Relativistic GKLS master equation?

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The celebrated GKLS master equation, widely called just Lindblad equation, is the universal dynamical equation of non-relativistic open quantum systems in their Markovian approximation. It is not necessary and perhaps impossible that GKLS equations possess sensible relativistic forms. In a lucid talk on black hole information loss paradox, David Poulin conjectured a Lorentz invariant GKLS master equation. It remained unpublished. Poulin passed away at heights of his activity. But the equation is really puzzling. A closer look uncovers a smartly hidden defect which leaves us without Lorentz invariant Markovian master equations. They, in view of the present author, should not exist.

I. INTRODUCTION

Dissipative relativistic phenomena are real. A natural example are pions. If we regard the dynamics of pions in itself, it is relativistic and dissipative: the pionic state decays toward the pionic vacuum state. The dynamics is the reduced dynamics of a unitary quantum field theory (QFT, Standard Model), and as such, it is non-Markovian: the time-derivative $d\rho(t)/dt$ of the pionic state depends on the history of ρ before t.

Long ago and far from the context of QFT, a very powerful mathematical theorem [1, 2] proved (see also [3, 4]) that non-relativistic Markovian evolution of quantum states can always be expressed by a very specific structure of a number of operators A_n :

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_{n} \left(A_n \rho A_n^{\dagger} - \frac{1}{2} \{ A_n^{\dagger} A_n, \rho \} \right).$$
(1)

Popularity of this GKLS master equation, many times referred just as Lindblad master equation after one of the inventors, has been and is remarkably extending in many fields in non-relativistic quantum physics. It is understood as a Markovian effective equation of open quantum systems [5] whose exact dynamics is non-Markovian. I used to share this view. The only exact Markovian evolutions are unitary. Exact non-unitary (e.g. dissipative) Markovian evolutions do not exist. Lorentz invariant Markovian dissipation is impossible.

An unexpected push came from David Poulin proposing a relativistic GKLS equation in his 2017 talk [6]. The proposal is impressive and has been shaking my firm judgement that relativistic GKLS equations are nonexisting.

II. POULIN'S OBSERVATION

Consider a quantized free scalar field φ of mass m and its canonical momentum π . The Hamiltonian H reads

$$H = \frac{1}{2} \int \left(\pi^2 + (\nabla \varphi)^2 + m^2 \varphi^2 \right) d\mathbf{x}$$
$$= \int \omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} d\mathbf{k}.$$
(2)

The state ρ evolves by the Schrödinger (-von-Neumann) equation of motion

$$\frac{d\rho}{dt} = -i[H,\rho]. \tag{3}$$

Lorentz invariance relies simply on the fact that $H = P_0$ where

$$P_{\mu} = \int k_{\mu} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} d\mathbf{k} \tag{4}$$

is a 4-vector (of total energy-momentum).

One can modify the free unitary dynamics by a nonunitary (e.g.: dissipative) mechanism represented by a superoperator \mathcal{D} :

$$\frac{d\rho}{dt} = -i[H,\rho] + \mathcal{D}\rho, \qquad (5)$$

where the dissipator \mathcal{D} has the GKLS structure (1). Poulin's proposal is this:

$$\mathcal{D}\rho = \gamma \int \left(2\pi^{-}\rho\pi^{+} - \{\pi^{+}\pi^{-},\rho\}\right) d\mathbf{x}$$
$$= \gamma \int \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger}\rho a_{\mathbf{k}} - \frac{1}{2}\{a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}},\rho\}\right) d\mathbf{k}, \qquad (6)$$

where π^{\pm} are the positive and negative frequency parts of π . The argument of Lorentz invariance is the same as above. One can write \mathcal{D} in the form

$$\mathcal{D} = \gamma \int \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger} \otimes a_{\mathbf{k}} - \frac{1}{2} (a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \otimes I + I \otimes a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}) \right) d\mathbf{k}$$
(7)

and argue that $\mathcal{D} = \mathcal{P}_0$ where

$$\mathcal{P}_{\mu} = g \int k_{\mu} \left(a_{\mathbf{k}}^{\dagger} \otimes a_{\mathbf{k}} - \frac{1}{2} (a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \otimes I + I \otimes a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}) \right) d\mathbf{k}$$
(8)

is a 4-vector.

With the new dissipative mechanism the bosons are decaying and for long time the system's state becomes the vacuum. The stable equilibrium vacuum state is supposed to be approached along a relativistic invariant Markovian evolution by construction. Poulin notes that the dynamics, unlike in standard QFT, is non-local on range 1/m. The resulting acausality is of short range provided m is large. This can, in certain theories, be a bearable anomaly.

However, the forthcoming analysis uncovers that the eq. (5) is not Lorentz invariant. The next section formulates the condition of boost invariance in Markovian dissipative quantum fields, like the proposed one. A lapse of Poulin's argument is detected.

III. CONDITION OF BOOST INVARIANCE

Let us recapitulate the condition of invariance under Lorentz boosts in standard QFT, with interaction V. Let us evolve the system dynamically for a small time δt and perform a boost with small velocity $\delta \mathbf{v}$. Or, apply the boost first and let the system evolve after it. If the dynamics is Lorentz invariant then the resulting two states must coincide apart from the spatial shift $\delta \mathbf{v} \delta t$ in the second state. The mathematical condition of this invariance (i.e.: interchangeability of dynamical evolution and boost) is the following:

$$[\mathbf{K}, H+V] = i\mathbf{P},\tag{9}$$

where **K** is the generator of boosts and **P** is the spatial part of P_{μ} in (4). The closed expression of **K** exists [7] but in practice we use the boost action on the operator basis $a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}$. The small boost acts like this:

$$a_{\mathbf{k}} + i\delta \mathbf{v}[\mathbf{K}, \sqrt{\omega_{\mathbf{k}}}a_{\mathbf{k}}] = \sqrt{\omega_{\mathbf{k}'}}a_{\mathbf{k}'} \tag{10}$$

and similarly for $a_{\mathbf{k}}^{\dagger}$, where $\mathbf{k}' = \mathbf{k} - \delta \mathbf{v} \omega_{\mathbf{k}}$ is the boosted **k**. Hence, the boost of any operator is equivalent with the boost of the (covariant) creation/annihilation operators. We have $[\mathbf{K}, H] = i\mathbf{P}$, and $[\mathbf{K}, V] = 0$ for non-derivative interaction, the condition (9) is satisfied.

In the proposed eq. (5) the Hamiltonian interaction term $-i[V,\rho]$ is replaced by the dissipative term $\mathcal{D}\rho$. The second term $[\mathbf{K}, V]$ of the condition (9) becomes non-vanishing:

$$(\mathbf{K} \otimes I)\mathcal{D} - \mathcal{D}(I \otimes \mathbf{K}) = i\boldsymbol{\mathcal{P}},\tag{11}$$

where \mathcal{P} is the spatial part of \mathcal{P}_{μ} in (8). The condition (9) of boost invariance becomes violated.

Now we put the argument of Sec. II under scrutiny. The proposal assumes that the boost generator is the standard Hermitian generator \mathbf{K} , acting as in eq. (10). This cannot be true. Since the time-evolution is not unitary the boosts cannot be unitary either (Fig. 1)!



FIG. 1: In frame (t, x), a single-boson non-relativistic localized state is prepared at location A (t = 0, x > 0) at rest. For t > 0, the boson is starting to decay. The initial local system at A reaches B in an irreversible process. If the initial state ρ defined at t = 0 were unitary equivalent with ρ' defined at t' = 0—where (t', x') is a different Lorentz frame— then the evolution of our local boson should be reversible, which is not the case.

The boost generator might become a superoperator \mathbf{K} to satisfy the condition of invariance, i.e.: the interchangeability between dynamical evolution and boost. The superoperator counterpart of the mathematical condition (9) of boost invariance is straightforward. But, in the next section we show that it is useless to search for the covariant boost. The eq. (5) cannot be Lorentz invariant.

IV. DISPROOF OF LORENTZ INVARIANCE

The dissipative term does not prevent us from using an interaction picture. We use an unconventional interaction picture where H evolves the state and \mathcal{D}^{\dagger} evolves the field:

$$\frac{d\rho}{dt} = -i[H,\rho], \qquad (12)$$

$$\partial_t \varphi(t, \mathbf{x}) = \mathcal{D}^{\dagger} \varphi(t, \mathbf{x}).$$
 (13)

The generator H of the unitary evolution and the generator \mathcal{D} of the dissipative evolution are commuting hence the constant H governs the state evolution. Now, the evolution of the state is standard Lorentz invariant. What about the evolution of the field? The initial condition reads:

$$\varphi(0,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} d\mathbf{k} + \text{H.c.} \quad (14)$$

From the relationships $\mathcal{D}^{\dagger}a = -\gamma a$ and $\mathcal{D}^{\dagger}a^{\dagger} = -\gamma a^{\dagger}$, the solution follows easily:

$$\varphi(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x} - \gamma\omega_{\mathbf{k}}t} d\mathbf{k} + \text{H.c.} \quad (15)$$

One would prove or disprove the boost invariance of the solutions. But we have a simpler tool, the field equation:

$$\partial_t^2 \varphi(t, \mathbf{x}) = \gamma^2 (m^2 - \nabla^2) \varphi(t, \mathbf{x}), \qquad (16)$$

which is manifest non-invariant. This is not surprising since Sec. III found already a flaw in the argument supporting Lorentz invariance of the proposal in Sec. II.

V. DIGRESSION: CLASSICAL AND QUANTUM WHITE NOISE

A naive Lorentz invariant field theory appeared in [8] first, where

$$\mathcal{D}\rho = g^2 \int \left(\varphi\rho\varphi - \frac{1}{2}\{\varphi^2, \rho\}\right) d\mathbf{x}.$$
 (17)

This is a Lindblad form (1) and the corresponding dynamics is Lorentz invariant indeed. It can be derived from the coupling $g\phi\xi$ to an external Lorentz invariant classical white noise field of ultra-local correlation

$$\langle \xi(x)\xi(y)\rangle = g\delta(x-y),$$
 (18)

after taking the average over this random field. The features of \mathcal{D} are unphysical, it is creating bosons at infinite rate which is a trivial consequence of the white noise. Unfortunately, $\xi(x)$ is the only possible Lorentz invariant white noise, or, in other words, the only Lorentz invariant classical Markovian process on the continuum.

We can construct a Lorentz invariant quantum white noise b(x) as well. It is a trivial relativistic generalization of quantum white noise b(t) introduced for damped quantum systems [9] and extensively used e.g. in quantum optics [10]. The canonical commutator is ultra-local bosonic:

$$[b(x), b^{\dagger}(y)] = \delta(x - y).$$
⁽¹⁹⁾

We use b(x) as an auxiliary field to construct a unitary QFT. Poulin's impressive proposal corresponds to the coupling

$$\sqrt{2\gamma}(\pi^+ b + \pi^- b^\dagger). \tag{20}$$

Assuming that the initial state of the *b*-field is the vacuum state, we evolve the composite state $\rho \otimes |vac\rangle \langle vac|$ unitarily and trace out the auxiliary field. We mentioned

in Sec. I that in standard QFTs the reduced dynamics are non-Markovian. But the auxiliary *b*-field is exceptional, it is ultra-local, non-propagating, e.t.c., so we get a Markovian evolution for ρ of the φ -field. This is exactly Poulin's dissipative dynamics (5) in interaction picture:

$$\frac{d\rho}{dt} = \gamma \int \left(2\pi^- \rho \pi^+ - \{\pi^+ \pi^-, \rho\}\right) d\mathbf{x},\qquad(21)$$

which is not Lorentz invariant according to Secs. III-IV.

How is it possible? The coupling was Lorentz invariant, the reduction is Lorentz invariant, then where has Lorentz invariance been lost? Sure, Lorentz invariance of the reduced dynamics is undermined by the non-locality of π^{\pm} in the otherwise Lorentz invariant coupling (20). Weinberg [7] warns us about the importance of locality condition. It is this condition that makes the combination of Lorentz invariance and quantum mechanics so restrictive.

VI. CLOSING REMARKS

For long time there have been one only context with the interest and unfulfilled desire for relativistic GKLS equations. The assumption of a tiny fundamental and spontaneous decoherence in massive degrees of freedom was realized by the non-relativistic GKLS equations [11, 12], but the relativistic extensions are missing up till now. Efforts [14–19], mostly related to the structure (17), are always leading to unphysical features, like, e.g., the mentioned vacuum instability, or just presence of tachyons.

Poulin's motivation was not different in that he assumed a tiny fundamental dissipative mechanism. He did it directly in the relativistic realm. The proposal is smartly hiding its defect. To point it out took quite a time for the present author.

Free pions decay exponantially, they follow a Markovian Lorentz invariant effective dynamics. Their exact dynamics cannot be Markovian. Any Markovian irreversible field process —whether quantized or classical is underlain by instantaneous jumps and they do not exist relativistically.

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