# Quantum Mechanics as a Theory of Incompatible Symmetries 

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#### Abstract

It is increasingly becoming recognized that incompatible variables, which play an essential role in quantum mechanics ( QM ), are not in fact unique to QM . Here we add a new example, the "Arrow" system, to the growing list of classical systems that possess incompatible variables. We show how classical probability theory can be extended to include any system with incompatible variables in a general incompatible variables (GIV) theory. We then show how the QM theory of elementary systems emerges naturally from the GIV framework when the fundamental variables are taken to be the symmetries of the states of the system. This result follows primarily because in QM the symmetries of the Poincaré group play a double role, not only as the operators which transform the states under symmetry transformations but also as the fundamental variables of the system. The incompatibility of the QM variables is then seen to be just the incompatibility of the corresponding space-time symmetries. We also arrive at a clearer understanding of the Born Rule: although not primarily derived from symmetry rather it is simply a free Pythagorean construction for accommodating basic features of classical probability theory in Hilbert spaces - it is Poincare symmetry that allows the Born Rule to take on its familiar form in QM, in agreement with Gleason's theorem. Finally, we show that any probabilistic system (classical or quantal) that possesses incompatible variables will show not only uncertainty, but also interference in its probability patterns. Thus the GIV framework provides the basis for a broader perspective from which to view QM: quantal systems are a subset of the set of all systems possessing incompatible variables (and hence showing uncertainty and interference), namely the subset in which the incompatible variables are incompatible symmetries.


## 1 Introduction

Quantum mechanics ( QM ) is now a century old, and yet, despite its agreement with experiment to high precision, its physical interpretation remains controversial, leading some to suggest that QM is incomplete. In this paper we take the view that an important first step towards a better understanding of QM is the recognition that
incompatible variables are not unique to $Q M$, and that any system with incompatible variables will show uncertainty and interference. Incompatible variables, along with the concomitant uncertainty and interference, have long been considered the exclusive hallmark of QM, but examples of purely classical, nonquantal systems that exhibit this supposedly "quantal" behaviour have now been found, such as Kirkpatrick's card game models [1] and the Hegstrom-Adshead model of MBCT [2].

The late Steven Weinberg suggested [3] it would be useful, in order to test QM, to find a larger, more general theory in which QM appears as a special case. We have done precisely that: because QM includes incompatible variables, we have constructed a general framework to describe any system with incompatible variables, in order to identify the specific features that define a genuinely quantal system. We call our theory the general incompatible variables (GIV) theory. Full details of our formulation of the GIV theory, including proofs, will be published elsewhere, but in this paper we briefly preview our conclusion, namely that what distinguishes QM from other GIV theories is symmetry: the incompatible variables of QM are incompatible symmetries or the generators of incompatible symmetries.

Different textbooks provide varying numbers of postulates of quantum mechanics, which typically boil down to a technical preamble, to the effect that observables are represented by Hilbert space operators $\widehat{O}$ that are not only linear but also Hermitian (to give orthogonal eigenstates and real eigenvalues), and that the eigenvalues $o_{i}$ of these operators represent the possible outcomes of measurements,

$$
\begin{equation*}
\widehat{O}\left|o_{i}\right\rangle=o_{i}\left|o_{i}\right\rangle \tag{1.1}
\end{equation*}
$$

followed by just two central postulates, often described as "underivable". The first postulate, commonly known as the "Born postulate", or "Born Rule", is that the probability of each outcome $o_{i}$ is given by the square of the overlap of the state vector $|\psi\rangle$ with the eigenstate $\left|o_{i}\right\rangle$ corresponding to that outcome,

$$
\begin{equation*}
\mathrm{P}_{\psi}\left(o_{i}\right)=\left|\left\langle o_{i} \mid \psi\right\rangle\right|^{2}=\cos ^{2} \omega_{\psi o_{i}} \quad \text { "Born postulate" } \tag{1.2}
\end{equation*}
$$

where $\mathrm{P}_{\psi}\left(o_{i}\right)$ is the probability of obtaining the outcome $o_{i}$ when the system is in state $|\psi\rangle$, and $\omega_{\psi o_{i}}$ is an angle relating the normalized state vectors $\left|o_{i}\right\rangle$ and $|\psi\rangle$ in a single Hilbert space containing all possible state vectors for the system of interest. The second postulate, which we will refer to as the "commutator postulate", is that the operators $\hat{O}$ are chosen to obey commutation relations such as the familiar

$$
\begin{equation*}
\left[\hat{x}, \hat{p}_{x}\right]=i \hbar \quad \text { "commutator postulate" } \tag{1.3}
\end{equation*}
$$

for position and momentum. The "commutator postulate" reflects the incompatibility of the fundamental variables of QM , and it is this incompatibility that gives rise to the supposedly "quantal" phenomena of uncertainty and interference.

Before proceeding any further, we wish to clarify that we use the terms "compatible" and "incompatible" in their familiar, dictionary-definition sense, which in QM takes a somewhat complicated form, namely that two variables A and $B$ are compatible if the corresponding operators $\hat{A}$ and $\widehat{B}$ commute, which then, according to a well-known theorem of QM [4], implies the alternative and equivalent definition that A and B are compatible if and only if there exists a complete orthonormal set of common eigenstates $\left(a_{i}, b_{j}\right)$ of A and B (see also Appendix A of reference [2]). Conversely, the variables A and B are incompatible if the corresponding operators do not commute and thus do not have a complete set of common eigenstates. We emphasize this because some authors - such as Hughes [5] and Kirkpatrick [1], as we see later - use a more restricted meaning of incompatibility that not only requires the variables to be not compatible (with noncommuting operators and no complete set of common eigenstates) but also, in addition, requires them to be representable in one and the same Hilbert space.

Frank Wilczek [6] has lamented the fact that QM does not appear to have a guiding principle based on symmetry, unlike relativity (based on the equivalence of different inertial frames) or gauge theory (based on the equivalence of different potentials). But in fact an attempt has been made to derive QM from symmetry in a 1995 paper [7], by Aage Bohr (the late son of Niels Bohr) and Ole Ulfbeck, that deserves to be much more widely known. Bohr and Ulfbeck aimed to show that, contrary to what we all tell our undergraduates, the two central postulates identified above - the "Born postulate" and the "commutator postulate" - have little to do with either wave character or quantization, but instead result from the Poincaré symmetry of spacetime. Their account of the derivation of the "commutator postulate" from symmetry represents a major insight. But their attempt to derive the "Born postulate" $\mathrm{P}_{\psi}\left(o_{i}\right)=\left|\left\langle o_{i} \mid \psi\right\rangle\right|^{2}$ from symmetry does not work because it tacitly assumes that $|\psi\rangle$ and $\left|o_{i}\right\rangle$ are represented in a single Hilbert space, but ignores Gleason's theorem [8], which shows that, if $|\psi\rangle$ and $\left|o_{i}\right\rangle$ are represented in a single Hilbert space, the only probability expression that works is the Born Rule, symmetry or no symmetry. In fact Weinberg pointed out in 2015 [9] that it is unclear where the "Born postulate" comes from, and that attempts to "derive" it have hitherto been circular. Most of these attempts are based upon deep QM principles, usually in the context of specific interpretations of QM, most notably by Deutsch [10] and Wallace [11], based on the many-worlds
interpretation, and by Zurek [12], based on decoherence by entanglement with the environment. But it is less widely known that just as incompatible variables are not unique to QM , the Born Rule is also not unique to QM , since Brumer and Gong [13] have derived an analogue of the Born Rule within purely classical mechanics.

We take the view that the "Born postulate" is not really a "postulate" as such, but rather is best seen as simply a free Pythagorean construction for accommodating basic features of classical probability theory in Hilbert spaces. Although commonly associated with QM, Hilbert spaces - which are intrinsically Pythagorean in nature [5] - are not unique to QM and can be applied to any probabilistic system, including the familiar coin toss [3]. One can always choose to use the Born Rule $\mathrm{P}_{\psi}\left(o_{i}\right)=\left|\left\langle o_{i} \mid \psi\right\rangle\right|^{2}$ as a construction (since it relies only on the truth of Pythagoras' theorem), provided that $|\psi\rangle$ and $\left|o_{i}\right\rangle$ are vectors in the same Hilbert space and that the eigenvectors $\left|o_{i}\right\rangle$ form a complete orthonormal set in that space. But Gleason's theorem goes further, proving that the Born Rule is not merely a convenient choice of probability rule but actually the only possible choice if all vectors describing the system belong to a single Hilbert space.

So what distinguishes QM from classical systems with incompatible variables? And what kind of formalism do non-QM GIV theories use? In QM, all possible state vectors for a system are represented in a single Hilbert space, and the probability of obtaining a particular result when measuring any one of the variables is given by a single Born Rule, as in equation (1.2). But Hughes realized [5] that if two variables A and B are not compatible (i.e. incompatible in the familiar dictionary-definition sense mentioned earlier) they cannot in general (except in special cases), be represented together in just one Hilbert space. This is because both sets of eigenvectors (those of A and those of B) would have to form complete orthonormal sets in that Hilbert space, and this is not in general possible for incompatible variables, because the condition of orthogonality for these vectors imposes severe and unnecessary restrictions on the form of the probability functions that would arbitrarily exclude many possible GIV systems, as we demonstrate in Section 2. To get around this problem, Hughes introduced the idea of using a separate Hilbert space for each of the non-compatible variables [5], but did not develop this approach any further because QM has what he described as the "remarkable feature" of containing variables that are not compatible and yet can be represented in the same Hilbert space. He therefore chose to define "incompatible" variables as those that are not only not compatible, but also representable in a single Hilbert space - and Kirkpatrick's work [1] also tacitly assumes a single Hilbert space. We prefer to describe variables that are not compatible and also representable in a single Hilbert space as "quantum incompatible", while using the
terms "ordinary incompatible", "general incompatible", or simply "incompatible", for the familiar meaning of just not compatible (but not necessarily representable in a single Hilbert space).

In this paper we provide an explanation, in terms of symmetry, for the "remarkable feature" of QM that its variables can be represented in a single Hilbert space, despite being incompatible in the ordinary sense. Following Hughes [5], our GIV theory uses many Hilbert spaces, one for each incompatible variable: the eigenvectors of A are orthogonal in Hilbert space $\mathcal{H}_{A}$ (in which the eigenvectors of A form the axes), but the vectors representing the eigenstates of B are not in general orthogonal in Hilbert space $\mathcal{H}_{A}$; similarly the eigenvectors of B are orthogonal in Hilbert space $\mathcal{H}_{B}$ (in which the eigenvectors of B form the axes), but the vectors representing the eigenstates of A are not in general orthogonal in $\mathcal{H}_{B}$. Each of our multiple Hilbert spaces uses what we call a restricted version of the Born Rule $\mathrm{P}_{\Psi}\left(o_{i}\right)=\left|\left\langle o_{i} \mid \psi\right\rangle\right|^{2}$ for the probability of getting the outcome $o_{i}$ when making a measurement on the state $|\psi\rangle$. In the restricted Born Rule, $|\psi\rangle$ can be any vector in the Hilbert space, but $\left|o_{i}\right\rangle$ is restricted to being one of the fixed set of orthonormal vectors defining the axes of that particular Hilbert space; this is in contrast to the usual version of the Born Rule, in which not only $|\psi\rangle$ but also $\left|o_{i}\right\rangle$ can be any vector within the Hilbert space. So probabilities of A outcomes can in general only be calculated in Hilbert space $\mathcal{H}_{A}$, and similarly probabilities of B outcomes can only be calculated in Hilbert space $\mathcal{H}_{B}$, if the variables A and B are incompatible.

So why is QM unique among GIV theories in using only one Hilbert space? Is there something special about the incompatible variables of QM? And what is the source of their incompatibility? The source of the incompatibility of variables in certain classical systems is typically rather mundane and obvious, such as the fact that an arrow cannot be pointing in two directions at the same time, as we discuss in Section 2. But the source of the incompatibility of the QM variables is much more profound: the QM variables are symmetries, and the incompatibility of the QM variables is actually the incompatibility of the corresponding symmetries. This kind of incompatibility is easy to visualize and understand: for example, an object having cylindrical symmetry as its highest symmetry cannot be cylindrically symmetric about both the $x$-axis and the $y$-axis at the same time, leading to the non-commutation $\left[\hat{J}_{x}, \hat{J}_{y}\right]=i \hbar \hat{J}_{z}$ of the angular momentum operators $\hat{J}_{x}$ and $\hat{J}_{y}$ that are the generators of rotations about the respective axes, and to the non-existence of a complete set of simultaneous eigenstates for $\hat{J}_{x}$ and $\hat{J}_{y}$. Similarly, as pointed out by Bohr \& Ulfbeck [7] and elaborated on in our forthcoming paper, a spacetime object cannot have both Lorentz symmetry and translational symmetry in the
$x$-direction at the same time, leading $[7,14]$ to the non-commutation $\left[\widehat{K}_{x}, \hat{p}_{x}\right]=\frac{i}{c} \widehat{H}$ of the generator $\widehat{K}_{x}$ of Lorentz transformations and the corresponding momentum $\hat{p}_{x}$ (generator of translations), which in the non-relativistic regime becomes $\left[\hat{x}, \hat{p}_{x}\right]=i \hbar$, with the operator $\hat{x}=(\hbar / m c) \widehat{K}_{x}$ playing the double role of the (approximate) generator of Lorentz transformations about the spacetime point $(0, \mathrm{t})$ and also the position coordinate $x$ of a particle.

The fact that the eigenstates of incompatible variables cannot in general be represented together in just one single Hilbert space has, to our knowledge, been noticed only by Hughes [5]. But what has not, to our knowledge, been noticed until now is that the eigenstates of incompatible variables can be represented in the same Hilbert space (and indeed must be representable in the same Hilbert space) if the incompatible variables are incompatible symmetries (as in QM )!

We show in our forthcoming paper (see also Section 3 below) that if the incompatible variables are symmetries, the many Hilbert space formalism of GIV theories can then be collapsed into the familiar QM formalism of a single Hilbert space with just a single, unrestricted Born Rule. And why is it uniquely symmetry that allows the many Hilbert spaces to be combined into one? We argue that when the fundamental variables are symmetries, as in QM, the many Hilbert spaces of GIV theory are then related to one another by rigid transformations (as opposed to non-rigid transformations in the general case), which ensures that the vectors representing the eigenvectors of a variable B are orthogonal not only in Hilbert space $\mathcal{H}_{B}$, but also in Hilbert space $\mathcal{H}_{A}$, enabling the many Hilbert spaces to be superimposed and amalgamated into one.

So symmetry is behind the simple, single Hilbert space formalism of QM. The role of symmetry in the "commutator postulate" is also clear, as shown by Bohr and Ulfbeck and others. But does symmetry also play a role in the "Born postulate", as Bohr and Ulfbeck [7] hoped to show? The Born Rule is not primarily a consequence of symmetry, since its unrestricted form for a single-Hilbert-space theory is required by Gleason's theorem [5, 8] and its restricted form in a many-Hilbert-space theory is a free construction. But when the incompatible variables are incompatible symmetries, the many-Hilbert-space description of GIV theory, in which only restricted forms of the Born Rule apply, becomes physically equivalent to a single Hilbert space description, in which the unrestricted Born Rule applies. So although the Born Rule is not derived from symmetry, it is symmetry that allows the restricted Born Rule to take on its familiar unrestricted form, in agreement with Gleason's theorem.

In this paper we summarize the reasoning behind all these conclusions, in advance of our forthcoming paper that provides more detailed proofs. In Section 2
we use a simple classical system that we call the "Arrow" as a specific example to illustrate the use of many Hilbert spaces for a system with incompatible variables, and to highlight the role of symmetry in enabling a single Hilbert space to be used. In Section 3 we outline how our treatment of the Arrow can be generalized in our GIV theory, and in Section 4 we use GIV theory to demonstrate that any system, classical or quantal, will show uncertainty and interference if it has incompatible variables.

## 2 The "Arrow": a non-QM system with incompatible variables

The coin toss is a non-QM system with a single variable ("sidedness") that has just two possible values, heads or tails. Our "Arrow" is a simple non-QM system that is like the coin toss, but with not just one but an infinite number of two-valued variables, all incompatible with each other. The Arrow is an ordinary directed object, such as a meter stick, with an arrow-head painted on the "head" end to clearly distinguish it from the "tail" end. The center of the Arrow is fixed at the origin, and its head can be rotated about an axis through the origin, perpendicular to the Arrow, but the Arrow always remains perpendicular to the rotation axis, similar to the rotation of a roulette wheel. We prepare the Arrow in an initial state by orienting it along some initial direction represented by a vector $\overrightarrow{\boldsymbol{n}}$ in physical space. To make a measurement of the variable $A$, we set up a measurement apparatus analogous to a roulette wheel that has only two slots for the ball: the two "slots" for the Arrow are positioned at opposite ends of a vector $\overrightarrow{\boldsymbol{a}}$ that passes through the origin at an angle $\theta_{n_{+} a_{+}}$to $\overrightarrow{\boldsymbol{n}}$. The Arrow is then caused (by random "kicks" or "spins") to rotate with damping until it finally settles into one of two possible orientations: (1) Arrow head in $\overrightarrow{\boldsymbol{a}}$ direction ( $\mathrm{A}=a_{+}$), which we designate state $\left(a_{+}\right)$, or (2) Arrow head in $-\overrightarrow{\boldsymbol{a}}$ direction ( $\mathrm{A}=a_{-}$), which we designate state (a_). How this is accomplished is unimportant, but it could be done by attaching a small magnet to the tip of the Arrow and placing two additional magnets at appropriate points along the $\overrightarrow{\boldsymbol{a}}$ and $-\overrightarrow{\boldsymbol{a}}$ directions. On release from its initial orientation $\overrightarrow{\boldsymbol{n}}$, the Arrow rotates and eventually clicks into place with the Arrow head pointing in either the $\overrightarrow{\boldsymbol{a}}$ or $-\overrightarrow{\boldsymbol{a}}$ "slots", giving one of only two possible outcomes for the variable A, namely $a_{+}$or $a_{-}$; and the probabilities of getting $a_{+}$ would depend in some way on the initial orientation of $\overrightarrow{\boldsymbol{n}}$, i.e. on the angle $\theta_{n_{+} a_{+}}$. The variable A has only two values, like heads or tails in the coin toss, or spin-up and spin-down in a QM spin $1 / 2$ system, so it is analogous to a roulette wheel where the ball can only land at either the " 12 o'clock" position or the " 6 o'clock" position, the two positions being $180^{\circ}$ apart (Fig. 1).


Fig. 1 Arrow system set up to measure variable A

One could also measure another variable B , with values $b_{+}$and $b_{-}$, by instead placing magnets in the " 10 o'clock" and " 4 o'clock" positions (also $180^{\circ}$ apart, Fig. 2); and in fact there are an infinite number of variables A, B, C....., one for each direction in the plane of the Arrow. These variables are all mutually incompatible, as the Arrow ends up pointing in a different direction for each variable, e.g. A and B are incompatible because when the Arrow comes to rest in the $\pm \overrightarrow{\boldsymbol{a}}$ direction in the measurement of A, it cannot at the same time be pointing in the $\pm \overrightarrow{\boldsymbol{b}}$ direction, as in a measurement of $B$. There is therefore no state such as $\left(a_{+}, b_{+}\right)$for which we could predict the values of both $\mathrm{A}\left(=a_{+}\right)$and $\mathrm{B}\left(=b_{+}\right)$with $100 \%$ certainty, in accord with the definition of incompatibility. Measurement of the incompatible variables A and B would in general also involve two different pieces of apparatus, with magnets along different axes, but could be done in a single apparatus using electromagnets in the different directions that could be switched on and off according to which variable was being measured.


Fig. 2 Arrow system set up to measure variable B

If the Arrow is prepared in the state $\left(n_{+}\right)$(i.e. pointing in the positive $\overrightarrow{\boldsymbol{n}}$ direction), the probability of obtaining the outcome $a_{+}$upon measuring A will be some function $f_{a_{+}}$of the angle $\theta_{n_{+} a_{+}}$between the $n_{+}$and $a_{+}$directions. We will use the notation $\mathrm{P}_{\left(n_{+}\right)}\left(a_{+}\right)$for the probability of obtaining the outcome $a_{+}$when starting from the state $\left(n_{+}\right)$, so we have

$$
\begin{equation*}
\mathrm{P}_{\left(n_{+}\right)}\left(a_{+}\right)=f_{a_{+}}\left(\theta_{n_{+} a_{+}}\right) \tag{2.1a}
\end{equation*}
$$

At this stage, we make no assumptions about any symmetries, so there is no reason to assume the same probability function $f$ for measuring the different eigenvalues of A , so we write the probability of obtaining the outcome $a_{-}$when starting from the state $\left(n_{+}\right)$as a different function $f_{a_{-}}$of the angle $\theta_{n_{+} a_{-}}$between the $n_{+}$and $a_{-}$ directions,

$$
\begin{equation*}
\mathrm{P}_{\left(n_{+}\right)}\left(a_{-}\right)=f_{a_{-}}\left(\theta_{n_{+} a_{-}}\right) \tag{2.1b}
\end{equation*}
$$

There is also no reason to assume the same probability functions $f$ for measuring the different variables A and B , so we write the corresponding probabilities for obtaining the respective outcomes $b_{+}$and $b_{-}$upon measuring B as

$$
\begin{align*}
& \mathrm{P}_{\left(n_{+}\right)}\left(b_{+}\right)=f_{b_{+}}\left(\theta_{n_{+} b_{+}}\right)  \tag{2.2a}\\
& \mathrm{P}_{\left(n_{+}\right)}\left(b_{-}\right)=f_{b_{-}}\left(\theta_{n_{+} b_{-}}\right) \tag{2.2b}
\end{align*}
$$

We thus assume for now that $f_{a_{+}}, f_{a_{-}}, f_{b_{+}}$and $f_{b_{-}}$are all different functions of the respective angles, and symmetry will be introduced later.

So what is the form of the various probability functions $f(\theta)$ ? The different functions $f$ could in general be almost any functions of the respective angles $\theta$, restricted only by the basic requirements of classical probability, namely normalization, exclusivity, value range 0 to 1 , and the additional constraints $f(0)=1$ and $f(\pi)=0$ arising from the linear geometry of the Arrow and the requirement that a measurement of A performed upon the eigenstate ( $a_{ \pm}$) must give the outcome $a_{ \pm}$with certainty. Many possible functions satisfy these minimal requirements, including the functions $f(\theta)=1-\frac{\theta}{\pi}, f(\theta)=1-\left(\frac{\theta}{\pi}\right)^{2}, f(\theta)=\cos ^{2} \frac{\theta}{2}$ and more.

We now consider how to represent the Arrow states in Hilbert space. If the variables A and B were compatible, an appropriate Hilbert space would use the four-dimensional orthonormal basis $\left\{\left|a_{+}, b_{+}\right\rangle,\left|a_{+}, b_{-}\right\rangle,\left|a_{-}, b_{+}\right\rangle,\left|a_{-}, b_{-}\right\rangle\right\}$. But with A and B incompatible, a set of simultaneous eigenstates of A and B such as this does not exist. So we try instead to use a two-dimensional Hilbert space $\mathcal{H}_{\mathrm{A}}$ with $\left\{\left|a_{+}^{A}\right\rangle,\left|a_{-}^{A}\right\rangle\right\}$ as an orthonormal basis. Suppose we prepare the Arrow in the state $\left(b_{+}\right)$(i.e. pointing in the positive $\overrightarrow{\boldsymbol{b}}$ direction in physical space, at an angle $\theta_{b_{+} a_{+}}$to the positive $\overrightarrow{\boldsymbol{a}}$ direction and an angle $\theta_{b_{+} a_{-}}$to the negative $\overrightarrow{\boldsymbol{a}}$ direction). We then represent the state $\left(b_{+}\right)$in the Hilbert space $\mathcal{H}_{\mathrm{A}}$ using the Pythagorean construction used by Hughes [5] to obtain

$$
\begin{equation*}
\left|b_{+}^{A}\right\rangle=f_{a_{+}}^{1 / 2}\left(\theta_{b_{+} a_{+}}\right)\left|a_{+}^{A}\right\rangle+f_{a_{-}}^{1 / 2}\left(\theta_{b_{+} a_{-}}\right)\left|a_{-}^{A}\right\rangle \tag{2.3a}
\end{equation*}
$$

where the probabilities of getting the results $a_{+}$and $a_{-}$when starting from state $\left(b_{+}\right)$are $\mathrm{P}_{\left(b_{+}\right)}\left(a_{+}\right)=f_{a_{+}}\left(\theta_{b_{+} a_{+}}\right)$and $\mathrm{P}_{\left(b_{+}\right)}\left(a_{-}\right)=f_{a_{-}}\left(\theta_{b_{+} a_{-}}\right)$respectively. If instead we prepare the Arrow in the state ( $b_{-}$) (i.e. pointing in the negative $\overrightarrow{\boldsymbol{b}}$ direction, at an angle $\theta_{b_{-} a_{+}}$to the positive $\overrightarrow{\boldsymbol{a}}$ direction and an angle $\theta_{b_{-} a_{-}}$to the negative $\overrightarrow{\boldsymbol{a}}$ direction), we can similarly represent the state ( $b_{-}$) in the Hilbert space $\mathcal{H}_{\mathrm{A}}$ as

$$
\begin{equation*}
\left|b_{-}^{A}\right\rangle=-f_{a_{+}}^{1 / 2}\left(\theta_{b_{-} a_{+}}\right)\left|a_{+}^{A}\right\rangle+f_{a_{-}}^{1 / 2}\left(\theta_{b_{-} a_{-}}\right)\left|a_{-}^{A}\right\rangle \tag{2.3b}
\end{equation*}
$$

where we have introduced the minus sign as a choice of arbitrary phase that will be useful later. Taking into account the normalization of the probabilities, equations (2.3) can then be written

$$
\begin{align*}
& \left|b_{+}^{A}\right\rangle=f_{a_{+}}^{1 / 2}\left(\theta_{b_{+} a_{+}}\right)\left|a_{+}^{A}\right\rangle+\left(1-f_{a_{+}}\left(\theta_{b_{+} a_{+}}\right)\right)^{1 / 2}\left|a_{-}^{A}\right\rangle  \tag{2.4a}\\
& \left|b_{-}^{A}\right\rangle=-\left(1-f_{a_{-}}\left(\theta_{b_{-} a_{-}}\right)\right)^{1 / 2}\left|a_{+}^{A}\right\rangle+f_{a_{-}}^{1 / 2}\left(\theta_{b_{-} a_{-}}\right)\left|a_{-}^{A}\right\rangle \tag{2.4b}
\end{align*}
$$

As we have seen, the ( $a_{ \pm}$) states of the Arrow are represented in the Hilbert space $\mathcal{H}_{A}$ by the orthonormal basis $\left\{\left|a_{+}^{A}\right\rangle,\left|a_{-}^{A}\right\rangle\right\}$ that forms the axes, and the ( $b_{ \pm}$) states are represented in $\mathcal{H}_{A}$ by the vectors $\left|b_{ \pm}^{A}\right\rangle$ in equations (2.4). The states $\left(b_{ \pm}\right)$are of course orthogonal in the Hilbert space $\mathcal{H}_{B}$, since the orthonormal basis $\left\{\left|b_{+}^{B}\right\rangle\right.$, $\left.\left|b_{-}^{B}\right\rangle\right\}$ forms the axes of $\mathcal{H}_{B}$, but if we evaluate $\left\langle b_{-}^{A} \mid b_{+}^{A}\right\rangle$ using equations (2.4), we see that $\left|b_{+}^{A}\right\rangle$ and $\left|b_{-}^{A}\right\rangle$ are can only be orthogonal in Hilbert space $\mathcal{H}_{A}$ if
$\left(1-f_{a_{+}}\left(\theta_{b_{+} a_{+}}\right)\right)^{1 / 2} f_{a_{-}}^{1 / 2}\left(\theta_{b_{-} a_{-}}\right)-\left(1-f_{a_{-}}\left(\theta_{b_{-} a_{-}}\right)\right)^{1 / 2} f_{a_{+}}^{1 / 2}\left(\theta_{b_{+} a_{+}}\right)=0$
which is not in general satisfied unless some of the angles and probability functions $f$ are equal. Attempting to represent the ( $b_{ \pm}$) states in Hilbert space $\mathcal{H}_{A}$ and using the Born rule $\mathrm{P}_{\Psi}\left(b_{ \pm}\right)=\left|\left\langle b_{ \pm}^{A} \mid \psi^{\mathrm{A}}\right\rangle\right|^{2}$ will therefore yield incorrect results for the probability of outcomes of measurements of the variable $B$, because the basic exclusivity rule of probability is violated if the $\left|b_{ \pm}^{A}\right\rangle$ states are not orthogonal (e.g. $\mathrm{P}_{\left(b_{+}\right)}\left(b_{-}\right)=\left|\left\langle b_{-}^{A} \mid b_{+}^{A}\right\rangle\right|^{2} \neq 0$ ).

But why are the $\left(b_{ \pm}\right)$states not in general orthogonal in $\mathcal{H}_{A}$ when the variables A and B are incompatible? Because the condition (2.5) for orthogonality can be satisfied only for certain very restricted values of the probability functions: specifically, we must have $f_{a_{+}}\left(\theta_{b_{+} a_{+}}\right)=f_{a_{-}}\left(\theta_{b_{-} a_{-}}\right)$, which is only possible when certain symmetries are present. And why is non-orthogonality not an issue when the variables A and B are compatible? Because when the variables are compatible, the eigenstates of A are also eigenstates of B , so the $\left(b_{ \pm}\right)$states and ( $a_{ \pm}$) states are one and the same, namely $\left(a_{ \pm}, b_{ \pm}\right)$or ( $a_{ \pm}, b_{\mp}$ ).

Returning to equations (2.4) and introducing the $\mathrm{C}_{2}$ symmetry of the system (the Arrow itself) and of the apparatus (the magnets in specific directions that are used to make measurements of the state of the arrow) results in the following symmetries between the angles (Fig. 3),

$$
\begin{align*}
& \theta_{b_{+} a_{+}}=\theta_{b_{-} a_{-}}=\theta  \tag{2.6a}\\
& \theta_{a_{-} b_{+}}=\theta_{a_{+} b_{-}}=\pi-\theta \tag{2.6b}
\end{align*}
$$

so that equations (2.4) now become

$$
\begin{align*}
& \left|b_{+}^{A}\right\rangle=f_{a_{+}}^{1 / 2}(\theta)\left|a_{+}^{A}\right\rangle+\left(1-f_{a_{+}}(\theta)\right)^{1 / 2}\left|a_{-}^{A}\right\rangle  \tag{2.7a}\\
& \left|b_{-}^{A}\right\rangle=-\left(1-f_{a_{-}}(\theta)\right)^{1 / 2}\left|a_{+}^{A}\right\rangle+f_{a_{-}}^{1 / 2}(\theta)\left|a_{-}^{A}\right\rangle \tag{2.7b}
\end{align*}
$$



Fig. 3 Symmetries between angles as a result of $\mathrm{C}_{2}$ apparatus symmetry in the Arrow system

However, $\left|b_{+}^{A}\right\rangle$ and $\left|b_{-}^{A}\right\rangle$ are still not orthogonal. Although the angles show these symmetries, the probabilities do not necessarily show the same symmetries, e.g. $\mathrm{P}_{\left(b_{+}\right)}\left(a_{+}\right)=f_{a_{+}}(\theta)$ might be different from $\mathrm{P}_{\left(b_{-}\right)}\left(a_{-}\right)=f_{a_{-}}(\theta)$ if there was an apparatus asymmetry between the $a_{+}$and $a_{-}$directions. An example of an apparatus asymmetry would be having the magnets in the $a_{+}$direction stronger than those in the $a_{-}$direction, resulting in the probability functions $f_{a_{+}}$and $f_{a_{-}}$ being unequal. But even with a $\mathrm{C}_{2}$ apparatus symmetry between the $a_{+}$and $a_{-}$ directions, the probability functions $f_{a_{+}}$and $f_{a_{-}}$still cannot be equated unless the $\mathrm{C}_{2}$ symmetry is extended to the spacetime that includes the two directions, giving

$$
\begin{equation*}
f_{a_{+}}(\theta)=f_{a_{-}}(\theta) \equiv f_{A}(\theta) \tag{2.8}
\end{equation*}
$$

which results in equations (2.7) becoming

$$
\begin{align*}
& \left|b_{+}^{A}\right\rangle=f_{A}^{1 / 2}(\theta)\left|a_{+}^{A}\right\rangle+\left(1-f_{A}(\theta)\right)^{1 / 2}\left|a_{-}^{A}\right\rangle  \tag{2.9a}\\
& \left|b_{-}^{A}\right\rangle=-\left(1-f_{A}(\theta)\right)^{1 / 2}\left|a_{+}^{A}\right\rangle+f_{A}^{1 / 2}(\theta)\left|a_{-}^{A}\right\rangle \tag{2.9b}
\end{align*}
$$

We see that the vectors representing the eigenstates $\left(b_{+}\right)$and $\left(b_{-}\right)$of B are now orthogonal not only in Hilbert space $\mathcal{H}_{B}\left(\left\langle b_{-}^{B} \mid b_{+}^{B}\right\rangle=0\right)$ but also in Hilbert space
$\mathcal{H}_{A}\left(\left\langle b_{-}^{A} \mid b_{+}^{A}\right\rangle=0\right.$, as in equation (2.5)) but only because symmetry has been introduced!

In the absence of symmetry, the basis $\left\{\left|b_{+}^{A}\right\rangle,\left|b_{-}^{A}\right\rangle\right\}$ in $\mathcal{H}_{A}$ is rotated with respect to the basis $\left\{\left|a_{+}^{A}\right\rangle,\left|a_{-}^{A}\right\rangle\right\}$ by a non-rigid rotation, in which $\left|b_{+}^{A}\right\rangle$ and $\left|b_{-}^{A}\right\rangle$ are rotated through different angles, making them non-orthogonal in $\mathcal{H}_{A}$ (Fig. 4).


Fig. 4 In the absence of symmetry, the basis $\left\{\left|b_{+}^{A}\right\rangle,\left|b_{-}^{A}\right\rangle\right\}$ in the Arrow system is in general rotated with respect to the basis $\left\{\left|a_{+}^{A}\right\rangle,\left|a_{-}^{A}\right\rangle\right\}$ by a non-rigid rotation

The two basis sets can be written in column vector form as

$$
\begin{equation*}
\boldsymbol{a}_{+}^{A}=\binom{1}{0}, \boldsymbol{a}_{-}^{A}=\binom{0}{1} \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{b}_{+}^{A}=\binom{f_{a_{+}}^{1 / 2}(\theta)}{\left(1-f_{a_{+}}(\theta)\right)^{1 / 2}}, \quad \boldsymbol{b}_{-}^{A}=\binom{-\left(1-f_{a_{-}}(\theta)\right)^{1 / 2}}{f_{a_{-}}^{1 / 2}(\theta)} \tag{2.11}
\end{equation*}
$$

and are connected by

$$
\begin{equation*}
\boldsymbol{b}_{ \pm}^{A}=\mathbf{M}_{\mathrm{ab}} \boldsymbol{a}_{ \pm}^{A} \tag{2.12}
\end{equation*}
$$

where the rotation matrix $\mathbf{M}_{\mathbf{a b}}$ has the form

$$
\mathbf{M}_{\mathbf{a b}}=\left(\begin{array}{cc}
\cos \alpha_{+} & -\sin \alpha_{-}  \tag{2.13}\\
\sin \alpha_{+} & \cos \alpha_{-}
\end{array}\right)
$$

with the respective rotation angles given by

$$
\begin{equation*}
\alpha_{ \pm}=\tan ^{-1} \frac{\left(1-f_{a_{ \pm}}(\theta)\right)^{1 / 2}}{f_{a_{ \pm}}^{1 / 2}(\theta)} \tag{2.14}
\end{equation*}
$$

Introduction of $\mathrm{C}_{2}$ symmetry enables $f_{a_{+}}$and $f_{a_{-}}$to be equated, so that $\left|b_{+}^{A}\right\rangle$ and $\left|b_{-}^{A}\right\rangle$ are now obtained by the rotation of $\left|a_{+}^{A}\right\rangle$ and $\left|a_{-}^{A}\right\rangle$ through the same angle $\alpha$, making the vectors representing $\left(b_{+}\right)$and $\left(b_{-}\right)$orthogonal not only in $\mathcal{H}_{B}$ but also in $\mathcal{H}_{A}$ (Fig. 5).


Fig. 5 In the presence of $\mathrm{C}_{2}$ symmetry, the basis $\left\{\left|b_{+}^{A}\right\rangle,\left|b_{-}^{A}\right\rangle\right\}$ of the Arrow system is rotated with respect to the basis $\left\{\left|a_{+}^{A}\right\rangle,\left|a_{-}^{A}\right\rangle\right\}$ by a rigid rotation

A key point that will be very important later is that, when symmetry is introduced, the rotation matrix becomes unitary,

$$
\mathbf{M}_{\mathbf{a b}}=\left(\begin{array}{cc}
\cos \alpha & -\sin \alpha  \tag{2.15}\\
\sin \alpha & \cos \alpha
\end{array}\right)
$$

and multiplication by a unitary matrix preserves both lengths and angles, thus effecting a rigid rotation.

Once the vectors representing $\left(b_{+}\right)$and $\left(b_{-}\right)$become orthogonal in both $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$ (after introduction of $\mathrm{C}_{2}$ symmetry), probabilities of $b_{i}$ outcomes can then be correctly calculated in either $\mathcal{H}_{A}$ or $\mathcal{H}_{B}$ (as can probabilities of $a_{i}$ outcomes), so it is no longer necessary to have two Hilbert spaces and it becomes possible to combine the Hilbert spaces $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$ into one (see also Section 3 for more details). And if all the states $\left(a_{i}\right)$ and $\left(b_{j}\right)$ can now be represented in just one Hilbert space thanks to $\mathrm{C}_{2}$ symmetry, it means that according to Gleason's theorem we must now write probabilities in the Born form as $\mathrm{P}_{\left(b_{j}\right)}\left(a_{i}\right)=\left|\left\langle a_{i} \mid b_{j}\right\rangle\right|^{2}=$ $\cos ^{2} \omega(\theta)$ ! But it is important to distinguish clearly between the angle $\theta$ between the $b_{j}$ and $a_{i}$ directions in physical space and the angle $\omega(\theta)$ between the corresponding vectors $\left|b_{i}\right\rangle$ and $\left|a_{i}\right\rangle$ in Hilbert space: although the angle $\omega$ is a function of $\theta$, the fact that we now have Born "cosine-squared" dependence on $\omega$ does not in any way imply a "cosine-squared" dependence of $f(\theta)$ on $\theta$. However, having $\left|b_{+}^{A}\right\rangle$ and $\left|b_{-}^{A}\right\rangle$ orthogonal when $\mathrm{C}_{2}$ symmetry is introduced does place an additional constraint $f(\theta)+f(\pi-\theta)=1$ on the form of the probability function (see equations (2.6) and Fig. 2) that now excludes $f(\theta)=1-\left(\frac{\theta}{\pi}\right)^{2}$ as a probability function, but still allows $f(\theta)=1-\frac{\theta}{\pi}$ and $f(\theta)=\cos ^{2} \frac{\theta}{2}$. So introducing enough symmetry into the Arrow system to produce a single Hilbert space with a single unrestricted Born Rule is one thing, but uniquely predicting the form of $f(\theta)$ is another. What would happen if we introduced more symmetry?

Extension of the $\mathrm{C}_{2}$ symmetry of the system-plus-apparatus also to $\mathrm{C}_{2}$ symmetry of spacetime enabled the probability functions of the two eigenstates to be equated, $f_{a_{+}}(\theta)=f_{a_{-}}(\theta)=f_{A}(\theta)$, but the variables $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$. could still have different probability functions, $f_{A}\left(\theta_{A}\right) \neq f_{B}\left(\theta_{B}\right) \neq f_{C}\left(\theta_{C}\right)$. If we now extend the symmetry of spacetime from mere $\mathrm{C}_{2}$ symmetry to complete isotropy of space, every direction $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$. becomes equivalent, so we must now take $f_{A}, f_{B}, f_{C} \ldots$ to be all the same function $f$ :

$$
\begin{equation*}
f_{A}(\theta)=f_{B}(\theta)=f_{C}(\theta)=\ldots \equiv f(\theta) \tag{2.16}
\end{equation*}
$$

But in order to fully incorporate isotropy of space, it is not enough to simply use the same probability function for all directions $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$; isotropy of space has the further implication that the results of measurements of any variable A (or B or C...) must be unchanged if the entire experiment, i.e. the system (the Arrow along the $\overrightarrow{\boldsymbol{n}}$ axis) together with the apparatus (the magnets along the $\overrightarrow{\boldsymbol{a}}$ axis), is rotated to a new orientation in space. In other words, we must ensure that physical rotation of the experiment through any angle $\alpha$ about the $y$-axis (i.e. perpendicular to the arrow) is a symmetry transformation.

Symmetry transformations leave a system "looking the same", so they have to preserve lengths and angles in physical space, which means they must be rigid; they must also leave experimental results unchanged, which means that the rigidity of symmetry transformations in physical space must carry over to a similar rigidity in Hilbert space, in order to conserve probabilities (which depend on angles in Hilbert space). Wigner proved in his famous theorem that symmetry transformations in Hilbert space are effected by operators that are linear and unitary. But Wigner's proof $[14,15]$ implicitly assumes a single Hilbert space, so his theorem is not (except in special cases) applicable to general incompatible variables, which require multiple Hilbert spaces. However, there is another symmetry condition that we expect to be obeyed in every separate Hilbert space of the GIV theory. This is the Haag-Wigner condition [16,17], which states that the operators that effect symmetry transformations in a Hilbert space must depend only on the relation between the two frames of reference in physical space-time, not on the intrinsic properties or absolute position in space-time of either frame, otherwise the homogeneity and isotropy of space-time would be violated. The simplest way to satisfy this condition is to require the symmetry operator to be linear; and a linear symmetry operator in a Hilbert space in which all vectors representing physical states are required to be normalized will automatically also be unitary [18], thus ensuring that the transformation effected by the operator is rigid, as required of a symmetry transformation.

In our forthcoming paper (see also Section 3 of this paper) we satisfy the Haag-Wigner condition for symmetry operators by requiring them to be linear (and therefore unitary). In the case of the Arrow system, we obtain a unitary rotation operator from the Haag-Wigner condition by requiring the operator to depend only upon $\theta$, the rotation angle in physical space, and not upon the angle giving the Arrow's initial orientation. We then show that this procedure yields exclusively the simple form

$$
\begin{equation*}
\omega(\theta)=\frac{1}{2} \theta \tag{2.17}
\end{equation*}
$$

and also the equivalence of all of the Hilbert spaces $\mathcal{H}_{A}, \mathcal{H}_{B}, \mathcal{H}_{C} \ldots$. Complete isotropy of space thus restricts the probability function to the unique form

$$
\begin{equation*}
f(\theta)=\cos ^{2} \omega(\theta)=\cos ^{2} \frac{\theta}{2} \tag{2.18}
\end{equation*}
$$

and requires only a single Hilbert space, which is precisely the result for the quantum mechanics of a spin $1 / 2$ system - and yet this result emerges for an entirely classical probabilistic system possessing incompatible variables (our Arrow) when we apply symmetry constraints! Hence we have the situation that in principle we can construct a classical Arrow system with almost any choice for the probability
function $f(\theta)$ (and with most choices not having the symmetries we have discussed), whereas a spin $1 / 2$ particle apparently has no choice but to incorporate the symmetries of spacetime.

## 3 The General Incompatible Variables (GIV) theory

The Arrow model introduced in Section 2 provided a means to uncover and highlight the role of symmetry in probabilistic theories of systems possessing incompatible variables. We now generalize our treatment of the classical Arrow system to provide a general framework - the General Incompatible Variables (GIV) formalism - for the treatment of any system, whether QM or non-QM, that has incompatible variables. We will see that QM is just a special case of GIV theories in which the fundamental variables are symmetries. The GIV formalism is basically the standard classical probability theory of Kolmogorov extended to include incompatible variables with the introduction of a separate Hilbert space for each incompatible variable. We first demonstrate the GIV formalism for the simple case of just two incompatible variables, $A$ and $B$, each of which has just two possible measurement outcomes, $a_{1}$ and $a_{2}$ for A, and $b_{1}$ and $b_{2}$ for B. Each state $(\psi)$ of the system can be thought of as having "components" in each of the separate Hilbert spaces:

$$
\begin{equation*}
(\psi)=\left(\left|\Psi^{A}\right\rangle,\left|\psi^{B}\right\rangle\right) \tag{3.1}
\end{equation*}
$$

The fundamental pure states (also called value states or eigenstates) of this system are then given in the GIV formalism as

$$
\begin{align*}
& \left(a_{i}\right)=\left(\left|a_{i}^{A}\right\rangle,\left|a_{i}^{B}\right\rangle\right), \quad i=1,2  \tag{3.2a}\\
& \left(b_{i}\right)=\left(\left|b_{i}^{A}\right\rangle,\left|b_{i}^{B}\right\rangle\right), \quad i=1,2 \tag{3.2b}
\end{align*}
$$

which requires two Hilbert spaces, $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$, where $\mathcal{H}_{A}$ is spanned by the orthonormal basis $\left\{\left|a_{1}^{A}\right\rangle,\left|a_{2}^{A}\right\rangle\right\}$ and $\mathcal{H}_{B}$ by the orthonormal basis $\left\{\left|b_{1}^{B}\right\rangle,\left|b_{2}^{B}\right\rangle\right\}$. Note that the vectors representing the states $\left(b_{1}\right)$ and $\left(b_{2}\right)$ are orthogonal in $\mathcal{H}_{B}$, but are (generally) not orthogonal in $\mathcal{H}_{A}$ (as we saw in Fig. 4 for the Arrow), so the probabilities of the outcomes $b_{1}$ and $b_{2}$ cannot in general be calculated in $\mathcal{H}_{A}$, they can only be calculated in $\mathcal{H}_{B}$. Similarly, the vectors representing the states $\left(a_{1}\right)$ and $\left(a_{2}\right)$ are orthogonal in $\mathcal{H}_{A}$, but are (generally) not orthogonal in $\mathcal{H}_{B}$, so the probabilities of the outcomes $a_{1}$ and $a_{2}$ can only be calculated in $\mathcal{H}_{A}$. Choosing corresponding axes of $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$ to be parallel (or superimposed), we write the basis vectors of $\mathcal{H}_{A}$ as

$$
\begin{equation*}
\boldsymbol{a}_{1}^{A}=\binom{1}{0}, \boldsymbol{a}_{2}^{A}=\binom{0}{1} \tag{3.3a}
\end{equation*}
$$

the basis vectors of $\mathcal{H}_{B}$ as

$$
\begin{equation*}
\boldsymbol{b}_{1}^{B}=\binom{1}{0}, \boldsymbol{b}_{2}^{B}=\binom{0}{1} \tag{3.3b}
\end{equation*}
$$

and the variables A and B as the matrices

$$
\begin{align*}
& \mathbf{A}_{\mathrm{A}}=\left(\begin{array}{cc}
a_{1} & 0 \\
0 & a_{2}
\end{array}\right)  \tag{3.4a}\\
& \mathbf{B}_{\mathrm{B}}=\left(\begin{array}{cc}
b_{1} & 0 \\
0 & b_{2}
\end{array}\right) \tag{3.4b}
\end{align*}
$$

where the subscripts A and B indicate that the matrix $\mathbf{A}_{\mathrm{A}}$ acts as an operator in $\mathcal{H}_{A}$, and the matrix $\mathbf{B}_{\mathrm{B}}$ acts as an operator in $\mathcal{H}_{B}$. We then have the eigenvalue equations

$$
\begin{array}{ll}
\mathbf{A}_{\mathrm{A}} \boldsymbol{a}_{i}^{A}=a_{i} \boldsymbol{a}_{i}^{A}, & i=1,2 \\
\mathbf{B}_{\mathrm{B}} \boldsymbol{b}_{i}^{B}=b_{i} \boldsymbol{b}_{i}^{B}, & i=1,2 \tag{3.5b}
\end{array}
$$

in matrix notation, and

$$
\begin{align*}
& \hat{A}_{A}\left|a_{i}^{A}\right\rangle=a_{i}\left|a_{i}^{A}\right\rangle, i=1,2  \tag{3.6a}\\
& \hat{B}_{B}\left|b_{i}^{B}\right\rangle=b_{i}\left|b_{i}^{B}\right\rangle, i=1,2 \tag{3.6b}
\end{align*}
$$

in Dirac notation. This notation is extended in an obvious way when there are additional incompatible variables $\mathrm{C}, \mathrm{D}, \ldots$ and more than two possible values for the variables. Finally, we note that equations (3.3) exhibit a simple feature that will become crucial later on, namely that

$$
\begin{align*}
& \boldsymbol{a}_{1}^{A}=\boldsymbol{b}_{1}^{B}=\binom{1}{0}  \tag{3.7a}\\
& \boldsymbol{a}_{2}^{A}=\boldsymbol{b}_{2}^{B}=\binom{0}{1} \tag{3.7b}
\end{align*}
$$

and, more generally,

$$
\begin{equation*}
\boldsymbol{a}_{i}^{A}=\boldsymbol{b}_{i}^{B}=\boldsymbol{c}_{i}^{C}=\ldots, i=1,2,3 \ldots \ldots \tag{3.8}
\end{equation*}
$$

We would like to point out that although the use of multiple Hilbert spaces, each with a separate Born rule, may seem an unnecessary complication, it is actually an almost trivial construction that can be applied to any probabilistic theory - and it is essential in the case of incompatible variables, because calculating the outcomes of measurements of B in Hilbert space $\mathcal{H}_{A}$ and of A in $\mathcal{H}_{B}$ cannot be done correctly in the general case, as we have seen. Introducing the multiple spaces also provides the most direct route to quantum mechanics as a special case, as we will show.

We now consider how symmetry transformations work in the multiple Hilbert spaces of a GIV system with incompatible variables A, B, C.... if the system has the symmetry of a group

$$
\begin{equation*}
G=\{E, X, Y, Z \ldots\} \tag{3.9}
\end{equation*}
$$

of symmetries $\mathrm{E}, \mathrm{X}, \mathrm{Y}, \mathrm{Z} \ldots$... where E is the identity element of the group. Applying the symmetry transformation X to the system in any admissible pure state, say $\left(a_{i}\right)$, will transform the state into another admissible pure state [17], say $\left(a_{i}^{\prime}\right)$ :

$$
\begin{equation*}
\mathrm{X}\left(a_{i}\right)=\left(a_{i}^{\prime}\right) \tag{3.10}
\end{equation*}
$$

Since the GIV states are sequences of vectors in the Hilbert spaces $\mathcal{H}_{A}, \mathcal{H}_{B}$, $\mathcal{H}_{C} \ldots$, the effect of X is to transform the sequence of vectors

$$
\begin{equation*}
\left(a_{i}\right)=\left(\left|a_{i}^{A}\right\rangle,\left|a_{i}^{B}\right\rangle,\left|a_{i}^{C}\right\rangle \ldots \ldots\right) \tag{3.11}
\end{equation*}
$$

into the sequence

$$
\begin{equation*}
\left(a_{i}^{\prime}\right)=\left(\left|a_{i}^{\prime A}\right\rangle,\left|a_{i}^{\prime B}\right\rangle,\left|a_{i}^{\prime C}\right\rangle \ldots \ldots\right) \tag{3.12}
\end{equation*}
$$

So equation (3.10) in fact represents a sequence of equations written in terms of operators acting in the individual Hilbert spaces as

$$
\begin{align*}
\hat{X}_{A}\left|a_{i}^{A}\right\rangle & =\left|a_{i}^{\prime A}\right\rangle  \tag{3.13a}\\
\hat{X}_{B}\left|a_{i}^{B}\right\rangle & =\left|a_{i}^{\prime B}\right\rangle  \tag{3.13b}\\
\hat{X}_{C}\left|a_{i}^{C}\right\rangle & =\left|a_{i}^{\prime C}\right\rangle \tag{3.13c}
\end{align*}
$$

where $\hat{X}_{A}$ is the operator corresponding to the symmetry X acting in $\mathcal{H}_{A}, \hat{X}_{B}$ is the corresponding operator in $\mathcal{H}_{B}$, etc., and similar equations are obtained for any of the symmetries $\mathrm{E}, \mathrm{X}, \mathrm{Y}, \mathrm{Z} \ldots \ldots$. in the group G . Since the vector $\left|a_{i}^{\prime A}\right\rangle$ is a vector in $\mathcal{H}_{A}$, it can be expanded in terms of the orthonormal basis $\left\{\left|a_{k}^{A}\right\rangle\right\}$ of $\mathcal{H}_{A}$, so that equation (3.13a) becomes

$$
\begin{equation*}
\hat{X}_{A}\left|a_{i}^{A}\right\rangle=\sum_{k}\left|a_{k}^{A}\right\rangle\left\langle a_{k}^{A}\right| \hat{X}_{A}\left|a_{i}^{A}\right\rangle=\sum_{k}\left[\Gamma_{\mathrm{A}}(\mathrm{X})\right]_{k i}\left|a_{k}^{A}\right\rangle \tag{3.14}
\end{equation*}
$$

where the expansion coefficients are the elements

$$
\begin{equation*}
\left[\Gamma_{\mathrm{A}}(\mathrm{X})\right]_{k i}=\left\langle a_{k}^{A}\right| \hat{X}_{A}\left|a_{i}^{A}\right\rangle \tag{3.15}
\end{equation*}
$$

of a square matrix $\Gamma_{A}(X)$. Similar matrices $\boldsymbol{\Gamma}_{A}(E), \Gamma_{A}(Y) \ldots$. can be defined for all the symmetry elements, and the set of matrices

$$
\begin{equation*}
\Gamma_{\mathrm{A}}=\left\{\boldsymbol{\Gamma}_{\mathrm{A}}(\mathrm{E}), \boldsymbol{\Gamma}_{\mathrm{A}}(\mathrm{X}), \boldsymbol{\Gamma}_{\mathrm{A}}(\mathrm{Y}), \boldsymbol{\Gamma}_{\mathrm{A}}(\mathrm{Z}) \ldots\right\} \tag{3.16}
\end{equation*}
$$

together form a representation of the group G (using the eigenstates of $\hat{A}$ as basis vectors), provided they satisfy the group multiplication table - but this is only possible if the operators $\hat{X}_{A}, \hat{Y}_{A}, \hat{Z}_{A} \ldots$. are linear operators as we discuss in detail in our forthcoming paper.

We would like to be able to require linearity of the symmetry operators $\hat{X}_{A}, \hat{Y}_{A}, \hat{Z}_{A} \ldots$ operating in Hilbert space $\mathcal{H}_{A}$ (and similarly for the symmetry operators $\hat{X}_{B}, \hat{Y}_{B}, \hat{Z}_{B} \ldots$ operating in Hilbert space $\mathcal{H}_{B}$, and for the corresponding symmetry operators in all the other multiple Hilbert spaces $\mathcal{H}_{C}, \mathcal{H}_{D} \ldots$.) to ensure that the set of matrices in equation (3.16) will indeed be a representation of the group G. However, we cannot use Wigner's theorem here to assign linearity and unitarity to all the symmetry operators in the multiple Hilbert spaces of GIV theory, because the proof of Wigner's theorem [14,15] assumes a single Hilbert space - along with its unrestricted Born expression for probability - which is the very thing we wish to prove! We instead follow the procedure used in Section 2 for the Arrow, and again appeal to the Haag-Wigner condition [16,17], which requires symmetry operators to depend only on the relation between two frames of reference, and not on the intrinsic properties of either; this condition is most easily satisfied by operators that are linear, and linear operators possessing an inverse and operating in a Hilbert space with vectors normalized to unit magnitude are automatically unitary [18]. The Haag-Wigner condition thus leads us to require linearity (and hence also unitarity) for the symmetry operators $\hat{X}_{A}, \hat{Y}_{A}, \hat{Z}_{A} \ldots$ operating in Hilbert space $\mathcal{H}_{A}$ (and similarly for the symmetry operators $\widehat{X}_{B}, \widehat{Y}_{B}, \hat{Z}_{B} \ldots$ operating in Hilbert space $\mathcal{H}_{B}$, and similarly for the symmetry operators in all the other Hilbert spaces), and then to investigate where this requirement leads - and we will find that it leads to QM .

The linearity-unitarity of the matrices $\boldsymbol{\Gamma}_{\mathrm{A}}(\mathrm{X}), \Gamma_{\mathrm{A}}(\mathrm{Y}), \boldsymbol{\Gamma}_{\mathrm{A}}(\mathrm{Z}) \ldots$ now confirms that they do indeed represent the operators $\hat{X}_{A}, \hat{Y}_{A}, \hat{Z}_{A} \ldots$ (up to phase factors), so we now have the following sets of linear-unitary representatives of the group $\mathrm{G}=$ $\{\mathrm{E}, \mathrm{X}, \mathrm{Y}, \mathrm{Z} \ldots\}:$

$$
\begin{align*}
& \Gamma_{\mathrm{A}}=\left\{\mathbf{E}_{\mathrm{A}}, \mathbf{X}_{\mathrm{A}}, \mathbf{Y}_{\mathrm{A}}, \mathbf{Z}_{\mathrm{A}} \ldots\right\}  \tag{3.17a}\\
& \Gamma_{\mathrm{B}}=\left\{\mathbf{E}_{\mathrm{B}}, \mathbf{X}_{\mathrm{B}}, \mathbf{Y}_{\mathrm{B}}, \mathbf{Z}_{\mathrm{B}} \ldots\right\}  \tag{3.17b}\\
& \Gamma_{\mathrm{C}}=\left\{\mathbf{E}_{\mathrm{C}}, \mathbf{X}_{\mathrm{C}}, \mathbf{Y}_{\mathrm{C}}, \mathbf{Z}_{\mathrm{C}} \ldots\right\} \tag{3.17c}
\end{align*}
$$

where the notation has been simplified by writing the matrices as $\mathbf{X}_{A}, \mathbf{Y}_{A} \ldots$ rather than $\Gamma_{A}(\mathrm{X}), \Gamma_{\mathrm{A}}(\mathrm{Y}) \ldots$. The respective representations $\Gamma_{\mathrm{A}}, \Gamma_{\mathrm{B}}, \Gamma_{\mathrm{C}} \ldots$ use the eigenstates of the respective fundamental variables $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$ as basis vectors.

We next investigate what happens in the GIV framework when we take the fundamental variables A, B, C... to be the symmetries X, Y, Z... of the system, so that the group $G=\{E, X, Y, Z \ldots\}$ now becomes

$$
\begin{equation*}
G=\{E, A, B, C \ldots\} \tag{3.18}
\end{equation*}
$$

The fundamental pure states $\left(a_{i}\right),\left(b_{i}\right),\left(c_{i}\right) \ldots$ are the eigenstates of the respective variables $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$, which are now symmetries - and as eigenstates (i.e. "value states") of symmetry operators, they are now states of definite symmetry. So whereas previously a symmetry X changed a general GIV state $\left(a_{i}\right)$ into a different state $\left(a_{i}^{\prime}\right)$ as in equation (3.10), $\mathrm{X}\left(a_{i}\right)=\left(a_{i}^{\prime}\right)$, the fundamental pure states $\left(a_{i}\right)$ are now invariant under the symmetry A, so equation (3.10) now becomes

$$
\begin{equation*}
\mathrm{A}\left(a_{i}\right)=\left(a_{i}\right) \tag{3.19}
\end{equation*}
$$

and equations (3.13) now become the set of eigenvalue equations

$$
\begin{align*}
& \hat{A}_{A}\left|a_{i}^{A}\right\rangle=a_{i}^{A}\left|a_{i}^{A}\right\rangle  \tag{3.20a}\\
& \hat{A}_{B}\left|a_{i}^{B}\right\rangle=a_{i}^{B}\left|a_{i}^{B}\right\rangle  \tag{3.20b}\\
& \hat{A}_{C}\left|a_{i}^{C}\right\rangle=a_{i}^{C}\left|a_{i}^{C}\right\rangle \tag{3.20c}
\end{align*}
$$

in their respective Hilbert spaces, with generally complex eigenvalues of unit magnitude since the symmetry operators are unitary, and similarly $\mathrm{B}\left(b_{i}\right)=\left(b_{i}\right)$ represents the set of equations

$$
\begin{align*}
& \hat{B}_{A}\left|b_{i}^{A}\right\rangle=b_{i}^{A}\left|b_{i}^{A}\right\rangle  \tag{3.21a}\\
& \hat{B}_{B}\left|b_{i}^{B}\right\rangle=b_{i}^{B}\left|b_{i}^{B}\right\rangle  \tag{3.21b}\\
& \hat{B}_{C}\left|b_{i}^{C}\right\rangle=b_{i}^{C}\left|b_{i}^{C}\right\rangle \tag{3.21c}
\end{align*}
$$

For connected Lie groups $\mathrm{G}[14]$ the operators $\hat{A}_{A}, \hat{A}_{B} \ldots, \hat{B}_{A}, \widehat{B}_{B} \ldots$ can be taken to be either the symmetry operators themselves or their generators, because the eigenstates of the symmetry operators are also eigenstates of the generators, e.g. in the case of rotational symmetry in QM , the angular momentum operator $\hat{J}_{z}$ is the generator of rotations $\hat{R}^{z}(\phi)=e^{-(i / \hbar) \phi \hat{J}_{z}}$ through an angle $\phi$ about the $z$-axis, and the angular momentum states $|J, M\rangle$ are eigenstates of both $\hat{R}^{z}(\phi)$ and $\hat{J}_{z}$.

Applying the linearity-unitarity requirement to all the symmetry operators $\hat{A}_{A}, \hat{A}_{B} \ldots, \hat{B}_{A}, \hat{B}_{B} \ldots$ in each of the multiple Hilbert spaces now gives us the following sets of linear-unitary representatives of the group $G=\{E, A, B, C \ldots\}$ :

$$
\begin{equation*}
\Gamma_{\mathrm{A}}=\left\{\mathbf{E}_{\mathrm{A}}, \mathbf{A}_{\mathrm{A}}, \mathbf{B}_{\mathrm{A}}, \mathbf{C}_{\mathrm{A}} \ldots\right\} \tag{3.22a}
\end{equation*}
$$

$$
\begin{align*}
& \Gamma_{\mathrm{B}}=\left\{\mathbf{E}_{\mathrm{B}}, \mathbf{A}_{\mathrm{B}}, \mathbf{B}_{\mathrm{B}}, \mathbf{C}_{\mathrm{B}} \ldots\right\}  \tag{3.22b}\\
& \Gamma_{\mathrm{C}}=\left\{\mathbf{E}_{\mathrm{C}}, \mathbf{A}_{\mathrm{C}}, \mathbf{B}_{\mathrm{C}}, \mathbf{C}_{\mathrm{C}} \ldots\right\} \tag{3.22c}
\end{align*}
$$

which are equations (3.17) with the general symmetries $\mathrm{X}, \mathrm{Y}, \mathrm{Z} \ldots$.. replaced by the symmetries $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$ which are now also the variables.

We focus here on equations (3.22) as they apply to irreducible representations, which we hereafter refer to as "irreps" [19], of the group G. We will see that these irreps describe elementary systems, in line with Wigner's discovery [15-17] that the manifold of states of an elementary particle constitute a representation space for an irrep of the appropriate group of symmetries of nature (the Poincaré group for relativistic elementary particles, the full rotation group for electrons in one-electron atoms, and so on). A familiar example is provided by the hydrogenic $\mathrm{s}, \mathrm{p}, \mathrm{d} .$. electrons: their angular wavefunctions are the spherical harmonics, which form the basis functions for the respective irreps $\Gamma^{(0)}, \Gamma^{(1)}, \Gamma^{(2)} \ldots$ of the full rotation group $\mathrm{R}_{3}$. In standard QM there is only one Hilbert space, and only one irrep, for a given elementary physical system (e.g. only one irrep $\Gamma^{(0)}$ for an s-electron). But in GIV theory, there are many Hilbert spaces $\mathcal{H}_{A}, \mathcal{H}_{B}, \mathcal{H}_{C} \ldots$, and equally many irreps $\Gamma_{\mathrm{A}}, \Gamma_{\mathrm{B}}, \Gamma_{\mathrm{C}} \ldots$, all applying to the same physical system, each based on a different set of basis vectors (the eigenvectors of the incompatible variables A, B, C... respectively). So the representations in equations (3.22) are now taken to be the irreps $\Gamma_{\mathrm{A}}, \Gamma_{\mathrm{B}}, \Gamma_{\mathrm{C}} \ldots$ based on the sets of basis vectors $\left\{\left|a_{i}^{A}\right\rangle\right\}$, $\left\{\left|b_{i}^{B}\right\rangle\right\},\left\{\left|c_{i}^{C}\right\rangle\right\} \ldots$ of the Hilbert spaces $\mathcal{H}_{A}, \mathcal{H}_{B}, \mathcal{H}_{C} \ldots$ respectively.

To make further progress and to be consistent with well-known experimental facts, we now require these irreps $\Gamma_{A}, \Gamma_{B}, \Gamma_{\mathrm{C}} \ldots$ in the different Hilbert spaces to be equivalent. The existence of "equivalent" matrix representations describing the effect of symmetry operations on alternate sets of basis functions is not unusual in standard QM, and arises naturally when variables are incompatible. For example, the basis vectors for irrep $\Gamma^{(0)}$ of the full rotation group $\mathrm{R}_{3}$ are the vectors $|J, M\rangle$, which are normally taken to be the eigenstates $\left|J, M_{z}\right\rangle$ of $\hat{J}^{2}$ and $\hat{J}_{z}$. But the eigenstates $\left|J, M_{x}\right\rangle$ of $\hat{J}_{x}$, which does not commute with $\hat{J}_{z}$, would form equally valid or "equivalent" basis vectors for the irrep $\Gamma^{(J)}$, as would the eigenstates $\left|J, M_{y}\right\rangle$ of $\hat{J}_{y}$, leading to an identical character table and thence identical physical results. Another example of the eigenstates of incompatible variables forming alternate but "equivalent" basis functions for irreps occurs in the Poincaré group, where the eigenvectors $|k\rangle$ of the spatial translation operator are normally taken to be the basis vectors for the irreps, but one could alternatively (and equivalently) use the eigenvectors $|K\rangle$ of the Lorentz boost operator, which does not commute
with the translation operator. In standard QM, the definition of the equivalence of two representations $\Gamma_{\varphi}$ and $\Gamma_{\varphi}$, based respectively on two different sets of basis vectors $\left\{\boldsymbol{\varphi}_{i}\right\}$ and $\left\{\boldsymbol{\varphi}_{i}^{\prime}\right\}$, is that corresponding matrix representatives $\boldsymbol{\Gamma}_{\varphi}(\mathrm{X})$ and $\boldsymbol{\Gamma}_{\varphi_{\prime}}(\mathrm{X})$ for a given symmetry operation X of the group are related by a similarity transformation

$$
\begin{equation*}
\boldsymbol{\Gamma}_{\varphi^{\prime}}(\mathrm{X})=\mathbf{S} \boldsymbol{\Gamma}_{\varphi}(\mathrm{X}) \mathbf{S}^{-1} \tag{3.23}
\end{equation*}
$$

or, in simplified notation,

$$
\begin{equation*}
\mathbf{X}^{\prime}=\mathbf{S} \mathbf{X} \mathbf{S}^{-1} \tag{3.24}
\end{equation*}
$$

where $\mathbf{S}$ is the matrix that relates individual basis vectors $\boldsymbol{\varphi}_{i}$ of $\Gamma_{\varphi}$ to the corresponding individual basis vectors $\boldsymbol{\varphi}_{\boldsymbol{i}}^{\prime}$ of $\Gamma_{\varphi}$

$$
\begin{equation*}
\boldsymbol{\varphi}_{i}^{\prime}=\mathbf{S} \boldsymbol{\varphi}_{i} \quad(i=1,2,3 \ldots) \tag{3.25}
\end{equation*}
$$

and $\boldsymbol{\varphi}_{i}$ and $\boldsymbol{\varphi}_{i}^{\prime}$ are column vectors in the single Hilbert space of standard QM.
So we require that the irreps $\Gamma_{A}, \Gamma_{B}, \Gamma_{C}, \ldots$ in equations (3.22), based respectively on the eigenvectors of the incompatible variables $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$, all be equivalent to one other, where our generalized definition of equivalence in GIV theory is as follows: the irrep $\Gamma_{A}$ in Hilbert space $\mathcal{H}_{A}$ is equivalent to the corresponding irrep $\Gamma_{B}$ in Hilbert space $\mathcal{H}_{B}$ if (1) the dimensions $n_{A}$ of $\mathcal{H}_{A}$ and $n_{B}$ of $\mathcal{H}_{B}$ are equal, $n_{A}=n_{B}=n$, and (2) the irreps are related by a generalized similarity transformation

$$
\begin{equation*}
\Gamma_{B}=S_{A B} \Gamma_{A} \mathbf{S}_{\mathrm{AB}}^{-1} \tag{3.26}
\end{equation*}
$$

that relates every matrix representative $\Gamma_{A}$ in the irrep $\Gamma_{A}$ to the corresponding matrix representative $\boldsymbol{\Gamma}_{\mathrm{B}}$ in the irrep $\Gamma_{B}$. We call equation (3.26) a generalized similarity transformation because it is analogous to the regular similarity transformation relation (3.23) that defines equivalence in QM , except that the matrix $\mathbf{S}_{\mathrm{AB}}$ in (3.26) does not operate within a single Hilbert space, as in QM, but instead maps matrices $\boldsymbol{\Gamma}_{\mathrm{A}}$ that act in Hilbert space $\mathcal{H}_{A}$ onto the corresponding matrices $\boldsymbol{\Gamma}_{\mathbf{B}}$ that act in Hilbert space $\mathcal{H}_{B}$. In our forthcoming paper, we justify this generalized equivalence requirement based on requiring equal Casimir invariants for $\Gamma_{A}$ and $\Gamma_{B}$ to avoid violating well-established experimental observations, such as the fact that an elementary particle has a fixed value of its spin J , independent of the spatial direction of its spin quantization axis, e.g. an elementary particle cannot be a spin zero particle for measurements in the $z$ direction and a spin one particle for measurements in the $x$ direction!

Equation (3.26) relates every matrix representative $\boldsymbol{\Gamma}_{\mathrm{A}}$ in the irrep $\Gamma_{\mathrm{A}}$ to the corresponding matrix representative $\Gamma_{B}$ in the irrep $\Gamma_{B}$ by the mappings

$$
\begin{align*}
& \mathbf{A}_{\mathbf{B}}=\mathbf{S}_{\mathbf{A B}} \mathbf{A}_{\mathbf{A}} \mathbf{S}_{\mathbf{A B}}^{-1}  \tag{3.27a}\\
& \mathbf{B}_{\mathrm{B}}=\mathbf{S}_{\mathbf{A B}} \mathbf{B}_{\mathbf{A}} \mathbf{S}_{\mathbf{A B}}^{-1}  \tag{3.27b}\\
& \mathbf{C}_{\mathbf{B}}=\mathbf{S}_{\mathbf{A B}} \mathbf{C}_{\mathbf{A}} \mathbf{S}_{\mathbf{A B}}^{-1} \tag{3.27c}
\end{align*}
$$

(compare equation (3.24)), where the similarity transformation matrix $\mathbf{S}_{\mathbf{A B}}$ relates vectors in $\mathcal{H}_{A}$ to the corresponding vectors in $\mathcal{H}_{B}$ (compare equation (3.25)):

$$
\begin{align*}
\boldsymbol{a}_{i}^{B} & =\mathbf{S}_{\mathrm{AB}} \boldsymbol{a}_{i}^{A}  \tag{3.28a}\\
\boldsymbol{b}_{i}^{B} & =\mathbf{S}_{\mathrm{AB}} \boldsymbol{b}_{i}^{A}  \tag{3.28b}\\
\boldsymbol{c}_{i}^{B} & =\mathbf{S}_{\mathrm{AB}} \boldsymbol{c}_{i}^{A} \tag{3.28c}
\end{align*}
$$

Similarly, the matrix $\mathbf{S}_{\mathbf{A C}}$ converts the matrices $\mathbf{A}_{\mathbf{A}}$ to $\mathbf{A}_{\mathbf{C}}, \mathbf{B}_{\mathbf{A}}$ to $\mathbf{B}_{\mathbf{C}}, \mathbf{C}_{\mathbf{A}}$ to $\mathbf{C}_{\mathbf{C}} \cdots$ and the vectors $\boldsymbol{a}_{i}^{A}$ to $\boldsymbol{a}_{i}^{C}, \boldsymbol{b}_{i}^{A}$ to $\boldsymbol{b}_{i}^{C} \ldots$ by equations analogous to equations (3.27) and (3.28) respectively.

At first sight it might appear too much of a generalization to allow a similarity transformation from one Hilbert space to another. But the matrices that map vectors between two Hilbert spaces are in fact identical to perfectly respectable matrices that act purely within one Hilbert space. The key to this realization is the important "parallel axes" relationship

$$
\begin{equation*}
\boldsymbol{a}_{i}^{A}=\boldsymbol{b}_{i}^{B}=\boldsymbol{c}_{i}^{C}=\ldots, i=1,2,3 \ldots . \tag{3.29}
\end{equation*}
$$

noted in equation (3.8), which expresses nothing more than the fact that the respective coordinate axes of Hilbert spaces $\mathcal{H}_{A}, \mathcal{H}_{B}, \mathcal{H}_{C} \ldots$ are the eigenvectors of $\mathbf{A}_{\mathrm{A}}, \mathbf{B}_{\mathrm{B}}, \mathbf{C}_{\mathrm{C}} \ldots$ respectively and are chosen to be parallel. Starting with the definition of $\mathbf{S}_{\mathbf{A B}}$ in equation (3.28a) and inserting $\boldsymbol{a}_{i}^{A}=\boldsymbol{b}_{i}^{B}$ to give

$$
\begin{equation*}
\boldsymbol{a}_{i}^{B}=\mathbf{S}_{\mathrm{AB}} \boldsymbol{a}_{i}^{A}=\mathbf{S}_{\mathrm{AB}} \boldsymbol{b}_{i}^{B}=\mathbf{M}_{\mathbf{b a}} \boldsymbol{b}_{i}^{B} \tag{3.30}
\end{equation*}
$$

we see that the matrix $\mathbf{S}_{\mathrm{AB}}$, which acts between Hilbert spaces to carry the vector $\boldsymbol{a}_{i}^{A}$ in $\mathcal{H}_{A}$ into the corresponding vector $\boldsymbol{a}_{i}^{B}$ in $\mathcal{H}_{B}$, is the same as the matrix $\mathbf{M}_{\text {ba }}$, which acts purely within a single Hilbert space to map a vector $\boldsymbol{b}_{i}^{B}$ into a vector $\boldsymbol{a}_{i}^{B}$

$$
\begin{equation*}
\mathbf{S}_{\mathrm{AB}}=\mathbf{M}_{\mathbf{b a}} \tag{3.31}
\end{equation*}
$$

This matrix, or rather its inverse, $\mathbf{M}_{\mathbf{a b}}=\mathbf{M}_{\mathbf{b} \mathbf{a}}^{-1}$, is familiar from equation (2.12) in our treatment of the Arrow system, and the form of $\mathbf{S}_{\mathbf{B A}}=\mathbf{S}_{\mathbf{A B}}^{\mathbf{1}}=\mathbf{M}_{\mathrm{ab}}$ is simply that of $\mathbf{M}_{\mathrm{ab}}$ in equations (2.13) and (2.14).

We now remind ourselves of the reason for using two Hilbert spaces, namely that when the variables A and B are incompatible, the vectors $\boldsymbol{b}_{i}^{A}$ are in general non-orthogonal in $\mathcal{H}_{A}$ because they are related to the vectors $\boldsymbol{a}_{i}^{A}$ by a non-rigid rotation, effected by a matrix $\mathbf{M}_{\mathbf{a b}}$,

$$
\begin{equation*}
\boldsymbol{b}_{i}^{A}=\mathbf{M}_{\mathbf{a b}} \boldsymbol{a}_{i}^{A} \tag{3.32}
\end{equation*}
$$

in which $\boldsymbol{a}_{1}^{A}$ and $\boldsymbol{a}_{2}^{A}$ are rotated by different angles to give $\boldsymbol{b}_{1}^{A}$ and $\boldsymbol{b}_{2}^{A}$ respectively, as in equations (2.13) and (2.14) for the Arrow; this non-orthogonality problem does not arise if the variables A and B are compatible, because the value state $\left(a_{i}\right)$ of A and the value state $\left(b_{j}\right)$ of B are then the same state $\left(a_{i}, b_{j}\right)$, so there is no rotation involved. The reason for the non-rigidity of the rotation (3.32) is that the form of the probabilities $\mathrm{P}_{\left(b_{i}\right)}\left(a_{j}\right)=\left|\left\langle a_{j}^{A} \mid b_{i}^{A}\right\rangle\right|^{2}$ in GIV theories can be wideranging, restricted only by the basic rules of probability (namely exclusivity, normalization, etc. - see Section 2) for general incompatible variables A and B. But in QM the incompatible variables A and B are symmetries of space-time, which imposes more severe restrictions on the probabilities. The result (as we show next) is that the rotation (3.32) becomes a rigid rotation, in which $\boldsymbol{a}_{1}^{A}$ and $\boldsymbol{a}_{2}^{A}$ are rotated by the same angles to give $\boldsymbol{b}_{1}^{A}$ and $\boldsymbol{b}_{2}^{A}$, so that the vectors $\boldsymbol{b}_{i}^{A}$ are now orthogonal in $\mathcal{H}_{A}$ as well as in $\mathcal{H}_{B}$. Furthermore, since $\mathbf{M}_{\mathbf{a b}}=\mathbf{S}_{\mathbf{B A}}$, this means that $\mathbf{S}_{\mathbf{B A}}$ will then also effect a rigid rotation in which the vectors $\boldsymbol{b}_{1}^{B}$ and $\boldsymbol{b}_{2}^{B}$, and all other pure state vectors $\boldsymbol{\Psi}^{B}$ in $\mathcal{H}_{B}$, are rotated by the same angle to give the corresponding vectors $\boldsymbol{b}_{1}^{A}, \boldsymbol{b}_{2}^{A}$ and $\boldsymbol{\Psi}^{A}$ in $\mathcal{H}_{A}$, so that the two Hilbert spaces $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$ become superimposable (and therefore equivalent) for all pure states, making separate Hilbert spaces unnecessary.

The key to recovering the single Hilbert space formalism of standard QM from the multi-Hilbert space formalism of GIV theory is therefore the requirement that the matrix $\mathbf{S}_{\mathbf{B A}}=\mathbf{S}_{\mathbf{A B}}^{-1}$ be unitary, in order to effect a rigid rotation between all pure state vectors in $\mathcal{H}_{B}$ and the corresponding vectors in $\mathcal{H}_{A}$. But in the general case, $\mathbf{S}_{\mathbf{B A}}$ is not unitary. So what, then, does it take to make $\mathbf{S}_{\mathbf{B A}}$ unitary, so that a single Hilbert space can be used? We have already seen that symmetry operators are unitary, but the matrix $\mathbf{S}$ that effects a similarity transformation between two equivalent irreps as in equations (3.27) is not necessarily itself a symmetry operation of the group. There are some cases where the $\mathbf{S}$ matrices are symmetry operators, as in the case of symmetry operations in the same class, such as rotations about the $x, y$ and $z$ axes in the full rotation group $\mathrm{R}_{3}$, which are related in a similarity transformation by the symmetry operation $\mathrm{C}_{3}$. But some of the most interesting incompatible variables are definitely not related by symmetry, such as position (the approximate generator of Lorentz boosts) and momentum (the
generator of spatial translations). For variables such as these, the $\mathbf{S}$ matrices are not symmetry operators, so we cannot use that to infer their unitary.

But if the incompatible variables A and B are themselves symmetries, we can always use the fact that one of the similarity transformation equations (3.27) is a diagonalization of a unitary matrix, and the matrix that diagonalizes a unitary operator is always itself unitary [20]. Specifically equation (3.27b), which we rewrite here as

$$
\begin{equation*}
\mathbf{B}_{\mathrm{B}}=\mathbf{S}_{\mathrm{AB}} \mathbf{B}_{\mathrm{A}} \mathbf{S}_{\mathbf{A B}}^{-1} \tag{3.33}
\end{equation*}
$$

is a diagonalization of the unitary matrix $\mathbf{B}_{\mathrm{A}}$ (unitary because it is a symmetry operator) to give the diagonal unitary matrix $\mathbf{B}_{\mathbf{B}}$, so the matrix $\mathbf{S}_{\mathbf{A B}}$ must be unitary even if it is not itself a symmetry operator. Similarly, all the other $\mathbf{S}$ matrices $\mathbf{S}_{\mathbf{A C}}, \mathbf{S}_{\mathbf{B C}} \ldots$ are also unitary. This diagonalization argument - which can be used only if the variables are symmetries (or the Hermitian generators of symmetries) guarantees that the $\mathbf{S}$ matrices, and thence (from equation (3.31)) the corresponding $\mathbf{M}$ matrices, are unitary (even in cases where $\mathbf{S}$ is not a symmetry operator) thus making the value states of both A and B orthogonal in both $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$, and enabling the pure states of the two Hilbert spaces with only restricted Born Rules to be combined into one Hilbert space with an unrestricted Born Rule, as in standard QM. Mixed states, which can also be represented as sequences of vectors in the many-Hilbert-space formulation, cannot be represented as vectors in a single Hilbert space because mixed states and pure states do not transform in the same way under symmetry transformations, as we discuss in detail in our forthcoming paper.

Equations such as (3.33) are valid whether or not the variables A and B are symmetries: if the variables are not symmetries, equation (3.33) is simply a transformation relating matrix operators in different Hilbert spaces, but if the variables are symmetries, it becomes, for any elementary system, a generalized equivalence relation between matrix representatives in irreps $\Gamma_{A}$ and $\Gamma_{B}$ of the symmetry group of the system in Hilbert spaces $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$ respectively. So why can't the diagonalization argument be used when the variables are not symmetries? Because unless the variables are symmetries, there is in general no reason to suppose that any of the matrices being diagonalized are unitary (although there may be cases of "accidental" unitarity), so there is in general no guarantee that any of the $\mathbf{S}$ matrices that do the diagonalization are unitary.

In fact, one cannot in general expect the matrix $\mathbf{S}$ to be unitary unless at least some symmetry is present, either in the form of the incompatible variables being related by symmetry (in which case $\mathbf{S}$ is a symmetry operator and therefore unitary), or in the form of the incompatible variables themselves being symmetries
(in which case $\mathbf{S}$ is unitary because it diagonalizes a unitary operator). The classical Arrow system is a case in which the incompatible variables A and B are not themselves symmetries, but are related by symmetry; position and momentum are incompatible variables that are themselves symmetries (generators of Lorentz and translational symmetry respectively), but are not related by symmetry; and the $x, y$ and $z$ components of angular momentum are incompatible variables that are both themselves symmetries (generators of rotational symmetry) and related by symmetry.

QM is thus a special case of GIV theories in which all the fundamental variables are themselves symmetries. Some of these symmetries are incompatible, leading to the "commutator postulate"; and the "Born postulate" also follows because the fact that the variables are themselves symmetries allows even those that are not related by symmetry to be represented in a single Hilbert space with an unrestricted Born Rule. Thus QM truly comes from symmetry!

## 4 Uncertainty and interference in classical systems

We now use our GIV theory to show that any probabilistic system -"quantal" or classical - that has incompatible variables will obey an uncertainty principle and exhibit interference. We demonstrate this for the general case of two incompatible variables, each of which has just two possible measurement outcomes, $a_{1}$ and $a_{2}$ for A , and $b_{1}$ and $b_{2}$ for B , as in equations (3.1) to (3.6), recalling that in GIV theory one must use Hilbert space $\mathcal{H}_{A}$ to calculate the probability of A outcomes, and Hilbert space $\mathcal{H}_{B}$ to calculate the probability of B outcomes.

We take uncertainty principle to have the general meaning that certainty about the anticipated result of an experiment to measure a variable $A$ can only be bought at the expense of uncertainty in the anticipated result of an experiment to measure another variable $B$ with which it is incompatible [5]. For incompatible variables A and B , the result of an experiment to measure A when in state $\left(a_{1}\right)$ is certain, always giving the outcome $a_{1}$ and never the outcome $a_{2}$.

$$
\begin{align*}
& \mathrm{P}_{\left(a_{1}\right)}\left(a_{1}\right)=\left|\left\langle a_{1}^{A} \mid a_{1}^{A}\right\rangle\right|^{2}=1  \tag{4.1a}\\
& \mathrm{P}_{\left(a_{1}\right)}\left(a_{2}\right)=\left|\left\langle a_{2}^{A} \mid a_{1}^{A}\right\rangle\right|^{2}=0 \tag{4.1b}
\end{align*}
$$

but the result of an experiment to measure B when in state $\left(a_{1}\right)$ is uncertain, yielding either $b_{1}$ or $b_{2}$, not with certainty but with probabilities

$$
\begin{align*}
& \mathrm{P}_{\left(a_{1}\right)}\left(b_{1}\right)=\left|\left\langle b_{1}^{B} \mid a_{1}^{B}\right\rangle\right|^{2}  \tag{4.2a}\\
& \mathrm{P}_{\left(a_{1}\right)}\left(b_{2}\right)=1-\left|\left\langle b_{1}^{B} \mid a_{1}^{B}\right\rangle\right|^{2} \tag{4.2b}
\end{align*}
$$

This uncertainty arises because common eigenstates of the form $\left(a_{i}, b_{j}\right)$ simply do not exist. By contrast, if A and B are compatible, a complete set of common eigenstates of A and B does exist, so we then have certainty for both variables,

$$
\begin{array}{ll}
\mathrm{P}_{\left(a_{1}, b_{1}\right)}\left(a_{1}\right)=1, & \mathrm{P}_{\left(a_{1}, b_{1}\right)}\left(a_{2}\right)=0 \\
\mathrm{P}_{\left(a_{1}, b_{1}\right)}\left(b_{1}\right)=1, & \mathrm{P}_{\left(a_{1}, b_{1}\right)}\left(b_{2}\right)=0 \tag{4.3b}
\end{array}
$$

Uncertainty can arise even in purely classical systems such as our Arrow, for which states such as $\left(a_{+}, b_{+}\right)$do not exist because the Arrow cannot be pointing in both the A and B directions at the same time.

The term interference has two related meanings. We focus first on the meaning in the context of a pair of incompatible variables $A$ and $B$, where it means that measurement of B can interfere with (affect the outcome of) an experiment to measure A. We consider first, for purposes of comparison, what happens in the case of compatible variables. If we perform a direct measurement of the variable A on the pure state $\psi=\left(a_{1}, b_{1}\right)$, we obtain the anticipated result $\mathrm{A}=a_{1}$ with certainty,

$$
\begin{equation*}
\left(a_{1}, b_{1}\right) \xrightarrow{\text { observe } \mathrm{A}}\left(a_{1}, b_{1}\right), \text { Outcome A }=a_{1} \tag{4.4a}
\end{equation*}
$$

If we perform an indirect measurement of A on state $\left(a_{1}, b_{1}\right)$, in which we first observe B and then observe A , we also obtain the expected outcome $\mathrm{A}=a_{1}$ with certainty,

$$
\begin{equation*}
\left(a_{1}, b_{1}\right) \xrightarrow{\text { Observe B }}\left(a_{1}, b_{1}\right) \xrightarrow{\text { Observe A }}\left(a_{1}, b_{1}\right), \text { Outcome A }=a_{1} \tag{4.4b}
\end{equation*}
$$

We see that measuring B before measuring A does not interfere with the measurement of A : the same outcome, $\mathrm{A}=a_{1}$, is obtained in both direct and indirect measurements, reflecting that fact that the operators for compatible variables commute, $[\hat{A}, \hat{B}]=0$. Turning now to the case of incompatible variables, and considering measurements of the variable A on an arbitrary state $\psi$, either directly, or via a prior measurement of $B$, we obtain the following diagrammatic equations,

$$
\begin{gather*}
\psi \xrightarrow{\text { Observe A }}\left\{\begin{array}{l}
\left(a_{1}\right) \\
\left(a_{2}\right)
\end{array}\right.  \tag{4.5a}\\
\psi \xrightarrow{\text { Observe B }}\left\{\begin{array}{l}
\left(b_{1}\right) \xrightarrow{\text { Observe A }}\left\{\begin{array}{l}
\left(a_{1}\right) \\
\left(a_{2}\right)
\end{array}\right. \\
\left(b_{2}\right) \xrightarrow{\text { Observe A }}\left\{\begin{array}{l}
\left(a_{1}\right) \\
\left(a_{2}\right)
\end{array}\right.
\end{array}\right. \tag{4.5b}
\end{gather*}
$$

noting that observation of B causes a transition from the state $(\psi)$ to an eigenstate $\left(b_{i}\right)$ of B , and observation of A then causes a transition to an eigenstate $\left(a_{i}\right)$ of A. For the direct measurement in (4.5a) the probability of obtaining the result $a_{i}$ from a starting state $\psi$ is given by

$$
\begin{equation*}
\mathrm{P}_{\psi}^{\mathrm{dir}}\left(a_{i}\right)=\left|\left\langle a_{i}^{A} \mid \Psi^{A}\right\rangle\right|^{2}, \quad i=1,2 \tag{4.6a}
\end{equation*}
$$

The indirect measurement of A in (4.5b) yields the result $a_{i}$ with a probability $\mathrm{P}_{\psi}^{\text {indir }}\left(a_{i}\right)$ given by the sum of the two sub-processes via $b_{1}$ and $b_{2}$ respectively:

$$
\begin{align*}
\mathrm{P}_{\psi}^{\text {indir }}\left(a_{i}\right) & =\mathrm{P}_{b_{1}}\left(a_{i}\right) \mathrm{P}_{\psi}\left(b_{1}\right)+\mathrm{P}_{b_{2}}\left(a_{i}\right) \mathrm{P}_{\psi}\left(b_{2}\right) \\
& =\left|\left\langle a_{i}^{A} \mid b_{1}^{A}\right\rangle\right|^{2}\left|\left\langle b_{1}^{B} \mid \psi^{B}\right\rangle\right|^{2}+\left|\left\langle a_{i}^{A} \mid b_{2}^{A}\right\rangle\right|^{2}\left|\left\langle b_{2}^{B} \mid \psi^{B}\right\rangle\right|^{2} \tag{4.6b}
\end{align*}
$$

The probability $\mathrm{P}_{\psi}^{\mathrm{dir}}\left(a_{i}\right)$ of obtaining $a_{i}$ by the direct measurement certainly has a different algebraic form from the probability $\mathrm{P}_{\psi}^{\text {indir }}\left(a_{i}\right)$ of obtaining $a_{i}$ by the indirect measurement, but it is not in general immediately obvious that their numerical values differ until one inserts values for the individual amplitudes for a specific case. A convenient specific case is provided by considering the probability of obtaining the outcome $\mathrm{A}=a_{2}$ from a starting state $(\psi)=\left(a_{1}\right)$, which must be zero in a direct measurement,

$$
\begin{equation*}
\mathrm{P}_{\left(a_{1}\right)}^{\mathrm{dir}}\left(a_{2}\right)=\left|\left\langle a_{2}^{A} \mid a_{1}^{A}\right\rangle\right|^{2}=0 \tag{4.7a}
\end{equation*}
$$

but cannot also be zero in an indirect measurement,

$$
\begin{equation*}
\mathrm{P}_{\left(a_{1}\right)}^{\operatorname{indir}}\left(a_{2}\right)=\left|\left\langle a_{2}^{A} \mid b_{1}^{A}\right\rangle\right|^{2}\left|\left\langle b_{1}^{B} \mid a_{1}^{B}\right\rangle\right|^{2}+\left|\left\langle a_{2}^{A} \mid b_{2}^{A}\right\rangle\right|^{2}\left|\left\langle b_{2}^{B} \mid a_{1}^{B}\right\rangle\right|^{2} \neq 0 \tag{4.7b}
\end{equation*}
$$

because this would require at least some of the matrix elements in (4.7b) to be zero, and it is readily shown that this is not possible if the variables A and B are incompatible. For example, $\left\langle a_{2}^{A} \mid b_{1}^{A}\right\rangle=0$ would require $\left|b_{1}^{A}\right\rangle=\left|a_{1}^{A}\right\rangle$ (to within a phase factor), which can only be the case if A and B are compatible, and similar considerations apply to all the other matrix elements.

The inequality of $\mathrm{P}_{\psi}^{\text {dir }}\left(a_{i}\right)$ and $\mathrm{P}_{\psi}^{\text {indir }}\left(a_{i}\right)$ becomes more obvious in cases where symmetry is present. Applying the general equations (4.6) to our classical Arrow system, for which the index $i$ is + or - , and considering just the probability of obtaining the result $a_{+}$, we have

$$
\begin{equation*}
\mathrm{P}_{\psi}^{\mathrm{dir}}\left(a_{+}\right)=\left|\left\langle a_{+}^{A} \mid \psi^{A}\right\rangle\right|^{2} \tag{4.8}
\end{equation*}
$$

for the direct measurement. We now make use of the fact that in the Arrow the eigenstates $\left(a_{+}\right)$and ( $a_{-}$) of A are related to one another by $\mathrm{C}_{2}$ symmetry, as are the eigenstates $\left(b_{+}\right)$and $\left(b_{-}\right)$of B : we saw in equations (2.9) that this symmetry
results in the eigenstates of B being orthogonal not only in Hilbert space $\mathcal{H}_{B}$ but also in Hilbert space $\mathcal{H}_{A}$, so they form a complete orthonormal set in $\mathcal{H}_{A}$, allowing us to insert the closure relation

$$
\begin{equation*}
\left|b_{+}^{A}\right\rangle\left\langle b_{+}^{A}\right|+\left|b_{-}^{A}\right\rangle\left\langle b_{-}^{A}\right|=\mathbf{1} \tag{4.9}
\end{equation*}
$$

into $\left\langle a_{+}^{A} \mid \psi^{A}\right\rangle$ in equation (4.6a), giving

$$
\begin{equation*}
\mathrm{P}_{\psi}^{\mathrm{dir}}\left(a_{+}\right)=\left|\left\langle a_{+}^{A} \mid b_{+}^{A}\right\rangle\left\langle b_{+}^{A} \mid \psi^{A}\right\rangle+\left\langle a_{+}^{A} \mid b_{-}^{A}\right\rangle\left\langle b_{-}^{A} \mid \psi^{A}\right\rangle\right|^{2} \tag{4.10}
\end{equation*}
$$

Since the $\mathrm{C}_{2}$ symmetry makes the eigenstates of B orthogonal in $\mathcal{H}_{A}$ as well as in $\mathcal{H}_{B}$, a single Hilbert space can now be used (see the discussion following equations (3.32)), so we drop the superscripts to give

$$
\begin{align*}
& \mathrm{P}_{\psi}^{\mathrm{dir}}\left(a_{+}\right)=\left|\left\langle a_{+} \mid b_{+}\right\rangle\left\langle b_{+} \mid \psi\right\rangle+\left\langle a_{+} \mid b_{-}\right\rangle\left\langle b_{-} \mid \psi\right\rangle\right|^{2}  \tag{4.11a}\\
& \mathrm{P}_{\Psi}^{\text {indir }}\left(a_{+}\right)=\left|\left\langle a_{+} \mid b_{+}\right\rangle\right|^{2}\left|\left\langle b_{+} \mid \psi\right\rangle\right|^{2}+\left|\left\langle a_{+} \mid b_{-}\right\rangle\right|^{2}\left|\left\langle b_{-} \mid \psi\right\rangle\right|^{2} \tag{4.11b}
\end{align*}
$$

for the direct and indirect measurements in the Arrow. It is now much clearer that $\mathrm{P}_{\psi}^{\mathrm{dir}}\left(a_{+}\right) \neq \mathrm{P}_{\Psi}^{\text {indir }}\left(a_{+}\right)$, since equations (4.11) are of the form

$$
\begin{align*}
& \mathrm{P}_{\psi}^{\mathrm{dir}}\left(a_{+}\right)=\left|R_{1}+R_{2}\right|^{2}=R_{1}^{2}+R_{2}^{2}+2 R_{1} R_{2}  \tag{4.12a}\\
& \mathrm{P}_{\Psi}^{\mathrm{indir}}\left(a_{+}\right)=R_{1}^{2}+R_{2}^{2} \tag{4.12b}
\end{align*}
$$

which is reminiscent of the second meaning of interference, in the context of waves overlapping in phase or out of phase to produce constructive or destructive interference. This type of interference is said to occur in wave optics when the interaction of two light beams yields a resultant irradiance that deviates from the sum of the component irradiances [21]. The irradiance at a given location is proportional to the probability of detecting a photon at that location, which is proportional to the square of the amplitude of the electric field of the light detected at that location. Interference occurs simply because amplitudes are additive, resulting in the square of the sum being different from the sum of the squares. The use of the term "amplitude" for the square root of a probability may give the impression that interference only occurs for systems that are literally physical waves, since square roots can be positive or negative like the amplitude of the peaks and troughs of a wave. But interference can actually occur in any probabilistic system, wave-like or not, that has incompatible variables, and we follow Kirkpatrick [1] in broadening the wave optics definition of interference in terms of resultant irradiance deviating from the sum of component irradiances to the more general statement that interference occurs when the probability of a process deviates from the sum of the probabilities of the sub-processes, which is
exactly what we see in the inequality of the results of direct and indirect measurements in our classical Arrow system in equations (4.11) and (4.12).

Richard Feynman famously said [22] that QM is about "adding amplitudes", to which one might add "before squaring them", because it is "adding amplitudes before squaring them" that is responsible for the wave-like interference effect that has hitherto been considered the hallmark of QM. But we see from equations (4.12) that there is in fact a direct parallel between interference in the probability patterns of our purely classical Arrow system and corresponding probability patterns in the famous double-slit electron diffraction experiments that gave birth to the QM notion of wave-particle duality. It is therefore clear that although interference ("adding amplitudes") is a prominent feature of QM, it is not unique to QM and is not necessarily indicative of actual physical waves, rather it is a general feature of all probability theories with incompatible variables. However, interference is particularly easy to recognize in QM, because what distinguishes QM from other probability theories is the fact that symmetry enables all the fundamental variables to be represented by orthogonal axes in a single Hilbert space, which in turn allows use of a closure relation to arrive at expressions of the form (4.12a), in which probabilities show interference even in the absence of actual physical waves. Symmetries present in the double-slit system, for example, lead to an enhanced impression of wave character, as we will discuss in more detail in our forthcoming paper.

## 5 Conclusion

It is increasingly becoming realized that incompatible variables - with concomitant uncertainty and interference - are not the exclusive preserve of QM, nor are they on a fundamental level the consequences of wave character or quantization [7]. We add to the growing list of classical systems with incompatible variables [1, 2] our "Arrow" system, a multi-variable analogue of the coin toss that shows uncertainty and interference similar to that in double-slit diffraction experiments. Inspired by Weinberg's suggestion [3] that it would be useful to find a larger, more general theory in which QM appears as a special case, we have constructed a general incompatible variables (GIV) theory, to describe any system with incompatible variables, and we find that what distinguishes QM from other GIV theories is symmetry: the incompatible variables of QM are simply incompatible symmetries. Specifically, the fundamental variables of QM - angular momentum, energy, linear momentum, and (approximate) position - are respectively the generators of the rotational, time-translational, space-translational and Lorentz boost symmetries of the Poincaré group.

Considering the two central postulates of QM , the role of Poincaré symmetry in the "commutator postulate" $\left[\widehat{x}, \widehat{p}_{x}\right]=i \hbar$ is abundantly clear [7,14], but the origin of the "Born postulate" $\mathrm{P}_{\psi}\left(o_{i}\right)=\left|\left\langle o_{i} \mid \psi\right\rangle\right|^{2}$ has hitherto remained obscure, despite Gleason's theorem [8] showing that the Born Rule is the only probability rule that works if all variables are represented in a single Hilbert space. However, Hughes [5] realized that two incompatible variables A and B cannot in general be represented in just one Hilbert space, because the vectors representing the eigenstates of B , while orthogonal in a Hilbert space $\mathcal{H}_{B}$ (in which the eigenvectors of B form the axes) are not in general (except in very special cases) orthogonal in a Hilbert space $\mathcal{H}_{A}$ (in which the eigenvectors of A form the axes), and, so far as we are aware, Hughes is the only one to have noticed this.

But the central point of this paper, which has not, to our knowledge, been noticed by anyone until now, is that symmetries allow incompatible variables to be represented together in a single Hilbert space, as we highlight using our simple classical Arrow system.

We follow Hughes [5] in using a separate Hilbert space for calculating the outcomes of a measurement of each incompatible variable in our GIV theory, and we introduce the new concept of a restricted Born Rule $\mathrm{P}_{\psi}\left(o_{i}\right)=\left|\left\langle o_{i} \mid \psi\right\rangle\right|^{2}$ in which $\left|o_{i}\right\rangle$ is restricted to being one of the orthonormal axes of that particular Hilbert space, in contrast to the usual unrestricted Born Rule, in which not only $|\psi\rangle$ but also $\left|o_{i}\right\rangle$ can be any vector within the Hilbert space. We then show that if the fundamental variables are symmetries, as in QM, the many Hilbert spaces of GIV theory are then related to one another by rigid transformations (as opposed to nonrigid transformations in the general case), which allows the many Hilbert spaces to be superimposed and combined into one single Hilbert space, with a single, unrestricted Born Rule for all of the variables.

QM is thus a special case of GIV theories, one in which all the fundamental variables are symmetries. The two central postulates of QM are no longer "underivable", but in fact come from these symmetries, some of which are incompatible. The derivation of the "commutator postulate" from Poincaré symmetry has been presented in the most detail by Bohr \& Ulfbeck [7]. But there has hitherto been no truly satisfactory "derivation" of the "Born postulate", and the attempt by Bohr \& Ulfbeck [7] to derive it too from symmetry did not succeed. However, our work in this paper now provides a much clearer understanding of the Born Rule: although it is not primarily derived from symmetry - rather it is simply a free Pythagorean construction for accommodating basic features of classical probability theory in Hilbert spaces - it is Poincaré symmetry that allows the fundamental variables of QM to be represented in a single Hilbert space, thus
enabling the restricted Born Rule to take on its familiar unrestricted form, in agreement with Gleason's theorem. So QM does truly come from symmetry!

Our forthcoming paper contains fuller derivations of all these conclusions; provides a diagrammatic understanding of the commutator $\left[\hat{x}, \widehat{p}_{x}\right]$, analogous to the more familiar understanding of the commutator $\left[\hat{\jmath}_{x}, \hat{j}_{y}\right]$; and discusses the implications of our conclusions for a new interpretation of QM as a theory of incompatible symmetries. Future work might extend our discussion of GIV systems to include fields, interactions, composite systems, and entanglement.

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