# Conjugacy classes of maximal cyclic subgroups of metacyclic p-groups

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#### Abstract

In this paper, we set  $\eta(G)$  to be the number of conjugacy classes of maximal cyclic subgroups of a finite group G. We compute  $\eta(G)$  for all metacyclic p-groups. We show that if G is a metacyclic p-group of order  $p^n$  that is not dihedral, generalized quaternion, or semi-dihedral, then  $\eta(G) \geq n-2$ , and we determine when equality holds.

Keywords: group covering, metacyclic group 2020 Mathematics Subject Classification: 20D15

### 1 Introduction

Unless otherwise stated, all groups in this paper are finite, and we will follow standard notation from [6]. As in [3] and [4], we set  $\eta(G)$  to be the number of conjugacy classes of maximal cyclic subgroups of a group G. For p=2, we have that  $\eta(G)=3$  when G is a dihedral 2-group, a generalized quaternion 2-group, or a semi-dihedral group. In [1], the second and third authors along with Yiftach Barnea and Mikhail Ershov have shown that for every prime  $p\geq 5$  there are infinitely many p-groups with  $\eta=p+2$  and for p=3 there are infinitely many 3-groups with  $\eta=9$ . This answers negatively Question 5.0.9 from [9] which asked whether  $\eta(G)$  grows with the order of G when G is a p-group and p is odd.

On the other hand, it is rare for this to occur. Indeed, the only 2-groups (in fact the only p-groups) that have  $\eta = 3$  are the Klein 4-group, the dihedral groups, the generalized quaternion groups, and the semi-dihedral groups. To

see this, we know that  $\eta(G) \geq \eta(G/G')$  (see [3]), and for p-groups  $\eta(G/G') \geq p+1$  when G/G' is not cyclic (see [4]). Thus,  $\eta=3$  can only occur when p=2. Also, in [4], we show that  $\eta(G/G')=3$  if and only if  $G/G'\cong C_2\times C_2$ . It is well known that if G is a 2-group of order at least 8 and |G:G'|=4, then G is either dihedral, generalized quaternion, or semi-dihedral. (See Problem 6B.8 of [6].)

Now, dihedral groups, generalized quaternion groups, and semi-dihedral groups are examples of metacyclic groups. I.e., groups G with a normal subgroup N so that N and G/N are both cyclic groups. This motivated us to investigate the invariant  $\eta$  for all metacyclic p-groups. Indeed this project began before the results of [1] were known and we were originally curious as to whether we would find another family of metacyclic p-groups with fixed  $\eta$ . However, we prove the following:

**Theorem 1.1** Let G be a metacyclic p-group of order  $p^n$  that is not a dihedral group, generalized quaternion group, or semi-dihedral group. Then  $\eta(G) \geq n-2$ .

In fact, we compute  $\eta(G)$  for every metacyclic p-group G. Thus, we list the metacyclic p-groups where equality occurs in Theorem 1.1. King in [7] gave a description of all metacyclic p-groups. We will give this description of these groups in Section 3. In particular, King divided the metacyclic p-groups into two families of groups which he called positive type and negative type. The negative type groups only occur when p=2, so if p is an odd prime, then all of the metacyclic p-groups are of positive type. We have the following result for the metacyclic groups of positive type.

**Theorem 1.2** Let G be a metacyclic group of positive type. Then  $\eta(G) = \eta(G/G')$ .

We note that Rogério in [8] has a formula to compute  $\eta(A)$  for an abelian group A. His formula involves the Euler  $\phi$ -function and a second number theoretic function. When G is a metacyclic abelian p-group, we prove in [4] a formula for  $\eta(G)$  that is only in terms of the sizes of the direct factors of G. Notice in Theorem 1.2 that G/G' will be a metacyclic abelian p-group, and so, our formula will compute  $\eta(G/G')$  and hence,  $\eta(G)$ .

When G is a metacyclic p-group of negative type, it is not usually the case that  $\eta(G)$  and  $\eta(G/G')$  are equal. However, we will find that there usually

is a proper quotient whose value of  $\eta$  equals  $\eta(G)$ . We will also see for most metacyclic groups of negative type that the formula for  $\eta$  is dependent on the formula for  $\eta$  that we found for the metacyclic abelian p-groups.

The authors would like to thank Emanuele Pacifici for a number of helpful conversations while working on this paper.

#### 2 Preliminaries

In our preprint [3], we prove two results that we need in this paper. The first is a criteria for determining when the quotient of a p-group G has the same value for  $\eta$  as  $\eta(G)$ . Given a prime p, we set  $G^{\{p\}} = \{g^p \mid g \in G\}$ . I.e.,  $G^{\{p\}}$  is the set of p-th powers in G.

**Theorem 2.1** Let N be a normal subgroup of the p-group G. Then  $\eta(G/N) \leq \eta(G)$ . Furthermore,  $\eta(G/N) = \eta(G)$  if and only if  $N \subseteq G^{\{p\}}$  and for all  $x \in G \setminus G^{\{p\}}$  every element of xN is conjugate to a generator of  $\langle x \rangle$ . In particular, if  $\eta(G/N) = \eta(G)$ , then  $G^{\{p\}}$  is a union of N-cosets and  $G^{\{p\}}N = G^{\{p\}}$ .

This second Proposition relates  $\eta(G)$  to the number of G-orbits of maximal cyclic subgroups of a normal subgroup.

**Proposition 2.2** Let N be a normal subgroup of a group G and let  $\eta^*(N)$  be the number of G-orbits on the N-conjugacy classes of maximal cyclic subgroups of N. Then  $\eta(G) \geq \eta^*(N)$ . In particular,

- (i) if N is central in G, then  $\eta(G) \ge \eta(N)$ .
- (ii) if |G:N|=k, then  $\eta(G) \ge \eta(N)/k$ .

Let p be a prime, and let a and b be positive integers. We take  $k = \max(a, b)$  and  $l = \min(a, b)$ . We set  $g_p(a, b) = p^{(l-1)}((k-l)(p-1) + p + 1)$ . In [4], we prove the following lemma.

**Lemma 2.3** If p is a prime and a and b are positive integers so that  $G = C_{p^a} \times C_{p^b}$ , then  $g_p(a,b) = \eta(G)$ .

We close this section with an easy lemma that computes  $g_2$  for small values and gives a lower bound for larger values. We remark that when p = 2, this function is much easier to work with.

**Lemma 2.4** Suppose  $k \geq l$ . Then the following hold:

- 1. If l = 1, then  $g_2(k, 1) = k + 2$ .
- 2. If l = 2, then  $q_2(k, 2) = 2(k + 1)$ .
- 3. If l = 3, then  $q_2(k, 3) = 4k$ .
- 4. If  $l \ge 4$ , then  $g_2(k, l) \ge 4k + 2l$ .

**Proof.** We have  $g_2(a, b) = g_2(k, l) = 2^{l-1}(k - l + 3)$ . Conclusions (1), (2), and (3) are immediate. We focus on (4). Begin with  $g_2(4, 4) = 24$ ; so the result holds for  $g_2(4, 4)$ . Next,  $g_2(l, l) - 6l = 3 \cdot 2^{l-1} - 6l$  is clearly increasing when  $l \geq 3$ . Thus, we have  $g_2(l, l) \geq 4l + 2l$  when  $l \geq 3$ . Let k = l + m for  $m \geq 0$ . Then  $g_2(k, l) = g_2(l + m, m) = 2^{l-1}(m + 3)$  and 4k + 2l = 4(l + m) + 2l = 6l + 4m. Fixing  $l \geq 4$ , we note that  $2^{l-1}(m+3) - 6l - 4m$  will be an increasing function in m. We conclude that  $g_2(k, l) \geq 4k + 2l$  for  $l \geq 4$ .  $\square$ 

# 3 Metacyclic p-Groups

For the rest of the paper, we will focus on metacyclic p-groups. A finite metacyclic p-group can be described as follows. This description is taken from [7],

$$G_p(\alpha, \beta, \epsilon, \delta, \pm) = \langle x, y \mid x^{p^{\alpha}} = 1, y^{p^{\beta}} = x^{p^{\alpha - \epsilon}}, x^y = x^r \rangle$$

where  $r=p^{\alpha-\delta}+1$  (positive type) or  $r=p^{\alpha-\delta}-1$  (negative type). The integers  $\alpha$ ,  $\beta$ ,  $\delta$ ,  $\epsilon$  satisfy  $\alpha,\beta>0$  and  $\delta,\epsilon$  nonnegative, furthermore  $\delta\leq \min\{\alpha-1,\beta\}$  and  $\delta+\epsilon\leq\alpha$ . When G has negative type, only  $\epsilon=0$  or 1 occur. For p odd

$$G \cong G_p(\alpha, \beta, \epsilon, \delta, +).$$

In other words, the negative type only occurs when p=2; when p is odd, only the positive type occurs. Metacyclic 2-groups can be of either positive type or negative type. We note that dihedral, semi-dihedral and generalized quaternion groups are all of negative type.

If p=2, then in addition  $\alpha-\delta>1$  and

$$G \cong G_2(\alpha, \beta, \epsilon, \delta, +)$$
 or  $G \cong G_2(\alpha, \beta, \epsilon, \delta, -)$ .

Note, the above presentation does not guarantee nonisomorphic groups for different parameters (see [2]). However, the parameters do determine some structural information about G. For example,  $|G| = p^{\alpha+\beta}$  and  $G' = \langle x^{p^{\alpha-\delta}} \rangle$  if G is of positive type and  $G' = \langle x^2 \rangle$  if G is of negative type. All elements of G can be written as  $y^b x^a$  for some integers a and b. Also if G is of positive type then  $Z(G) = \langle x^{p^{\delta}}, y^{p^{\delta}} \rangle$  and  $|Z(G)| = p^{\alpha+\beta-2\delta}$ , if G is of negative type  $Z(G) = \langle x^{2^{\alpha-1}}, y^{2^{\max\{1,\delta\}}} \rangle$ , [2, Prop. 2.5]. Note that if G is of positive type and  $\delta = 0$ , then G will be abelian.

As we mentioned above, the dihedral groups, the generalized quaternion groups, and the semi-dihedral groups are the only p-groups G that satisfy  $\eta(G) = 3$ . These are also precisely the 2-groups of maximal class. We have also mentioned that they are metacyclic. In terms of our notation, the dihedral groups are  $G_2(\alpha, 1, 0, 0, -)$ , the generalized quaternion groups are  $G_2(\alpha, 1, 1, 0, -)$ , and the semi-dihedral groups are  $G_2(\alpha, 1, 0, 1, -)$ .

For Lemmas 3.1 and 3.3, we are writing  $G_p(\alpha, \beta, \epsilon, \delta, \pm)$  as  $G_p(\alpha, \beta, \epsilon, \delta, \gamma)$  where we take  $\gamma = +$  when G is of positive type and  $\gamma = -$  when G is of negative type. We consider quotients of G. Note that this lemma would not be well defined if  $\delta = 0$  and would not say anything if  $\delta = 1$ .

**Lemma 3.1** Suppose G is  $G_p(\alpha, \beta, \epsilon, \delta, \gamma)$  with  $\delta \geq 2$ . Then  $N = \langle x^{p^{\alpha-\delta+1}} \rangle$  is a normal subgroup of G and G/N is isomorphic to

$$G_p(\alpha - \delta + 1, \beta, (\epsilon - \delta + 1)^*, 1, \gamma)$$

where  $(\epsilon - \delta + 1)^* = \epsilon - \delta + 1$  when  $\epsilon \ge \delta - 1$  and  $(\epsilon - \delta + 1)^* = 0$  when  $\epsilon < \delta - 1$ .

**Proof.** Set  $Z = \langle x^{p^{\alpha-1}} \rangle \leq Z(G)$ . We first prove that G/Z is isomorphic to  $G_p(\alpha-1,\beta,\epsilon-1,\delta-1,\gamma)$  when  $\epsilon \geq 1$  and  $G_p(\alpha-1,\beta,0,\delta-1,\gamma)$  when  $\epsilon = 0$ . We know that  $G/Z = \langle xZ,yZ \rangle$  where xZ has order  $p^{\alpha-1}$ . Observe that  $(yZ)^{p^{\beta}} = y^{p^{\beta}}Z = x^{p^{\alpha-\epsilon}}Z$ . When  $\epsilon \geq 1$ , we have

$$x^{p^{\alpha-\epsilon}}Z = x^{p^{(\alpha-1)-(\epsilon-1)}}Z$$

and when  $\epsilon = 0$ , we have

$$x^{p^{\alpha-\epsilon}}Z = x^{p^{\alpha}}Z = Z.$$

Also,

$$(xZ)^{yZ} = x^y Z = x^{p^{\alpha-\delta}+\gamma} Z = x^{p^{(\alpha-1)-(\delta-1)}+\gamma} Z.$$

Hence, G/Z satisfies the hypotheses for  $G_p(\alpha-1, \beta, \epsilon-1, \delta-1, \gamma)$  when  $\epsilon \geq 1$  and  $G_p(\alpha-1, \beta, 0, \delta-1, \gamma)$  when  $\epsilon = 0$ .

We know that  $X=\langle x\rangle$  is a cyclic, normal subgroup of G. Observe that N is contained in X and so is characteristic. This implies that N is normal in G. Observe that  $Z \leq N$  and we have shown that  $G/Z \cong G_p(\alpha-1,\beta,\epsilon-1,\delta-1,\gamma)$  or  $G_p(\alpha-1,\beta,0,\delta-1,\gamma)$ . If  $\delta=2$ , then N=Z, and we have the desired result. Otherwise, we have  $\delta\geq 3$ . Using induction, we have  $G/N\cong (G/Z)/(N/Z)$  is isomorphic to either

$$G_p((\alpha-1)-(\delta-1)+1,\beta,(\epsilon-1)-(\delta-1)+1,1,\gamma) \cong G_p(\alpha-\delta+1,\beta,\epsilon-\delta+1,1,\gamma)$$

or

$$G_p((\alpha - 1) - (\delta - 1) + 1, \beta, 0, 1, \gamma) \cong G_p(\alpha - \delta + 1, \beta, 0, 1, \gamma). \square$$

We consider the metacyclic groups of positive type and use Theorem 2.1 and Lemma 2.3. Thus, we first analyze G/G'.

**Lemma 3.2** Suppose  $G = G_p(\alpha, \beta, \epsilon, \delta, +)$ .

(i) If 
$$\delta \geq \epsilon$$
 or  $\delta < \epsilon$  and  $\alpha \geq \beta + \epsilon$ , then  $G/G' = C_{p^{\alpha - \delta}} \times C_{p^{\beta}}$ .

(ii) If 
$$\delta < \epsilon$$
 and  $\alpha < \beta + \epsilon$ , then  $G/G' = C_{p^{\alpha - \epsilon}} \times C_{p^{\beta + \epsilon - \delta}}$ .

**Proof.** Now  $G' = \langle x^{p^{\alpha-\delta}} \rangle$ , so  $|G'| = p^{\delta}$ . Also  $|G| = p^{\alpha+\beta}$ , so  $|G: G'| = p^{\alpha+\beta-\delta}$ .

If  $\delta \geq \epsilon$ , then  $\langle y \rangle \cap G' = \langle x^{p^{\alpha-\epsilon}} \rangle = \langle x \rangle \cap \langle y \rangle$ . We see that xG' has order  $p^{\alpha-\delta}$ , and yG' has order  $p^{\beta}$  and  $G/G' = \langle xG' \rangle \times \langle yG' \rangle$  yielding the desired result.

Now suppose  $\delta < \epsilon$ . In this case, we see that  $G' < \langle x^{p^{\alpha-\epsilon}} \rangle = \langle x \rangle \cap \langle y \rangle$ . We see that xG' has order  $p^{\alpha-\delta}$  and yG' has order  $p^{\beta+\epsilon-\delta}$ . Since  $G' < \langle x \rangle \cap \langle y \rangle$ , we do not have that G/G' is a direct product of  $\langle xG' \rangle$  and  $\langle yG' \rangle$ . We see that G/G' is abelian and generated by xG' and yG', so every element of G/G' has order  $\leq \max\{p^{\alpha-\delta}, p^{\beta+\epsilon-\delta}\}$ . If  $\alpha \geq \beta + \epsilon$ , then  $\alpha - \delta \geq \beta + \epsilon - \delta$ . In this case, xG' has the largest order of any element in G/G', and so we get  $G/G' = C_{p^{\alpha-\delta}} \times C_{p^{\beta}}$  since  $|G/G'| = p^{\alpha+\beta-\delta}$ . On the other hand, if  $\alpha < \beta + \epsilon$ , then  $\alpha - \delta < \beta + \epsilon - \delta$ . In this case, yG' has the largest order of any element

in G/G' and we get  $G/G' = C_{p^{\alpha-\epsilon}} \times C_{p^{\beta+\epsilon-\delta}}$ .  $\square$ 

Given an element  $g \in G$ , we write cl(g) to denote the conjugacy class of g in G.

**Lemma 3.3** Let  $G = G_p(\alpha, \beta, \epsilon, \delta, \gamma)$ . If  $g = y^{pl+a}x^m$  for integers l, m, and a so that  $a \in \{1, \ldots, p-1\}$ , then cl(g) = gG'.

**Proof.** We first claim that  $G = \langle x, g \rangle$ . We know that  $G = \langle x, y \rangle$ . Obviously,  $\langle x, g \rangle \leq G$ . Observe that  $y^{pl+a} = gx^{-m} \in \langle x, g \rangle$ . Since the order of y is a power of p, this implies that  $y \in \langle x, g \rangle$ . We conclude that  $G = \langle x, y \rangle \leq \langle x, g \rangle \leq G$ . This proves the claim.

Because  $\langle x \rangle$  is normal in G, we obtain  $G = \langle x \rangle \langle g \rangle$ . Observe that  $\langle g \rangle \leq C_G(g)$ . By Dedekind's lemma (see Lemma X.3 on page 328 of [6]), it follows that  $C_G(g) = (C_G(g) \cap \langle x \rangle) \langle g \rangle = C_{\langle x \rangle}(g) \langle g \rangle$ . Since x centralizes  $x^m$ , we have

$$C_{\langle x \rangle}(g) = C_{\langle x \rangle}(y^{pl+a}x^m) = C_{\langle x \rangle}(y^{pl+a}) = C_{\langle x \rangle}(y) = \langle x^{p^t} \rangle,$$

where  $t = \delta$  if  $\gamma = +$  and  $t = \alpha - 1$  when  $\gamma = -$ . We see that  $C_G(g) = \langle g, x^{p^t} \rangle$ . We deduce that

$$|G:C_G(g)|=|\langle x\rangle:\langle x^{p^t}\rangle|=p^t=|G'|.$$

Since  $\operatorname{cl}(g) \subseteq gG'$ , we conclude that  $\operatorname{cl}(g) = gG'$ .  $\square$ 

Given a group G and a prime p, we define  $G^p = \langle G^{\{p\}} \rangle$ . I.e.,  $G^p$  is the subgroup generated by  $G^{\{p\}}$ . In a similar fashion, we define  $G^4 = \langle g^4 \mid g \in G \rangle$ . Following the literature, we say that a finite p-group G is powerful if (i)  $G' \leq G^p$  when p is odd and (ii)  $G' \leq G^4$  when p = 2. If G is a powerful p-group, then it is known that  $G^p = G^{\{p\}}$ , i.e. the set of p-powers of elements of G is equal to the subgroup the p-powers generate. (See Section 2 of [5] and in particular Propostion 2.6 of that citation.)

We claim that metacyclic p-groups of positive type are powerful. Let G be  $G_p(\alpha, \beta, \epsilon, \delta, +)$ , then  $G' = \langle x^{p^{\alpha-\delta}} \rangle$ . As  $\alpha - \delta \geq 1$  it follows immediately that G is powerful when p is odd. For p = 2, we note that  $\alpha - \delta \geq 2$  so again we have that G is powerful.

When G is of positive type, we extend Lemma 3.3.

**Lemma 3.4** Let  $G = G_p(\alpha, \beta, \epsilon, \delta, +)$  and  $g \in G \setminus G^{\{p\}}$ . Then cl(g) = gG'.

**Proof.** Let  $g \in G$  then  $g = y^n x^m$  for some integers n and m. As G is powerful, it follows that if  $g \in G \setminus G^{\{p\}}$ , then  $g \notin G^p$ , and thus, one of n and m is not divisible by p. When n is not divisible by p, we obtain the conclusion by Lemma 3.3.

We now suppose that  $g = y^n x^m$  where m is not divisible by p. We want to prove that  $\operatorname{cl}(g) = gG'$ . We know that  $\operatorname{cl}(g) \subseteq gG'$ . It suffices to prove that  $|\operatorname{cl}(g)| \ge |gG'| = |G'| = p^{\delta}$ . On the other hand, we know that y acts as an automorphism of order  $p^{\delta}$  on  $\langle x \rangle$ , so x has  $p^{\delta}$  distinct images under powers of y. Thus, if  $1 \le a, b \le p^{\delta}$ , then  $x^{y^a} = x^{y^b}$  if and only if a = b. Since m is coprime to p, we see that  $(x^{y^a})^m = (x^{y^b})^m$  if and only if a = b. Hence, we have that  $g^{y^a} = g^{y^b}$  if and only if  $(y^n x^m)^{y^a} = (y^n x^m)^{y^b}$  and this occurs if and only if a = b. We deduce that a = b0 has at least a = b1 distinct conjugates under a = b2 and so a = b3 as desired. This proves the lemma. a = b3

We now prove that if G is metacyclic of positive type, then  $\eta(G) = \eta(G/G')$ . Combining this fact with Lemmas 2.3 and 3.2, we are able to compute  $\eta(G)$  for all primes p.

Corollary 3.5 Suppose G is  $G_p(\alpha, \beta, \epsilon, \delta, +)$ . Then  $\eta(G) = \eta(G/G')$ .

**Proof.** As G is powerful, by Theorem 2.1, we need to show that for all  $g \in G \setminus G^{\{p\}}$  every element of gG' is conjugate to a generator of  $\langle g \rangle$ , this follows from Lemma 3.4.  $\square$ 

For the record, we explicitly record the value of  $\eta(G)$  when G is a metacyclic group of positive type.

Corollary 3.6 Suppose G is  $G_p(\alpha, \beta, \epsilon, \delta, +)$ .

- (i) If  $\delta \geq \epsilon$  or  $\delta < \epsilon$  and  $\alpha \geq \beta + \epsilon$ , then  $\eta(G) = g_p(\alpha \delta, \beta)$ .
  - (a) If  $\beta \leq \alpha \delta$ , then  $\eta(G) = p^{\beta 1}((\alpha \delta \beta)(p 1) + p + 1)$ .
  - (b) If  $\beta > \alpha \delta$ , then  $\eta(G) = p^{\alpha \delta 1}((\beta \alpha + \delta)(p 1) + p + 1)$ .
- (ii) If  $\delta < \epsilon$  and  $\alpha < \beta + \epsilon$ , then  $\eta(G) = g_p(\alpha \epsilon, \beta + \epsilon \delta) = p^{\alpha \epsilon 1}((\beta \alpha + 2\epsilon \delta)(p 1) + p + 1)$ .

**Proof.** Using Corollary 3.5, we have  $\eta(G) = \eta(G/G')$ . If  $\delta \geq \epsilon$  or  $\delta < \epsilon$  and  $\alpha \geq \beta + \epsilon$ , then in view of Lemma 3.2, we see that  $G/G' = C_{p^{\alpha-\delta}} \times C_{p^{\beta}}$ 

and  $\eta(G) = g_p(\alpha - \delta, \beta)$ . The remainder of (i) follows from the definition of  $g_p$ . Suppose  $\delta < \epsilon$  and  $\alpha < \beta + \epsilon$ . Applying Lemma 3.2, we see that  $G/G' = C_{p^{\alpha - \epsilon}} \times C_{p^{\beta + \epsilon - \delta}}$ . Observe that  $\alpha < \beta + \epsilon$  yields  $\alpha - \epsilon < \beta < \beta + \epsilon - \delta$  as we are assuming  $\delta < \epsilon$ . In light of the definition of  $g_p$ , we obtain conclusion (ii).  $\square$ 

When G is metacyclic of positive type, we show that  $\eta(G) \geq \alpha + \beta$ .

Corollary 3.7 If G is  $G_p(\alpha, \beta, \epsilon, \delta, +)$ , then  $\eta(G) \geq \alpha + \beta$ .

**Proof.** We consider separately the cases given in Corollary 3.6. We use the fact that  $2^{\beta-1} \geq \beta$  for  $\beta$  a positive integer. First, (i)(a), where  $\alpha - \delta \geq \beta$ ,

$$\eta(G) = p^{\beta-1}((\alpha - \delta - \beta)(p-1) + p + 1) 
\geq 2^{\beta-1}(\alpha - \delta - \beta + 3) 
\geq \alpha - \delta - \beta + 3\beta 
> \alpha + \beta$$

since  $\beta \geq \delta$ .

Now, case (i)(b), so  $\alpha - \delta < \beta$ . First assume  $\alpha - \delta > 1$ , then

$$\eta(G) = p^{\alpha-\delta-1}((\beta-\alpha+\delta)(p-1)+p+1) 
\geq 2^{\alpha-\delta-1}(\beta-\alpha+\delta+3) 
\geq 2(\beta-\alpha+\delta)+3(\alpha-\delta) 
= 2\beta+(\alpha-\delta) 
\geq \beta+\alpha+(\beta-\delta) 
\geq \beta+\alpha$$

since  $\beta \geq \delta$ . If  $\alpha - \delta = 1$  then  $p \geq 3$ , also note  $\alpha = 1 + \delta \leq 1 + \beta$ . So, we have

$$\eta(G) \ge 2(\beta - \alpha + \delta) + 4 = 2\beta + 2 > \beta + \alpha.$$

Case (ii) follows similarly to (i)(a), we have  $\alpha - \epsilon \leq \beta + \epsilon - \delta$ ,

$$\eta(G) = p^{\alpha - \epsilon - 1}((\beta - \alpha + 2\epsilon - \delta)(p - 1) + p + 1)$$

$$\geq 2^{\alpha - \epsilon - 1}(\beta - \alpha + 2\epsilon - \delta + 3)$$

$$\geq \beta - \alpha + 2\epsilon - \delta + 3(\alpha - \epsilon)$$

$$= \beta + \alpha + (\alpha - \epsilon - \delta)$$

$$\geq \beta + \alpha$$

## 4 Metacyclic Groups of Negative Type

The goal of this section is to compute  $\eta$  when G is a metacyclic group of negative type. We begin by looking at quotients of G. We begin with a preliminary lemma that is useful in understanding the quotients.

Using the notation of Section 3 and applying Theorem 2.1, we have that if  $G = G_2(\alpha, \beta, \epsilon, \delta, -)$  with  $\delta \geq 1$  and  $N = \langle x^{2^{\alpha - \delta + 1}} \rangle$ , then  $\eta(G) \geq \eta(G/N)$ . We now show that in fact this is an equality. We remind the reader that  $\alpha - \delta \geq 2$  when p = 2.

We now prove the promised equality between  $\eta(G)$  and  $\eta(G/N)$ .

**Theorem 4.1** Let  $G = G_2(\alpha, \beta, \epsilon, \delta, -)$  where  $\delta \geq 1$ . Then  $\eta(G) = \eta(G/N)$  where  $N = \langle x^{2^{\alpha - \delta + 1}} \rangle$ .

**Proof.** Note that N does not make sense if  $\delta = 0$ ; so that it is why we assume  $\delta \geq 1$ . Also, if  $\delta = 1$ , then N = 1; so the conclusion is trivial in this case. Hence, we will assume  $\delta \geq 2$ .

We first prove that  $\eta(G) = \eta(G/Z)$  where  $Z = \langle x^{2^{\alpha-1}} \rangle$ . Recall from Theorem 2.1 that to prove  $\eta(G) = \eta(G/Z)$ , we need to prove that  $Z \subseteq G^{\{2\}}$  and if  $g \in G \setminus G^{\{2\}}$ , then every element of gZ is conjugate to a generator of  $\langle g \rangle$ . Observe that  $Z \subseteq G^{\{2\}}$ . Since  $x^{2^{\alpha-1}}$  is the only nonidentity element of Z, it suffices to prove that if  $g \notin G^{\{2\}}$ , then  $\langle g \rangle$  and  $\langle gx^{2^{\alpha-1}} \rangle$  are conjugate. We know from [2] that  $G' = \langle x^2 \rangle$ .

We prove the claim by working by induction on  $\delta$ . We begin with the case that  $\delta = 2$ . We know that  $x^y = x^{2^{\alpha-2}-1}$ . It follows that

$$(x^2)^y = (x^y)^2 = (x^{2^{\alpha-2}-1})^2 = x^{2^{\alpha-1}-2} = (x^{-2})x^{2^{\alpha-1}}.$$

Observe that this yields that  $(x^{-2})^y = x^2 x^{2^{\alpha-1}}$ . Using this fact and the observation that  $x^{2^{\alpha-1}}$  is central, we then have

$$(x^2)^{y^2} = (x^{-2}x^{2^{\alpha-1}})^y = x^2x^{2^{\alpha-1}}x^{2^{\alpha-1}} = x^2.$$

It follows that  $x^2$  and  $y^2$  commute. Let  $A = \langle x^2, y^2 \rangle$ , and observe that  $G' \leq A$ , so A is a normal, abelian subgroup of G.

We know that every element of G has the form  $y^k x^m$  where  $0 \le k \le 2^{\beta} - 1$ and  $0 \le m \le 2^{\alpha} - 1$  are integers. Notice that if 4 divides both k and m, then  $q \in A^{\{2\}} \subset G^{\{2\}}$ . Also,  $x^2, y^2 \in G^{\{2\}}$ .

If  $g = y^{2l+1}x^m$  for integers l and m, then we can appeal to Lemma 3.3 to see that g is conjugate to  $gx^{2^{\alpha-1}}$  and so,  $\langle g \rangle$  and  $\langle gx^{2^{\alpha-1}} \rangle$  are conjugate, as

Since  $x^y = x^{2^{\alpha-2}-1}$ , we have  $x^{y^2} = x^{2^{2\alpha-4}-2^{\alpha-1}+1}$ . Since  $\delta > 2$ , we know that  $\alpha \geq 4$  (this is using the fact that  $\alpha - \delta \geq 2$ ), so  $2\alpha - 4 \geq \alpha$ . Hence, we have  $x^{y^2} = x^{-2^{\alpha-1}+1}$ . In addition,  $x^{2^{\alpha-1}}$  has order 2, so  $x^{-2^{\alpha-1}} = x^{2^{\alpha-1}}$ . Thus, we have shown  $x^{y^2} = x^{2^{\alpha-1}+1}$ .

Suppose now that  $g = y^{2l}x^{2h+1}$  for integers l and h. From above, we have

$$g^{y^2} = (y^{2l}x^{2h+1})^{y^2} = y^{2l}(x^{y^2})^{2h+1} = y^{2l}(x^{2^{\alpha-1}+1})^{2h+1} = y^{2l}x^{2h+1}x^{2^{\alpha-1}} = gx^{2^{\alpha-1}}$$

We deduce that  $\langle g \rangle$  and  $\langle gx^{2^{\alpha-1}} \rangle$  are conjugate, as desired. We have shown that  $x^{y^2} = xx^{2^{\alpha-1}}$ . This implies that  $x^{-1}y^{-2}x = x^{2^{\alpha-1}}y^{-2}$ . Inverting, we obtain  $(y^2)^x = y^2x^{2^{\alpha}-1}$ . Now, suppose that  $g = y^{2l}x^{2h}$ . We can assume from above that either l is odd or h is odd. Assume first that l is odd. We have

$$g^{x} = (y^{2l}x^{2h})^{x} = ((y^{2})^{x})^{l}x^{2h} = (y^{2}x^{2^{\alpha-1}})^{l}x^{2h} = y^{2l}x^{2h}x^{2^{\alpha-1}} = gx^{2^{\alpha-1}}.$$

We obtain  $\langle g \rangle$  and  $\langle gx^{2^{\alpha-1}} \rangle$  are conjugate, as desired.

We are left with the case that  $g = y^{4l}x^{2(2h+1)}$  for integers h and l. We claim that  $g \in G^{\{2\}}$ . Notice that there is an integer k so that  $\langle g \rangle = \langle y^{4k} x^2 \rangle$ and that  $g \in G^{\{2\}}$  if and only if  $y^{4k}x^2 \in G^{\{2\}}$ . We show that  $y^{4k}x^2 \in G^{\{2\}}$ . We have  $x^{y^2} = xx^{2^{\alpha-1}}$ . It follows that  $xy^2 = y^2xx^{2^{\alpha-1}}$  and

$$(y^{2k}x)^2 = y^{2k}xy^{2k}x = y^{2k}y^{2k}xx^{2^{\alpha-1}k}x = y^{4k}x^2x^{2^{\alpha-1}k}.$$

When k is even, we see that  $(y^{2k}x)^2 = y^{4k}x^2$ . Now assume that k is odd. We have

$$(y^{2k}xx^{2^{\alpha-2}})^2 = y^{2k}xx^{2^{\alpha-2}}y^{2k}xx^{2^{\alpha-2}} = y^{2k}y^{2k}xx^{2^{\alpha-1}k}xx^{2^{\alpha-2}2}$$
$$= y^{4k}x^2x^{2^{\alpha-1}(k+1)} = y^{4k}x^2.$$

Note that we are using the fact that  $x^{2^{\alpha-2}}$  commutes with both x and  $y^2$ here. Thus, this yields  $g \in G^{\{2\}}$ . We conclude for all elements  $g \in G \setminus G^{\{2\}}$ that g and  $gx^{2^{\alpha-1}}$  are conjugate and we have proved that  $\eta(G) = \eta(G/Z)$ when  $\delta = 2$ .

We now assume that  $\delta > 2$ . Let  $M = \langle x^2, y \rangle$ . Since  $x^y = x^{2^{\alpha - \delta} - 1}$ , we see that  $(x^2)^y = (x^{2^{\alpha - \delta} - 1})^2 = (x^2)^{2^{(\alpha - 1) - (\delta - 1)} - 1}$ . Also,  $y^{2^\beta} = x^{2^{\alpha - \epsilon}} = (x^2)^{2^{(\alpha - 1) - \epsilon}}$ . Observe that  $(x^2)^{2^{(\alpha - 1) - 1}} = x^{2^{\alpha - 1}}$ . We conclude that  $M = G_2(\alpha - 1, \beta, \epsilon, \delta - 1, -)$ . Let  $g \in G \setminus G^{\{2\}}$ . If  $g \in M$ , then  $g \in M \setminus M^{\{2\}}$ . By induction, we have that g is conjugate to  $g(x^2)^{2^{(\alpha - 1) - 1}}$ , and so, g and  $gx^{2^{\alpha - 1}}$  are conjugate. Thus, we may assume that  $g \notin M$ . This implies that  $g = y^l x^{2m+1}$  for integers l and m. We know that g induces an automorphism of g of order g. It follows that  $g^{2^{\delta - 1}}$  induces an automorphism of g of order g. Since g induces that g induces an automorphism of g of order g. Since g induces that g is the unique automorphism of g that has order g and is a square. Hence, we see that  $g^{2^{\delta - 1}} = g^{2^{\delta - 1}}$ . We conclude that  $g^{2^{\delta - 1}} = g^{2^{\delta - 1}}$ . This completes the proof of the claim that g in g in

We now work to prove  $\eta(G) = \eta(G/N)$ . We work by induction on  $\delta$ . If  $\delta = 2$ , then N = Z, and the above claim yields the result. We assume that  $\delta \geq 3$ . We have that  $\eta(G) = \eta(G/Z)$ . By induction,  $\eta(G/Z) = \eta((G/Z)/(N/Z))$ , and the First Isomorphism Theorem implies that  $G/N \cong (G/Z)/(N/Z)$ , so  $\eta(G/N) = \eta((G/Z)/(N/Z))$ , and we have the desired equality.  $\square$ 

In light of Theorem 4.1 and Lemma 3.1, we see that if we can compute  $\eta$  for  $G_2(\alpha, \beta, \epsilon, \delta, -)$  when  $\delta = 0, 1$ , then we can compute  $\eta$  for all metacyclic 2-groups of negative type. There are a number of cases to consider when  $\delta = 0$  or 1, and then using these cases, we will compute  $\eta$  when  $\delta \geq 2$ . Recall that the dihedral 2-groups are the groups of the form  $G_2(\alpha, 1, 0, 0, -)$ , the generalized quaternion 2-groups are of the form  $G_2(\alpha, 1, 1, 0, -)$ , and the semi-dihedral groups are of the form  $G_2(\alpha, 1, 0, 1, -)$ . Also, it is known that  $G_2(\alpha, \beta, 1, 0, -)$  and  $G_2(\alpha, \beta, 1, 1, -)$  are isomorphic for all  $\alpha \geq 3$  and  $\beta \geq 2$ . Since  $\delta \leq \beta$ , it follows that dihedral, generalized quaternion, and semi-dihedral are the only groups of negative type where  $\beta = 1$ .

Thus, we need to analyze the negative metacyclic 2-groups of type

$$G_2(\alpha,\beta,\epsilon,\delta,-)$$

with  $\beta \geq 2$ . We recall a few facts about the classification of such groups. In particular, for negative type  $\epsilon$  is either 0 or 1 only. Also the parameters satisfy:  $\alpha \geq \delta + 2$  and  $\beta \geq \delta$  when  $\epsilon = 0$  and  $\beta \geq \delta + 1$  when  $\epsilon = 1$ .

When  $\delta = 0$  or 1, there is a particular abelian normal subgroup M of G. For this subgroup M, we determine which maximal cyclic subgroups of M are maximal in G and how many maximal cyclic subgroups of G lie outside of M. This yields the following result. Recall that  $\eta^*(M)$  is the number of G-orbits on the M-conjugacy classes of maximal cyclic subgroups of M.

**Proposition 4.2** Suppose G is  $G_2(\alpha, \beta, \epsilon, \delta, -)$  where  $\delta = 0$  or 1 and  $\beta \geq 2$ . Let  $M = \langle x, y^2 \rangle$ . Then M is a normal abelian subgroup of G and the following holds:

- (i) If  $\delta = 0$ , then  $\eta(G) = \eta^*(M) + 1$  and every maximal cyclic subgroup of M is maximal cyclic in G except  $\langle y^2 \rangle$ .
- (ii) If  $\delta = 1$ , then  $\eta(G) = \eta^*(M)$  and every maximal cyclic subgroup of M is maximal cyclic in G except  $\langle y^2 \rangle$  and  $\langle y^2 x^{2^{\alpha-1}} \rangle$ .

**Proof.** As M is a subgroup of index 2 in G it follows that M is normal in G. Let  $Y = \langle y^2 \rangle$ . Observe that  $y^2$  centralizes  $\langle x \rangle$  and is obviously central in  $\langle y \rangle$ ; so  $Y = \langle y^2 \rangle$  is central in G. Now, M is central-by-cyclic, so M is abelian.

We now prove that there are exactly two conjugacy classes of maximal cyclic subgroups of G outside of M. Since  $\langle x \rangle$  is normal in G and  $G = \langle x \rangle \langle y \rangle = \langle x \rangle \langle xy \rangle$ , we see that  $C_G(\langle y \rangle) = C_{\langle x \rangle}(\langle y \rangle) \langle y \rangle = \langle x^{2^{\alpha-1}} \rangle \langle y \rangle$  and  $C_G(\langle xy \rangle) = \langle x^{2^{\alpha-1}} \rangle \langle xy \rangle$ . It follows that both  $\langle y \rangle$  and  $\langle xy \rangle$  lie in conjugacy classes of size  $|\langle x \rangle : \langle x^{2^{\alpha-1}} \rangle| = 2^{\alpha-1}$ . It is not difficult to see now that every cyclic subgroup of G outside of M is conjugate to either  $\langle y \rangle$  or  $\langle xy \rangle$ .

(i) For  $\delta = 0$  we show that every maximal cyclic subgroup of M is a maximal cyclic subgroup of G except  $\langle y^2 \rangle$  which lies in exactly 2 different conjugacy classes of maximal cyclic subgroups of G, namely  $\langle y \rangle$  and  $\langle xy \rangle$ .

Observe that yY acts on M/Y inverting every element. Thus, M/Y is a cyclic subgroup of index 2 in G/Y. We have  $(yY)^2 = Y$ , so G/Y is a dihedral group. It follows that if  $g \in G \setminus M$ , then  $(gY)^2 = Y$  and so,  $g^2 \in Y$ . Hence, Y is the only maximal cyclic subgroup of M that is not maximal cyclic in G. Notice that  $Y \leq \langle y \rangle$ . Also, we know that  $\langle yY \rangle$  and  $\langle xyY \rangle$  are in different conjugacy classes of subgroups of G/Y, so  $\langle y \rangle$  and  $\langle xy \rangle$  are in different conjugacy classes of G. Since  $x^y = x^{-1}$ , so  $xy = yx^{-1}$ . It follows that  $(yx)^2 = yxyx = y(yx^{-1})x = y^2$ .

(ii) For  $\delta = 1$  we show that the only maximal cyclic subgroups of M that are not maximal in G are  $\langle y^2 \rangle$  and  $\langle y^2 x^{2^{\alpha-1}} \rangle$ . Again there are exactly 2 different conjugacy classes of maximal cyclic subgroups outside of M given by  $\langle y \rangle$  and  $\langle xy \rangle$ . Note that  $\langle y \rangle$  contains  $\langle y^2 \rangle$  and  $\langle xy \rangle$  contains  $\langle y^2 x^{2^{\alpha-1}} \rangle$ .

Note that M/Y is cyclic in G/Y of order  $2^{\alpha}$ . Also,  $(yY)^2 = Y$  and  $(xY)^{yY} = x^{2^{\alpha-1}-1}Y = (xY)^{2^{\alpha-1}-1}$ . It follows that G/Y is isomorphic to a semi-dihedral group. Let  $Z = \langle x^{2^{\alpha-1}}, Y \rangle$ , and observe that Z/Y = Z(G/Y). Notice that if  $g \in G \setminus M$ , then  $(gY)^2 \in Z/Y$ . This implies that  $g^2 \in Z$ . Observe that  $\langle y^2 \rangle$  and  $\langle y^2 x^{2^{\alpha-1}} \rangle$  are central (and hence normal) in G. It follows that the square of any conjugate of y will be  $y^2$ . Since  $\delta = 1$ , we have  $x^y = x^{2^{\alpha-1}-1}$ , so  $xy = yx^{2^{\alpha-1}-1}$ . We have  $(yx)^2 = yxyx = y(yx^{2^{\alpha-1}-1})x = y^2x^{2^{\alpha-1}}$ . This implies that the square of any conjugate of xy will be  $y^2x^{2^{\alpha-1}}$ . Hence, any other subgroup of M that is maximal cyclic in M will be maximal cyclic in G.  $\square$ 

We now work to compute  $\eta$  for the groups with negative type and  $\delta$  equal to 0 or 1. We will first handle the case when  $\epsilon = 0$  and  $\beta = 2$ . For the following lemma recall that  $\alpha \geq \delta + 2$  when p = 2, so when  $\delta = 1$  we must have  $\alpha \geq 3$ .

**Lemma 4.3** Suppose G is  $G_2(\alpha, 2, 0, \delta, -)$ . Then

(i) 
$$\eta(G) = \alpha + 3$$
 if  $\delta = 0$  and

(ii) 
$$\eta(G) = \alpha + 2$$
 if  $\delta = 1$ .

**Proof.** Following Proposition 4.2, we take  $M = \langle x, y^2 \rangle$ ; so M is abelian. We have  $M \cong C_{2^{\alpha}} \times C_2$  and  $\eta(M) = \alpha + 2$  by Lemma 2.3. We claim that all subgroups of M are normal in G. To see this, note that if K is a subgroup of M then (1) K is a subgroup of  $\langle x \rangle$ , (2)  $K = \langle x^a, y^2 \rangle$  for some integer  $1 \leq a \leq 2^{\alpha} - 1$  or (3)  $K = \langle x^a y^2 \rangle$  for some integer  $1 \leq a \leq 2^{\alpha} - 1$ . When  $\delta = 0$ , we know that  $x^y = x^{-1}$ , so  $(x^a)^y = (x^a)^{-1}$  and  $(x^a y^2)^y = (x^a y^2)^{-1}$  for every integer a. When  $\delta = 1$ , we have  $(x^a)^y = x^{a(2^{\alpha-1}-1)}$ . The observation is that  $\langle x^a \rangle = \langle x^{a(2^{\alpha-1}-1)} \rangle$ ,  $\langle x^a, y^2 \rangle = \langle x^{a(2^{\alpha-1}-1)}, y^2 \rangle$ , and  $\langle x^a y^2 \rangle = \langle x^{a(2^{\alpha-1}-1)}y^2 \rangle$ . This proves the claim. Therefore  $\eta^*(M) = \eta(M)$  and the result follows from Proposition 4.2.  $\square$ .

We continue with the case where  $\epsilon = 0$ . We now consider the case that  $\beta \geq 3$ . Recall that  $g_p(a,b) = p^{(l-1)}((k-l)(p-1)+p+1)$  where p a prime, a and b are positive integers, and we take  $k = \max(a,b)$  and  $l = \min(a,b)$ . Recall also that  $g_p(a,b) = \eta(C_{p^a} \times C_{p^b})$ . The following can be viewed as an improvement on Proposition 2.2(ii).

**Theorem 4.4** Suppose G is  $G_2(\alpha, \beta, 0, \delta, -)$  with  $\beta \geq 3$ . As previously let  $M = \langle x, y^2 \rangle$ . Then the following hold:

- 1. If  $\delta = 1$ , then  $\eta(G) = \eta(M)/2 + 2 = g_2(\alpha, \beta 1)/2 + 2$ .
- 2. If  $\delta = 0$ , then  $\eta(G) = \eta(M)/2 + 3 = g_2(\alpha, \beta 1)/2 + 3$ .

**Proof.** Note that we are assuming  $\delta$  is 0 or 1. As in Proposition 4.2, we let  $M = \langle x, y^2 \rangle$ ; so it follows that M is abelian. In particular, since we are assuming that  $\epsilon = 0$ , we have  $M \cong \langle x \rangle \times \langle y^2 \rangle = C_{2^{\alpha}} \times C_{2^{\beta-1}}$ . Using Lemma 2.3, we obtain  $\eta(M) = g_2(\alpha, \beta - 1)$ . Let k be the maximum of  $\alpha$  and  $\beta - 1$  and let k be the minimum of k and k and k and k and k and k are k be the minimum of k and k are k be the maximum of k and k are k be the maximum of k and k are k be the maximum of k and k are k be the maximum of k and k are k be the maximum of k and k are k be the maximum of k and k are k be the maximum of k and k are k are k and k are k are k are k and k are k are k are k are k are k are k and k are k and k are k are

It is not difficult to see that  $\langle x \rangle$ ,  $\langle y^2 \rangle$ , and  $\langle y^2 x^{2^{\alpha-1}} \rangle$  are maximal cyclic subgroups of M that are normal in G. We claim that  $\langle y^{2(2^{\beta-2})}x \rangle$  is a maximal cyclic subgroup of M that is normal in G. It is easy to see that it is maximal cyclic. When  $\delta=0$ , we see that  $(\langle y^{2(2^{\beta-2})}x \rangle)^y=\langle y^{2(2^{\beta-2})}x^{-1}\rangle=\langle (y^{2(2^{\beta-2})}x)^{-1}\rangle$ , and when  $\delta=1$ , we have  $(\langle y^{2(2^{\beta-2})}x \rangle)^y=\langle y^{2(2^{\beta-2})}x^{2^{\alpha}-1}\rangle=\langle (y^{2(2^{\beta-2})}x)^{2^{\alpha}-1}\rangle$ . This proves that it is normal in G.

We will prove that all the other maximal cyclic subgroups of M will be in conjugacy classes of size 2 in G. Thus,  $\eta^*(M) = (\eta(M) - 4)/2 + 4 = \eta(M)/2 - 2 + 4 = g_2(\alpha, \beta - 1)/2 + 2$ .

Let C be a maximal cyclic subgroup of M. It is not difficult to see that C will be generated by an element of the form  $y^{2l}x$  or one of the form  $y^2x^l$ . When  $\delta=0$ , we have that  $(y^{2l}x)^y=y^{2l}x^{-1}$  and  $(y^2x^l)^y=y^2x^{-l}$ . For C to be normal, we need this conjugate to be in C. When the generator is  $y^{2l}x$ , we need  $y^{2l}x^{-1}=(y^{2l}x)^k=y^{2lk}x^k$  for some integer k. This implies that  $y^{2l-2lk}=x^{k+1}$ . Since  $\epsilon=0$ , we have that  $y^{2l-2lk}=x^{k+1}=1$ . We see that we must have  $2^{\alpha}$  dividing k+1 and  $2^{\beta}$  must divide 2l(1-k). Thus, there is an integer r so that  $k+1=2^{\alpha}r$ , and thus,  $k=2^{\alpha}r-1$ . We obtain that  $2^{\beta-1}$  must divide  $l(1-(2^{\alpha}r-1))=l(2-2^{\alpha}r)=2l(1-2^{\alpha-1}r)$ . Since we know that  $\alpha\geq 2$ , this implies that  $2^{\beta-2}$  must divide l. It follows that  $\langle x\rangle$  and  $\langle y^{2^{\beta-1}}x\rangle$  are the only two maximal cyclic subgroups of M that are normal in G that are generated by an element of the form  $y^{2l}x$  when  $\delta=0$ .

When the generator is  $y^2x^l$ , we need  $y^2x^{-l}=(y^2x^l)^k=y^{2k}x^{lk}$  for some integer k. This implies that  $y^{2-2k}=x^{lk+l}=1$ . This implies that  $2^{\beta}$  divides 2(1-k) and so,  $2^{\beta-1}$  divides 1-k. Hence, there is an integer r so that  $1-k=r2^{\beta-1}$ , and hence,  $k=1-r2^{\beta-1}$ . We see that  $2^{\alpha}$  divides l(1+k)=1

 $l(1+(1-r2^{\beta-1}))=l(2-r2^{\beta-1})=2l(1-r2^{\beta-2})$ . Since  $\beta\geq 3$ , we deduce that  $2^{\alpha-1}$  must divide l. It follows that  $\langle y^2\rangle$  and  $\langle y^2x^{2^{\alpha-1}}\rangle$  are the only maximal cyclic subgroups of M that are normal in G that are generated by an element of the form  $y^2x^l$  when  $\delta=0$ . This proves the result when  $\delta=0$ .

Now we suppose that  $\delta=1$ . Recall that  $\alpha\geq \delta+2$ , so  $\alpha\geq 3$ . We have that  $(y^{2l}x)^y=y^{2l}x^{2^{\alpha-1}-1}$  and  $(y^2x^l)^y=y^2x^{l(2^{\alpha-1}-1)}$ . For C to be normal, we need this conjugate to be in C. Suppose the generator is  $y^{2l}x$ . We need  $y^{2l}x^{2^{\alpha-1}-1}=(y^{2l}x)^k=y^{2lk}x^k$  for some integer k. This implies that  $y^{2l-2lk}=x^{k-2^{\alpha-1}+1}=1$ . We deduce that  $2^\alpha$  must divide  $k-2^{\alpha-1}+1$ , and so, there is an integer r so that  $k-2^{\alpha-1}+1=2^\alpha r$ . We obtain  $k=2^\alpha r+2^{\alpha-1}-1$ . We have that  $2^\beta$  divides  $2l(1-k)=2l(1-2^\alpha r-2^{\alpha-1}+1)$ . It follows that  $2^{\beta-2}$  divides  $l(1-2^{\alpha-1}r-2^{\alpha-2})$ . Since  $\alpha\geq 3$ , we see that  $2^{\beta-2}$  divides l. We conclude that  $\langle x\rangle$  and  $\langle y^{2^{\beta-1}}x\rangle$  are the only two maximal cyclic subgroups of M that are normal in G that are generated by an element of the form  $y^{2l}x$  when  $\delta=1$ .

When the generator is  $y^2x^l$ , we need  $y^2x^{l(2^{\alpha-1}-1)}=(y^2x^l)^k=y^{2k}x^{lk}$  for some integer k. We see that  $y^{2-2k}=x^{lk-l(2^{\alpha-1}-1)}=1$ . It follows that  $2^\beta$  divides 2(1-k), and so,  $2^{\beta-1}$  divides 1-k. There is an integer r so that  $1-k=2^{\beta-1}r$  which yields  $k=1-2^{\beta-1}r$ . We now determine that  $2^\alpha$  divides  $l(k-2^{\alpha-1}+1)=l(1-2^{\beta-1}r-2^{\alpha-1}+1)=2l(1-2^{\beta-2}r-2^{\alpha-2})$ . Since  $\alpha\geq 3$  and  $\beta\geq 3$ , we have that  $2^{\alpha-1}$  divides l. We conclude that  $\langle y^2\rangle$  and  $\langle y^2x^{2^{\alpha-1}}\rangle$  are the only maximal cyclic subgroups of M that are normal in G that are generated by an element of the form  $y^2x^l$  when  $\delta=1$ . This proves the result when  $\delta=1$ .  $\square$ 

In this next corollary, recall that  $\delta \leq \beta$ , so when  $\beta = 2$ , we must have  $\delta = 2$ . We are able to use Theorem 4.4 to compute  $\eta$  for groups of negative type where  $\delta \geq 2$ .

Corollary 4.5 Suppose G is  $G_2(\alpha, \beta, \epsilon, \delta, -)$  with  $\delta \geq 2$ , then

1. 
$$\eta(G) = \alpha - \delta + 3 = \alpha + 1 \text{ if } \beta = 2.$$

2. 
$$\eta(G) = g_2(\alpha - \delta + 1, \beta - 1)/2 + 2$$
 if  $\beta \ge 3$ .

**Proof.** By Theorem 4.1, we have that  $\eta(G) = \eta(G/N)$  where  $N = \langle x^{2^{\alpha-\delta+1}} \rangle$ . Applying Lemma 3.1, we see that  $G/N \cong G_2(\alpha - \delta + 1, \beta, 0, 1, -)$ . Using Lemma 4.3, we see that  $\eta(G/N) = \alpha - \delta + 1 + 2 = \alpha + 3 - \delta$  when  $\beta = 2$ . Since  $2 \le \delta \le \beta = 2$ , we see that  $\delta = 2$ , and so,  $\eta(G) = \alpha + 1$ . When  $\beta \ge 3$ , we

apply Theorem 4.4 to see that  $\eta(G) = \eta(G/N) = g_2(\alpha - \delta + 1, \beta - 1)/2 + 2$ .  $\square$ 

We now compute  $\eta$  for groups of negative type with  $\delta=0$  and  $\epsilon=1$ . We first handle the case where  $\beta=2$ .

**Lemma 4.6** Suppose G is  $G_2(\alpha, 2, 1, 0, -)$  then  $\eta(G) = \alpha + 2$ .

**Proof.** Define  $M = \langle x, y^2 \rangle$ . By Proposition 4.2, we know that M is a normal abelian subgroup of G. First note that  $(x^{2^{\alpha-2}}y^2)^2 = x^{2^{\alpha-1}}y^4 = x^{2^{\alpha-1}}x^{2^{\alpha-1}} = x^{2^{\alpha}} = 1$ . Thus,  $M = \langle x \rangle \times \langle x^{2^{\alpha-2}}y^2 \rangle \cong C_{2^{\alpha}} \times C_2$  and  $\eta(M) = \alpha + 2$ . Consideration of the maximal cyclic subgroups of M shows that all are normal except  $\langle (1, x^{2^{\alpha-2}}y^2) \rangle$  and  $\langle (x^{2^{\alpha-1}}, x^{2^{\alpha-2}}y^2) \rangle$  which are conjugate in G via g. To see that these two subgroups are conjugate, observe that g has three subgroups of order 2 and that  $\langle x^{2^{\alpha-1}} \rangle = \langle y^{2^{\beta}} \rangle$  is central in g and that g and that g is cyclic. Either g normalizes both of the other two subgroups of order 2 or it permutes them. However, if g were to normalize them, they would be normal in g and since they have order 2, that would imply that they would be central in g. This however would contradict the fact that the center of g is cyclic. Thus g is g in g in

We continue with the groups of negative type where  $\delta = 0$  and  $\epsilon = 1$ . We next consider  $\beta \geq 3$  and  $\alpha = 2$ .

**Lemma 4.7** Suppose G is  $G_2(2, \beta, 1, 0, -)$  with  $\beta \geq 3$ . Then  $\eta(G) = \beta + 2$ .

**Proof.** Define  $M = \langle x, y^2 \rangle$ . By Proposition 4.2, we know that M is a normal abelian subgroup of G. Note that  $(xy^{2^{\beta-1}})^2 = x^2y^{2^{\beta}} = x^2x^2 = x^4 = 1$ . So  $M = \langle xy^{2^{\beta-1}} \rangle \times \langle y^2 \rangle \cong C_2 \times C_{2^{\beta}}$  and  $\eta(M) = \beta + 2$ . Consideration of the maximal cyclic subgroups of M shows that all are normal except for  $\langle (xy^{2^{\beta-1}}, 1) \rangle$  and  $\langle (xy^{2^{\beta-1}}, y^{2^{\beta}}) \rangle$  which are conjugate in G via g. The proof that these two subgroups are conjugate is similar to the proof of Lemma 4.6. In particular, Z(G) is cyclic, M has three subgroups of order 2, and if g normalized these two subgroups, then it would centralize them and contradict the fact that Z(G) is cyclic. Thus g = g + 1. The result follows from Proposition 4.2.  $\square$ 

We conclude by computing  $\eta$  when  $\delta = 0$ ,  $\epsilon = 1$ ,  $\alpha \geq 3$ , and  $\beta \geq 3$ . Note this also covers the cases  $\delta = 1$ ,  $\epsilon = 1$  and  $\alpha, \beta \geq 3$ .

**Theorem 4.8** Suppose G is  $G_2(\alpha, \beta, 1, 0, -)$  with  $\alpha \geq 3$  and  $\beta \geq 3$ . Let  $M = \langle x, y^2 \rangle$ .

- 1. If  $\alpha \geq \beta$ , then  $\eta(G) = \eta(M)/2 + 3 = g_2(\alpha, \beta 1)/2 + 3$ .
- 2. If  $\alpha < \beta$ , then  $\eta(G) = \eta(M)/2 + 3 = g_2(\alpha 1, \beta)/2 + 3$ .

**Proof.** As in Proposition 4.2, we let  $M=\langle x,y^2\rangle$ ; so it follows that M is abelian. We know that  $|M|=2^{\alpha+\beta-1}$ , that x has order  $2^{\alpha}$  and  $y^2$  has order  $2^{\beta}$ . Suppose  $\alpha \geq \beta$ , then  $M \cong C_{2^{\alpha}} \times C_{2^{\beta-1}}$ , and so  $\eta(M)=g_2(\alpha,\beta-1)$ . Let  $w=y^2x^{2^{\alpha-\beta}}$ . Observe that  $w^{2^{\beta-2}}=(y^2x^{2^{\alpha-\beta}})^{2^{\beta-2}}=y^{2^{\beta-1}}x^{2^{\alpha-2}} \not\in \langle x\rangle$  and  $w^{2^{\beta-1}}=(y^2x^{2^{\alpha-\beta}})^{2^{\beta-1}}=y^{2^{\beta}}x^{2^{\alpha-1}}=x^{2^{\alpha-1}}x^{2^{\alpha-1}}=1$ . It follows that  $M=\langle x\rangle \times \langle w\rangle$ .

If  $\beta \geq \alpha + 1$ , then  $M \cong C_{2^{\alpha-1}} \times C_{2^{\beta}}$ , and so  $\eta(M) = g_2(\alpha - 1, \beta)$ . Let  $u = y^{2^{\beta-\alpha+1}}x$ . We compute  $u^{2^{\alpha-2}} = (y^{2^{\beta-\alpha+1}}x)^{2^{\alpha-2}} = y^{2^{\beta-1}}x^{2^{\alpha-2}} \notin \langle y \rangle$  and  $u^{2^{\alpha-1}} = (y^{2^{\beta-\alpha+1}}x)^{2^{\alpha-1}} = y^{2^{\beta}}x^{2^{\alpha-1}} = x^{2^{\alpha-1}}x^{2^{\alpha-1}} = 1$ . We deduce that  $M = \langle u \rangle \times \langle y \rangle$ .

In both cases, we will show that  $\eta^*(M) = \eta(M)/2 + 2$ , and we obtain the conclusion by applying Proposition 4.2. Notice that a maximal cyclic subgroup of M will be generated either by an element of the form  $y^{2l}x$  for some integer l or by an element of the form  $y^2x^l$  for some integer l. Observe that  $\langle x \rangle$  and  $\langle y^2 \rangle$  are maximal cyclic subgroups of M that are normal in G.

We next show that  $\langle y^{2^{\beta-1}}x\rangle$  and  $\langle y^2x^{2^{\alpha-2}}\rangle$  are normal subgroups in G. Since M is abelian and has index 2 in G, it suffices to show that y normalizes these subgroups. We compute  $(y^{2^{\beta-1}}x)^y = y^{2^{\beta-1}}x^{-1} = (y^{2^{\beta-1}}x)^{-1}$ . Since y conjugates the generator of  $\langle y^{2^{\beta-1}}x\rangle$  to its inverse, this implies that  $\langle y^{2^{\beta-1}}x\rangle$  is normal in G.

We now turn to  $\langle y^2 x^{2^{\alpha-2}} \rangle$ . We begin with the observation that  $(y^2 x^{2^{\alpha-2}})^4 = y^8$ . Since  $\beta \geq 3$ , we see that  $x^{2^{\alpha-1}} = y^{2^{\beta}} \in \langle y^2 x^{2^{\alpha-2}} \rangle$ . Conjugating yields  $(y^2 x^{2^{\alpha-2}})^y = y^2 x^{-2^{\alpha-2}}$ . Note that  $x^{-2^{\alpha-2}} = x^{2^{\alpha-2}} x^{2^{\alpha-1}}$ . We have  $(y^2 x^{2^{\alpha-2}})^y = y^2 x^{2^{\alpha-2}} x^{2^{\alpha-1}}$ . Since both  $y^2 x^{2^{\alpha-2}}$  and  $x^{2^{\alpha-1}}$  lie in  $\langle y^2 x^{2^{\alpha-2}} \rangle$ , we conclude that  $(y^2 x^{2^{\alpha-2}})^y$  lies in  $\langle y^2 x^{2^{\alpha-2}} \rangle$ . We deduce that  $\langle y^2 x^{2^{\alpha-2}} \rangle$  is normal in G.

We prove that the remaining maximal cyclic subgroups of M lie in orbits of size 2. We have noted that a maximal cyclic subgroup C of M will have a generator of the form  $y^{2l}x$  or of the form  $y^2x^l$  for some integer l. If C has a generator of the form  $y^{2l}x$ , then for C to be normal we need  $(y^{2l}x)^y = y^{2l}x^{-1} \in C$ . This implies that  $y^{2l}x^{-1} = (y^{2l}x)^k$  for some integer k. We have  $y^{2l-2lk} = x^{k+1} = u \in \langle x \rangle \cap \langle y^2 \rangle = \langle x^{2^{\alpha-1}} \rangle$ . Hence, u is either 1 or  $x^{2^{\alpha-1}}$ . If

u=1, then  $2^{\alpha}$  divides k+1 and  $2^{\beta+1}$  divides 2l(1-k). We see that there is an integer r so that  $k+1=2^{\alpha}r$ , and hence,  $k=2^{\alpha}r-1$ . This implies that  $2^{\beta+1}$  divides  $2l(1-k)=2l(1-2^{\alpha}r+1)=4l(1-2^{\alpha-1}r)$ . Since  $\alpha\geq 2$ , this yields  $2^{\beta-1}$  divides l. When  $u=x^{2^{\alpha-1}}$ , we obtain that  $k+1\equiv 2^{\alpha-1}\pmod{2^{\alpha}}$ . Hence, there is an integer r so that  $k+1=2^{\alpha-1}+r2^{\alpha}$ , and so,  $k=2^{\alpha-1}+r2^{\alpha}-1$ . We see that  $2l(1-k)\equiv 2^{\beta}\pmod{2^{\beta+1}}$ . This implies that  $2^{\beta+1}$  divides  $2l(1-k)-2^{\beta}=2l(1-2^{\alpha-1}-r2^{\alpha}+1)-2^{\beta}=4l(1-2^{\alpha-2}-r2^{\alpha-1})-2^{\beta}$ . We deduce that  $2^{\beta-2}$  divides l. We conclude that  $\langle x \rangle$  and  $\langle y^{2^{\beta-1}}x \rangle$  are the only maximal cyclic subgroups of M having the form  $\langle y^{2l}x \rangle$  that are normal in G.

We now suppose that C has a generator of the form  $y^2x^l$ . We need  $(y^2x^l)^y=y^2x^{-l}\in C$ . Hence, we have that  $y^2x^{-l}=(y^2x^l)^k=y^{2k}x^{lk}$  for some integer k. We have  $y^{2-2k}=x^{lk+l}=u$ . As in the previous paragraph, we see that u is either 1 or  $x^{2^{\alpha-1}}$ . If u=1, then we have that  $2^{\beta+1}$  divides 2(1-k), and so, there is an integer r so that  $1-k=2^{\beta}r$ . We determine that  $2^{\alpha}$  divides  $l(k+1)=l(1-2^{\beta}r+1)=2l(1-2^{\beta-1}r)$ . It follows that  $2^{\alpha-1}$  divides l. Now, suppose that  $u=x^{2^{\alpha-1}}$ . We must have that  $2(1-k)\equiv 2^{\beta}\pmod{2^{\beta+1}}$  and  $2^{\beta+1}$  and  $2^{\beta+1}$ . This implies that  $2^{\alpha-1}$  divides  $2^{\beta+1}$ . We then obtain that  $2^{\alpha}$  divides  $2^{\beta+1}r$ . This implies that  $2^{\alpha-1}$  divides  $2^{\beta+1}r$ . We then obtain that  $2^{\alpha}$  divides  $2^{\beta+1}r$ . This implies that  $2^{\alpha-1}$  divides  $2^{\beta-1}r$  and  $2^{\beta-1}r$  divides  $2^{\beta-1}r$  di

We now see that the number of G-orbits of maximal cyclic subgroups of M is  $(\eta(M)-4)/2+4=\eta(M)/2-2+4=\eta(M)+2$ , which completes the proof of the result.  $\square$ 

We close by proving that when G is metacyclic of minus type that is not dihedral, generalized quaternion, or semi-dihedral, then  $\eta(G) \geq \alpha + \beta - 2$  and we determine when equality occurs. We first handle when  $\delta$  equals 0 or 1. In this case, we have  $\eta(G) \geq \alpha + \beta$ .

**Proposition 4.9** Suppose  $G = G_2(\alpha, \beta, \epsilon, \delta, -)$  with  $\delta = 0$  or 1 and  $\beta \geq 2$ . Then  $\eta(G) \geq \alpha + \beta$ .

**Proof.** (i) Suppose  $\epsilon = 0$ . Denote  $l = \min(\alpha, \beta - 1)$  and  $k = \max(\alpha, \beta - 1)$ .

First, consider  $l \geq 3$ . Then  $\beta \geq 4$  and by Theorem 4.4 and Lemma 2.4

$$\eta(G) \ge g_2(\alpha, \beta - 1)/2 + 2 \ge 2k + 2 \ge \alpha + \beta.$$

Next, assume l=2. So  $\beta \geq 3$  and by Theorem 4.4 and Lemma 2.4

$$\eta(G) \ge g_2(\alpha, \beta - 1)/2 + 2 = k + 3 \ge \alpha + \beta.$$

Finally, set l=1. As  $\alpha \geq 2$ , we have  $\beta=2$ . The result follows from Lemma 4.3.

(ii) Now suppose  $\epsilon = 1$ . Assume  $\alpha \ge \beta$ , then  $l = \min(\alpha, \beta - 1) = \beta - 1$  and  $k = \max(\alpha, \beta - 1) = \alpha$ . If  $l \ge 3$ , then  $\beta \ge 4$  and  $\alpha \ge 4$ , so we can assume  $\delta = 0$ . Applying Theorem 4.8 and Lemma 2.4 yields

$$\eta(G) = g_2(\alpha, \beta - 1)/2 + 3 \ge 2k + 3 \ge \alpha + \beta.$$

If l=2, then  $\beta=3$ , and we again appeal to Theorem 4.8 to obtain

$$\eta(G) = g_2(\alpha, \beta - 1)/2 + 3 = g_2(k, 2)/2 + 3 = k + 4 \ge \alpha + \beta.$$

If l=1, then  $\beta=2$ . If  $\alpha=2$  then  $\delta=0$  and if  $\alpha\geq 3$  we can assume  $\delta=0$ . Thus we apply Lemma 4.6.

Finally, suppose  $\epsilon = 1$  and  $\alpha < \beta$ . We set  $l = \min(\alpha - 1, \beta) = \alpha - 1$  and  $k = \max(\alpha - 1, \beta) = \beta$ . When  $l \ge 3$ , we apply Theorem 4.8 and Lemma 2.4 to get

$$\eta(G) = g_2(\alpha - 1, \beta)/2 + 3 \ge 2k + 3 \ge \alpha + \beta.$$

If l=2, then  $\alpha=3$  and  $\beta>3$ . Apply Theorem 4.8 with Lemma 2.4 to give

$$\eta(G) = g_2(\beta, 2)/2 + 3 = \beta + 4 \ge \alpha + \beta.$$

If l=1, then  $\alpha=2$  and  $\delta=0$ , the result follows from Lemma 4.7.  $\square$ 

We now have the case where  $\delta \geq 2$ .

**Proposition 4.10** Suppose  $G = G_2(\alpha, \beta, \epsilon, \delta, -)$  with  $\delta \geq 2$ . Then  $\eta(G) \geq \alpha + \beta - 2$ . Equality holds if and only if  $\beta = \delta$  and either (i)  $\beta = 3$  or (ii)  $\beta \geq 4$  and  $\alpha - \beta = 2$ .

**Proof.** Set  $l = \min(\alpha - \delta + 1, \beta - 1)$  and  $k = \max(\alpha - \delta + 1, \beta - 1)$ . We consider various cases according to the value of l.

First, suppose  $l \geq 4$ . Then by Corollary 4.5 and Lemma 2.4

$$\eta(G) = g_2(\alpha - \delta + 1, \beta - 1)/2 + 2 
= g_2(k, l)/2 + 2 \ge 2k + l + 2 
= (k + l) + k + 2 
\ge \alpha - \delta + \beta + \beta - 1 + 2 
> \alpha + \beta + 1$$

since  $\delta \leq \beta$ .

Now consider l=3. We use Corollary 4.5 and Lemma 2.4 to find an exact value for  $\eta(G)$ .

$$\eta(G) = g_2(\alpha - \delta + 1, \beta - 1)/2 + 2 = g_2(k, 3)/2 + 2 = 2k + 2.$$

If  $\alpha - \delta + 1 > \beta - 1 = 3$ , then  $\delta \le 4$  and

$$\eta(G) = 2(\alpha - \delta + 1) + 2 = \alpha + (\alpha - \delta + 2) + (-\delta + 2) > \alpha + \beta - 2.$$

On the other hand, when  $\beta - 1 \ge \alpha - \delta + 1 = 3$ , we obtain  $\beta \ge 4$  and  $\alpha - \delta = 2$ , so  $\alpha - 2 \le \beta$  and

$$\eta(G) = 2(\beta - 1) + 2 = 2\beta \ge \beta + \alpha - 2$$

with equality if and only if  $\beta = \delta$ .

Next suppose l=2. Since  $\alpha-\delta+1\geq 2+1=3$ , we must have  $\beta=3$ . Applying Corollary 4.5 and Lemma 2.4,

$$\eta(G) = g_2(\alpha - \delta + 1, \beta - 1)/2 + 2 = g_2(k, 2)/2 + 2 
= k + 3 = \alpha - \delta + 4 
> \alpha + 1 = \alpha + \beta - 2$$

with equality if and only if  $\delta = 3 = \beta$ .

Lastly consider l=1. In this case  $\beta=2$  and the result follows from Corollary 4.5.  $\square$ 

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