

A non-trivial PT-symmetric continuum Hamiltonian and its Eigenstates and Eigenvalues

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In this paper, a non-trivial system governed by a continuum PT-symmetric Hamiltonian is discussed. We show that this Hamiltonian is iso-spectral to the simple harmonic oscillator. We find its eigenfunctions and the path in the complex plane along which these functions form an orthonormal set. We also find the hidden symmetry operator, \mathcal{C} , for this system. All calculations are performed analytically and without approximation.

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I. INTRODUCTION

Over the last two decades there has been considerable interest in PT-symmetric systems after Carl M. Bender and collaborators discovered that a quantum Hamiltonian need not be Hermitian to have real eigenvalues, but could also be PT-symmetric.[1] Here, P stands for parity, T for time reversal (complex conjugation). It was found that there is a hidden symmetry operator for such systems which does not exist in Hermitian systems. The corresponding operator, \mathcal{C} , is necessary in order to define a new inner product for which the eigenstates of the Hamiltonian have positive norm.

Since this discovery, PT-symmetric quantum theory has entered the main stream of physics investigations and the number of articles written by many authors on PT-symmetric systems is legion.[2] The consistency of PT-symmetric quantum mechanics has been shown; [3] PT-symmetry has appeared in quantum optics [4, 5] as well as technology [6–8]. Recently, even time-dependent systems have been studied.[9] An excellent text on PT-symmetry by Bender was published in 2019 (with many contributors and dozens of references) which will be our main reference for this article.[10]

Here, we will present a non-trivial system governed by a PT-symmetric Hamiltonian in the continuum. In contrast to non-trivial systems studied in the past, such as the ix^3 oscillator, for which perturbation theory has been required, we will perform all calculations without approximation. We have organized the paper as follows. In the next section, we will review the relevant and important facts on PT-symmetric quantum theory. The Hamiltonian will then be presented. Next, we present a mathematical theorem which will allow us to further analyze this Hamiltonian and show that it is isospectral to the simple harmonic oscillator. In a further section, we will derive the exact eigenstates of the Hamiltonian and demonstrate the path in the complex plane along which one must integrate to ensure the states form an orthonormal set. Finally, we will derive the hidden symmetry operator for this system.

II. BRIEF SUMMARY OF PT-SYMMETRIC QUANTUM THEORY

A PT-symmetric Hamiltonian satisfies

$$[H, \mathcal{PT}] = 0, \quad (1)$$

where P is parity and T is time reversal. The latter means complex conjugation in order to preserve the fundamental commutation relation $[x, p] = i$ under PT (we take $\hbar = 1$ throughout): under PT, $p \rightarrow p$ and $x \rightarrow -x$. Examples

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of (time-independent) potentials might be $V(x) = ix^3$ or $V(x) = x^2(ix)^n$. Eigenstates of such a Hamiltonian will be denoted here as $\phi_n(x)$; thus,

$$H\phi_n(x) = E_n\phi_n(x). \quad (2)$$

Such states do not have norms which are preserved in time. The so-called PT-norms are preserved,

$$N = \int \phi_n(x) \mathcal{PT} \phi_n(x) dx,$$

but they can be *negative* which is not allowed for a probability. A norm may however be defined as follows. There is a hidden symmetry which is expressed by a new operator, \mathcal{C} , which does not exist in Hermitian quantum theory. The action of this new operator is

$$\mathcal{C}\phi_n(x) = (-1)^n \phi_n(x), \quad (3)$$

corresponding to the PT-norm ± 1 . Thus, the CPT-norm,

$$N = \int \phi_n(x) \mathcal{CPT} \phi_n(x) \quad (4)$$

is real and *positive* for all n . The method of finding this operator is typically to note that it must satisfy

$$[H, \mathcal{C}] = 0 = [\mathcal{C}, \mathcal{PT}] \quad \mathcal{C}^2 = 1. \quad (5)$$

These equations have been solved perturbatively or by semi-classical means in some non-trivial cases.[11, 12] It is known that the C-operator may be written as,

$$\mathcal{C} = e^{Q(x,p)} \mathcal{P}, \quad (6)$$

where the operator $Q(x, p)$ is *even* in position, x , and *odd* in the momentum, p , in order to satisfy the above requirements. We will freely use this representation in what follows. Also, it has been pointed out that for a PT-symmetric Hamiltonian, there exists an iso-spectral Hermitian Hamiltonian, h with corresponding eigenstates $\psi_n(x)$ both related to their PT counterparts by a similar exponential transformation:

$$h = e^{-Q/2} H e^{Q/2}, \quad \psi_n(x) = e^{-Q/2} \phi_n(x). \quad (7)$$

The Schrödinger equation

$$h\psi_n(x) = E_n\psi_n(x)$$

is easily shown to be identical to that in the PT-symmetric realm, (2), by direct substitution; the eigenvalues, E_n are the same in both. Eigenstates of h are orthonormal in the usual way, as are the eigenstates of H , but in the PT-symmetric realm, it is with respect to the CPT-inner product:

$$\int \phi_n(x) \mathcal{CPT} \phi_m(x) dx = \delta_{nm}. \quad (8)$$

Finally, one must note that this latter integral need not be evaluated along the real axis, but may need to be along a curve in the complex x -plane. We will address this need in our example.

III. THE HAMILTONIAN

The PT-symmetric Hamiltonian we will discuss is

$$H = \frac{1}{2} p s^4 p + 4\epsilon^2 s^2 + \frac{x^2}{2s^2}, \quad (9)$$

where $s = 1 + 2i\epsilon x$, ϵ is a real parameter and as usual, $p = -id/dx$ is the momentum operator. We will use the notation s for $1 + 2i\epsilon x$ throughout as well as its complex conjugate $\bar{s} = 1 - 2i\epsilon x$. That this Hamiltonian is PT-symmetric is manifest because $PTs = s$; and it will reduce to the simple harmonic oscillator at $\epsilon = 0$. In fact, this PT-symmetric Hamiltonian is iso-spectral to the harmonic oscillator with eigenvalues $n + 1/2$ (for convenience, we will take the

oscillator frequency to be $\omega = 1$ and particle mass $m = 1$.) In order to demonstrate this last assertion, we will need a mathematical lemma which we now state (this lemma will be used several times in what follows):

LEMMA - For any differentiable function $U(x)$, real parameter ϵ and operator $F = x^2 p + p x^2$,

$$e^{\epsilon F} U(x) = \frac{1}{s} U(x/s). \quad (10)$$

In order to prove this lemma, we begin with the expansion of the exponential.

$$e^{\epsilon F} U(x) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} F^{(n)} U(x) \equiv \sum_{n=0}^{\infty} f_n(x). \quad (11)$$

Before continuing, we note that $f_0(x) = U(x)$. Now, the $n + 1$ th term in this series satisfies

$$f_{n+1}(x) = \frac{\epsilon}{(n+1)} F f_n(x),$$

or, after substituting the form of the operator F ,

$$f_{n+1}(x) = -\frac{\mu}{n+1} [x f_n(x) + x^2 f'_n(x)]. \quad (12)$$

with $\mu = 2i\epsilon$. We note that this is a differential-difference equation. To solve this we multiply this result by z^{-n} and sum over all n from 0 to ∞ and define

$$g(x, z) = \sum_{n=0}^{\infty} f_n(x) z^{-n}. \quad (13)$$

$g(x, z)$ constitutes the z -transform of $f_n(x)$ which is invertable in terms of a contour integral. We will not need the explicit form of f_n , but only the properties listed below of $g(x, z)$ itself:

$$\lim_{z \rightarrow \infty} g(x, z) = f_0(x) = U(x), \quad e^{\epsilon F} U(x) = g(x, 1). \quad (14)$$

Now, according to Eq.(12,13), $g(x, z)$ satisfies the first-order PDE

$$z^2 \frac{\partial g(x, z)}{\partial z} - \mu x^2 \frac{\partial g(x, z)}{\partial x} = \mu x g(x, z). \quad (15)$$

This PDE is solvable by standard means - the exact, general solution is

$$g(x, z) = \frac{1}{x} \Phi\left(\frac{2i\epsilon x z}{z + 2i\epsilon x}\right). \quad (16)$$

We must satisfy the initial condition $g(x, \infty) = U(x)$; hence

$$U(x) = \lim_{z \rightarrow \infty} g(x, z) = \frac{1}{x} \Phi(2i\epsilon x) \equiv \frac{1}{x} \Phi(A), \quad (17)$$

and thus

$$\Phi(A) = \frac{A}{2i\epsilon} U\left(\frac{A}{2i\epsilon}\right),$$

or

$$g(x, z) = \frac{z}{z + 2i\epsilon x} U\left(\frac{xz}{z + 2i\epsilon x}\right).$$

The series sum is $g(x, 1)$; hence the lemma follows

$$e^{\epsilon F} U(x) = g(x, 1) = \frac{1}{s} U(x/s) \quad (18)$$

The simple harmonic oscillator Hamiltonian is $h = \frac{1}{2}p^2 + \frac{1}{2}x^2$. With this lemma in hand, we may now show that

whenever ψ_n is a solution of the harmonic oscillator $h\psi_n = E_n\psi_n$ then the corresponding PT-symmetric H above satisfies $H\phi_n = E_n\phi_n$ with the same E_n . To simplify the coming algebra, let $w = 1/s$, $z = wx$ and s is defined as before. Now, since the action of p on x is $-i$ it follows that

$$pw = -2\epsilon w^2, \quad pz = -iw^2 \quad (19)$$

The Hamiltonian (9) in this notation is

$$H = \frac{1}{2}pw^{-4}p + 4\epsilon^2w^{-2} + \frac{1}{2}z^2. \quad (20)$$

Let us suppose that the transformation between ϕ and ψ is given by Eq.(7) and the lemma as

$$\phi(x) = e^{Q/2}\psi(x) = e^{\epsilon F}\psi(x) = w\psi(z), \quad (21)$$

in the current notation and where the transformation operator is $Q = 2\epsilon F$ (later, we will show this explicitly in another way). We find using the above that

$$p\phi = \psi pw + w\frac{d\psi}{dz}pz = -iw^3\frac{d\psi}{dz} - 2\epsilon w^2\psi. \quad (22)$$

We then find, using straightforward algebra that

$$\frac{1}{2}pw^{-4}p\phi = w\left[-\frac{1}{2}\frac{d^2\psi}{dz^2} - 4\epsilon^2w^{-2}\psi\right]. \quad (23)$$

Then it follows that

$$H\phi = w\left[-\frac{1}{2}\frac{d^2\psi}{dz^2} + \frac{1}{2}z^2\psi(z)\right] \quad (24)$$

The expression in square brackets is just the usual harmonic oscillator Hamiltonian acting on ψ . Thus, if ψ is an eigenstate of the oscillator, then $\phi = \exp(\epsilon F)\psi$ is an eigenstate of the PT-symmetric Hamiltonian with the same eigenvalue. We will later write down the states $\phi_n(x)$ explicitly.

The Hamiltonian (9) may be obtained from the harmonic oscillator directly. Reversing the transformation given in (7) we have

$$H = e^{Q/2}he^{-Q/2} = e^{Q/2}\left[\frac{1}{2}p^2 + \frac{1}{2}x^2\right]e^{-Q/2} \quad (25)$$

The right-hand side may be written in terms of nested commutators with $Q = 2\epsilon F = 2\epsilon[x^2p + px^2]$:

$$H = e^{\epsilon F}he^{-\epsilon F} = h + \frac{\epsilon}{1!}[F, h] + \frac{\epsilon^2}{2!}[F, [F, h]] + \dots \quad (26)$$

or

$$H = h + \frac{\epsilon}{1!}C_1 + \frac{\epsilon^2}{2!}C_2 + \dots, \quad (27)$$

where $C_n = [F, C_{n-1}]$. As one calculates commutators, one notices two types of terms: powers of x , or powers of x sandwiched between two factors of momentum, p .

$$[F, x^n] = -2inx^{n+1}, \quad [F, px^n p] = i(8 - 2n)px^{n+1}p - 2nix^{n-1} \quad (28)$$

It turns out that there are only a finite number of terms bounded by ps , a finite grouping of powers of x and an infinite series in powers of x alone. This series may be easily summed, and combining all terms yields the Hamiltonian (9).

IV. THE EIGENFUNCTIONS OF H

The eigenfunctions of H may be found directly from the transformation (7). Recall that the wavefunctions of the simple harmonic oscillator are

$$\psi_n(x) = A_n H_n(x) e^{-x^2/2}, \quad (29)$$

where the $A_n = (\sqrt{\pi}2^n n!)^{-1/2}$, $H_n(x)$ are Hermite polynomials satisfying $H_n(-x) = (-1)^n H_n(x)$. Thus, by our lemma

$$\phi_n(x) = e^{\epsilon F} \psi_n(x) = \frac{A_n}{s} H_n(x/s) e^{-x^2/2s^2} \quad (30)$$

We note that due to the symmetry property of the H_n noted above $\mathcal{PT}\phi_n(x) = (-1)^n \phi_n(x)$; that is, these are eigenstates of PT with eigenvalues ± 1 .

We may now write down the C-operator for this system and find its action on these states. According to Eq. (6), the C-operator should be

$$\mathcal{C} = e^Q \mathcal{P} = e^{2\epsilon F} \mathcal{P}. \quad (31)$$

We have already shown the action of the exponential operator above in the lemma; we need only double the coefficient of F . Hence,

$$\mathcal{C}\phi_n(x) = e^{2\epsilon F} \phi(-x) = \frac{1}{t} \phi_n(-x/t), \quad (32)$$

where $t = 1 + 4i\epsilon x$ (similar to s). In order to find the explicit action of \mathcal{C} , we need only calculate the following:

$$\mathcal{C}\left[\frac{1}{s}\right] = \left[\frac{1}{t}\right] \left[\frac{1}{1 - 2i\epsilon x/t}\right] = \frac{1}{s}, \quad (33)$$

$$\mathcal{C}\left[\frac{x}{s}\right]^n = \left[\frac{-x/t}{1 - 2i\epsilon x/t}\right]^n = (-1)^n [x/s]^n. \quad (34)$$

It then follows immediately from the explicit form of $\phi_n(x)$ above that

$$\mathcal{C}\phi_n(x) = (-1)^n \phi_n(x), \quad (35)$$

as it should according to the general theory. The action of \mathcal{CPT} is thus to multiply the state ϕ_n by unity and thus the CPT-norm of the state will be positive.

The last property we need to demonstrate is the orthogonality of the states in (30). We begin with the known orthonormality of the harmonic oscillator states (which are real-valued):

$$\delta_{nm} = \int_{-\infty}^{\infty} \psi_n(q) \psi_m(q) dq. \quad (36)$$

Transforming these states to eigenstates of H by Eq.(7) using our lemma, we have

$$\delta_{nm} = \int_{-\infty}^{\infty} \frac{dq}{\bar{s}^2} \phi_n(q/\bar{s}) \phi_m(q/\bar{s}), \quad (37)$$

with $\bar{s} = 1 - 2i\epsilon q$. In order to make sense of this orthogonality relation, we change variables from the real number, q , which runs over all real values, to the complex number z which is parametrized by q :

$$z = x + iy = \frac{q}{1 - 2i\epsilon q}, \quad dz = \frac{dq}{\bar{s}(q)^2} \quad (38)$$

The inner product integral now reads,

$$\delta_{nm} = \oint \phi_n(z) \phi_m(z) dz, \quad (39)$$

where the integration path in the complex z -plane begins at $z = i/2\epsilon$ ($q = -\infty$) passes through the origin and once again ends at $z = i/2\epsilon$ ($q = \infty$) as shown in the figure.[13]

V. SUMMARY

We have found a non-trivial PT-symmetric continuum Hamiltonian, Eq.(9) which is isospectral to the simple harmonic oscillator with eigenvalues $n + 1/2$. The corresponding eigenstates, Eq.(30), are PT-symmetric and form an orthonormal set of states, with path of integration a closed curve in the complex plane. We have written down the action of the hidden symmetry operator, \mathcal{C} , Eqs.(32,35) and have explicitly shown that its action on any one of the states is to simply multiply it by its PT-norm; that is, the \mathcal{CPT} norm is real and positive. In contrast to other systems studied in the continuum, we made no approximations.

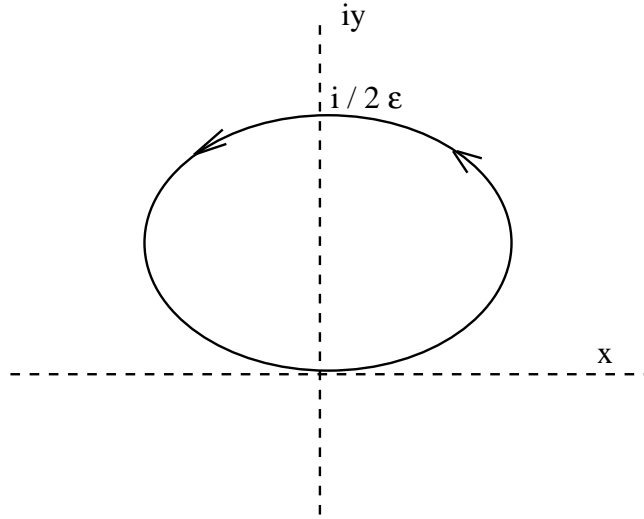


FIG. 1: The path required of the inner product of states Eq.(39)

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Data availability statement

No new data were created or analyzed in this study.

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 - [13] The length of this path is a finite number, $\pi/2\epsilon$, which only becomes infinite in the Hermitian limit $\epsilon \rightarrow 0$.