LATTICE SEQUENCE SPACES AND SUMMING MAPPINGS

DAHMANE ACHOUR AND TOUFIK TIAIBA

Abstract. The objective of this study is to advance the theory concerning positive summing operators. Our focus lies in examining the space of positive strongly p-summable sequences and the space of positive unconditionally p-summable sequences. We utilize these in conjunction with the Banach lattice of positive weakly p-summable sequences to present and characterize the classes of positive strongly (p,q)-summing operators, positive (p,q)-summing, and positive Cohen (p,q)-nuclear operators. Additionally, we describe these classes in terms of the continuity of an associated tensor operator that is defined between tensor products of sequences spaces.

1. Introduction and background

The spaces of sequences with values in a Banach lattice are intimately related to the summability of operators between Banach lattices. For example, the positive (p,q)-summing operators, introduced by Blasco [4], are the continuous operators which take positive weakly p-summable sequences $\ell_{p,|\omega|}(X)$ into q-summable sequences $\ell_q(E)$ (see also [26]). In [1], Achour-Belacel introduced the notion of positive strongly (p,q)-summing operators to characterize those operators whose adjoints are positive (q^*,p^*)-summing operators.

In [17] and [5] the authors introduce the space of positive strongly p-summable sequences $\ell_p^\pi(X)$ (initially introduced by Cohen for Banach spaces [8]), as well as the space of positive unconditionally p-summable sequences $\ell_{p,|\omega|}^u(X)$.

Tensor products have proved to be a useful tool for the theory of operator ideals. Indeed, the excellent monograph [9] deals with the tensor product point of view of the theory and provides many applications to the study of the structure of several spaces of summing linear operators.

The following characterizations provide nice examples of how tensor products come into the theory of summing operators:

- An operator $T: X \to Y$ is absolutely p-summing (see [20]) if and only if $I \otimes T : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_{\Delta_p} Y$ is continuous, where Δ_p satisfies $\Delta_p(\sum_{i=1}^n e_i \otimes x_i) = (\sum_{i=1}^n \|x_i\|^p)^{1/p}$. and ε is the injective tensor norm (see [9]).
- Let $1 < p < \infty$. An operator $T: X \to Y$ is Cohen p-nuclear (p-dominated) if and only if $I \otimes T : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_\pi Y$ is continuous, π is the projective norm (see [8] and [9]).
- An operator $T: X \to Y$ is strongly p-summing if and only if $I \otimes T : \ell_p \otimes_{\Delta_p} X \to \ell_p \otimes_\pi Y$ is continuous (see [2]).

The interplay between tensor products and positive summing operators have not been explored yet. In this paper, first we utilize sequences in Banach lattice spaces to define and characterize certain classes of positive summing operators. Then we describe these classes in terms of the

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continuity of the canonically defined tensor product operator $I\otimes T:\ell_p\otimes_\alpha X\to\ell_p\otimes_\beta Y$ for adequate p and tensor norms α and β.

Our results are presented as follows. After this introductory section, Section 2 is devoted to providing new properties of the positive (p,q)-summing operators. Particularly, we prove that these operators are continuous operators transform positive lattice unconditionally p-summable sequences $\ell^u_{p,|\omega|}(X)$ into q-summable sequences $\ell_q(E)$. In Section 3, utilizing the Banach lattice of positive strongly p-summable sequences, we present a novel characterization of positive strongly (p,q)-summing operators. Furthermore, we demonstrate that this class is equivalent to the class of (p,q)-majorizing operators introduced in [7]. In Section 4, we study the notion of positive Cohen (p,q)-nuclear operators. We explore the summability properties of these operators by defining their corresponding operators between spaces of positive weakly p-summable sequences $\ell_{p,|\omega|}(X)$ and strongly positive strongly p-summable sequences $\ell^\pi_p(Y)$. In the final section, we describe these classes in terms of the continuity of an associated tensor operator that is defined between tensor products of sequences spaces.

We use standard notation for Banach lattices (see [18] and [22]). If X is an ordered set, the usual order on $X^{\mathbb{N}}$ is defined by $x=(x_n)_n\geq 0\Leftrightarrow x_n\geq 0$ for each $n\in\mathbb{N}$. If X is a vector lattice and $\|x\|\leq\|y\|$ whenever $|x|\leq|y|$ ($|x|=\sup\{x,-x\}$) we say that X is a Banach lattice. If the lattice is complete, we say that X is a Banach lattice. Note that this implies obviously that for any $x\in X$ the elements x and $|x|$ have the same norm. We denote by $X_+=\{x\in X,x\geq 0\}$. An element x of X is a positive if $x\in X_+$. For $x\in X$ let $x^+:=\sup\{x,0\}, x^-:=\sup\{-x,0\}$ be the positive part, the negative part of x, respectively. For any $x\in X$, we have the following properties $x=x^+-x^-$ and $|x|=x^++x^-$.

The dual X^* of a Banach lattice X is a complete Banach lattice endowed with the natural order

$$x_1^*\leq x_2^*\Longleftrightarrow\langle x_1^*,x\rangle\leq\langle x_2^*,x\rangle,\ \forall x\in X_+$$

where $\langle .,.\rangle$ denotes the bracket of duality.

By a sublattice of a Banach lattice X we mean a linear subspace A of X so that $\sup\{x,y\}=x\wedge y$ belongs to A whenever $x,y\in A$. The canonical embedding $i_E:X\to X^{**}$ such that $\langle i_E(x),x^*\rangle=\langle x^*,x\rangle$ of X into its second dual X^{**} is an order isometry from X into a sublattice of X^{**}, see [18, Proposition 1.a.2]. If we consider X as a sublattice of X^{**} we have for $x_1,x_2\in X$

$$x_1\leq x_2\Longleftrightarrow\langle x_1,x^*\rangle\leq\langle x_2,x^*\rangle,\ \forall x^*\in X_+^*.$$

Throughout this paper X and Y Banach lattices, E and F Banach spaces. An operator $T:X\to Y$ which preserves the lattice operations is called lattice homomorphism, that is, $T(x_1\vee x_2)=T(x_1)\vee T(x_2)$ for all $x_1,x_2\in X$. An one-to-one, surjective lattice homomorphism is called lattice isomorphism. The space of all bounded linear operators from E to F is denoted by $\mathcal{L}(E,F)$ and it is a Banach space with the usual supremum norm. The continuous dual space $\mathcal{L}(E,\mathbb{K})$ of E is denoted by E^*, whereas B_E denotes the closed unit ball of E. The symbol $E\equiv F$ means that E and F are isometrically isomorphic.

Let $1\leq p\leq\infty$, we write p^* the conjugate index of p, that is $1/p+1/p^*=1$. As usual $\ell_p(E)$ denotes the vector space of all absolutely p-summable sequences, with the usual norm $\|\cdot\|_p$ and $\ell_{p,\omega}(E)$ the space of all weakly p-summable sequences with the norm $\|(x_n)_n\|_{p,\omega}=\sup_{x^*\in B_{E^*}}\|(\langle x_n,x^*\rangle)_n\|_p$. The closure in $\ell_{p,\omega}(E)$ of the set of all sequences in E which have only a finite number of non-zero terms, is a Banach space with respect to the norm $\|\cdot\|_{p,\omega}$. We denote this space by $\ell^u_p(E)$. The space

$$(c_0)_\omega(E):=\{(x_n)_n\subset E:(\langle x_n,\xi\rangle)_n\in c_0,\forall\xi\in E^*\}$$

is a closed subspace of $\ell_{\infty,\omega}(E)$ (see [15, §19.4]). It is well known that $\ell_{p,w}(E)$ is an isometrically isomorphic to $\mathcal{L}(\ell_{p^*}, E)$ for $1 < p \leq \infty$ and $\ell_{1,\omega}(E)$ is an isometrically isomorphic to $\mathcal{L}(c_0, E)$. We denote by $\ell_p\langle E\rangle$ the space of all strongly p-summable sequences (Cohen strongly p-summable sequences, see [8]), that is the space of all sequences $(x_n)_{n=1}^{\infty}$ in E such that $(x_n^*(x_n))_n \in \ell_1$, for any $(x_n^*)_{n=1}^{\infty} \in \ell_{p^*,\omega}(E^*)$, which is a Banach space with the norm $\|(x_n)_{n=1}^{\infty}\|_{\langle p\rangle} := \sup_{\|(x_n^*)_{n=1}^{\infty}\|_{p^*,\omega}\leq 1} \left|\sum_{n=1}^{\infty} x_n^*(x_n)\right| = \sup_{\|(x_n^*)_{i=1}^{\infty}\|_{p^*,\omega}\leq 1} \sum_{n=1}^{\infty} |x_n^*(x_n)|$. The following fact, discussed in [3], is well-known

$$\ell_{p^*,\omega}(E^*) \equiv [\ell_p\langle E\rangle]^* \text{ and } [\ell_{p^*}\langle E^*\rangle] \equiv [\ell_{p,\omega}(E)]^*. \tag{1.1}$$

Moreover, it is well-known that

$$[\ell_p(E)]^* \equiv \ell_{p^*}(E^*) \text{ for } 1 \leq p < \infty \text{ and } [c_0(E)]^* \equiv \ell_1(E^*). \tag{1.2}$$

Sequences in Banach lattice spaces. Consider the case where E is replaced by a Banach lattice X. The space of positive weakly p-summable sequences was introduced in [17] by

$$\ell_{p,|\omega|}(X) = \left\{(x_n)_n \in X^{\mathbb{N}} : \sum_{n=1}^{\infty} \langle x^*, |x_n|\rangle^p < +\infty,\ \forall x^* \in X_+^*\right\}$$

and

$$\|(x_n)_n\|_{p,|\omega|} = \sup_{x^*\in B_{X_+^*}} (\sum_{n=1}^{\infty} \langle x^*, |x_n|\rangle^p)^{\frac{1}{p}}$$

Also, $(c_0)_{|\omega|}(X)$ is a closed vector lattice subspace of $\ell_{\infty,|\omega|}(X)$. Then $(\ell_{p,|\omega|}(X), \|\cdot\|_{p,|\omega|})$ is a Banach lattice (see [5] and [17]). Moreover, we have
(I_0) $\|(x_n)_n\|_{p,\omega} \leq \|(x_n)_n\|_{p,_{|\omega|}}$ for all $(x_n) \in \ell_{p,|\omega|}(X)$.
(I_1) If $(x_n)_n \geq 0$, we have

$$\|(x_n)_n\|_{p,_{|\omega|}} = \|(x_n)_n\|_{p,\omega}. \tag{1.3}$$

We define

$$\ell_p^{\varepsilon}(X^*) = \ell_{p,|\omega|}(X^*) = \left\{(x_n^*)_n \in (X^*)^{\mathbb{N}} : \sum_{n=1}^{\infty} \langle x, |x_n^*|\rangle^p < +\infty,\ \forall x \in X_+\right\}$$

and

$$\|(x_n^*)_n\|_{p,|\omega|} = \sup_{x\in B_{X_+}} \left(\sum_{n=1}^{\infty} \langle |x_n^*|, x\rangle^p\right)^{\frac{1}{p}} \tag{1.4}$$

Then $\ell_{p,|\omega|}(X^*)$ with this norm is a Banach lattice (see [5] and [17]).
Let $\ell_{p,|\omega|}^u(X)$ denote the closed sublattice of $\ell_{p,|\omega|}(X)$ defined by

$$\ell_{p,|\omega|}^u(X) = \left\{(x_n)_n \in X^{\mathbb{N}} : \lim_n \|(x_k)_{k=n+1}^{\infty}\|_{p,|\omega|} = 0\right\}.$$

In this case we say that $(x_n)_n$ is positive unconditionally p-summable sequences.
Let

$$\ell_p^{\pi}(X) = \left\{(x_n)_n \in X^{\mathbb{N}} : \sum_{n=1}^{\infty} |\langle x_n^*, |x_n|\rangle| < +\infty,\ \forall (x_n^*)_n \in (\ell_{p^*,|\omega|}(X^*))^+\right\}$$

and

$$\|(x_n)_n\|_{\ell_p^\pi(X)} = \sup_{(x_n^*)_n \in B_{[\ell_{p^*,|\omega|}(X^*)]^+}} \sum_{n=1}^{\infty} \langle x_n^*, |x_n| \rangle. \tag{1.5}$$

In this case we say that $(x_n)_n$ is positive strongly p-summable sequences. Then $\ell_p^\pi(X)$ with this norm is a Banach lattice ([5]). For convenience let us denote $\ell_1\langle X\rangle = \ell_1^\pi(X) = \ell_1(X)$.
By (I_0) and (I_1), we have
(I_0') $\|(x_n)_n\|_{\ell_p^\pi(X)} \leq \|(x_n)_n\|_{\ell_p\langle X\rangle}$ for all $(x_n)_n \in \ell_p\langle X\rangle$.
(I_1') if $(x_n)_n \geq 0$ then $(x_n)_n \in \ell_p^\pi(X)$ if and only if $(x_n)_n \in \ell_p\langle X\rangle$, and

$$\|(x_n)_n\|_{\ell_p^\pi(X)} = \|(x_n)_n\|_{\ell_p\langle X\rangle}. \tag{1.6}$$

Moreover, we have the following results due to [11, 17].

Proposition 1.1. [11, Proposition 3.1 and Proposition 3.2] *Let X be a Banach lattice and $1 < p < \infty$*

(a)

$$\ell_p^\pi(X^*) = \left\{ (x_n^*)_n \in (X^*)^{\mathbb{N}} : \sum_{n=1}^{\infty} |\langle x_n, |x_n^*| \rangle| < +\infty, \forall (x_n)_n \in (\ell_{p^*,|\omega|}(X))^+ \right\}$$

and for each $(x_n^*)_n \in \ell_p^\pi(X^*)$,

$$\|(x_n)_n\|_{\ell_p^\pi(X^*)} = \sup_{(x_n)_n \in B_{[\ell_{p^*,|\omega|}(X)]^+}} \sum_{n=1}^{\infty} \langle x_n, |x_n^*| \rangle.$$

(b)

$$\ell_p^\pi(X^*) = \left\{ (x_n^*)_n \in (X^*)^{\mathbb{N}} : \sum_{n=1}^{\infty} |\langle x_n, |x_n^*| \rangle| < +\infty, \forall (x_n)_n \in (\ell^u_{p^*,|\omega|}(X))^+ \right\}$$

and for each $(x_n^*)_n \in \ell_p^\pi(X^*)$,

$$\|(x_n^*)_n\|_{\ell_p^\pi(X^*)} = \sup_{(x_n)_n \in B_{[\ell^u_{p^*,\omega}(X)]^+}} \sum_{n=1}^{\infty} \langle x_n, |x_n^*| \rangle.$$

Lemma 1.2. [17] *Let X be a Riesz space (i.e., vector lattice), (Y, C) an ordered vector space such that $Y = C - C$ and $T : X_+ \longrightarrow C$ a positive, homogeneous and additive bijection. Then Y is a lattice space and T can be uniquely extended to a lattice isomorphism from X onto Y.*

Theorem 1.3. *Let X be a Banach lattice and $1 < p < \infty$.*

(i) *The Banach lattice $\ell_{p^*,|\omega|}(X^*)$ is lattice and isometrically isomorphic to $\left[\ell_p^\pi(X)\right]^*$*
(ii) [11, Corollary 3.3 and Corollary 3.4] *The Banach lattice $\left[\ell_{p^*}^\pi(X^*)\right]$ is lattice and isometrically isomorphic to $\left[\ell^u_{p,|\omega|}(X)\right]^*$*

Proof. (i) Let $1 < p < \infty$, we define the mapping

$$T : \ell_{p^*,|\omega|}(X^*) \longrightarrow \left[\ell_p^\pi(X)\right]^*$$
$$x^* = (x_n^*)_n \longmapsto T(x^*) = T_{x^*}$$

where T_{x^*} is the linear functional defined by

$$T_{x^*} : \ell_p^\pi(X) \longrightarrow \mathbb{K}$$
$$(x_n)_n \longmapsto T_{x^*}((x_n)_n) = \sum_{n=1}^{\infty} x_n^*(x_n).$$

The map T is clearly a positive map from $\left(\ell_{p^*,|\omega|}(X^*)\right)_+$ to $\left[\ell_p^\pi(X)\right]_+^*$, and it is homogeneous, additive and injective.To see that it is surjective, note that if $S \in \left[\ell_p^\pi(X)\right]_+^*$ and

$$I_n : X \longrightarrow \ell_p\langle X\rangle$$
$$x \longmapsto (0, ..., x, 0, ...)$$

$x_n^* = S \circ I_n \in X_+^*$ for all $n \in \mathbb{N}$.
Then

$$\begin{aligned} T_{(S\circ I_n)_n}((x_n)_n) &= \sum_{n=1}^{\infty}(S \circ I_n)(x_n) \\ &= \sum_{n=1}^{\infty} S(I_n(x_n)) \\ &= S(I_1(x_1)) + \cdots + S(I_n(x_n)) + \cdots \\ &= S((x_1, 0, ...)) + \cdots + S((0, ..., x_n, ...)) + \cdots \\ &= S((x_n)_n) \end{aligned}$$

For $x \in B_X^+$, we get

$$\begin{aligned} \left(\sum_{n=1}^{\infty}\langle x_n^*, x\rangle|^{p^*}\right)^{\frac{1}{p^*}} &= \left(\sum_{n=1}^{\infty} |S \circ I_n)(x)|^{p^*}\right)^{\frac{1}{p^*}} \\ &= \sup_{(\alpha_n)\in B_{\ell_p}^+} |\sum_{n=1}^{\infty} \alpha_n S \circ I_n(x)| \\ &= \sup_{(\alpha_n)\in B_{\ell_p}^+} |S((\alpha_n x)_n)| \\ &\leq \|S\| \sup_{(\alpha_n)\in B_{\ell_p}^+} \|(\alpha_n x)_n\|_{\ell_p\langle X\rangle} \end{aligned}$$

We need to estimate the latter expression. Note that

$$\begin{aligned} \|(\alpha_n x)_n\|_{\ell_p\langle X\rangle} &= \sup_{(x_n^*)_n\in B_{[\ell_{p^*,|\omega|}(X^*)]^+}} \sum_n \langle x_n^*, |\alpha_n x_n|\rangle \\ &= \sup_{(x_n^*)_n\in B_{[\ell_{p^*,|\omega|}(X^*)]^+}} \sum_n |\alpha_n|\langle x_n^*, |x_n|\rangle \\ &\leq \|(\alpha_n)_n\|_p. \sup_{(x_n^*)_n\in B_{[\ell_{p^*,|\omega|}(X^*)]^+}} \|(x_n)_n\|_{p^*}^{\omega} \\ &\leq \|(\alpha_n)_n\|_p \end{aligned}$$

Then

$$\begin{aligned}\left(\sum_{n=1}^{\infty}|S\circ I_n)(x)|^{p^*}\right)^{\frac{1}{p^*}} &\leq \|S\| \sup_{(\alpha_n)\in B^+_{\ell_p}} \|(\alpha_n)_n\|_p\\ &= \|S\|\end{aligned}$$

Hence, by (1.4), $(x_n^*)_n \in (\ell_{p^*,|\omega|}(X^*))_+$.

Since $\ell_{p^*,|\omega|}(X^*)$ is a Riesz space and $\left[\ell_p^\pi(X)\right]^* = \left[\ell_p^\pi(X)\right]^*_+ - \left[\ell_p^\pi(X)\right]^*_+$, it follows from Lemma 1.2 that $\left[\ell_p^\pi(X)\right]^*$ is a lattice space and that T is a lattice isomorphism from $\ell_{p^*,|\omega|}(X^*)$ onto $\left[\ell_p^\pi(X)\right]^*$.

For $(x_n^*)_n \in \ell_{p^*,|\omega|}(X^*)$,

$$\begin{aligned}\|T((x_n^*)_n)\|_{\left[\ell_p^\pi(X)\right]^*} &= \|T((x_n^*)_n)\|_{\mathcal{L}(\ell_p^\pi(X),\mathbb{K})}\\ &= \||T((x_n^*)_n)|\|_{\left[\ell_p^\pi(X)\right]^*}\\ &= \|T(|(x_n)_n|)\|_{\left[\ell_p^\pi(X)\right]^*}\\ &= \|(|x_n^*|)_n\|_{\ell_{p^*,|\omega|}(X^*)}\\ &= \|(x_n^*)_n\|_{\ell_{p^*,|\omega|}(X^*)}.\end{aligned}$$

This means that T is an isometry from $\ell_{p^*,|\omega|}(X^*)$ onto $\left[\ell_p^\pi(X)\right]^*$. □

2. Positive (p,q)-summing operators generated by $\ell^u_{p,|\omega|}(X)$

Let $1 \leq q \leq p < \infty$. An operator $T : X \longrightarrow F$ is said to be positive (p,q)-summing [4] if there exists a constant $C > 0$ such that for every $x_1, x_2, ..., x_n \in X$, we have

$$\|(T(x_i))_{i=1}^n\|_{\ell_p(F)} \leq C \|(x_i)_{i=1}^n\|_{\ell_{q,|\omega|}(X)}. \tag{2.1}$$

For $q < p = \infty$,

$$\sup \|T(x_i)\| \leq C \|(x_i)_{i=1}^n\|_{\ell_{q,|\omega|}(X)}.$$

We shall denote by $\Lambda_{p,q}(X,F)$ the space of positive (p,q)-summing operators. This space becomes a Banach space with the norm $\|\cdot\|_{\Lambda_{p,q}}$ given by the infimum of the constants verifying (2.1). For $p = \infty$ and $1 \leq q < \infty$ we consider $\Lambda_{\infty,q}(X,Y) = \mathcal{L}(X,F)$ and $\|T\|_{\Lambda_{\infty,q}} = \|T\|$.

Now, we give characterizations of these classes in terms of transformations of lattice vector-valued sequences.

Proposition 2.1. [4, Proposition 2] *Let $T : X \longrightarrow F$ be an operator and $1 \leq q \leq p \leq \infty$, the following are equivalent.*

(1) $T \in \Lambda_{p,q}(X,F)$.

(2) *The associated operator $\widehat{T} : \ell_{q,|\omega|}(X) \longrightarrow \ell_p(F)$ given by $\widehat{T}((x_i)_{i=1}^\infty) = (T(x_i))_{i=1}^\infty, (x_i)_{i=1}^\infty \in \ell_{q,|\omega|}(X)$ is well-defined and continuous.*

In this case $\|T\|_{\Lambda_{p,q}} = \|\hat{T}\|$.

Theorem 2.2. *From a continuous linear $T \in \mathcal{L}(X,F)$ and $1 \leq q \leq p \leq \infty$, the following condition are equivalent.*

(i) $T \in \Lambda_{p,q}(X,F)$.

(ii) The sequence $(T(x_i))_i \in \ell_p(F)$ whenever $(x_i)_i \in \ell^u_{q,|\omega|}(X)$.

(iii) The induced map

$$\widehat{T}:\ell^u_{q,|\omega|}(X)\to\ell_p\left(F\right),\widehat{T}\left((x_i)_{i=1}^{\infty}\right)=(T(x_i))_{i=1}^{\infty}.$$

is a well-defined continuous linear operator and $\|T\|_{\Lambda_{p,q}}=\|\widehat{T}\|$.

Proof. $(i)\Rightarrow(ii)$. Let $x=(x_i)_{i=1}^{\infty}\in\ell^u_{q,|\omega|}(X)$, we have

$$\|(T(x_i))_{i=1}^{n}\|_{\ell_p(F)}\le C\left\|(x_i)_{i=1}^{n}\right\|_{\ell_{q,|\omega|}(X)},$$

for all $n\in\mathbb{N}$. So, if $m_1>m_2$,

$$\begin{aligned}\|(T(x_i))_{i=1}^{m_1}-(T(x_i))_{i=1}^{m_2}\|_{\ell_p(F)}&=\left\|(T(x_i))_{i=m_2+1}^{m_1}\right\|_{\ell_p(F)}\\&\le C\left\|(x_i)_{i=m_2+1}^{m_1}\right\|_{\ell_{q,|\omega|}(X)}.\end{aligned}$$

We conclude that $(y_n)_{n=1}^{\infty}$ with $y_n=(T(x_i))_{i=1}^{n}$ is Cauchy in $\ell_p(F)$ and so converges to some $(z_i)_{i=1}^{\infty}\in\ell_p(F)$.
Given $\varepsilon>0$, we can find $N_o\in\mathbb{N}$ so that

$$n\ge N_0\Rightarrow\|(T(x_i))_{i=1}^{n}-(z)_{i=1}^{\infty}\|_{\ell_p(F)}<\varepsilon.$$

So, for a fixed $i_0\in\mathbb{N}$, we have

$$\|T(x_{i_0})-z_{i_0}\|<\varepsilon.$$

We conclude that $T(x_{i_0})=z_{i_0}$. Hence $(T(x_i))_{i=1}^{\infty}=(z_i)_{i=1}^{\infty}\in\ell_p(F)$.

$(ii)\Rightarrow(iii)$. Is it clear that T is linear implies that $\widehat{T}$ is linear, for show that $\widehat{T}$ is continuous we show that $\widehat{T}$ is a closed graph. Suppose that the sequence $(T(x_i)_{i=1}^{\infty})\in\ell_p(F)$ whenever $(x_i)_{i=1}^{\infty}\in\ell^u_{q,|\omega|}(X)$ and let $\left((x_i,\widehat{T}(x_i))\right)_{i=1}^{\infty}$ a convergent sequence in the Cartesian product $\ell^u_{q,|\omega|}(X)\times\ell_p(F)$ that is, $\left((x_i,\widehat{T}(x_i)\right)\longrightarrow(x,y)$. So

$$x_k\longrightarrow x=(x_n)_{n=1}^{\infty}\in\ell^u_{q,|\omega|}(X),\tag{2.2}$$

and

$$\hat{T}(x_i)\longrightarrow y=(y_n)_{n=1}^{\infty}\in\ell_p(F).\tag{2.3}$$

From (2.2), for all $\varepsilon>0$ there exist $k>0$ such that

$$\begin{aligned}|x^*(x_i^k-x_i)|^q&\le(|x^*||(x_i^k-_i)|)^q\le\sum_{i=1}^{\infty}(x^*|(x_i^k-x_i)|)^q\le\sup_{x^*\in B_{X^*_+}}\left(\sum_{i=1}^{\infty}(x^*|(x_i^k-x_i)|)^q\right)\\&\le\|x^k-x\|^q_{\ell^u_{q,|\omega|}}\\&\le\varepsilon^q\end{aligned}$$

whenever $k>k_0$, $x^*\in B_{X^*_+}$ and for all $i\in\mathbb{N}$ in this way, by Hanh-Banach Theorem, we get

$$\|x_i^k-x_i\|^q=\sup_{x^*\in B_{X^*}}|x^*(x_i^k-x_i)|^q\le2^q\sup_{x^*\in B_{X^*_+}}|x^*(x_i^k-x_i)|^q\le2^q\xi^q.\tag{2.4}$$

whenever $k>k_0$ and for all $i\in\mathbb{N}$, then we have $x_i^k\longrightarrow x_i\in X$ for all $k\longrightarrow\infty$. How T is continuous, we find

$$\lim_k T(x_i^k)=T(x_i),\ \text{for all}\quad i\in\mathbb{N}$$

From (2.3), for all $\varepsilon > 0$ there exist $k^{'} > 0$ such that

$$\begin{aligned}\|T(x_i^{k^{'}}) - y_i\|^p \leq \sum_{i=1}^{\infty} \|T(x_i^{k^{'}}) - y_i\|^p &\leq \|(T(x_i^{k^{'}}))_{i=1}^{\infty} - (y_i)_{i=1}^{\infty}\|_p^p \\ &= \|\widehat{T}(x^{k^{'}}) - y\|_p^p \\ &\leq \varepsilon^p,\end{aligned}$$

whenever $k^{'} > k_0^{'}$ and for all $i \in \mathbb{N}$, we find

$$\lim_k T(x_i^{k^{'}}) = y_i, \quad \text{for all} \quad i \in \mathbb{N}. \tag{2.5}$$

From (2.4), (2.5) and uniqueness of the limit, it follows that

$$\widehat{T}(x) = (T(x_i)_{i=1}^{\infty}) = (y_i)_{i=1}^{\infty} = y.$$

This implies that the linear mapping $\widehat{T}$ has a closed graph.

$(iii) \Rightarrow (i)$ Is straightforward.

□

3. Positive strongly (p, q)-summing operators generated by $\ell_p^{\pi}(Y)$

Let $1 \leq q \leq p \leq \infty$ Recall that an operator $T \in \mathcal{L}(E, Y)$ is called positive strongly (p, q)-summing [1] if there exists a constant $C > 0$ such that for all finite sets $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset (Y^*)^+$, we have

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| \leq C \|(x_i)_{i=1}^n\|_q . \|(y_i^*)_{i=1}^n\|_{p^*,\omega}. \tag{3.1}$$

We shall denote by $\mathcal{D}_{p,q}^+(E, Y)$ the space of positive strongly (p, q)-summing operators or $\mathcal{D}_p^+(E, Y)$ if $p = q$, the space of positive strongly p-summing operators. This space becomes a Banach space with the norm $\|\cdot\|_{\mathcal{D}_{p,q}^+}$ given by the infimum of the constants verifying (3.1).

Lemma 3.1. *$T \in \mathcal{D}_{p,q}^+(E, Y)$ if and only if there exists a constant $C > 0$ such that for all finite sets $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset (Y^*)^+$, we have*

$$\|(T(x_i)_{i=1}^n\|_{\ell_p^{\pi}(Y)} \leq C \|(x_i)_{i=1}^n\|_q. \tag{3.2}$$

Proof. Let $T \in \mathcal{D}_{p,q}^+(E, Y)$ then there exists a constant $C > 0$ such that for all finite sets $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset (Y^*)^+$, we have

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* | \rangle \leq C \|(x_i)_{i=1}^n\|_q . \|(y_i^*)_{i=1}^n\|_{p^*,\omega}.$$

For each $z \in Y$ and $z^* \in Y^*$,

$$|z^*|(|z|) = \sup \{|g^*(z)| : |g^*| \leq |z^*|\}. \tag{3.3}$$

Now let $(y_i^*)_{i=1}^n \in (\ell_{p^*,|\omega|}(Y^*))^+$ and $\varepsilon > 0$, from (3.3) there exists for each $1 \leq i \leq n$, a $g_i^* \in Y^*$ such that $|g_i^*| \leq |y_i^*|$ and

$$|y_i^*|(|T(x_i)|) \leq |g_i^*(T(x_i))| + \frac{\varepsilon}{n}.$$

Note that $(g_i^*)_{i=1}^n \in (\ell^n_{p^*,|\omega|}(Y^*))^+$. Then

$$\begin{aligned}\sum_{i=1}^{n}\langle|T(x_i)|, y_i^*\rangle = \sum_{i=1}^{n}\langle|T(x_i)|, |y_i^*|\rangle &\leq \sum_{i=1}^{n}|\langle T(x_i), g_i^*\rangle| + \varepsilon \\ &\leq C\,[\|(x_i)_{i=1}^n\|_q.\|(g_i^*)_{i=1}^n\|_{p^*,\omega}] + \varepsilon \\ &\leq C\,[\|(x_i)_{i=1}^n\|_q.\|(y_i^*)_{i=1}^n\|_{p^*,\omega}] + \varepsilon.\end{aligned}$$

Then

$$\|(T(x_i)_{i=1}^n\|_{\ell_p^\pi(Y)} \leq C\|(x_i)_{i=1}^n\|_q + \varepsilon.$$

Conversely, directly by $|\langle T(x_i), y_i^*\rangle| \leq \langle|T(x_i)|, y_i^*\rangle$ for every i. □

As in classical cases, the natural approach to presenting the summability properties of positive strongly (p,q)-summing operators by defining the corresponding operator between appropriate lattice sequence spaces..

Proposition 3.2. *Let $T : E \longrightarrow Y$ be an operator and $1 \leq q \leq p \leq \infty$, the following are equivalent:*

(1) *$T \in \mathcal{D}^+_{p,q}(E,Y)$.*

(2) *The associated operator $\widehat{T} : \ell_q(E) \longrightarrow \ell_p^\pi(Y)$ given by $\widehat{T}\,((x_i)_{i=1}^\infty) = (T(x_i))_{i=1}^\infty, (x_i)_{i=1}^\infty \in \ell_q(E)$ is well-defined and continuous.*

In this case $\|T\|_{\mathcal{D}^+_{p,q}} = \|\widehat{T}\|$.

Proof. For the necessity, let $T \in \mathcal{D}^+_{p,q}(E,Y)$, $(x_i)_{i=1}^\infty \in \ell_q(E)$ and $(y_i^*)_{i=1}^\infty \in (\ell_{p^*,|\omega|}(Y^*))^+$. So we have

$$\begin{aligned}\textstyle\sum_{i=1}^\infty \langle|T(x_i)|, y_i^*\rangle &= \sup_n \textstyle\sum_{i=1}^n \langle|T(x_i)|, y_i^*\rangle \\ &\leq \|T\|_{\mathcal{D}^+_{p,q}} \sup_n \|(x_i)_{i=1}^n\|_q.\|(y_i^*)_{i=1}^n\|_{p^*,\omega} \\ &\leq \|T\|_{\mathcal{D}^+_{p,q}} \|(x_i)_{i=1}^\infty\|_q.\|(y_i^*)_{i=1}^\infty\|_{p^*,\omega}.\end{aligned}$$

Since this holds for all $(y_i^*)_{i=1}^\infty \in (\ell_{p^*,|\omega|}(Y^*))^+$, we obtain

$$\|(T(x_i))_{i=1}^\infty\|_{\ell_p^\pi(Y)} = \sup_{\|(y_i^*)_{i=1}^\infty\|_{p^*,\omega} \leq 1} \sum_{i=1}^\infty \langle|T(x_i)|, y_i^*\rangle \leq \|T\|_{\mathcal{D}^+_{p,q}} \|(x_i)_{i=1}^\infty\|_q\,,$$

and then $\widehat{T}$ is continuous with norm $\leq \|T\|_{\mathcal{D}^+_{p,q}}$.

In order to prove sufficiency, suppose $\widehat{T}$ is well-defined and continuous and assume that $T \notin \mathcal{D}^+_{p,q}(E,Y)$. Then for each $n \in \mathbb{N}$, we may choose a finite sequence $(x_{i,n})_{i=1}^{m_n} \subset E$ such that

$$\left\|(x_{i,n})_{i=1}^{m_n}\right\|_q \leq 1 \quad \text{and} \quad \|(T(x_{i,n}))_{i=1}^{m_n}\|_{\ell_p^\pi(Y)} > 2^n,$$

which implies

$$\sum_{i=1}^{m_n}\langle|T(x_{i,n})|, y_{i,n}^*\rangle > 2^{2n} \tag{3.4}$$

for some $(y_{i,n}^*)_{i=1}^{m_n} \in (\ell_{p^*,|\omega|}(Y^*))^+$ such that $\left\|(y_{i,n}^*)_{i=1}^{m_n}\right\|_{p^*,\omega} \leq 1$. Let $(z_j)_{j=1}^\infty$ be the sequence

$$\begin{aligned}&\left(\left(\frac{x_{i,1}}{2^1}\right)_{i=1}^{m_1}, \left(\frac{x_{i,2}}{2^2}\right)_{i=1}^{m_2}, ..., \left(\frac{x_{i,n}}{2^n}\right)_{i=1}^{m_n}, ...\right) \\ &= \left(\frac{x_{1,1}}{2^1}, \frac{x_{2,1}}{2^1}, ..., \frac{x_{m_1,1}}{2^1}, \frac{x_{1,2}}{2^2}, \frac{x_{2,2}}{2^2}, ..., \frac{x_{m_2,2}}{2^2}, ..., \frac{x_{1,n}}{2^n}, \frac{x_{2,n}}{2^n}, ..., \frac{x_{m_n,n}}{2^n}, ...\right).\end{aligned}$$

We have

$$\left\|(z_j)_j^{\infty}\right\|_q = \left(\sum_{j=1}^{+\infty}\sum_{i=1}^{m_j}\left\|\frac{x_{i,j}}{2^j}\right\|^q\right)^{\frac{1}{q}} = \left(\sum_{j=1}^{+\infty}\frac{1}{2^{jq}}\left\|(x_{i,j})_{i=1}^{m_j}\right\|_q^q\right)^{\frac{1}{q}} \leq \left(\sum_{j=1}^{+\infty}\frac{1}{2^{jq}}\right)^{\frac{1}{q}} \leq 1.$$

Then, $(z_j)_{j=1}^{\infty} \in \ell_q(E)$. However, $\widehat{T}((z_j)_{j=1}^{\infty}) \notin \ell_p^{\pi}(Y)$. In order to see this, consider the sequences

$$(\varphi_j)_{j=1}^{\infty} = \left(\left(\frac{y_{i,1}^*}{2^1}\right)_{i=1}^{m_1}, \left(\frac{y_{i,2}^*}{2^2}\right)_{i=1}^{m_2}, ..., \left(\frac{y_{i,n}^*}{2^n}\right)_{i=1}^{m_n}, ...\right).$$

Clearly $(\varphi_j)_{j=1}^{\infty} \in B_{(\ell_{p^*,|\omega|}(Y^*))^+}$. Then

$$\begin{aligned}\left\|(\varphi_j)_{j=1}^{\infty}\right\|_{p^*,|\omega|} &= \sup_{y\in B_{Y_+}}\left(\sum_{j=1}^{+\infty}\sum_{i=1}^{m_j}\langle x, \frac{y_{i,j}^*}{2^j}\rangle^{p^*}\right)^{\frac{1}{p^*}} = \sup_{y\in B_{Y_+}}\left(\sum_{j=1}^{+\infty}\frac{1}{2^{jp^*}}\sum_{i=1}^{m_j}\langle x, y_{i,j}^*\rangle^{p^*}\right)^{\frac{1}{p^*}}\\ &\leq \left(\sum_{j=1}^{+\infty}\frac{1}{2^{jp^*}}\right)^{\frac{1}{p^*}} \leq 1.\end{aligned}$$

By (3.4) it turns out that

$$\begin{aligned}\|\widehat{T}((z_j)_{j=1}^{\infty})\|_{\ell_p^{\pi}(Y)} = \|(T(z_j))_{j=1}^{\infty})\|_{\ell_p^{\pi}(Y)} &= \sup_{\left\|(\xi_j)_{j=1}^{\infty}\right\|_{p^*,\omega}\leq 1}\sum_{j=1}^{\infty}\langle|T(z_j)|, \xi_j\rangle\\ &\geq \sum_{j=1}^{\infty}\langle|T(z_j)|, \varphi_j\rangle\\ &= \sum_{j=1}^{\infty}\frac{1}{2^{2j}}\sum_{i=1}^{m_j}\langle|T(x_{i,j})|, y_{i,j}^*\rangle = \infty\end{aligned}$$

which according (1.5) is a contradiction with the fact that $\widehat{T}$ maps $\ell_q(X)$ (continuously) into $\ell_p^{\pi}(Y)$. Since

$$\|(T(x_i))_{i=1}^{\infty})\|_{\ell_p^{\pi}(Y)} = \|\widehat{T}((x_i)_{i=1}^{\infty})\|_{\ell_p^{\pi}(Y)} \leq \|\widehat{T}\| \left\|(x_i)_{i=1}^{\infty}\right\|_q,$$

we have $\|T\|_{\mathcal{D}_{p,q}^+} \leq \|\widehat{T}\|$. □

In the following result, we characterize the class of positive summing linear operators and positive strongly summing linear operators by utilizing the adjoint operator. For the proof of this result, we will utilize the duality of lattice sequence spaces. Theorem 3.3 was established in [1, Theorem 4.6], and the proof provided there is direct. Using Theorem 1.3, (1.2), Proposition 3.2 and taking into account that the adjoint of the $\widehat{T} : \ell_q(E) \longrightarrow \ell_p^{\pi}(Y)$ can be identified with the operator $\widehat{T^*} : \ell_{p^*,|\omega|}(Y^*) \longrightarrow \ell_{q^*}(E^*); \widehat{T^*}((y_i^*)_i) = (T^*(y_i^*))_i$, we provide an alternative proof of the results in Theorem 3.3.

Theorem 3.3. *Let $T : E \longrightarrow Y$ be an operator and $1 \leq q \leq p \leq \infty$.*

1. *The operator T belongs to $\Lambda_{p,q}(X, F)$ if and only if its adjoint T^* belongs to $\mathcal{D}_{q^*,p^*}^+(F^*, X^*)$. Furthermore, $\|T\|_{\Lambda_{p,q}} = \|T^*\|_{\mathcal{D}_{q^*,p^*}^+}$.*
2. *The operator T belongs to $\mathcal{D}_{p,q}^+(E, Y)$ if and only if its adjoint T^* belongs to $\Lambda_{q^*,p^*}(Y^*, E^*)$. Furthermore, $\|T\|_{\mathcal{D}_{p,q}^+} = \|T^*\|_{\Lambda_{q^*,p^*}}$.*

Proof. (1) Let $T \in \mathcal{L}(X, F)$ and $T^* \in \mathcal{L}(F^*, X^*)$ its adjoint. Suppose that $T \in \Lambda_{p,q}(X, F)$, then by Theorem (2.2) $\widehat{T} : \ell^u_{q,|\omega|}(X) \longrightarrow \ell_p(F)$ is continuous with $\|T\|_{\Lambda_{p,q}} = \|\widehat{T}\|$. By (1.2) and Theorem (1.3), the following diagram commutes

$$\begin{array}{ccc} \ell_p(F)^* & \xrightarrow{\widehat{T}^*} & \ell^u_{q,|\omega|}(X)^* \\ J_1 \uparrow & & \uparrow J_2 \\ \ell_{p^*}(F^*) & \xrightarrow{\widehat{T^*}} & \ell^\pi_{q^*}(X^*) \end{array}$$

i.e. $\widehat{T}^* \circ J_1 = J_2 \circ \widehat{T^*}$, where J_1 is an isometric isomorphism and J_2 is an isometric lattice isomorphism such that $J_i((z_n^*)_n)((z_n)_n) = f((z_n)_n) = \sum_{n=1}^{\infty} \langle z_n, z_n^* \rangle, i = 1, 2$, with the inverse I_i, defined by $I_i(f) = (f \circ I_n)_n = (z_n^*)_n$. In fact, the map $\widehat{T^*}$, defined by $(y_n^*)_n \mapsto (T^*(y_n^*))_n$, let $(y_n^*)_n \in \ell_{p^*}(F^*)$, then for all $(x_n)_n \in \ell^u_{q,|\omega|}(X)$,

$$\begin{aligned} (\widehat{T}^* \circ J_1)((y_n^*)_n)((x_n)_n) &= J_1((y_n^*)_n)\left(\widehat{T}((x_n)_n)\right) \\ &= J_1((y_n^*)_n)((T(x_n))_n) \\ &= \sum_{n=1}^{\infty} \langle y_n^*, T(x_n) \rangle \\ &= \sum_{n=1}^{\infty} \langle T^*(y_n^*), x_n \rangle \\ &= J_2((T^*(y_n^*))_n)((x_n)_n) \\ &= J_2(\widehat{T^*}((y_n^*)_n)((x_n)_n) \\ &= (J_2 \circ \widehat{T^*})((y_n^*)_n)((x_n)_n). \end{aligned}$$

i.e., $\widehat{T}^* \circ J_1 = J_2 \circ \widehat{T^*}$. Then, $\widehat{T}$ is well-defined and continuous if and only if $\widehat{T^*}$ is well-defined and continuous. Consequently, from Proposition 3.2, it follows that T is positive (p, q)-summing, if and only if its adjoint $T^* \in \mathcal{L}(F^*, X^*)$ is strongly positive (q^*, p^*)-summing. Furthermore, $\|T\|_{\Lambda_{p,q}} = \|T^*\|_{\mathcal{D}^+_{q^*,p^*}} = \|\widehat{T}\|$.

(2) Let $T \in \mathcal{D}^+_{p,q}(E, Y)$. Then, by Proposition (3.2), the operator $\widehat{T} : \ell_q(E) \longrightarrow \ell^\pi_p(Y)$ is continuous with $\|T\|_{\mathcal{D}^+_{p,q}} = \|\widehat{T}\|$. Using Theorem (1.3), and taking into account that the adjoint of the operator $\widehat{T} : \ell_q(E) \longrightarrow \ell^\pi_p(Y)$ can be identified with the operator

$$\widehat{T^*} : \ell_{p^*,|\omega|}(Y^*) \longrightarrow \ell_{q^*}(E^*) \text{ given by } \widehat{T^*}((y_i^*)_i) = (T^*(y_i^*))_i,$$

it follows that $\widehat{T}$ and $\widehat{T^*}$ are well-defined and continuous. Therefore, from Proposition (2.1) it follows that T is positive strongly (p, q)-summing if and only if its adjoint T^* is positive (q^*, p^*)-summing, and $\|T\|_{\mathcal{D}^+_{p,q}} = \|T^*\|_{\Lambda_{q^*,p^*}} = \|\widehat{T}\|$. □

Corollary 3.4. *Let $1 \leq q \leq p \leq \infty$.*

(1) *The operator $T \in \mathcal{L}(E, Y)$ belongs to $\mathcal{D}^+_{p,q}(E, Y)$ if and only if T^{**} belongs to $\mathcal{D}^+_{p,q}(E^{**}, Y^{**})$. Furthermore,*

$$\|T\|_{\mathcal{D}^+_{p,q}} = \|T^{**}\|_{\mathcal{D}^+_{p,q}}.$$

(2) *The operator* $T \in \mathcal{L}(X,F)$ *belongs to* $\Lambda_{p,q}(X,F)$ *if and only if* T^{**} *belongs to* $\Lambda_{p,q}(X^{**},F^{**})$ *. Furthermore,*

$$\|T\|_{\Lambda_{p,q}} = \|T^{**}\|_{\Lambda_{p,q}}.$$

We say that an operators $T : E \longrightarrow Y$ is called positive (p,q)-majorizing (see [7] for $p = q$) if there exists a constant $C > 0$ such that

$$\left(\sum_{i=1}^{n} |\langle T(z_i), y_i^*\rangle|^{q^*}\right)^{\frac{1}{q^*}} \leq C\|(y_i^*)_{i=1}^n\|_{p^*,\omega} \tag{3.5}$$

for all $(z_i)_{i=1}^n$ in B_E and $(y_i^*)_{i=1}^n$ in $(Y^*)^+$. The space of all positive (p,q)-majorizing from E to Y is denoted by $\Upsilon_{p,q}(E,Y)$. this space becomes a Banach space with the norm $\|\cdot\|_{\Upsilon_{p,q}}$ given by the infimum of the constants C satisfying (3.5). In [7], the authors proved the duality relationships between positive p-summing operators and positive p-majorizing operators. It was known [1] that an operator $T : X \longrightarrow F$ is positive p-summing if and only if T^* is positive strongly p^*-summing. Similarly, an operator $T : E \longrightarrow Y$ is positive strongly p-summing if and only if T^* is positive p^*-summing. In the following, we directly prove that the concept of positive strongly p-summing and the concept of positive p-majorizing are equivalent.

Theorem 3.5. *Let* $T : E \longrightarrow Y$ *be an operator, the following are equivalent:*

(1) T *is positive* (p,q)*-majorizing.*
(2) T *is positive strongly* (p,q)*-summing.*

Proof. Suppose that T is positive (p,q)-majorizing, given any finite sequence $(x_i)_{i=1}^n$ in E and $(y_i^*)_{i=1}^n$ in $(Y^*)^+$, we get

$$\begin{aligned}\sum_{i=1}^{n} |\langle T(x_i), y_i^*\rangle| &= \sum_{i=1}^{n} \|x_i\||\langle T(\frac{x_i}{\|x_i\|}), y_i^*\rangle| \\ &\leq \left(\sum_{i=1}^{n} \|x_i\|^q\right)^{\frac{1}{q}} \cdot \left(\sum_{i=1}^{n} |\langle T(\frac{x_i}{\|x_i\|}), y_i^*\rangle|^{q^*}\right)^{\frac{1}{q^*}} \\ &\leq \|T\|_{\Upsilon_p}.\|(x_i)_{i=1}^n\|_q.\|(y_i^*)_{i=1}^n\|_{p^*,|\omega|}.\end{aligned}$$

This implies that T is positive strongly (p,q)-summing and $\|T\|_{D_{p,q}^+} \leq \|T\|_{\Upsilon_{p,q}}$.
Conversely, assume that T is positive strongly (p,q)-summing. Let $(z_i)_{i=1}^n$ be a finite sequence in B_E and $(y_i^*)_{i=1}^n$ in $(Y^*)^+$, we have

$$\begin{aligned}\left(\sum_{i=1}^{n} |\langle T(z_i), y_i^*\rangle|^{q^*}\right)^{\frac{1}{q^*}} &= \sup_{(\lambda_i)_{i=1}^n \in B_{\ell_q}} |\sum_{i=1}^{n} \lambda_i\langle T(z_i), y_i^*\rangle| \\ &= \sup_{(\lambda_i)_{i=1}^n \in B_{\ell_q}} |\sum_{i=1}^{n} \langle T(\lambda_i z_i), y_i^*\rangle| \\ &\leq \|T\|_{D_{p,q}^+}. \sup_{(\lambda_i)_{i=1}^n \in B_{\ell_q}} \|(\lambda_i z_i)_{i=1}^n\|_q.\|(y_i^*)_{i=1}^n\|_{p^*,|\omega|} \\ &\leq \|T\|_{D_{p,q}^+}. \sup_{(\lambda_i)_{i=1}^n \in B_{\ell_q}} \|(\lambda_i)_{i=1}^n\|_q.\|(y_i^*)_{i=1}^n\|_{p^*,|\omega|} \\ &= \|T\|_{D_{p,q}^+}.\|(y_i^*)_{i=1}^n\|_{p^*,|\omega|}.\end{aligned}$$

This means that T is positive (p,q)-majorizing and $\|T\|_{\Upsilon_{p,q}} \leq \|T\|_{D^+_{p,q}}$. □

Corollary 3.6. *$T \in \mathcal{L}(E,Y)$ is positive p-majorizing if and only if T is positive strongly p-summing.*

4. Positive Cohen (p,q)-nuclear operators

Cohen [8] introduced the concept of p-nuclear operators, which was extended to the Cohen (p,q)-nuclear operators by Apiola [3]. Let $1 < p,q \leq \infty$. An operator $T \in \mathcal{L}(E,F)$ is Cohen (p,q)-nuclear if $(T(x_n))_n \in \ell_p\langle F\rangle$ whenever $(x_n)_n \in \ell_{q,\omega}(E)$. We denote the space of Cohen (p,q)-nuclear operators by $\mathcal{CN}_{(p,q)}(E,F)$. According to [3, 8], the following conditions are equivalent for a linear operator $T \in \mathcal{L}(E,F)$.

$$T \in \mathcal{CN}_{(p,q)}(E,F) \Longleftrightarrow \widehat{T} \in \mathcal{L}(\ell_{q,\omega}(E), \ell_p\langle F\rangle), \tag{4.1}$$

where $\widehat{T}((x_n)_n) = (T(x_n))_n$, for every $(x_n)_n \in \ell_{q,\omega}(E)$.

In this section, we introduce the positive Cohen (p,q)-nuclear operators. For $p = q$, these operators are closely linked to positive strongly p-summing and positive p-summing operators, as stated in Kwapien's Factorization Theorem (see [16, Proposition 2]). Here, we distinguish three cases:

Definition 4.1. *Let $1 \leq q \leq p < \infty$ and X, Y be Banach lattices, E and F be Banach spaces.*

(a) [23] *An operator T from a Banach lattice X to a Banach space F is left positive Cohen (p,q)-nuclear if there exists a constant $C > 0$ such that for all $(x_i)_{i=1}^n \subset X$ (of positive elements), we have*

$$\|(T(x_i))_{i=1}^n\|_{\ell_p\langle F\rangle} \leq C\|(x_i)_{i=1}^n\|_{q,|\omega|}. \tag{4.2}$$

(b) *An operator T from a Banach space E to a Banach lattice Y is right positive Cohen (p,q)-nuclear if there exists a constant $C > 0$ such that for all $(x_i)_{i=1}^n \subset E$, we have*

$$\|(T(x_i))_{i=1}^n\|_{\ell_p^\pi(Y)} \leq C\|(x_i)_{i=1}^n\|_{q,\omega}. \tag{4.3}$$

(c) *An operator T from a Banach lattice X to a Banach lattice Y is positive Cohen (p,q)-nuclear if there exists a constant $C > 0$ such that for all $(x_i)_{i=1}^n \subset X$, we have $\|(T(x_i))_{i=1}^n\|_{\ell_p^\pi(Y)} \leq C\|(x_i)_{i=1}^n\|_{q,|\omega|}$. (see* [7, Definition 3.1] *for $r = 1$).*

The class of all positive Cohen (p,q)-nuclear operators from X to Y (respectively X to F and E to Y) is denoted by $\mathcal{CN}^+_{p,q}$ (respectively $\mathcal{CN}^{left,+}_{p,q}(X,F)$ and $\mathcal{CN}^{right,+}_{p,q}$) we put $\|T\|_{\mathcal{CN}^+_p} = \inf C$.

The proof of the following results follows similar lines as in Proposition 3.2 and Proposition 3.22 in [23] and is omitted.

Proposition 4.2. *Let $1 \leq q \leq p < \infty$ and X, Y be Banach lattices, E and F be Banach spaces.*

(1) [23, Proposition 3.22] *$T \in \mathcal{CN}^{left,+}_{p,q}(X,F)$; if and only if $\widehat{T} : \ell_{q,|\omega|}(X) \longrightarrow \ell_p\langle F\rangle$ is is a well-defined continuous linear operator.*

(2) *$T \in \mathcal{CN}^{right,+}_{p,q}(E,Y)$, if and only if $\widehat{T} : \ell_{q,\omega}(E) \longrightarrow \ell^\pi_p(Y)$ is is a well-defined continuous linear operator.*

(3) *$T \in \mathcal{CN}^+_{p,q}(X,Y)$, if and only if $\widehat{T} : \ell_{q,|\omega|}(X) \longrightarrow \ell^\pi_p(Y)$ is is a well-defined continuous linear operator.*

A result by Apiola states that the adjoint of a Cohen (p,q)-nuclear linear operator is Cohen (q^*,p^*)-nuclear linear operator. When $p=q$, this result appears in [8]. Utilizing Theorem 1.3, (1.1) and (1.2) and taking into account that the adjoint of the operators $\widehat{T}:\ell_{q,|\omega|}(X)\longrightarrow\ell_p\langle F\rangle$, $\widehat{T}:\ell_{q,\omega}(E)\longrightarrow\ell_p^{\pi}(Y)$ and $\widehat{T}:\ell_{q,|\omega|}(X)\longrightarrow\ell_p^{\pi}(Y)$ can be identified with the operators

$$\widehat{T^*}:\ell_{p^*,\omega}\left(F^*\right)\longrightarrow\ell_{q^*}^{\pi}(X^*),\widehat{T^*}:\ell_{q,|\omega|}\left(Y^*\right)\longrightarrow\ell_{q^*}\left\langle E^*\right\rangle \text{ and } \widehat{T^*}:\ell_{p^*,|\omega|}\left(Y^*\right)\longrightarrow\ell_{q^*}^{\pi}(X^*),$$

defined as $\widehat{T^*}((x_n^*)_n)=(T^*(x_n^*))_n$, we extend this to positive Cohen (p,q)-nuclear operators.

Theorem 4.3. *Let $1\leq q\leq p<\infty$ and X, Y be Banach lattices, E and F be Banach spaces .*

(1) *The operator T belongs to $\mathcal{CN}_{p,q}^{left,+}(X,F)$ if and only if its adjoint T^* belongs to $\mathcal{CN}_{q^*,p^*}^{right,+}(F^*,X^*)$. Furthermore,*

$$\|T\|_{\mathcal{CN}_{p,q}^{left,+}}=\|T^*\|_{\mathcal{CN}_{q^*,p^*}^{right,+}}.$$

(2) *The operator T belongs to $\mathcal{CN}_{p,q}^{right,+}(E,Y)$ if and only if its adjoint T^* belongs to $\mathcal{CN}_{q^*,p^*}^{left,+}(Y^*,E^*)$. Furthermore,*

$$\|T\|_{\mathcal{CN}_{p,q}^{right,+}}=\|T^*\|_{\mathcal{CN}_{q^*,p^*}^{left,+}}.$$

(3) *The operator T belongs to $\mathcal{CN}_{p,q}^{+}(X,Y)$ if and only if its adjoint T^* belongs to $\mathcal{CN}_{q^*,p^*}^{+}(Y^*,X^*)$. Furthermore,*

$$\|T\|_{\mathcal{CN}_{p,q}^{+}}=\|T^*\|_{\mathcal{CN}_{q^*,p^*}^{+}}.$$

Remark 4.4. *In a recent paper* [7, Definition 3.1]*, the authors introduced the concept of positive (p,q)-dominance, where $1/p+1/q=1/r$, defined between Banach lattices. Within this framework, both the Pietsch's Domination Theorem and Kwapien's Factorization Theorem are established. This concept precisely aligns with the positive Cohen p-nuclear concept presented here when $r=1$. Thus, by referring the reader to the papers* [7, Theorem 3.3 and Theorem 3.7]*, we can also derive the well-known theorems, namely Pietsch's Domination Theorem and Kwapien's Factorization Theorem, for the other two concepts proposed here (for left and right positive Cohen p-nuclear). Notice that Kwapien's Factorization Theorem ensures that positive Cohen p-nuclear are closely related to positive strongly p-summing and positive p-summing operators.*

5. Tensor characterizations

Now we are interested to characterize the aforementioned classes using abstract summability properties linked to the continuity of tensor product operators defined within vector-valued sequence spaces.

The Wittstock injective tensor Product and Fremlin projective tensor Product. For Banach lattices X and Y, let $X\otimes Y$ denote the algebraic tensor product of X and Y . For each $u=\sum_{i=1}^{n}x_i\otimes y_i\in X\otimes Y$, define $T_u:X^*\to Y$ by $T_u(x^*)=\sum_{i=1}^{n}x^*(x_i)y_i$ for each $x^*\in X^*$. The injective cone on $X\otimes Y$ is defined to be

$$C_i=\left\{u=\sum_{i=1}^{n}x_i\otimes y_i\in X\otimes Y:T_u(x^*)\in Y_+,\forall x^*\in X_+^*\right\}.$$

Wittstock [24, 25] introduced the positive injective tensor norm on $X\otimes Y$ as follows:

$$\|u\|_i=\inf\left\{\sup\left\{\|T_v(x^*)\|:x^*\in B_{X_+^*}\right\}:v\in C_i,u\pm u\in C_i\right\}$$

Let $X\widetilde{\otimes}_i Y$ denote the completion of $X\otimes Y$ with respect to $\|\cdot\|_i$. Then $X\widetilde{\otimes}_i Y$ with C_i as its positive cone is a Banach lattice (also see [19, Sect. 3.8]), called the Wittstock injective tensor product of X and Y. The projective cone on $X\otimes Y$ is defined to be

$$C_p=\left\{\sum_{i=1}^{n} x_i\otimes y_i : x_i\in X_+, y_i\in Y_+, n\in\mathbb{N}\right\}.$$

Fremlin [13, 14] introduced the positive projective tensor norm on $X\otimes Y$ as follows:

$$\|u\|_{|\pi|}=\sup\left\{\left|\sum_{i=1}^{n}\phi(x_i,y_i)\right| : u=\sum_{i=1}^{n} x_i\otimes y_i\in X\otimes Y, \phi\in M\right\}$$

where M is the set of all positive bilinear functional ϕ on $X\times Y$ with $\|\phi\|\leq 1$. Let $X\widehat{\otimes}_F Y$ denote the completion of $X\otimes Y$ with respect to $\|\cdot\|_{|\pi|}$. Then $X\widehat{\otimes}_F Y$ with C_p as its positive cone is a Banach lattice (also see [19, Sect. 3.8]), called the Fremlin projective tensor product of X and Y. Let p be real numbers such that $1<p<\infty$, then, due to [5, 17,] we have

(P_1) $\ell^u_{p,|\omega|}(X)$ is isometrically lattice isomorphic to $\ell_p\widetilde{\otimes}_i X$.

(P_2) $\ell^\pi_p(X)$ is isometrically lattice isomorphic to $\ell_p\widehat{\otimes}_F X$.

Let $\ell_p\widehat{\otimes}_\epsilon E$ and $\ell_p\widehat{\otimes}_\pi E$ denote the Grothendieck injective and projective tensor product of ℓ_p with a Banach space E, respectively (see Ryan [21]). It is well known that the space $\ell^u_{p,\omega}(E)$ is isometrically isomorphic to $\ell_p\widehat{\otimes}_\epsilon E$ whereas $\ell_p(X)$ is isometrically isomorphic to $\ell_p\otimes_{\Delta_p} X$.(see [**?**, 12.9] and $\ell_p\langle E\rangle$ is isometrically isomorphic to $\ell_p\widehat{\otimes}_\pi E$ (see [8, Proposition 2.2.5 and Proposition 2.2.6], [12, Corrolary 3.9] and [6]). Given a linear operator $T:X\to Y$, its associated tensor product operator $I\otimes T:\ell_p\otimes X\to\ell_p\otimes Y$ is defined by

$$I\otimes T(\sum_{i=1}^{n} e_i\otimes x_i):=\sum_{i=1}^{n} e_i\otimes T(x_i),$$

and this map is clearly linear.

We apply now Theorem 2.2 and (P_1) to the class of positive (p,q)-summing operators to get new characterizations in terms of tensor product transformations**.**

Corollary 5.1. *Let $1<p<\infty$ and $T\in\mathcal{L}(X,F)$. The following are equivalent:*

1. *T is positive (p,q)-summing operator.*
2. *The induced linear operator $I\otimes T:\ell_q\widetilde{\otimes}_i X\to\ell_p\widehat{\otimes}_\pi F$ is continuous.*

In this case $\|T\|_{\Lambda_{p,q}}=\|I\otimes T\|$.

According to (P_1) and Theorem 3.2, we obtain characterizations in terms of tensor product transformations for the class of positive strongly (p,q)-summing operators

Corollary 5.2. *Let $1<p\leq\infty$ and $T\in\mathcal{L}(E,Y)$. The following are equivalent:*

1. *T is positive strongly p-summing.*
2. *The induced linear operator $I\otimes T:\ell_p\widehat{\otimes}_{\Delta_p} E\to\ell_p\widehat{\otimes}_F Y$ is continuous.*

In this case $\|T\|_{\mathcal{D}^+_{p,q}}=\|I\otimes T\|$.

It is known from [8, Theorem 2.1.3] that $T\in\mathcal{L}(E,F)$ is Cohen p-nuclear if and only if the mapping $I\otimes T:\ell_p\widehat{\otimes}_\epsilon E\to\ell_p\widehat{\otimes}_\pi F$ is continuous. Utilizing Proposition 4.2, (P_1) and (P_2), we extend this result as follows.

Corollary 5.3. *Let $1 \leq q \leq p < \infty$ and X, Y be Banach lattices, E and F be Banach spaces.*

(a_1) $T \in \mathcal{L}(X, F)$ is left positive Cohen (p, q)-nuclear if and only if the mapping $I \otimes T : \ell_q \widetilde{\otimes}_i X \to \ell_p \widehat{\otimes}_\pi F$ is continuous.

(a_2) $T \in \mathcal{L}(E, Y)$ is positive right Cohen (p, q)- nuclear if and only if the mapping $I \otimes T : \ell_q \widehat{\otimes}_\epsilon E \to \ell_p \widehat{\otimes}_F Y$ is continuous.

$(a_3) = T \in \mathcal{L}(X, Y)$ is positive Cohen (p, q)- nuclear if and only if the mapping $I \otimes T : \ell_q \widetilde{\otimes}_i X \to \ell_p \widehat{\otimes}_F Y$ is continuous.

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Laboratory of Functional Analysis and Geometry of Spaces, Faculty of Mathematics and Computer Science. University Mohamed Boudiaf of M'sila-Algeria PB 166 Ichebilia, M'sila 28000, Algeria, ID: https://orcid.org/0000-0003-0916-862X

Email address: dahmane.achour@univ-msila.dz

Laboratory of Functional Analysis and Geometry of Spaces, Faculty of Mathematics and Computer Science. University Mohamed Boudiaf of M'sila-Algeria PB 166 Ichebilia, M'sila 28000, Algeria, ID: https://orcid.org/0009-0004-0040-5224

Email address: toufik.tiaiba@univ-msila.dz