

Quantum space, ground space traversal, and how to embed multi-prover interactive proofs into unentanglement

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June 13, 2022

Abstract

A celebrated result in classical complexity theory is Savitch’s theorem, which states that non-deterministic polynomial-space computations (NPSPACE) can be simulated by deterministic poly-space computations (PSPACE). In this work, we initiate the study of a quantum analogue of NPSPACE, denoted Streaming-QCMASPACE (SQCMASPACE), in which an exponentially long classical proof is streamed to a poly-space quantum verifier. We first show that a quantum analogue of Savitch’s theorem is unlikely to hold, in that $\text{SQCMASPACE} = \text{NEXP}$. For completeness, we also introduce the companion class Streaming-QMASPACE (SQMASPACE) with an exponentially long streamed *quantum* proof, and show $\text{SQMASPACE} = \text{QMA}_{\text{EXP}}$ (the quantum analogue of NEXP). Our primary focus, however, is on the study of exponentially long streaming *classical* proofs, where we next show the following two main results.

The first result shows that, in strong contrast to the classical setting, the solution space of a quantum constraint satisfaction problem (i.e. a local Hamiltonian) is *always* connected when exponentially long proofs are permitted. For this, we show how to simulate any Lipschitz continuous path on the unit hypersphere via a sequence of *local* unitary gates, at the expense of blowing up the circuit size. This shows that quantum error-correcting codes can be unable to detect one codeword erroneously evolving to another if the evolution happens sufficiently slowly, and additionally answers an open question of [Gharibian, Sikora, ICALP 2015] regarding the Ground State Connectivity problem.

Our second main result is that any SQCMASPACE computation can be embedded into “unentanglement”, i.e. into a quantum constraint satisfaction problem with unentangled provers. Formally, we show how to embed SQCMASPACE into the Sparse Separable Hamiltonian problem of [Chailloux, Sattath, CCC 2012] (known to be QMA(2)-complete for $1/\text{poly}$ promise gap), at the expense of scaling the promise gap with the streamed proof size. As a corollary, we obtain the first systematic construction for obtaining QMA(2)-type upper bounds on arbitrary multi-prover interactive proof systems, where the QMA(2) promise gap scales exponentially with the number of bits of communication in the interactive proof. At the heart of our construction is a new technique for exploiting unentanglement to simulate quadratic Boolean functions, which in some sense allows *history* states to encode the *future*.

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1 Introduction

Computational complexity theory studies the resources required to solve a given computational problem. The resources of time and space, in particular, are very well-studied, revealing certain interesting discrepancies. For example, while the question of whether non-deterministic poly-time (NP) equals deterministic poly-time (P) remains a central open problem in the field, in the context of *space*, the answer is well-known: In 1970, Savitch [Sav70] gave his celebrated result that non-deterministic poly-space computations (NPSpace) could be simulated by deterministic poly-space computations (PSPACE), yielding $\text{PSPACE} = \text{NPSpace}$.

Motivated by the prospect of a quantum analogue of Savitch’s theorem, in this work, we initiate the study of a “non-deterministic” quantum analogue of PSPACE, which we call SQCMASPACE. To define the latter, recall that NPSpace may be viewed as a PSPACE machine which receives an *exponential* length proof $y \in \{0, 1\}^{2^n}$. Of course, a PSPACE verifier cannot even write down y given its limited memory, so a natural way to formalize this idea is to allow y to be *streamed*, bit by bit. This is the approach we take¹ in defining SQCMASPACE.

Definition 1.1 (SQCMASPACE (informal; see Definition 2.2)). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in $\text{SQCMASPACE}(p, q, r)$ for polynomially-bounded functions p, q, r , if there exist thresholds $\alpha(n), \beta(n)$ satisfying $\alpha(n) - \beta(n) \geq 2^{-r(n)}$, and a polynomial-space uniform family of quantum circuits $\{Q_n\}$ such that, for any input $x \in \Sigma^n$:

- If $x \in A_{\text{yes}}$, there exists a streaming proof $y \in \{0, 1\}^{2^{p(n)}}$ such that Q_n accepts (x, y) with probability at least α .
- If $x \in A_{\text{no}}$, then for all streaming proofs $y \in \{0, 1\}^{2^{p(n)}}$, Q_n accepts (x, y) with probability at most β .

To avoid cluttering the introduction, we leave our formal definition of *streaming proof* to Section 2 (Definition 2.1 therein), and instead make do with the following intuitive definition: To “stream” the next bit y_i to the verifier, we imagine the prover applies either “proof gate” I (if $y_i = 0$) or X (if $y_i = 1$), for I and X the single-qubit identity and Pauli X (i.e. NOT) gates, respectively, to a designated qubit k in the verifier’s memory, which is initialized to $|0\rangle_k$. The verifier then copies² this bit into its main memory via Controlled-NOT (CNOT), and the prover subsequently uncomputes bit y_i by re-applying I or X to k , respectively. In other words, there is no separate proof—we view the entire computation as a sequence of gates on the verifier’s memory, some gates of which (the “proof” gates) are *a priori* unknown. For clarity, this is similar to how communication is modelled in quantum interactive proofs, where prover and verifier take turns acting on a shared “message register” (see e.g. Kitaev and Watrous [KW00]).

In Section 1.3, we survey previous works studying quantum notions of PSPACE. Most relevant to our discussion at this point, however, is the work of Fefferman and Remscrem [FR21], which defines a quantum variant of NPSpace denoted QMASPACE, and which differs from SQCMASPACE in three respects: The first two differences are that QMASPACE has a *poly*-length proof which is *quantum*, whereas SQCMASPACE has an *exponential* length streamed proof, which is *classical*. The third difference is that whereas $\text{QMASPACE} = \text{PSPACE}$ [FR21], here we show $\text{SQCMASPACE} =$

¹One can in principle consider alternative definitions of SQCMASPACE. For example, Definition 1.1 allows only one streaming pass of the proof, but one could consider multiple passes. An even stronger access model might allow the ability to query arbitrary single bits of the proof. For our results here, however, a single-pass streaming model suffices, e.g. this definition already captures NEXP, as we show in Theorem 1.5.

²The verifier can also simulate the choice *not* to copy the bit into memory, if desired. See the discussion after Definition 2.1.

NEXP (Theorem 1.5, stated shortly). To the best of our knowledge, the current work is the first to formalize and study a quantum analogue of NPSPACE which allows an *exponentially* long classical proof.

Broader theme. Beyond initiating the study of SQCMASPACE itself, the broader theme of this work asks: “*What can one say about exponentially long proofs verified by poly-space quantum verifiers?*” For example, can allowing an exp-length proof “trivialize” a problem which is provably hard for poly-length proofs? Can exponential length proofs be encoded into *poly-size* history state³ constructions? Here, we give positive answers to both of these questions, for which we now set up the background.

Question 1: Exp- versus poly-length proofs, and the solution space of constraint satisfaction problems (CSPs). In 2006, Gopalan, Kolaitis, Maneva and Papadimitriou [GKMP06] initiated the study of *reconfiguration problems* for SAT, which ask: Given two solutions x and y to a SAT formula ϕ , is there a path in the hypercube from x to y on which all intermediate vertices z are also solutions? Alternatively, in graph theoretic terms, is the solution space of ϕ *connected*? Reference [GKMP06] showed that this decision problem is PSPACE-complete, which in particular implies the problem is not *trivial* — the solution space can be either connected or disconnected, and deciding between the two is hard.

In the quantum setting, one can ask analogous questions about the “solution space” of quantum CSPs, and this has implications for the study of quantum error-correcting codes. To begin, the quantum generalization of a MAX- k -SAT instance ϕ is a k -local Hamiltonian H . H is a $2^n \times 2^n$ Hermitian operator acting on n qubits, but specified *succinctly* via a sum of “local clauses” H_i acting on k qubits (analogous to how ϕ is specified locally via an AND of k -local disjunctions), i.e. $H = \sum_i H_i$. The smallest eigenvalue of H , $\lambda_{\min}(H)$, is the *ground state energy* of H (for ϕ , this encodes the maximum number of simultaneously satisfiable clauses), and the corresponding space of eigenvectors the *ground space* (for ϕ , this encodes the space of optimal assignments). In 2002, Kitaev [KSV02] gave his now celebrated “quantum Cook-Levin theorem”, which showed that estimating the ground state energy of H , known as the k -local Hamiltonian problem (k -LH), is complete⁴ for Quantum-Merlin Arthur (QMA).

With the definition of local Hamiltonians (i.e. quantum CSPs) in hand, we can now state the quantum analogue of reconfiguration, defined as follows.

Definition 1.2 (Ground State Connectivity (GSCON) [GS18] (informal); see Definition 2.15). Given a k -local Hamiltonian H with ground states $|\psi\rangle$ and $|\phi\rangle$ (represented succinctly via quantum circuits), and parameters m, l , does there exist a sequence of l -local unitaries U_1, \dots, U_m such that:

1. ($|\psi\rangle$ mapped to $|\phi\rangle$) $U_m \cdots U_1 |\phi\rangle \approx |\psi\rangle$, and
2. (intermediate states have low energy) $\forall i \in [m], U_i \cdots U_1 |\psi\rangle$ has low energy relative to H ?

In words, GSCON asks whether there exists a sequence of m l -local unitaries that map $|\psi\rangle$ to $|\phi\rangle$ such that intermediate states have low energy (i.e. are also approximate “solutions”) with respect to H . Here, the use of *local* unitaries U_i is crucial, and generalizes the notion of following a path on the hypercube for SAT (which would involve flipping one bit of an assignment per step, or in quantum terms, applying a local X gate). Thus, GSCON asks: Is the ground space of H “connected”?

³A history state [KSV02] is the quantum analogue of a tableau in the Cook-Levin theorem [Coo71; Lev73].

⁴QMA is a quantum analogue of Merlin-Arthur (MA), except with a quantum proof and quantum verifier.

Recall now that in the classical setting, the solution space of a SAT formula can be either connected or disconnected [GKMP06]. In this work, we ask the analogous fundamental question about the structure of ground spaces of local Hamiltonians:

Question 1.3. *Can ground spaces of local Hamiltonians be either “connected” or “disconnected”?*

It is known that if only *poly*-length sequences of local gates are allowed, the answer to this question is YES — namely, GSCON with a polynomial sequence of 2-local unitaries ($m = \text{poly}(n), l = 2$) and inverse polynomial spectral gap is⁵ QCMA-complete [GS18]. However, even in the classical case, in the worst case a connecting path in the hypercube might be *exponentially* long! (Indeed, this is what makes the PSPACE-completeness result of [GKMP06] possible.) Thus, to answer Question 1.3, we must allow sequences of *exponentially* many local gates, i.e. GSCON with $m = \exp(n)$, denoted $\text{GSCON}_{\text{exp}}$.

In addition to this fundamental structural motivation, there are two additional reasons why Question 1.3 is interesting:

- First, from a complexity theory perspective, an instance of $\text{GSCON}_{\text{exp}}$ is straightforwardly in SQCMASPACE—roughly, in step i , the prover streams gate U_i to the verifier, who applies it to map its current state from $U_{i-1} \cdots U_1 |\psi\rangle$ to $U_i \cdots U_1 |\psi\rangle$. Once the proof is fully received, the verifier randomly chooses to check one of the two conditions in the GSCON definition, and accepts if the condition is met. Thus, *if* a “quantum Savitch’s” theorem were to hold, i.e. $\text{SQCMASPACE} = \text{PSPACE}$, then we would immediately obtain $\text{GSCON}_{\text{exp}} \in \text{SQCMASPACE} = \text{PSPACE}$, resolving an open question of [GS18].
- Second, and perhaps most interesting, is the connection to quantum error-correcting codes. For example, in a stabilizer code [Got97], the set of valid codewords is the ground space of a local Hamiltonian H . In this case, one desires the ground space of H to be “disconnected” in the following sense. Let $|\psi\rangle$ be a codeword of H . Then, any sufficiently short sequence of local gates (think of these as local errors “corrupting” $|\psi\rangle$) should ideally take one *out* of the ground space, so that measuring the Hamiltonian catches the corrupting process with non-negligible probability. Indeed, this is precisely what quantum codes typically achieve. What is much less obvious, however, is what happens with *exponential length* corrupting processes — by allowing an exponential-length sequences of local gates, can we stealthily map from $|\psi\rangle$ to some other codeword $|\phi\rangle$ while remaining *exponentially close* to the ground space? If so, then a single measurement of the Hamiltonian during this corrupting process is highly unlikely to detect that we are no longer in state $|\psi\rangle$!

Question 2: Exp-length proofs, poly-size history states, and QMA(2). Our next question asks: *Can exponential length proofs be encoded into poly-size history state/circuit-to-Hamiltonian constructions?* Here, a circuit-to-Hamiltonian construction is the quantum analogue of the Cook-Levin construction [Coo71; Lev73], i.e. a map from quantum circuits V to local Hamiltonians H , such that the ground space of H encodes the action of V . The basic premise is captured by Kitaev’s 5-local construction, which maps a QMA verification circuit $V = V_m \cdots V_1$ (for 1- and 2-qubit gates V_i) to a local Hamiltonian $H = H_{\text{in}} + H_{\text{prop}} + H_{\text{out}} + H_{\text{stab}}$. Intuitively, each of H_{in} , H_{prop} , and H_{out} plays a role analogous to its classical cousin in the Cook-Levin construction — H_{in} ensures V ’s computation is initialized correctly, H_{prop} that in time step t the gate V_t is applied, and H_{out} that

⁵Quantum-Classical Merlin-Arthur (QCMA) is a quantum analogue of MA, except with a classical proof and quantum verifier.

rejecting computations are penalized. Then, the “ideal” quantum assignment perfectly satisfying H_{in} and H_{prop} is the *history state*

$$|\psi_{\text{hist}}\rangle = \frac{1}{\sqrt{m+1}} \sum_{t=0}^m V_t \cdots V_1 |\psi_{\text{proof}}\rangle_A |0 \cdots 0\rangle_B |t\rangle_C \quad (1)$$

(the quantum analogue of a “tableau”), where in the context of QMA, register A starts with the quantum proof $|\psi_{\text{proof}}\rangle$, B is the ancilla space, and C is the clock keeping track of time.

Returning to the question at hand, the naive approach to encoding an exponentially long proof (given explicitly) into history state $|\psi_{\text{hist}}\rangle$ would result in an *exponential* size proof register A , which is too large for our purposes. However, in our definition of SQCMASPACE, the proof is not given explicitly, but *streamed* via application of local gates. While this may seem *a priori* more difficult to work with, it has a distinct benefit — since all gates V_t encoding streamed proof bits (i.e. “proof gates”) are “part of” the verification circuit itself, we can directly encode them into the history state’s *superposition/sum* over time steps (requiring only poly-space), thus obviating the need for a separate proof register, A ! Of course, now we are out of the frying pan into the fire, for there remains a serious problem — the propagation term $H_{\text{prop}} = \sum_{t=1}^m H_t$, which explicitly encodes each gate V_t into its corresponding local propagation term, H_t , needs to be fully specified in *advance*. However, by definition of streaming proof, the gates V_t which are proof gates are *not* known in advance. Can correct propagation still somehow be enforced? To put it more “dramatically”, can a *history* state be used to encode the *future*?

In and of itself, this seems paradoxical. Yet, there *is* a setting in which special cases of classical proofs can be “compressed” into an exponentially smaller number of qubits—QMA(2) (Definition 2.6). Informally, QMA(2) is defined as QMA, except where the verifier is promised to get a proof in *tensor product* across some prespecified partition L versus R of the qubits, i.e. an “unentangled” proof of form $|\psi_1\rangle_L \otimes |\psi_2\rangle_R$. In this setting, Blier and Tapp [BT12] first showed that the NP-complete problem 3-SAT could be verified using just *log-size* “unentangled” proofs, log-space quantum verification, and $1/\text{poly}$ promise gap, i.e. in $\text{PQMA}_{\log}(2)$. Next, Pereszlényi [Per12] showed a similar result for verifying the NEXP-complete language SUCCINCT-3-COLORING via poly-size unentangled proofs and $1/\text{exp}$ promise gap, i.e. in $\text{PreciseQMA}(2)$ (thus obtaining $\text{PreciseQMA}(2) = \text{NEXP}$). (Further related works in Section 1.3.) However, these constructions are expressly tailored to the problems being reduced from, and *a priori* have nothing to do with streaming. Moreover, to-date, no systematic constructions were known for embedding “long” classical proofs into “small” unentangled quantum systems. We thus ask:

Question 1.4. *Can unentanglement be exploited to compress streaming proofs into exponentially smaller⁶ history state constructions?*

1.1 Our Results

We divide our results into three parts: SQCMASPACE, ground space traversal, and embedding streaming proofs into unentanglement.

1. The complexity of SQCMASPACE. We first show that a quantum analogue of Savitch’s theorem for SQCMASPACE is highly unlikely to hold, even in the setting of *constant* promise gap.

⁶For clarity, “smaller” refers to the number of qubits in the history state. Thus, if the proof has length $f(n)$, then the history state should be a $O(\log(f(n)))$ -qubit state.

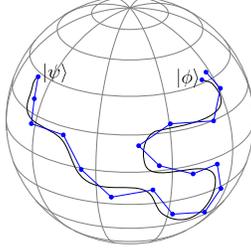


Figure 1: (Color online) Simplified illustration of the Universal quantum path following lemma with f in black (smooth), $|\psi\rangle = f(0)$, $|\phi\rangle = f(1)$, and the path of intermediate states $|\psi_t\rangle$ in blue (piece-wise linear). In the actual construction, each linear segment is itself further subdivided and likewise approximately simulated.

Theorem 1.5. SQCMASPACE with $2^{\text{poly}(n)}$ proof bits, $\text{poly}(n)$ ancilla qubits, completeness 1, and soundness $1/2$, equals NEXP, i.e. $\text{SQCMASPACE}(\text{poly}, \text{poly}, 1) = \text{NEXP}$.

For completeness, we also define the analogous class SQMASPACE (Definition 2.7), which takes an exponential length streamed *quantum* proof, and show its equality to QMA_{EXP} (quantum analogue of NEXP):

Theorem 1.6. SQMASPACE with $2^{\text{poly}(n)}$ proof bits, $\text{poly}(n)$ ancilla qubits, completeness $2/3$, and soundness $1/3$, equals QMA_{EXP} , i.e. $\text{SQMASPACE}(\text{poly}, \text{poly}, 1) = \text{QMA}_{\text{EXP}}$. With $\text{poly}(n)$ proof bits, $O(\log(n))$, ancilla bits, it equals QMA, i.e. $\text{SQMASPACE}(\log, \log, 0) = \text{QMA}$.

2. Ground space traversal. Our second result reveals that Question 1.3 has an arguably surprising resolution — in strong contrast to the classical case, in which the solution space for a SAT instance can be connected or disconnected, in the quantum setting, ground spaces of local Hamiltonians are *always* connected.

At the heart of this result is a new technical lemma showing how to simulate any Lipschitz continuous path on the hypersphere by an exponentially long sequence of *local* quantum gates (i.e. gates on a typical gate-based quantum computer). For this, define a *path* between an initial state $|\psi\rangle$ to final state $|\phi\rangle$ as any Lipschitz continuous function on the unit hypersphere, i.e. $f : [0, 1] \mapsto S^{d-1}$, with $f(0) = |\psi\rangle$ and $f(1) = |\phi\rangle$ (illustration in Figure 1; formal definitions in Section 4). We show⁷:

Lemma 1.7 (Universal quantum path following lemma). *Set $d := 2^n$, and let $f : [0, 1] \rightarrow S^{d-1}$ be a K -Lipschitz continuous path. For every $\varepsilon > 0$, there exists a sequence of $M \in O((\frac{n^2 d^2}{\varepsilon})^{2n})$ 2-local unitaries $U = U_M \cdots U_1$ which “ ε -approximately simulates” the path f as follows. Define $|\psi_t\rangle = U_t \cdots U_1 |\psi_0\rangle$ for $t \in \{0, \dots, M\}$ and $|\psi_0\rangle := f(0)$. Then, for all t ,*

$$\| |\psi_t\rangle - f(t/M) \|_2 \leq \varepsilon. \quad (2)$$

With Lemma 1.7 in hand, we resolve Question 1.3 by showing that in the quantum setting, ground spaces of local Hamiltonians are always connected in the following sense.

Theorem 1.8. *Let $H \in \text{Herm}(\mathbb{C}^d)$, $d = 2^n$ with $0 \preceq H \preceq I$, $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$ with $\langle \psi | H | \psi \rangle \leq \eta$ and $\langle \phi | H | \phi \rangle \leq \eta$. For any $\Delta \geq 2^{-\text{poly}(n)}$, there exists a sequence of 2-local unitary gates $U = U_m \cdots U_1$ with $m \leq 2^{\text{poly}(n)}$ such that*

⁷For simplicity in stating the bound on M in Lemma 1.7, we assume $K \in \Theta(1)$, as this suffices for our applications. However, Lemma 1.7 also holds for non-constant K with $M \in O(K(\frac{n^2 d^2}{\varepsilon})^{2n})$ if $0 < K \leq 1$ and $M \in O(2^{O(n)}(\frac{K^2 n^2 d^2}{\varepsilon})^{2n})$ if $K > 1$.

(1) $\|U|\psi\rangle - |\phi\rangle\|_2 \leq \Delta$, and

(2) for all $i \in [m]$, $\langle \psi_i | H | \psi_i \rangle \leq \eta + \Delta$, where $|\psi_i\rangle := U_i \cdots U_1 |\psi\rangle$.

In words, even if we wish to remain *exponentially* close to the ground space of H throughout the local evolution from $|\psi\rangle$ to $|\phi\rangle$, this can be achieved, at the expense of exponentially blowing up the length of the evolution. Returning to our motivating example of error correcting codes, we conclude: For any H , if the ground space of H encodes a quantum error-correcting code, and $|\psi\rangle$ and $|\phi\rangle$ are any pair of code words, then Theorem 1.8 says one can stealthily corrupt $|\psi\rangle$ into $|\phi\rangle$ via a sequence of 2-qubit gates, so that at any point in the evolution, we are exponentially close to the code space, and thus the corruption is unlikely to be caught via measurement of H . The trade-off is that, again, this evolution path “hugging” the code space is exponentially long.

As an immediate corollary, we are now able to answer an open question of [GS18].

Corollary 1.9 (Informal; formally Corollary 5.1). *GSCON with exponentially many gates and inverse polynomial promise gap is in P.*

This follows since by Theorem 1.8, all GSCON instances in the parameter regime above are YES instances. Thus, allowing an exponentially long proof *trivializes* GSCON, which is otherwise QCMA-complete in the setting of poly-length proofs [GS18].

As a sanity check, we also strengthen a result of [GS18] by showing that even an *unbounded* number of 1-local gates (as opposed to 2-local gates in Corollary 5.1) with constant promise gap do not suffice to trivialize GSCON.

Theorem 1.10 (Informal; formally Theorem 5.2). *GSCON is PSPACE-complete for 1-local gates, constant promise gap, and an unbounded number of gates.*

The previous PSPACE-hardness result of [GS18] required inverse exponential promise gap and an exponential bound on the number of gates.

3. Embedding streaming proofs into unentanglement. We next resolve Question 1.4 in the positive, showing that streaming proofs can be systematically compressed into exponentially smaller history states.

The formalization of this goes via the Sparse Separable Hamiltonian (SSH) problem (Definition 2.5), which informally is identical to the k -local Hamiltonian problem, except for two key differences: (1) H is sparse, rather than local, and (2) proofs are restricted to be in tensor product form. A bit more formally: Given a sparse Hamiltonian H (Definition 2.4) on n qubits and bipartition L versus R of $[n]$, does there exist $|\psi_1\rangle_L \otimes |\psi_2\rangle_R$ such that

$$\langle \psi_1 |_L \otimes \langle \psi_2 |_R H | \psi_1 \rangle_L \otimes | \psi_2 \rangle_R \quad (3)$$

is “small”, or does it hold that for all $|\psi_1\rangle_L \otimes |\psi_2\rangle_R$, $\langle \psi_1 |_L \otimes \langle \psi_2 |_R H | \psi_1 \rangle_L \otimes | \psi_2 \rangle_R$ is “large”? Note that, in general, optimizations over tensor product states $|\psi_1\rangle_L \otimes |\psi_2\rangle_R \in \mathbb{C}^{d^2}$ are harder than optimizations over *all* $|\psi\rangle \in \mathbb{C}^{d^2}$, i.e. without the tensor product requirement. For example, if H in Equation (3) had *polynomial* dimension, then maximizing Equation (3) is NP-hard [Gur03], whereas maximizing $\langle \psi | H | \psi \rangle$ over all $|\psi\rangle \in \mathbb{C}^{d^2}$ is an eigenvalue problem, and thus efficiently solvable in the dimension of H . In other words, the optimal solution to a tensor product optimization is *not* necessarily an eigenvector of H , and this makes the design and analysis of unentangled proof

systems challenging⁸.

We now state our main technical result. A key parameter is the *promise gap* of the Sparse Separable Hamiltonian problem. Chailloux and Sattath [CS12] show SSH is QMA(2)-complete (Definition 2.6) for inverse polynomial promise gap. We show:

Lemma 1.11 ((Informal) Embedding lemma; formally Lemma 6.1). *Let $p, q, r, m, \alpha, \beta : \mathbb{R} \mapsto \mathbb{R}$, where p, q, r are poly-bounded. Let Q be a quantum circuit consisting of m 2-qubit gates, taking in (1) input $x \in \Sigma^n$, (2) a classical streaming proof $y \in \{0, 1\}^{2^p}$, and (3) q ancilla qubits initialized to all zeroes. We are promised that either there exists a streaming proof y causing Q to accept with probability at least α , or all streaming proofs are accepted with probability at most β , for $\alpha - \beta \geq 2^{-r}$. Then, there exists a poly-time many-one reduction from (Q, x) to a Separable Sparse Hamiltonian H instance with norm $\|H\|_\infty \in \text{poly}(m, 2^r)$, and with thresholds α', β' , such that:*

1. H acts on $O(q + \log m)$ qubits.
2. The promise gap scales as $|\alpha' - \beta'| \in \Omega\left(\frac{1}{m2^r}\right)$.

In words, any quantum verification Q with q qubits as workspace and taking in a classical proof of length 2^p can be compressed to a Separable Sparse Hamiltonian instance on $O(q + p)$ qubits and with promise gap scaling⁹ as $1/2^p$. Moreover, the mapping (1) preserves the space required up to poly overhead, and (2) embeds the proof of length 2^p bits into $\sim p$ qubits. To the best of our knowledge, this is the first such systematic method for compressing arbitrary classical proofs via unentanglement.

Applications of the Embedding Lemma. Lemma 6.1 immediately applies to arbitrary SQCMASPACE verifiers. Here, however, we focus on the application to MIP:

Corollary 1.12 ((Informal) Reducing MIP to unentanglement; formally, Corollary 7.2). *There exists a poly-time many-one reduction from any classical multi-prover interactive protocol (MIP, Definition 2.10) with p provers, r rounds, u space, and t bits of communication per round, to an instance of Separable Sparse Hamiltonian on $\tilde{O}(u)$ qubits with promise gap scaling dominated by scaling 2^{-tr} . (The tilde in \tilde{O} hides polylogarithmic factors.)*

For context, recall that MIP with two provers, one round and polynomially many bits of communication equals NEXP [BFL90; FL92] (formal restatement in Theorem 2.11). As for NP, it is contained in MIP with 2 provers, 1 round, and logarithmic bits of communication (see Section 2.2). In words, Corollary 1.12 says that any MIP protocol can be reduced to an SSH instance, with the key parameter being the number of bits t of communication; this is what dictates the promise gap of the SSH instance H we obtain. Note we also preserve the space used by the MIP protocol (which is important for Corollary 7.3 for the case of NP, where the MIP uses log-space).

With Lemma 6.1 in hand, we next show various QMA(2)-type containments. For this, we first show that the specific Hamiltonian construction H output by the Embedding Lemma can be decided in QMA(2) using appropriate parameters:

⁸For example, Marriott-Watrous [MW05] *strong* error reduction for QMA (i.e. without increasing the proof size) fails for QMA(2), since it crucially leverages the fact that for QMA, the optimal assignment is an eigenvector. The attainment of the “standard” notion of *weak* error reduction (i.e. via parallel repetition) by Harrow and Montanaro [HM13] was considered a breakthrough.

⁹This statement assumes the verification time m , proof length 2^p , and promise gap 2^r are polynomially related, which is a reasonable setting. Of course, in general, these relationships need not hold. What we *can* assume without loss of generality is that $m \geq 2^p$ to allow Q to read the entire proof. This means the two potentially dominating terms are m and 2^r , which is why these appear in the norm and promise gap of H in Lemma 6.1.

Lemma 1.13 (Informal; see Lemma 7.5). *Let H be the Sparse Separable Hamiltonian instance produced by the Embedding Lemma, acting on n qubits and with promise gap g . Then, H can be decided by a QMA(2) verifier acting on $O(n)$ qubits and with promise gap $O(g)$.*

As an aside, at present we are curiously unable to show Lemma 7.5 *without* exploiting the specific structure¹⁰ of H from the Embedding Lemma.

Finally, by combining Lemma 6.1 and Lemma 7.5, we obtain the following two main corollaries:

Corollary 1.14 (Informal; see Corollary 7.7). *SQCMASPACE with proof length 2^p , q ancilla qubits, and promise gap $1/2^r$ is contained in QMA(2) with $q + \log p$ proof and ancilla qubits, and promise gap $1/2^{p+r}$.*

Above, note that p and r are polynomially bounded, i.e. logarithmic p and r are allowed.

Corollary 1.15 (Informal; see Corollary 7.8). *MIP with t bits of communication per round, space u , v random bits, p provers, r rounds, and completeness/soundness c and s , respectively, is contained in QMA(2) with $u + v + \log(tr \log(pt))$ proof and ancilla qubits, and promise gap $2^{-tr \log(pt) + \log(c-s)}$.*

Thus, we obtain the first systematic QMA(2)-type bounds on arbitrary multi-prover interactive protocols. Above, the QMA(2) verifier requires the same amount of ancilla space as the MIP, and the QMA(2) promise gap depends exponentially on the total amount of communication but only polynomially on the MIP promise gap. As a bonus, we also immediately rederive (Corollary 7.6) in a unified fashion the results $\text{NP} = \text{PQMA}_{\log}(2)$ [BT12] and $\text{NEXP} = \text{PreciseQMA}(2)$ [Per12].

Finally, as a last application of the Embedding Lemma, we return to our study of GSCON by showing NEXP-completeness for a variant of GSCON:

Theorem 1.16 (Informal; formally Theorem B.3). *GSCON is NEXP-complete with a sparse Hamiltonian, an inverse exponential promise gap, and an exponential number of 2-local gates which may not act across a given L versus R cut of the qubits (i.e. all intermediate states are product across the L versus R cut).*

1.2 Techniques

The proof of $\text{SQCMASPACE} = \text{NEXP}$ (Theorem 1.5) follows easily from the PCP characterization of NEXP [BFL90]; see Section 3.1. As for $\text{SQMASPACE} = \text{QMA}_{\text{EXP}}$ (Theorem 1.6), the obstacle is to show that (weak) error reduction holds for SQMASPACE. This is because with only poly-size ancilla space, the verifier seemingly can only repeat the verification a polynomial number of times, which is not enough to amplify an exponentially small promise gap. We overcome this by forcing the streamed proof itself to repeatedly replenish the verifier’s ancilla, and run a pair of counters to both ensure the prover sends correctly initialized ancilla qubits all set to zero, along with sufficiently many “good” proofs accepted with high probability.

The main technical contributions of this work, however, are the Universal quantum path following lemma (Lemma 1.7) and the Embedding lemma (Lemma 6.1), which we now discuss.

1. Universal Quantum Path Following Lemma. Recall Lemma 1.7 shows how to simulate any Lipschitz continuous path on the unit hypersphere via an exponentially long sequence of local gates. To show this, we first “discretize” the given path f into a dense enough sequence of points

¹⁰In other words, given an arbitrary sparse Hamiltonian H of potentially exponential norm, it is not clear to us how one would verify it in QMA(2) with (say) $1/\text{exp}$ promise gap. (For example, Quantum Phase Estimation (QPE) would seemingly fail — see Section 7.2 for a brief discussion.)

$|\psi_1\rangle, \dots, |\psi_N\rangle$ so that each consecutive pair of points $|\psi_j\rangle$ and $|\psi_{j+1}\rangle$ is “close”. Thus, if *global* (i.e. n -local) unitaries were allowed, a “small rotation” (i.e. close to identity) would suffice to exactly map $|\psi_j\rangle$ to $|\psi_{j+1}\rangle$. However, here we are restricted to 2-local gates, and the typical approach [MI00] to simulate global rotations using 2-local gates would yield intermediate states possibly very far from $|\psi_j\rangle$ and $|\psi_{j+1}\rangle$ (and more generally, from the desired path f). Hence, we devise a general decomposition of global unitaries close to identity into 2-local gates close to identity. Specifically, we give an approximate decomposition $e^{itH} \approx \prod_j e^{it_j H_j}$, where $\sum_j |t_j|$ is bounded by $2^{\text{poly}(n)} |t|^{1/2n}$. Basically, we can decompose a unitary with a short *pulse time* into many local unitaries with short pulse times, which allows us to map a quantum state along each segment $|\psi_j\rangle$ to $|\psi_{j+1}\rangle$.

For that, we first write $H = \sum_j \alpha_j P_j$ in the Pauli basis (i.e. each P_j is a tensor product of the Pauli matrices and identity) and apply the Suzuki decomposition [Suz76] (Lemma 4.9) $e^{iH} = \prod_j e^{i\alpha_j P_j} + O(t^2)$, where $\sum_j |\alpha_j| \leq t$. Clinton, Bausch, and Cubitt [CBC21] give an exact 2-local decomposition for the $e^{i\alpha_j P_j}$ terms with bounded pulse times. We provide an alternative construction with a simpler analysis, and which requires a polynomial number of gates to implement a Hamiltonian interaction (compared to exponential in [CBC21]), at the cost of a slightly worse pulse time bound compared to [CBC21].

In terms of application, recall GSCON asks whether there exists a sequence of local unitaries mapping ground state $|\psi\rangle$ of H to orthogonal ground state $|\phi\rangle$, while remaining in low energy space. Since we can apply Lemma 1.7 to *arbitrary* Lipschitz continuous paths, we can apply it to path

$$f(t) = \cos(t\pi/2)|\psi\rangle + \sin(t\pi/2)|\phi\rangle, \quad (4)$$

where note that for all t , $f(t)$ is also a ground state¹¹ of H . Thus, Lemma 1.7 allows us to “follow” this path via 2-qubit gates, yielding a suitable gate sequence $U_m \cdots U_1$ for GSCON. In general, this sequence requires an exponential number of gates, and in return achieves exponential precision.

2. The Embedding Lemma. Lemma 6.1 shows how to compress any quantum verification Q with q qubits as workspace and taking in a streaming classical proof of length 2^p into a Separable Sparse Hamiltonian instance on $O(q + p)$ qubits and promise gap scaling as $1/2^p$. So, let $Q = V_m \cdots V_1$ be a quantum verifier taking in streaming proof y . Recall we formalized “streaming” by partitioning the gates $\{V_i\}$ into two sets: “Proof gates” indexed by $P \subseteq [m]$, and “computation gates” indexed by $[m] \setminus P$. Our goal is to design a Hamiltonian H so that, when there exists proof y accepted by Q , then an appropriately defined history state $|\psi_{\text{hist}}\rangle$ has low energy against H . The problem is that we do not know the proof gates $\{V_i\}$ with $i \in P$ while computing the reduction—the verifier Q only learns this information in the *future*. To overcome this, at a very high level, we instead demand an appropriately defined *unentangled* history state of form $|\psi_{\text{hist}}\rangle_L \otimes |\psi_{\text{hist}}\rangle_R$. We then exploit this “unentanglement” to logically simulate a quadratic Boolean function across the two copies of $|\psi_{\text{hist}}\rangle$, in turn allowing the history state to decide “on-the-fly” whether it wishes to stream proof bit 0 or 1 in step $t \in P$.

Formally, we define our Hamiltonian as (details in Section 6) $\tilde{H} = \Delta_{\text{in}} \tilde{H}_{\text{in}} + \Delta_{\text{prop}} \tilde{H}_{\text{prop}} + \Delta_{\text{sym}} \tilde{H}_{\text{sym}} + \tilde{H}_{\text{out}}$ for some weights $\Delta_{\text{in}}, \Delta_{\text{prop}}, \Delta_{\text{sym}} \geq 0$. Briefly, \tilde{H}_{in} and \tilde{H}_{out} ensure that in any candidate proof $|\psi\rangle_L \otimes |\phi\rangle_R$, both $|\psi\rangle_L$ and $|\phi\rangle_R$ are initialized correctly at time $t = 0$ and accept at time $t = m$. \tilde{H}_{sym} enforces that a low energy state is symmetric under exchange with respect to

¹¹Note Theorem 1.8 also applies when $|\psi\rangle$ and $|\phi\rangle$ are not ground states, but just low energy states.

the cut L versus R . The key ingredient, however, is hiding in \tilde{H}_{prop} , and is the FLUX gadget¹²,

$$(H_t^I)_L \otimes (H_t^{iX})_R + (H_t^{iX})_L \otimes (H_t^I)_R, \quad (5)$$

used to encode *future* streamed proof gates (i.e. for time steps $t \in P$).

This gadget works as follows. A propagation term H_t^I or H_t^{iX} enforces that at time $t \in P$, the local proof $|\psi\rangle$ applies proof gate I (to simulate streaming bit 0) or proof gate iX (to simulate streaming bit 1), respectively. Since we do not know in advance which of these two gates will be applied, we run a thought experiment — imagine we had two parallel universes, where universe L streams bit 0, or universe R streams bit 1. This can be simulated via term $(H_t^I)_L \otimes (H_t^{iX})_R$ — namely, since the tensor product is multiplicative, this constraint is satisfied, i.e.

$$((H_t^I)_L \otimes (H_t^{iX})_R) |\psi\rangle_L \otimes |\phi\rangle_R = H_t^I |\psi\rangle_L \otimes H_t^{iX} |\phi\rangle_R = 0, \quad (6)$$

only if either universe L (i.e. $|\psi\rangle_L$) applies gate I or universe R (i.e. $|\psi\rangle_R$) applies gate iX , or both. The keyword here is “or”, in that the tensor product allows us to simulate the Boolean OR function between universes. Of course, we have not yet achieved anything, since neither universe has any choice in which bit it streams! This brings us back to the FLUX gadget — observe that the “+” in Equation (5) acts as a Boolean “AND”. In other words, to satisfy the gadget, universe L can apply I (this annihilates the first term, $(H_t^I)_L \otimes (H_t^{iX})_R$) and R can apply I (this annihilates the second term, $(H_t^{iX})_L \otimes (H_t^I)_R$). Similarly, both can instead choose to apply iX to satisfy the gadget. The conclusion is that both universes can freely decide which proof bit to stream at time $t \in P$, so long as they stream the *same* bit! Indeed, this works because we have exploited unentanglement to simulate the quadratic Boolean function EQUALS: $(x \vee \bar{y}) \wedge (\bar{x} \vee y) \leftrightarrow x = y$ for $x, y \in \{0, 1\}$.

The next challenge is to prove soundness of the construction, where recall tensor product optimizations are difficult to analyze since optimal solutions do not correspond to eigenvectors (and thus, standard techniques from the study of k -LH cannot be directly employed). Indeed, this step is rather involved (a step-by-step derivation of the construction is in Section 6.1). For example, the careful reader might wonder why we chose iX to stream bit 1 rather than simply X — it turns out use of X breaks soundness of the FLUX gadget. Even when we use iX , without the symmetry constraint \tilde{H}_{sym} , soundness again breaks via simultaneous cheating across *multiple* FLUX gadgets.

To overcome these obstacles, very briefly, our analysis first exploits the large weight Δ_{sym} to enforce any low energy state to look like $|\psi\rangle_L \otimes |\psi\rangle_R$ for some $|\psi\rangle$. To next force $|\psi\rangle$ to look like an actual history state, two ingredients smoothly fit together. First, since we use iX instead of X in the FLUX gadget, it turns out that for any choice of assignment $|\psi_1\rangle_L$ on L , a low energy state $|\psi_2\rangle_R$ on system R must implement at time $t \in P$ the operator

$$U(a_t, b_t) = \frac{1}{\sqrt{a_t^2 + b_t^2}} (a_t iX + b_t I). \quad (7)$$

for some $a_t, b_t \geq 0$. Now, due to the i in iX , $U(a_t, b_t)$ turns out to be *unitary*. Thus, conditioned on any fixed assignment on L , we can “invert” $U(a_t, b_t)$ by applying Kitaev’s change-of-basis operator [KSV02], thus diagonalizing what we call the “residual propagation term on R ”,

$$\langle \psi_1 | H_t^I | \psi_1 \rangle (H_t^{iX})_R + \langle \psi_1 | H_t^{iX} | \psi_1 \rangle (H_t^I)_R. \quad (8)$$

The second ingredient is to show that by setting Δ_{prop} large enough, we can extract a “proper”

¹²This gadget will allow the history state to “encode the future”; the name is thus a reference to the “flux capacitor”, which makes time travel possible in the film *Back to the Future*.

propagation Hamiltonian hiding under this “residual operator on R ” over *all* time steps. This allows us to force any low energy state of \tilde{H} to indeed be of form $|\psi_{\text{hist}}\rangle_L \otimes |\psi_{\text{hist}}\rangle_R$ — which is *almost* what we want.

The final problem is that for any $t \in P$, $|\psi_{\text{hist}}\rangle$ is currently forced to apply a unitary of form $U(a_t, b_t)$ from Equation (7) for some a_t, b_t . What we *actually* want is for the FLUX gadget to act like a “switch”—either $a_t = 0$ and $b_t \gg 0$ (streaming proof bit 0) or $a_t \gg 0$ and $b_t = 0$ (streaming proof bit 1). By carefully exploiting the structure of $U(a_t, b_t)$ itself, we finally show that any low energy $|\psi_{\text{hist}}\rangle_L \otimes |\psi_{\text{hist}}\rangle_R$ can be “rounded” to obtain a state closeby which perfectly satisfies this desired “switch” behavior for all $t \in P$.

1.3 Related Work

GSCON. In the classical setting, Gopalan, Kolaitis, Maneva, and Papadimitriou [GKMP06] show the problem of determining whether two solutions of a Boolean formula are connected through its solution space is in P or PSPACE-complete, depending on the types of constraints allowed in the formula. The GSCON problem was introduced by Gharibian and Sikora [GS18], who show that GSCON with $m = \text{poly}(n)$ ($l = 2$)-local unitaries is QCMA-complete. For $m = \exp(n)$ and $l = 1$, it is PSPACE-complete. If the Hamiltonian is given as a succinct circuit description, GSCON is NEXP-complete. Gosset, Mehta, and Vidick [GMV17] show the surprising result that QCMA-completeness holds even for *commuting* local Hamiltonians (an analogous result for QMA-completeness of k -LH on *commuting* Hamiltonians remains an open question). Nagaj, Hangleiter, Eisert, and Schwarz [NHES21] next show QCMA-completeness for stoquastic Hamiltonians. Watson, Bausch, and Gharibian [WBG20] study GSCON with a *translationally invariant Hamiltonian* on a 1D chain of qudits (i.e. there exists a single 2-local Hamiltonian acting on each pair of neighbors in the chain) and prove QCMA_{EXP} -completeness (QCMA_{EXP} is QCMA with exponentially large proof and exponential-time quantum verifier). We remark that the EXP in QCMA_{EXP} arises due to the translation-invariance, which causes the encoding size of the problem to be exponentially smaller than the size of the chain.

QMA(2). The complexity class $\text{QMA}(k)$ (QMA with k unentangled proofs) was first introduced by Kobayashi, Matsumoto, and Yamakami [KMY03]. Blier and Tapp [BT12] show that $\text{NP} \subseteq \text{QMA}_{\log}(2)$ (QMA(2) but with log-sized proofs and a log-space verifier) with perfect completeness and $1 - 1/\text{poly}$ soundness. Aaronson et al. [ABDFS08] give a $\text{QMA}_{\log}(\tilde{O}(\sqrt{n}))$ protocol for 3-SAT with a constant soundness gap (as opposed to $1/\text{poly}$ in [BT12]). They further argue that assuming a weak version of the Additivity Conjecture from quantum information theory, $\text{QMA}(k) = \text{QMA}(2)$ for all $k \geq 2$ and $\text{QMA}(2)$ can be amplified to exponentially small error. Harrow and Montanaro [HM13] prove this statement by developing a protocol for a *product test* that allows a quantum verifier to check if a state is a product state across n cuts, given two copies. It also follows that 3-SAT has a $\text{QMA}(2)$ protocol with proof size $\tilde{O}(\sqrt{n})$. We remark that it remains an open problem whether $\text{QMA}(2)$ is equal to NEXP, though an oracle separation to coNP exists [KMY03]. Gharibian, Santha, Sikora, Sundaram and Yirka [GSSSY18] define quantum generalizations of the Polynomial Hierarchy, QCPH and QPH (using classical and quantum proofs, respectively, and quantum verifiers in both cases), and show that (1) if $\text{QCPH} = \text{QPH}$, then $\text{QMA}(2)$ is in the Counting Hierarchy, and (2) unless $\text{QMA}(2) = \text{Q}\Sigma_3$ ($\text{Q}\Sigma_3$ the third level of QPH), $\text{QMA}(2)$ is strictly contained in NEXP.

Chen and Drucker [CD10] improve upon [ABDFS08] with a $\text{BellQMA}_{\log}(\tilde{O}(\sqrt{n}))$ protocol for 3-SAT, where $\text{BellQMA}(k)$ is defined as $\text{QMA}(k)$ without entangled measurements. $\text{QMA}(2)$ permits an inverse polynomial gap, however with an exponentially small gap it is equal to NEXP

as shown by Pereszlényi [Per12]. With a linear number of provers and an exponential soundness gap, BellQMA equals NEXP as well. Kinoshita [Kin18] proves that QMA(2) with postselection also equals NEXP. Chiesa and Forbes [CF13] give a tight soundness analysis of the protocol of [BT12], showing a soundness gap $\Omega(n^{-1})$, notably without using a PCP. They further improve upon [CD10] by providing a smooth trade-off between the number of provers k and the soundness gap $\Omega(k^2/n)$. Chailloux and Sattath [CS12] show the Separable Sparse Hamiltonian problem with $1/\text{poly}$ promise gap is complete for QMA(2). Sparsity is crucial here, as [CS12] shows the Separable *Local* Hamiltonian problem is in QMA.

Space-bounded quantum computation. Watrous [Wat99; Wat03] initiates the study of space-bounded quantum computation and shows $\text{BQSPACE}(s(n)) \subseteq \text{SPACE}(O(s(n)^2))$, where BQSPACE is the space-bounded variant of BQP with intermediate measurements. It follows that $\text{BQPSPACE} = \text{PSPACE}$. Fefferman and Lin [FL18] prove that QMA with an inverse exponentially small gap, denoted PreciseQMA, is equal to PSPACE, by showing that $\text{BQUSPACE}(s(n))$ (like BQSPACE but with only unitary gates) equals QMA with a poly-time verifier, $O(s(n))$ space and proof size, and $2^{-O(s(n))}$ soundness gap. Consequently, the precise local Hamiltonian problem (inverse exponential precision) is PSPACE-complete. Fefferman and Remscrem [FR21] improve upon these results by showing $\text{BQUSPACE}(s) = \text{BQSPACE}(s) = \text{QUMASPACE}(s) = \text{QMASPACE}(s)$. (For clarity, recall QMASPACE receives a *poly-sized quantum* proof, whereas in this work SQCMASPACE takes an *exponential size classical* proof.) Notably, they are able to eliminate intermediate measurements, which is nontrivial in the space-bounded setting as deferred measurements require a fresh ancilla for each measurement.

1.4 Open questions

First, while we have given characterizations for both SQCMASPACE and SQMASPACE, our focus has primarily been on *classical* streamed proofs. Discovering further properties of *quantum* streamed proofs is thus left as a natural open question.

Next, via the Universal Quantum Path Following Lemma (Lemma 1.7), we showed that GSCON with exponentially many gates and inverse poly promise gap is in P (Corollary 5.1). However, what remains unclear is the complexity of GSCON with exponentially many gates and inverse *exponential* promise gap. Then, depending on the exact size of the gap and number of unitaries allowed, Lemma 1.7 does not necessarily apply, and indeed, in Theorem A.3 we show that GSCON in this setting is PSPACE-hard. The only progress we are able to make here is Theorem B.3, which requires a *sparse* (versus local) Hamiltonian and predefined L versus R cut across which gates may not act (whereas originally GSCON has no such restriction). Second, whereas the classical analogue of GSCON, S, T -CONN, satisfies a dichotomy theorem (i.e. is either in P or PSPACE-complete depending on the constraints allowed) [GKMP06], a similar result remains unknown for GSCON.

In terms of unentanglement, the Embedding Lemma (Lemma 6.1) recovers the result of [BT12] for NP with *log-size* QMA(2) proofs, and in particular, also with an inverse poly promise gap. Whether this gap can be improved to *constant* while maintaining a log-size proof remains open. Next, can an analogue of Lemma 7.5 be shown *without* assuming the structure on H guaranteed by the Embedding Lemma? Recall our proof of Lemma 7.5 crucially leveraged the latter. Finally, the complexity of QMA(2) remains frustratingly open — is $\text{QMA}(2) = \text{NEXP}$? What other natural complete problems are there for QMA(2) beyond the (inverse poly-gapped) Separable Sparse Hamiltonian [CS12]?

1.5 Organization

Section 2 begins with all relevant definitions. In Section 3, we give our no-go result for a quantum analogue of Savitch’s theorem and analyze streaming quantum proofs. Section 4 gives our first main result, the Universal Quantum Path Following Lemma. This is then applied in Section 5 to show Theorem 1.8, i.e. that the low energy space of any Hamiltonian is always “connected” in the presence of exponentially long local gate sequences. Section 6 gives our second main result, the Embedding Lemma, with applications in Section 7. Appendices A and B study variants of GSCON with exponentially many local gates.

2 Definitions

We begin by defining SQCMASPACE. Remarks: First, the definition of SQCMASPACE will allow inverse exponential promise gap, although we show in Theorem 1.5 that without loss of generality, we may assume a constant promise gap. Second, intermediate measurements are not free in our model, and so only a polynomial number of simulated measurements via the principle of deferred measurement [NC11] can be made in polynomial space. To begin, we model a *streaming* classical proof via the quantum circuit model as follows.

Definition 2.1 (Streaming classical proof). Let U be a quantum circuit acting on an n_1 -qubit input register R_1 , n_2 -qubit ancilla register R_2 , and 1-qubit proof register R_3 , for some $n_1, n_2 > 0$. Registers R_2 and R_3 are initialized to all zeroes. At a high level, the idea is to stream a classical proof in register R_3 one bit at a time. To do so, we view the entire execution of U as a sequence of 1- and 2-qubit gates, but where certain 1-qubit gates on R_3 are *a priori* unknown. Formally:

1. There are two main phases in the circuit, which repeat until the circuit completes. In iteration i :
 - (a) (Computation phase) A sequence of 1- and 2-qubit gates acts solely on registers R_1 and R_2 .
 - (b) (Proof phase)
 - i. (Compute) Single-qubit gate $W_i \in \{I, X\}$ is applied to R_3 , for X the Pauli NOT gate.
 - ii. (Copy) R_3 is classically copied into R_2 via CNOT gate (controlled from R_3 onto R_2).
 - iii. (Uncompute) W_i is applied to R_3 to return R_3 to $|0\rangle$.

Remarks. Above, we view each gate W_i as being applied dynamically by the prover, i.e. each time the computation phase ends, the prover supplies the next bit. In principle, this can be embedded into an interactive proof, although this is possibly overkill, as all communication is one-way (from prover to verifier). Further clarifications: (1) Each time the computation phase is run, the sequence of gates applied need not be the same as in the previous computation phase. (2) For simplicity, we may assume without loss of generality that the computation ends with a proof streaming phase in which $W_i = I$. (3) Without loss of generality, in Step 1(b)ii we assume there is a fixed qubit in R_2 , say q , to which the content of R_3 is copied each time. If U does *not* wish to use the next proof bit, it may set q to $|+\rangle$ just before Step 1(b)ii, so that the CNOT gate of 1(b)ii acts invariantly.

Definition 2.2 (Streaming-QCMASPACE (SQCMASPACE(p, q, r))). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in SQCMASPACE(p, q, r) for polynomially-bounded functions p, q, r , if there exist

thresholds $\alpha(n), \beta(n)$ satisfying $\alpha(n) - \beta(n) \geq 2^{-r(n)}$, and a $q(n)$ -space uniform family of quantum circuits $\{Q_n\}$ with properties as follows. Q_n takes as input a string $x \in \Sigma^n$, a classical streaming proof $y \in \{0, 1\}^{2^{p(n)}}$, and $q(n)$ ancilla qubits in state $|0\rangle^{\otimes q(n)}$. We say Q_n accepts (x, y) with probability p if on input (x, y) , measuring Q_n 's dedicated output wire in the standard basis yields 1 with probability p . Then:

- (Completeness) If $x \in A_{\text{yes}}$, there exists a streaming proof $y \in \{0, 1\}^{2^{p(n)}}$ such that Q_n accepts (x, y) with probability at least α .
- (Soundness) If $x \in A_{\text{no}}$, for all streaming proofs $y \in \{0, 1\}^{2^{p(n)}}$, Q_n accepts (x, y) with probability at most β .

Finally, let the input, ancilla, and proof registers be denoted R_1, R_2, R_3 respectively. To enforce that R_1 and R_3 are not used as ancilla, we require that Q_n only acts on R_1 and R_3 via CNOTs with the control in R_1 or R_3 and the target in R_2 .

Note the use of term ‘‘polynomially-bounded’’—thus, $r = \log n$ is allowed above. For clarity, a polynomial-space Turing machine is bounded only in its workspace tape length; its output tape is unbounded to allow for outputting the (exponential length) quantum circuit Q_n .

Remark 2.3. Throughout this paper, for SQCMASPACE and all other complexity classes below, we slightly abuse notation and use SQCMASPACE(p, q, r) to mean SQCMASPACE($O(p), O(q), O(r)$) (i.e. we omit explicitly writing the Big-Oh each time).

Definition 2.4 (Sparse Hamiltonian (e.g. [CS12])). A Hermitian operator $H \in \text{Herm}((\mathbb{C}^2)^{\otimes n})$ is *row-sparse* if each row of H has at most $\text{poly}(n)$ non-zero entries, and if there exists an efficient classical algorithm mapping row index $i \in [2^n]$ to a sequence of all non-zero entries H_{ij} of H .

Definition 2.5 (Separable Sparse Hamiltonian (SSH(g)) [CS12]). Let $g : \mathbb{N} \mapsto \mathbb{R}$ be an efficiently computable function. Given as input a sparse Hamiltonian H , a bipartition L versus R of the n qubits H acts on, and threshold parameters α, β satisfying $\beta - \alpha \geq 1/g(n)$, decide:

- (YES case) If there exists $|\psi_1\rangle_L |\psi_2\rangle_R$ such that $\langle \psi_1 |_L \langle \psi_2 |_R H | \psi_1 \rangle_L | \psi_2 \rangle_R \leq \alpha$, output YES.
- (NO case) If for all $|\psi_1\rangle_L |\psi_2\rangle_R$, $\langle \psi_1 |_L \langle \psi_2 |_R H | \psi_1 \rangle_L | \psi_2 \rangle_R \geq \beta$, output NO.

Chailloux and Sattath show that the Separable Sparse Hamiltonian problem with inverse polynomial gap, SSH($1/\text{poly}$), is QMA(2)-complete, for QMA(2) defined next.

Definition 2.6 (QMA(2, p, q, r) [KMY03]). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in QMA(2, p, q, r) for polynomially bounded functions p, q, r if there exist thresholds $\alpha(n), \beta(n)$ satisfying $\alpha(n) - \beta(n) \geq 2^{-r(n)}$, and a poly-time uniform family of quantum circuits $\{Q_n\}$ with properties as follows. Q_n takes as input a string $x \in \Sigma^n$, a quantum proof $|\psi_1\rangle_L \otimes |\psi_2\rangle_R \in \mathbb{C}^{2^{p(n)}} \otimes \mathbb{C}^{2^{q(n)}}$, and $q(n)$ ancilla qubits in state $|0\rangle^{\otimes q(n)}$. We say Q_n accepts (x, y) with probability p_{acc} if on input $(x, |\psi_1\rangle_L |\psi_2\rangle_R)$, measuring Q_n 's dedicated output wire in the standard basis yields 1 with probability p_{acc} . Then:

- (Completeness) If $x \in A_{\text{yes}}$, there exists a $|\psi_1\rangle_L |\psi_2\rangle_R$ such that Q_n accepts $(x, |\psi_1\rangle_L |\psi_2\rangle_R)$ with probability at least α .
- (Soundness) If $x \in A_{\text{no}}$, for all $|\psi_1\rangle_L |\psi_2\rangle_R$, Q_n accepts $(x, |\psi_1\rangle_L |\psi_2\rangle_R)$ with probability at most β .

Caution: We are using slightly non-standard notation above, in that the promise gap scales as 2^{-r} , whereas typically in the literature the parameter r would define a $1/r$ gap. This is to align with our definition of (e.g.) SQCMASPACE, which can have an exponentially small promise gap. Next, by setting p, q, r appropriately, Definition 2.6 captures the variants of QMA(2) studied thus far in the literature (as far as we are aware): When $p, q \in \text{poly}(n)$ and $r \in \log n$, we recover QMA(2) [KMY03], $p, q, r \in \text{poly}(n)$ yields PreciseQMA(2) [Per12], and $p, q, r \in \log(n)$ gives PQMA_{log}(2) [BT12] (for PQMA_{log}(2), only $c = 1$ versus $s = 1 - 1/\text{poly}(n)$ is known, i.e. error reduction to arbitrary c and s remains open without blowing up the proof size superlogarithmically). Note that even when $q \in \log(n)$, the circuit Q_n may still consist of $\text{poly}(n)$ gates. Harrow and Montanaro [HM13] have shown that error reduction holds for QMA(2), i.e. we may assume α and β are exponentially close to 1 and 0, respectively.

2.1 Streaming-QMASPACE

For our streaming version of QMASPACE, the previous setup of SQCMASPACE does not suffice, as (e.g.) single-qubit gates do not suffice to generate arbitrary quantum proofs. Hence, we define SQCMASPACE with an exponentially long proof that is swapped into the proof register bit-by-bit.

Definition 2.7 (Streaming-QMASPACE (SQMASPACE(p, q, r))). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in SQMASPACE(p, q, r) for polynomially-bounded functions p, q, r , if there exist thresholds $\alpha(n), \beta(n)$ satisfying $\alpha(n) - \beta(n) \geq 2^{-r(n)}$, and a $q(n)$ -space uniform family of quantum circuits $\{Q_n\}$ with properties as follows. Q_n takes as input a string $x \in \Sigma^n$, a $2^{p(n)}$ -qubit proof $|\psi\rangle$ in register \mathcal{Y} , a $q(n)$ -bit ancilla register \mathcal{X} initialized to $|0\rangle^{\otimes q(n)}$, and is of form (see Figure 2)

$$Q_n = \prod_{i=m}^1 ((V_i)_{\mathcal{X}} \cdot \text{SWAP}_{\mathcal{X}_1, \mathcal{Y}_i}) \cdot (V_0)_{\mathcal{X}}. \quad (9)$$

Then,

- (Completeness) If $x \in A_{\text{yes}}$: $\exists |\psi\rangle \in \mathcal{Y} : \langle 0^q | \langle \psi | (Q_n^\dagger |1\rangle \langle 1|_{\mathcal{X}_1} Q_n) |0^q\rangle | \psi \rangle \geq \alpha(n)$.
- (Soundness) If $x \in A_{\text{no}}$: $\forall |\psi\rangle \in \mathcal{Y} : \langle 0^q | \langle \psi | (Q_n^\dagger |1\rangle \langle 1|_{\mathcal{X}_1} Q_n) |0^q\rangle | \psi \rangle \leq \beta(n)$.

As in Definition 2.2, we do not allow Q_n to alter the contents of its input register (to avoid using said register as additional ancilla space).

As in SQCMASPACE, we allow an inverse exponential promise gap. Later in Theorem 3.2, we show that weak error reduction holds for SQMASPACE. Note that proof streaming is modeled here by swapping the proof bits one by one into a designated ancilla register of the verifier.

Since we will reduce SQMASPACE to QMA_{EXP}, we also define the latter now, as well as its complete problem. QMA_{EXP} is defined the same as QMA but with an exponential-time uniform circuit family.

Definition 2.8 (QMA_{EXP}(p, q)). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in QMA_{EXP}($p(n), q(n)$) for polynomially bounded functions p, q if there exists an exponential time uniform family of quantum circuits $\{Q_n\}$ with properties as follows. Q_n takes as input a string $x \in \Sigma^n$, a quantum proof $|\psi_1\rangle \in (\mathbb{C}^2)^{\otimes 2^{p(n)}}$, and ancilla qubits in state $|0\rangle^{\otimes 2^{q(n)}}$. We say Q_n accepts $(x, |\psi\rangle)$ with probability p_{acc} if on input $(x, |\psi\rangle)$, measuring Q_n 's dedicated output wire in the standard basis yields 1 with probability p_{acc} . Then:

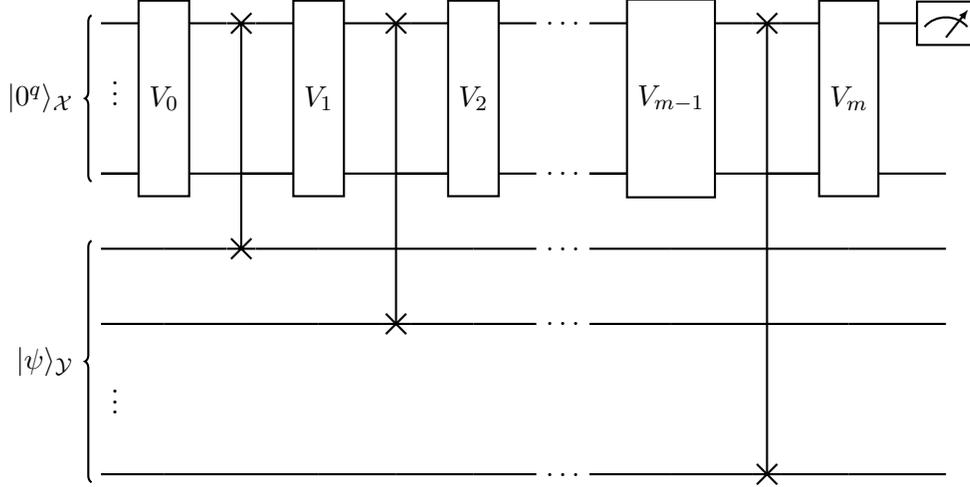


Figure 2: SQMASPACE circuit. The gates from \mathcal{X} to \mathcal{Y} are SWAP gates. For simplicity, we do not depict the input register, whose contents without loss of generality are not altered by Q_n .

- (Completeness) If $x \in A_{\text{yes}}$, there exists a $|\psi\rangle$ such that Q_n accepts $(x, |\psi\rangle)$ with probability at least $2/3$.
- (Soundness) If $x \in A_{\text{no}}$, for all $|\psi\rangle$, Q_n accepts $(x, |\psi\rangle)$ with probability at most $1/3$.

We write QMA_{EXP} to mean $\text{QMA}_{\text{EXP}}(\text{poly}, \text{poly})$. The 1D translationally invariant Hamiltonian problem is complete for QMA_{EXP} [DS09]. Here, “1D translationally invariant” means the *same* local constraint $H_{i,i+1}$ is repeated on all consecutive qubits i on the chain. Formally:

Definition 2.9 (1D-TIH [DS09]). Fix a constant d . The input is the length of the chain N encoded in binary, a single Hamiltonian constraint $H \in \text{Herm}(\mathbb{C}^d \otimes \mathbb{C}^d)$ specified with a constant number of bits, and polynomials α, β such that $\beta - \alpha \geq 1/\text{poly}(N)$. The full Hamiltonian is thus $H^{(N)} := \sum_{i=1}^{N-1} H_{i,i+1}$ (i.e. H acts on each pair of qudits on a line). Decide:

- (YES case) If $\lambda_{\min}(H^{(N)}) \leq \alpha(N)$, output YES.
- (NO case) If $\lambda_{\min}(H^{(N)}) \geq \beta(N)$, output NO.

Note that, crucially, the length of the chain N is exponential in the encoding size of the input.

2.2 Multi-prover interactive proofs

Definition 2.10 ($\text{MIP}(t(n), u(n), v(n), p(n), r(n), c(n), s(n))$ (introduced in [BGKW88], as stated in [FV15])). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in $\text{MIP}(t, p, r, c, s)$ if there exist polynomial t and polynomially bounded functions u and v , and a classical verifier V using $\text{poly}(n)$ time, $u(n)$ space, $v(n)$ bits of randomness, and interacting with p non-communicating provers via r rounds of interaction, where each round consists of $t(n)$ bits of communication between verifier and provers, and where $n = |x|$ is the size of input x , such that

- If $x \in A_{\text{yes}}$, then there exists a strategy for the provers that is accepted by the verifier with probability at least c .
- If $x \in A_{\text{no}}$, any strategy of the provers is accepted by the verifier with probability at most s .

Theorem 2.11 ([BFL90; FL92]). *For any polynomial r ,*

$$\text{MIP}(\text{poly}, \text{poly}, \text{poly}, \text{poly}, \text{poly}, 2/3, 1/3) = \text{MIP}(\text{poly}, \text{poly}, \text{poly}, 2, 1, 1, 2^{-r}) = \text{NEXP}. \quad (10)$$

Next, $\text{NP} \subseteq \text{MIP}(\log, \log, \log, 2, 1, 1, 1 - 1/\text{poly}(n))$ —this follows from the standard 3-SAT multi-prover protocol, in which the verifier asks prover 1 for the assignment (x, y, z) to a random clause, and prover 2 for the assignment to one of the bits x, y , or z uniformly at random. By applying the PCP theorem, one immediately strengthens this inclusion to the case of *constant* soundness for MIP. More generally, $\text{MA} = \text{MIP}(\log, \text{poly}, \text{poly}, 2, 1, 1, 1 - 1/\text{poly}(n))$. (The forward inclusion follows by applying the Cook-Levin theorem to the MA verifier to obtain a 3-SAT formula ϕ , and using MIP’s randomness to choose $\text{poly}(n)$ random bits for ϕ (this is why the space requirement increases from \log to poly for NP versus MA, respectively) and then run the interactive protocol for NP above. The reverse inclusion simply has the MA verifier receive a brute force concatenation of all answers the provers would send to the verifier possible questions, which will have $\text{poly}(n)$ total length.)

2.3 Probabilistically checkable proofs

Definition 2.12 (PCP[$r(n), q(n)$] [AS98]). A language L is in PCP[$r(n), q(n)$] if there exists verifier Turing machine M that behaves as follows:

1. M receives input x , a proof $y \in \{0, 1\}^*$, and a random string $z \in \{0, 1\}^{r(n)}$ on separate tapes.
2. M computes indices $i_1, \dots, i_{q(n)}$ without accessing y (it may access z) in polynomial time.
3. M copies proof bits $y_{i_1}, \dots, y_{i_{q(n)}}$ to its work tape.
4. M accepts or rejects in polynomial time without accessing y .

M must also satisfy the following conditions for all $x \in \{0, 1\}^n$:

- If $x \in L$, $\exists y : \Pr_z[M(x, y, z) = 1] = 1$, where $z \in \{0, 1\}^{r(n)}$ is chosen uniformly at random.
- If $x \notin L$, $\forall y : \Pr_z[M(x, y, z) = 1] \leq 1/2$.

Theorem 2.13 (PCP Theorem [ALMSS98]). $\text{NP} = \text{PCP}[O(\log(n)), O(1)]$.

Theorem 2.14 ([BFL90]). $\text{NEXP} = \text{PCP}[O(\text{poly}(n)), O(\text{poly}(n))]$.

2.4 Ground State Connectivity Problem

The *ground state connectivity* problem (GSCON) introduced by Gharibian and Sikora [GS18] intuitively asks the following question: Given a Hamiltonian H and ground states $|\psi\rangle$ and $|\phi\rangle$, does there exist a sequence of local gates that maps $|\psi\rangle$ to $|\phi\rangle$, such that all intermediate states have low energy with respect to H ? Formally, it is defined as follows.

Definition 2.15 (GSCON($H, k, \eta_1, \eta_2, \eta_3, \eta_4, \Delta, l, m, U_\psi, U_\phi$) [GS18]).

Input:

- A k -local Hamiltonian $H \in \text{Herm}(\mathcal{B}^{\otimes n})$, where $\mathcal{B} := \mathbb{C}^2$.
- $\eta_1, \eta_2, \eta_3, \eta_4, \Delta \in \mathbb{R}$ and integer $m \geq 0$, such that $\eta_2 - \eta_1 \geq \Delta$ and $\eta_4 - \eta_3 \geq \Delta$.

- Polynomial size quantum circuits U_ϕ, U_ψ generating “starting” and “target” states $|\phi\rangle$ and $|\psi\rangle$ (on input $|0^n\rangle$), respectively, satisfying $\langle\psi|H|\psi\rangle \leq \eta_1$ and $\langle\phi|H|\phi\rangle \leq \eta_1$.

Output:

YES: There exists a sequence of l -local unitaries U_1, \dots, U_m such that:

- (Intermediate states remain in low energy space) For all $i \in [m]$ and intermediate states $|\psi_i\rangle := U_i \cdots U_1 |\psi\rangle$, it holds that $\langle\psi_i|H|\psi_i\rangle \leq \eta_1$, and
- (Final state is close to target state) $\| |\psi_m\rangle - |\phi\rangle \|_2 \leq \eta_3$.

NO: For all l -local sequences of unitaries U_1, \dots, U_m , either:

- (Intermediate state obtains high energy) There exists $i \in [m]$ and an intermediate state $|\psi_i\rangle$ such that $\langle\psi_i|H|\psi_i\rangle \geq \eta_2$, or
- (Final state far from target state) $\| |\psi_m\rangle - |\phi\rangle \|_2 \geq \eta_4$.

We assume U_ψ and U_ϕ to be given as sequences of gates from a universal gate set. The numeric parameters are specified with rational entries using $O(\text{poly}(n))$ bits of precision. Note that $|\psi\rangle$ and $|\phi\rangle$ are not necessarily required to be ground states.

This definition is quite flexible as it allows all parameters to be specified. For 2-local unitaries, a 5-local Hamiltonian, polynomial m and Δ , GSCON is QCMA-complete.

Theorem 2.16 ([GS18]). *There exists a polynomial p such that GSCON is QCMA-complete for $m = O(p(n))$, $\Delta = \Theta(1/m^5)$, $l = 2$, and $k \geq 5$, where n denotes the number of qubits H acts on.*

Choosing different parameters leads to PSPACE-completeness.

Theorem 2.17 ([GS18]). *GSCON is PSPACE-complete for $m = 2^n$, $\Delta = 2^{-(2n+4)}$, $l = 1$, $k = 3$, where n denotes the number of qubits H acts on.*

This result is a consequence of the fact that S, T -CONN is PSPACE-complete [GKMP06].

Definition 2.18 (S, T -CONN). Given a 3-CNF formula ϕ and solutions $x, y \in \{0, 1\}^n$ to ϕ , does there exist a sequence of strings x_1, \dots, x_m , such that

- $x_1 = x$, and $x_m = y$, and
- for all $i \in [m]$, the Hamming distance between x_i and x_{i+1} is at most 1, and
- for all $i \in [m]$, x_i is a solution to ϕ ?

Observe the similarity between S, T -CONN and GSCON: ϕ corresponds to H , x to $|\psi\rangle$, y to $|\phi\rangle$, and x_i to $|\psi_i\rangle$. We are interested in the power of GSCON with $l = 2$ and $m = 2^{\text{poly}(n)}$, and denote this class $\text{GSCON}_{\text{exp}}$.

Definition 2.19 ($\text{GSCON}_{\text{exp}}$). $\text{GSCON}_{\text{exp}}$ is the union over all $\text{GSCON}(\dots)$, where $l = 2$, $m = O(2^{p(n)})$ and $\Delta = \Omega(2^{-p(n)})$ for some polynomial p .

3 Quantum analogues of NPSPACE

In this paper, we investigate the power of $\text{GSCON}_{\text{exp}}$. To show the containment $\text{GSCON} \in \text{QCMA}$ for polynomial m and Δ , [GS18] construct a QCMA-verifier that receives classical approximations of the unitaries U_1, \dots, U_m as proof. That technique no longer works for $\text{GSCON}_{\text{exp}}$. The paths through the Hamiltonian's low energy space can be of exponential length and therefore intermediate states can no longer be expressed succinctly. For that reason, we conjecture that $\text{GSCON}_{\text{exp}}$ may not even be contained in PSPACE.

It holds that $\text{GSCON}_{\text{exp}}$ is PSPACE-hard (see Appendix A) (under polynomial-time reductions) and contained in NEXP. The containment is trivial, since a NEXP-verifier can choose the unitary sequence nondeterministically. Also, hardness is not implied by Theorem 2.17, since 2-local unitaries can flip two bits at the same time.

A natural question is whether there exists a better upper bound than NEXP. We do not know the answer to that question. One intuitive candidate would be a quantum analogue of NPSPACE, which we model as a variant of QCMA with an exponentially long proof and polynomially many qubits. However, we argue in Section 3.1 that any such construction equals NEXP.

Lastly, we show in Section 5, that for sufficiently large $m = 2^{\text{poly}(n)}$, we can map $|\psi\rangle$ to $|\phi\rangle$ while remaining close to the span of $|\psi\rangle$ and $|\phi\rangle$. We can make the distance to the span arbitrarily small by increasing m . We conclude that $\text{GSCON}_{\text{exp}}$ does not have any NO-instances with $m = 2^{\text{poly}(n)}$ and $\Delta = 2^{-o(n)}$ for sufficiently large n .

3.1 Streaming-QCMASPACE vs. NEXP and Savitch's theorem

We now show our no-go theorem for a quantum analogue of Savitch's theorem. For this, recall the definition of SQCMASPACE:

Definition 2.2 (Streaming-QCMASPACE ($\text{SQCMASPACE}(p, q, r)$)). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in $\text{SQCMASPACE}(p, q, r)$ for polynomially-bounded functions p, q, r , if there exist thresholds $\alpha(n), \beta(n)$ satisfying $\alpha(n) - \beta(n) \geq 2^{-r(n)}$, and a $q(n)$ -space uniform family of quantum circuits $\{Q_n\}$ with properties as follows. Q_n takes as input a string $x \in \Sigma^n$, a classical streaming proof $y \in \{0, 1\}^{2^{p(n)}}$, and $q(n)$ ancilla qubits in state $|0\rangle^{\otimes q(n)}$. We say Q_n accepts (x, y) with probability p if on input (x, y) , measuring Q_n 's dedicated output wire in the standard basis yields 1 with probability p . Then:

- (Completeness) If $x \in A_{\text{yes}}$, there exists a streaming proof $y \in \{0, 1\}^{2^{p(n)}}$ such that Q_n accepts (x, y) with probability at least α .
- (Soundness) If $x \in A_{\text{no}}$, for all streaming proofs $y \in \{0, 1\}^{2^{p(n)}}$, Q_n accepts (x, y) with probability at most β .

Finally, let the input, ancilla, and proof registers be denoted R_1, R_2, R_3 respectively. To enforce that R_1 and R_3 are not used as ancilla, we require that Q_n only acts on R_1 and R_3 via CNOTs with the control in R_1 or R_3 and the target in R_2 .

We observe that a PCP verifier (see Definition 2.12) can easily be simulated in SQCMASPACE.

Theorem 1.5. *SQCMASPACE with $2^{\text{poly}(n)}$ proof bits, $\text{poly}(n)$ ancilla qubits, completeness 1, and soundness $1/2$, equals NEXP, i.e. $\text{SQCMASPACE}(\text{poly}, \text{poly}, 1) = \text{NEXP}$.*

Proof. The containment $\text{SQCMASPACE} \subseteq \text{NEXP}$ is trivial. To show $\text{NEXP} \subseteq \text{SQCMASPACE}$, we use the fact that $\text{NEXP} = \text{PCP}[\text{poly}, \text{poly}]$ (see Theorem 2.14). We construct a SQCMASPACE verifier Q that simulates a $\text{PCP}[r, q]$ verifier. Let $T = 2^{\text{poly}(n)}$ be an upper bound on the largest proof bit index accessed by M .

1. Q generates the random string $z \in \{0, 1\}^{r(n)}$ by constructing a state $|+\rangle^{\otimes r(n)}$ and then measuring it in standard basis. This can be done with the usual deferred measurement technique (e.g., [NC11]) since $r(n)$ is polynomial (it is nontrivial to simulate an exponential number of measurements).
2. Q simulates the index computation of M and stores the indices $i_1, \dots, i_{q(n)}$ in ancilla space.
3. For $j = 1, \dots, T$, Q applies W_j to an ancilla $|0\rangle_c$, which maps it to $|y_j\rangle_c$. If $j = i_k$ for some k , copy y_j to a fresh ancilla. Afterwards, W_j is applied again to reset the ancilla c back to $|0\rangle_c$.
4. Simulate M with the stored proof bits to accept or reject.

Since the measured string $z \in \{0, 1\}^{r(n)}$ is distributed uniformly at random, we have

$$\Pr[Q_n^y \text{ accepts } |x\rangle] = \Pr_z[M(x, y, z) = 1].$$

Note that mapping $|y_j\rangle_c$ back to $|0\rangle_c$ is no issue because the circuit is entirely classical after generating the random string. \square

We remark that above theorem really only uses quantum computation to generate randomness. It follows that the soundness gap in $\text{SQCMASPACE}(\text{poly}, \text{poly}, \text{poly})$ can be reduced to a constant.

Corollary 3.1. $\text{SQCMASPACE}(\text{poly}, \text{poly}, 1) = \text{SQCMASPACE}(\text{poly}, \text{poly}, \text{poly})$.

We leave as open problem the question whether direction error reduction is possible, i.e. without the detour via PCP and NEXP .

An alternative interpretation of Theorem 1.5 is that Savitch's theorem [Sav70], which implies $\text{PSPACE} = \text{NPSpace}$, has likely no quantum analogue because the space-bounded variant of BQP , denoted $\text{BQ}_{\text{U}}\text{PSPACE}$, equals PSPACE , as shown by Fefferman and Lin [FL18]. $\text{BQ}_{\text{U}}\text{PSPACE}$ is defined as BQP with polynomial-space uniformly generated quantum circuits (i.e. like SQCMASPACE without a proof). Watrous [Wat99; Wat03; Wat08] gave an earlier definition of BQPSPACE based on quantum Turing machines. The main difference between these definitions is that the quantum Turing machines may perform an exponential number of intermediate measurements, whereas that is not possible with a $\text{BQ}_{\text{U}}\text{PSPACE}$ verifier (the subscript 'U' indicates the verifier may only perform unitary operations). The usual deferred measurement approach does not work because it requires fresh ancillae for each measurement. Both definitions nevertheless equal PSPACE . Recently, Fefferman and Remscrem [FR21] proved that even $\text{QMASPACE} = \text{PSPACE}$, where the QMASPACE verifier is an exponentially long quantum circuit that receives a polynomially-sized proof and is allowed to perform an unrestricted number of intermediate measurements. Hence, a variant of SQCMASPACE with exponentially long circuit, but only polynomially sized proof, would also equal PSPACE .

3.2 Streaming-QMASPACE vs. QMA_{EXP}

Next, we characterize the power of SQMASPACE , which recall is defined as:

Definition 2.7 (Streaming-QMASPACE (SQMASPACE(p, q, r))). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in SQMASPACE(p, q, r) for polynomially-bounded functions p, q, r , if there exist thresholds $\alpha(n), \beta(n)$ satisfying $\alpha(n) - \beta(n) \geq 2^{-r(n)}$, and a $q(n)$ -space uniform family of quantum circuits $\{Q_n\}$ with properties as follows. Q_n takes as input a string $x \in \Sigma^n$, a $2^{p(n)}$ -qubit proof $|\psi\rangle$ in register \mathcal{Y} , a $q(n)$ -bit ancilla register \mathcal{X} initialized to $|0\rangle^{\otimes q(n)}$, and is of form (see Figure 2)

$$Q_n = \prod_{i=m}^1 ((V_i)_{\mathcal{X}} \cdot \text{SWAP}_{\mathcal{X}_1, \mathcal{Y}_i}) \cdot (V_0)_{\mathcal{X}}. \quad (9)$$

Then,

- (Completeness) If $x \in A_{\text{yes}}$: $\exists |\psi\rangle \in \mathcal{Y} : \langle 0^q | \langle \psi | (Q_n^\dagger |1\rangle \langle 1|_{\mathcal{X}_1} Q_n) |0^q\rangle | \psi \rangle \geq \alpha(n)$.
- (Soundness) If $x \in A_{\text{no}}$: $\forall |\psi\rangle \in \mathcal{Y} : \langle 0^q | \langle \psi | (Q_n^\dagger |1\rangle \langle 1|_{\mathcal{X}_1} Q_n) |0^q\rangle | \psi \rangle \leq \beta(n)$.

As in Definition 2.2, we do not allow Q_n to alter the contents of its input register (to avoid using said register as additional ancilla space).

We first show that SQMASPACE can be amplified to a constant promise gap¹³.

Theorem 3.2. SQMASPACE(p, q, r) \subseteq SQMASPACE($p', q', 1$), where $q'(n) = q(n) + O(r(n))$ and $p'(n) = O(p(n) + r(n) + \log q(n))$.

Proof. Let Q_n be an SQMASPACE(p, q, r) verifier with m gates, which is given some input x . Our goal is to amplify its completeness/soundness via repetition. The main challenge is that there are not enough ancillas for the usual deferred measurement approach as we require 2^q repetitions with $O(q)$ space. Our solution is to use the proof itself as ancillas for each repetition and count the number of accepting computations.

We construct a verifier V with the following registers: Counter C_{acc} to count accepting computations, C_{init} to count correct initializations of ancillas, and \mathcal{X} to simulate Q_n . Formally, V is given as (explanation in words below):

$$V := \prod_{i=R}^1 \left[C-U_{\mathcal{X}_1=1, C_{\text{acc}}}^+ \cdot \prod_{j=m}^1 \left(V_i S_{\mathcal{X}_1, \mathcal{Y}_{i(q+m)+q+j}} \right) \cdot V_0 \cdot C-U_{\mathcal{X}=0^q, C_{\text{init}}}^+ \cdot \prod_{j=q}^1 \left(S_{\mathcal{X}_1, \mathcal{X}_j} S_{\mathcal{X}_1, \mathcal{Y}_{i(q+m)+j}} \right) \right] \quad (11)$$

Here, $C-U^+$ indicates a controlled increment operation (e.g. $C-U_{\mathcal{X}_1=1, C_{\text{acc}}}^+$ increments the counter in C_{acc} if $\mathcal{X}_1 = 1$) and S a SWAP gate. The indices are upside down to indicate that the leftmost term has the largest index. In words, V operates in R rounds as follows:

1. (Refresh ancilla qubits) Swap q streamed proof bits into \mathcal{X} .
2. (Condition on ancilla being properly initialized) Increment C_{init} conditioned on $\mathcal{X} = 0^q$, i.e. apply unitary

$$|0^q\rangle \langle 0^q|_{\mathcal{X}} \otimes U_{C_{\text{init}}}^+ + (I - |0^q\rangle \langle 0^q|)_{\mathcal{X}} \otimes I_{C_{\text{init}}}. \quad (12)$$

3. (Run the actual verification) Simulate Q_n .
4. Increment C_{acc} conditioned on $\mathcal{X}_1 = 1$ (output qubit of Q_n).

¹³In fact, the completeness/soundness thresholds can be made to be exponentially close to 1/0 in the length of the proof.

Letting $\delta := (\alpha - \beta)/2$ and $t := (\alpha - \delta)R$, define the accepting projector as

$$\Pi_{\text{acc}} := \sum_{i \geq t} |i\rangle\langle i|_{C_{\text{acc}}} \otimes |R\rangle\langle R|_{C_{\text{init}}}. \quad (13)$$

Note that $\lceil \log_2(R) \rceil$ bits are required for each counter. Hence, V acts on $q' = q + 2\lceil \log_2(R) \rceil$ ancillas.

Completeness: In the YES-case, we assume the honest prover sends $|\phi\rangle = (|0^q\rangle|\psi\rangle)^{\otimes R}$, where $|\psi\rangle$ is a proof accepted by Q_n with probability $\geq \alpha$. We can view C_{acc} as the sum of independent random variables corresponding to the outcomes of each round. Then by Hoeffding's inequality,

$$\Pr[-C_{\text{acc}} + \alpha R \geq \delta R] \leq e^{-2\delta^2 R} \leq 1/3 \quad (14)$$

for $R \geq \ln(1/3)/2\delta^2$ with $\log(R) = O(r)$.

Soundness: In the NO-case, we first argue that we can assume the proof's ancilla bits are initialized to 0. We split the proof register \mathcal{Y} into A for the ancillas, B for the actual proof, and write $|\phi\rangle = \sum_{z \in \{0,1\}^{qR}} a_z |z\rangle_A |\phi_z\rangle_B$. Then $\Pi_{\text{acc}} V |\phi\rangle = a_{0^{qR}} |0^{qR}\rangle_A |\phi_{0^{qR}}\rangle_B$ as only $A = |0^{qR}\rangle$ causes $C_{\text{init}} = r$. Hence, we can assume $A = |0^{qR}\rangle$ and get the POVM

$$P_{\text{acc}} = \langle 0^{q'+qR} |_{A, C_{\text{init}}, C_{\text{acc}}} (V^\dagger \Pi_{\text{acc}} V) |0^{q'+qR}\rangle_{A, C_{\text{init}}, C_{\text{acc}}} \quad (15)$$

$$= \sum_{z \in \{0,1\}^R, |z| \geq t} \bigotimes_{i=1}^R P_{z_i}, \quad (16)$$

where $|z|$ denotes the Hamming weight and P_1, P_0 the accepting/rejecting POVM of Q_n . Since P_0 and $P_1 = I - P_0$ commute, P_{acc} has an eigenbasis $\{|\phi_{i_1}, \dots, \phi_{i_R}\rangle\}_{i_1, \dots, i_R}$, where $\{|\phi_i\rangle\}_i$ is an eigenbasis of P_0 (and P_1). Therefore, P_{acc} has an eigenvector with maximum eigenvalue of the form $|\psi_1\rangle \otimes \dots \otimes |\psi_R\rangle$, where Q_n accepts each $|\psi_i\rangle$ with probability $\leq \beta$ by assumption. Since we have projected A onto all-zeroes, the optimal proof is of form $\bigotimes_{i=1}^R |0^q\rangle |\psi_i\rangle$, and we can apply Hoeffding's inequality again:

$$\Pr[C_{\text{acc}} - \beta R \geq \delta R] \leq e^{-2\delta^2 R} \leq 1/3. \quad (17)$$

□

Corollary 3.3. $\text{SQMASPACE}(\text{poly}, \text{poly}, 1) = \text{QMA}_{\text{EXP}}$.

Proof. Containment is trivial. The other direction follows by amplifying the verification circuit V of a 1D-TIH instance (1D translationally invariant Hamiltonian, Definition 2.9), which is complete for QMA_{EXP} with a promise gap of $1/\exp$ [GI13]. We use the standard verifier from Kitaev's "quantum Cook-Levin theorem" [KSV02] that picks a random index i , and then measures using a (potentially rescaled) 2-local constraint H as POVM (recall in TIH all terms on the chain are identical). V then has a promise gap of $1/\exp(n)$, where n is the input size. Note it is straightforward to implement V as an $\text{SQMASPACE}(\text{poly}, \text{poly}, \text{poly})$ circuit because we only need to measure one Hamiltonian term (selected at random). Hence, the required qubits can be swapped into the ancilla space when streamed and measured at the end of the computation. Theorem 3.2 completes the proof. □

Corollary 3.4. $\text{SQMASPACE}(\log, \log, 1) = \text{QMA}$.

Proof. The proof is analogous to Corollary 3.3, but instead using QMA-completeness of the (non-translationally invariant) local Hamiltonian problem [KSV02]. \square

4 Universal Quantum Path Following Lemma

In this section, we give a general construction for simulating *any* Lipschitz continuous path f on the unit hypersphere via a sequence of 2-local gates. We begin with definitions.

Definition 4.1 (Paths and Lipschitz continuity). For any $d \geq 2$, consider unit hypersphere $S^{d-1} := \{|\psi\rangle \in \mathbb{C}^d \mid \|\psi\|_2 = 1\}$. A *path* is any function $f : [0, 1] \rightarrow S^{d-1}$. We say f is K -Lipschitz continuous if for all $a, b \in [0, 1]$, $\|f(a) - f(b)\|_2 \leq K|a - b|$.

We measure the distance between two paths by the metric $d(f, g) := \max_{t \in [0, 1]} \|f(t) - g(t)\|_2$ for $\|\cdot\|_2$ the Euclidean norm.

The main result of this section is the following.

Lemma 1.7 (Universal quantum path following lemma). *Set $d := 2^n$, and let $f : [0, 1] \rightarrow S^{d-1}$ be a K -Lipschitz continuous path. For every $\varepsilon > 0$, there exists a sequence of $M \in O((\frac{n^2 d^2}{\varepsilon})^{2n})$ 2-local unitaries $U = U_M \cdots U_1$ which “ ε -approximately simulates” the path f as follows. Define $|\psi_t\rangle = U_t \cdots U_1 |\psi_0\rangle$ for $t \in \{0, \dots, M\}$ and $|\psi_0\rangle := f(0)$. Then, for all t ,*

$$\| |\psi_t\rangle - f(t/M) \|_2 \leq \varepsilon. \quad (2)$$

In words, any Lipschitz continuous path f on the unit hypersphere can be approximately simulated to any desired precision ε by applying a sequence of M 2-local unitaries (see Figure 1 for an illustration). The main idea behind the proof is to first discretize f sufficiently finely, and then to locally simulate f between each consecutive pair of discrete points via a sequence of “small rotations”. Here, by “small rotations”, we mean unitaries close to identity, which can also be written as $U = e^{iH}$ with small $\|H\|_\infty$. We can write $H = \sum_j \alpha_j H_j$ in Pauli basis (i.e. each H_j is a tensor product of I, X, Y, Z) with small α_j . Applying a result due to Suzuki (Lemma 4.9), we have $e^{iH} \approx \prod_j e^{i\alpha_j H_j}$. Next, a construction of Clinton, Bausch, and Cubitt [CBC21] is used to decompose the $e^{i\alpha_j H_j}$ into 2-local unitaries U_1, \dots, U_m (m independent of α_j) such that $U_k \rightarrow I$ as $\alpha_j \rightarrow 0$ (Section 4.2).¹⁴ Section 4.3 combines the Suzuki and CBC decompositions and Section 4.4 applies that result to complete the proof of Lemma 1.7.

4.1 Technical Lemmas

We state a collection of technical results used in the proof of Lemma 1.7.

4.1.1 Norms

Lemma 4.2 ([Gol96, Equation 2.2.5]). *For all $v \in \mathbb{C}^d$, $\|v\|_2 \leq \|v\|_1 \leq \sqrt{d}\|v\|_2$.*

Lemma 4.3. *For all $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$, $\| |\psi\rangle - |\phi\rangle \|_2 = \sqrt{2 - 2\operatorname{Re}(\langle \phi | \psi \rangle)}$.*

For operators $M \in \mathcal{L}(\mathbb{C}^d)$, the corresponding *operator norm*, usually called the *spectral norm*, is defined as

$$\|M\|_\infty := \max_{v \in \mathbb{C}^d} \frac{\|Mv\|_2}{\|v\|_2}. \quad (18)$$

¹⁴Decompositions of arbitrary unitaries into 2-local gates are well known (e.g., [NC11]), but to the best of our knowledge, they do not provide bounds on the distance from I .

It holds that $\|M\|_\infty = \sqrt{\lambda_{\max}(M^\dagger M)}$. For $M \succcurlyeq 0$, we have $\|M\|_\infty = \lambda_{\max}(M)$, where $\lambda_{\max}(M)$ denotes the largest eigenvalue of M . We write $M = O(f(d))$ if $\|M\|_\infty = O(f(d))$ for some function f .

The *Frobenius norm* is defined as

$$\|M\|_F := \sqrt{\text{Tr}(M^\dagger M)} = \sqrt{\sum_{i=1}^d \sum_{j=1}^d |m_{ij}|^2}, \quad (19)$$

where m_{ij} denote the entries of M . Note that the Frobenius norm is the same as the Euclidean norm of M viewed as a d^2 -dimensional vector.

Lemma 4.4 ([Gol96, Equation 2.3.7]). $\|M\|_\infty \leq \|M\|_F \leq \sqrt{d}\|M\|_\infty$

We define the *trace norm* for operators $M \in \mathcal{L}(\mathbb{C}^d)$ as $\|M\|_{\text{tr}} := \text{Tr}(|M|) = \text{Tr}\sqrt{M^\dagger M}$, where $|\cdot|$ and $\sqrt{\cdot}$ are applied as operator functions.

Lemma 4.5. Let $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$. Then, $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\text{tr}} = 2\sqrt{1 - |\langle\psi|\phi\rangle|^2}$.

4.1.2 Unitaries and Hamiltonians

Lemma 4.6. For all $x \in \mathbb{R}$, it holds that $|1 - e^{ix}| \leq |x|$.

Lemma 4.7. Let $H \in \text{Herm}(\mathbb{C}^d)$. For $U = e^{iH}$, $\|U - I\|_\infty \leq \|H\|_\infty$.

Proof. Let $H = \sum_{j=1}^d \lambda_j |\psi_j\rangle\langle\psi_j|$ be the spectral decomposition of H . Then, the spectral decomposition of $U - I$ is given by

$$I - e^{iH} = \sum_{j=1}^d |\psi_j\rangle\langle\psi_j| - \sum_{j=1}^d e^{i\lambda_j} |\psi_j\rangle\langle\psi_j| = \sum_{j=1}^d (1 - e^{i\lambda_j}) |\psi_j\rangle\langle\psi_j|. \quad (20)$$

Therefore,

$$\|I - U\|_\infty = \max_j |1 - e^{i\lambda_j}| \leq \max_j |\lambda_j| = \|H\|_\infty, \quad (21)$$

where the inequality follows from Lemma 4.6. \square

Lemma 4.8. Let $U = U_m \cdots U_1$ and $V = V_m \cdots V_1$ be unitary matrices. For a submultiplicative norm $\|\cdot\|$, it holds that $\|U - V\| \leq \sum_{i=1}^m \|U_i - V_i\|$.

Lemma 4.9 (Suzuki). Let $H = \sum_{j=1}^m H_j$ be a sum of Hermitian operators such that $\sum_{j=1}^m \|H_j\|_\infty \leq t \leq 1$ and $s \in \mathbb{N}$. Then

$$e^{iH} = \left(\prod_{j=1}^m e^{iH_j/s} \right)^s + O\left(\frac{t^2}{s}\right). \quad (22)$$

Proof. Follows directly from [Suz76, Theorem 3]. \square

This lemma can also be used for Hamiltonian simulation, for if H is a k -local Hamiltonian, then the $e^{iH_j/n}$ terms are k -local gates. Therefore, we can simulate the evolution e^{iH} with only local gates. The well-known Lie-Trotter product formula follows directly from Equation (22)

$$e^{iH_1+iH_2} = \lim_{n \rightarrow \infty} \left(e^{iH_1/n} e^{iH_2/n} \right)^n. \quad (23)$$

4.2 Decomposition of Pauli Interactions

Next, we show how to decompose operators e^{itH} for $H \in \{I, X, Y, Z\}^{\otimes n}$ into 2-local gates of the form $e^{it_j H_j}$, such that the total evolution time $\sum_j |t_j|$ is bounded by $O(t^{1/n})$. This result is originally due to Clinton, Bausch, and Cubitt [CBC21]. For clarity, here we give an alternative construction of their decomposition (still using Lemmas from [CBC21]), with a simpler analysis of pulse time bounds, and with an exponential improvement in the number of gates required — see Remark 4.12 below for details. The main insight we use in the decomposition is as follows.

Lemma 4.10 ([CBC21, Lemmas 7 and 9]). *Let $U = e^{itH}$ for a Hamiltonian $H = \frac{1}{2i}[h_1, h_2]$, where h_1 and h_2 anti-commute and square to identity. For $0 \leq t \leq \pi/2$, there exist $t_1, t_2 \in \mathbb{R}$ with*

$$|t_1| + |t_2| \leq \sqrt{2t}, \quad (24)$$

and

$$U = e^{it_1 h_1} e^{it_2 h_2} e^{it_2 h_1} e^{it_1 h_2}. \quad (25)$$

We can also use Lemma 4.10 with negative $t \geq -\pi/2$ by applying the lemma to $-t$ and then using the inverse of the resulting decomposition ($(e^{itH})^\dagger = e^{-itH}$).

To apply this to Pauli interactions, we observe that X, Y, Z pairwise anti-commute, square to identity, and

$$[X, Y] = 2iZ, \quad [X, Z] = 2iY, \quad [Y, Z] = 2iX. \quad (26)$$

Hence, we can apply Lemma 4.10 to decompose e^{itH} for $n = 2^k + 1$ and

$$H = P_1 \otimes \cdots \otimes P_n \in \{I, X, Y, Z\}^{\otimes n} \quad (27)$$

into two $2^{k-1} + 1$ local evolutions as follows. Let $j = 2^{n-1} + 1$, assume $P_j = Z$, and set

$$h_1 = P_1 \otimes \cdots \otimes P_{j-1} \otimes X_j \otimes I_{j+1, \dots, n}, \quad (28)$$

$$h_2 = I_{1, \dots, j-1} \otimes Y_j \otimes P_{j+1} \otimes \cdots \otimes P_n. \quad (29)$$

Then,

$$[h_1, h_2] = P_1 \otimes \cdots \otimes P_{j-1} \otimes XY_j \otimes P_{j+1} \otimes \cdots \otimes P_n \quad (30)$$

$$- P_1 \otimes \cdots \otimes P_{j-1} \otimes YX_j \otimes P_{j+1} \otimes \cdots \otimes P_n \quad (31)$$

$$= P_1 \otimes \cdots \otimes P_{j-1} \otimes [X, Y]_j \otimes P_{j+1} \otimes \cdots \otimes P_n, \quad (32)$$

$$= 2iH. \quad (33)$$

The cases $P_j = X$ or $P_j = Y$ are analogous due to Equation (26). The decomposition is depicted in Figure 3 (tensor products between the Pauli operators are omitted for conciseness).

For $n \neq 2^k + 1$, we cannot always choose the split j to be exactly in the middle, but the resulting interactions will still be at most $(2^{k-1} + 1)$ -local, provided $n \leq 2^k + 1$. Applying this decomposition recursively we show:

Lemma 4.11. *Let $H \in \{I, X, Y, Z\}^{\otimes n}$ with $n \in (2^{k-1} + 1, 2^k + 1]$, and $t \in \mathbb{R}$ with $8|t|^{1/2^k} \leq \pi/2$. There exists a decomposition of $e^{itH} = \prod_{j=1}^m e^{it_j H_j}$, where the H_j are 2-local Pauli matrices, $m \leq 4^k = O(n^2)$, and $\sum_{i=1}^m |t_i| = O(n^2 |t|^{1/2^k}) = O(n^2 |t|^{1/2n})$.*

For clarity, the last equality of the claim, $O(n^2 |t|^{1/2^k}) = O(n^2 |t|^{1/2n})$, holds since $8|t|^{2^{-k}} \leq \pi/2$ implies $|t| < 1$.

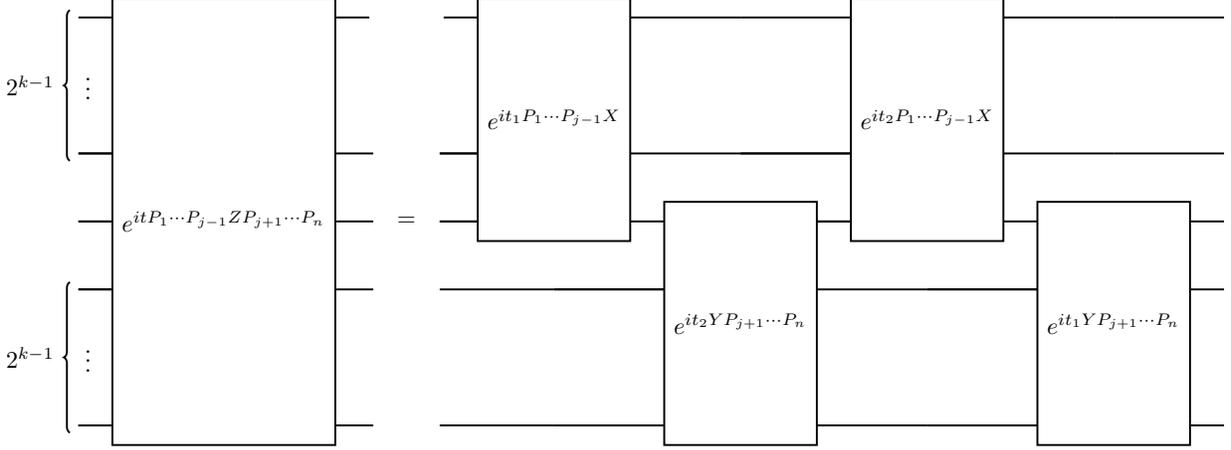


Figure 3: Decomposition of Pauli interactions.

Proof of Lemma 4.11. We construct the 2-local H_1, \dots, H_m by applying Lemma 4.10 recursively as outlined above. $e^{itH} = \prod_{j=1}^m e^{it_j H_j}$ follows from the correctness of Lemma 4.10. After each recursion, we have interactions that are at most $(2^{k-1} + 1)$ -local. Hence, after k recursions, only 2-local interactions remain and we are done. Since each recursion increases the number of interactions by a factor of 4, we have $m \leq 4^k = O(n^2)$.

By Lemma 4.10, a recursion constructs new interactions, each with a pulse time of at most

$$|t_1| + |t_2| \leq \sqrt{2t}. \quad (34)$$

This gives us a recurrence for an upper bound on the individual pulse times after r recursions:

$$T(r) = \begin{cases} |t|, & \text{if } r = 0 \\ \sqrt{2T(r-1)}, & \text{if } r > 0 \end{cases}. \quad (35)$$

Using induction on r , we show that

$$T(r) = 2^{1-2^{-r}} |t|^{2^{-r}}. \quad (36)$$

For $r = 0$, we have $T(0) = |t|$. For $r > 0$, we have

$$T(r) = \sqrt{2T(r-1)} = \left(2 \cdot 2^{1-2^{-r+1}} |t|^{2^{-r+1}}\right)^{1/2} = 2^{1/2} \cdot 2^{1/2-2^{-r}} |t|^{2^{-r}} = 2^{1-2^{-r}} |t|^{2^{-r}}. \quad (37)$$

Hence, the individual pulse times after k recursions are bounded by $T(k) \leq 8|t|^{2^{-k}}$. The total pulse time is then bounded by $mT(k) = O(n^2 |t|^{1/2n})$. \square

Remark 4.12. Our decomposition only uses polynomially many gates, whereas it appears to us that the construction given in [CBC21] uses exponentially many. This might be of interest for physical applications. We also only require their Lemmas 7 and 9, without having to use the more complex Lemmas 8 and 10. Their decomposition has a total pulse time of $O(|t|^{1/n})$.

4.3 General Decomposition

Next, we show how to use Lemma 4.11 to decompose general unitaries of small norm.

Lemma 4.13. *Let $U = e^{iH}$ for Hermitian $H \in \text{Herm}(\mathbb{C}^d)$, $d = 2^n$ with $\|H\|_\infty =: \varepsilon < (\pi/16)^{2n}$. There exists an approximate decomposition $U = U_m \cdots U_1 + O(d^2\varepsilon^2)$ into $m \leq 2^{O(n)}$ 2-local unitaries, such that*

$$\sum_{j=1}^m \|I - U_j\|_\infty = O\left(n^2 d^2 \varepsilon^{1/2n}\right) \quad (38)$$

Proof. We write H in the Pauli basis:

$$H = \sum_{j=1}^{d^2} \alpha_j P_j, \quad (39)$$

where $\alpha_j \in \mathbb{R}$ and $P_j \in \{I, X, Y, Z\}^{\otimes n}$ for all $j \in [d^2]$. Then, by Lemma 4.4

$$\sqrt{d}\|H\|_\infty \geq \|H\|_F = \sqrt{\text{Tr}(H^\dagger H)} = \sqrt{d \sum_{j=1}^{d^2} \alpha_j^2}. \quad (40)$$

Therefore, $|\alpha_j| \leq \varepsilon$ for all $j \in [d^2]$. It holds that

$$\sum_{j=1}^{d^2} \|\alpha_j P_j\|_\infty = \sum_{j=1}^{d^2} |\alpha_j| \leq d \sqrt{\sum_{j=1}^{d^2} \alpha_j^2} \leq d\|H\|_\infty \leq 1, \quad (41)$$

where the first inequality follows by Lemma 4.4 (note $\sum_{j=1}^{d^2} |\alpha_j|$ is a sum of d^2 terms), and the second by Equation (40). Applying Lemma 4.9 with $s = 1$, we have

$$U = e^{iH} = \prod_{j=1}^{d^2} e^{i\alpha_j P_j} + O(d^2\varepsilon^2). \quad (42)$$

Lemma 4.11 allows us to decompose each term $e^{i\alpha_j P_j}$ into $m_j = O(n^2)$ 2-local unitaries

$$e^{i\alpha_j P_j} = \prod_{k=1}^{m_j} e^{it_{j,k} H_{j,k}} \quad (43)$$

with an evolution time of $\sum_{k=1}^{m_j} |t_{j,k}| = O(n^2 |\alpha_j|^{1/2n})$. We get the complete decomposition

$$U = e^{iH} = \prod_{j=1}^{d^2} \prod_{k=1}^{m_j} e^{it_{j,k} H_{j,k}} + O(d^2\varepsilon^2), \quad (44)$$

with a total evolution time of

$$O\left(n^2 \sum_{j=1}^{d^2} |\alpha_j|^{1/2n}\right) = O\left(n^2 d^2 \varepsilon^{1/2n}\right). \quad (45)$$

Equation (38) follows from Lemma 4.7 as

$$\|I - e^{it_{j,k}H_{j,k}}\|_\infty \leq \|t_{j,k}H_{j,k}\|_\infty = t_{j,k}. \quad (46)$$

□

We remark, that we usually choose $\varepsilon \ll (\pi/16)^{2n}$ in order to make Equation (38) small. Furthermore, the above decomposition is approximate. It appears to be an open question whether a similar result is achievable with an exact decomposition.

4.4 Approximating paths via local unitaries

We are almost ready to prove 1.7; the last ingredient we require is the following lemma. It uses the above decomposition to (approximately) map between two close vectors $|\psi\rangle$ and $|\phi\rangle$ while bounding the distance of intermediate states from $|\psi\rangle$.

Lemma 4.14. *Let $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$ be unit vectors with $d = 2^n$. Let $\| |\psi\rangle - |\phi\rangle \|_2 \leq \varepsilon < (\pi/16)^{2n}$. There exists a sequence of 2-local unitaries $U = U_m \cdots U_1$ with $m \leq 2^{O(n)}$, such that*

$$(1) \quad \| |\phi\rangle - U|\psi\rangle \|_2 = O(d^2\varepsilon^2), \text{ and}$$

$$(2) \quad \text{for all } i \in [m], \quad \| |\psi\rangle - U_i \cdots U_1 |\psi\rangle \|_2 = O(n^2 d^2 \varepsilon^{1/2n}).$$

Proof. Let $\theta = \cos^{-1}(\text{Re}(\langle \psi | \phi \rangle))$ be the angle between $|\psi\rangle$ and $|\phi\rangle$. After a suitable change of basis W , we have

$$W|\psi\rangle = |0\rangle, \quad (47)$$

$$W|\phi\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle. \quad (48)$$

Hence, we only need to apply the rotation matrix (extended to d dimensions)

$$R(\theta) = \left(\begin{array}{cc|c} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ \hline 0 & 0 & I \end{array} \right) \quad (49)$$

to map from $W|\psi\rangle$ to $W|\phi\rangle$. Let $V = W^\dagger R(\theta)W$. V has the same eigenvalues as $R(\theta)$, namely $e^{i\theta}, e^{-i\theta}, 1$. Hence, $V = e^{iH}$ for $\|H\|_\infty = |\theta|$.

To apply Lemma 4.13, we need to bound θ . We have for $|\theta| < 1$,

$$\| |\psi\rangle - |\phi\rangle \|_2 = \sqrt{2 - 2\text{Re}(\langle \phi | \psi \rangle)} = \sqrt{2 - 2\cos \theta} \geq \sqrt{\theta^2 - \theta^4/12} \geq |\theta|/2, \quad (50)$$

where the first equality follows from Lemma 4.3, and the first inequality via Taylor expansion. Hence, $|\theta| \leq 2\varepsilon$. Let $U = U_m \cdots U_1$ from Lemma 4.13. Properties (1) and (2) of the claim follow from Lemma 4.8. □

We now restate and prove 1.7.

Lemma 1.7 (Universal quantum path following lemma). *Set $d := 2^n$, and let $f : [0, 1] \rightarrow S^{d-1}$ be a K -Lipschitz continuous path. For every $\varepsilon > 0$, there exists a sequence of $M \in O((\frac{n^2 d^2}{\varepsilon})^{2n})$*

2-local unitaries $U = U_M \cdots U_1$ which “ ε -approximately simulates” the path f as follows. Define $|\psi_t\rangle = U_t \cdots U_1 |\psi_0\rangle$ for $t \in \{0, \dots, M\}$ and $|\psi_0\rangle := f(0)$. Then, for all t ,

$$\| |\psi_t\rangle - f(t/M) \|_2 \leq \varepsilon. \quad (2)$$

Proof. The idea is to first discretize f into a sufficiently large number $N' + 1$ of points, and subsequently apply Lemma 4.14 to simulate f along each interval $[i/N', (i+1)/N']$. To begin, Definition 4.1 says that for any $i \in \{0, \dots, N' - 1\}$,

$$\| f(i/N') - f((i+1)/N') \|_2 \leq K/N' =: \delta. \quad (51)$$

We will shortly set N' as needed, but it will be sufficiently large so that $\delta < (\pi/16)^{2n}$. Thus, to the i th interval $[i/N', (i+1)/N']$ we can apply Lemma 4.14 to obtain a sequence of 2-local unitaries $U_i = U_{i,m_i} \cdots U_{i,1}$ with $m_i \leq 2^{O(n)}$ such that for all i ,

$$\| f((i+1)/N') - U_i \cdot f(i/N') \|_2 = O(d^2 \delta^2), \text{ and} \quad (52)$$

$$\forall j \in \{1, \dots, m_i\}, \| f(i/N') - U_{i,j} \cdots U_{i,1} f(i/N') \|_2 = O(n^2 d^2 \delta^{1/2n}). \quad (53)$$

Letting $U = U_{N'} \cdots U_1$, we have $M = \sum_{i=0}^{N'-1} m_i \leq N' 2^{O(n)}$. It remains to choose N' so as to bound the point-wise error to ε as in Equation (2).

The analysis proceeds in two stages. First, Equation (52) and Lemma 4.8 imply that for any $t \in \{0, \dots, M\}$ such that $t/M = i/N'$ for some $i \in \{0, \dots, N' - 1\}$ (these are the $N' + 1$ points obtained after our first round of discretizing f),

$$\| |\psi_t\rangle - f(t/M) \|_2 \in O(N' d^2 \delta^2). \quad (54)$$

Second, we “subdivided” each interval $[i/N', (i+1)/N']$ into intermediate points $U_{i,j} \cdots U_{i,1} f(i/N')$ via Lemma 4.14. Equation (53) says each of these intermediate points is at most $O(n^2 d^2 \delta^{1/2n})$ -far from the start point of that interval, $f(i/N')$. Combining the two errors, we have that for any $t \in \{0, \dots, M\}$,

$$\| |\psi_t\rangle - f(t/M) \|_2 \in O(N' d^2 \delta^2 + n^2 d^2 \delta^{1/2n}). \quad (55)$$

(Note N' does not appear on the $n^2 d^2 \delta^{1/2n}$ term, as this error does not accumulate from one interval $[i/N', (i+1)/N']$ to the next.) To bound this by $\varepsilon > 0$, it suffices to set

$$N' \in \begin{cases} \Omega \left(K \left(\frac{n^2 d^2}{\varepsilon} \right)^{2n} \right) & \text{if } 0 < K \leq 1 \\ \Omega \left(\left(\frac{K^2 n^2 d^2}{\varepsilon} \right)^{2n} \right) & \text{if } K > 1, \end{cases} \quad (56)$$

and thus $M \leq N' 2^{O(n)} \in O(K (\frac{n^2 d^2}{\varepsilon})^{2n})$ if $0 < K \leq 1$ and $M \in O(2^{O(n)} (\frac{K^2 n^2 d^2}{\varepsilon})^{2n})$ if $K > 1$. \square

5 Applying Quantum Path Following to $\text{GSCON}_{\text{exp}}$

In the previous section, we show how to implement general paths with 2-local unitaries. We apply this approach to construct unitary sequences for GSCON instances. Note, however, that these sequences have exponential length. Suppose we are given a GSCON instance, where $l = 2$, and for simplicity the starting state $|\psi\rangle$ and the target state $|\phi\rangle$ are orthogonal ground states of H (as opposed to just low energy states). To determine whether we have a YES-instance, we need

to check whether there exists a sequence of 2-local unitaries that maps $|\psi\rangle$ to $|\phi\rangle$ but keeps the energy of intermediate states low. Certainly, states in the span of $|\psi\rangle$ and $|\phi\rangle$ are also ground states. Hence, we can apply Lemma 1.7 to the path $f(t) := \cos(t\pi/2)|\psi\rangle + \sin(t\pi/2)|\phi\rangle$ to obtain a suitable unitary sequence. This approach also works if $|\psi\rangle$ and $|\phi\rangle$ are *not* orthonormal ground states, which is proven in the theorem below.

Theorem 1.8. *Let $H \in \text{Herm}(\mathbb{C}^d)$, $d = 2^n$ with $0 \preceq H \preceq I$, $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$ with $\langle \psi | H | \psi \rangle \leq \eta$ and $\langle \phi | H | \phi \rangle \leq \eta$. For any $\Delta \geq 2^{-\text{poly}(n)}$, there exists a sequence of 2-local unitary gates $U = U_m \cdots U_1$ with $m \leq 2^{\text{poly}(n)}$ such that*

$$(1) \|U|\psi\rangle - |\phi\rangle\|_2 \leq \Delta, \text{ and}$$

$$(2) \text{ for all } i \in [m], \langle \psi_i | H | \psi_i \rangle \leq \eta + \Delta, \text{ where } |\psi_i\rangle := U_i \cdots U_1 |\psi\rangle.$$

Proof. The idea is to first rotate $|\psi\rangle$ onto a ground state $|\mu\rangle$ and then use the same method to rotate $|\mu\rangle$ to $|\phi\rangle$. This will leave all intermediate states at an energy of at most η (in practice it might exceed η since the construction is approximate). Let $|\lambda_1\rangle, \dots, |\lambda_d\rangle$ be an orthonormal eigenbasis of H . We assume $|\mu\rangle = |\lambda_1\rangle$ is a ground state. We can write

$$|\psi\rangle = \cos(\theta)|\mu\rangle + \sin(\theta)|\nu\rangle,$$

where $|\nu\rangle \in \text{Span}\{|\lambda_2\rangle, \dots, |\lambda_d\rangle\}$.

We argue that increasing the amplitude on $|\mu\rangle$ and decreasing the amplitude on $|\nu\rangle$ cannot increase the energy. Let $|\psi'\rangle = \cos(\theta')|\mu\rangle + \sin(\theta')|\nu\rangle$ with $|\cos \theta'| > |\cos \theta|$. Then, for $\lambda = \lambda_{\min}(H)$,

$$\langle \psi | H | \psi \rangle = \cos^2(\theta)\lambda + \sin^2(\theta)\langle \nu | H | \nu \rangle \tag{57}$$

$$= \cos^2(\theta)\lambda + (1 - \cos^2(\theta))\langle \nu | H | \nu \rangle \tag{58}$$

$$< \cos^2(\theta')\lambda + (1 - \cos^2(\theta'))\langle \nu | H | \nu \rangle \tag{59}$$

$$= \langle \psi' | H | \psi' \rangle, \tag{60}$$

where the inequality follows since $\lambda \leq \langle \nu | H | \nu \rangle$. Define path $f(t) := \cos((1-t)\theta)|\mu\rangle + \sin((1-t)\theta)|\nu\rangle$. The path f is Lipschitz continuous for $K = \pi/2$, and by the above argument, $f(t)^\dagger H f(t) \leq \eta$ (assuming $\theta \in [0, \pi/2]$ without loss of generality). Define a path g from $|\mu\rangle$ to $|\phi\rangle$ in the same way and concatenate both to path h . The proof is complete by applying Lemma 1.7 to h with $\varepsilon = \Delta$. \square

Note that the theorem easily generalizes to any $0 \preceq H \preceq 2^{\text{poly}(n)}I$.

We can now partially answer an open question of Gharibian and Sikora [GS18], which asked about the complexity of GSCON with exponential m and $l = 2$. Specifically, Reference [GS18] showed that for exponential m , $l = 1$, and inverse exponential gap Δ , GSCON is PSPACE-complete, and conjectured that the analogous $l = 2$ case is NEXP-complete. Here, on the one hand, we will later show (Theorem 5.2 below) that the $l = 1$ PSPACE-completeness result holds even for *constant* gap Δ . However, in strong contrast, in the $l = 2$ case Theorem 1.8 says that for any *subexponential* Δ , GSCON is poly-time decidable:

Corollary 5.1. *GSCON with $m = 2^{\text{poly}(n)}$, $l = 2$, and subexponential Δ does not have NO-instances for sufficiently large n .*

In the following, we investigate how the above result relates to the 1-local case (Section 5.1), the classical case (Section 5.2), and the Traversal Lemma from [GS18] (Section 5.3).

5.1 Relation to the 1-Local Case

We have seen that any $\text{GSCON}_{\text{exp}}$ instance with 2-local gates ($l = 2$) becomes either a YES-instance or an invalid instance if we make m sufficiently large. This raises the question of what happens in the 1-local case (i.e. $l = 1$). We show that GSCON with $l = 1$ and $m = \infty$ (i.e. gate sequences may be arbitrarily long) is still PSPACE-complete. Therefore, arbitrarily long sequences does not change the hardness of 1-local GSCON , which is also PSPACE-complete for bounded m .

Theorem 5.2. *GSCON is PSPACE-complete for $l = 1, k = 3, \eta_1 = \eta_3 = 0, \eta_2 = 1/8, \eta_4 = 1/2, \Delta = 1/8$ and unbounded m .*

Proof. Lemmas 5.4 and 5.6 below show hardness and containment. \square

5.1.1 Hardness

Lemma 5.3 ([GS18, Lemma 6.2]). *GSCON is PSPACE-hard for $k = 3, \eta_1 = \eta_3 = 0, \eta_2 = 2^{-(2n+4)}, \eta_4 = 1/4, \Delta = 2^{-(2n+4)}, l = 1$, and $m = 2^n$, where n denotes the number of qubits H acts on.*

We can strengthen this statement as follows.

Lemma 5.4. *GSCON is PSPACE-hard for $l = 1, k = 3, \eta_1 = \eta_3 = 0, \eta_2 = 1/8, \eta_4 = 1/2, \Delta = 1/8$ and unbounded m .*

Proof. As in [GS18], we reduce $L \in \text{PSPACE}$ to $S, T\text{-CONN}(1)$ (see Definition 5.7). Let (ϕ, x, y) be an $S, T\text{-CONN}$ instance. We set $|\psi\rangle = |x\rangle, |\phi\rangle = |y\rangle, H = \sum_i H_i$, where $H_i = |z_i\rangle\langle z_i| \otimes I$ for the unsatisfying assignment z_i of clause c_i in ϕ . Completeness follows from Lemma 5.3.

To prove soundness, let (ϕ, x, y) be a no-instance and consider a sequence of 1-local unitaries U_1, \dots, U_m . We show that $(H, |x\rangle, |y\rangle)$ is a no instance for GSCON . All intermediate states are of the form

$$|\psi_i\rangle = \bigotimes_{j=1}^n (\alpha_{i,j}^0 |0\rangle + \alpha_{i,j}^1 |1\rangle). \quad (61)$$

For $i = 1, \dots, m$ define $x_i = x_{i,1} \cdots x_{i,n}$ with

$$x_{i,j} = \begin{cases} 0 & \text{if } |\alpha_{i,j}^0|^2 \geq \frac{1}{2} \\ 1 & \text{else} \end{cases}. \quad (62)$$

Hence, $|\langle x_i | \psi_i \rangle| \geq 2^{-n/2}$. Since $|\psi_i\rangle$ and $|\psi_{i+1}\rangle$ differ in only one qubit, we have $h(x_i, x_{i+1}) \leq 1$. Assume for contradiction that $\| |\psi_m\rangle - |\phi\rangle \|_2 \leq \eta_4$. Then,

$$\| |\psi_m\rangle - |y\rangle \|_2 = \sqrt{2 - 2\text{Re}(\langle \psi_m | y \rangle)} \leq \eta_4. \quad (63)$$

Hence,

$$|\langle \psi_m | y \rangle| \geq \text{Re}(\langle \psi_m | y \rangle) \geq 1 - \frac{\eta_4^2}{2} > \sqrt{\frac{1}{2}}. \quad (64)$$

Therefore, we have $|\alpha_{m,j}^{y_j}| > \sqrt{1/2}$ for all $j \in [n]$ and thus $y = x_m$. Hence, there exists an x_i such that x_i does not satisfy a clause c_j of ϕ . It follows that $\langle x_i | H_j | x_i \rangle = 1$. W.l.o.g., H_j operates on qubits 1, 2, 3. Then, $x_{i,1} x_{i,2} x_{i,3} = z_j$ and $\langle \psi_i | H_j | \psi_i \rangle = |\alpha_{m,1}^{x_{i,1}} \alpha_{m,2}^{x_{i,2}} \alpha_{m,3}^{x_{i,3}}|^2 \geq 1/8 = \eta_2$. Thus, $(H, |\psi\rangle, |\phi\rangle)$ is a no-instance. \square

5.1.2 Containment

Lemma 5.5 ([GS18, Lemma 6.3]). *For all nonnegative constants c_1 and c_2 , GSCON with $l = 1$ is in PSPACE for $m \leq 2^{n^{c_1}}$ and $\Delta \geq 1/2^{n^{c_2}}$, where n denotes the number of qubits H acts on.*

GSCON is also contained in PSPACE for unbounded m .

Lemma 5.6. *GSCON \in PSPACE with $l = 1$ for $\Delta \geq 2^{-\text{poly}(n)}$ and unbounded m .*

Proof. We use the same NPSPACE algorithm given in [GS18, Algorithm 2] used to prove Lemma 5.5 with some sufficiently large m' , except here we must account for the fact that m can be *unbounded*, unlike in [GS18]. That m is unbounded does not affect the soundness analysis, which follows from Lemma 5.5. For completeness, however, we show that the sequence U_1, \dots, U_m can be shortened to some $U'_1, \dots, U'_{m'}$ with $m' \leq 2^{\text{poly}(n)}$. For this, note that every intermediate state is of the form

$$|\psi_i\rangle = \left(\bigotimes_{j=1}^n V_{i,j} \right) |\psi\rangle =: V_i |\psi\rangle. \quad (65)$$

For all i , there exists at most one j for which $V_{i,j} \neq V_{i+1,j}$. The NPSPACE algorithm, however, stores approximations $V'_{i,j}$ in $2^{p(n)}$ bits such that for all $j \in [n]$, $\|V'_{i,j} - V_{i,j}\|_\infty \leq \varepsilon/n$ for some $\varepsilon \geq 2^{-\text{poly}(n)}$ (i.e. the V' are taken from an ε -net; see [GS18, Lemma 3.1]). Therefore, $\|V_i - V'_i\|_\infty \leq \varepsilon$, where $V'_i = \bigotimes_{j=1}^n V'_{i,j}$. Thus, $\|\psi_i - \psi'_i\|_2 \leq \varepsilon$ and $\langle \psi'_i | H | \psi'_i \rangle \leq \eta_1 + 2\varepsilon$ if $\|H\|_\infty \leq 1$. There are $m' := 2^{n \cdot p(n)}$ possibilities for each V_i . Hence, the sequence can be shortened to $V''_1, \dots, V''_{m'}$. We can assume that V'_i and V'_{i+1} only differ on a single qubit, which allows us to construct the sequence $U'_1, \dots, U'_{m'}$ of 1-local unitaries with $V''_i = U'_i V''_{i-1}$. Choosing sufficiently small ε , the NPSPACE algorithm accepts when given the sequence $U'_1, \dots, U'_{m'}$. \square

5.2 Locality in S, T -CONN

We have shown that GSCON with $m = \infty$ becomes trivial for $l = 2$, but remains PSPACE-complete for $l = 1$. Does a similar result hold classically, i.e. for S, T -CONN? We define l -local S, T -CONN and show that a classical analogue of Theorem 1.8 does not hold.

Definition 5.7 (S, T -CONN(l)). Given a 3-CNF formula ϕ and solutions $x, y \in \{0, 1\}^n$ to ϕ , does there exist a sequence of strings x_1, \dots, x_m , such that

- $x_1 = x$, and $x_m = y$, and
- for all $i \in [m]$, the Hamming distance between x_i and x_{i+1} is at most l , and
- for all $i \in [m]$, x_i is a solution to ϕ ?

Theorem 5.8. *S, T -CONN(l) is PSPACE-complete for all $l \geq 1$.*

Proof. S, T -CONN(l) \subseteq PSPACE follows from Savitch's theorem. S, T -CONN(1) is PSPACE-complete [GKMP06]. Let $L \in$ PSPACE and reduce L to a S, T -CONN(1)-instance (ϕ, x, y) . Construct a 3-CNF formula ϕ' from ϕ by creating $l - 1$ copies of each variable with equality constraints with corresponding solutions x' and y' . We show that $(\phi, x, y) \in S, T$ -CONN(1) iff $(\phi', x', y') \in S, T$ -CONN(l).

If $(\phi, x, y) \in S, T$ -CONN(1), then there exists a sequence of solutions $x = x_1, \dots, x_m = y$ with $h(x_i, x_{i+1}) = 1$. By adding $l - 1$ copies of each variable, we get a sequence of valid solutions $x' = x'_1, \dots, x'_m = y'$ with $h(x'_i, x'_{i+1}) = l$. Hence, $(\phi', x', y') \in S, T$ -CONN(l).

If $(\phi', x', y') \in S, T\text{-CONN}(l)$, then there exists a sequence of solutions $x' = x'_1, \dots, x'_m = y'$ with $h(x'_i, x'_{i+1}) \leq l$. All l copies of each variable must be equal in each solution. Hence, between x'_i and x'_{i+1} all copies of exactly one variable are changed. Hence, we can convert x'_1, \dots, x'_m to solutions of $\phi x = x_1, \dots, x_m = y$. Thus, $(\phi, x, y) \in S, T\text{-CONN}(1)$. \square

5.3 Relation to the Traversal Lemma

Reference [GS18] uses the *Traversal Lemma* as an important tool to show that GSCON is QCMA-complete. Two states $|u\rangle, |w\rangle \in \mathcal{B}^{\otimes n}$ are said to be k -orthogonal if for all k -local unitaries U , we have $\langle w|U|v\rangle = 0$. Two subspaces $S, T \subseteq \mathcal{B}^{\otimes n}$ are called k -orthogonal if any pair of vectors $|v\rangle \in S, |w\rangle \in T$ is k -orthogonal.

Lemma 5.9 (Traversal Lemma [GS18]). *Let $S, T \subseteq \mathcal{B}^{\otimes n}$ be k -orthogonal subspaces. Let $|v\rangle \in S, |w\rangle \in T$ and consider a sequence of unitaries U_1, \dots, U_m with*

$$\| |w\rangle - U_m \cdots U_1 |v\rangle \|_2 \leq \varepsilon < 1/2.$$

Let $|v_i\rangle := U_i \cdots U_1 |v\rangle$ and $P := I - \Pi_S - \Pi_T$. Then, there exists an $i \in [m]$ such that

$$\langle v_i | P | v_i \rangle \geq \left(\frac{1 - 2\varepsilon}{2m} \right)^2.$$

Reference [GS18] provides an example for which the Traversal Lemma is tight by explicitly constructing a gate sequence to map $|000\rangle$ to $|111\rangle$ with $\langle v_i | P | v_i \rangle \leq \Delta$ for $m = O(1/\Delta^2)$. Our Theorem 1.8 constructs such a sequence in general, although it is not as tight, because m is only polynomial in Δ^{-1} and exponential in n .

6 Embedding streaming proofs into unentanglement

In this section, we state and prove the Embedding Lemma (Lemma 6.1), which shows how to embed any quantum circuit verifying a streaming proof into unentanglement (more accurately, into a Sparse Separable Hamiltonian problem (Definition 2.5)).

Lemma 6.1 (Embedding lemma). *Let $p, q, r, m, \alpha, \beta : \mathbb{R} \mapsto \mathbb{R}$ be efficiently computable functions, where p, q, r are polynomially bounded. Let Q_n be a quantum circuit consisting of $m(n)$ 1- and 2-qubit gates, taking in (1) input $x \in \Sigma^n$, (2) a classical streaming proof $y \in \{0, 1\}^{2^{p(n)}}$, and (3) $q(n)$ ancilla qubits in state $|0\rangle^{\otimes q(n)}$, such that $m(n) \geq 2^{p(n)}$ and $q(n) \geq p(n)$ for all sufficiently large n . Define thresholds $\alpha(n), \beta(n)$ satisfying $\alpha(n) - \beta(n) \geq 2^{-r(n)}$. We are promised that either:*

- (YES) *There exists¹⁵ a streaming proof $y \in \{0, 1\}^{2^{p(n)}}$ such that Q_n accepts (x, y) with probability at least α .*
- (NO) *For all streaming proofs $y \in \{0, 1\}^{2^{p(n)}}$, Q_n accepts (x, y) with probability at most β .*

There exists a poly(n)-time mapping from (Q_n, x) to a sparse Hamiltonian H on $O(q(n) + \log(m(n)))$ qubits, partition (L, R) of the qubits H acts on, and threshold parameters $\alpha'(n)$ and $\beta'(n)$ satisfying $\alpha'(n) - \beta'(n) \geq ((m(n) + 1)2^{r(n)})^{-1}$ such that:

¹⁵To clarify, we are slightly abusing notation here for simplicity. Formally, Definition 2.1 defines y as being “part of the circuit” Q_n . Section 6.1 will reflect this by using notation $Q_n(y)$ (i.e. the circuit Q_n with proof gates according to y). In the statement of the lemma, however, we say for simplicity, in the usual wording, “there exists a proof y such that...”.

- If (Q_n, x) is a YES case, there exists $|\psi_1\rangle_L|\psi_2\rangle_R$ such that $\langle\psi_1|_L\langle\psi_2|_RH|\psi_1\rangle_L|\psi_2\rangle_R \leq \alpha'$.
- If (Q_n, x) is a NO case, then for all $|\psi_1\rangle_L|\psi_2\rangle_R$, $\langle\psi_1|_L\langle\psi_2|_RH|\psi_1\rangle_L|\psi_2\rangle_R \geq \beta'$.

The norm of H scales as $\|H\|_\infty \in \text{poly}(m(n), 2^{r(n)})$.

Note that verification circuits Q_n in which the classical proof y is fully specified (as opposed to streamed), such as for NP or QCMA, are also covered by Lemma 6.1 *so long as* the ancilla space is large enough to store the entire proof y . (In this case, as each bit of y in Lemma 6.1 is streamed, we save it to a fresh ancilla qubit. Once the entire proof is recorded, we run the (say) QCMA circuit Q_n on y . Thus, there is no loss of generality in streaming the proof.)

Organization of section. Section 6.1 first sets up the proof ingredients. For pedagogical purposes, an effort is made to derive each of the ingredients as a response to a roadblock which arises when using a simpler construction. The full formal proof combining all ingredients is in Section 6.2.

6.1 Proof setup and ingredients

Let $Q_n(y) = V_m \cdots V_1$ be the quantum circuit in Lemma 6.1 for input size n given streaming proof y , which recall acts on registers R_1 (input of size n), R_2 (ancilla of $q(n) \in \text{poly}(n)$ qubits), R_3 (streaming classical proof, single qubit). We write $Q_n(y)$, as opposed to simply Q_n , because the set $\{V_i\}$ includes both computation and proof unitaries (cf. Definition 2.1), of which the latter are *a priori* unknown. This is in contrast to, say, QMA verification, where Q_n is fixed given just n .

Setup. Next, we recall and slightly adapt the definitions of history state and the Feynman-Kitaev circuit-to-Hamiltonian construction [KSV02] to our setting. As is common in the study of circuit-to-Hamiltonian mappings, without loss of generality¹⁶ we do not need to explicitly encode the input register, R_1 . We will, however, keep the naming conventions for R_2, R_3, R_4 for consistency.

We define the history state as

$$|\psi_{\text{hist}}(y)\rangle = \frac{1}{\sqrt{m+1}} \sum_{t=0}^m V_t \cdots V_1 |0 \cdots 0\rangle_{R_2} |0\rangle_{R_3} |t\rangle_{R_4}, \quad (66)$$

where R_4 denotes the clock register. As with $Q_n(y)$, we write $|\psi_{\text{hist}}(y)\rangle$ to stress the proof y is now embedded into the circuit Q_n , rather than given directly via a separate proof register (as it would be in the setting of QMA). Also, since $m(n) \in \Omega(2^{p(n)})$ necessarily (otherwise the circuit does not have time to see each bit of proof y), the clock register R_4 is encoded in binary (as opposed to unary, as in [KSV02]) to potentially handle $p(n)$ polynomial in n . This makes the Hamiltonian terms defined below $\log(m(n))$ -local.

Next, we define the Feynman-Kitaev circuit-to-Hamiltonian construction elements as

$$H_{\text{in}} := (I - |0 \cdots 0\rangle\langle 0 \cdots 0|)_{R_2} \otimes |1\rangle\langle 1|_{R_3} \otimes |0\rangle\langle 0|_{R_4} \quad (67)$$

$$H_{\text{out}} := |0\rangle\langle 0|_{\text{out}} \otimes |m\rangle\langle m|_{R_4} \quad (68)$$

$$H_{\text{prop}} := \sum_{t=1}^m H_t, \text{ where } H_t \text{ is defined as} \quad (69)$$

$$H_t := -V_t \otimes |t\rangle\langle t-1|_{R_4} - V_t^\dagger \otimes |t-1\rangle\langle t|_{R_4} + I \otimes (|t\rangle\langle t| + |t-1\rangle\langle t-1|)_{R_4}, \quad (70)$$

¹⁶This is because, as per Definition 2.1, R_1 is treated as a read-only register, and thus as classical control. Since we will be designing a Hamiltonian whose terms depend on the gates in Q_n , the poly-time Turing machine computing the reduction can simply hardcode the gates on-the-fly conditional on the bits of x .

where in H_{out} , $|0\rangle\langle 0|_{\text{out}}$ projects onto the dedicated output wire of Q_n (say, the first qubit of R_2).

Finally, define for 1- or 2-qubit unitary U the operator H_t^U as H_t , but with V_t replaced with U . Let $P \subseteq [m]$ denote the set of time steps for which $V_i = W_i$ or $V_i = W_i^\dagger$ (corresponding to Steps 1(b)i and 1(b)iii of Definition 2.1, respectively), i.e. in which a proof bit is written or uncomputed. We shall refer to such V_i as *proof gates*. Let y^* denote an optimal streamed proof, i.e. accepted by Q_n with the maximum probability p^* possible.

The construction. The basic goal of our construction is simple—design Hamiltonian H so that $|\psi_{\text{hist}}(y^*)\rangle$ is its ground state. Unfortunately, since we do not know the proof gates in advance, we cannot embed the action of $Q_n(y)$ into H_{prop} . To overcome this, we weaken our optimism—we instead design H to so that $|\psi_{\text{hist}}(y^*)\rangle_L \otimes |\psi_{\text{hist}}(y^*)\rangle_R$ is a low-energy state (in the sense of Definition 2.5) of H . We then use unentanglement across the two copies to logically simulate Boolean functions, allowing the history state to decide “on-the-fly” whether it wishes to stream proof bit 0 or 1 in the next round. We proceed in a sequence of attempts, each time pushing the current setup as far as possible before it breaks down, and then adding the next work-around. The full final construction is stated succinctly in Section 6.2. For clarity, throughout we assume the Hamiltonians we design act on bipartition L versus R of the Hilbert space.

Attempt 1: The foundation. Define:

$$\tilde{H}_{\text{in}} = (H_{\text{in}})_L \otimes I_R + I_L \otimes (H_{\text{in}})_R \quad (71)$$

$$\tilde{H}_{\text{prop}} = \sum_{t=1}^m \tilde{H}_t, \quad \text{where } \tilde{H}_t \text{ is defined as} \quad (72)$$

$$\tilde{H}_t = \begin{cases} (H_t^I)_L \otimes (H_t^X)_R + (H_t^X)_L \otimes (H_t^I)_R & \text{if } t \in P \\ (H_t)_L \otimes I_R + I_L \otimes (H_t)_R & \text{if } t \notin P \end{cases} \quad (73)$$

$$\tilde{H}_{\text{out}} = (H_{\text{out}})_L \otimes I_R + I_L \otimes (H_{\text{out}})_R \quad (74)$$

$$\tilde{H} = \tilde{H}_{\text{in}} + \tilde{H}_{\text{prop}} + \tilde{H}_{\text{out}}. \quad (75)$$

Completeness will hold straightforwardly for this and all subsequent iterations of the construction (see proof of Lemma 6.1 in Section 6.2), but the intuition is as follows. Recall our goal is for $|\psi_{\text{hist}}(y^*)\rangle_L \otimes |\psi_{\text{hist}}(y^*)\rangle_R$ to be a low-energy state of \tilde{H} . Then, the “+” in \tilde{H}_{in} and \tilde{H}_{out} simulates a logical “AND”, forcing both L and R registers to be correctly initialized and to accept in the final time step, respectively. \tilde{H}_t is split into two cases: When $t \notin P$, we know V_t and hence can directly force both L and R to implement it via the “+”. When $t \in P$, however, we do not know V_t , but only that $V_t \in \{I, X\}$ acting on R_3 . In this case, the “ \otimes ” in \tilde{H}_t simulates a logical “OR”, and \tilde{H}_t itself simulates identity $(x \vee \bar{y}) \wedge (\bar{x} \vee y) \leftrightarrow x = y$ for $x, y \in \{0, 1\}$; denote this construction of \tilde{H}_t as the FLUX gadget. Intuitively, if (say) $|\psi_{\text{hist}}\rangle_L$ chooses to apply $V_t = I$ to annihilate $(H_t^I)_L$, then $|\psi_{\text{hist}}\rangle_R$ must also apply $V_t = I$ to annihilate $(H_t^I)_R$.

We now address the various shortcomings of this construction, beginning with the fact that the FLUX gadget itself is not sound.

Obstacle 1: Fooling the FLUX gadget. Let $|\psi_1\rangle_L |\psi_2\rangle_R$ be an arbitrary state. To force a dishonest prover to simulate an honest one, ideally, \tilde{H}_t with $t \in P$ should act approximately like a “switch”, meaning

$$\langle \psi_1 |_L \langle \psi_2 |_L \tilde{H}_t | \psi_1 \rangle_R | \psi_2 \rangle_R \approx 0 \text{ iff } (\langle \psi_1 | H_t^I | \psi_1 \rangle \approx 0 \text{ and } \langle \psi_1 | H_t^X | \psi_1 \rangle \approx 1 \text{ (or vice versa)}) . \quad (76)$$

To formally study this idea, define for $a, b \in \mathbb{R}$ the operator-valued function

$$G(a, b) := aH_t^X + bH_t^I, \quad (77)$$

so that

$$\langle \psi_2 | G(\langle \psi_1 | H_t^I | \psi_1 \rangle, \langle \psi_1 | H_t^X | \psi_1 \rangle) | \psi_2 \rangle = \langle \psi_1 | \langle \psi_2 | (H_t^I \otimes H_t^X + H_t^X \otimes H_t^I) | \psi_1 \rangle | \psi_2 \rangle. \quad (78)$$

The problem is that for *any* $a, b \in \mathbb{R}$, $aH_t^X + bH_t^I$ has null vector $|\phi\rangle_{R_1 R_2} |+\rangle_{R_3} (|t-1\rangle + |t\rangle)_{R_4}$ for any $|\phi\rangle$, clearly violating the intended behavior of applying either I or X to $|0\rangle_{R_3}$ in step t . Moreover, we cannot simply force R_3 set to $|0\rangle$ or $|1\rangle$, as the projector onto the latter space is simply identity.

Attempt 2: Make it complex. Suppose instead of using I and X to encode proof bit 0 and 1, we instead use more general unitaries $U, V \in U(\mathbb{C}^2)$ applied to some initial state $|\phi\rangle$ (generalizing the use of $|0\rangle$ in R_3). In other words, an honest prover prepares $U|\phi\rangle$ to encode logical proof bit 0, and $V|\phi\rangle$ for proof bit 1. The FLUX gadget is thus generalized to (for $t \in P$)

$$\tilde{H}_t(U, V) := (H_t^U)_L \otimes (H_t^V)_R + (H_t^V)_L \otimes (H_t^U)_R. \quad (79)$$

(Observe the initial state $|\phi\rangle$ is not explicitly encoded here; this would instead be enforced by setting R_3 to $|\phi\rangle$ at time step 0 of the history state. We will shortly choose $|\phi\rangle = |0\rangle$ anyway, which is enforced by our present choice of \tilde{H}_{in} .) Next, Equation (77) is generalized to

$$G(a, b) := aH_t^V + bH_t^U. \quad (80)$$

The reason soundness breaks following Equation (77) is captured by the following sufficient condition.

Lemma 6.2. *Let U, V, G be defined as in Equation (80). If there exist unit vectors $|\gamma_1\rangle, |\gamma_2\rangle \in \mathbb{C}^2$ such that*

1. $V^\dagger U |\gamma_1\rangle = |\gamma_1\rangle$,
2. $VU^\dagger |\gamma_2\rangle = |\gamma_2\rangle$, and
3. $U |\gamma_1\rangle = |\gamma_2\rangle$,

then there exists non-zero $|\eta\rangle$ acting on $R_2 R_3 R_4$ such that for all $a, b \in \mathbb{R}$, $G(a, b)|\eta\rangle = 0$.

Proof. Assume such $|\gamma_1\rangle, |\gamma_2\rangle$ exist. Then, $U|\gamma_1\rangle = V|\gamma_1\rangle$ and $U^\dagger|\gamma_2\rangle = V^\dagger|\gamma_2\rangle$. For any $|v\rangle$ acting on R_2 , define

$$|\eta\rangle_{R_2 R_3 R_4} := |v\rangle_{R_2} (|\gamma_1\rangle_{R_3} |t-1\rangle_{R_4} + |\gamma_2\rangle_{R_3} |t\rangle_{R_4}). \quad (81)$$

Then,

$$G(a, b)|\eta\rangle \propto (a + b)|v\rangle_{R_2} \otimes \left(-U|\gamma_1\rangle|t\rangle - U^\dagger|\gamma_2\rangle|t-1\rangle + |\gamma_2\rangle|t\rangle + |\gamma_1\rangle|t-1\rangle \right)_{R_3 R_4} = 0, \quad (82)$$

where the last equality uses $U|\gamma_1\rangle = |\gamma_2\rangle$. \square

As a sanity check, we may apply this to Equation (77) by setting $|\gamma_1\rangle = |\gamma_2\rangle = |+\rangle$, $U = X$ and $V = I$, for which the preconditions of Lemma 6.2 hold.

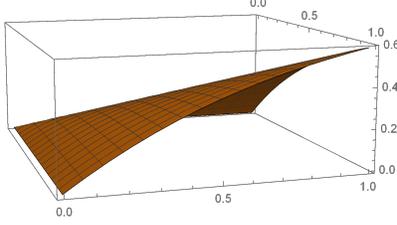


Figure 4: Above, the horizontal axes correspond to a and b , and the vertical axis to the minimum eigenvalue of $G(a, b)$.

Now that we understand the bottleneck, we can work around it. Call $(|\phi\rangle, U, V)$ a *valid* encoding if $\langle\phi|V^\dagger U|\phi\rangle = 0$, i.e. U and V map $|\phi\rangle$ to orthogonal states, which may be viewed as logical 0 and 1. For simplicity, pick $|\phi\rangle = |0\rangle$. First, by Item 1 of Lemma 6.2, $V^\dagger U$ should not have a 1-eigenvector, and second, condition $\langle 0|V^\dagger U|0\rangle = 0$ and the fact that $V^\dagger U$ is unitary enforce

$$V^\dagger U = \begin{pmatrix} 0 & e^{i\theta_1} \\ e^{i\theta_2} & 0 \end{pmatrix} \quad (83)$$

for some $\theta_1, \theta_2 \in \mathbb{R}$. Thus, set $|\phi\rangle = |0\rangle$, $U = I$, and $V = iX$. We now have that, restricted to $\text{Span}(|t-1\rangle, |t\rangle)$ on R_4 ,

$$\lambda_{\min}(G(a, b)) = a + b - \sqrt{a^2 + b^2}. \quad (84)$$

The behavior of $\lambda_{\min}(G(a, b))$ is depicted graphically in Figure 4, from which one immediately sees that $\lambda_{\min}(G(a, b)) = 0$ only if at least one of $a = 0$ or $b = 0$. By Equation (78), this almost gets us what we want—in order to have a chance at annihilating the residual operator $G(a, b)$, the first proof must correctly encode either I or iX as the t th gate.¹⁷ We just need to check that *both* a and b are not zero (otherwise, the second proof becomes unconstrained at time t).

Lemma 6.3. *For $U = I$ and $V = iX$, restricted to $\text{Span}(|t-1\rangle, |t\rangle)$ on R_4 we have for any unit $|\psi\rangle$*

$$\langle\psi|H_t^U|\psi\rangle + \langle\psi|H_t^V|\psi\rangle \geq 2 - \sqrt{2} \approx 0.586. \quad (85)$$

Proof. The claim follows by plugging Equation (84) into the sequence of observations,

$$\min_{\text{unit } |\psi\rangle} \langle\psi|(H_t^U + H_t^V)|\psi\rangle = \min_{\text{unit } |\psi\rangle} \langle\psi|G(1, 1)|\psi\rangle = \lambda_{\min}(G(1, 1)) = 2 - \sqrt{2}. \quad (86)$$

□

In words, defining $a = \langle\psi_1|H_t^U|\psi_1\rangle$ and $b = \langle\psi_1|H_t^V|\psi_1\rangle$ (cf. Equation (78)), Lemma 6.3 says that $a + b \geq 2 - \sqrt{2} \gg 0$, i.e. we cannot have $a = b = 0$.

The fix. For all $t \in P$, update our current construction so that

$$\tilde{H}_t := \tilde{H}_t(I, iX) = (H_t^I)_L \otimes (H_t^{iX})_R + (H_t^{iX})_L \otimes (H_t^I)_R \quad (87)$$

To recap, to annihilate \tilde{H}_t for $t \in P$, the first proof $|\psi_1\rangle$ must simulate either I ($a = 0$) or iX ($b = 0$) at time t . If $a = 0$ (resp. $b = 0$), then $|\psi_2\rangle$ must annihilate $G(0, c)$ (resp. $G(c, 0)$) for

¹⁷Note the use of iX at a time step t in Definition 2.2 only produces a global phase i , and so does not affect the distribution obtained when measuring the output of the circuit.

$c \geq 2 - \sqrt{2}$, meaning the history state must simulate application of iX (resp. I) at time t . (These pieces will be formally combined in Section 6.2.)

Obstacle 2: Skipping time steps. Thus far, we have characterized how a *single* FLUX gadget \tilde{H}_t for $t \in P$ acts in isolation. In particular, when there is a single FLUX gadget, we have shown that it is sound, forcing $|\psi_1\rangle \otimes |\psi_2\rangle$ to correctly act like a “switch” at time t .

The next step is to analyze whether soundness holds in the presence of *multiple* FLUX gadgets, which requires analysis of \tilde{H}_{prop} *as a whole*. To do so, define $M(a, b) = aiX + bI$, and rewrite

$$G(a, b) = -M(a, b) \otimes |t\rangle\langle t-1| - M^\dagger(a, b) \otimes |t-1\rangle\langle t| + (a+b)I \otimes (|t\rangle\langle t| + |t-1\rangle\langle t-1|). \quad (88)$$

It will be helpful to view this as a “dynamic” choice of propagation Hamiltonian, where $M(a, b)$ is applied in step t . For convenience, we often omit the a, b term and write M henceforth.

The standard approach [KSV02] for analyzing a propagation Hamiltonian H_{prop} is to apply a change of basis that maps H_{prop} to a tri-diagonal matrix encoding a 1D random walk. Unfortunately, this change of basis requires a unitary gate to be applied at each step t , and M above is not unitary. However, since we chose $V = iX$ (as opposed to $V = X$),

$$U(a, b) := \frac{1}{\sqrt{a^2 + b^2}} M \quad (89)$$

is unitary. (Aside: It is not necessarily true that $a^2 + b^2 = 1$.) Plugging this into Equation (88), we have

$$G(a, b) = -\sqrt{a^2 + b^2} U(a, b) \otimes |t\rangle\langle t-1| - \sqrt{a^2 + b^2} U^\dagger(a, b) \otimes |t-1\rangle\langle t| + (a+b)I \otimes (|t\rangle\langle t| + |t-1\rangle\langle t-1|). \quad (90)$$

Recall now Kitaev’s [KSV02] change of basis unitary W , which for circuit $V_m \cdots V_1$ acting on $R_2 R_3$ as in Equation (66), is defined as $W = \sum_{t=1}^m V_1^\dagger \cdots V_t^\dagger \otimes |t\rangle\langle t|_{R_4}$, except where for $t \in P$, we now replace V_t with $U_t(a, b)$. Then, restricted to $\text{Span}(|t-1\rangle, |t\rangle)$ on R_4 ,

$$WG(a, b)W^\dagger = \begin{pmatrix} a+b & -\sqrt{a^2 + b^2} \\ -\sqrt{a^2 + b^2} & a+b \end{pmatrix}. \quad (91)$$

So, for example, if we chain together time steps $t \in P$ (compute proof bit), $t+1$ (copy proof bit), $t+2 \in P$ (uncompute proof bit), the joint propagation Hamiltonian under conjugation by W is:¹⁸

$$\begin{pmatrix} \ddots & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1+a_i+b_i & -\sqrt{a_i^2 + b_i^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{a_i^2 + b_i^2} & 1+a_i+b_i & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1+a_j+b_j & -\sqrt{a_j^2 + b_j^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{a_j^2 + b_j^2} & 1+a_j+b_j & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \ddots \end{pmatrix} \quad (92)$$

¹⁸ W only acts on space R . Recall that $G(a_t, b_t)$ is the residual operator on timestep t after applying $|\psi_1\rangle$ (see Equation (78)), where a_t, b_t are functions in $|\psi_1\rangle$.

Two remarks: First, $a_i, b_i \geq 0$ since $H_t^U \succeq 0$ for all unitary U . Second, as a sanity check, when the first prover is honest, we have $a_i = 0$ and $b_i = 1$ or vice versa for all i , reducing us to Kitaev’s original 1D random walk matrix on the second prover’s space, as expected.

Reading the fine print: How to break soundness, again. Equation (84) and Lemma 6.3 together imply that in isolation, the FLUX gadget is sound. However, that analysis was done *restricted to* $\text{Span}(|t-1\rangle, |t\rangle)$ in R_4 . A cheating prover, on the other hand, *a priori* is under no obligation to place any reasonable weight on time steps $|t-1\rangle$ and $|t\rangle$ in the history state. Indeed, we now sketch a cheating strategy which breaks soundness when multiple FLUX gadgets are chained together.

Roughly, the intuition is as follows. Normally, Kitaev’s propagation Hamiltonian acts logically as follows: \tilde{H}_t (Equation (70)) ensures that the weight on consecutive time steps $t-1$ and t is identical. By chaining together all \tilde{H}_t , we thus obtain that all time steps must have equal amplitude, and moreover this must be non-zero (otherwise we cannot have a unit vector). The reason this breaks down in our current setting is that each \tilde{H}_t has *two* ways of being satisfied—either $|\psi_L\rangle$ has equal amplitude on steps $t-1$ and t , or $|\psi_R\rangle$ does (or possibly both). So let us give a simple example of how to exploit this. Suppose there are m time steps, with time step 1 and m being proof bit computation steps (i.e. so that \tilde{H}_1 and \tilde{H}_m encode the FLUX gadget). We claim that any unit vector of form

$$|\psi_L\rangle \otimes |\psi_R\rangle := (|\phi_L\rangle_{R_2 R_3} |m\rangle_{R_4}) \otimes (|\phi_R\rangle_{R_2 R_3} |0\rangle_{R_4}) \quad (93)$$

is in the null space of \tilde{H}_{prop} , violating soundness. To see why, note that $|\psi_L\rangle$ trivially annihilates all terms \tilde{H}_t except $t = m$, since it only has support on $|m\rangle_{R_4}$. As for \tilde{H}_m , while this is not annihilated by $|\psi_L\rangle$, it *is* annihilated by $|\psi_R\rangle$, since the latter only has support on $|0\rangle_{R_4}$. Note that $|\psi_L\rangle$ reciprocates this favor for $|\psi_R\rangle$ at time $t = 0$, in that the former annihilates \tilde{H}_0 , allowing $|\psi_R\rangle$ to “hide” all its amplitude on $|0\rangle_{R_4}$.

The fix. The silver lining is that this loophole is highly asymmetric—in our simple example, $|\psi_L\rangle$ and $|\psi_R\rangle$ had their supports on disjoint sets of time steps in R_4 . To close this loophole, we thus force $|\psi_1\rangle \approx |\psi_2\rangle$ by adding the projector onto the complement of the symmetric subspace (with respect to the L versus R cut) to our Hamiltonian:

$$\tilde{H}_{\text{sym}} := I - P_{LR}^{\text{sym}} \quad \text{for} \quad P_{LR}^{\text{sym}} = \frac{1}{2} \left(I_{LR} + \sum_{xy} |xy\rangle \langle yx|_{LR} \right) \quad (94)$$

Note \tilde{H}_{sym} is sparse (Definition 2.4); this is the second of two places necessitating our construction to be sparse. Moreover, any $|\psi_L\rangle \otimes |\psi_R\rangle$ satisfying $\tilde{H}_{\text{sym}} |\psi_L\rangle \otimes |\psi_R\rangle = 0$ must have $|\psi_L\rangle = |\psi_R\rangle$ by definition of the symmetric subspace.

The final ingredient. With symmetry in hand, we give the final ingredient, Lemma 6.4. For this, define for any $t \in P$ (cf. Equation (77) and Equation (78))

$$a_t := \langle \psi_1 | \Delta H_t^I | \psi_1 \rangle \quad b_t := \langle \psi_1 | \Delta H_t^{iX} | \psi_1 \rangle \quad (95)$$

for Δ defined as needed. The following lemma shows that in the case of perfect symmetry, we may compute a polynomial Δ in m such that, for all $t \in P$, $a_t + b_t$ cannot be “too small”, *even if* the adversary can cheat across multiple FLUX gadgets.

Lemma 6.4 (Full support lemma). *Define \tilde{H}_{prop} as in Equation (75), except with the update of Equation (87). Assume perfect symmetry, i.e. $|\psi_1\rangle = |\psi_2\rangle$, and the notation of Equation (95). For any¹⁹ $\delta \geq 0$ and $\Delta \geq 1$ satisfying $\Delta > \max(36\delta, (8m^4)/c)$ (for $c \in O(1)$ from Equation (102)), the following holds: If*

$$\langle \psi_1 | \langle \psi_1 | \Delta \tilde{H}_{\text{prop}} | \psi_1 \rangle | \psi_1 \rangle \leq 2, \quad (96)$$

then for all $t \in P$, $a_t + b_t \geq \delta$.

Proof. As in the claim, assume $\langle \psi_1 | \langle \psi_1 | \Delta \tilde{H}_{\text{prop}} | \psi_1 \rangle | \psi_1 \rangle \leq 2 =: \mu$. Next, for sake of contradiction, assume there exists $t^* \in P$ with $0 \leq a_{t^*} + b_{t^*} \leq \delta$ (recall $H_t^I, H_t^{iX} \succeq 0$ for all t). To highlight²⁰ the single place in which the symmetry assumption is used, we run as much as of the proof as possible in full generality (i.e. not requiring $|\psi_1\rangle = |\psi_2\rangle$). We aim to set Δ so as to achieve a contradiction.

Step 1: Bounding the weight on time steps $t^* - 1$ and t^* . Recalling that R_4 is the clock register, let

$$S_t := \text{Span}(I_{R_2, R_3} \otimes |t-1\rangle_{R_4}, I_{R_2, R_3} \otimes |t\rangle_{R_4}).$$

Then, for all unit vectors $|\phi\rangle \in S_{t^*}$, Lemma 6.3 says $\langle \phi | (H_{t^*}^I + H_{t^*}^{iX}) | \phi \rangle \geq 2 - \sqrt{2}$. Writing $|\psi_1\rangle = a|\phi_1\rangle + b|\phi_2\rangle$, for $|a|^2 + |b|^2 = 1$ and unit vectors $|\phi_1\rangle \in S_{t^*}, |\phi_2\rangle \in S_{t^*}^\perp$, observe

$$\delta \geq a_{t^*} + b_{t^*} = \Delta \langle \psi_1 | (H_{t^*}^I + H_{t^*}^{iX}) | \psi_1 \rangle = \Delta |a|^2 \langle \phi_1 | (H_{t^*}^I + H_{t^*}^{iX}) | \phi_1 \rangle \geq \Delta |a|^2 (2 - \sqrt{2}),$$

where the second equality follows since $H_{t^*}^I$ and $H_{t^*}^{iX}$ are only supported on S_{t^*} by definition, and the last inequality by Lemma 6.3. We conclude that

$$|a| \leq \sqrt{\frac{\delta}{\Delta(2 - \sqrt{2})}} =: \delta', \quad (97)$$

implying the weights of $|\psi_1\rangle$ on time steps $t^* - 1$ and t^* are each at most δ' as well, i.e. writing

$$|\psi_1\rangle = \sum_{t=0}^m |\eta_t\rangle_{R_2, R_3} |t\rangle_{R_4} \quad (98)$$

for vectors $|\eta_t\rangle$, we have $\| |\eta_{t^*-1}\rangle \|_2, \| |\eta_{t^*}\rangle \|_2 \leq \delta'$.

Step 2: Decomposing the analysis into computation and proof phases. We next decompose the analysis into proof and computation phases. By Definition 2.1, we may partition the set of time steps $\{1, \dots, m\}$ into sets of contiguous time steps $T_1, T_2, T_3, \dots, T_{m'}$ for $m' \leq m$ as follows. To begin, T_1 is set of time steps corresponding to the first time the first computation phase is run (Step 1(a) of Definition 2.1), T_2 the gate W_1 (proof phase, Step 1(b)i), T_3 the single CNOT gate (proof phase, Step 1(b)ii), T_4 the second W_1 gate (proof phase, Step 1(b)iii). The pattern now repeats itself until we have accounted for all time steps. For simplicity²¹, we assume $T_{m'} = \{m\}$ is the final time step, which corresponds to an execution of Step 1(b)iii (proof phase, uncompute).

¹⁹In our use of Lemma 6.4, δ and Δ will be functions in parameters such as m , i.e. $\delta \in O(1/\text{poly}(m))$ and $\Delta \in \Omega(\text{poly}(m))$.

²⁰Recall that a theme of the present work is to highlight precisely which parts of our construction require unentanglement.

²¹This keeps the notation simpler; the proof approach applies analogously even without this assumption.

Consider now

$$\langle \psi_1 | \langle \psi_2 | \Delta \tilde{H}_{\text{prop}} | \psi_1 \rangle | \psi_2 \rangle = \langle \psi_1 | \Delta \sum_{t \notin P} H_t | \psi_1 \rangle + \langle \psi_2 | \Delta \sum_{t \notin P} H_t | \psi_2 \rangle + \langle \psi_1 | \langle \psi_2 | \sum_{t \in P} \Delta \tilde{H}_t | \psi_1 \rangle | \psi_2 \rangle. \quad (99)$$

We focus on the terms involving $|\psi_1\rangle$. Define $T_{\text{comp}} := \{T_i \mid i \text{ is odd}\}$ (Steps 1(a) and 1(b)ii of Definition 2.1) and $T_{\text{proof}} := \{T_i \mid i \text{ is even}\}$ (Steps 1(b)i and 1(b)iii of Definition 2.1), and for any $T \subseteq \{0, \dots, m\}$, define shorthand $H_T := \sum_{t \in T} H_t$ for $T \in T_{\text{comp}}$ and $\tilde{H}_T := \sum_{t \in T} \tilde{H}_t$ for $T \in T_{\text{proof}}$ (note the former acts on L or R , the latter on both L and R). Then, by definition

$$\langle \psi_1 | \Delta \sum_{t \notin P} H_t | \psi_1 \rangle + \langle \psi_1 | \langle \psi_2 | \sum_{t \in P} \Delta \tilde{H}_t | \psi_1 \rangle | \psi_2 \rangle = \sum_{T \in T_{\text{comp}}} \langle \psi_1 | \Delta H_T | \psi_1 \rangle + \sum_{T \in T_{\text{proof}}} \Delta \langle \psi_1 | \langle \psi_2 | \tilde{H}_T | \psi_1 \rangle | \psi_2 \rangle. \quad (100)$$

As an aside, note that for any distinct sets $A, B \in T_{\text{comp}}$, H_A and H_B have support on disjoint sets of time steps. (This is because A and B must have at least one proof phase $C \in T_{\text{proof}}$ between them.) Moreover, although $\bigcup_{T \in T_{\text{comp}}} T \neq [L]$ (since we are missing all proof time steps P), nevertheless the *Hamiltonian* $\sum_{T \in T_{\text{comp}}} H_T$ has support on all time steps in $\{0, \dots, L\}$. (This is because each $C \in T_{\text{proof}}$ is a singleton, and each H_t has support on both $|t\rangle$ and $|t-1\rangle$.)

Step 3: Triggering a chain reaction. With the decomposition of Step 2 in mind, we can now sketch the remaining proof approach at a high level.

1. Recall $t^* \in P$, i.e. is in a proof phase, and that from Equations (97) and (98) that $\|\eta_{t^*-1}\|_2 \leq \delta'$ and $\|\eta_{t^*}\|_2 \leq \delta'$.
2. Since $\|\eta_{t^*}\|_2$ is small, we claim this triggers a ‘‘chain reaction’’ causing *all* $\|\eta_t\|_2$ for $t \geq t^*$ to be small. An identical argument also applies to $t^* - 1$ and all $t \leq t^* - 1$. (For brevity, we show the claim only for the former case; the latter case follows analogously.)
3. Thus, all amplitudes of $|\psi_1\rangle$ are small, contradicting the fact that $|\psi_1\rangle$ is a unit vector.

To make this formal, and in particular to show the claim in the second point above, we treat proof and computation phases separately.²² Consider any $T \in T_{\text{comp}} \cup T_{\text{proof}}$, and suppose $t^* + 1$ is the smallest time step in T . We show that if $\|\eta_{t^*}\|_2$ is small, so is $\|\eta_t\|_2$ for all $t \in T$. For brevity, define for any $T \in T_{\text{comp}} \cup T_{\text{proof}}$ the projector $\Pi_T := \sum_{t \in T} |t\rangle\langle t|_{R_4}$, and $|\psi_T\rangle := \Pi_T |\psi_1\rangle$.

Case 1: $T \in T_{\text{comp}}$. Suppose $T = \{t^* + 1, t^* + 2, \dots, t^* + s\}$ for some s . Then, H_T has support on time steps $\{t^*, \dots, t^* + s\}$. Now suppose $\|\eta_{t^*}\|_2 \leq \varepsilon$ for arbitrary $\varepsilon \geq 0$. Since all time steps in T are computation steps, we may use the well-known facts [KSV02] that:

1. (Fact 1) The null space of H_T is the span of all states of form

$$\sum_{t=t^*}^{t^*+s} U_t \cdots U_{t^*+2} U_{t^*+1} |\phi_{\text{init}}\rangle |t\rangle, \quad (101)$$

where U_t is the t th computation gate applied in computation phase T , and for any initial unit vector $|\phi_{\text{init}}\rangle$.

²²In Kitaev’s original circuit-to-Hamiltonian construction [KSV02], this claim is achieved in one elegant stroke by analyzing the eigenvalues of a random walk matrix which is unitarily equivalent to the propagation Hamiltonian. In our setting, however, we also have operators \tilde{H}_t acting on both $|\psi_1\rangle, |\psi_2\rangle$, i.e. we are not looking at the spectral properties of a propagation Hamiltonian acting solely on $|\psi_1\rangle$.

2. (Fact 2) The eigenvalues of H_T are precisely $\lambda_t = 2(1 - \cos[\pi t/(s+1)])$ for $0 \leq t \leq s$, and so the smallest non-zero eigenvalue is

$$2(1 - \cos(\pi/(s+1))) \geq c/s^2 \text{ for some } c \in \Theta(1). \quad (102)$$

Defining $T' := T \cup \{t^*\}$, consider now $|\psi_{T'}\rangle = \sum_{t=t^*}^{t^*+s} |\eta_t\rangle|t\rangle$ (recall $|\psi_{T'}\rangle$ is $|\psi_1\rangle$ projected onto time steps in T') for vectors $\{|\eta_t\rangle\}$ of possibly *differing* norms. We claim that $\| |\eta_t\rangle \|_2$ is small for all $t \in \{t^*, \dots, t^* + s\}$.

To see this, by assumption, $\| |\eta_{t^*}\rangle \|_2 \leq \varepsilon$ and $\langle \psi_{T'} | \Delta H_T | \psi_{T'} \rangle \leq \mu$. Writing $|\psi_{T'}\rangle$ in terms of its components in the null space ($|\psi_{T',0}\rangle$) and support ($|\psi_{T',+}\rangle$) of H_T , respectively, i.e.

$$|\psi_{T'}\rangle = \sum_{t=t^*}^{t^*+s} |\eta_t\rangle|t\rangle = \sum_{t=t^*}^{t^*+s} |\eta_{t,0}\rangle|t\rangle + \sum_{t=t^*}^{t^*+s} |\eta_{t,+}\rangle|t\rangle =: |\psi_{T',0}\rangle + |\psi_{T',+}\rangle, \quad (103)$$

we have

$$\mu \geq \langle \psi_{T'} | \Delta H_T | \psi_{T'} \rangle = \langle \psi_{T',+} | \Delta H_T | \psi_{T',+} \rangle \geq \frac{c\Delta \| |\psi_{T',+}\rangle \|_2^2}{s^2}, \quad (104)$$

where the last inequality follows from Fact 2. Thus, $\| |\psi_{T',+}\rangle \|_2^2 \leq s^2\mu/(c\Delta)$. But by Fact 1, all $|\eta_{t,0}\rangle$ have the same norm with $\| |\eta_{t^*,0}\rangle \|_2 \leq \varepsilon$ (since $\| |\eta_{t^*}\rangle \|_2 \leq \varepsilon$ by assumption), and each $|\eta_{t,+}\rangle$ has norm $\| |\eta_{t,+}\rangle \|_2 \leq s\sqrt{\mu/(c\Delta)}$. By the triangle inequality, we conclude that for all $t \in \{t^*, \dots, t^* + s\}$,

$$\| |\eta_t\rangle \|_2 \leq \varepsilon + s\sqrt{\frac{\mu}{c\Delta}}. \quad (105)$$

Case 2: $T \in T_{\text{proof}}$. For concreteness, suppose $T = \{t^* + 1\}$, so that H_T has support on time steps $T' := \{t^*, t^* + 1\}$. Now suppose $\| |\eta_{t^*}\rangle \|_2 \leq \varepsilon$ for arbitrary $\varepsilon \geq 0$. Letting F_{t^*+1} denote an arbitrary Feynman-Kitaev propagation term (Equation (70)) for arbitrary unitary U_{t^*+1} at time $t^* + 1$,

$$\langle \psi_1 | F_{t^*+1} | \psi_1 \rangle = \langle \psi_{T'} | F_{t^*+1} | \psi_{T'} \rangle = \frac{1}{2} \| |\eta_{t^*}\rangle \|_2^2 + \frac{1}{2} \| |\eta_{t^*+1}\rangle \|_2^2 - \text{Re}(\langle \eta_{t^*+1} | U_{t^*+1} | \eta_{t^*} \rangle) \quad (106)$$

$$\geq \frac{1}{2} (\| |\eta_{t^*}\rangle \|_2 - \| |\eta_{t^*+1}\rangle \|_2)^2, \quad (107)$$

where the inequality follows by the Cauchy-Schwarz inequality and unitary invariance of the Euclidean norm. Suppose $\| |\eta_{t^*+1}\rangle \|_2 = \| |\eta_{t^*}\rangle \|_2 + \zeta$ for $\zeta \in \mathbb{R}$. By Equation (107), $\langle \psi_1 | F_{t^*+1} | \psi_1 \rangle \geq \zeta^2/2$. And now we use the assumption that $|\psi_1\rangle = |\psi_2\rangle$ to obtain that

$$\mu \geq \Delta \langle \psi_1 | \langle \psi_2 | (H_{t^*+1}^I \otimes H_{t^*+1}^{iX} + H_{t^*+1}^{iX} \otimes H_{t^*+1}^I) | \psi_1 \rangle | \psi_2 \rangle \geq \frac{\Delta \zeta^4}{2}, \quad (108)$$

where we have substituted $H_{t^*+1}^I$ or $H_{t^*+1}^{iX}$ for F_{t^*+1} , as appropriate. We conclude that

$$\| |\eta_{t^*+1}\rangle \|_2 \leq \varepsilon + \zeta \leq \varepsilon + \left(\frac{2\mu}{\Delta} \right)^{\frac{1}{4}}. \quad (109)$$

Step 4: Combining all bounds. By Equations (97) and (98), there exists a $t^* \in P$ with $\| |\eta_{t^*-1}\rangle \|_2 \leq \delta'$ and $\| |\eta_{t^*}\rangle \|_2 \leq \delta'$ for $\delta' := \sqrt{\delta/(\Delta(2-\sqrt{2}))}$. Running the chain reaction upwards from t^* (respectively, downwards from $t^* - 1$):

- Each time we encounter a computation phase $T \in T_{\text{comp}}$, Equation (105) says we increase our

norm by at most an additive factor of $s\sqrt{\mu/(c\Delta)}$.

- Each time we encounter a proof phase $T \in T_{\text{proof}}$, Equation (109) says we increase our norm by at most an additive factor of $(2\mu/\Delta)^{\frac{1}{4}}$.

We hence have the (naive) upper bound

$$1 = \|\psi_1\|_2 \leq \sum_{t=0}^m \|\eta_t\|_2 \leq 2\sqrt{\frac{\delta}{\Delta(2-\sqrt{2})}} + (m-2) \left(m\sqrt{\frac{\mu}{c\Delta}} + \left(\frac{2\mu}{\Delta}\right)^{\frac{1}{4}} \right), \quad (110)$$

where the first inequality follows by the triangle inequality, and the second²³ since $s \leq m$ in Equation (105). Since $\delta \geq 0$, $\Delta \geq 1$, and $\mu = 2 \in \Theta(1)$, choosing $\Delta > \max(36\delta, (8m^4)/c)$ yields a contradiction, completing the proof. \square

6.2 Final proof: Combining all ingredients

With the ingredients of Section 6.1 in hand, we are ready to restate and prove the main lemma of this section.

Lemma 6.1 (Embedding lemma). *Let $p, q, r, m, \alpha, \beta : \mathbb{R} \mapsto \mathbb{R}$ be efficiently computable functions, where p, q, r are polynomially bounded. Let Q_n be a quantum circuit consisting of $m(n)$ 1- and 2-qubit gates, taking in (1) input $x \in \Sigma^n$, (2) a classical streaming proof $y \in \{0, 1\}^{2^{p(n)}}$, and (3) $q(n)$ ancilla qubits in state $|0\rangle^{\otimes q(n)}$, such that $m(n) \geq 2^{p(n)}$ and $q(n) \geq p(n)$ for all sufficiently large n . Define thresholds $\alpha(n), \beta(n)$ satisfying $\alpha(n) - \beta(n) \geq 2^{-r(n)}$. We are promised that either:*

- (YES) There exists¹⁵ a streaming proof $y \in \{0, 1\}^{2^{p(n)}}$ such that Q_n accepts (x, y) with probability at least α .
- (NO) For all streaming proofs $y \in \{0, 1\}^{2^{p(n)}}$, Q_n accepts (x, y) with probability at most β .

There exists a poly(n)-time mapping from (Q_n, x) to a sparse Hamiltonian H on $O(q(n) + \log(m(n)))$ qubits, partition (L, R) of the qubits H acts on, and threshold parameters $\alpha'(n)$ and $\beta'(n)$ satisfying $\alpha(n)' - \beta(n)' \geq ((m(n) + 1)2^{r(n)})^{-1}$ such that:

- If (Q_n, x) is a YES case, there exists $|\psi_1\rangle_L |\psi_2\rangle_R$ such that $\langle \psi_1 |_L \langle \psi_2 |_R H | \psi_1 \rangle_L | \psi_2 \rangle_R \leq \alpha'$.
- If (Q_n, x) is a NO case, then for all $|\psi_1\rangle_L |\psi_2\rangle_R$, $\langle \psi_1 |_L \langle \psi_2 |_R H | \psi_1 \rangle_L | \psi_2 \rangle_R \geq \beta'$.

The norm of H scales as $\|H\|_\infty \in \text{poly}(m(n), 2^{r(n)})$.

Proof. To reduce clutter, we omit the dependence on n when referring to functions $p, q, r, m, \alpha, \beta$. We assume all notation and definitions of Section 6.1. Define

$$\tilde{H} = \Delta_{\text{in}} \tilde{H}_{\text{in}} + \Delta_{\text{prop}} \tilde{H}_{\text{prop}} + \Delta_{\text{sym}} \tilde{H}_{\text{sym}} + \tilde{H}_{\text{out}}, \quad (111)$$

²³This is a naive bound, since for each phase we are charging both $s\sqrt{\mu/(c\Delta)}$ and $(2\mu/\Delta)^{\frac{1}{4}}$ for simplicity, rather than introducing additional notation to carefully account for each type of phase.

where for convenience we restate definitions

$$\tilde{H}_{\text{in}} = (H_{\text{in}})_L \otimes I_R + I_L \otimes (H_{\text{in}})_R \quad (112)$$

$$\tilde{H}_{\text{prop}} = \sum_{t=1}^m \tilde{H}_t, \quad \text{where } \tilde{H}_t \text{ is defined as} \quad (113)$$

$$\tilde{H}_t = \begin{cases} (H_t^I)_L \otimes (H_t^{iX})_R + (H_t^{iX})_L \otimes (H_t^I)_R & \text{if } t \in P \\ (H_t)_L \otimes I_R + I_L \otimes (H_t)_R & \text{if } t \notin P \end{cases} \quad (114)$$

$$\tilde{H}_{\text{out}} = (H_{\text{out}})_L \otimes I_R + I_L \otimes (H_{\text{out}})_R \quad (115)$$

$$\tilde{H}_{\text{sym}} = I - P_{LR}^{\text{sym}} \quad \text{for} \quad P_{LR}^{\text{sym}} = \frac{1}{2} \left(I_{LR} + \sum_{xy} |xy\rangle\langle yx|_{LR} \right), \quad (116)$$

and $\Delta_{\text{in}}, \Delta_{\text{prop}}, \Delta_{\text{sym}}$ are set as follows. Set $M := (m+1)2^r$. Then, define²⁴ $\Delta_{\text{in}} = M^{31}$, $\Delta_{\text{prop}} = 72M^{31}$, and $\Delta_{\text{sym}} = M^{66+2k}$, where $q(n) \in O(n^k)$ for some $k \in O(1)$ (recall q is the poly-bounded number of ancilla qubits in circuit Q_n). Next, set $\alpha' = 2^{\frac{1-\alpha}{m+1}}$ and $\beta' = 2^{\frac{1-\beta}{m+1}} - \frac{1}{M}$, where recall $\alpha - \beta \geq 2^{-r}$ by assumption. Observe \tilde{H} acts on $O(q(n) + \log(m(n)))$ qubits (workspace and clock register encoded in binary, respectively). Importantly, \tilde{H} is sparse (in the sense of Definition 2.4; here we use the fact that although H_{prop} has m terms, which may be exponential, each such term has support on only 2 basis states in the clock register in Section 6.1). For clarity, this means our reduction does *not* output the explicit Hamiltonian \tilde{H} , but rather the classical algorithm of Definition 2.4 which produces entries of \tilde{H} on demand. Finally, the norm of \tilde{H} is $\|\tilde{H}\|_{\infty} \in \text{poly}(m, 2^r)$, as claimed.

Correctness. Assume (Q_n, x) is a YES case. Let $Q_n = V'_m \cdots V'_2 V'_1$. For each $t \in P$ with $V'_t = X$ (i.e. a proof bit of 1 is streamed at time t), define $V_t := iX$, and for all $t \notin P$, define $V_t := V'_t$. Recall the history state of Equation (66), i.e.

$$|\psi_{\text{hist}}(y)\rangle = \frac{1}{\sqrt{m+1}} \sum_{t=0}^m V_t \cdots V_1 |0 \cdots 0\rangle_{R_2} |0\rangle_{R_3} |t\rangle_{R_4}, \quad (117)$$

where y is implicitly encoded by the choice of gates V'_t for $t \in P$. It is straightforward to verify

$$\tilde{H}_{\text{in}} |\psi_{\text{hist}}\rangle \otimes |\psi_{\text{hist}}\rangle = \tilde{H}_{\text{prop}} |\psi_{\text{hist}}\rangle \otimes |\psi_{\text{hist}}\rangle = \tilde{H}_{\text{sym}} |\psi_{\text{hist}}\rangle \otimes |\psi_{\text{hist}}\rangle = 0, \quad \text{and} \quad (118)$$

$$\langle \psi_{\text{hist}} | \otimes \langle \psi_{\text{hist}} | \tilde{H}_{\text{out}} | \psi_{\text{hist}}\rangle \otimes |\psi_{\text{hist}}\rangle \leq \frac{2(1-\alpha)}{m+1} = \alpha', \quad (119)$$

where the factor 2 appears since \tilde{H}_{out} contains two copies of H_{out} . Thus, completeness holds.

Assume next that (Q_n, x) is a NO case. Assume, for sake of contradiction, there exists $|\psi_1\rangle_L |\psi_2\rangle_R$ such that $\langle \psi_1 |_L \langle \psi_2 |_R \tilde{H} | \psi_1\rangle_L | \psi_2\rangle_R \leq \beta'$. The soundness analysis proceeds in steps. Throughout, recall $\tilde{H}_{\text{in}}, \tilde{H}_{\text{prop}}, \tilde{H}_{\text{sym}}, \tilde{H}_{\text{out}} \succeq 0$.

Step 1: Enforcing symmetry. We first show that, up to small additive error, we may assume $|\psi_1\rangle = |\psi_2\rangle$. By assumption,

$$\langle \psi_1 |_L \langle \psi_2 |_R \Delta \tilde{H}_{\text{sym}} | \psi_1\rangle_L | \psi_2\rangle_R \leq \langle \psi_1 |_L \langle \psi_2 |_R \tilde{H} | \psi_1\rangle_L | \psi_2\rangle_R \leq \beta',$$

²⁴We have not attempted to optimize these parameters.

from which we have $\langle \psi_1 | \langle \psi_2 | \tilde{H}_{\text{sym}} | \psi_1 \rangle | \psi_2 \rangle \leq \beta' / \Delta_{\text{sym}}$. Since the spaces L and R have the same dimension, we may write $|\psi_2\rangle = a|\psi_1\rangle + b|\psi_1^\perp\rangle$ for $|a|^2 + |b|^2 = 1$ and $|\psi_1^\perp\rangle$ orthogonal to $|\psi_1\rangle$. We thus have

$$\frac{\beta'}{\Delta_{\text{sym}}} \geq \langle \psi_1 | \langle \psi_2 | \tilde{H}_{\text{sym}} | \psi_1 \rangle | \psi_2 \rangle \quad (120)$$

$$= |b|^2 \langle \psi_1 | \langle \psi_1^\perp | \tilde{H}_{\text{sym}} | \psi_1 \rangle | \psi_1^\perp \rangle \quad (121)$$

$$= \frac{1}{2} |b|^2 \langle \psi_1 | \langle \psi_1^\perp | \left(I_{LR} - \sum_{xy} |xy\rangle \langle yx|_{LR} \right) | \psi_1 \rangle | \psi_1^\perp \rangle \quad (122)$$

$$= \frac{1}{2} |b|^2, \quad (123)$$

where the third statement follows by substituting the definition of \tilde{H}_{sym} , and the fourth since $\sum_{xy} |xy\rangle \langle yx|$ is the SWAP operator (and so $(\sum_{xy} |xy\rangle \langle yx|) |\psi_1\rangle | \psi_1^\perp \rangle = | \psi_1^\perp \rangle | \psi_1 \rangle$). Applying identity $\| |u\rangle \langle u| - |v\rangle \langle v| \|_{\text{tr}} = 2\sqrt{1 - |\langle u|v\rangle|^2}$ (Lemma 4.5), we conclude

$$\| | \psi_1 \rangle \langle \psi_1 | \otimes | \psi_2 \rangle \langle \psi_2 | - | \psi_1 \rangle \langle \psi_1 | \otimes | \psi_1 \rangle \langle \psi_1 | \|_{\text{tr}} \leq 2\sqrt{\frac{2\beta'}{\Delta_{\text{sym}}}} \leq \frac{2\sqrt{2}}{M^{33}} =: \gamma_1. \quad (124)$$

Step 2: Extracting a history state which is “good enough”. We next treat \tilde{H}_{in} and \tilde{H}_{prop} simultaneously. Similar to Step 1, $\langle \psi_1 | \langle \psi_2 | (\Delta_{\text{in}} \tilde{H}_{\text{in}} + \Delta_{\text{prop}} \tilde{H}_{\text{prop}}) | \psi_1 \rangle | \psi_2 \rangle \leq \beta'$ by assumption. Combining this with Equation (124), the Hölder inequality, and the triangle inequality yields

$$\langle \psi_1 | \langle \psi_1 | \left(\Delta_{\text{in}} \tilde{H}_{\text{in}} + \Delta_{\text{prop}} \tilde{H}_{\text{prop}} \right) | \psi_1 \rangle | \psi_1 \rangle \leq \beta' + 2\sqrt{\frac{2\beta'}{\Delta_{\text{sym}}}} (\Delta_{\text{in}} q + 2\Delta_{\text{prop}} m) \quad (125)$$

$$\leq 1 + O\left(\frac{1}{M}\right) \quad (126)$$

$$=: \zeta, \quad (127)$$

where (1) we are implicitly writing H_{in} as a sum of 1-local terms as is standard, e.g. via trick $I - |00\rangle \langle 00| \preceq |1\rangle \langle 1| \otimes I + I \otimes |1\rangle \langle 1|$, and so $\|\tilde{H}_{\text{in}}\|_\infty \leq q$, (2) since for any t , $\|H_t\|_\infty = 1$, implying $\|\tilde{H}_{\text{prop}}\|_\infty \leq 2m$ by the triangle inequality, and (3) we use that $q(n) \in O(n^k)$.

Our strategy for this step is now as follows. We first exploit Lemma 6.4 to extract from \tilde{H}_{prop} a “proper” Feynman-Kitaev propagation Hamiltonian (i.e. in the form of Equation (69)). We then couple the latter with \tilde{H}_{in} and Equation (127) to argue that $|\psi_1\rangle$ must be close to a history state. This history state will not be exactly what we need, but we will show in the next step that it is “good enough”.

To begin, recall

$$\langle \psi_1 | \langle \psi_1 | \Delta_{\text{prop}} \tilde{H}_{\text{prop}} | \psi_1 \rangle | \psi_1 \rangle = \langle \psi_1 | \left(2 \sum_{t \notin P} \Delta_{\text{prop}} H_t + \sum_{t \in P} G(a_t, b_t) \right) | \psi_1 \rangle \quad (128)$$

$$a_t = \langle \psi_1 | \Delta_{\text{prop}} H_t^I | \psi_1 \rangle \geq 0 \quad (129)$$

$$b_t = \langle \psi_1 | \Delta_{\text{prop}} H_t^{iX} | \psi_1 \rangle \geq 0 \quad (130)$$

for $G(a_t, b_t)$ from Equation (90). We now show how to extract a “proper” Feynman-Kitaev propa-

gation Hamiltonian from the right hand side of Equation (128).

Lemma 6.5. *Assume the notation above, and that $\langle \psi_1 | \langle \psi_1 | \Delta_{\text{prop}} \tilde{H}_{\text{prop}} | \psi_1 \rangle | \psi_1 \rangle \leq 2$. Suppose that $\delta' \geq 0$ and $\Delta_{\text{prop}} \geq 1$ satisfy $\Delta_{\text{prop}} > \max(36\sqrt{2}\delta', (8m^4)/c)$ (for $c \in O(1)$ from Equation (102)). For all $t \in P$, define F_t to be the Feynman-Kitaev propagation term (Equation (70)) for unitary $U(a_t, b_t)$ from Equation (89). Then,*

$$2\Delta_{\text{prop}} \sum_{t \notin P} H_t + \sum_{t \in P} G(a_t, b_t) \succeq \delta' \left(\sum_{t \notin P} H_t + \sum_{t \in P} F_t \right). \quad (131)$$

Proof. Consider first the case of $t \in P$. Recall

$$\begin{aligned} G(a_t, b_t) &= -\sqrt{a^2 + b^2} U(a_t, b_t) \otimes |t\rangle\langle t-1| - \sqrt{a^2 + b^2} U^\dagger(a_t, b_t) \otimes |t-1\rangle\langle t| \\ &\quad + (a_t + b_t) I \otimes (|t\rangle\langle t| + |t-1\rangle\langle t-1|). \end{aligned} \quad (132)$$

Set $\delta = \sqrt{2}\delta'$. Then, we have by Lemma 6.4 that $a_t + b_t \geq \sqrt{a_t^2 + b_t^2} \geq \delta'$ (here we use $\|\cdot\|_1 \geq \|\cdot\|_2 \geq \|\cdot\|_1/\sqrt{2}$ for \mathbb{C}^2). Thus, defining $s_1 := \sqrt{a^2 + b^2} - \delta'$ and $s_2 := a + b - \delta'$, we may rewrite

$$\begin{aligned} G(a_t, b_t) &= \delta' F_t + [-s_1 U(a_t, b_t) \otimes |t\rangle\langle t-1| - s_1 U^\dagger(a_t, b_t) \otimes |t-1\rangle\langle t| \\ &\quad + s_2 I \otimes (|t\rangle\langle t| + |t-1\rangle\langle t-1|)]. \end{aligned} \quad (133)$$

Since $a + b \geq \sqrt{a^2 + b^2}$ for all $a, b \geq 0$, we have $s_2 \geq s_1 \geq 0$, implying $G(a_t, b_t) - \delta' F_t \succeq 0$. Similarly for $t \notin P$, since $\Delta_{\text{prop}} \in \omega(\delta')$ by assumption, we have $(2\Delta_{\text{prop}} - \delta')H_t \succeq 0$, from which the claim follows. \square

To apply Lemma 6.5, set $\delta' = M^{31}$. By Section 6.2, $\langle \psi_1 | \langle \psi_1 | (\Delta_{\text{prop}} \tilde{H}_{\text{prop}}) | \psi_1 \rangle | \psi_1 \rangle \leq 2$. Thus, Equation (128) and Lemma 6.5 yield

$$\langle \psi_1 | \left(2\Delta_{\text{prop}} \sum_{t \notin P} H_t + \sum_{t \in P} G(a_t, b_t) \right) | \psi_1 \rangle \geq \langle \psi_1 | \left(\delta' \left(\sum_{t \notin P} H_t + \sum_{t \in P} F_t \right) \right) | \psi_1 \rangle =: \delta' \langle \psi_1 | H_{\text{prop}} | \psi_1 \rangle. \quad (134)$$

Note H_{prop} is a standard Feynman-Kitaev propagation Hamiltonian over all m time steps. So, set $\Delta_{\text{in}} = \delta'$, and combine Equation (127), Equation (128), and Equation (134) to obtain

$$\zeta \geq \langle \psi_1 | \langle \psi_1 | \Delta_{\text{in}} \tilde{H}_{\text{in}} + \Delta_{\text{prop}} \tilde{H}_{\text{prop}} | \psi_1 \rangle | \psi_1 \rangle \quad (135)$$

$$\geq \delta' \left(\langle \psi_1 | \langle \psi_1 | \tilde{H}_{\text{in}} | \psi_1 \rangle | \psi_1 \rangle + \langle \psi_1 | H_{\text{prop}} | \psi_1 \rangle \right) \quad (136)$$

$$\geq \langle \psi_1 | \delta' (H_{\text{in}} + H_{\text{prop}}) | \psi_1 \rangle, \quad (137)$$

where the last inequality follows since $\langle \psi_1 | \langle \psi_1 | \tilde{H}_{\text{in}} | \psi_1 \rangle | \psi_1 \rangle = 2\langle \psi_1 | H_{\text{in}} | \psi_1 \rangle$ and since $H_{\text{in}} \succeq 0$. Since $H_{\text{in}} + H_{\text{prop}}$ is a standard Feynman-Kitaev construction, it is known²⁵ (Lemma 3 of [GK12]) that its smallest non-zero eigenvalue scales as $\Omega(1/m^3)$. Moreover, for the null space of $H_{\text{in}} + H_{\text{prop}}$, since H_{in} requires time step $t = 0$ to be initialized to $|0 \cdots 0\rangle_{R_2} |0\rangle_{R_3} |0\rangle_{R_4}$, we have that *conditioned*

²⁵More accurately, Lemma 3 of [GK12] shows this lower bound for $H_{\text{in}} + H_{\text{prop}} + H_{\text{stab}}$, but the H_{stab} term is easily omitted while retaining the bound.

on any $|\psi_1\rangle$ on system L , $H_{\text{in}} + H_{\text{prop}}$ in system R has *unique* null vector

$$|\psi_{\text{hist}}\rangle = \frac{1}{\sqrt{m+1}} \sum_{t=0}^m V_t \cdots V_1 |0 \cdots 0\rangle_{R_2} |0\rangle_{R_3} |t\rangle_{R_4}, \quad (138)$$

with unitaries V_t for $t \in P$ defined as $V_t = U(a_t, b_t)$. (The uniqueness follows since there is no proof register in our setting, in contrast to the setting of the local Hamiltonian problem for QMA.) Note $|\psi_{\text{hist}}\rangle$ is *not* our desired history state $|\psi_{\text{hist}}(y)\rangle$ (Equation (66)), since unitaries $U(a_t, b_t)$ do not necessarily simulate the honest action of applying I or X for the proof bit at step t . (Step 3 will show, however, that $|\psi_{\text{hist}}\rangle$ is nevertheless “good enough”.)

Finally, we combine these observations to confirm $|\psi_1\rangle$ can be made close to $|\psi_{\text{hist}}\rangle$ for our choice of δ' . Write $|\psi_1\rangle = a|\psi_{\text{hist}}\rangle + b|\psi_{\text{hist}}^\perp\rangle$ for $|a|^2 + |b|^2 = 1$. Then, by Equation (137),

$$\frac{\zeta}{\delta'} \geq |b|^2 \langle \psi_{\text{hist}}^\perp | H_{\text{in}} + H_{\text{prop}} | \psi_{\text{hist}}^\perp \rangle \geq \frac{|b|^2 c}{m^3} \quad (139)$$

for some $c \in \Theta(1)$ (recall $H_{\text{in}} + H_{\text{prop}}$ has min non-zero eigenvalue $\Omega(1/m^3)$). Thus,

$$\| |\psi_1\rangle\langle\psi_1|^{\otimes 2} - |\psi_{\text{hist}}\rangle\langle\psi_{\text{hist}}|^{\otimes 2} \|_{\text{tr}} = 2\sqrt{1 - |\langle\psi_1|\psi_{\text{hist}}\rangle|^4} \leq 4\sqrt{\frac{m^3}{c\delta'}} \zeta \leq \frac{8}{\sqrt{c}} \frac{1}{M^{14}} =: \gamma_2, \quad (140)$$

where the second statement holds since $m^3\zeta/(c\delta') < 1$, and since $\Delta_{\text{in}} = \delta'$. One comment is important here: Above there is the subtlety that $|\psi_{\text{hist}}\rangle$ is conditioned on $|\psi_1\rangle$, so it would be more accurate to write $|\psi_{\text{hist}}(\psi_1)\rangle$. Thus, what the trace distance bound above shows is that any low-energy $|\psi_1\rangle$ (in the sense of Equation (127)) must be close to the history state $|\psi_{\text{hist}}(\psi_1)\rangle$ it defines.

Step 3. Why $|\psi_{\text{hist}}\rangle$ is good enough. We have shown that for any $t \in P$, there exist scalars $a_t, b_t \geq 0$, such that $|\psi_{\text{hist}}\rangle$ applies unitary $V_t = U(a_t, b_t)$ at time t . Recall that

$$U(a_t, b_t) = \frac{1}{\sqrt{a_t^2 + b_t^2}} (a_t iX + b_t I). \quad (141)$$

In the honest case, recall that for all $t \in P$ the history state would choose $|\psi_1\rangle$ on system L so that either $a_t = 0$ and $b_t = 1$ (corresponding to streaming proof bit 0 in step t) or $a_t = 1$ and $b_t = 0$ (corresponding to streaming proof bit 1 in step t). We now argue that for any low-energy $|\psi_{\text{hist}}\rangle$, this must *approximately* hold.

First, by Equations (124), (140), the Hölder inequality, and the triangle inequality, for all \tilde{H}_t ,

$$\langle \psi_{\text{hist}} | \langle \psi_{\text{hist}} | \Delta_{\text{prop}} \tilde{H}_t | \psi_{\text{hist}} \rangle | \psi_{\text{hist}} \rangle \leq \beta' + (\gamma_1 + \gamma_2) \|\Delta_{\text{prop}} \tilde{H}_t\|_\infty \leq \beta' + 2\Delta_{\text{prop}} (\gamma_1 + \gamma_2), \quad (142)$$

where the last statement holds since $\|\tilde{H}_t\|_\infty \leq 2$. But

$$\langle \psi_{\text{hist}} | \langle \psi_{\text{hist}} | \Delta_{\text{prop}} \tilde{H}_t | \psi_{\text{hist}} \rangle | \psi_{\text{hist}} \rangle = 2\Delta_{\text{prop}} \langle \psi_{\text{hist}} | H_t^I | \psi_{\text{hist}} \rangle \langle \psi_{\text{hist}} | H_t^{iX} | \psi_{\text{hist}} \rangle \quad (143)$$

$$= \frac{8\Delta_{\text{prop}}}{(m+1)^2} \left(1 - \frac{b_t}{\sqrt{a_t^2 + b_t^2}} \right) \left(1 - \frac{a_t}{\sqrt{a_t^2 + b_t^2}} \right). \quad (144)$$

Assume without loss of generality that $b_t \geq a_t \geq 0$. Then, combining Equation (142) with Equation

(144) and rearranging yields

$$\frac{b_t}{\sqrt{a_t^2 + b_t^2}} \geq 1 - \frac{m+1}{2} \sqrt{\frac{\beta'}{2\Delta_{\text{prop}}} + \gamma_1 + \gamma_2} =: 1 - \varepsilon \quad (145)$$

for $\varepsilon \geq 0$, where our parameter choices ensure $\varepsilon \ll 1$. From this, we also conclude

$$\frac{a}{\sqrt{a^2 + b^2}} \leq \sqrt{1 - (1 - \varepsilon)^2} \leq \sqrt{2\varepsilon}. \quad (146)$$

We conclude that when $b \geq a$, it must be that $|\psi_{\text{hist}}\rangle$ applied a unitary close to I at time t , i.e.

$$\|U(a_t, b_t) - I\|_\infty = \left\| \frac{a}{\sqrt{a^2 + b^2}} iX + \left(\frac{b}{\sqrt{a^2 + b^2}} - 1 \right) I \right\|_\infty \leq \sqrt{2\varepsilon} + \varepsilon \leq 4\sqrt{\varepsilon}, \quad (147)$$

where the second statement follows since $1 \geq b/(\sqrt{a^2 + b^2}) \geq 1 - \varepsilon$, and the last since $\varepsilon \leq \sqrt{2\varepsilon}$ for small ε . An essentially identical calculation shows that in the complementary case when $a_t \geq b_t \geq 0$, $\|U(a_t, b_t) - iX\|_\infty \leq 4\sqrt{\varepsilon}$. (Note $a_t = b_t$ is impossible, as otherwise Equation (145) yields a contradiction for small ε .)

Finally, recalling the definition of $|\psi_{\text{hist}}\rangle$ from Equation (138), we “round” to an *honest* circuit $V' = V'_m \cdots V'_1$ as follows. For $t \notin P$, set $V'_t = V_t$, and for $t \in P$, set $V'_t = I$ if $b_t > a_t$ and $V'_t = iX$ if $b_t < a_t$. Then, for all $t \in [m]$, we have $\|V_t - V'_t\|_\infty \leq 4\sqrt{\varepsilon}$, from which we conclude via standard bounds that

$$\|V_m \cdots V_1 - V'_m \cdots V'_1\|_\infty \leq 4m\sqrt{\varepsilon} = 4m \sqrt{\frac{m+1}{2} \sqrt{\frac{\beta'}{2\Delta_{\text{prop}}} + \gamma_1 + \gamma_2}} =: \gamma_3. \quad (148)$$

There is a minor subtlety we should clarify at this point. By construction, for any $t \in P$, V' applies either I or iX , as desired. Then, Definition 2.1 has the additional structure that each $W_i \in \{I, X\}$ in a “compute” proof phase (Step 1(b)i) is subsequently undone by a matching $W_i^\dagger \in \{I, X\}$ in the “uncompute” proof phase (Step 1(b)iii). Let $t, t+2 \in P$ be an arbitrary pair of such “compute” and “uncompute” steps, respectively. Then, our construction only enforces that $V'_t, V'_{t+2} \in \{I, iX\}$, but not that $V'_{t+2} = (V'_t)^\dagger$. However, this is without loss of generality, since any streaming proof which deviates from this pattern can easily be simulated by a “proper” streaming proof without increasing the proof length²⁶. Thus, deviating from this pattern cannot increase the best acceptance probability over all streamed proofs y .

Step 4: The contradiction. Recall that (Q_n, x) is a NO case, and that we have assumed, for sake of contradiction, that $\langle \psi_1 | \langle \psi_2 | \tilde{H} | \psi_1 \rangle | \psi_2 \rangle \leq \beta'$. The former implies that for any streaming proof y , Q_n accepts with probability at most β . But V' is by construction the verifier Q_n , except with all gates at times $t \in P$ “rounded” to the closest gate in $\{iX, I\}$. Thus, V' simulates Q_n on *some* streaming proof y , implying V' also accepts with probability at most β . Since $|\psi_{\text{hist}}\rangle$ encodes circuit V with

²⁶For example, suppose at step t and $t+2$, V' applies iX and I . This corresponds to classically streaming bit 1 in step t , but not uncomputing register R_3 from $|1\rangle$ back to $|0\rangle$ in step $t+2$. Logically, this just has the effect of negating the standard basis, so that when the next proof bit is streamed, iX and I now correspond to streaming bits 0 and 1, respectively (as opposed to 1 and 0).

$\|V - V'\|_\infty \leq \gamma_3$ (Equation (148)), we conclude that $\Pr(V \text{ accepts}) \leq \beta + \gamma_3$. Thus,

$$\langle \psi_{\text{hist}} | \langle \psi_{\text{hist}} | \tilde{H}_{\text{out}} | \psi_{\text{hist}} \rangle | \psi_{\text{hist}} \rangle = 2 \langle \psi_{\text{hist}} | H_{\text{out}} | \psi_{\text{hist}} \rangle \geq 2 \frac{1 - \beta}{m + 1} - \frac{2\gamma_3}{m + 1}, \quad (149)$$

which by the Hölder inequality, Equation (124), and Equation (140) implies

$$\beta' \geq \langle \psi_1 | \langle \psi_2 | \tilde{H} | \psi_1 \rangle | \psi_2 \rangle \geq \langle \psi_1 | \langle \psi_2 | \tilde{H}_{\text{out}} | \psi_1 \rangle | \psi_2 \rangle \geq 2 \frac{1 - \beta}{m + 1} - \frac{2\gamma_3}{m + 1} - 2(\gamma_1 + \gamma_2), \quad (150)$$

where we have used $\|\tilde{H}_{\text{out}}\|_\infty = 2$. Combining Equation (150) with Equations (124), (140), and (148), we obtain $\frac{2\gamma_3}{m+1} + 2(\gamma_1 + \gamma_2) < 1/M = \frac{1}{(m+1)2^{r+1}}$, obtaining the desired contradiction. \square

7 Applications of the Embedding Lemma

In this section, we apply the Embedding Lemma (Lemma 6.1) to obtain various corollaries. These proceed in two steps. Section 7.1 first reduces problems from various complexity classes into instances of Separable Sparse Hamiltonian (SSH). Section 7.2 then shows how the exact structure of the SSH instances from Lemma 6.1 can be exploited to obtain various upper bounds of form $\text{QMA}(2, p, q, r)$ for appropriate p, q, r .

7.1 Reductions to Separable Sparse Hamiltonian (SSH)

The first corollary is immediate by recalling that without loss of generality, a SQCMASPACE circuit has $m \in \Theta(2^p)$.

Corollary 7.1. *There exists a poly-time many-one reduction from any SQCMASPACE(p, q, r) instance to an instance of Separable Sparse Hamiltonian on $O(q + \log p)$ qubits with promise gap $\Omega(2^{-p-r})$.*

The second corollary requires slightly more work, but still goes by combining Lemma 6.1 with completely standard techniques.

Corollary 7.2. *There exists a poly-time many-one reduction from any MIP(t, u, v, p, r, c, s) protocol to an instance of Separable Sparse Hamiltonian on $O(u + v + \log(tr \log(pt)))$ qubits with promise gap scaling as $\Omega\left([2^{tr \log(pt)}(c - s)]^{-1}\right)$.*

Proof sketch. Apply the standard trick of concatenating, for all possible sequences of questions from the verifier V to the provers, the corresponding sequence of all answers from the provers. This will be the proof y to be streamed, and it has length $|y| = pt2^{tr}$. Without loss of generality, we may assume y first records, in order, all possible answers from the provers to the verifier's first round of questions (call this "block 1"), followed by all possible answers from the provers to the second round of questions ("block 2"), etc. (Note the questions in a given round can depend on the answers from all previous rounds). Thus, given streaming access to y , a SQCMASPACE verifier Q can straightforwardly simulate V as follows: For each round t of the MIP protocol, Q simulates V to select its questions. It then streams block t of y , storing only the answers to the questions selected for round t . It then proceeds to round $t + 1$.

Let us analyze Q 's parameters. First, note that Q 's time complexity increases to $\Theta(|y|) \in \Theta(pt2^{tr})$ —this is because under Definition 2.1, Q 's total gate count also counts the gates used to stream bits, of which there are $|y|$. For clarity, of these $\Theta(pt2^{tr})$ time steps, only $\text{poly}(n)$ are used

to simulate computation steps of V . We now discuss space complexity. For this, random bits can be simulated via the principle of deferred measurement [NC11]. This requires the use of a fresh ancilla qubit for each measurement. Since V uses v random bits, and u ancilla space, Q 's overall space requirement is $u + v$. Summarizing, for this construction, Q has parameters $p = tr \log(pt)$, $q \in O(u + v)$, $r \in O(\log(c - s))$. Applying Lemma 6.1 to Q now yields the claim. \square

From Corollary 7.2, we immediately obtain the following, since $\text{NP} \subseteq \text{MIP}(\log, \log, 2, 1, 1, 1 - 1/\text{poly}(n))$ (Section 2).

Corollary 7.3. *Any instance of an NP language is reducible under poly-time many-one reductions to an instance of Sparse Separable Hamiltonian on $O(\log(n))$ qubits with completeness 1 and soundness $1 - 1/\text{poly}(n)$.*

Note the completeness 1 arises since $\alpha' = 0$ in the proof of Lemma 6.1 (since the MIP has completeness 1 and thus $\alpha = 1$), implying the history state is a null state of \tilde{H} . Observe the instance of Sparse Separable Hamiltonian above can be decided in NP, since it acts on $\log n$ qubits (and so the NP verifier can explicitly write out the matrix for the Hamiltonian). The corollary for NEXP follows analogously by recalling $\text{NEXP} = \text{MIP}(\text{poly}, \text{poly}, \text{poly } 2, 1, 1, 2^{-r})$ for any desired polynomial r (Theorem 2.11).

Corollary 7.4. *Any instance of a NEXP language is reducible under poly-time many-one reductions to an instance of Sparse Separable Hamiltonian on $O(\text{poly}(n))$ qubits with completeness 1 and soundness $1 - 1/\exp(n)$.*

As was the case for NP above, here the instance of Sparse Separable Hamiltonian is decidable in NEXP, since it acts on $\text{poly}(n)$ qubits.

7.2 Containment in $\text{QMA}(2, p, q, r)$

Next, by combining Corollary 7.1, Corollary 7.2, Corollary 7.3 and Corollary 7.4 with the following lemma, we immediately obtain containment in $\text{QMA}(2, p, q, r)$ for various appropriate p, q, r .

Lemma 7.5. *Assume the notation of Lemma 6.1, and let \tilde{H} be the Sparse Separable Hamiltonian instance produced by the latter. Then, \tilde{H} can be decided in²⁷ $\text{QMA}(2, q + \log m, q + \log m, r \log m)$, i.e. with proof and ancilla space scaling as $O(q + \log m)$, and promise gap as $O(1/(2^r m))$.*

In words, the $\text{QMA}(2)$ verifier preserves (up to linear overhead) both the number of qubits \tilde{H} acts on and its promise gap. The proof of Lemma 7.5 exploits the structure of the Hamiltonian produced by the Embedding Lemma, together with standard ideas. Curiously, at present we do *not* know²⁸ how to show the analogue of Lemma 7.5 for *arbitrary* sparse Hamiltonians (i.e. satisfying Definition 2.4 and having worst-case exponential norm, but not promised to be of the form produced by Lemma 6.1).

Proof of Lemma 7.5. We will need to explicitly reference the definitions below, reproduced for

²⁷Recall from Remark 2.3 that we omit big-Oh notation when listing class parameters, including for QMA.

²⁸Briefly, a natural approach is via phase estimation, as done in [CS12] for $\text{QMA}(2)$. However, the issue is that phase estimation requires exponential time in general to obtain exponential precision, which may be required in our setting since the weights $\Delta_{\text{in}}, \Delta_{\text{prop}}, \Delta_{\text{sym}}$ scale as $\text{poly}(m)$, which can be exponential in n .

convenience:

$$\tilde{H}_{\text{in}} = (H_{\text{in}})_L \otimes I_R + I_L \otimes (H_{\text{in}})_R \quad (151)$$

$$\tilde{H}_{\text{prop}} = \sum_{t=1}^m \tilde{H}_t, \quad \text{where } \tilde{H}_t \text{ is defined as} \quad (152)$$

$$\tilde{H}_t = \begin{cases} (H_t^I)_L \otimes (H_t^{iX})_R + (H_t^{iX})_L \otimes (H_t^I)_R & \text{if } t \in P \\ (H_t)_L \otimes I_R + I_L \otimes (H_t)_R & \text{if } t \notin P \end{cases} \quad (153)$$

$$\tilde{H}_{\text{out}} = (H_{\text{out}})_L \otimes I_R + I_L \otimes (H_{\text{out}})_R \quad (154)$$

$$\tilde{H}_{\text{sym}} = I - P_{LR}^{\text{sym}} \quad \text{for} \quad P_{LR}^{\text{sym}} = \frac{1}{2} \left(I_{LR} + \sum_{xy} |xy\rangle\langle yx|_{LR} \right), \quad (155)$$

The relevant facts regarding $\tilde{H} = \Delta_{\text{in}}\tilde{H}_{\text{in}} + \Delta_{\text{prop}}\tilde{H}_{\text{prop}} + \Delta_{\text{sym}}\tilde{H}_{\text{sym}} + \tilde{H}_{\text{out}}$ are:

1. n is the input size to circuit Q_n , and all functions m, p, q, r are parameterized in terms of n .
2. $\Delta_{\text{in}}, \Delta_{\text{prop}}, \Delta_{\text{sym}}$ are fixed polynomials in m (the number of gates in the circuit Q_n , where recall $m \geq 2^p$ by assumption to allow enough time to process the full streamed proof),
3. \tilde{H} acts on $O(q + \log m)$ qubits,
4. the promise gap scales as $|\alpha' - \beta'| \in \Omega((m2^r)^{-1})$ (recall Q_n had promise gap 2^{-r}), and
5. in the YES case, there exists $|\psi_1\rangle_L |\psi_2\rangle_R$ such that $\langle \psi_1 | \langle \psi_2 | \tilde{H} | \psi_1 \rangle_L | \psi_2 \rangle_R \leq \alpha'$, and in the NO case, for all $|\psi_1\rangle_L |\psi_2\rangle_R$, $\langle \psi_1 | \langle \psi_2 | \tilde{H} | \psi_1 \rangle_L | \psi_2 \rangle_R \geq \beta'$.

We construct a QMA(2, $q + \log m, q + \log m, r \log m$) verifier V deciding whether \tilde{H} is a YES or NO instance.

Constructing V . We use Kitaev's original approach for placing the k -local Hamiltonian problem in QMA [KSV02, Proposition 14.2]: Pick a random ‘‘term’’ (defined shortly) of \tilde{H} and measure it against the claimed proof $|\psi\rangle = |\psi_1\rangle_L |\psi_2\rangle_R$. The catch is that unlike in [KSV02], the ‘‘terms’’ of \tilde{H} are not k -local, so a slight bit more work is required to ensure V can implement these measurements.

To begin, define the ‘‘terms’’ of \tilde{H} as precisely the set of summands (with appropriate weights) on Equation (151) (e.g. $\Delta_{\text{in}} H_{\text{in}} \otimes I$ is a term), Equation (153) (e.g. for any $t \in P$, $\Delta_{\text{prop}} H_t^I \otimes H_t^{iX}$ and $\Delta_{\text{prop}} H_t^{iX} \otimes H_t^I$ are each terms), and Equation (154) (e.g. $I \otimes H_{\text{out}}$), as well as $\Delta_{\text{sym}} \tilde{H}_{\text{sym}}$. Then, there are precisely $K := 2m + 5$ terms, on which we fix an arbitrary ordering. By construction, for all $i \in \{1, \dots, K\}$, each term is a projector Π_i up to scaling w_i — for example, since H_t^I and H_t^{iX} are projectors, so is $\Delta_{\text{prop}} H_t^I \otimes H_t^{iX}$ up to scaling Δ_{prop} . (In this case, $\Pi_i = H_t^I \otimes H_t^{iX}$ and $w_i = \Delta_{\text{prop}}$.)

We thus write $\tilde{H} = \sum_{i=1}^K w_i \Pi_i$ with $0 \leq w_i \leq \text{poly}(m)$, and define total weight $W := \sum_{i=1}^K w_i$. V now acts as follows given proof $|\psi\rangle = |\psi_1\rangle_L |\psi_2\rangle_R$:

1. Randomly select index $i \in [K]$ with probability $p_i = w_i/W$.
2. Apply two-outcome projective measurement $M_0 := \Pi_i, M_1 := I - \Pi_i$ to $|\psi\rangle$.
3. Accept on outcome M_1 , reject on outcome M_0 .

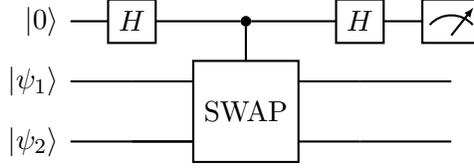


Figure 5: The circuit for the SWAP test. The SWAP gate has action $|x\rangle|y\rangle \mapsto |y\rangle|x\rangle$ for any standard basis states $|x\rangle, |y\rangle$. Note the inputs $|\psi_1\rangle$ and $|\psi_2\rangle$ are in tensor product. Measuring the first wire in the standard basis yields output 0 with probability $(1 + |\langle\psi_1|\psi_2\rangle|^2)/2$, and postselecting on 0 projects $|\psi_1\rangle|\psi_2\rangle$ onto the symmetric subspace.

The probability that V accepts $|\psi\rangle$ is

$$\Pr[V \text{ accepts } |\psi\rangle] = \sum_{i=1}^K p_i \langle\psi|(I - \Pi_i)|\psi\rangle = 1 - \frac{1}{W} \langle\psi|\tilde{H}|\psi\rangle. \quad (156)$$

Therefore, V accepts with probability at least $1 - \alpha'/W$ in the YES case and at most $1 - \beta'/W$ in the NO case. Thus, V has promise gap

$$\frac{\beta' - \alpha'}{W} \in \Omega\left(\frac{1}{m2^r} \cdot \frac{1}{\text{poly}(m)}\right) \in \Omega\left(\frac{1}{2^r \text{poly}(m)}\right), \quad (157)$$

where we have used the fact that $W \in \text{poly}(m)$ (since there are $2m + 5$ terms, each with weight $w_i \in \text{poly}(m)$).

Efficiency of V . It remains to argue that V can be implemented efficiently, which in our setting means using $O(q + \log m)$ ancilla qubits and $\text{poly}(n)$ gates. For Step 1 of V (picking random $i \in [K]$), here is one approach to sample from distribution $\{p_i\}$ efficiently: Choose $j \in \{1, \dots, W\}$ uniformly at random, where recall $W \in \text{poly}(m) \in \exp(n)$ in the worst case. Then, compute the smallest $K' \in [K]$ such that $j \leq \sum_{i=1}^{K'} w_i$, and output K' . Both steps can clearly be done with $O(\log m)$ qubits, and K' can be computed in time $\text{poly}(n)$ since there are only a constant number of distinct weight values w_i in our construction.

As for Step 2 (projective measurements), the simplest measurement corresponds to term \tilde{H}_{sym} , for which $M_1 = P^{\text{sym}}$ (i.e. the projector onto the symmetric subspace) and $M_0 = I - M_1$. This measurement is well-known to be efficiently implemented by the SWAP test [BCWW01] (Figure 5), which outputs 0 with probability $\langle\psi_1|\langle\psi_2|P^{\text{sym}}|\psi_1\rangle|\psi_2\rangle = (1 + |\langle\psi_1|\psi_2\rangle|^2)/2$. The SWAP test clearly uses $O(q + \log m)$ qubits and is computable in time $\text{poly}(n)$.

As for the remaining terms of \tilde{H} , we show how to efficiently implement for $t \notin P$ the measurement corresponding to projector H_t from Equation (70):

$$H_t := -\frac{1}{2}V_t \otimes |t\rangle\langle t-1|_{R_4} - \frac{1}{2}V_t^\dagger \otimes |t-1\rangle\langle t|_{R_4} + \frac{1}{2}I \otimes (|t\rangle\langle t| + |t-1\rangle\langle t-1|)_{R_4} \quad (158)$$

The measurement of all remaining terms then follows similarly. Above, recall that V_t is a 2-qubit unitary, but the problem is the clock register R_4 , which requires $O(\log m)$ qubits, which can scale as $\text{poly}(n)$ in the worst case. However, this is easy to overcome — informally, V efficiently applies a change of basis to R_4 to map $|t-1\rangle$ and $|t\rangle$ to $|0\rangle$ and $|1\rangle$ (expressed in binary), respectively. Since the latter two differ only on their least significant bit, we have that under the change of basis,

H_t can be implemented by a 3-local measurement (two qubits for V_t , one qubit for the clock).

More formally, define permutation U which swaps $|t-1\rangle_{R_4}$ with $|0\rangle_{R_4}$, swaps $|t\rangle_{R_4}$ with $|1\rangle_{R_4}$, and otherwise acts invariantly on any $|x\rangle_{R_4}$ for $x \notin \{0, 1, t-1, t\}$. This permutation can clearly be implemented efficiently classically (and thus quantumly) with linear overhead space overhead, and in $\text{poly}(n)$ time. Let B denote the last qubit of R_4 , and A all other qubits of R_4 . Then, expanding R_4 out in binary:

$$\begin{aligned} UH_tU^\dagger &= |0 \cdots 0\rangle\langle 0 \cdots 0|_A \otimes \left(-\frac{1}{2}V_t \otimes |1\rangle\langle 0|_B - \frac{1}{2}V_t^\dagger \otimes |0\rangle\langle 1|_B + \frac{1}{2}I \otimes (|1\rangle\langle 1| + |0\rangle\langle 0|)_B \right) \\ &=: |0 \cdots 0\rangle\langle 0 \cdots 0|_A \otimes H'_t. \end{aligned}$$

A measurement corresponding to projector UH_tU^\dagger can be efficiently implemented, since H'_t is now a 3-qubit operator. (For example, V first measures A in the standard basis, and conditioned on obtaining outcome $|0 \cdots 0\rangle_A$, measures H'_t .) Thus, V applies UH_tU^\dagger to $U|\psi\rangle$ to complete the measurement. Again, each of these takes $O(q + \log m)$ space and $\text{poly}(n)$ time, as required. \square

With Lemma 7.5 in hand, the following corollary is immediate, and recovers the results of Blier and Tapp [BT12] for NP and Pereszlényi for NEXP [Per12]. Below, recall $\text{PQMA}_{\log}(2) = \text{QMA}(2, \log n, \log n, \log n)$, i.e. $\text{QMA}(2)$ with log-size proof and ancilla and $1/\text{poly}$ promise gap (technically, $\text{PQMA}_{\log}(2)$ also has perfect completeness by definition, which also matches the result we obtain below).

Corollary 7.6. $\text{NP} = \text{PQMA}_{\log}(2)$ (cf. [BT12]) and $\text{NEXP} = \text{PreciseQMA}(2)$ (cf. [Per12]).

Proof. The containments $\text{PQMA}_{\log}(2) \subseteq \text{NP}$ and $\text{PreciseQMA}(2) \subseteq \text{NEXP}$ are trivial. The containment $\text{NP} \subseteq \text{PQMA}_{\log}(2)$ follows by mapping NP to a log-size SSH instance via Corollary 7.3, followed by application of Lemma 7.5 to verify the SSH instance in $\text{QMA}(2, \log n, \log n, \log n) = \text{PQMA}_{\log}(2)$. $\text{NEXP} \subseteq \text{PreciseQMA}(2)$ follows analogously by combining Corollary 7.4 with Lemma 7.5. \square

Via analogous arguments, we also obtain the following immediate corollaries.

Corollary 7.7. $\text{SQCMASPACE}(p, q, r) \subseteq \text{QMA}(2, q + \log p, q + \log p, p + r)$.

In words, SQCMASPACE with proof length 2^p , q ancilla qubits, and promise gap $1/2^r$ is contained in $\text{QMA}(2)$ with $q + \log p$ proof and ancilla qubits, and promise gap $1/2^{p+r}$.

Corollary 7.8. *It holds that*

$$\text{MIP}(t, u, v, p, r, c, s) \subseteq \text{QMA}(2, u + v + \log(tr \log(pt)), u + v + \log(tr \log(pt)), tr \log(pt) + \log(c - s)). \quad (159)$$

In words, MIP with t bits of communication per round, space u, v random bits, p provers, r rounds, and completeness/soundness c and s , respectively, is contained in $\text{QMA}(2)$ with $u + v + \log(tr \log(pt))$ proof and ancilla qubits, and promise gap $2^{-tr \log(pt) + \log(c-s)}$. In more words, the amount of space is preserved, and the promise gap depends exponentially on the total amount of communication but only polynomially on the MIP promise gap.

Acknowledgements

We thank Rolando Somma for pointing us to [CBC21] and for interesting discussions, and Chinmay Nirke for feedback on this manuscript. SG acknowledges support from DFG grants 450041824 and 432788384.

A GSCON_{exp} is PSPACE-hard

During the proof of Theorem 2.16, [GS18, Lemma 5.2] shows the following result, which we restate here for completeness.

Lemma A.1. *Let $A \in \text{Herm}(\mathcal{B}^{\otimes n})$ be a k' -local Hamiltonian. Consider the following promise problem Π' .*

YES: There exists a sequence $(U_i)_{i=1}^{m'}$ of l -local unitaries such that $\langle \psi_A | A | \psi_A \rangle \leq \alpha$ for $|\psi_A\rangle = U_{m'} \cdots U_1 |0\rangle^{\otimes n}$.

NO: $\lambda_{\min}(A) \geq \beta$.

Π' is polynomial-time reducible to GSCON with $m = 2m' + 2$, $\eta_1 = \alpha$, $\eta_2 = \beta/(16m^2)$, $\eta_3 = 0$, $\eta_4 = 1/4$, $l = 2$, $k = k' + 2$, $\Delta = \eta_2 - \eta_3$, if $\Delta > 0$.

Proof. The basic idea is to construct a Hamiltonian H by adding three ‘‘GO’’ qubits to A , such that traversing the low energy space of H forces one to simulate a protocol, which first prepares state $|\psi_A\rangle$ using local gates, then checks that $|\psi_A\rangle$ is indeed low energy, and finally uncomputes $|\psi_A\rangle$.

Define $H \in \text{Herm}(\mathcal{B}^{\otimes(n+3)})$ acting on a *Hamiltonian* register h and *GO* register G :

$$H := A_h \otimes P_G, \quad P := I - |000\rangle\langle 000| - |111\rangle\langle 111|$$

H is k -local, as P can be written 2-locally [GS18]. The initial and final states are defined as $|\psi\rangle := |0\rangle^{\otimes n} |0\rangle^{\otimes 3}$ and $|\phi\rangle := |0\rangle^{\otimes n} |1\rangle^{\otimes 3}$. $\Pi = (H, \eta_1, \eta_2, \eta_3, \eta_4, \Delta, l, m, |\psi\rangle, |\phi\rangle)$ is now a valid instance of GSCON, and can be computed in polynomial time.

Correctness: Suppose Π' is a YES instance, i.e. there exists a sequence $(U_i)_{i=1}^{m'}$ of l -local unitaries, such that $\langle \psi_A | A | \psi_A \rangle \leq \alpha$ for $|\psi_A\rangle = U_{m'} \cdots U_1 |\psi\rangle$. We show that Π is also a YES instance by constructing a sequence $(V_i)_{i=1}^m$ of l -local unitaries, such that $|\phi\rangle = V_m \cdots V_1 |\psi\rangle$ and $\langle \psi_i | H | \psi_i \rangle \leq \eta_1$ with $|\psi_i\rangle := V_i \cdots V_1 |\psi\rangle$ for all $i \in [m]$. $V_m \cdots V_1$ implement the following steps:

1. *Prepare $|\psi_A\rangle$:* Apply $(U_{m'} \cdots U_1)_h$.
2. *Begin checking $|\psi_A\rangle$:* Apply $(X \otimes X \otimes I)_G$.
3. *Finish checking $|\psi_A\rangle$:* Apply $(I \otimes I \otimes X)_G$.
4. *Uncompute $|\psi_A\rangle$:* Apply $(U_1^\dagger \cdots U_{m'}^\dagger)_h$.

This sequence has length $m = 2m' + 2$ and maps $|\psi\rangle$ to $|\phi\rangle$ as desired. All intermediate states (besides the state after Step 2) $|\psi_i\rangle$ are in the nullspace of H , as P maps their register G to 0. After Step 2, we have state $|a_2\rangle = |\psi_A\rangle_h |110\rangle_G$. By assumption, it holds that $\langle a_2 | H | a_2 \rangle = \langle \psi_A | A | \psi_A \rangle \leq \alpha = \eta_1$.

Soundness: Suppose Π' is a NO instance. Let S and T be the image of projectors $I_h \otimes |000\rangle\langle 000|_G$ and $I_h \otimes |111\rangle\langle 111|_G$, respectively. S, T are 2-orthogonal, and $|\psi\rangle \in S$, $|\phi\rangle \in T$. Now fix any sequence $(V_i)_{i=1}^m$ of two-qubit unitaries. If $\| |\psi_m\rangle - |\phi\rangle \|_2 \geq 1/4 = \eta_4$, Π is already a NO instance for GSCON. Otherwise, we can apply the Traversal Lemma (Lemma 5.9) with $\varepsilon = 1/4$ to conclude that there exists an $i \in [m]$, such that

$$\langle \psi_i | P' | \psi_i \rangle \geq \left(\frac{1}{4m} \right)^2 = \frac{\eta_2}{\beta},$$

where $|\psi_i\rangle := V_i \cdots V_1 |\psi\rangle$ and $P' = I - \Pi_S - \Pi_T = I_h \otimes P_G$. Then, Π is a NO instance because

$$\langle \psi_i | H | \psi_i \rangle = \langle \psi_i | A \otimes P | \psi_i \rangle \geq \beta \langle \psi_i | I_h \otimes P | \psi_i \rangle = \beta \langle \psi_i | P' | \psi_i \rangle \geq \eta_2,$$

where the first inequality follows since, by assumption, $\lambda_{\min}(A) \geq \beta$. \square

To prove PSPACE-hardness, we combine Lemma A.1 with two further insights. Firstly, $2^{r(n)}$ unitary 2-local gates are sufficient to construct any state $|\psi\rangle \in \mathcal{B}^n$ exactly (starting in $|0^n\rangle$) for some polynomial r [NC11]. Therefore, the problem Π defined in Lemma A.1 is equivalent to the problem whether $\lambda_{\min}(H) \leq \alpha$ or $\lambda_{\min}(H) \geq \beta$ for $m' = 2^{r(n)}$.

Secondly, QMA with an inverse exponential promise gap (i.e. $c - s = 2^{-\text{poly}(n)}$), denoted PreciseQMA, was shown by Fefferman and Lin to be PSPACE-complete [FL18]. They also show that k -LH with inverse exponential gap, denoted *precise* k -LH, is PSPACE-complete. Their construction leads to the following lemma, which allows us to reduce PSPACE to a precise k -LH instance with thresholds α and β , such that we can apply Lemma A.1 to solve it in $\text{GSCON}_{\text{exp}}$ with $m = 2^{r(n)}$.

Lemma A.2. *Any problem Π in PSPACE is poly-time reducible to a k -LH instance with $\beta/\alpha \geq 2^{p(n)}$ with $\alpha \geq 2^{-\text{poly}(n)}$, where $p(n)$ is a freely chosen polynomial.*

Proof. This proof is based on [FL18, Theorem 24]. Π can be reduced to PreciseQMA with completeness c and soundness s , such that

$$1 - c = \varepsilon, \quad 1 - s = -\varepsilon + 2^{-g(n)},$$

for some polynomial $g(n)$ depending on Π and any $\varepsilon = 2^{-q(n)}$ for some polynomial $q(n)$ of our choice [FL18].

The corresponding PreciseQMA verifier uses $T \leq h(n, \log(1/\varepsilon))$ unitaries, for some polynomial $h(x, y)$ [FL18]. Hence, PreciseQMA with completeness c and soundness s can be reduced to a 3-local Hamiltonian instance with thresholds

$$\alpha = \frac{1 - c}{T + 1} = \frac{\varepsilon}{T + 1}, \quad \beta = \frac{1 - s}{T^3} = \frac{2^{-g(n)} - \varepsilon}{T^3}.$$

We can then choose a polynomial $q(n) \geq 2g(n)$ such that

$$\frac{\beta}{\alpha} = \frac{T + 1}{T^3} \frac{2^{-g(n)} - \varepsilon}{\varepsilon} \geq T^{-2} \varepsilon^{-1} 2^{-g(n)-1} = \frac{2^{q(n)-g(n)-1}}{h^2(n, q(n))} \geq 2^{p(n)}.$$

\square

Theorem A.3. $\text{GSCON}_{\text{exp}}$ is PSPACE-hard.

Proof. Follows directly from Lemmas A.1 and A.2 and the above discussion. \square

B SEPARABLE SPARSE $\text{GSCON}_{\text{exp}}$ IS NEXP-COMPLETE

Based on the fact that the separable sparse Hamiltonian problem is NEXP-complete (Corollary 7.4), we introduce a variant of GSCON that is also NEXP-complete.

Definition B.1 (Separable sparse ground state connectivity). SEPARABLE SPARSE GSCON is defined as GSCON, but the input $H \in \text{Herm}(\mathcal{B}^{\otimes n})$ is a sparse Hamiltonian (instead of a local one)

with a bipartition (L, R) of the qubits H acts on, and every unitary U_1, \dots, U_m acts either on L or on R . SEPARABLE SPARSE GSCON_{exp} is defined analogously.

An analogue of Lemma A.1 for the separable sparse case also follows from the proof of [GS18, Lemma 5.2].

Lemma B.2. *Let $A \in \text{Herm}(\mathcal{B}^{\otimes n})$ be a sparse Hamiltonian with a bipartition of the qubits A acts on into (L, R) . Consider the following promise problem Π' .*

YES: *There exists a sequence $(U_i)_{i=1}^{m'}$ of l -local unitaries acting either on L or R such that $\langle \psi_A | A | \psi_A \rangle \leq \alpha$ for $|\psi_A\rangle = U_{m'} \cdots U_1 |0\rangle^{\otimes n}$.*

NO: *For all unit vectors $|\psi_1\rangle, |\psi_2\rangle$, $\langle \psi_1 | L \langle \psi_2 | R A | \psi_1 \rangle_L | \psi_2 \rangle_R \geq \beta$.*

Π' is polynomial-time reducible to SEPARABLE SPARSE GSCON with $m = 2m' + 2$, $\eta_1 = \alpha$, $\eta_2 = \beta/(16m^2)$, $\eta_3 = 0$, $\eta_4 = 1/4$, $l = 2$, $\Delta = \eta_2 - \eta_3$, if $\Delta > 0$.

Proof. The proof is almost the same as Lemma A.1, so we just mention the differences. Here, Π is a SEPARABLE SPARSE GSCON instance. We choose the bipartition as $(L, R + G)$, i.e. the gates may act on R and G simultaneously. In the end, we need to show $\langle \psi_i | H | \psi_i \rangle \geq \eta_2$. For that, note that $|\psi_i\rangle$ is a product state of the form $|\psi_i\rangle = |\gamma_1\rangle_L \otimes |\gamma_2\rangle_{RG}$, where we can further decompose

$$|\gamma_2\rangle = \sum_{x \in \{0,1\}^3} a_x |\gamma_{2,x}\rangle_R |x\rangle_G.$$

Then for $X := \{0,1\}^3 \setminus \{000, 111\}$,

$$\langle \psi_i | H | \psi_i \rangle = \sum_{x \in X} |a_x|^2 \langle \gamma_1 | \langle \gamma_{2,x} | A | \gamma_1 \rangle | \gamma_{2,x} \rangle \geq \sum_{x \in X} |a_x|^2 \beta = \beta \langle \psi_i | P' | \psi_i \rangle \geq \eta_2,$$

where the first inequality holds by assumption of the NO case. □

Theorem B.3. SEPARABLE SPARSE GSCON_{exp} is NEXP-complete.

Proof. Containment of SEPARABLE SPARSE GSCON_{exp} in NEXP is trivial. Hardness is analogous to Theorem A.3 using Lemma B.2 and Corollary 7.4. □

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