|$ is odd, and since n is even, $|V(A)| - |R'|$ must be odd as well. Lastly, we use the absorbing property of A to find a loose ab-path P_A with $V(P_A) = V(A) \smallsetminus R'$. The union of P^* and P_A gives us a loose Hamilton cycle in G as desired. □

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A Complementary proofs

Proof of Lemma 6.5. The proof is almost identical to the proof of [20, Lemma 7.3] with $k = 2$. We also make use of the following result from [53].

Lemma A.1 ([53, Lemma 10.7])**.** *For any sufficiently large s, there exists a bipartite graph R with vertex parts X and $Y \cup Z$, where Y and Z are disjoint, with $|X| = 3s$ and $|Y| = |Z| = 2s$, and maximum degree 100, such that if we remove any s vertices from Z, the resulting bipartite graph has a perfect matching.*

We construct T by starting with the bipartite graph R from Lemma A.1 with $s = \lceil m/2 \rceil$ and potentially deleting one vertex from Z to ensure it has size m. Let G be an m-vertex graph with maximum degree 4 and no independent set of size $m/2$ (e.g. take the square of a cycle on m vertices) and add to T the edges of G, placed on the vertex set Z. Thus, T has at most $7s \leqslant 4m$ vertices, and its maximum degree is at most $100 + 4 = 104$.

Consider a set $Z' \subseteq Z$ with $|Z'| < m/2$ such that $V(T) \smallsetminus Z'$ has even cardinality. Since G has no independent set of size s, we can construct a matching M_1 in $T[Z \smallsetminus Z']$ by repeatedly taking away edges one by one until precisely s vertices are left. These remaining s vertices of Z along with X and Y induce a subgraph in T which contains a perfect matching M_2 by Lemma A.1. Therefore, $M_1 \cup M_2$ is a perfect matching of $T - Z'$. $\square$

Proof of Claim 6.8. We first prove a claim that allows us to split the contracted backbone of an absorber into three subgraphs and consider each of them separately.

Claim A.2. *Let H be a 3-graph consisting of two linear 3-graphs H_1 and H_2 intersecting in a single vertex. Then if $\alpha \geqslant 1/2$ and $m_3(H_i) \leqslant \alpha$ for $i \in \{1, 2\}$, we have $m_3(H) \leqslant \alpha$.*

Proof. Let $H' \subseteq H$ with $v(H') > 3$ and let $H'_i = H' \cap H_i$. Note that $e(H') = e(H'_1) + e(H'_2)$ and $v(H') \geqslant v(H'_1) + v(H'_2) - 1$. It suffices to consider only cases where H' intersects both H_1 and H_2 in at least one edge (and thus each H'_i has at least three vertices). If this is not the case for some $i \in \{1, 2\}$, we have $m_3(H') \leqslant m_3(H'_{3-i}) \leqslant \alpha$. If $v(H'_i) > 3$, by $m_3(H_i) \leqslant \alpha$ it follows that $e(H'_i) \leqslant \alpha v(H'_i) - 3\alpha + 1$. If $v(H'_i) = 3$, then $e(H'_i) = 1$, and $e(H'_i) \leqslant \alpha v(H'_i) - 3\alpha + 1$ holds as well. Therefore,

$$\frac{e(H') - 1}{v(H') - 3} \leqslant \frac{e(H'_1) + e(H'_2) - 1}{v(H'_1) + v(H'_2) - 4} \leqslant \frac{\alpha(v(H'_1) + v(H'_2)) - 6\alpha + 1}{v(H'_1) + v(H'_2) - 4} \leqslant \alpha,$$

where the last inequality follows from $\alpha \geqslant 1/2$. $\square$

Consider the components of the contracted backbone of an absorber obtained after removing the edges of $A^2_{x_7y_7}$ (see Figure 3b). Denote the component which contains x', respectively y', by $S_{x'}$, respectively $S_{y'}$. By Claim A.2, it is enough to show that the 3-density of $A^2_{x_7y_7}$ and of $S_{x'}$ are each at most $2/3$.

To do this, consider a 3-graph H that is a subgraph of either $A^2_{x_7y_7}$ or $S_{x'}$. It suffices to consider only subgraphs with no isolated vertices and with $e(H) \geqslant 2$. If $e(H) = 2$, we have $v(H) \geqslant 5$ since $A^2_{x_7y_7}$ and $S_{x'}$ are both linear hypergraphs, so $\frac{e(H)-1}{v(H)-3} \leqslant \frac{2}{3}$. If $e(H) = 3$, then $v(H) \geqslant 6$ since the union of any two edges contains at least 5 vertices, and an extra edge requires an extra vertex (otherwise it must have at least two vertices in common with at least one of the first two edges). Thus, $\frac{e(H)-1}{v(H)-3} \leqslant \frac{2}{3}$ in this case as well. If $e(H) = 4$, we have $v(H) \geqslant 8$ since $A^2_{x_7y_7}$ and

$S_{x'}$ each have 9 vertices and 5 edges, and in each of them there is no edge with two vertices of degree 1. Thus, $\frac{e(H)-1}{v(H)-3} \leqslant \frac{3}{5} < \frac{2}{3}$ again. Finally, $e(H) = 5$ implies H is either $A^2_{x_7y_7}$ or $S_{x'}$ and so $v(H) = 9$ and $\frac{e(H)-1}{v(H)-3} = \frac{4}{6} = \frac{2}{3}$. $\square$